Tyler Gillette

## Spring 2021 MATH 76 Activity 9

## SEQUENCES AND SERIES

You will need a calculator to complete this activity. You can use your phone if needed.

# 1. Sequences

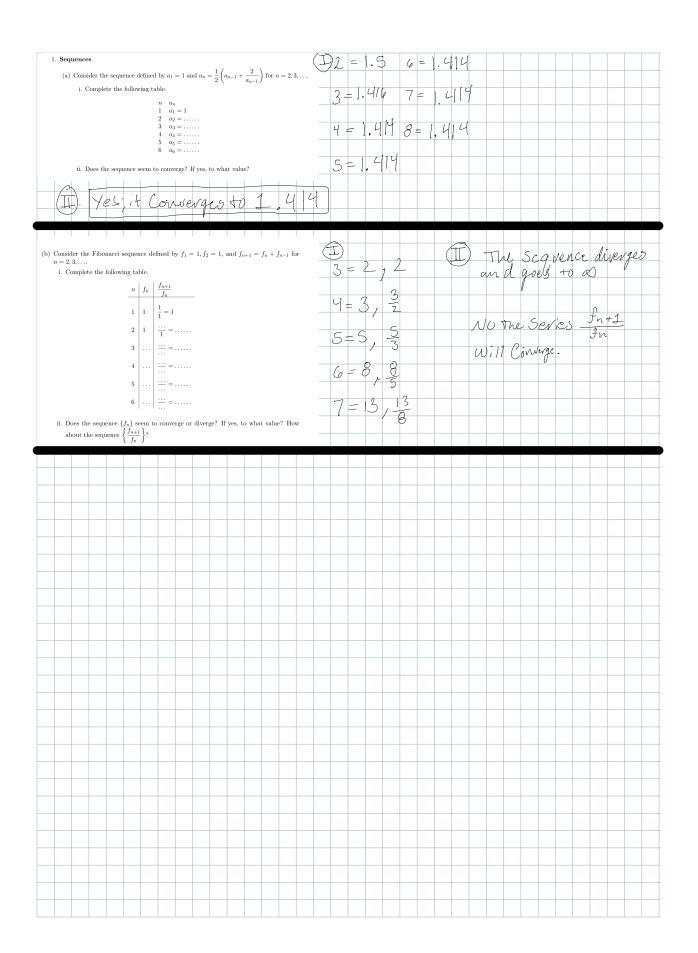
- (a) Consider the sequence defined by  $a_1 = 1$  and  $a_n = \frac{1}{2} \left( a_{n-1} + \frac{2}{a_{n-1}} \right)$  for  $n = 2, 3, \dots$ 
  - i. Complete the following table.

$$n$$
  $a_n$   
 $1$   $a_1 = 1$   
 $2$   $a_2 = \dots$   
 $3$   $a_3 = \dots$   
 $4$   $a_4 = \dots$   
 $5$   $a_5 = \dots$   
 $6$   $a_6 = \dots$ 

- ii. Does the sequence seem to converge? If yes, to what value?
- (b) Consider the Fibonacci sequence defined by  $f_1 = 1$ ,  $f_2 = 1$ , and  $f_{n+1} = f_n + f_{n-1}$  for  $n = 2, 3, \ldots$ 
  - i. Complete the following table.

$$\begin{array}{c|cccc}
n & f_n & \frac{f_{n+1}}{f_n} \\
\hline
1 & 1 & \frac{1}{1} = 1 \\
2 & 1 & \frac{\cdots}{1} = \cdots \\
3 & \cdots & \frac{\cdots}{\cdots} = \cdots \\
4 & \cdots & \frac{\cdots}{\cdots} = \cdots \\
5 & \cdots & \frac{\cdots}{\cdots} = \cdots \\
6 & \cdots & \frac{\cdots}{\cdots} = \cdots
\end{array}$$

ii. Does the sequence  $\{f_n\}$  seem to converge or diverge? If yes, to what value? How about the sequence  $\left\{\frac{f_{n+1}}{f_n}\right\}$ ?



(c) In addition to the techniques of computing limits that you have previously learned, there are results like the **Squeeze theorem** that are useful to find limits.

Compute the limit (if it exists) of the following sequences as n approaches  $\infty$ .

i. 
$$a_n = \frac{(-1)^n}{2^n}$$

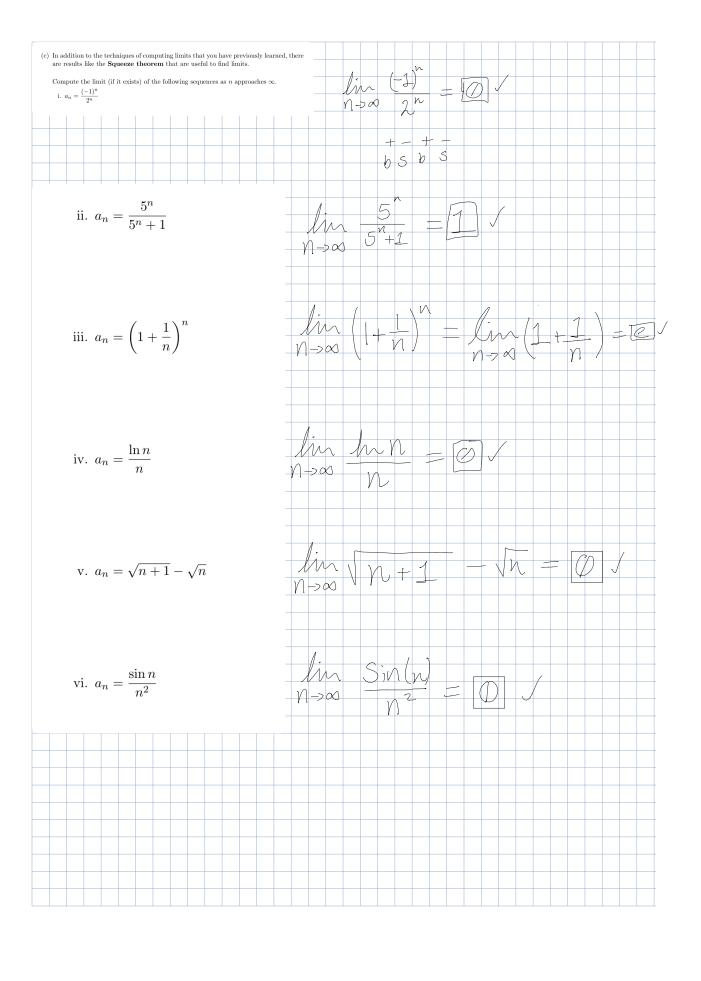
ii. 
$$a_n = \frac{5^n}{5^n + 1}$$

iii. 
$$a_n = \left(1 + \frac{1}{n}\right)^n$$

iv. 
$$a_n = \frac{\ln n}{n}$$

v. 
$$a_n = \sqrt{n+1} - \sqrt{n}$$

vi. 
$$a_n = \frac{\sin n}{n^2}$$



#### 2. Series

## Understanding the infinite series "process"

Start the "process" with a sequence  $\{x_k\} = \{x_1, x_2, x_3, \ldots\}$ , and compute the partial sums

$$S_1 = x_1$$

$$S_2 = x_1 + x_2$$

$$\vdots = \vdots$$

$$S_n = x_1 + x_2 + \ldots + x_n = \sum_{k=1}^{n} x_k$$

The numbers  $S_1, S_2, \ldots, S_n$  form a sequence and the limit of that sequence,  $\lim_{n \to \infty} S_n = \sum_{k=1}^{\infty} x_k$ ,

is the infinite series  $\sum_{k=1}^{\infty} x_k$ . If  $\lim_{n\to\infty} S_n$  exists and is a real number then the infinite series

$$\sum_{k=1}^{\infty} x_k$$
 converges. Otherwise it diverges.

Now finding the limit  $\lim_{n\to\infty} S_n$  is not trivial if  $S_n$  is cannot be expressed as a function of n. There are a few cases when this is possible. For example

• 
$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$
 when  $r \neq 0, 1$ 

• when the sum telescopes 
$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$$

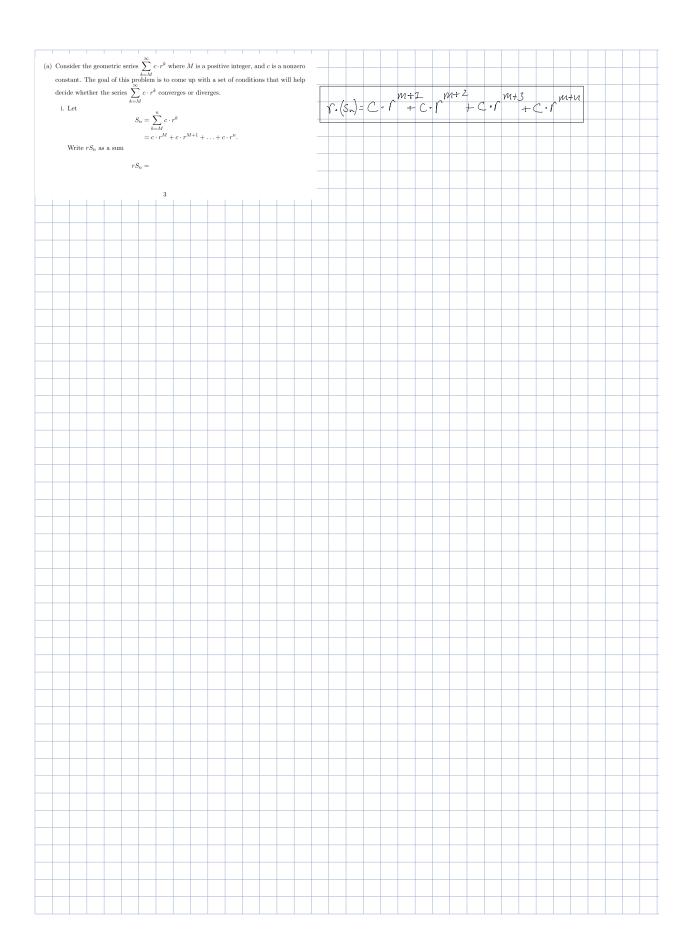
When it is not trivial to express  $S_n$  as a function of n, which is often the case, **a convergence** test will be used to determine if the infinite series  $\sum_{k=1}^{\infty} x_k$  converges or diverges.

- (a) Consider the geometric series  $\sum_{k=M}^{\infty} c \cdot r^k$  where M is a positive integer, and c is a nonzero constant. The goal of this problem is to come up with a set of conditions that will help decide whether the series  $\sum_{k=M}^{\infty} c \cdot r^k$  converges or diverges.
  - i. Let

$$S_n = \sum_{k=M}^n c \cdot r^k$$
$$= c \cdot r^M + c \cdot r^{M+1} + \dots + c \cdot r^n.$$

Write  $rS_n$  as a sum

$$rS_n =$$



ii. Write  $S_n - rS_n$  as a sum

$$S_n - rS_n =$$

- iii. Using your result in part ii., what is a formula for  $S_n$ ?
- iv. Next we want to evaluate  $\lim_{n\to\infty} S_n$ . Compute
  - A.  $\lim_{n\to\infty} r^{n+1}$  when |r|<1. Choose a value of r that satisfies the condition |r|<1 and test the limit
  - B.  $\lim_{n\to\infty} r^{n+1}$  when |r|>1. Choose a value of r that satisfies the condition |r|>1 and test the limit. Note that if r=1, the formula found in part iii. does not hold and if r=-1 the limit  $\lim_{n\to\infty} r^{n+1}$  does not exist.
  - C.  $\lim_{n\to\infty} S_n$

- v. Write the conditions under which the infinite series  $\sum_{k=M}^{\infty} c \cdot r^k$  converges. To what limit does it converge to?
- vi. Write the conditions under which the infinite series  $\sum_{k=M}^{\infty} c \cdot r^k$  diverges.

