Math 76 Exercises –3.2B More Trigonometric Integrals

1. Evaluate each integral. Check by differentiating.

(a) $\int \tan x \, dx = -\int \frac{-\sin x}{\cos x} \, dx$

$u = \cos x$
$du = -\sin x \, dx$

$= -\int \frac{1}{u} \, du$

$= -\ln|\cos x| + C$ ← Either formula is okay, but you should memorize it!

$= \ln(|\cos x|^{-1}) + C$

$= \boxed{\ln|\sec x| + C}$

(b) $\frac{1}{4}\int 4\sec^3(4x) \, dx$

$t = 4x$
$dt = 4\,dx$

$= \frac{1}{4}\int \sec^3(t) \, dt$

$= \frac{1}{4}\int \sec(t)\sec^2(t) \, dt$

$u = \sec t$ $\quad v = \tan t$
$du = \sec t \tan t \, dt$ $\quad dv = \sec^2 t \, dt$

$= \frac{1}{4}\left[\sec(t)\tan(t) - \int \sec t \tan^2 t \, dt\right]$

$= \frac{1}{4}\left[\sec(t)\tan(t) - \int \sec t(\sec^2 t - 1) \, dt\right]$

$= \frac{1}{4}\left[\sec(t)\tan(t) - \int \sec^3(t) \, dt + \int \sec t \, dt\right]$

$= \frac{1}{4}\sec t \tan t - \frac{1}{4}\int \sec^3 t \, dt + \frac{1}{4}\ln|\sec t + \tan t|$

$\frac{1}{2}\int \sec^3 t \, dt = \frac{1}{4}\sec t \tan t + \frac{1}{4}\ln|\sec t + \tan t| + C$

So $\frac{1}{4}\int \sec^3 t \, dt = \frac{1}{8}\sec t \tan t + \frac{1}{8}\ln|\sec t + \tan t| + C$

$= \boxed{\frac{1}{8}\sec(4x)\tan(4x) + \frac{1}{8}\ln|\sec(4x) + \tan(4x)| + C}$

(c) $\int \tan^2 x \sec x \, dx$

$= \int (\sec^2 x - 1) \sec x \, dx$

$= \int \sec^3 x - \sec x \, dx$

$= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| - \ln|\sec x + \tan x| + C$

(See previous problem)

$= \boxed{\frac{1}{2} \sec x \tan x - \frac{1}{2} \ln|\sec x + \tan x| + C}$

(d) $\int \sin^4 x \cos^4 x \, dx$

$\sin x \cos x = \frac{1}{2} \sin(2x)$

$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$

$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$

$= \int (\sin x \cos x)^4 \, dx$

$= \int \left(\frac{1}{2} \sin(2x)\right)^4 dx$

$= \frac{1}{16} \int \sin^4(2x) \, dx$

$= \frac{1}{16} \int (\sin^2(2x))^2 \, dx$

$= \frac{1}{16} \int \left(\frac{1}{2}(1 - \cos(4x))\right)^2 dx$

$= \frac{1}{64} \int (1 - 2\cos(4x) + \cos^2(4x)) \, dx$

$= \frac{1}{64} \left(x - \frac{1}{2} \sin(4x) + \int \cos^2(4x) \, dx\right)$

$= \frac{1}{64} \left(x - \frac{1}{2} \sin(4x) + \int \frac{1}{2}(1 + \cos(8x)) \, dx\right)$

$= \frac{1}{64} \left(x - \frac{1}{2} \sin(4x) + \frac{1}{2}\left(x + \frac{1}{8} \sin(8x)\right)\right) + C$

$= \boxed{\frac{3}{128} x - \frac{1}{128} \sin(4x) + \frac{1}{1024} \sin(8x) + C}$

2. Find the length of the curve $y = \ln(\cos x)$ from $x = 0$ to $x = \frac{\pi}{3}$.

$$y' = \frac{-\sin x}{\cos x} = -\tan x$$

$$1 + (y')^2 = 1 + \tan^2 x = \sec^2 x$$

$$L = \int_0^{\frac{\pi}{3}} \sqrt{1+(y')^2}\,dx$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{\sec^2 x}\,dx$$

$$= \int_0^{\frac{\pi}{3}} \sec x\,dx$$

sec x is positive when $0 \le x \le \frac{\pi}{3}$!

$$= \ln|\sec x + \tan x|\Big|_0^{\frac{\pi}{3}}$$

$$= \ln\left|\sec\tfrac{\pi}{3} + \tan\tfrac{\pi}{3}\right| - \ln|\sec 0 + \tan 0|$$

$$= \ln|2+\sqrt{3}| - \ln|1+0|$$

$$= \ln(2+\sqrt{3}) - 0$$

$$= \boxed{\ln(2+\sqrt{3})}$$