Math 76 Exercises – 7.2 Calculus of Parametric Curves

1. For the curve represented by the parametric equations

$$x = 5t^3 - 3t^2 + 1$$
$$y = t^2 + 4t,$$

(a) Find $\frac{dy}{dx}$ as a function of t.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \boxed{\frac{2t+4}{15t^2-6t}}$$

(b) Find the equation of the tangent line to the curve at the point where $t = 1$.

$$m = \left.\frac{dy}{dx}\right|_{t=} = \frac{2+4}{15-6} = \frac{6}{9} = \frac{2}{3}$$

Point of tangency is $(x(1), y(1)) = (5-3+1, 1+4) = (3,5)$.

$$y - 5 = \frac{2}{3}(x-3)$$
$$y - 5 = \frac{2}{3}x - 2$$
$$\boxed{y = \frac{2}{3}x + 3}$$

2. For the curve represented by the parametric equations

$$x = 9 - t^2$$
$$y = t^3 - 6t,$$

(a) Find the x- and y-intercept(s) of the curve.

(i) Set $y=0$: $t^3 - 6t = 0$

$$t(t^2-6) = 0$$
$$t(t+\sqrt{6})(t-\sqrt{6}) = 0$$
$$t = 0,\ t = -\sqrt{6},\ t = \sqrt{6}.$$

$$x(0) = 9 - 0^2 = 9$$
$$x(\sqrt{6}) = x(-\sqrt{6}) = 9 - 6 = 3.$$

So x-intercepts are $\boxed{(9,0), (3,0)}$

(ii) Set $x=0$: $9 - t^2 = 0$

$$t^2 = 9$$
$$t = \pm 3$$

$$y(3) = 27 - 18 = 9$$
$$y(-3) = -27 + 18 = -9$$

So y-intercepts are $\boxed{(0,9), (0,-9)}$

(b) Find the points at which the curve has a horizontal tangent line.

$$\frac{dy}{dt} = 3t^2 - 6 \overset{set}{=} 0 \qquad x(\sqrt{2}) = 9-(\sqrt{2})^2 = 9-2 = 7$$

$$t^2 = 2 \qquad y(\sqrt{2}) = 2\sqrt{2} - 6\sqrt{2} = -4\sqrt{2}$$

$$t = \pm\sqrt{2} \qquad x(-\sqrt{2}) = 9-(-\sqrt{2})^2 = 9-2 = 7$$

$$y(-\sqrt{2}) = -2\sqrt{2} + 6\sqrt{2} = 4\sqrt{2}$$

So the points are $\boxed{(7, -4\sqrt{2}),\ (7, 4\sqrt{2})}$

(c) Find the points at which the curve has a vertical tangent line.

$$\frac{dx}{dt} = -2t \overset{set}{=} 0 \qquad x(0) = 9 - 0^2 = 9$$

$$t = 0 \qquad y(0) = 0^3 - 6 \cdot 0 = 0$$

So there is a vertical tangent at $\boxed{(9, 0)}$

(d) Find the values of t for which the curve is concave up.

(e) Find the values of t for which the curve is concave down.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2-6}{-2t} = -\frac{3}{2}t + \frac{3}{t}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = -\frac{3}{2} - \frac{3}{t^2}$$

$$y''(x) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-\frac{3}{2} - \frac{3}{t^2}}{-2t}$$

$$= \frac{3}{4t} + \frac{3}{2t^3} = \frac{3t^2+6}{4t^3} \overset{set}{=} 0 \Rightarrow 3t^2 + 6 = 0 \Rightarrow t^2 = -2$$

No solution.

Note that $y''(x)$ is undefined at $t = 0$. So the concavity can change where $t = 0$.

$y''(x)$ $- - - -$ | $+ + + +$

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Sure enough, $y''(x) < 0$ for $t < 0$ and $y''(x) > 0$ for $t > 0$.

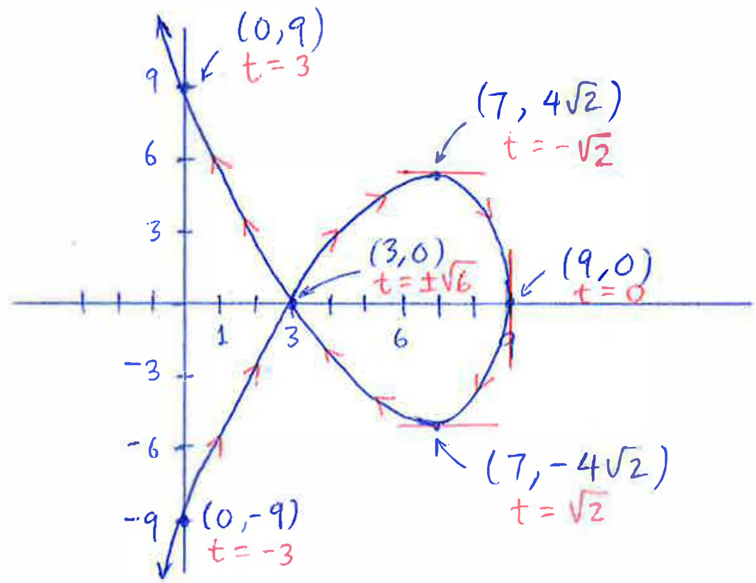
So the curve is $\boxed{\text{concave down for } t<0 \text{ and concave up for } t>0}$

(f) There is a point (x, y) at which the curve crosses itself. Can you find it? What are the values of t at this point?

From part (a) we have the point $(3, 0)$ at both $t = \sqrt{6}$ and $t = -\sqrt{6}$.

(g) Sketch a graph of the curve. Include arrows showing the direction of increasing t-values.

t	x	y
-3	0	-9
$-\sqrt{6}$	3	0
$-\sqrt{2}$	7	$4\sqrt{2}$
0	9	0
$\sqrt{2}$	7	$-4\sqrt{2}$
$\sqrt{6}$	3	0
3	0	9



(h) Note that $x(t)$ is an even function and $y(t)$ is an odd function (verify!). What does this tell you about the symmetry of the graph of the curve?

$x(-t) = 9 - (-t)^2 = 9 - t^2 = x(t)$, so $x(t)$ is an even function.

$y(-t) = (-t)^3 - 6(-t) = -t^3 + 6t = -y(t)$, so $y(t)$ is an odd function.

This means that for each value of t, we have both (a, b) and $(a, -b)$ on the graph, where $a = x(t) = x(-t)$ and $b = y(t)$ (so $-b = -y(t) = y(-t)$).

So the graph is $\boxed{\text{symmetric about the x-axis}}$