Math 76 Exercises –3.4C More Partial Fractions

1. Write out the partial fraction decomposition of each function.

(a) $f(x) = \dfrac{5x^2+8}{(x^2+3x+3)^2}$ $= \dfrac{Ax+B}{x^2+3x+3} + \dfrac{Cx+D}{(x^2+3x+3)^2}$

Notice that $3^2-4(1)(3) = -3 < 0$, so x^2+3x+3 is irreducible.

$(Ax+B)(x^2+3x+3)+Cx+D = 5x^2+8$

$Ax^3 = 0x^3 \Rightarrow A=0.$

$(3A+B)x^2 = 5x^2 \Rightarrow 3\cdot 0 + B = 5 \Rightarrow B=5.$

$(3A+3B+C)x = 0x \Rightarrow 3\cdot 0 + 3\cdot 5 + C = 0 \Rightarrow C=-15$

$3B+D = 8 \Rightarrow 3\cdot 5 + D = 8 \Rightarrow D=-7.$

So $f(x) = \boxed{\dfrac{5}{x^2+3x+3} - \dfrac{15x+7}{(x^2+3x+3)^2}}$

(b) $g(x) = \dfrac{4x+1}{x^4+2x^2}$ $= \dfrac{4x+1}{x^2(x^2+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+2}$

$Ax(x^2+2)+B(x^2+2)+(Cx+D)x^2 = 4x+1.$

$x=0$: $0+2B+0 = 4\cdot 0+1 \Rightarrow 2B=1 \Rightarrow B=\frac{1}{2}$

$Ax(x^2+2)+\frac{1}{2}(x^2+2)+(Cx+D)x^2 = 4x+1.$

$Ax^3+Cx^3 = 0x^3 \Rightarrow A+C=0$

$\frac{1}{2}x^2+Dx^2 = 0x^2 \Rightarrow D=-\frac{1}{2}$

$2Ax = 4x \Rightarrow 2A=4 \Rightarrow A=2 \Rightarrow C=-2.$

So $g(x) = \boxed{\dfrac{2}{x} + \dfrac{1}{2}\cdot\dfrac{1}{x^2} - \dfrac{2x+\frac{1}{2}}{x^2+2}}$

(c) $h(x) = \dfrac{x+1}{(x-4)^2(x^2+9)} = \dfrac{A}{x-4} + \dfrac{B}{(x-4)^2} + \dfrac{Cx+D}{x^2+9}$

$$A(x-4)(x^2+9) + B(x^2+9) + (Cx+D)(x-4)^2 = x+1.$$

$x=4$: $\quad 0 + 25B + 0 = 5 \Rightarrow \underline{B = \frac{1}{5}}$

$$A(x-4)(x^2+9) + \tfrac{1}{5}(x^2+9) + (Cx+D)(x^2-8x+16) = x+1$$

$$Ax^3 + Cx^3 = 0x^3 \Rightarrow A + C = 0. \Rightarrow C = -A. \quad (*)$$

$$-36Ax^2 + \tfrac{1}{5}x^2 + Dx^2 - 8Cx^2 = 0x^2$$

$$\Rightarrow -36A - 8C + D = -\tfrac{1}{5} \overset{(*)}{\Rightarrow} -36A + 8A + D = -\tfrac{1}{5}$$

$$9Ax + 16Cx - 8Dx = 1 \qquad \Rightarrow -28A + D = -\tfrac{1}{5} \quad (**)$$

$$\Rightarrow 9A + 16C - 8D = 1 \overset{(*)}{\Longrightarrow} 9A - 16A - 8D = 1$$

$$\Rightarrow -7A - 8D = 1 \quad (**)$$

Using $(**)$ we have $\quad -28A + D = -\frac{1}{5}$

$$(-4)(-7A - 8D) = (-4)(1)$$

$$28A + 32D = -4$$

$$33D = -\tfrac{1}{5} - 4 = -\tfrac{21}{5}$$

$$\underline{D = -\tfrac{21}{165}}$$

So $\; -28A - \dfrac{21}{165} = -\dfrac{1}{5} \Rightarrow -28A = -\dfrac{1}{5} + \dfrac{21}{165} = \dfrac{-12}{165}$

$$\Rightarrow A = \frac{12}{28\cdot 165} = \frac{3}{7\cdot 165} = \underline{\frac{1}{385}}$$

$$\boxed{h(x) = \frac{1}{385}\cdot\frac{1}{x-4} + \frac{1}{5}\cdot\frac{1}{(x-4)^2} - \frac{\frac{1}{385}x + \frac{21}{165}}{x^2+9}} \qquad \Rightarrow \underline{C = -\frac{1}{385}}$$

(d) $j(x) = \dfrac{x^3}{x^3+8}$

$$\begin{array}{r} 1 \\ x^3+8 \,\overline{)\, x^3 \qquad} \\ -(x^3+8) \\ \hline -8 \end{array}$$

$$= 1 - \frac{8}{x^3+8}$$

$$= 1 - \frac{8}{(x+2)(x^2-2x+4)} = 1 - \left(\frac{A}{x+2} + \frac{Bx+C}{x^2-2x+4} \right)$$

$$A(x^2-2x+4) + (Bx+C)(x+2) = 8$$

$\underline{x=-2}$ $\quad A((-2)^2 - 2(-2) + 4) + 0 = 8$

$$12A = 8 \Rightarrow \underline{\underline{A = \tfrac{2}{3}}}$$

$$\tfrac{2}{3}(x^2-2x+4) + Bx^2 + (2B+C)x + 2C = 8$$

$$\tfrac{2}{3}x^2 + Bx^2 = 0x^2 \Rightarrow \underline{\underline{B = -\tfrac{2}{3}}}$$

$$\tfrac{2}{3}\cdot 4 + 2C = 8 \Rightarrow 2C = 8 - \tfrac{8}{3} = \tfrac{16}{3} \Rightarrow \underline{\underline{C = \tfrac{8}{3}}}$$

So $\quad j(x) = 1 - \dfrac{2}{3} \cdot \dfrac{1}{x+2} - \dfrac{-\frac{2}{3}x + \frac{8}{3}}{x^2-2x+4}$

$$= \boxed{1 - \frac{2}{3} \cdot \frac{1}{x+2} + \frac{2}{3} \cdot \frac{x-4}{x^2-2x+4}}$$

2. Evaluate each integral.

(a) $\displaystyle\int \frac{5x^2+8}{(x^2+3x+3)^2}\,dx = \int \frac{5}{x^2+3x+3} - \frac{15x+7}{(x^2+3x+3)^2}\,dx$

By completing the square in the denominator, we get

$$x^2+3x+3 = x^2+3x+\tfrac{9}{4}+3-\tfrac{9}{4} = \left(x+\tfrac{3}{2}\right)^2+\tfrac{3}{4}$$

$$= u^2+\tfrac{3}{4}, \text{ where}$$

$u = x+\frac{3}{2}$ $\quad x = u-\frac{3}{2}$

$du = dx$.

So the integral is $\displaystyle\boxed{\int \frac{5\,du}{u^2+\frac{3}{4}}} - \boxed{\int \frac{15(u-\frac{3}{2})+7}{(u^2+\frac{3}{4})^2}\,du}$.

(1) (2)

Let's take the integrals one at a time:

(1) $\displaystyle\int \frac{5}{u^2+\frac{3}{4}}\,du = 5\int \frac{1}{\frac{3}{4}\left(\frac{4}{3}u^2+1\right)}\,du = \frac{20}{3}\int \frac{1}{\left(\frac{2}{\sqrt{3}}u\right)^2+1}\,du$

$\displaystyle = \frac{20}{3}\cdot\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2}{\sqrt{3}}u\right)+C = \boxed{\frac{10\sqrt{3}}{3}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x+\frac{3}{2}\right)\right)+C}$

(2) $\displaystyle\int \frac{15(u-\frac{3}{2})+7}{(u^2+\frac{3}{4})^2}\,du = 15\int \frac{u}{(u^2+\frac{3}{4})^2}\,du + \left(7-\frac{45}{2}\right)\int \frac{1}{(u^2+\frac{3}{4})^2}\,du$

$\displaystyle\frac{15}{2}\int \frac{2u}{(u^2+\frac{3}{4})^2}\,du = \frac{15}{2}\cdot\left(-\frac{1}{u^2+3/4}\right)$

(Let $t = u^2+\frac{3}{4}$; $dt = 2u\,du$)

$\displaystyle = -\frac{15}{2}\cdot\frac{1}{x^2+3x+3}+C$

(next page)

(2a, cont.) $\left(7-\frac{45}{2}\right)\int \frac{1}{(u^2+\frac{3}{4})^2}\,du = -\frac{31}{2}\int \frac{1}{(u^2+\frac{3}{4})^2}\,du$

$u=\frac{\sqrt{3}}{2}\tan\theta$

$du=\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$

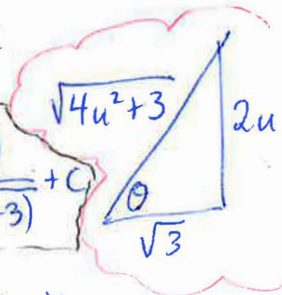
$$= -\frac{31}{2}\int \frac{1}{(\frac{3}{4}\tan^2\theta+\frac{3}{4})^2}\,\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$$

$$= -\frac{31}{2}\cdot\left(\frac{4}{3}\right)^2\cdot\frac{\sqrt{3}}{2}\int\frac{\sec^2\theta}{(\tan^2\theta+1)^2}\,d\theta$$

$$= \frac{-31\cdot 4\sqrt{3}}{9}\int\frac{\sec^2\theta}{\sec^4\theta}\,d\theta = -\frac{31\cdot 4\sqrt{3}}{9}\int\cos^2\theta\,d\theta$$

$$= -\frac{31\cdot 4\sqrt{3}}{9}\cdot\frac{1}{2}\int 1+\cos(2\theta)\,d\theta = -\frac{31\cdot 2\sqrt{3}}{9}\Big(\theta+\underbrace{\tfrac{1}{2}\sin(2\theta)}_{=\sin\theta\cos\theta}\Big)+C$$

$$= -\frac{62\sqrt{3}}{9}\left(\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)+\frac{2\sqrt{3}\,u}{\sqrt{4u^2+3}}\right)+C$$



$$\boxed{= -\frac{62\sqrt{3}}{9}\left(\tan^{-1}\left(\frac{2(x+\frac{3}{2})}{\sqrt{3}}\right)+\frac{2\sqrt{3}\,(x+\frac{3}{2})}{\sqrt{4(x^2+3x+3)}}\right)+C}$$

Put the steps together to get the final answer. (whew!)

(b) $\int \frac{4x+1}{x^4+2x^2}\,dx$

$$= \int \frac{2}{x}+\frac{1}{2}\cdot\frac{1}{x^2}-\frac{2x+\frac{1}{2}}{x^2+2}\,dx = 2\ln|x|-\frac{1}{2}\cdot\frac{1}{x}-\boxed{\int\frac{2x+\frac{1}{2}}{x^2+2}\,dx}$$

$$\int\frac{2x+\frac{1}{2}}{x^2+2}\,dx = \int\frac{2x}{x^2+2}\,dx+\frac{1}{2}\int\frac{1}{x^2+2}\,dx$$

$$= \ln(x^2+2)+\frac{1}{2}\cdot\frac{1}{2}\sqrt{2}\int\frac{1}{(\frac{1}{\sqrt{2}}x)^2+1}\,\frac{1}{\sqrt{2}}\,dx$$

$$= \ln(x^2+2)+\frac{\sqrt{2}}{4}\tan^{-1}\left(\frac{1}{\sqrt{2}}x\right)+C.$$

So the final answer is

$$\boxed{2\ln|x|-\frac{1}{2x}-\ln(x^2+2)-\frac{\sqrt{2}}{4}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C}$$

(c) $\displaystyle\int \frac{x+1}{(x-4)^2(x^2+9)}\,dx = \frac{1}{385}\int \frac{1}{x-4}\,dx + \frac{1}{5}\int \frac{1}{(x-4)^2}\,dx$

$$- \frac{1}{385\cdot 2}\int \frac{2x}{x^2+9}\,dx - \frac{21}{165}\int \frac{1}{x^2+9}\,dx$$

$$= \boxed{\frac{1}{385}\ln|x-4| - \frac{1}{5}\cdot\frac{1}{x-4} - \frac{1}{770}\ln(x^2+9) - \frac{21}{165}\cdot\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C}$$

(d) $\int \frac{x^3}{x^3+8}\,dx = \int 1 - \frac{2}{3}\frac{1}{x+2} + \frac{2}{3}\cdot\frac{x-4}{x^2-2x+4}\,dx$

$$= x - \frac{2}{3}\ln|x+2| + \frac{2}{3}\boxed{\int \frac{x-4}{x^2-2x+4}\,dx}$$

$$\int \frac{x-4}{x^2-2x+4}\,dx = \int \frac{x-4}{x^2-2x+1+3}\,dx$$

$$= \int \frac{x-4}{(x-1)^2+3}\,dx.$$

$u = x-1 \iff x = u+1$

$du = dx.$

Note that $x^2-2x+4 = (x-1)^2+3$
$= u^2+3.$

$$= \int \frac{u-3}{u^2+3}\,du$$

$$= \frac{1}{2}\int \frac{2u}{u^2+3}\,du - 3\int \frac{1}{u^2+3}\,du$$

$$= \frac{1}{2}\ln(u^2+3) - 3\cdot\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C$$

$$= \frac{1}{2}\ln(x^2-2x+4) - \sqrt{3}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + C.$$

So our final answer is

$$\boxed{x - \frac{2}{3}\ln|x+2| + \frac{1}{3}\ln(x^2-2x+4) - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + C}$$