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Spring 2021 MATH 76
Activity 12

$$L(x) = f(a) + f'(a)(x-a)$$

$$Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

TAYLOR POLYNOMIALS

1. LINEAR AND QUADRATIC APPROXIMATIONS
   Compute the linear approximation centered at a defined by

$$L(x) = f(a) + f'(a)(x-a)$$

   and the quadratic approximation centered at a defined by

$$Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

   for the following functions when available:

   (a) $f(x) = x^{3/2}$ with $a = 1$

   (b) $f(x) = \sqrt{x}$ with $a = 4$

   (c) $f(x) = \cos(x)$ with $a = \pi/4$

   (d) $f(x) = x^{1/3}$ with $a = 8$

   (e) $f(x) = \tan^{-1}(x)$ with $a = 0$

(a) $f(x) = x^{3/2}$ with $a = 1$

$L(x) = f(a) + f'(a)(x-a)$

$Q(x) = \boxed{f(a) + f'(a)(x-a)} + \frac{f''(a)}{2}(x-a)^2$

$$\boxed{\begin{aligned} f(x) &= x^{\frac{3}{2}} \\ f'(x) &= \frac{3\sqrt{x}}{2} \\ f''(x) &= \frac{3}{4\sqrt{x}} \end{aligned}}$$

$a = 1$

$$L = (1)^{\frac{3}{2}} + \frac{3\sqrt{1}}{2}(x-1)$$

$$L = 1 + \frac{3}{2}x - \frac{3}{2}$$

$$\boxed{L = \frac{3}{2}x - \frac{1}{2}}$$

$$Q(x) = (1)^{\frac{3}{2}} + \frac{3\sqrt{1}}{2}(x-1) + \frac{3}{4\sqrt{1}} \cdot \frac{1}{2}(x-1)^2$$

$$Q(x) = 1 + \frac{3}{2}(x-1) + \frac{3}{8}(x-1)^2$$

$$\rightarrow \boxed{Q(x) = -\frac{1}{2} + \frac{3}{2}x + \frac{3}{8}(x^2 - 2x - 1)}$$

$$Q(x) = \underline{-\frac{1}{2}} + \frac{3}{2}x + \underline{\frac{3}{8}x^2} - \frac{3}{4}x \underline{- \frac{3}{8}}$$

$$\boxed{Q(x) = \frac{3}{8}x^2 + \frac{3}{4}x - \frac{7}{8}}$$

(b) $f(x) = \sqrt{x}$ with $a = 4$

$L(x) = f(a) + f'(a)(x-a)$

$Q(x) = \boxed{f(a) + f'(a)(x-a)} + \frac{f''(a)}{2}(x-a)^2$

$$f(x) = \sqrt{x}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f''(x) = -\frac{1}{4x^{\frac{3}{2}}}$$

$$\boxed{L = 2 + \frac{1}{4}x + 1} \checkmark$$

$$\boxed{Q(x) = 2 + \frac{1}{4}x + 1 - \frac{1}{64}x^2 + \frac{1}{16}x - \frac{1}{64}} \checkmark$$

(c) $f(x) = \cos(x)$ with $a = \pi/4$

$L(x) = f(a) + f'(a)(x-a)$

$Q(x) = \boxed{f(a) + f'(a)(x-a)} + \frac{f''(a)}{2}(x-a)^2$

$$f(x) = \cos(x)$$

$$f'(x) = -\sin(x)$$

$$f''(x) = -\cos(x)$$

$$\boxed{L = \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)} \checkmark$$

$$\boxed{Q(x) = \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) - \frac{\cos\left(\frac{\pi}{4}\right)}{2}\left(x - \frac{\pi}{4}\right)^2} \checkmark$$

(d) $f(x) = x^{1/3}$ with $a = 8$

$L(x) = f(a) + f'(a)(x-a)$

$Q(x) = \boxed{f(a) + f'(a)(x-a)} + \frac{f''(a)}{2}(x-a)^2$

$f(x) = x^{\frac{1}{3}}$

$f'(x) = \frac{1}{3x^{\frac{2}{3}}}$

$f''(x) = -\frac{2}{9x^{\frac{5}{3}}}$

$\boxed{L = 2 + \frac{1}{12}(x-8)}$ ✓

$\boxed{Q(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2}$ ✓

(e) $f(x) = \tan^{-1}(x)$ with $a = 0$

$L(x) = f(a) + f'(a)(x-a)$

$Q(x) = \boxed{f(a) + f'(a)(x-a)} + \frac{f''(a)}{2}(x-a)^2$

$f(x) = \text{ATan}(x)$

$f'(x) = \frac{1}{x^2+1}$

$f''(x) = -\frac{2x}{(x^2+1)^2}$

$\boxed{L = x}$ ✓

$\boxed{Q(x) = x}$ ✓

2. TAYLOR POLYNOMIALS

Let f be a function whose derivatives $f', f'', \ldots, f^{(n)}$ are all defined at a. The n^{th} order Taylor polynomial centered at a is defined as

$$\begin{aligned} p_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \end{aligned}$$

(a) Compute the 7^{th} Taylor polynomial centered at $a = 0$ for the functions

i. $f(x) = \sin x$

ii. $f(x) = \cos x$

iii. $f(x) = e^x$

iv. $f(x) = \dfrac{1}{1-x}$

i. $f(x) = \sin x$ $\quad a = \emptyset$

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$f = \sin x = \emptyset$

$f' = \cos x = 1$

$f^2 = -\sin x = \emptyset$

$f^3 = -\cos x = -1$

$f^4 = \sin x = \emptyset$

$f^5 = \cos x = 1$ –

$f^6 = -\sin x = 0$

$f^7 = -\cos x = -1$

$$\boxed{P_7(x) = x - \frac{1}{3!}(x-\emptyset)^3 + \frac{1}{5!}(x-\emptyset)^5 - \frac{1}{7!}(x-0)^7}$$ ✓

ii. $f(x) = \cos x$ $\quad a = \emptyset$

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$f = \cos x = 1$

$f' = -\sin x = \emptyset$

$f^2 = -\cos x = -1$

$f^3 = \sin x = 0$

$f^4 = \cos x = 1$

$f^5 = -\sin x = \emptyset$

$f^6 = -\cos x = -1$

$f^7 = \sin x = \emptyset$

$$\boxed{P_7(x) = 1 - \frac{1}{2!}(x-\emptyset)^2 + \frac{1}{4!}(x-\emptyset)^4 - \frac{1}{6!}(x-\emptyset)^6}$$ ✓

$$f(a) + f'(a)(x-a) + \frac{f'''}{\#!}(x-a)^{\#}$$

iii. $f(x) = e^x$

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$f(x) = e^x$

$\downarrow$

$f^7(x) = e^x$

$$\boxed{P_7(x) = 1 + (x-\emptyset) + \frac{1}{2!}(x-0)^2 + \frac{1}{3!}(x-\emptyset)^3 \ldots \frac{1}{7!}(x-\emptyset)^7}$$

iv. $f(x) = \dfrac{1}{1-x}$

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$f = \dfrac{1}{1-x}$

$f' = \dfrac{1}{(x-1)^2}$

$f^2 = -\dfrac{2}{(x-1)^3}$

$f^3 = \dfrac{6}{(x-1)^4}$

$f^4 = -\dfrac{24}{(x-1)^5}$

$f^5 = \dfrac{120}{(x-1)^6}$

$f^6 = \dfrac{720}{(1-x)^7}$

$f^7 = \dfrac{5040}{(x-1)^8}$

$$\boxed{P_7(x) = \frac{1}{0!} + \frac{(x-0)}{1!}^2 - \frac{2}{2!}(x-0)^2 + \frac{6}{3!}(x-0)^3 - \frac{24}{4!}(x-0)^4 + \frac{120}{5!}(x-0)^5 + \frac{720}{6!}(x-0)^6 + \frac{5040}{7!}(x-0)^7}$$

✓

(b) Following the patterns observed in the previous question, write out,

i. the $(2n+1)$th Taylor polynomial centered at $a = 0$ for $f(x) = \sin x$

ii. the $(2n)$th Taylor polynomial centered at $a = 0$ for $f(x) = \cos x$

iii. the nth Taylor polynomial centered at $a = 0$ for $f(x) = e^x$

iv. the nth Taylor polynomial centered at $a = 0$ for $f(x) = \dfrac{1}{1-x}$

(c) Let $n \to \infty$ in the expressions obtained in the previous question and write out the functions as a power series.

i. $f(x) = \sin x = \sum_{k=0}^{\infty} \ldots\ldots\ldots$

ii. $f(x) = \cos x = \sum_{k=0}^{\infty} \ldots\ldots\ldots$

iii. $f(x) = e^x = \sum_{k=0}^{\infty} \ldots\ldots\ldots$

iv. $f(x) = \dfrac{1}{1-x} = \sum_{k=0}^{\infty} \ldots\ldots\ldots$

i. the $(2n+1)$th Taylor polynomial centered at $a = 0$ for $f(x) = \sin x$

$$\boxed{P_{2n+1}(x)\; x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots + \frac{(-1)^{2n+1}}{(2n+1)!}x^{2n+1}}$$

ii. the $(2n)$th Taylor polynomial centered at $a = 0$ for $f(x) = \cos x$

$$\boxed{P_{2n}(x) = 1 - \frac{1}{2!}(x-0)^2 + \frac{1}{4!}(x-0)^4 - \frac{1}{6!}(x-0)^6 + \ldots - \frac{(-1)^{n-1}}{(2n)!}x^{2n}}$$ ✓

iii. the nth Taylor polynomial centered at $a = 0$ for $f(x) = e^x$

$$\boxed{P_n(x) = 1 + (x-0) + \frac{1}{2!}(x-0)^2 + \frac{1}{3!}(x-0)^3 \ldots \frac{1}{7!}(x-0)^7 + \ldots + \frac{1}{n!}x^n}$$ ✓

iv. the nth Taylor polynomial centered at $a = 0$ for $f(x) = \dfrac{1}{1-x}$

$$\boxed{\begin{aligned} P_n(x) &= \frac{1}{0!} + \frac{(x-0)^2}{1!} - \frac{2}{2!}(x-0)^2 + \frac{6}{3!}(x-0)^3 - \frac{24}{4!}(x-0)^4 + \\ &\frac{120}{5!}(x-0)^5 + \frac{720}{6!}(x-0)^6 + \frac{5040}{7!}(x-0)^7 + \ldots + X^n \end{aligned}}$$

i. $f(x) = \sin x = \sum_{k=0}^{\infty} \ldots\ldots\ldots$

$$P_{2n+1}(x)\; x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots + \frac{(-1)^{2n+1}}{(2n+1)!}x^{2n+1}$$

$$\boxed{\sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{(2k+1)!}x^{2k+1}}$$ ✓

ii. $f(x) = \cos x = \sum_{k=0}^{\infty} \ldots\ldots\ldots$

$$\boxed{\sum_{k=0}^{\infty} - \frac{(-1)^{k-1}}{(2k)!} x^{2k}}$$ ✓

iii. $f(x) = e^x = \sum_{k=0}^{\infty} \ldots\ldots\ldots$

$$\boxed{\sum_{k=0}^{\infty} \frac{1}{k!} x^{k}}$$ ✓

iv. $f(x) = \dfrac{1}{1-x} = \sum_{k=0}^{\infty} \ldots\ldots\ldots$

$$\boxed{\sum_{k=0}^{\infty} - x^{k}}$$ ✓

3. REMAINDER

The remainder R_n is the difference between p_n and the original function f, and if there exists a positive number M such that $|f^{(n+1)}(x)| \leq M$ then $|R_n(x)| \leq M\dfrac{|x-a|^{n+1}}{(n+1)!}$

(a) Evaluate the maximum error that can be generated if $\cos(0.1)$ is approximated using the Taylor polynomial $p_{10}(0.1)$.

(b) What is the minimal value of n required to make at most an error of 10^{-5} when approximating $\sin(0.05)$ by $p_n(0.05)$. Use a calculator.

(a) Evaluate the maximum error that can be generated if $\cos(0.1)$ is approximated using the Taylor polynomial $p_{10}(0.1)$.

$\cos(0.1) \leq P_{10}(0.1)$

$|f^{(n+1)}(x)| \leq M$ then $|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}$

$\left|\sin_{11}(0.1)x\right| \Rightarrow R_{11} \frac{(0.1-0)^{11}}{11!} =$

$\boxed{2.5052 \cdot e-19}$ ✓

(b) What is the minimal value of n required to make at most an error of 10^{-5} when approximating $\sin(0.05)$ by $p_n(0.05)$. Use a calculator.

$10{,}000{,}0 \cdot .049979 \leq 0.00001 \cdot 10{,}000{,}0$

$10,\ 1$

$1000 \quad 100{,}000$

$10{,}000 \quad 10{,}000{,}000$

$4997.9 \leq 1$

$\frac{0.05^{n+1}}{(n+1)!}$ $\boxed{2.6042e-0.7}$ ✓