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Kelm

Spring 2021 MATH 76
Activity 6

TRIG SUBSTITUTION

When a is a positive fixed number, we choose $\theta = \sin^{-1}\left(\frac{x}{a}\right)$, or $\theta = \tan^{-1}\left(\frac{x}{a}\right)$, or $\theta = \sec^{-1}\left(\frac{x}{a}\right)$ when it is convenient. The different types of substitutions are highlighted in the table below, along with the values of x and corresponding values of θ that make the substitutions valid.

Expression	Trig sub	Trig identities	Simplification
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$a^2 - a^2\sin^2\theta = a^2\cos^2\theta$	$\sqrt{a^2 - x^2} = a\cos\theta, \quad -a \le x \le a$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$a^2 + a^2\tan^2\theta = a^2\sec^2\theta$	$\sqrt{a^2 + x^2} = a\sec\theta, \quad$ all x and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$	$\sqrt{x^2 - a^2} = \begin{cases} a\tan\theta, & x \ge a \text{ and } 0 \le \theta < \frac{\pi}{2} \\ -a\tan\theta, & x \le -a \text{ and } \frac{\pi}{2} < \theta \le \pi \end{cases}$

1. Use a trig substitution of x to simplify the following expressions.

   (a) $\sqrt{x^2 + 4}$

   (b) $\sqrt{16 - x^2}$ where $-4 \le x \le 4$

   (c) $\sqrt{x^2 - 9}$ where $x \ge 3$

   (d) $\sqrt{1 - 16x^2}$ where $-\frac{1}{4} \le x \le \frac{1}{4}$

   (e) $\sqrt{9x^2 + 16}$

(a) $\sqrt{x^2+4}$

$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$a^2+a^2\tan^2\theta = a^2\sec^2\theta$	$\sqrt{a^2+x^2} = a\sec\theta$, all x and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$= \sqrt{2^2 + x^2}$$

$$= \sqrt{2^2 + 2^2\tan^2\theta}$$

$$= \sqrt{2^2\sec^2\theta}$$

$$\boxed{= 2\sec\theta}$$

(b) $\sqrt{16-x^2}$ where $-4 \le x \le 4$

$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$a^2-a^2\sin^2\theta = a^2\cos^2\theta$	$\sqrt{a^2-x^2} = a\cos\theta$, $-a \le x \le a$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$= \sqrt{4^2 - x^2}$$

$$= \sqrt{4^2 - 4^2\sin^2\theta} \qquad = 4\sqrt{1-\sin^2\theta}$$

$$= \sqrt{4^2\cos^2\theta} \qquad = 4\sqrt{\cos^2\theta}$$

$$\boxed{= 4\cos\theta} \qquad = 4\cos\theta$$

(c) $\sqrt{x^2-9}$ where $x \ge 3$

$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$	$\sqrt{x^2-a^2} = \begin{cases} a\tan\theta, & x \ge a \text{ and } 0 \le \theta < \frac{\pi}{2} \\ -a\tan\theta, & x \le -a \text{ and } \frac{\pi}{2} < \theta \le \pi \end{cases}$

$$= \sqrt{x^2 - 3^2}$$

$$= \sqrt{3^2\sec^2\theta - 3^2}$$

$$= \sqrt{3^2\tan^2\theta}$$

$$\boxed{= 3\tan\theta}$$

(d) $\sqrt{1-16x^2}$ where $-\frac{1}{4} \le x \le \frac{1}{4}$

$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$a^2-a^2\sin^2\theta = a^2\cos^2\theta$	$\sqrt{a^2-x^2} = a\cos\theta$, $-a \le x \le a$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$= \sqrt{(\sqrt{1})^2 - (4x)^2} \qquad a^2 = (\sqrt{1})^2$$

$$= \sqrt{\sqrt{1}^2\sin^2\theta - (4x)^2} \qquad x^2 = (4x)^2$$

$$= \sqrt{\sqrt{1}^2\sin^2\theta - x^2}$$

$$= \sqrt{\sqrt{1}^2\cos^2\theta}$$

$$= \sqrt{1}\cos\theta$$

$$\boxed{= \cos\theta}$$

$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$a^2 + a^2\tan^2\theta = a^2\sec^2\theta$	$\sqrt{a^2+x^2} = a\sec\theta$, all x and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

(e) $\sqrt{9x^2+16}$

$$= \sqrt{9x^2 + 4^2}$$

$$= \sqrt{9}\,\sqrt{x^2 + \frac{16}{9}} = 3\sqrt{x^2 + \frac{16}{4}}$$

$$= 3\sqrt{\left(\frac{4}{2}\right)^2 \tan^2\theta + \left(\frac{4}{2}\right)^2}$$

$$= 3\sqrt{\left(\frac{4}{2}\right)^2 \sec^2\theta}$$

$$= 3\cdot\frac{4}{3}\sec\theta = \boxed{4\sec\theta}$$

2. Make the appropriate trig substitution of x in the following integrals and simplify. DO NOT FORGET dx. DO NOT INTEGRATE.

   (a) $\displaystyle\int \frac{1}{\sqrt{x^2+4}}\,dx$

   (b) $\displaystyle\int \frac{x^2}{\sqrt{x^2+9}}\,dx$

   (c) $\displaystyle\int \sqrt{9x^2-25}\,dx \quad \text{where } x \geq \frac{5}{3}$

   (d) $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx \quad \text{where } -4 < x < 4$

   (e) $\displaystyle\int \frac{\sqrt{x^2+16}}{x^4}\,dx$

(a) $\int \frac{1}{\sqrt{x^2+4}}\,dx$ $= \int \frac{1}{\sqrt{x^2+2^2}}\,dx$

$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$a^2+a^2\tan^2\theta = a^2\sec^2\theta$	$\sqrt{a^2+x^2} = a\sec\theta$, all x and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$X = 2\tan\theta \qquad dx = 2\sec^2\theta\,d\theta$

$$= \int \frac{1}{\sqrt{2^2\tan^2\theta + 2^2}}\,d\theta = \int \frac{1}{\sqrt{2^2\sec^2\theta}}\,2\sec^2\theta\,d\theta = \int \frac{\cancel{2}\sec^{\cancel{2}}\theta}{\cancel{2\sec\theta}}\,d\theta$$

$$\boxed{= \int \sec\theta\,d\theta}\ \checkmark$$

(b) $\int \frac{x^2}{\sqrt{x^2+9}}\,dx$ ✓ $= \int \frac{x^2}{\sqrt{x^2+3^2}}\,dx$

$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$a^2+a^2\tan^2\theta = a^2\sec^2\theta$	$\sqrt{a^2+x^2} = a\sec\theta$, all x and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$x = 3\tan\theta \qquad dx = 3\sec^2\theta\,d\theta$

$$= \int \frac{\cancel{3}\,9\tan^2\theta}{\cancel{3\sec\theta}}\,3\sec^2\theta\,d\theta\ \boxed{= \int 9\tan^2\theta\sec\theta\,d\theta}\ \checkmark$$

(c) $\int \sqrt{9x^2-25}\,dx$ where $x \geq \frac{5}{3}$ $= \int \sqrt{9x^2 - 5^2}\,dx$

$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$	$\sqrt{x^2-a^2} = \begin{cases} a\tan\theta, & x \geq a \text{ and } 0 \leq \theta < \frac{\pi}{2} \\ -a\tan\theta, & x \leq -a \text{ and } \frac{\pi}{2} < \theta \leq \pi \end{cases}$

$$= 3\int \sqrt{x^2 - 5^2}\,dx \qquad x = 5\sec\theta \qquad dx = 5\sec\theta\tan\theta\,d\theta$$

$$= 3\int \sqrt{5^2\sec^2\theta - 5^2}\,5\sec\theta\tan\theta\,d\theta = 75\int \sqrt{\sec^2\theta - 1}\,\sec\theta\tan\theta\,d\theta$$

$$= 75\int \tan^2\theta\sec\theta\,d\theta = 75\int (\sec^2\theta - 1)\tan\theta\sec\theta\,d\theta \qquad \begin{aligned} U &= \sec\theta \\ dU &= \sec\theta\tan\theta \end{aligned}$$

$$= 75\int (U^2 - 1)\,dU = \boxed{75\int (\sec^2\theta - 1)\,d\theta}\ \checkmark$$

$\sqrt{a^2-x^2}$	$x=a\sin\theta$	$a^2-a^2\sin^2\theta=a^2\cos^2\theta$	$\sqrt{a^2-x^2}=a\cos\theta,\quad -a\le x\le a$ and $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

(d) $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$ where $-4<x<4$

$$= \int \frac{x^2}{\sqrt{4^2-x^2}}\,dx$$

$x = 4\sin\theta$

$dx = 4\cos\theta\,d\theta$

$$= \int \frac{4^2\sin^2\theta}{\sqrt{4^2-4^2\sin^2\theta}}\,d\theta = \int \frac{4^2\sin^2\theta}{4\cos\theta}\,4\cos\theta\,d\theta$$

$$= 16\int \frac{\sin^2\theta}{\cancel{\cos\theta}}\,\cancel{\cos\theta}\,d\theta \quad \boxed{= 16\int \sin^2\theta\,d\theta} \quad \checkmark$$

$\sqrt{a^2+x^2}$	$x=a\tan\theta$	$a^2+a^2\tan^2\theta=a^2\sec^2\theta$	$\sqrt{a^2+x^2}=a\sec\theta,\quad$ all x and $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

(e) $\displaystyle\int \frac{\sqrt{x^2+16}}{x^4}\,dx$

$x = 4\tan\theta$

$dx = 4\sec^2\theta\,d\theta$

$$= \int \frac{\sqrt{x^2+4^2}}{x^4}\,dx = \int \frac{\sqrt{4^2\tan^2\theta+4^2}}{4^4\tan^4\theta}\,d\theta$$

$$= \int \frac{\sqrt{4^2\sec^2\theta}}{4^4\tan^4\theta}\,4\sec^2\theta\,d\theta = \int \frac{4\sec\theta}{4^4\tan^4\theta}\,4\sec^2\theta\,d\theta$$

$$= \frac{1}{16}\int \frac{\sec^3\theta}{\tan^4\theta}\,d\theta = \frac{1}{16}\int \frac{\cos\theta}{\sin^4\theta}$$

$u = \sin\theta$

$du = \cos\theta\,dx$

$$= \frac{1}{16}\int \frac{1}{u^4}\,dx \quad \boxed{= \frac{1}{16}\int \frac{1}{\sin^4\theta}\,d\theta} \quad \checkmark$$

3. Compute the following integrals by using a trig substitution of x. Do not forget dx.

(a) $\displaystyle\int \frac{1}{\sqrt{x^2 - 25}}\, dx \quad \text{where } x > 5$

(b) $\displaystyle\int \frac{x^2}{\sqrt{1 - x^2}}\, dx \quad \text{where } -1 < x < 1$

(c) $\displaystyle\int \sqrt{1 + 4x^2} dx$

X (a) $\displaystyle\int \frac{1}{\sqrt{x^2-25}}\,dx$ where $x>5$

$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$	$\sqrt{x^2-a^2} = \begin{cases} a\tan\theta, & x \geq a \text{ and } 0 \leq \theta < \frac{\pi}{2} \\ -a\tan\theta, & x \leq -a \text{ and } \frac{\pi}{2} < \theta \leq \pi \end{cases}$

$X = 5\sec\theta \quad dx = 5\sec\theta\tan\theta\, d\theta$

$U = \frac{x}{5} \quad dU = \frac{1}{5}dx \quad dx = 5\,dU$

$$= \int \frac{1}{\sqrt{x^2 - 5^2}}\,dx$$

$U = \sec^2\theta \quad dU = \text{Arcsec}\theta\, d\theta$

$d\theta = \sec\theta\tan\theta\, dU$

$$= \int \frac{1}{\sqrt{U^2-1}}\,dU = \int \frac{1}{\sqrt{\sec^2\theta - 1}}\,dU$$

$$= \int \frac{\sec\theta\tan\theta}{\sqrt{\sec^2\theta - 1}}\,dU = \int \frac{\sec\theta\cancel{\tan\theta}}{\cancel{\tan\theta}}\,dU = \int \sec\theta = \boxed{\sec\theta\tan\theta + C}$$ ✓

X (b) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$ where $-1<x<1$

$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$a^2 - a^2\sin^2\theta = a^2\cos^2\theta$	$\sqrt{a^2-x^2} = a\cos\theta$, $-a \leq x \leq a$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$X = \sin\theta \quad dx = \cos\theta\, d\theta$

$U = \text{Arcsin}\theta \quad dU = \cos\theta\, dx$

$$= \int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{\cos U \sin^2 U}{\sqrt{1-\sin^2 U}}\,d\theta = \int \frac{\cos U \sin^2 U}{\sqrt{\cos^2 U}}\,d\theta$$

$$= \int \frac{\cancel{\cos U}\sin^2 U}{\cancel{\cos U}}\,dU = \int \sin^2 U\, d\theta = -\frac{\cos U \sin U}{2} + \frac{1}{2}\int 1\, dU$$

$$= -\frac{\cos U \sin U}{2} + \frac{U}{2} + C = \boxed{\frac{\text{Arcsin}\theta}{2} - \frac{x\sqrt{1-x^2}}{2} + C}$$ ✓

$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$a^2+a^2\tan^2\theta = a^2\sec^2\theta$	$\sqrt{a^2+x^2} = a\sec\theta$, all x and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

(c) $\int \sqrt{1+4x^2}dx$

$$= 2\int \sqrt{\frac{1}{4} + x^2}\, dx \qquad x = \frac{1}{2}\tan\theta \qquad dx = \frac{1}{2}\sec^2\theta\, d\theta$$

$$= 2\int \sqrt{\frac{1}{4} + \left(\frac{1}{2}\right)^2\tan^2\theta}\ \frac{1}{2}\sec^2\theta\, d\theta$$

$$= \int \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\tan^2\theta}\ \sec^2\theta\, d\theta = \int \frac{1}{2}\sec\theta\sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int \sec\theta\sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int \sec\theta\tan\theta + \frac{1}{2}\int\sec\theta\, d\theta$$

$$= \frac{\ln(\tan\theta + \sec\theta)}{2} + \frac{\sec\theta\tan\theta}{2} + C$$

$$\boxed{= \frac{\ln|\tan\theta + \sec\theta| + \sec\theta\tan\theta}{2} + C} \checkmark$$