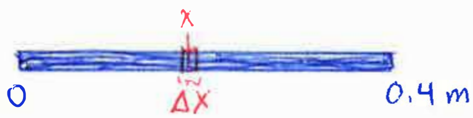
Math 76 Exercises –2.5A Mass and Work

1. A thin bar of length 40 cm has density $k(x) = 3x + 5$ kg/m at a distance of x meters from one end of the bar. What is the total mass of the bar?

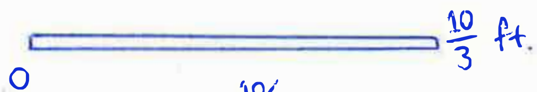


$40 \text{ cm} = 0.4 \text{ m}$

$$\text{mass} = \int_0^{0.4} (3x+5)\,dx = \frac{3}{2}x^2 + 5x \Big|_0^{2/5} = \frac{3}{\cancel{2}} \cdot \frac{\cancel{4}^{2}}{25} + \cancel{5} \cdot \frac{2}{\cancel{5}} - (0-0)$$

$$= \frac{6}{25} + 2 = \boxed{\frac{56}{25} \text{ kg}}$$

2. A thin bar of length 40 inches has weight density $k(x) = 4x^2 - 1$ lb./ft. at a distance of x feet from one end of the bar. What is the total weight of the bar?

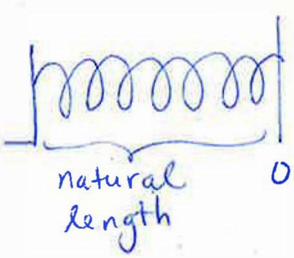


$40 \text{ in.} = \frac{40}{12} = \frac{10}{3} \text{ ft.}$

$$\text{weight} = \int_0^{10/3} (4x^2 - 1)\,dx = \frac{4}{3}x^3 - x \Big|_0^{10/3}$$

$$= \frac{4}{3} \cdot \frac{1000}{27} - \frac{10}{3} = \frac{4000 - 270}{81} = \frac{3730}{81} \approx \boxed{46 \text{ lb}}$$

3. A force of 10 lb. is needed to keep a certain spring stretched to 12 in. beyond its natural length. How much work is done in stretching the spring from its natural length to 18 in. beyond its natural length?



$F(x) = kx$ $\quad$ x = # of feet beyond natural length. 12 in. = 1 ft.

$10 = k \cdot 1$

$k = 10$.

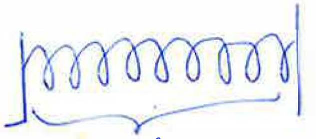
So $F(x) = 10x$

$$W = \int_0^{1.5} 10x\,dx = 5x^2 \Big|_0^{1.5} = 5x^2 \Big|_0^{3/2}$$

$$= 5 \cdot \frac{9}{4} - 0$$

$$= \frac{45}{4} = \boxed{11.25 \text{ ft.-lb.}}$$

4. Suppose the natural length of a spring is 20 cm. If 30 J of work is done in stretching a spring from a length of 35 cm to a length of 45 cm, what is the work done in stretching the spring from its natural length to 30 cm beyond its natural length?



length of spring (cm)	length (m)	meters beyond natural length
20	0.2	0
35	0.35	0.15
45	0.45	0.25

We know

$$\int_{0.15}^{0.25} kx\,dx = 30, \text{ so } 30 = \frac{k}{2}x^2\Big|_{15/100}^{25/100} = \frac{k}{2}x^2\Big|_{3/20}^{1/4} = \frac{k}{2}\left(\left(\frac{1}{4}\right)^2 - \left(\frac{3}{20}\right)^2\right)$$

$$= \frac{k}{50} \text{ (check)},$$

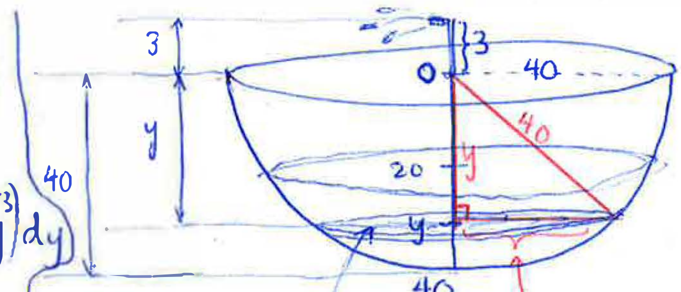
so $k = 30 \cdot 50 = 1500$. Now

$$W = \int_0^{0.3} 1500x\,dx = 750x^2\Big|_0^{3/10} = 750\frac{9}{100} = \frac{135}{2} = \boxed{67.5 \text{ J}}$$

5. A hemispherical water tank of radius 40 m has a water spout 3 m above the top of the tank. If the tank is filled to a depth of 20 m, what is the work done in pumping all the water out of the tank?

$$W = 9800\int_{20}^{40}(y+3)\cdot\pi(1600-y^2)\,dy$$

$$= 9800\pi\int_{20}^{40}(4800 + 1600y - 3y^2 - y^3)\,dy$$



Area of surface

$= \pi\cdot\left(\sqrt{1600-y^2}\right)^2$

$= \pi(1600-y^2)$

$$= 9800\pi\left(4800y + 800y^2 - y^3 - \frac{1}{4}y^4\right)\Big|_{20}^{40}$$

$$= 9800\pi\left(4800\cdot 40 + 800\cdot 1600 - 64000 - \frac{1}{4}40^4 - \left(4800\cdot 20 + 800\cdot 400 - 8000 - \frac{1}{4}\cdot 20^4\right)\right)$$

$$= 3920000000\pi \text{ J} \simeq \boxed{1.23\times 10^7 \text{ kJ}}$$

6. A hemispherical water tank of radius 40 ft. has a water spout 3 ft. above the top of the tank. If the tank is filled to a depth of 20 ft., what is the work done in pumping all the water out of the tank?

The setup for this problem is the same, except that the weight of water is 62.5 lb./ft.^{3} rather than 9800 N/m^3. So we have

$$W = 62.5 \int_{20}^{40} (y+3)\pi(1600-y^2)\,dy$$

$$= 62.5\pi \cdot 400{,}000$$

$$= 25{,}000{,}000\,\pi$$

$$\approx \boxed{7{,}853{,}981.634 \text{ ft.-lb.}}$$