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Spring 2021 MATH 76
Activity 10

SOME CONVERGENCE TESTS

Strategies for infinite series $\sum_{k=M}^{\infty} x_k$ for some positive integer M.

- With a quick glance does it look like the series terms don't converge to zero in the limit, i.e. does $\lim_{k\to\infty} x_k \neq 0$? If so, use the **Divergence Test.** Note that you should only use the Divergence Test if a quick glance suggests that the series terms may not converge to zero in the limit.

- Is the series a $p-$**series** $\left(\sum_{k=M}^{\infty} \frac{a}{k^p}\right)$ or a **geometric series** $\left(\sum_{k=M}^{\infty} ar^k \text{ or } \sum_{k=M}^{\infty} ar^{k-j}\right)$? If so use, the fact that a $p-$series will only converge if $p > 1$ and a geometric series will only converge if $|r| < 1$. Remember as well that some algebraic manipulations may be required to get a geometric series into the correct form.

- Does the series $\sum_{k=M}^{\infty} x_k$ look similar but not equal to a $p-$series or a geometric series? If so, then with either a $p-$series or a geometric series $\sum_{k=M}^{\infty} y_k$ **that you come up with**, try:

  1. the **Comparison Test**. In this case
     (a) if $\sum_{k=M}^{\infty} y_k$ is convergent, you need to justify that $x_k \leq y_k$
     (b) if $\sum_{k=M}^{\infty} y_k$ is divergent, you need to justify that $y_k \leq x_k$
  2. or the **Limit Comparison Test**. In this case
     (a) if $\lim_{k\to\infty} \frac{x_k}{y_k} \neq 0$ and a real number, $\sum_{k=M}^{\infty} x_k$ and $\sum_{k=M}^{\infty} y_k$ either both converge or both diverge
     (b) if $\lim_{k\to\infty} \frac{x_k}{y_k} = 0$ and $\sum_{k=M}^{\infty} y_k$ converges then $\sum_{k=M}^{\infty} x_k$ converges too
     (c) if $\lim_{k\to\infty} \frac{x_k}{y_k} = \infty$ and $\sum_{k=M}^{\infty} y_k$ diverges then $\sum_{k=M}^{\infty} x_k$ diverges too.

  Remember however, that in order to use the Comparison Test and the Limit Comparison Test the series terms all need to be positive. This is also applicable to a series with rational terms or with terms involving polynomials under radicals (i.e. a fraction involving only polynomials or polynomials under radicals).

- If $x_k = f(k)$ for some positive, decreasing function f and $\int_M^{\infty} f(x)dx$ is easy to evaluate, then the **Integral Test** may work.

The following table summarizes the tests.

Tests	Series	Convergent if	Divergent if	Comments
Geometric series	$\sum_{k=M}^{\infty} ar^k,\ a \neq 0$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	$\sum_{k=M}^{\infty} ar^k = \frac{ar^M}{1-r}$, if $\lvert r\rvert < 1$
$p-$series	$\sum_{k=M}^{\infty} \frac{a}{k^p}$	$p > 1$	$p \leq 1$	
Divergence test	$\sum_{k=M}^{\infty} x_k$	N/A	$\lim_{k\to\infty} x_k \neq 0$	Inconcl. if $\lim_{k\to\infty} x_k = 0$
Integral test	$\sum_{k=M}^{\infty} x_k,\ x_k = f(k)$ f is continuous, positive, decreasing	$\int_M^{\infty} f(x)dx$ conv.	$\int_M^{\infty} f(x)dx$ div.	
Comparison	$\sum_{k=M}^{\infty} x_k,\ x_k > 0$	$0 < x_k \leq y_k$ and $\sum_{k=M}^{\infty} y_k$ conv.	$0 < y_k \leq x_k$ and $\sum_{k=M}^{\infty} y_k$ div.	$\sum_{k=M}^{\infty} x_k$ is given, you supply $\sum_{k=M}^{\infty} y_k$
Limit Comparison	$\sum_{k=M}^{\infty} x_k$, $x_k > 0, y_k > 0$	$0 \leq \lim_{k\to\infty} \frac{x_k}{y_k} < \infty$ and $\sum_{k=M}^{\infty} y_k$ conv.	$\lim_{k\to\infty} \frac{x_k}{y_k} > 0$ or $\lim_{k\to\infty} \frac{x_k}{y_k} = \infty$ and $\sum_{k=M}^{\infty} y_k$ div.	$\sum_{k=M}^{\infty} x_k$ is given, you supply $\sum_{k=M}^{\infty} y_k$

Determine if the following infinite series converges or diverges and explain why. Try several tests for the same series.

1. $\sum_{k=1}^{\infty} \frac{k^3}{k^5+3}$

2. $\sum_{k=1}^{\infty} \frac{3^k}{4^k+4}$

1. $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{k^5+3}$

(1) $a_n = \frac{k^3}{k^5+3}$ $\quad a_n \le b_n$ ✓

$b_n = \frac{k^3}{k^5}$

$$\lim_{k\to\infty} \frac{\frac{k^3}{k^5+3}}{\frac{k^3}{k^5}} = \lim_{k\to\infty} \frac{\cancel{k^3}}{k^5+3} \cdot \frac{k^5}{\cancel{k^3}} \quad \frac{1k^5}{1k^5+0} = \boxed{\lim_{k\to\infty} 1}$$

Converges based on the Limit comp. test

(2) $a_n \le b_n$ ✓

$a_n = \frac{k^3}{k^5+3}$

$b_n = \frac{k^3}{k^5} = \frac{1}{k^2}$ $\quad \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \emptyset$

Converges based The Comparison Test

2. $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{4^k+4}$

(1) Limit Comparison test

X_k $a_n = \frac{3^k}{4^k+4}$

$$\lim_{k\to\infty} \frac{\cancel{3^k}}{4^k+4} \cdot \frac{4^k}{\cancel{3^k}} = \lim_{k\to\infty} \frac{\cancel{4^k}}{\cancel{4^k}+3} = \boxed{1}$$

Y_k $b_n = \frac{3^k}{4^k}$

Converges based on the Limit comp. Test

(2) $a_n = \frac{3^k}{4^k+4}$ $\quad a_n \le b_n$ ✓

$b_n = \frac{3^k}{4^k}$ $\quad \frac{3^k}{4^k} = \left(\frac{3}{4}\right)^k \quad \lim_{k\to\infty} \left(\frac{3}{4}\right)^k = \emptyset$

Converges based on the Comparison Test

3. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$

4. $\displaystyle\sum_{k=1}^{\infty} \frac{k^5}{k^4 + 3}$

5. $\displaystyle\sum_{k=1}^{\infty} \frac{k + 5}{k\sqrt{k + 3}}$

3. $\sum_{k=1}^{\infty} \frac{k}{2^k}$

1 $\boxed{\text{Ratio Test}}$

2 $\frac{k}{2^k}$ $\frac{2}{4}$, $\frac{6}{64}$, $\frac{20}{1048576}$ Always Positive, decreasing continuous.

$$\int_1^{\infty} \frac{k}{2^k}\,dx = -\frac{\ln(2)+1}{\ln^2(2)\cdot 2^k}\Big|_1^{\infty} = 0 + \frac{\ln(2)+1}{2\ln^2(2)}$$

Converges based on The integral Test

4. $\sum_{k=1}^{\infty} \frac{k^5}{k^4+3}$

1 $\lim_{k\to\infty} \frac{k^5}{k^4+3} = \lim_{k\to\infty} \frac{k^{\infty}}{k^{\infty}_{\text{smaller}}} = \infty$ diverges based on the divergence test

5. $\sum_{k=1}^{\infty} \frac{k+5}{k\sqrt{k+3}}$

1 $a_n = \frac{k+5}{k\sqrt{k+3}}$ $a_n \leq b_n$ $k\sqrt{k+3}$

$b_n = \frac{k}{k}$ $\sum_{k=1}^{\infty} \frac{k}{k} = \infty$ diverges based comp Test

2 $\lim_{k\to} \frac{k+5}{k\sqrt{k+3}} = \infty$ diverges based on divergence Test

6. $\displaystyle\sum_{k=1}^{\infty} \frac{3+\cos(k)}{e^k}$

7. Given that $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k^3+1} = 1.6865$, determine the value of $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k^3+1}$.

8. Find the **value** (or limit) of the infinite series $\displaystyle\sum_{k=1}^{\infty} \frac{1+2^k}{3^{k-1}}$, if it exists.

6. $\sum_{k=1}^{\infty} \frac{3+\cos(k)}{e^k}$

1

$a_n = \frac{3+\cos(k)}{e^k}$

$b_n = \frac{3}{e^k}$

$\lim_{k\to\infty} \frac{3+\cos(k)}{e^k} \cdot \frac{e^k}{3}$

$= \lim_{k\to\infty} \frac{3+\cos(k)}{3} = \lim_{k\to\infty} 1 + \frac{\cos(k)}{3}$ Converges based on Limit Comp Test.

2

$-1 \le \cos(k) \le 1$

$3 \quad 3 \quad 3$

$-2 \le \cos(k)+3 \le 4$

$\frac{-2}{e^k} \quad \frac{\cos(k)+3}{e^k} \quad \frac{4}{e^k}$

$-\frac{2}{e^k} \le \frac{\cos(k)+3}{e^k} \le \frac{4}{e^k}$

$= 4\sum_{k=1}^{\infty} \frac{1}{e^k}$ $r < 1$ Converges based on Geometric Series

7. Given that $\sum_{k=0}^{\infty} \frac{1}{k^3+1} = 1.6865$, determine the value of $\sum_{k=3}^{\infty} \frac{1}{k^3+1}$.

$1 + \frac{1}{2} + \frac{1}{9} = 1.6111111$

$1.6865 - 1.6111111 = \boxed{0.07539}$ ✓

8. Find the **value** (or limit) of the infinite series $\sum_{k=1}^{\infty} \frac{1+2^k}{3^{k-1}}$, if it exists.

$$= \frac{1}{3^{k-1}} + \frac{2^k}{3^{k-1}}$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{3^{k-1}} + \frac{2^k}{3^{k-1}} \right)$$

$$= \sum_{k=1}^{\infty} \frac{1}{(3^k)\frac{1}{3}} + \sum_{k=1}^{\infty} \frac{2^k}{(3^k)(\frac{1}{3})}$$

$$= 3\sum_{k=1}^{\infty} \frac{1}{3^k} + 3\sum_{k=1}^{\infty} \frac{2^k}{3^k}$$

$$\frac{\text{first term}}{1 - r}$$

$$= \frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{\frac{2}{3}}{1-\frac{2}{3}}$$

$$= \frac{3}{2} + 6$$

$$= \boxed{\frac{15}{2}}$$