Math 76 Exercises – 5.4 Comparison Tests

Determine, if possible, whether each of the following series converges or diverges. If you cannot apply any comparison tests, explain why not.

1. $\sum_{n=1}^{\infty} \frac{3}{1+n^2}$ Let $a_n = \frac{3}{1+n^2}$ and let $b_n = \frac{3}{n^2}$.

Since $0 \le a_n \le b_n$ for all $n \geq 1$ and $\sum b_n$ converges, the series $\sum a_n$ <u>converges</u> by the (direct) comparison test.

2. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2} + n^{4/5}}$. Let $a_n = \frac{1}{n^{3/2} + n^{4/5}}$ and $b_n = \frac{1}{n^{3/2}}$. $\frac{3}{2} > \frac{4}{5}$

We have $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n^{3/2}+n^{4/5}} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{n^{3/2}}{n^{3/2}} = 1$, which is finite and positive, so $\sum a_n$ and $\sum b_n$ are comparable. Since $\sum b_n$ is a p-series with $p = 3/2 > 1$, $\sum b_n$ converges. Thus $\sum a_n$ <u>converges</u> by the Limit Comparison Test.

3. $\sum_{n=4}^{\infty} \frac{7n-15}{n(n-3)} = \sum_{n=4}^{\infty} \frac{7n-15}{n^2-3n}$.

Let $a_n = \frac{7n-15}{n^2-3n}$ and $b_n = \frac{1}{n}$. (Dominant terms of a_n are $7n$ and n^2; $\frac{n}{n^2} = \frac{1}{n}$)

We have $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{7n-15}{n^2-3n} \cdot \frac{n}{1} = 7$, which is finite and positive, so the series are comparable. Since $\sum b_n$ diverges, so does $\sum a_n$, by the L.C.T.

So $\sum_{n=4}^{\infty} \frac{7n-15}{n(n-3)}$ <u>diverges</u>.

4. $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

$\frac{\cos(n)}{n^2}$ is not positive for all n; thus the Comparison Tests do not apply.

5. $\sum_{n=2}^{\infty} \frac{4^n}{5^n - n^2 + 2}$

Let $a_n = \frac{4^n}{5^n - n^2 + 2}$, $b_n = \frac{4^n}{5^n} = \left(\frac{4}{5}\right)^n$.

We have $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\cancel{4^n}}{5^n - n^2 + 2} \cdot \frac{5^n}{\cancel{4^n}} = \lim_{n\to\infty} \frac{5^n}{5^n} = 1$, so the series are comparable. Since $\sum b_n$ is a geometric series with $|r| = \frac{4}{5} < 1$, $\sum b_n$ converges. Therefore $\sum a_n$ converges.

6. $\sum_{n=0}^{\infty} \frac{4n-7}{\sqrt[3]{5n^{11} - n + 8}}$

Let $a_n = \frac{4n-7}{\sqrt[3]{5n^{11} - n + 8}}$. If we cross out the non-dominant terms of a_n we get $\frac{4n}{\sqrt[3]{5n^{11}}}$. So let $b_n = \frac{n}{\sqrt[3]{n^{11}}}$.

We have $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4n-7}{\sqrt[3]{5n^{11} - n + 8}} \cdot \frac{\sqrt[3]{n^{11}}}{n} = \lim_{n\to\infty} \frac{4n}{\sqrt[3]{5n^{11}}} \cdot \frac{\sqrt[3]{n^{11}}}{n} = \frac{4}{\sqrt[3]{5}}$ which is finite and positive, so the series are comparable. $\sum b_n = \sum \frac{n}{n^{11/3}} = \sum \frac{1}{n^{8/3}}$ converges, so $\sum a_n$ converges.

7. $\sum_{n=1}^{\infty} \frac{14}{3^n - 2^n}$

Let $a_n = \frac{14}{3^n - 2^n}$, and let $b_n = \frac{1}{3^n}$. We have

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{14}{3^n - 2^n} \cdot \frac{3^n}{1} = 14$, which is finite and positive, so $\sum a_n$ and $\sum b_n$ are comparable.

$\sum b_n = \sum \left(\frac{1}{3}\right)^n$ is a convergent geometric series, so $\sum a_n$ converges by L.C.T.

8. $\sum_{n=3}^{\infty} \frac{2}{n \ln n - 7}$

Let $a_n = \frac{2}{n \ln n - 7}$, $b_n = \frac{2}{n \ln n}$.

By in-class exercises 8.4 #2, $\sum b_n$ diverges.

Since $0 \le b_n \le a_n$ for all n with $n \ln n > 7$, $\sum a_n$ diverges by the (direct) comparison test.

9. $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{3n^2 - 1}$

Let $a_n = \frac{\sin(1/n)}{3n^2-1}$ and let $b_n = \frac{1}{3n^2-1}$.

Note that $\sum b_n$ converges by comparison (L.C.T.) with $\sum \frac{1}{n^2}$, and $0 \le a_n \le b_n$ for all n, so $\sum a_n$ converges by (direct) comparison test.

10. $\sum_{n=2}^{\infty} \frac{\sqrt{n^2 - 8}}{\sqrt[3]{n} + \sqrt[5]{7}}$

Let $a_n = \frac{\sqrt{n^2-8}}{\sqrt[3]{n} + \sqrt[5]{7}}$. If we cross out the non-dominant terms of a_n we get $\frac{\sqrt{n^2}}{\sqrt[3]{n}} = \frac{n}{n^{1/3}} = n^{2/3}$, which does not approach 0. In other words,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^{2/3} = \infty.$$

Therefore $\sum a_n$ diverges by the Divergence Test.