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Spring 2021 MATH 76
Activity 13

TAYLOR SERIES

1. Use $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, when $|x| < 1$, to write the Taylor series of the following functions about the given center. State the interval of convergence for each series.

   (a) $f(x) = \dfrac{1}{1+2x}$; center $a = 0$;

   (b) $f(x) = \dfrac{1}{1-x^4}$; center $a = 0$;

   (c) $f(x) = \dfrac{x}{4+x^2}$; center $a = 0$;

   (d) $f(x) = \dfrac{1}{(1-x)^2}$; center $a = 0$; *Hint:* $\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right) = \dfrac{1}{(1-x)^2}$

   (e) $f(x) = \dfrac{1}{x}$; center $a = 1$; *Hint:* $\dfrac{1}{x} = \dfrac{1}{1-(1-x)}$

2. (a) Use $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$ to write the Taylor series of the function $\dfrac{e^{x^2} - x^2 - 1}{x^4}$.

   (b) Use the Taylor series to find $\lim_{x \to 0} \dfrac{e^{x^2} - x^2 - 1}{x^4}$.

3. (a) Use $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$ to write the Taylor series of the function $\dfrac{\cos\sqrt{x} - 1}{2x}$.

   (b) Use the Taylor series to find $\lim_{x \to 0^+} \dfrac{\cos\sqrt{x} - 1}{2x}$.

4. Find the Taylor series of the following function about the center $a = 0$. Specify the interval of convergence.

   (a) $f(x) = \ln(1+x)$

   (b) $f(x) = \tan^{-1}(x)$

5. Find the interval of convergence of the following series.

   (a) $\sinh(x) = \sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{(2n+1)!}$

   (b) $\cosh(x) = \sum_{n=0}^{\infty} \dfrac{x^{2n}}{(2n)!}$

1. Use $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, when $|x| < 1$, to write the Taylor series of the following functions about the given center. State the interval of convergence for each series.

(a) $f(x) = \frac{1}{1+2x}$; center $a = 0$;

$\boxed{f(x) = \sum_{n=0}^{\infty} -2x^n}$ ✓ interval of convergence is $(-1, 1)$

(b) $f(x) = \frac{1}{1-x^4}$; center $a = 0$;

$\boxed{f(x) = \sum_{n=0}^{\infty} x^{4n} \quad (-1, 1)}$ ✓

(c) $f(x) = \frac{x}{4+x^2}$; center $a = 0$;

$f(x) = \frac{x}{4+x^2} = \frac{\frac{x}{4}}{1-\left(-\frac{x^2}{4}\right)}$

$\boxed{f(x) = \sum_{n=0}^{\infty} \left(-\frac{x^2}{4}\right)^n \quad (-2, 2)}$ ✓

(d) $f(x) = \frac{1}{(1-x)^2}$; center $a = 0$; *Hint:* $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$

$f(x) = \frac{1}{(1-x)^2}$ $\quad \int f(x) = \frac{1}{1-x}$

$\int f(x) = \sum_{n=0}^{\infty} x^n$

$\boxed{f(x) = \sum_{n=0}^{\infty} nx^{n-1} \quad (-1, 1)}$ ✓

(e) $f(x) = \frac{1}{x}$; center $a = 1$; *Hint:* $\frac{1}{x} = \frac{1}{1-(1-x)}$

$f(x) = \frac{1}{x} \quad a = 1 \quad \sum_{n=0}^{\infty} ar^n$

$\boxed{f(x) = \sum_{n=0}^{\infty} (1)(1-x)^n \quad (0, 2)}$ ✓

2. (a) Use $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ to write the Taylor series of the function $\frac{e^{x^2} - x^2 - 1}{x^4}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \frac{e^{x^2} - x^2 - 1}{x^4} = \boxed{\sum_{n=0}^{\infty} \frac{\frac{x^{2n}}{2n!}}{x^4} - \sum_{n=0}^{\infty} \frac{x^2 - 1}{x^4}}$$ ✓

(b) Use the Taylor series to find $\lim_{x \to 0} \frac{e^{x^2} - x^2 - 1}{x^4}$.

$$\lim_{x \to 0} \frac{e^{x^2} - x^2 - 1}{x^4} = \boxed{\frac{1}{2}}$$ ✓