Introduction to Physics

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1 Acceleration

So far we have mentioned this mystical property called *acceleration* and made a big deal out it. And it is a big deal, for it links cause and effect. But, we must understand it.

First, let us define it:

$$\vec{a} = \frac{\Delta \vec{v}}{\Delta t} \tag{1}$$

We recall that Δ means "change in." Therefore, the acceleration is defined according to the *change* in an object's velocity vector.

$$\vec{a} = \frac{\vec{v}_{\mathrm{f}} - \vec{v}_{\mathrm{o}}}{t} \tag{2}$$

In more general terms, the acceleration describes the *change* in the motion of a body (whether it changes speed or direction). (Notice that we have replaced Δt with t. We can do so as long as we remember that this time is the time it takes for the object to change its velocity vector from $\vec{v}_{\mathrm{o}}$ to $\vec{v}_{\mathrm{f}}$.)

Before moving on, let us solve for $\vec{v}_{\mathrm{f}}$ in the above equation:

$$\vec{a}t = \vec{v}_{\mathrm{f}} - \vec{v}_{\mathrm{o}} \quad \longrightarrow \quad \vec{a}t + \vec{v}_{\mathrm{o}} = \vec{v}_{\mathrm{f}}$$

But this is just one of our equations of motion. Therefore, one of our equations of motion that we have been referring comes directly from the definition of acceleration.

Recall that the velocity vector, like all vectors, is defined according to its length (which corresponds to the speed of the object) and its direction (which corresponds to the object's direction of travel). Therefore, an object is accelerating whenever

- it changes speed or
- it changes direction

Because we measure velocities in this class in meters per second, and the acceleration is defined as the change in acceleration per unit time, then we can see that acceleration is measured in meters per second per second, which is the same as meters per second squared or $\mathrm{m/s^2}$.

A falling object is a typical example of an accelerating object. When we release an object, it momentarily (at the very instant we release it) has no speed, so its velocity is 0. However, it will quickly gain velocity in the negative direction. For such falling bodies, the acceleration is roughly $9.8\ \mathrm{m/s^2}$, which we round to $10\ \mathrm{m/s^2}$ to make calculations easier. We call this special value of acceleration the *gravitational acceleration constant* and label it g. (Note that your online homework tool assumes $g = 9.8\ \mathrm{m/s^2}$, so do not use 10 for 9.8 when completing homework assignments.

Another way to think of the acceleration vector is to liken it to the growth of the velocity vector. When the velocity vector is getting bigger (the object is gaining speed), then the velocity vector is growing in the direction of the acceleration vector.

If the velocity vector is getting smaller (the object is slowing down), then the velocity vector is growing in the opposite direction it's pointing. In this case, the acceleration vector points in the *opposite* direction of the velocity vector.

In general,

- whenever an object is *speeding up*, its velocity vector and acceleration vector will point in the *same direction* and

- whenever an object is *slowing down*, its velocity vector and acceleration vector will point in opposite directions.

Figure 1 graphically displays the relationship between displacement, velocity, and acceleration for an object thrown vertically upward. In this case, the object will accelerate downward at $g = 10$ m/s^2. We see that the displacement changes rapidly at first (indicating a large velocity) but the change in displacement drops as the object climbs to his apex.

At the apex, the velocity of the object drops to 0 *but it is still accelerating!* Throughout its entire motion the object accelerates downwards at g.

After reaching the apex, the velocity of the object becomes negative, indicating that the object is traveling downwards.

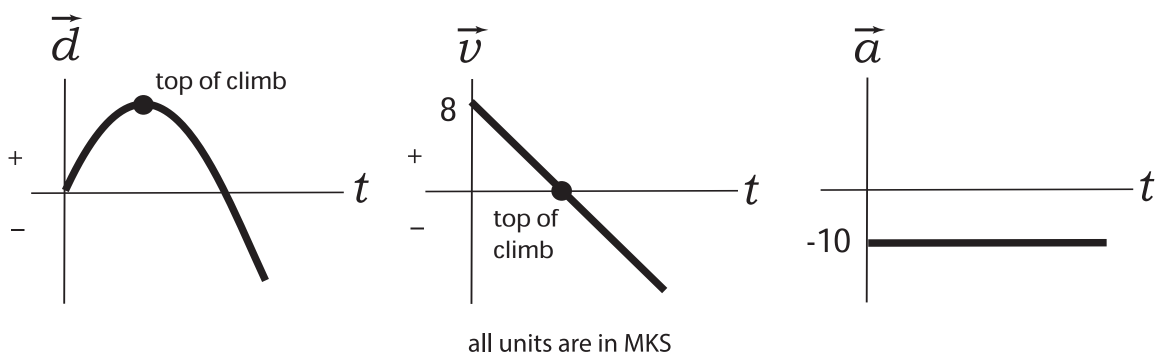

Figure 1: How displacement, velocity, and acceleration rate as functions of time. The velocity is the slope of the displacement vs time plot, and the acceleration is the slope of the velocity vs time plot.

Close examination of Figure 1 shows that

- At any point in time, the velocity of the object is the *slope* of the displacement versus time plot. That is, $\vec{v} = \frac{\Delta d}{\Delta t}$

- At any point in time, the acceleration of the object is the *slope* of the velocity versus time plot. That is, $\vec{a} = \frac{\Delta v}{\Delta t}$

The two conditions above fall naturally out of the definitions of these variables, with the numerator in the equation corresponding to the "rise" in the time plot and the denominator corresponding to the "run" of the time plot.

2 Unit 1: The Equations of Motion

Now that we have a reasonable handle on how to perform vector algebra, we can turn our attention to the tools we use to find the acceleration of the system.

We know that we have two types of tools at our disposal:

1. Equations of motion
2. Newton's second law

We will discuss Newton's second law later. For now, let's focus on the equations of motion. There are two:

$$\vec{d} = \vec{v}_{\mathrm{o}} t + \frac{1}{2}\vec{a}t^2$$

$$\vec{v}_{\mathrm{f}} = \vec{v}_{\mathrm{o}} + \vec{a}t$$

3 Linear motion

Now that we have addressed acceleration, we can begin to understand our equations of motion and how to use them to solve physics problems.

3.1 Displacement as a Function of Time

Let us focus on the first equation of motion, $\vec{d} = \vec{v}_{\mathrm{o}} t + \frac{1}{2}\vec{a}t^2$. For simplicity, we will assume that motion is constricted along one dimension. (We will discuss two-dimensional motion in a bit.) We know intuitively that if we travel at a constant velocity $\vec{v}$ for a time t, then our displacement at that time will be given by $\vec{d} = \vec{a}t$. We use this equation all the time. If we are traveling at 60 mph for 2 hours in a straight line, we know that we end up a distance of $(60)(2) = 120$ miles from our starting point during this time.

However, we made an assumption that our speed (and direction) were constant. On the other hand, suppose we started our journey at a speed of 60 mph but we were accelerating in the same direction of travel. In such a case, we realize that we would travel *farther* than 120 miles from our starting point. If we were accelerating in the opposite direction of travel (that is, decelerating, also commonly called "slowing down"), we know that we would travel much less than 120 miles.

So it appears that acceleration changes the distance we would normally travel. In vector-speak, the acceleration vector $\vec{a}$ shortens or lengthens the *displacement* $\vec{d}$ we would normally undergo if we were not accelerating.

But by how much?

Examining the first equation of motion, that is,

$$\vec{d} = \vec{v}_{\mathrm{o}} t + \frac{!}{2}\vec{a}t^2$$

we see that if $\vec{a} = 0$ then we end up with $\vec{d} = \vec{v}_{\mathrm{o}} t$, which is the usual distance-velocity-time equation we used for the simple case of constant velocity. Therefore, the $\frac{1}{2}\vec{a}t^2$ term must be the corrective factor for taking into account the effect of acceleration. And it is.

So at this point, we have a decent understanding of the equation of motion for displacement:

> For a body that is accelerating at a constant value $\vec{a}$, the displacement that takes place after a time t is given by summing (1) the displacement $\vec{v}_{\mathrm{o}} t$ that would normally take place without the acceleration and (2) the corrective favor $\frac{1}{2}\vec{a}t^2$ that takes into account acceleration.

Figure 2 illustrates the meaning of these terms.

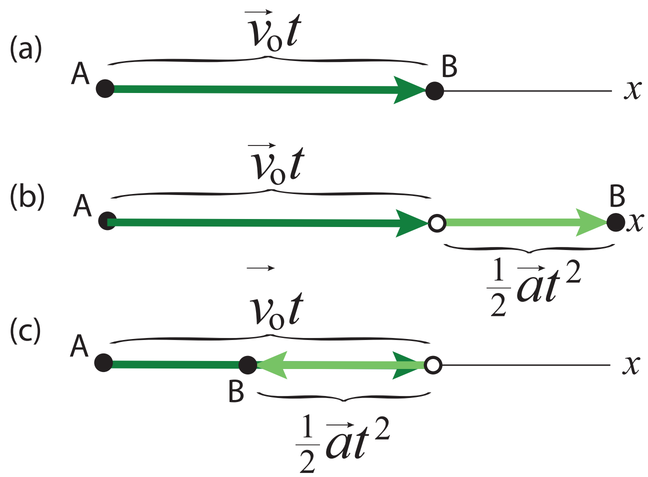

Figure 2: A visual representation of the $\vec{d} = \vec{v}_o t + \frac{1}{2}\vec{a}t^2$ equation of motion. The light green vectors correspond to the $\frac{1}{2}\vec{a}t$ corrections. In all three cases the resulting displacement $\vec{d}$ points from the starting point (Point A) to Point B.

3.1.1 Velocity equation of motion

What about the second equation of motion, $\vec{v}_f = \vec{v}_o + \vec{a}t$? This one relates the velocity of an object as it accelerates for a time t. According to this equation, if $\vec{a} = 0$ then $\vec{v}_f = \vec{v}_o$. This simply states that, for all time t, the final velocity remains the same as the initial velocity. However, since the acceleration $\vec{a}$ is defined as the change in the velocity per unit time, then this makes. If there is no acceleration, then the velocity does not change and $\vec{v}_f$ remains the same as $\vec{v}_o$.

However, if an object is acceleration, then its velocity vector *does* change. If the acceleration vector points in the same direction as the velocity vector (which means the object is speeding up), then we know that $\vec{v}_f$ will be longer than $\vec{v}_o$. On the flip side, if the acceleration vector is pointing in the *opposite* direction of the velocity vector (that is, the object is slowing down), then we know that $\vec{v}_f$ will be shorter than $\vec{v}_o$

By how much? By examining the equation of motion for velocity, we see that the $\vec{a}t$ term in the equation of motion is the corrective factor for finding the velocity of a body when it is accelerating.

In the next section, the meaning of these terms will become even more important when we examine two-dimensional motion. In the meantime, let us plow through a couple of examples to clarify how these equations of motion are used.

3.1.2 Example 1 – Throwing a ball

Suppose we throw a ball straight upwards and note that it takes 2 seconds to fall back to our hand. Assuming that the ball left our hand and returned to our hand at the same height above the floor, what can we say about the ball's motion?

This is a classic problem in linear motion and one for which you must understand. At first, it appears that we do not have enough information to solve this problem. But rather than fret about it, we move forward.

First, let us write down our equations of motion that govern the flight of the ball. Because the ball will be undergoing constant acceleration, then our equations of motion apply:

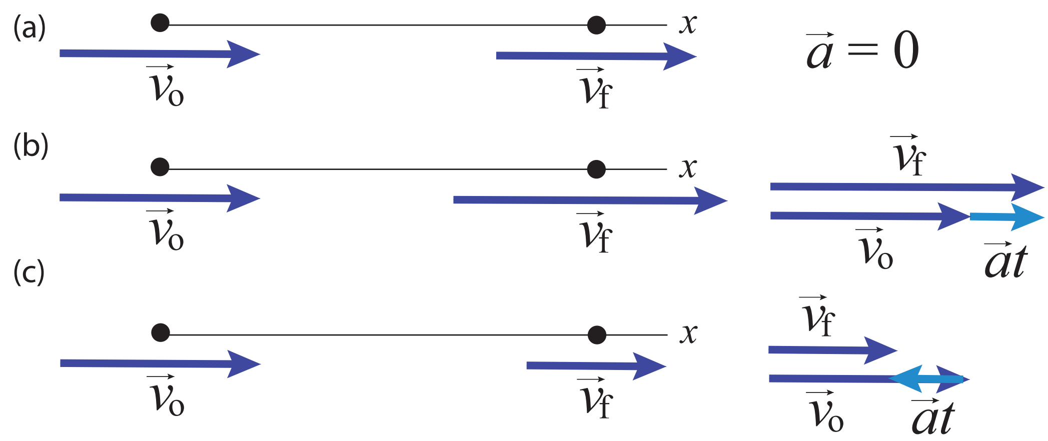

Figure 3: A visual representation of the $\vec{v}_\mathrm{f} = \vec{v}_\mathrm{o} + \vec{a}t$ equation of motion.

$$\vec{d} = \vec{v}_\mathrm{o}t + \frac{1}{2}\vec{a}t^2$$

$$\vec{v}_\mathrm{f} = \vec{v}_\mathrm{o} + \vec{a}t$$

In this call, all of the vectors in the above equations point along the vertical, so this is a one-dimensional problem. Because the ball is in free-fall (that is, only the force of gravity acts on it), then we know its acceleration; it is 10 m/s^2 *downwards*. What else do we know? Well, we said the time of flight was $t = 3$ seconds. We also know that the ball returns *to its same vertical position* from which it started. Therefore, its displacement $\vec{d} = 0$. (Make sure you understand this last point.) Let us see what our equations of motion tell us so far:

$$0 = \vec{v}_\mathrm{o}(2) + \frac{1}{2}(-10)(2)^2$$

$$\vec{v}_\mathrm{f} = \vec{v}_\mathrm{o} + (-10)(2)$$

Notice that I substituted -10 for the acceleration. The "10" part is the amount of acceleration; the negative sign is the direction.

Interestingly enough, the first equation simplifies to

$$0 = 2\vec{v}_\mathrm{o} - 20$$

$$\vec{v}_\mathrm{f} = \vec{v}_\mathrm{o} - 20$$

We can solve the first equation immediately for $\vec{v}_\mathrm{o}$, that is, $\vec{v}_\mathrm{o} = 10$ meters per second. The answer comes out positive, indicating that the original velocity pointed upward, which makes sense.

Now that we know $\vec{v}_\mathrm{o} = 10$, the second equation now reveals the final velocity in which the ball returns to our hand:

$$\vec{v}_\mathrm{f} = 10 - 20$$

Therefore, $\vec{v}_\mathrm{f} = -10$ m/s. Therefore, the ball will return at the same speed at which it left. (However, we assumed that the force of gravity was the only force acting on the ball and, therefore, that its acceleration was $g = 10$ m/s62. This assumption is not realistic, especially if the object has a large surface area.)

3.1.3 Example 2 – Braking car

Suppose a car traveling at 30 m/s^2 comes to stop in a distance of 90 meters. If we assume its acceleration is constant, how long did it take the car to come to stop?

Again, we are assuming a constant acceleration, so we can turn to our equations of motion:

$$\vec{d} = \vec{v}_\text{o} t + \frac{1}{2}\vec{a}t^2$$

$$\vec{v}_\text{f} = \vec{v}_\text{o} + \vec{a}t$$

We know a few things about this problem. If we assume that the car is traveling in the positive direction throughout its motion, then its initial (original) velocity $\vec{v}_\text{o} = 30$ m/s in the positive direction. Also, we know its final velocity $\vec{v}_\text{f} = 0$.

$$90 = 30t + \frac{1}{2}\vec{a}t^2$$

$$0 = 30 + \vec{a}t$$

This looks hopeless at first, but we recognize that we have two equations here with two unknowns ($\vec{a}$ and t). With two equations and two unknowns, we should be able to solve the problem. Solving first for $\vec{a}$ in the second equation we get

$$\vec{a} = \frac{-30}{t}$$

Since time is only positive, then this means our acceleration must be negative. This makes sense because we know that the acceleration vector must point in the opposite direction of our velocity vector for the car to slow down (decelerate).

Now, we substitute our result for $\vec{a}$ into the first equation:

$$90 = 30t + \frac{1}{2}\frac{-30}{t}t^2.$$

Therefore, $90 = 15t$, which means the time it takes to stop is $t = 4$ seconds.

We could have solved this equation easier by referring to another equation of motion:

$$\vec{v}_\text{f}^2 = \vec{v}_\text{o}^2 + 2ad.$$

This equation shows up repeatedly in text books. However, I try to avoid this equation as the *ad* term is actually a product of vectors, and we do not multiply vectors in this course. This equation will work in special cases but it wil also blow up in your face if you are not careful with its usage. I prefer to use the two equations of motion for which we are already familiar. These two equations will handle any problem involving constant acceleration thrown at them.

4 Two-dimensional motion

Now that we have discussed linear motion, let us turn to two-dimensional motion. Although there are numerous examples of two-dimensional motion in the real world, we will concentrate on a specific example called *projectile motion.*

First, let us define projectile motion:

> Projectile motion is a special example of two-dimensional motion where an object travels through space in free fall – that is, the only force acting on it is gravity.

When an object is undergoing projectile motion, its acceleration is constant in both amount and direction. The amount of acceleration is $a = 9.8\ \mathrm{m/s^2}$. The direction is toward the center of Earth, which for the vast majority of problems we encounter means *straight down*.

If we fire a cannonball into the air, its motion will be roughly projectile motion. The cannonball will be acted upon by air friction (a force) and gravity (another force). However, if we can ignore air friction, then we can consider the only force acting on it as gravity. Therefore, the cannonball will accelerate downward at all points along its path of travel.

When an object is undergoing projectile motion, it will travel along a path that is quadratic in nature, as shown by the heavy dashed line in Figure 4. Notice that the displacement and velocity vectors for the object do not necessarily line up along the horizontal or vertical. (The velocity vector will line up horizontally at the very top of the climb, but this only happens momentarily.)

By the way, an object falling in the absence of air friction is also in free-fall. Technically, we could say that the object is undergoing projectile motion. However, when we discuss projectile motion we usually refer to the more interesting case where the velocity and displacement of the object are not linear.

4.1 Displacement as a Function of Time

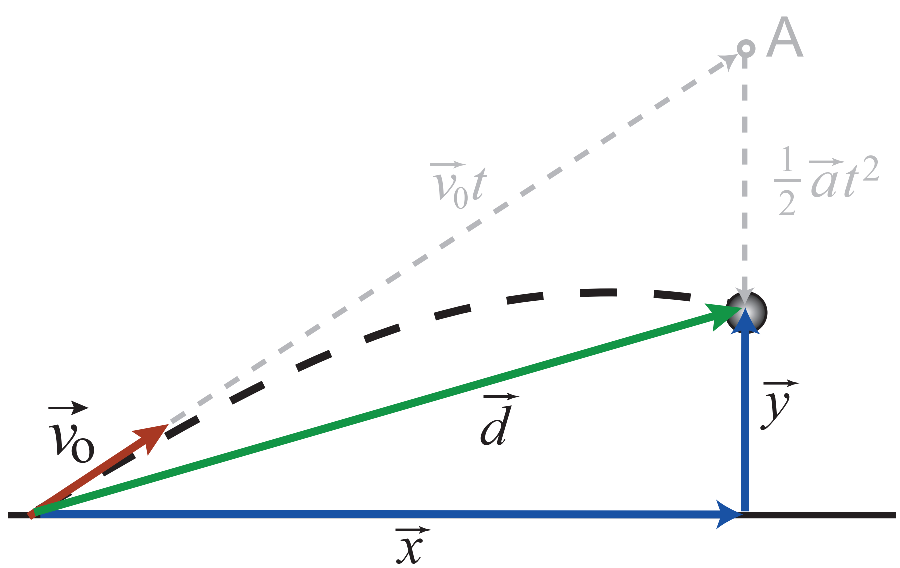

Figure 4: A typical projectile motion example. Note how the displacement and velocity vectors are illustrated here, with the displacement vector pointing from the start of the motion to the object's position and the velocity vector pointing along the object's direction of travel. Note also that the length of the velocity vector (the speed of the object) has changed.

We can see from Figure 4 that our displacement equation of motion describes the displacement of the object for all time, and that the $\vec{v}_o t$ and $\frac{1}{2}\vec{a}t^2$ terms retain their original meaning. If the object somehow did not accelerate, it would travel in a straight line along the gray dashed line, ending up at Point A at time t.

However, the object *is* accelerating downwards, so we expect the object to fall away from the gray dashed line. Its resulting motion will follow the heavy dashed line. Note that the distance from Point A (where the object would have ended up if it has not been acceleration) is above its actual location by an amount $\frac{1}{2}\vec{a}t^2$. In vector language, summing the displacement vector $\vec{v}_o t$ for no acceleration with the $\frac{1}{2}\vec{a}t^2$ correction for acceleration produces the net displacement of the object. In other words,

$$\vec{d} = \vec{v}_o t + \frac{1}{2}\vec{a}t^2,$$

which is our equation of motion for displacement.

4.2 Velocity as a Function of Time

Let us now turn our attention to the velocity equation of motion:

$$\vec{v}_f = \vec{v}_o + \vec{a}t \tag{3}$$

If the object is not accelerating, then $\vec{a} = 0$ and we get $\vec{v}_f = \vec{v}_o$, which means that for all time the velocity at that time is the same as the original velocity, which is exactly what zero acceleration means.

But when an object is accelerating, we know that the velocity changes with time. Examining the velocity equation of motion, we note that the $\vec{a}t$ term acts like a corrective factor. It takes the original velocity vector $\vec{v}_o$ and adjusts it.

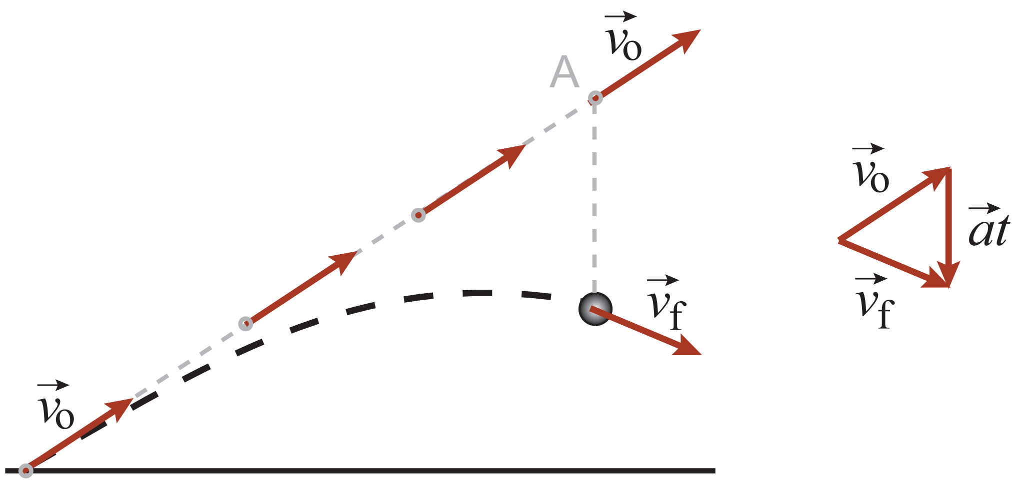

Figure 5: The equation of motion for velocity, $\vec{v}_f = \vec{v}_o + \vec{a}t$ also makes sense in two dimensions, with $\vec{a}t$ acting as the correcting term to the velocity vector. Because $\vec{v}_f$ points in the direction of travel for all time, we can see that the $\vec{a}t$ term not only accounts for the change in speed of the object (that is, the length of the velocity vector) but also the change in direction of the object (that is, the direction the velocity vector points).

At this point, you are probably desperate for some examples to help understand how these terms in the equations of motion help us solve two-dimensional motion problems. Okay, let's do a few.

4.2.1 Example 3 – Thelma and Louise

We will start with a prototypical projectile motion problem inspired by a film called *Thelma and Louise*. In this movie, two women drive a classic 1965 Ford Thunderbird off a cliff. If we assume they are driving at 60 m/s (roughly 120 mph) when they leave the road, how high was the cliff if it takes them 2 seconds to hit the ground? Also, at what speed and in which direction did they hit the ground?

We can visualize this problem as in Figure 6.[1]

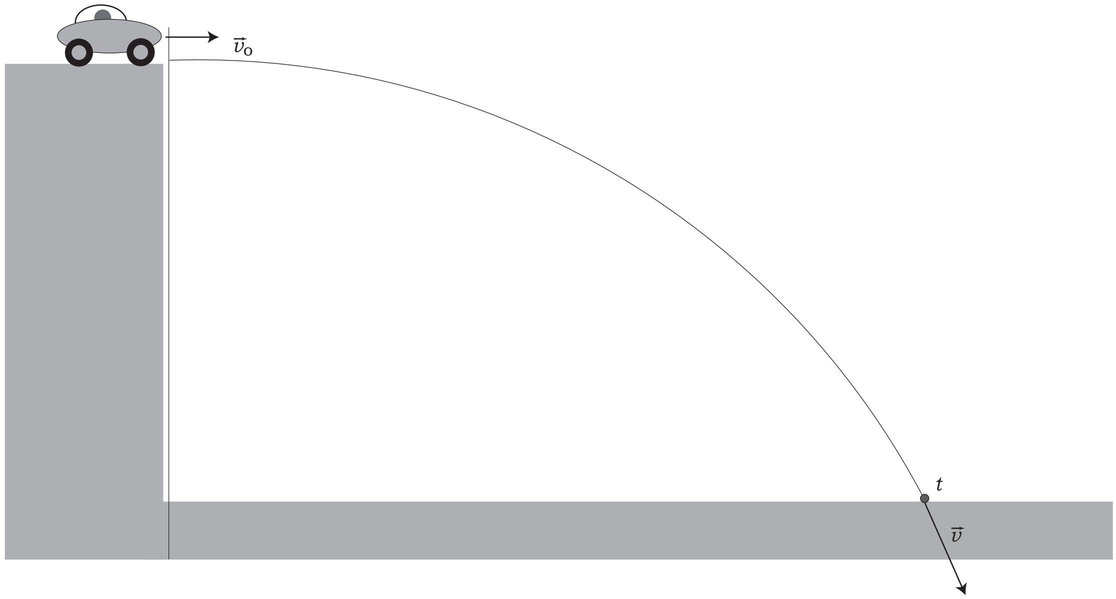

Figure 6: The *Thelma and Louise* problem, where a car is driven off a cliff.

This problem, which I have featured on numerous exams, looks imposing but is rather straightforward if you don't panic. As always, we begin with the equations of motion. (Write them down and you will score a couple of points for at least knowing these equations apply.)

$$\vec{d} = \vec{v}_\mathrm{o} t + \frac{1}{2}\vec{a}t^2$$

$$\vec{v}_\mathrm{f} = \vec{v}_\mathrm{o} + \vec{a}t$$

We can see what these variables refer by examining our previous figure, but illustrated with all the vectors in the above equations.

In the above figure, we note that the displacement vector $\vec{d}$ and final velocity vector $\vec{v}_\mathrm{f}$ are bad vectors. (They don't point along the horizontal or vertical). Therefore, we need to replace them with good vectors.

Both equations of motion are vector equations, so they apply in any direction we wish. Let us write them for the horizontal and vertical directions:

$$\vec{d}_\mathrm{x} = \vec{v}_\mathrm{ox} t + \frac{1}{2}\vec{a}_\mathrm{x} t^2$$

$$\vec{v}_\mathrm{fx} = \vec{v}_\mathrm{ox} + \vec{a}_\mathrm{x} t$$

$$\vec{d}_\mathrm{y} = \vec{v}_\mathrm{oy} t + \frac{1}{2}\vec{a}_\mathrm{y} t^2$$

$$\vec{v}_\mathrm{fy} = \vec{v}_\mathrm{oy} + \vec{a}_\mathrm{y} t$$

We already know the values of some of these variables. For one, we know the acceleration vector points down and has a magnitude of 10 m/s^2. We also know that the initial (original) velocity $\vec{v}_\mathrm{o}$ points along the horizontal direction. The vertical component of the displacement $\vec{d}_\mathrm{y}$ is simply the

[1]Do I look like an artist?

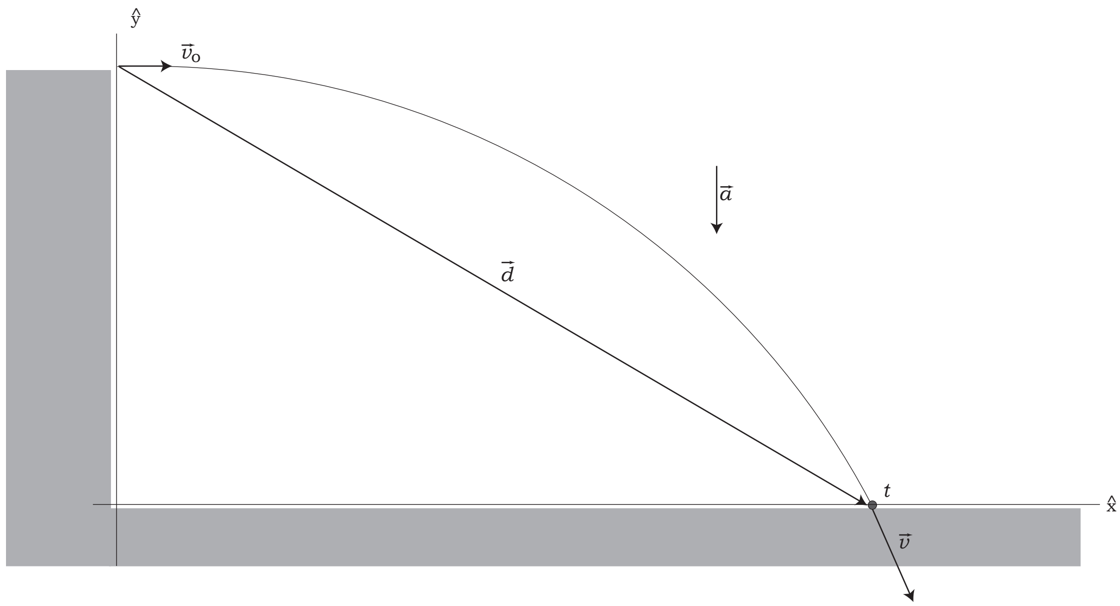

Figure 7: The *Thelma and Louise* problem, illustrated with the vectors involved in the equations of motion.

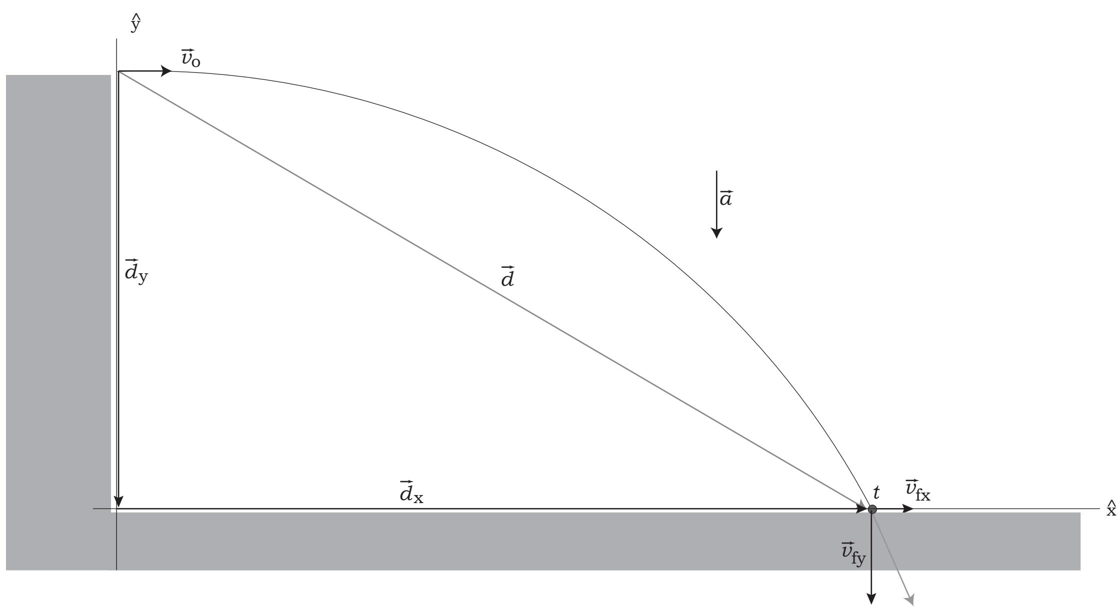

Figure 8: The *Thelma and Louise* problem, with the horizontal and vertical component vectors shown. Note that these horizontal and vertical (good) vectors sum to form the original vector. Be sure to verify this in your own mind.

height of the cliff and $\vec{d}_x$ is just the horizontal displacement from the cliff in which Thelma and Louise hit the ground.

Using our good vectors, we have for the equations of motion:

$$\vec{d}_x = (0)(2) + \frac{1}{2}(-10)(2)^2$$

$$\vec{d}_y = (0)(2) + \frac{1}{2}(-10)(2)^2$$

$$\vec{v}_{\mathrm{fx}} = \vec{v}_{\mathrm{ox}} + (0)(2)$$

$$\vec{v}_{\mathrm{fy}} = 0 + (-10)(2)$$

The first equation simply tells us that the horizontal displacement is $\vec{d}_{\mathrm{x}} = 120$ meters. This makes sense. Because there is no acceleration in the horizontal direction, then the original horizontal speed of the car (60 m/s) is maintained throughout the flight. Since the flight only lasts 2 seconds, we can easily see that this horizontal distance is 120 meters.

The second equation is for computing the vertical displacement $\vec{d}_{\mathrm{x}}$. We see that

$$\vec{d}_{\mathrm{y}} = -20$$

Therefore, the top of the cliff is elevated 20 meters above the landing spot, so the cliff height must be 20 meters.

What about the speed of the car and its direction of travel when it hits the ground? Both properties refer to the final velocity vector $\vec{v}_{\mathrm{f}}$; the speed of the car corresponds to the length of the final velocity vector and the direction the car is traveling refers to the direction of the final velocity vector. So finding this final velocity vector is important. We go back to our equations of motion written for the horizontal and vertical directions, which simplify to

$$\vec{v}_{\mathrm{fx}} = 60 \mathrm{and}$$

$$\vec{v}_{\mathrm{fy}} = -20$$

To find the speed, we recognize that these are the components of the final velocity vector, so the length of the final velocity vector is given by the Pythagorean theorem:

$$\vec{v}_{\mathrm{f}}^2 = \sqrt{60^2 + (-20)^2} \longrightarrow \vec{v}_{\mathrm{f}} = 63$$

Therefore, our car will hit the ground at 63 m/s.

In which direction will the car hit the ground? To answer that question, we need to find the direction $\vec{v}_{\mathrm{f}}$ points, which we define as θ in the figure below. This angle is found by applying the arctangent function:

$$\theta = \arctan\left(\frac{20}{60}\right) = 18.4^\circ$$

Therefore, when the car hits the ground, it will be traveling 18.4° below the horizontal. You will notice that the lengths of horizontal and vertical components of $\vec{v}_{\mathrm{f}}$ in the figure are way off. I will fix that later.

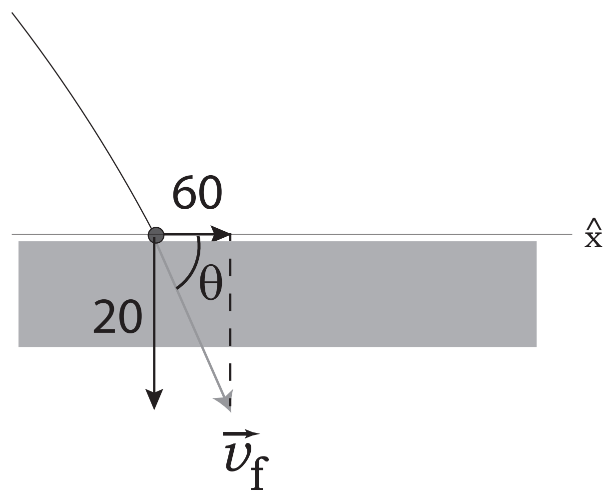

Figure 9: The final velocity in the *Thelma and Louise* problem. Note that we define the direction of travel with respect to the horizontal. The choice is yours as long as it is clear which angle you are calculating.