Math 76 Exercises – 5.2B Telescoping Series

Do the following series "telescope"? For each telescoping series, find the sum.

1. $\displaystyle\sum_{n=2}^{\infty}\left(\frac{3}{2n-1}-\frac{3}{2n+1}\right)$

$$S_k = \left(\frac{3}{3} - \cancel{\frac{3}{5}}\right) + \left(\cancel{\frac{3}{5}} - \cancel{\frac{3}{7}}\right) + \cdots + \left(\cancel{\frac{3}{2k-3}} - \cancel{\frac{3}{2k-1}}\right) + \left(\cancel{\frac{3}{2k-1}} - \frac{3}{2k+1}\right)$$

$$= 1 - \frac{3}{2k+1}. \qquad \text{(telescoping series)}$$

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(1 - \frac{3}{2k+1}\right) = 1.$$

$$\text{So} \quad \sum_{n=2}^{\infty}\left(\frac{3}{2n-1} - \frac{3}{2n+1}\right) = \boxed{1}$$

2. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+n}$

$$\frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$A(n+1) + Bn = 1$$

$$\underline{n=-1}: \quad 0 - B = 1 \Rightarrow B = -1$$

$$\underline{n=0}: \quad A + 0 = 1 \Rightarrow A = 1.$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2+n} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right).$$

$$S_k = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{k-1}} - \cancel{\frac{1}{k}}\right) + \left(\cancel{\frac{1}{k}} - \frac{1}{k+1}\right)$$

$$= 1 - \frac{1}{k+1}. \qquad \text{(telescoping series)}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2+n} = \lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(1 - \frac{1}{k+1}\right) = \boxed{1}$$

3. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{n+4}{(n+1)(n+2)}\right)$

$$\frac{n+4}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$$

$A(n+2) + B(n+1) = n+4$

$\underline{n=-2}$: $0 + B(-1) = 2 \Rightarrow B = -2$

$\underline{n=-1}$: $A + 0 = 3 \Rightarrow A = 3$.

$$= \sum_{n=0}^{\infty} \frac{3}{n+1} - \frac{2}{n+2}$$

$$S_k = \left(\frac{3}{1} - \frac{2}{2}\right) + \left(\frac{3}{2} - \frac{2}{3}\right) + \left(\frac{3}{3} - \frac{2}{4}\right) + \cdots + \left(\frac{3}{k+1} - \frac{2}{k+2}\right)$$

(does not telescope)

4. $\displaystyle\sum_{n=5}^{\infty}\left(\frac{2}{n-1} - \frac{2}{n+2}\right)$

denominators differ by 3 but count up by 1, so we need to write out many terms before we can see any telescoping...

$$S_k = \left(\frac{2}{4} - \cancel{\frac{2}{7}}\right) + \left(\frac{2}{5} - \cancel{\frac{2}{8}}\right) + \left(\frac{2}{6} - \cancel{\frac{2}{9}}\right) + \left(\cancel{\frac{2}{7}} - \cancel{\frac{2}{10}}\right) + \left(\cancel{\frac{2}{8}} - \cancel{\frac{2}{11}}\right)$$

$$+ \left(\cancel{\frac{2}{9}} - \cancel{\frac{2}{12}}\right) + \cdots + \left(\cancel{\frac{2}{k-6}} - \cancel{\frac{2}{k-3}}\right) + \left(\cancel{\frac{2}{k-5}} - \cancel{\frac{2}{k-2}}\right) + \left(\cancel{\frac{2}{k-4}} - \cancel{\frac{2}{k-1}}\right)$$

$$+ \left(\cancel{\frac{2}{k-3}} - \frac{2}{k}\right) + \left(\cancel{\frac{2}{k-2}} - \frac{2}{k+1}\right) + \left(\cancel{\frac{2}{k-1}} - \frac{2}{k+2}\right)$$

$$= \frac{1}{2} + \frac{2}{5} + \frac{1}{3} - \frac{2}{k} - \frac{2}{k+1} - \frac{2}{k+2}.$$ (telescoping series)

$$\lim_{k\to\infty} S_k = \frac{1}{2} + \frac{2}{5} + \frac{1}{3} - 0 - 0 - 0 = \frac{15+12+10}{30} = \frac{37}{30}.$$

So $\displaystyle\sum_{n=5}^{\infty}\left(\frac{2}{n-1} - \frac{2}{n+2}\right) = \boxed{\frac{37}{30}}$

5. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2n}-\frac{1}{2n+1}\right)$

$$S_k = \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\cdots+\left(\frac{1}{2k}-\frac{1}{2k+1}\right)$$

Denominators keep increasing, so no matching terms to cancel. Not a telescoping series.

6. $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n(n-2)}$

$$\frac{1}{n(n-2)} = \frac{A}{n} + \frac{B}{n-2}$$

$$A(n-2)+Bn = 1$$

$\underline{n=2}$: $0 + 2B = 1 \Rightarrow B = \frac{1}{2}$

$\underline{n=0}$: $A(-2)+0 = 1 \Rightarrow A = -\frac{1}{2}$.

$$= \sum_{n=3}^{\infty} \frac{1/2}{n-2} - \frac{1/2}{n} = \frac{1}{2}\sum_{n=3}^{\infty}\left(\frac{1}{n-2}-\frac{1}{n}\right)$$

$$S_k = \frac{1}{2}\left(\left(\underline{1}-\cancel{\frac{1}{3}}\right)+\left(\underline{\frac{1}{2}}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{6}}\right)+\left(\cancel{\frac{1}{5}}-\cancel{\frac{1}{7}}\right)\right.$$

$$\left.+\cdots+\left(\cancel{\frac{1}{k-5}}-\cancel{\frac{1}{k-3}}\right)+\left(\cancel{\frac{1}{k-4}}-\cancel{\frac{1}{k-2}}\right)+\left(\cancel{\frac{1}{k-3}}-\underline{\frac{1}{k-1}}\right)+\left(\cancel{\frac{1}{k-2}}-\underline{\frac{1}{k}}\right)\right)$$

$$= \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{k-1}-\frac{1}{k}\right)$$

$$\sum_{n=3}^{\infty}\frac{1}{n(n-2)} = \lim_{k\to\infty} S_k = \frac{1}{2}\left(1+\frac{1}{2}-0-0\right) = \frac{1}{2}\cdot\frac{3}{2} = \boxed{\frac{3}{4}}$$