Determine, if possible, whether each of the following series

- converges absolutely;
- converges conditionally; or
- diverges.

You may use any test we have learned so far, though for many — but not all — of these problems the ratio or root test is most helpful. If the ratio or root test does not apply, explain why not. Can you find another test to evaluate the series, in that case?

1. $\sum_{n=1}^{\infty} \frac{n^4}{4^n}$

Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^4}{4^{n+1}} \cdot \frac{4^n}{n^4}$

$= \lim_{n\to\infty} \frac{(n+1)^4 \cdot \cancel{4^n}}{\cancel{4^n} \cdot 4 \cdot n^4} = \lim_{n\to\infty} \frac{n^4 + (\text{middle terms}) + 1}{4n^4} = \frac{1}{4} < 1.$

Therefore $\sum \frac{n^4}{4^n}$ converges absolutely

[Moral: exponential functions grow faster than power functions, so $\frac{(\text{power})}{(\text{exponential})}$ goes to zero very fast!]

2. $\sum_{n=0}^{\infty} \frac{e^n}{n!}$

Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n}$

$= \lim_{n\to\infty} \frac{\cancel{e^n} \cdot e \cdot \cancel{n!}}{(n+1) \cdot \cancel{n!}\, \cancel{e^n}}$ ... $(n+1)! = (n+1)n(n-1)\cdots 2\cdot 1 = (n+1)n!$

$= \lim_{n\to\infty} \frac{e}{n+1} = 0 < 1.$

Therefore $\sum \frac{e^n}{n!}$ converges absolutely

[Moral: factorials grow faster than exponential functions, so $\frac{(\text{exponential})}{(\text{factorial})}$ goes to zero very fast!]

3. $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n(n+7)^n}{(5n-2)^n}$

Root test: $\displaystyle\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\left(\frac{n+7}{5n+2}\right)^n\right)^{\frac{1}{n}}$

$\displaystyle= \lim_{n\to\infty} \frac{n+7}{5n+2} = \frac{1}{5} < 1.$

Therefore $\displaystyle\sum \frac{(-1)^n(n+7)^n}{(5n-2)^n}$ converges absolutely

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(3n-1)!}{4n^n}$

Ratio Test: $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(3(n+1)-1)!}{\cancel{4}(n+1)^{n+1}} \cdot \frac{\cancel{4}n^n}{(3n-1)!}$

$\displaystyle= \lim_{n\to\infty} \frac{(3n+2)!}{(3n-1)!} \cdot \frac{n^n}{(n+1)^n(n+1)} = \lim_{n\to\infty} \underbrace{\frac{(3n+2)(3n+1)3n}{n+1}}_{\text{goes to } \infty} \cdot \underbrace{\left(\frac{n}{n+1}\right)^n}_{\text{Hmm...} *}$

$(3n+2)! = (3n+2)(3n+1)3n(3n-1)!$

* Let $L = \displaystyle\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$. This is an indeterminate form of type "1^∞". Use logarithms and l'Hôpital's Rule:

$\displaystyle\ln(L) = \lim_{n\to\infty} n \ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} \frac{\ln(n) - \ln(n+1)}{\frac{1}{n}} \overset{H}{=} \lim_{n\to\infty} \frac{\frac{1}{n} - \frac{1}{n+1}}{-\frac{1}{n^2}}$

$\displaystyle= \lim_{n\to\infty} -\frac{n^2}{n(n+1)}$ (check this algebra!) $= -1$. So $L = \frac{1}{e}$.

Therefore $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty > 1.$ So $\displaystyle\sum \frac{(3n-1)!}{4n^n}$ diverges.

5. $1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{3}\right)^3+\left(\frac{1}{4}\right)^4+\dots = \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^n.$

Root Test: $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty}\left(\left(\frac{1}{n}\right)^n\right)^{1/n} = \lim_{n\to\infty}\frac{1}{n} = 0 < 1.$

Therefore the series converges absolutely

Ratio Test? $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

6. $\sum_{n=0}^{\infty}\frac{(-1)^n(4n-7)}{\sqrt[3]{5n^{11}-n+8}}$

$$= \lim_{n\to\infty}\frac{4(n+1)-7}{\sqrt[3]{5(n+1)^{11}-(n+1)+8}}\cdot\frac{\sqrt[3]{5n^{11}-n+8}}{4n-7}$$

$$= \lim_{n\to\infty}\frac{4n}{\sqrt[3]{5n^{11}}}\cdot\frac{\sqrt[3]{5n^{11}}}{4n} = 1. \text{ FAILS}$$

[Moral: the Ratio Test does not tend to work on p-series or series that compare with p-series, e.g. rational functions]

Instead try for absolute convergence with Limit Comparison Test:

$|a_n| = \frac{4n-7}{\sqrt[3]{5n^{11}-n+8}}$. $b_n = \frac{n}{\sqrt[3]{n^{11}}} = \frac{n}{n^{11/3}} = \frac{1}{n^{8/3}} \rightsquigarrow \sum b_n$ conv.

$$\lim_{n\to\infty}\frac{|a_n|}{b_n} = \lim_{n\to\infty}\frac{4n-7}{\sqrt[3]{5n^{11}-n+8}}\cdot\frac{\sqrt[3]{n^{11}}}{n} = \lim_{n\to\infty}\frac{4n}{\sqrt[3]{5n^{11}}}\cdot\frac{\sqrt[3]{n^{11}}}{n} = \frac{4}{\sqrt[3]{5}}$$

finite and positive!

So since $\sum b_n$ converges, $\sum |a_n|$ converges.

Thus $\sum a_n$ converges absolutely

7. $\displaystyle\sum_{n=2}^{\infty} \frac{(n^2+7)^n}{(n+5)^{2n}}$ Root Test? $\displaystyle\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{n^2+7}{(n+5)^2} = 1.$ FAILS

Notice that $\dfrac{(n^2+7)^n}{(n+5)^{2n}}$ is "comparable" to $\dfrac{n^{2n}}{n^{2n}} = 1$

(check!). By the Divergence Test, $\sum 1$ diverges.

Therefore $\displaystyle\sum \frac{(n^2+7)^n}{(n+5)^{2n}}$ diverges

[Alternatively, can apply the Divergence Test directly by taking $\displaystyle\lim_{n\to\infty} \frac{(n^2+7)^n}{(n+5)^{2n}} = \lim_{n\to\infty} \left(\frac{n^2+7}{(n+5)^2}\right)^n$.

(Use logarithms and l'Hôpital as in # 4 to show that the above limit is equal to 1, which is not equal to 0.)]

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8. $\sum_{n=1}^{\infty} \frac{14}{3^n - 2^n}$

Ratio Test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\cancel{14}}{3^{n+1} - 2^{n+1}} \cdot \frac{3^n - 2^n}{\cancel{14}}$

$= \lim_{n\to\infty} \frac{3^n}{3^{n+1}} = \frac{1}{3} < 1.$

Therefore $\sum \frac{14}{3^n - 2^n}$ converges absolutely

9. $\sum_{n=3}^{\infty} \frac{(-1)^n 2}{n \ln n - 7}$

$|a_n| = b_n = \frac{2}{n \ln n - 7}$ is comparable to

$\frac{1}{n \ln n}$ (Check that $\lim_{n\to\infty} \frac{2}{n \ln n - 7} \cdot \frac{n \ln n}{1}$ is finite and positive). The series $\sum \frac{1}{n \ln n}$ diverges by the Integral Test (see 5.3 #3). Therefore $\sum |a_n|$ diverges, so $\sum a_n$ does not converge absolutely.

However, using the Alternating Series Test,

- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n \ln n} = 0$ and
- $\{b_n\}$ is decreasing (check).

Thus $\sum a_n$ converges.

Therefore $\sum a_n$ converges conditionally