Solving the Archetypal Problem in Classical Physics

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1 Review

In the first unit of this course we discussed determinism and how it applies to the archetypal problem in classical physics:

> Given the forces acting on a body, predict its motion;
> Given the motion of a body, surmise the forces acting on it.

We noted that finding the acceleration was key to linking the forces acting on a body with its resulting motion. We also described the tools used to extract the motion from a problem. We learned that, given enough information about the forces acting on a body we could find its acceleration using Newton's Second Law:

$$\vec{a} = \frac{\Sigma \vec{F}}{m}. \tag{1}$$

In other words, vector summing all the forces acting on the system and dividing by the mass of the system tells the resulting acceleration of the system.

We also learned that, as long as the acceleration was kept constant, the equations of motion could also produce the acceleration if there was enough information about the motion of the body.

In Unit 2, we turned our attention to these equations of motion for constant acceleration:

$$\vec{d} = \vec{v}_\text{o} t + \frac{1}{2}\vec{a}t^2 \qquad \vec{v} = \vec{v}_\text{o} + \vec{a}t \tag{2}$$

We learned what each term in the equations describes physically and even applied the equations to the case of two-dimensional motion focusing in particular on projectile motion.

In Unit 3, we turned our attention to the forces acting on a body. We learned that in a given problem there can be any number of forces acting on the body in question. We limited our scope to the following forces: Applied force, the gravitational force, the normal force, tension, and friction. We then learned how to apply Newton's second law to bodies being acted upon by such forces.

Finally, in this unit we will begin solving the archetypal problem. Let us start off with some examples.

1.1 Two bodies connected by a string

Consider the two carts attached by a string in Figure 1. Suppose that the cart in the figure has a mass $M = 3$ kilograms and the hanging mass $m = 2$ kg. The question is, "How long will take for cart M to travel $d = 2$ meters starting from rest?" To simplify our problem, we assume the surface is frictionless.

We should note right up front that whatever motion one mass experiences must be experienced by the other mass since they are tied by a string. Therefore, the magnitude of the acceleration for

M is the same as that of m. At any point in time, the distance they travel and the speeds they possess will also match.

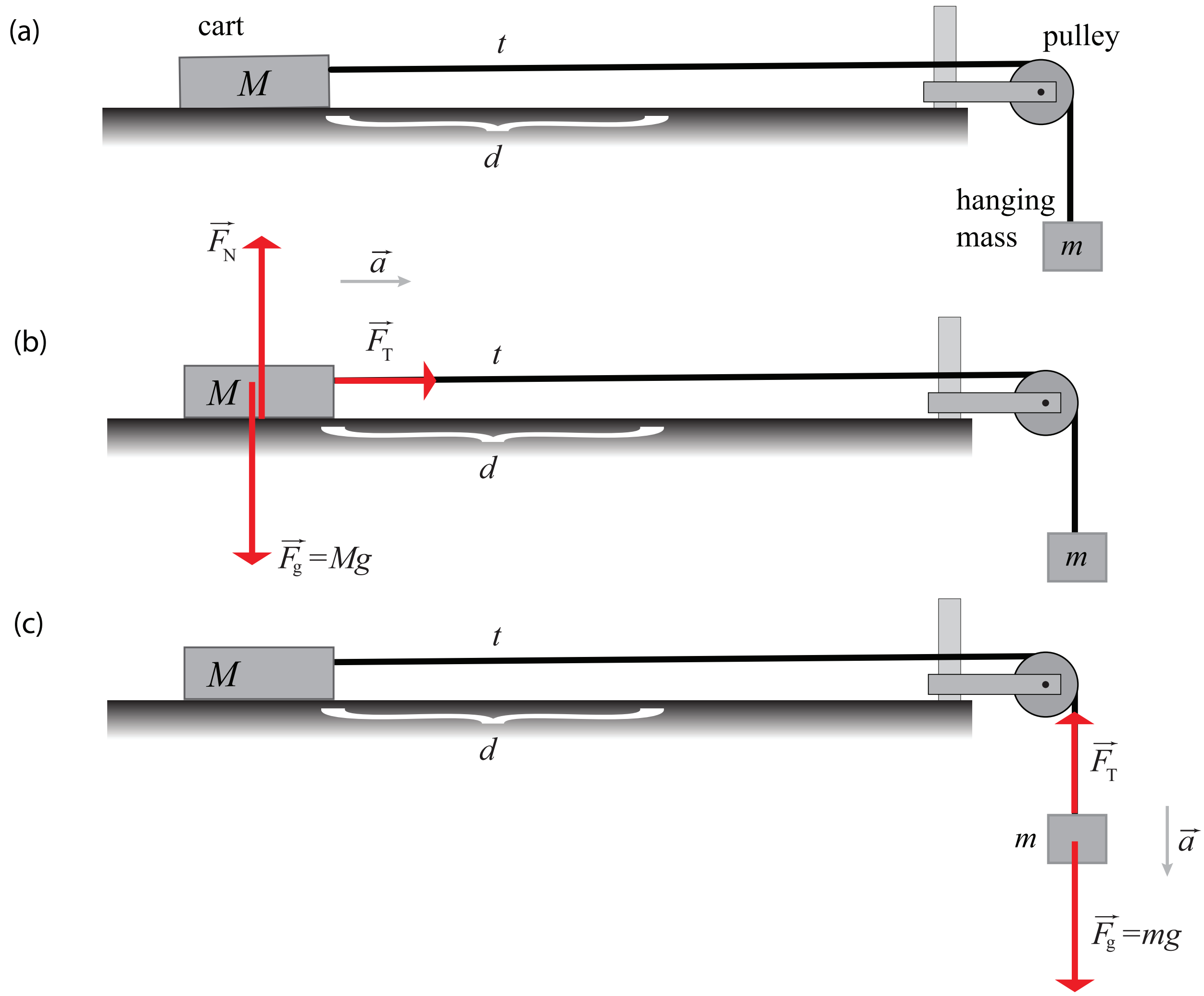

Figure 1: (a) One of the classical archetypal problems is that of two blocks connected by a string passing over a pulley with a frictionless surface. If we know the values for the masses M and m, how long will it take for the block to travel a distance d starting from rest? In part (b), we identify all the forces acting on mass M. In part (c) we turn our attention to the hanging mass m.

Newton's Second Law is a vector equation, so we can express it any direction we wish. If we first define the cart M as the system, then Newton's second law states:

$$\vec{a}_{\mathrm{x}} = \frac{\Sigma \vec{F}_{\mathrm{x}}}{M} \qquad \vec{a}_{\mathrm{y}} = \frac{\Sigma \vec{F}_{\mathrm{y}}}{M} \tag{3}$$

However, for the cart we know that $\vec{a}_{\mathrm{y}} = 0$ because it is not accelerating vertically. Therefore all of the acceleration is horizontal so can simply relabel $\vec{a}_{\mathrm{x}}$ as $\vec{a}$. Therefore

$$\vec{a} = \frac{\vec{F}_{\mathrm{T}}}{M} = \frac{\vec{F}_{\mathrm{T}}}{3} \qquad 0 = \frac{\vec{F}_{\mathrm{N}} - Mg}{M} = \frac{\vec{F}_{\mathrm{N}} - (3)(10)}{3} \tag{4}$$

The second equation tells us that the normal force $\vec{F}_{\mathrm{N}} = 30$ newtons. While true, this may or may not be useful to us. (It turns out to be useful when there is friction between the cart and the surface.) But, we trudge on. The first equation is simply

$$a = \frac{F_{\mathrm{T}}}{3} \tag{5}$$

Notice that we have replaced $\vec{a}$ with simply a noting that this acceleration points in the positive direction. Also, we replaced $\vec{F}_{\mathrm{T}}$ with F_{T}, again noting that this force points in the positive (upward) direction.

If we define the system as the hanging mass m we first note that whatever magnitude of acceleration cart M possesses, this must match the magnitude of the acceleration of the hanging mass m. Then Newton's Second Law states

$$\vec{a}_{\mathrm{y}} = \frac{\Sigma \vec{F}_{\mathrm{y}}}{m} = \frac{\vec{F}_{\mathrm{T}} - mg}{m}. \tag{6}$$

Now we can substitute values. However, to do so we must remove the vector symbols off the top of each variable, which requires that we insert the proper sign (plus or minus). With most of these variables, no problem. However, we should note that the hanging mass is accelerating *downwards* at some unknown magnitude a.[1] Therefore,

$$-a = \frac{\vec{F}_{\mathrm{T}} - (2)(10)}{2}. \tag{7}$$

Scanning over our work, we notice that we have two equations and two variables for solving for the acceleration. Solving for the tension force for cart M we get $\vec{F}_{\mathrm{T}} = 3a$, which we substitute into the force summation for the hanging mass:

$$-a = \frac{3a - 20}{2}. \tag{8}$$

Solving for the acceleration, we get $a = 4\ \mathrm{m/s^2}$. Our answer is positive, indicating we chose our original direction for the acceleration vector correctly.

So far, so good. We used the Newton's Second Law tool to extract the acceleration from the knowledge of the forces. Again, finding the acceleration is a vital step in solving the archetypal problem.

Now that we have this acceleration, we can use the equations of motion tools to find the time it takes to accelerate from rest to a distance $d = 2$ meters. (Note that by our normal convention the displacement $\vec{d}$ is positive since the object ends up to the right of where it starts.) We start with

$$\vec{d} = \vec{v}_{\mathrm{o}} t + \frac{1}{2}\vec{a}t^2. \tag{9}$$

Since cart M starts from rest, then $\vec{v}_{\mathrm{o}} = 0$, leaving us with (remembering that we already found $a = 4\ \mathrm{m/s^2}$)

$$2 = \frac{1}{2}(4)t^2. \tag{10}$$

Solving for time, we get $t = 1$ second. (The MKS/SI unit of time is seconds. Because we ensured that all of our units were expressed in the MKS/SI unit system, then our answer will also be expressed in the MKS/SI unit system.)

[1] Yes, I forgot to do this in class and ended up with a completely unreasonable value for the acceleration.

1.2 Example: Dragster

I have administered the dragster problem to my students on numerous occasions. It is a classic problem that appears, on its surface, as completely unsolvable.

The dragster problem involves a car starting from rest and accelerating a given distance. This distance used to a quarter-mile, but due to the lack of intestinal fortitude by the new breed of dragster wimps this distance has been reduced to only an eighth of a mile. We will simplify the distance of travel to 200 meters.

Hanging from the rear-view mirror of the dragster is a rock, as shown in Figure 2. (Why would a drag racer hang a rock from a rear view mirror? Because it's his lucky rock.) As the dragster accelerates down the track, the air freshener will sweep back like a pendulum, maintaining a fix angle θ with respect to the horizontal. Let us assume $\theta = 30^{\text{o}}$.

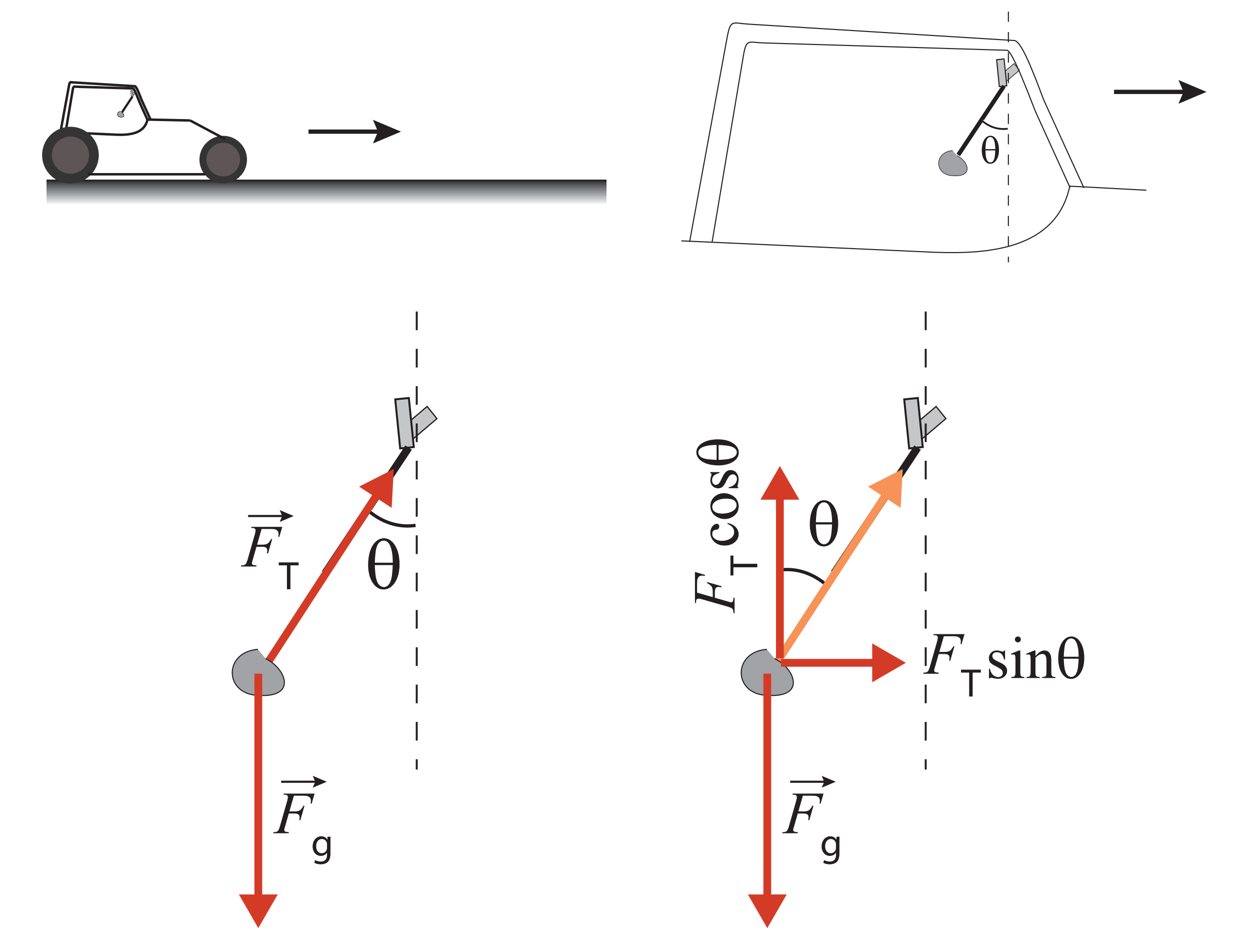

Figure 2: A dragster accelerating down a track. Attached to the rear view mirror is a rock, which subtends an angle θ during the acceleration. The dragster, starting from rest, will accelerate for a distance $d = 200$ meters. How long will it take for the dragster to reach the end of the track?

Our question is, "How long will it take for the car to reach the end of the track?" Rather than bemoan the lack of information provided to us (there isn't even any mention of the mass of the rock), we begin. We know we must find the acceleration of the rock somehow, and there are only two ways of doing so:

- Apply Newton's second law

- Apply the equations of motion

We can eliminate the use of the equations of motion right away. We know only the displacement ($\vec{d} = 200$ meters in the direction toward the right) and the initial velocity ($\vec{v}_\mathrm{o} = 0$). That along is not enough to find the acceleration because any acceleration can propel the dragster to 200 meters down the track starting from rest.

Therefore, we turn our attention to Newton's Second Law:

$$\vec{a} = \frac{\Sigma\vec{F}}{m} \tag{11}$$

This equation tells us what to do: Vector sum the forces acting on the system (the rock) and divide by the mass of the rock. We don't know the mass of the rock, so let's just call it m and hope something magical happens.

Because Newton's second law is a vector equation, it apples along any direction. Here, the horizontal and vertical directions are natural directions, so let's choose those. (Ideally, it is best to choose one of the directions along the acceleration vector so that one of the force summations equals 0.)

$$\vec{a}_\mathrm{x} = \frac{\Sigma\vec{F}_\mathrm{x}}{m} \qquad \vec{a}_\mathrm{y} = \frac{\Sigma\vec{F}_\mathrm{y}}{m} \tag{12}$$

We know that the dragster accelerates only horizontally, so $\vec{a}_\mathrm{y} = 0$. What forces act on the rock? There are only two: The tension force created by the string and gravity. The tension force $\vec{F}_\mathrm{T}$ points along the string, which is neither horizontal or vertical; therefore, we will need to find its horizontal and vertical components. Luckily, the gravitational force $\vec{F}_\mathrm{g}$ points straight down.

Note that we need to use a small bit of geometry to relabel our angle in the picture. After we do, the horizontal and vertical components of the tension force are fairly easy to pick off if remember the "Touchy Touchy Touchy Touchy Cosine Song."

$$\vec{F}_\mathrm{T,y} = F_\mathrm{T}\cos\theta \qquad \vec{F}_\mathrm{T,x} = F_\mathrm{T}\sin\theta \tag{13}$$

Vector summing forces along the horizontal and vertical we obtain

$$\vec{a}_\mathrm{x} = \frac{F_\mathrm{T}\sin(30^\circ)}{m} \qquad 0 = \frac{F_\mathrm{T}\cos(30^\circ) - m(10)}{m} \tag{14}$$

where we have used that $g = 10$ m/s^2. Next, we note that $\sin(30^\circ) = 0.50$ and $\cos(30^\circ) = 0.86$. Therefore

$$\vec{a}_\mathrm{x} = \frac{0.50F_\mathrm{T}}{m} \qquad 0 = \frac{0.86F_\mathrm{T} - m(10)}{m} \tag{15}$$

The second equation means that $F_\mathrm{T} = \frac{10m}{0.86}$. Substitution this result into the first equation we get

$$\vec{a}_\mathrm{x} = \frac{F_\mathrm{T}}{m} = \left(\frac{0.5}{m}\right)\left(\frac{10m}{0.86}\right) = 5.8\mathrm{m/s}^2 \tag{16}$$

We have found the acceleration! (Notice that the mass of the rock m cancels, so it was never needed to solve this problem.) That means we can now predict what will happen with the motion. In this case, we want to find out how long it takes to acceleration down the track. We refer to our displacement equation of motion, $\vec{d} = \vec{v}_\mathrm{o}t + \frac{1}{2}\vec{a}t^2$ and substitute what we know into the equation:

$$200 = (0)t + \frac{1}{2}(5.8)t^2 \quad \longrightarrow \quad 200 = 2.9t^2 \tag{17}$$

Solving for the time we find $t = \sqrt{200/2.9} = 8.3$ seconds.

How fast will the dragster be traveling when it reaches the end of the drag strip? We refer to our velocity equation of motion,

$$\vec{v} = \vec{v}_{\mathrm{o}} + \vec{a}t \quad \longrightarrow \quad \vec{v} = 0 + (5.8)(8.3) = 48 \text{ m/s (about 100 mph)} \tag{18}$$

1.3 Example: Sliding Block on Rough Surface

Consider another example problem. In this one, a block of mass $m = 3$ kg is sliding across the ground at 10 m/s when it encounters a rough patch of ground, as shown in Figure 3. After hitting this rough patch, the block slows to only 4 m/s in a distance of 2 meters. The question is, "What is the coefficient of kinetic friction between the rough patch of ground and the block?"

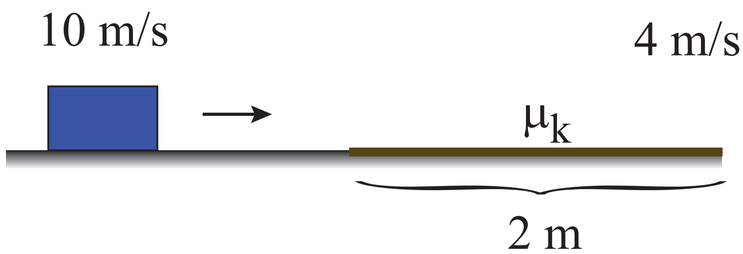

Figure 3: A dragster accelerating down a track. Attached to the rear view mirror is a rock, which subtends an angle θ during the acceleration. The dragster, starting from rest, will accelerate for a distance $d = 200$ meters. How long will it take for the dragster to reach the end of the track?

As always, the key is to find the acceleration of the block while it is sliding across the rough ground. We have two ways to find the acceleration:

- Apply Newton's second law
- Apply the equations of motion

In this instance, the object decelerates from a known initial speed to a final speed in a given distance. Only one acceleration will do that.[2] Therefore, the equations of motion should do the trick.

$$\vec{d} = \vec{v}_{\mathrm{o}}t + \frac{1}{2}\vec{a}t^2 \qquad \vec{v} = \vec{v}_{\mathrm{o}} + \vec{a}t \tag{19}$$

We can substitute our known values into the first equation and obtain

$$2 = 10t + \frac{1}{2}(-a)t^2 \quad \longrightarrow \quad 4 = 20t - at^2. \tag{20}$$

Note that we inserted $-a$ for $\vec{a}$. We don't know the amount of acceleration a but we can guess its direction – toward the left in the figure, which is in the negative direction. Note also that we multiplied through the equation by 2 to get rid of the fraction.

We only have one equation, but two unknowns (a and t). So, let's look at our velocity equation of motion:

$$\vec{v} = \vec{v}_{\mathrm{o}} + \vec{a}t \quad \longrightarrow \quad 4 = 10 + (-a)t \quad \longrightarrow \quad at = 6. \tag{21}$$

[2] As always, we are assuming a constant acceleration.

Now we have two equations, two unknowns. We can therefore solve for time t in the preceding equation and substitute this result into the displacement equation of motion:

$$4 = 20\Big(\frac{6}{a}\Big) - a\Big(\frac{36}{a^2}\Big) \qquad \longrightarrow \qquad 4a = 100 - 36a. \tag{22}$$

Solving for the acceleration, we get $a = 2.5\ \mathrm{m/s^2}$. Now that we have our acceleration, we can analyze the forces acting on the block. Newton's Second Law for the horizontal and vertical directions states

$$\vec{a}_{\mathrm{x}} = \frac{\Sigma\vec{F}_{\mathrm{x}}}{m} \qquad \vec{a}_{\mathrm{y}} = \frac{\Sigma\vec{F}_{\mathrm{y}}}{m} \tag{23}$$

Substituting what we know and what we learned we get

$$-2.5 = \frac{-F_{\mathrm{fr}}}{m} \qquad 0 = \frac{F_{\mathrm{N}} - mg}{m} \tag{24}$$

The vertical summation tells us that $F_{\mathrm{N}} = mg$. (Note that we discovered this by summing the forces – never assume it.) The horizontal summation produces $F_{\mathrm{fr}} = 2.5m$

At this point, we seem to be at a loggerhead. But we have forgotten that we know something else about our system. Because this is a friction problem, then we can relate the friction force to the normal force, that is $F_{\mathrm{fr}} = \mu_k F_{\mathrm{N}}$. Therefore, we can say that $F_{\mathrm{fr}} = \mu_k mg = 10\mu_k m$. Substituting this expression for the force of friction we get $\mu_k = \frac{2.5}{10} = 0.25$.

We are finished!