Spring 2021 MATH 76
Activity 7

PARTIAL FRACTIONS

Assume that $\dfrac{f(x)}{g(x)}$ is a rational integrand where the degree of $f(x)$ is **smaller** than the degree of $g(x)$.

- Simple linear factors $g(x) = (x - r_1)(x - r_2)\dots(x - r_n)$

  $$\frac{f(x)}{g(x)} = \frac{A_1}{x - r_1} + \frac{A_2}{x - r_2} + \dots + \frac{A_n}{x - r_n}$$

- Repeated linear factors $g(x) = (x - r)^n$

  $$\frac{f(x)}{g(x)} = \frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \dots + \frac{A_n}{(x - r)^n}$$

- Irreducible quadratic factor $g(x) = ax^2 + bx + c$ and cannot be factored ($b^2 - 4ac < 0$)

  $$\frac{f(x)}{g(x)} = \frac{Ax + B}{ax^2 + bx + c}$$

1. Write the following rational expressions in the appropriate partial fractions form. **Do not find the constants.**

   (a) $\dfrac{9x^3 + 30x - 20x^2 - 97}{(x - 2)(x - 3)(x^2 + 5)}$

   (b) $\dfrac{7x^2 + 75x - 150}{x^3 - 25x}$

   (c) $\dfrac{7x - 26}{x^2 - 6x - 16}$

   (d) $\dfrac{2 + x^4}{x^3 + 9x}$. Try a long division first.

2. Find the partial fraction decomposition of the following rational expressions. **Find the constants.**

   (a) $\dfrac{8x - 36}{(x-5)^2}$

   (b) $\dfrac{-3x - 23}{x^2 - x - 12}$

   (c) $\dfrac{3x^2 - 3x + 10}{(x-2)(x^2+4)}$

3. Evaluate the following integrals.

(a) $\int \frac{x-1}{x^2+x}dx$

(b) $\int \frac{2x-3}{x^3+x}dx$

(c) $\int \frac{4x^3-3x+5}{x^2-2x}dx.$

4. Here are some steps to evaluate $\int \frac{dx}{x^2 + 4x + 13}$.

(a) Verify that $x^2 + 4x + 13$ is irreducible.

(b) Write $x^2 + 4x + 13 = (x + \ldots)^2 + \ldots$

(c) Use a $u-$ substitution $u = x + \ldots$ to simplify the integrand.

(d) Check that for $a > 0$, $\int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C$ and use it to compute the integral $\int \frac{dx}{x^2 + 4x + 13}$.

5. The following integrals are given. Fill in the details to explain the answers.

(a) $\displaystyle\int \frac{2x+1}{x^2+4}dx = \ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

(b) $\displaystyle\int \frac{2x+2}{x^2+6x+10}dx = \ln(x^2+6x+10) - 4\tan^{-1}(x+3) + C$

(c) $\displaystyle\int \frac{2x+2}{(x-3)^2(x+1)}dx = -\frac{2}{x-3} + C$

(d) $\displaystyle\int \frac{3x}{x^3-x^2-2x}dx = \ln|x-2| - \ln|x+1| + C$