1. Find $1 + (f'(x))^2$ for the following functions $f(x)$:

   (a) $f(x) = 2e^x + \frac{1}{8}e^{-x}$

   $f'(x) = 2e^{x} - \frac{1}{8}e^{-x}$

   $1 + (f'(x))^2 = 1 + (2e^{x} - \frac{1}{8}e^{-x})^2$

   (b) $f(x) = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}$

   $f'(x) = x^{\frac{1}{2}} - \frac{1}{4}x^{-\frac{1}{2}}$

   $1 + (f'(x))^2 = 1 + (x^{\frac{1}{2}} - \frac{1}{4}x^{-\frac{1}{2}})^2$

2. Find $1 + (g'(y))^2$ for the following functions $g(y)$:

   (a) $g(y) = 3y^{4/3} - \frac{3}{32}y^{2/3}$

   $g'(y) = 4y^{\frac{1}{3}} - \frac{\cancel{3}}{\underset{16}{\cancel{32}}} \cdot \frac{\cancel{2}}{\cancel{3}} y^{-\frac{1}{3}}$

   $1 + (g'(y))^2 = 1 + (4y^{\frac{1}{3}} - \frac{1}{16}y^{-\frac{1}{3}})^2$

   (b) $g(y) = \frac{(y+2)^{3/2}}{3}$

   $g'(y) = \frac{1}{2}(y+2)^{\frac{1}{2}}$

   $1 + (g'(y))^2 = 1 + \frac{1}{4}(y+2)$

3. Simplify $\sqrt{1+(f'(x))^2}$ for each of the functions in #1. Simplify $\sqrt{1+(g'(y))^2}$ for each of the functions in #2. What do you notice?

1(a) $1+(f'(x))^2 = 1+(2e^x - \frac{1}{8}e^{-x})^2$

$= 1 + (4e^{2x} - 2\cdot\frac{1}{4}e^x e^{-x} + \frac{1}{64}e^{-2x})$

$= 1 + 4e^{2x} - \frac{1}{2} + \frac{1}{64}e^{-2x}$

$= 4e^{2x} + \frac{1}{2} + \frac{1}{64}e^{-2x}$

$= (2e^x + \frac{1}{8}e^{-x})^2$

minus turns into plus!

So $\sqrt{1+(f'(x))^2} = \boxed{2e^x + \frac{1}{8}e^{-x}}$

1(b) $1+(f'(x))^2 = 1 + (x^{\frac{1}{2}} - \frac{1}{4}x^{-\frac{1}{2}})^2$

$= 1 + (x - \frac{1}{2} + \frac{1}{16}x^{-1})$

$= x + \frac{1}{2} + \frac{1}{16}x^{-1} = (x^{\frac{1}{2}} + \frac{1}{4}x^{-\frac{1}{2}})^2$

So $\sqrt{1+(f'(x))^2} = \boxed{x^{\frac{1}{2}} + \frac{1}{4}x^{-\frac{1}{2}}}$

2(a) $1+(g'(y))^2 = 1 + (4y^{\frac{1}{3}} - \frac{1}{16}y^{-\frac{1}{3}})^2$

$= 1 + 16y^{2/3} - \frac{1}{2} + \frac{1}{256}y^{-2/3}$

$= 16y^{2/3} + \frac{1}{2} + \frac{1}{256}y^{-2/3} = (4y^{\frac{1}{3}} + \frac{1}{16}y^{-\frac{1}{3}})^2$

So $\sqrt{1+(g'(y))^2} = \boxed{4y^{\frac{1}{3}} + \frac{1}{16}y^{-\frac{1}{3}}}$

2(b) $1+(g'(y))^2 = 1 + \frac{1}{4}(y+2) = \frac{1}{4}y + \frac{3}{4} = \frac{1}{4}(y+3)$

So $\sqrt{1+(g'(y))^2} = \boxed{\frac{1}{2}\sqrt{y+3}}$ (does not follow pattern above)

4. (**) Find the length of the curve $g(y) = 3y^{4/3} - \frac{3}{32}y^{2/3}$ from $y = 1$ to $y = 8$. Simplify your answer.

From #2(a) and #3, $\sqrt{1+(g'(y))^2} = 4y^{\frac{1}{3}} + \frac{1}{16}y^{-\frac{1}{3}}$, so

$$L = \int_1^8 4y^{\frac{1}{3}} + \frac{1}{16}y^{-\frac{1}{3}}\,dy = 3y^{4/3} + \frac{3}{32}y^{2/3}\Big|_1^8$$

$$= 3\cdot 8^{4/3} + \frac{3}{32}\cdot 8^{2/3} - \left(3\cdot 1^{4/3} + \frac{3}{32}\cdot 1^{2/3}\right)$$

$$= 3\cdot 16 + \frac{3}{32}\cdot 4 - \left(3 + \frac{3}{32}\right)$$

$$= 48 + \frac{12}{32} - 3 - \frac{3}{32} = 45 + \frac{9}{32} = \boxed{\frac{1449}{32}}$$

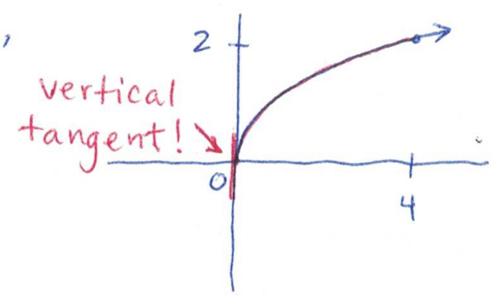
... Compare with original function $g(y)$. What do you notice?

5. (**) Consider the arc of the curve $y = \sqrt{x}$ from $x = 0$ to $x = 4$.

(a) What happens when you try to find the length of the curve? What's going on?

$y' = \frac{1}{2\sqrt{x}}$ is not defined at $x=0$,

So $\int_0^4 \sqrt{1+\frac{1}{4x}}\,dx$ is not a proper integral.



(b) Set up an integral for the length of the curve in terms of y. Is this a valid integral?

$y = \sqrt{x}$

$x = y^2 \quad (0 \le y \le 2)$

$x' = 2y$ is defined for all y on $[0,2]$, so

$$L = \boxed{\int_0^2 \sqrt{1+4y^2}\,dy}$$

6. (**) Suppose you know that the arc length of a certain smooth function $f(x)$ from $x = 0$ to $x = 2\pi$ is

$$L = \int_0^{2\pi} \sqrt{1 + 36\sin^2(2x)}\, dx.$$

What can we say about $f(x)$?

$$L = \int_0^{2\pi} \sqrt{1 + (f'(x))^2}\, dx, \text{ so}$$

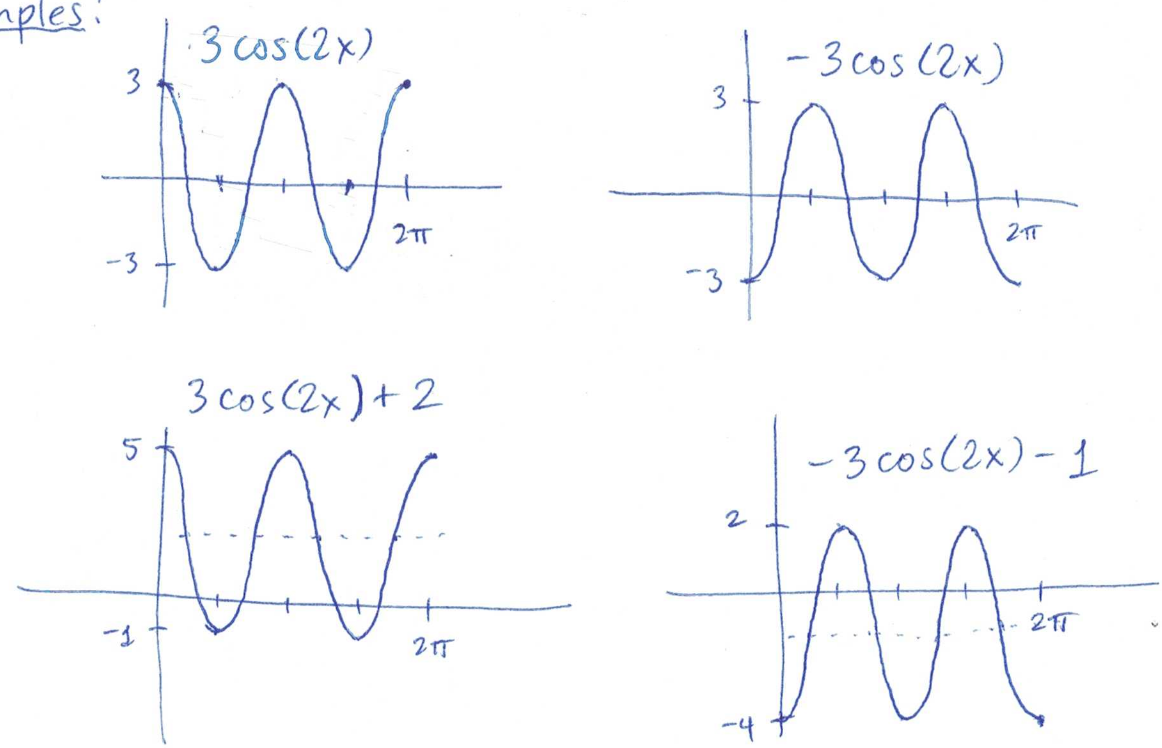
$$(f'(x))^2 = 36\sin^2(2x).$$

$$\text{So } f'(x) = \pm 6\sin(2x)$$

$$\text{So } f(x) = \boxed{\mp 3\cos(2x) + C}$$

$$\int 6\sin(2x)\,dx = -3\cos(2x) + C.$$

Examples:



Notice that they all have the same length!