1. Let $f(x) = \cos x$.

(a) Find a number M such that $|f^{(n+1)}(x)| \le M$ for all n and all x.

$f'(x) = -\sin x$
$f''(x) = -\cos x$
$f^{(3)}(x) = \sin x$
$f^{(4)}(x) = \cos x$
$f^{(5)}(x) = -\sin x$
$\vdots$

All derivatives are between -1 and 1.
So $|f^{(n+1)}(x)| \le 1$ for all n and all x. $\boxed{M = 1}$

(b) Find a bound for $R_n(x)$ for the Taylor polynomials $p_n(x)$ of $f(x)$ centered at $c = 0$.

$$|R_n(x)| \le \frac{M x^{n+1}}{(n+1)!} = \frac{x^{n+1}}{(n+1)!}$$

(c) What value of n will guarantee that $p_n(0.75)$ will give an approximation of $\cos(0.75)$ with an absolute error less than 0.001?

$x = 0.75$, so $|R_n(0.75)| \le \frac{(0.75)^{n+1}}{(n+1)!} \overset{\text{want}}{<} 0.001.$

$n = 4: \frac{(0.75)^5}{5!} \approx 0.001978$

$n = 5: \frac{(0.75)^6}{6!} \approx 0.000247 < 0.001.$

So need $\boxed{n = 5}$

$p_5(0.75) \approx \cos(0.75)$ with error < 0.001.

(d) What value of n will guarantee that $p_n(-0.2)$ will give an approximation of $\cos(-0.2)$ with an absolute error less than 0.001?

$x = -0.2$, so $|R_n(-0.2)| \le \frac{(0.2)^{n+1}}{(n+1)!} \overset{\text{want}}{<} 0.001.$

$n = 2: \frac{(0.2)^3}{3!} \approx 0.00133$

$n = 3: \frac{(0.2)^4}{4!} \approx 0.000067 < 0.001.$

$x = -0.2$ is closer to the center $c = 0$ than $x = 0.75$, so we expect a smaller n

So use $\boxed{n = 3}$ $p_3(-0.2) \approx \cos(-0.2)$ with error less than 0.001.

2. Use the remainder term to estimate the absolute error in approximating each of the following quantities with the 4th-order Taylor polynomial centered at 0.

$n=4$ $c=0$

(a) $\sin(0.3)$

$f(x)=\sin(x)$. We can use $M=1$ for all n and all x similar to #1.

$$|R_4(0.3)| \le \frac{(0.3)^5}{5!} = 0.00002025.$$

So $p_4(0.3) \approx \sin(0.3)$ with error less than 0.00002025.

(b) $\sqrt[4]{e}$

$f(x)=e^x$. $x=\frac{1}{4}$. $f^{(n+1)}(\frac{1}{4}) = e^{\frac{1}{4}} < 2$ (easier)

So we'll use $M=2$.

$f'(x)=e^x$

$f''(x)=e^x$

$f^{(3)}(x)=e^x$

$\vdots$

$$|R_n(\tfrac{1}{4})| \le \frac{2(\frac{1}{4})^{n+1}}{(n+1)!} \qquad |R_4(\tfrac{1}{4})| \le \frac{2(\frac{1}{4})^5}{5!}$$

$$\approx 0.0000163.$$

So $p_4(\frac{1}{4}) \approx e^{\frac{1}{4}}$ with error less than $\boxed{0.0000163\ldots}$

(c) $\frac{1}{e}$

$f(x)=e^x$ $x=-1$. $f^{(n+1)}(-1) = \frac{1}{e} < 1$ (easier),

So we'll use $M=1$.

$$|R_n(-1)| \le \frac{1(1)^{n+1}}{(n+1)!} \qquad |R_4(-1)| \le \frac{1^5}{5!} \approx 0.00833.$$

So $p_4(-1) \approx \frac{1}{e}$ with absolute error less than $\boxed{0.00833\ldots}$

-1 is further from $c=0$ than $\frac{1}{4}$, so we expect a bigger error

3. Compare the estimates in #2 with the actual absolute errors computed using a calculator. How close are your estimates?

(a) $p_4(x) = 0 + \frac{1\,x^1}{1!} + 0 + \frac{-1 \cdot x^3}{3!} + 0$

$= x - \frac{x^3}{6}$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin(x)$	0
1	$\cos(x)$	1
2	$-\sin(x)$	0
3	$-\cos(x)$	-1
4	$\sin(x)$	0

$p_4(0.3) = 0.3 - \frac{(0.3)^3}{6} = 0.2955$

$\sin(0.3) \approx 0.29552$ (calculator)

So absolute error $\approx |0.29552 - 0.2955| = 0.0000202067$

(Check: this is less than 0.00002025)

(b)

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^x	1
1	e^x	1
2	e^x	1
3	e^x	1
4	e^x	1

$p_4(x) = 1 + \frac{1\,x^1}{1!} + \frac{1 \cdot x^2}{2!} + \frac{1 \cdot x^3}{3!} + \frac{1 \cdot x^4}{4!}$

$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

$p_4\left(\frac{1}{4}\right) = 1 + \frac{1}{4} + \frac{\left(\frac{1}{4}\right)^2}{2} + \frac{\left(\frac{1}{4}\right)^3}{6} + \frac{\left(\frac{1}{4}\right)^4}{24}$

$\approx 1.2840169.$

$e^{\frac{1}{4}} \approx 1.2840254\ldots$ (calculator)

So absolute error $\approx |1.2840169 - 1.2840254\cdots| \approx 0.00000849$

(Check: this is less than $0.0000163\ldots$)

(c) $p_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$ (from part (b))

$p_4(-1) = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = 0.375$

$e^{-1} \approx 0.367879$ (calculator)

So absolute error $\approx |0.375 - 0.367879| \approx 0.00712$ (which is less than $0.00833\ldots$)

4. How many terms of each alternating series would be needed to estimate the sum with an absolute error less than 10^{-4}?

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{n}{5^n - 3} = -\frac{1}{5^1-3} + \frac{2}{5^2-3} - \frac{3}{5^3-3} + \cdots$

$b_n = \frac{n}{5^n - 3}$

$b_6 = \frac{6}{5^6 - 3} \approx 0.000384$

$b_7 = \frac{7}{5^7 - 3} \approx 0.0000896 < 10^{-4}$.

So $|R_6| \leq b_7 < 10^{-4}$.

So we need $\boxed{\text{6 terms}}$

(b) $\sum_{n=1}^{\infty}(-1)^n \frac{6n^2}{3^n + 1}$ $\qquad b_n = \frac{6n^2}{3^n + 1}$

Trial and error...

$b_{13} = \frac{6 \cdot 13^2}{3^{13} + 1} \approx 0.000636$

$b_{14} = \frac{6 \cdot 14^2}{3^{14} + 1} \approx 0.00024587$

$b_{15} = \frac{6 \cdot 15^2}{3^{15} + 1} \approx 0.000094 < 10^{-4}$.

So $|R_{14}| \leq b_{15} < 10^{-4}$. So we need $\boxed{\text{14 terms}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, n!}{n^n + 5n - 2}$ $\qquad b_n = \dfrac{n!}{n^n + 5n - 2}$

$$b_4 = \frac{4!}{4^4 + 5\cdot 4 - 2} \approx 0.08759$$

$\vdots$

$$b_{10} = \frac{10!}{10^{10} + 5\cdot 10 - 2} \approx 0.00036$$

$$b_{11} = \frac{11!}{11^{11} + 5\cdot 11 - 2} \approx 0.0001399$$

$$b_{12} = \frac{12!}{12^{12} + 5\cdot 12 - 2} \approx 0.0000537 < 10^{-4}.$$

$\boxed{11 \text{ terms}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(1/n)}{7n^4}$ $\qquad b_n = \dfrac{\sin\left(\frac{1}{n}\right)}{7n^4}$

$$b_4 = \frac{\sin\left(\frac{1}{4}\right)}{7\cdot 4^4} \approx 0.00013806$$

$$b_5 = \frac{\sin\left(\frac{1}{5}\right)}{7\cdot 5^4} \approx 0.00004541 < 10^{-4}.$$

So we need $\boxed{4 \text{ terms}}$

5. Estimate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(1/n)}{7n^4}$ with an absolute error less than 10^{-3}.

From #4(d) we see that $b_4 < 10^{-3}$, so we need only 3 terms to get an estimate with $|R_n| < 10^{-3}$. We have

$$S_3 = \frac{\sin(1)}{7} - \frac{\sin\left(\frac{1}{2}\right)}{7\cdot 2^4} + \frac{\sin\left(\frac{1}{3}\right)}{7\cdot 3^4} \approx 0.1165.$$

So $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin\left(\frac{1}{n}\right)}{7n^4} \approx \boxed{0.1165}$$