

Energy and Power

John R. Walkup, Ph.D.

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1 Review

(under construction)

2 Work

So far, we have learned to examine the forces acting on a system and then use that knowledge to predict the behavior of the system. Or, we have looked at the behavior of the system and used that knowledge to analyze the forces acting on the system.

Consider a huge (massive) refrigerator at rest on a kitchen floor. If the coefficient of friction between the refrigerator and the floor is high enough, I can push on the refrigerator until the cows come home and never get anything accomplished. The force that I am applying is equivalent to the frictional force that opposes me. But, in the end, the wall is still there — I haven't accomplished anything.

Suppose instead that I can push with enough force to move the refrigerator across the room at a constant velocity. In this case, the acceleration is 0 so the force that I apply must still equal the frictional force in magnitude (but opposite in direction).

In both cases, the acceleration of the refrigerator is 0, but somehow I have managed to accomplish something in the second scenario that I was unable to accomplish in the first scenario (namely, move the box across the room). Therefore, we now invent a new physics principle that can measure (loosely speaking) how much a force applied to an object actually accomplishes. We call this physical property *work* and denote it W .

If we are going to define this property, we realize that it must depend on at least two physical properties: The force F applied to the object and the distance d that the object travels while the force is being applied. We could simply multiply F and d , but we should also notice that our applied force was not the only one acting on the refrigerator: Friction, the normal force, and gravity also acted on it throughout its motion. Consider the normal force and gravity. While they acted on the refrigerator, they didn't help move it. And if anything, friction actually *opposed* the motion. Clearly, not only is it important how much force acts on the refrigerator while it moves, but the direction the force points must also be a factor.

If we take into account this direction, we obtain a reasonable definition of work:

$$W = Fd \cos \theta_{\vec{F}\vec{d}} \quad (1)$$

Here, $\theta_{\vec{F}\vec{d}}$ is the angle between the force vector \vec{F} and the displacement vector \vec{d} . We can apply this equation to find the work done by the force. Consider the example shown in Figure 1. In this case, the 100-N force acts through a distance of 2 meters. However, the angle between the force vector

and the displacement vector is 150° . (Make sure you understand why $\theta_{\vec{F}\vec{d}} = 150^\circ$.) Therefore, the work done by this force is given by $W = (100 \text{ N})(2 \text{ m}) \cos(150^\circ) = (100)(2)(-0.866) = -173 \text{ joules}$. The meaning of the negative sign will clear up shortly.

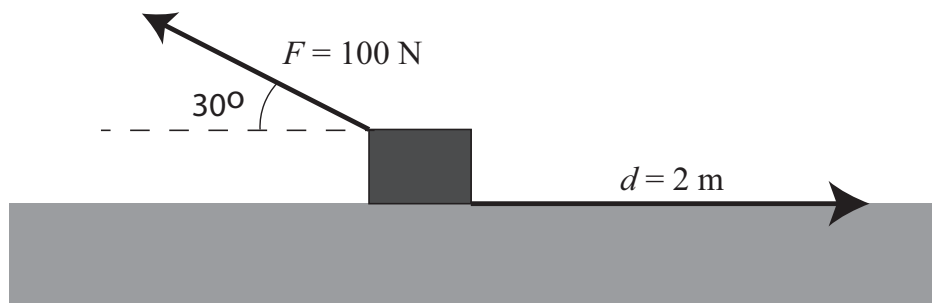


Figure 1: A force applied to an object as it travels through a distance d . Note that the object can slide to the right even though the force acts toward the left.

Before moving on, we should note some important aspects of work:

- Work is always done by a force. So when calculating work we must always ask, "by which force?"
- The values that we substitute in for F and d should always be positive numbers because the $\cos \theta_{\vec{F}\vec{d}}$ is solely responsible for generating the proper sign (plus or minus) on our result.
- The SI units of work (that is, the MKS units of work) would seemingly be newtons times meters, that is, $\text{N}\cdot\text{m}$. However, we give this unit a special name, the *joule*. The joule would be the amount of work accomplished when we lift an apple (which weighs about a newton) a height of 1 meters. Therefore, every time we lift an apple to eat while standing we do about 1 joule of work (roughly).
- The " $\vec{F}\vec{d}$ " subscript on $\theta_{\vec{F}\vec{d}}$ is merely reminder that this angle measures the angle between the vectors \vec{F} and \vec{d} . (Textbooks don't use these subscripts, but I find them helpful reminders. Your mileage may vary.)
- If the force applied to the object acts in the same direction as the displacement, then $\theta_{\vec{F}\vec{d}} = 0$ and therefore $\cos \theta_{\vec{F}\vec{d}} = 1$. In this case, our equation for work simplifies to $W = Fd$ and will be positively valued, which means the force is actually accomplishing something, that is, the force is trying to push the object in a certain direction and the object is indeed moving in that direction.
- If the force counteracts the motion of the object, then $\theta_{\vec{F}\vec{d}} = 180^\circ$ and so $\cos \theta_{\vec{F}\vec{d}} = -1$ therefore producing a negative work. A negative work means the object is moving in opposition to the force; we can say that the force is losing ground with respect to the object's motion.
- If the force applied to the object is perpendicular to the displacement vector, then $\theta_{\vec{F}\vec{d}} = 90^\circ$ and therefore $\cos \theta_{\vec{F}\vec{d}} = 0$. In this case, the work done by this force is 0. (The force acts on the object for sure, but it does no work.)

2.1 Example: Refrigerator

Consider a refrigerator of mass 100 kg that we are pushing across a flat kitchen floor at a constant velocity for $d = 10$ meters. Suppose the coefficient of kinetic friction is $\mu_k = 0.5$ and we have

managed to push hard enough on the refrigerator to get it moving. How much work does each of the forces acting on the block perform during this process?

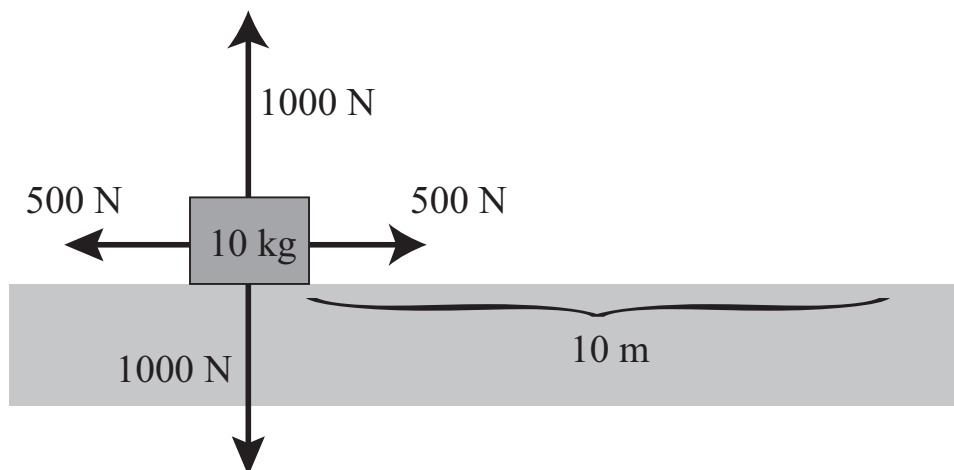


Figure 2: The forces acting on the refrigerator as the object is pushed through a distance of 10 meters.

Let us first examine the forces acting on the refrigerator. First, the force of gravity acting on the refrigerator is roughly $F_g = mg = (100 \text{ kg})(10 \text{ m/s}^2) = 1000$ newtons. Then, summing the forces along the vertical, we get $F_N - mg = 0$, so it isn't hard to see that the normal force $F_N = 1000$ newtons as well. Because the refrigerator is already moving across the kitchen surface (that is, the system is clearly the kinetic case), the force of friction acting on it is $F_{\text{fr}} = \mu_k F_N = (0.5)(1000 \text{ N}) = 500$ newtons. Because we are moving the refrigerator across at constant velocity (that is with zero acceleration) then the force we apply must also equal $F_A = 500$ newtons. (Remember, Newton's second law states that if an object is not accelerating along a certain direction, the forces that act along that direction must vector sum to 0.) How much work is done by our applied force (that is, \vec{F}_A)? We refer to our equation for work:

$$W = F_A d \cos \theta_{\vec{F}_A \vec{d}} \quad (2)$$

Here, the force we apply is $F_A = 500$ newtons and we apply this force through a distance $d = 10$ meters. But what about the angle $\theta_{\vec{F}_A \vec{d}}$? Since our applied force is in the same direction as the path we push the refrigerator, then $\theta_{\vec{F}_A \vec{d}} = 0$ and therefore $\cos \theta_{\vec{F}_A \vec{d}} = 1$. Therefore, the work we do is $W = (500)(10)(1) = 5000$ joules. The answer is positive because the force we apply is helping make the object travel in the same direction as the object travels.

The amount of friction acting on the refrigerator is also 500 newtons, but the frictional force *opposes* the motion. Therefore $\theta_{\vec{F}_{\text{fr}} \vec{d}} = 180^\circ$ and so $\cos \theta_{\vec{F}_{\text{fr}} \vec{d}} = -1$. The amount of work done by friction is therefore $W = (500)(10)(-1) = -5000$ joules. This work is negative because the refrigerator is moving *in opposition* to the frictional force.

Now, let us examine the work done by the other forces. The force of gravity is $F_g = mg = 1000$ newtons, but this force is perpendicular to the motion. (The force of gravity acts downward, but the displacement vector points horizontally.) Therefore $\theta_{\vec{F}_g \vec{d}} = 90^\circ$ and so the work done by gravity is $W = 0$. Likewise, the normal force acting on the refrigerator, much like gravity, is also 1000 newtons. It also points perpendicular to the surface, which in this case is straight up. Again, this force is perpendicular to the displacement and the work done by the normal force is therefore $W = 0$.

To summarize our results for this problem, we have found that

- the work done by the applied force is 5000 joules,
- the work done by gravity is 0,
- the work done by the normal force is 0, and
- the work done by friction is -5000 joules.

2.2 Example: Lowering a box

Suppose I have a box weighing 200 newtons sitting on the top of a shelf that I need to be lowered 2 meters to the shelf below. Suppose I do this by applying an upward force of 150 newtons while the box lowers. (Note that if I applied a 200 newton force upwards the box could drop at constant speed. However, the force I am applying is somewhat less than 200 newtons, so the box will accelerate downwards.)

How much work is done by gravity during this process? Gravity applies a force of magnitude 200 newtons, which points downwards. The force of gravity points in the same direction of the displacement, so $\theta_{\vec{F}\vec{d}} = 0$ and so $\cos\theta_{\vec{F}\vec{d}} = 1$. Therefore, the work done by gravity is $W = (200)(2)(1) = 400$ joules.

How much work do I do? I apply a force of 150 newtons, but in the *opposite* direction of the path. Therefore $\cos\theta_{\vec{F}\vec{d}} = \cos(180^\circ) = -1$. The work done by me is therefore $W = (150)(2)(-1) = -300$ joules. The negative answer is expected — I was pushing up on the object, but the motion of the object opposed me.

We should also note that the net work W_{net} done by all forces is $W_{\text{net}} = 400 + (-300) = 100$ joules. The concept of net work will become important later.

3 Energy

Physics is filled with numerous concepts related to measurement, such as momentum, acceleration, impulse, and even one called the jerk. (No fooling!) The nature of this course only touches on a few of these concepts. For our purposes, the most important concept to understand (besides the more obvious ones like distance and time) is energy. When we say that something has energy, we are saying that it has the ability to do work. So we define:

Energy — the ability to do work.

3.0.1 Kinetic energy

An example we will use is a nail in Styrofoam, as depicted in Figure 3. A brick that is traveling can drive the nail into the Styrofoam. In more formal terms, the brick can apply a force F on the nail, driving the nail through a distance d . Therefore, the brick, because of its motion, is said to have energy because it has the ability to perform work on the nail.

So if an object has kinetic energy (and therefore the ability to do work) whenever it is moving, how much of this kinetic energy does it have? Well, we have a formula for that,

$$\text{KE} = \frac{1}{2}mv^2. \quad (3)$$

An object can have energy due its motion (kinetic energy) or location near other charged objects.

The energy we discussed so far was a result of the motion of an object, such as the brick. This form of energy is called *kinetic energy*. There is another form of energy called potential energy that is a little harder to understand. Let us return to the brick falling on the nail.

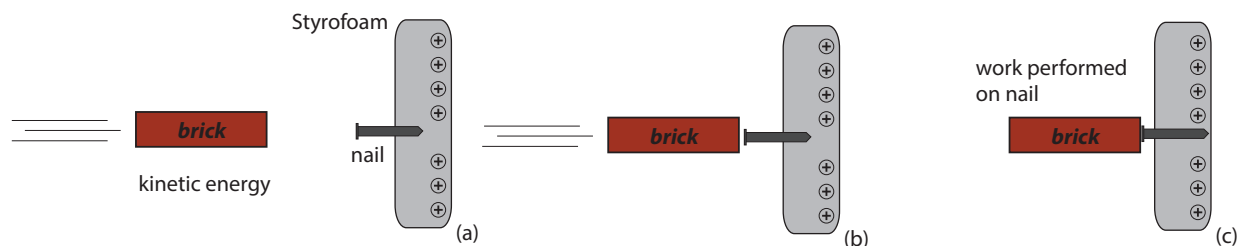


Figure 3: An object of mass m traveling at a speed v can do work on an object it encounters. It is therefore said to have *kinetic energy* of $KE = \frac{1}{2}mv^2$.

3.0.2 Potential energy

If we hold the brick over the top of the nail, we know that if we dropped the brick, it would fall towards the nail, picking up speed up to the point where it strikes the nail and performs work on it. Although not obvious, we can say that the brick, while being held above the nail, also has energy because in that position it can, once released, ultimately perform work on the nail. Yes, the work is not performed right away (the brick first has to fall), but this ability to do work is still there. We say then that the brick has energy due to its location above the nail.

Energy that is associated with an object's location, and not its movement, is called *potential energy*. We would call the energy of the brick in this situation *gravitational potential energy*, because the potential energy of the brick is due to the force of gravity acting on it. (This also means that if the system was placed in outer space far from any meaningful gravitational field, then it will not have any gravitational potential energy — releasing the brick does no work because the brick will simply float above the nail.)

So how much gravitational potential energy does an object possess? For one, the gravitational potential energy must depend on its mass. A lead brick raised over the top of the nail will clearly be able to do more work than a Styrofoam brick. The potential energy must also depend on the strength of the gravitational field, that is, g . Finally, the height the object is placed above the nail must also play a role. In all, the potential energy of an object of mass m placed a height h above a surface is given by

$$PE = mgh. \quad (4)$$

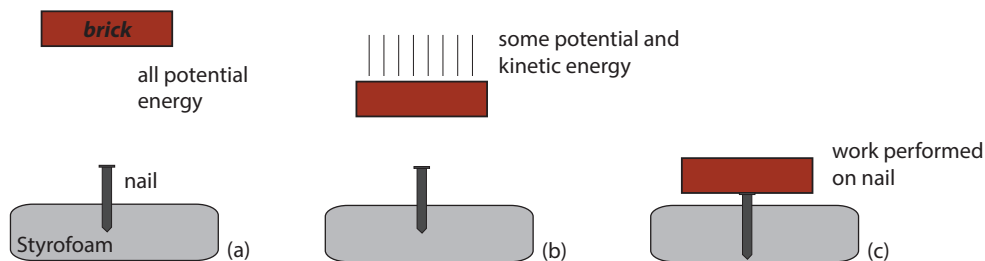


Figure 4: When a brick is placed in a gravitational field above another object (in this case, a nail) it can perform work once it is let go and allowed to fall, therefore striking the nail and driving a distance into the Styrofoam. The energy due to an object's location is called *potential energy*, which in this case is gravitational potential energy.

3.1 Total energy and conservation of energy

Notice that as the brick falls, it gains more speed. As such, we say that the total energy of the brick is conserved. This is an example of energy transformation — the conversion of energy from one form into another.

The total energy of a system is given by the sum of its kinetic and potential energy, that is

$$E = KE + PE. \quad (5)$$

This means that any change in total energy ΔE is given by

$$\Delta E = \Delta KE + \Delta PE. \quad (6)$$

If the total energy of the system remains constant, then $\Delta E = 0$ and therefore $\Delta KE = -\Delta PE$. This means that as the brick falls, any loss in potential energy (that is, ΔPE) will be compensated by a gain in kinetic energy (ΔKE).

But under which conditions is $\Delta E = 0$? Knowing the answer to this question is of paramount importance in introductory physics. We must examine the forces acting on the system and categorize them according to two types — conservative or non-conservative — and whether the work they do on a system is path-dependent.

Consider the block of mass m resting on a rough surface in Figure 5. I plan to move the block from point A to point B. Because the surface is rough, I must apply a force to the mass to move it between these two points. Which path do I take?

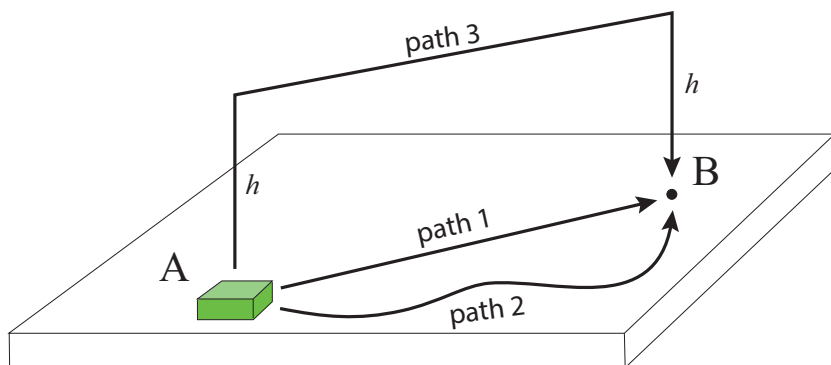


Figure 5: The work performed in moving the box from Point A to Point B can depend on the path chosen, depending on whether the force creating the work is conservative or non-conservative.

Inherently, we know that if we wanted to push the mass along the surface of the table with an applied force F_A , then pushing it along Path 1 is “better” than pushing it along Path 2. The magnitude of the force required to push the object is the same along either path, but the work done is not — Path 1 is shorter and so requires less work. What if we choose Path 3, where we lift the object up, shift its location to hover over point B, then lower it to Point B? The amount of work we perform along this path depends on the mass of the object, but we can see that there is no good reason to believe that the work we perform along Path 3 will be the same as that of Path 1 or Path 2. The work generated by our applied force therefore depends on the path we choose. The same goes for the frictional force — friction does more work on the object along Path 2 than Path 1 and does no work at all along Path 3. So the work done by friction is also path-dependent.

Now, consider gravity. The amount of work done by gravity is clearly 0 for Path 1 and Path 2 because the gravitational force is perpendicular to the displacement (and therefore $\cos \theta_{\vec{F}\vec{d}} = 0$ along both paths).

What about Path 3? The force of gravity initially does work to raise the object the distance h in the figure, so it does an amount of work $W = -mgh$ (this work is negative because the force of gravity opposes the displacement). Along the horizontal path the force of gravity does no work. As the box is lowered, the force of gravity does an amount of work $W = mgh$. Overall, the work done by gravity for the entire path is 0, which is the same as the other two paths. Regardless of which path we choose, the work done by gravity is the same. All that matters is where the object starts and where it ends. In this sense, the work done by gravity is *path independent*.

We can now define conservative and non-conservative forces.

- Conservative force — any force for which the work done by the force is *path-independent*.
- Non-conservative force — any force for which the work done by the force is *path-dependent*.

The number of forces that are conservative are few in number. As far as this course is concerned, they only comprise

- Force of gravity
- Force of an ideal spring

Unless told otherwise, all other forces (e.g., normal force, tension force, friction) are *non-conservative*.

4 Work energy theorems

So far we have discussed work and energy. Now it is time to connect these two physics principles by stating the work-energy theorems. For conservative forces (e.g., force of gravity, force caused by an ideal spring), we have

$$W_C = F_C d \cos \theta_{\vec{F}\vec{d}} = -\Delta\text{PE}. \quad (7)$$

This states that any conservative force that does positive work on a system tends to lower its potential energy. This is easy to see whenever you drop an object. In this case, the gravitational force (which is a conservative force) pushes the object through a distance h . The work done by this force is therefore $W = Fd = (mg)h$, which is a positive value because the force of gravity points in the same direction as the displacement. Therefore,

$$W_C = F_C d \cos \theta_{\vec{F}\vec{d}} = mgh = -\Delta\text{PE} \quad \longrightarrow \quad \Delta\text{PE} = -mgh. \quad (8)$$

According to this result, our potential energy drops by mgh as the object falls, which is exactly what happens. Before the object is released, it possesses a potential energy $\text{PE} = mgh$ and after it hits the ground it has no potential energy — its potential energy did indeed drop by mgh , just as the work-energy theorem states.

What if we raise the object a distance h ? In this case, the work we do is $W_C = -mgh$ (the negative sign comes in because the displacement opposes the force of gravity). In this case, the change in potential energy from the work-energy theorem states

$$W_C = F_C d \cos \theta_{\vec{F}\vec{d}} = -mgh = -\Delta\text{PE} \quad \longrightarrow \quad \Delta\text{PE} = mgh. \quad (9)$$

According to this, the potential energy rises by mgh , which is exactly what happens when we lift the object a distance h .

How do non-conservative forces affect the energy of the system? To answer this question, we turn to the work-energy theorem for non-conservative forces:

$$W_{\text{NC}} = F_{\text{NC}} d \cos \theta_{\vec{F}\vec{d}} = \Delta E \quad (10)$$

So, whenever a non-conservative force does positive work on a system, it increases the total energy of the system. By the same token, if the non-conservative forces does negative work, it lowers the total energy of the system.

When we drop an object, there are no non-conservative forces doing work on the object as it falls so the change in total energy $\Delta E = 0$. This means that whatever potential energy is lost must produce an increase in kinetic energy, and vice versa.

What about the case where we raise the object with our hand through a distance h ? When we do so, we apply an upward normal force F_N (which is non-conservative) to the object. Here, \vec{F}_N and \vec{d} point in the same direction, so $\theta_{\vec{F}\vec{d}} = 0$. The work energy theorem for non-conservative forces then states

$$W_{\text{NC}} = F_{\text{NC}} d \cos \theta_{\vec{F}\vec{d}} = F_N h(1) = \Delta E \quad \longrightarrow \quad \Delta E = F_N h. \quad (11)$$

This means the total energy of the object increased by an amount $F_N h$ when we raised it.

If we raised the object at constant velocity, then the acceleration is 0 and so $F_N = mg$, which means the total energy increased by mgh . (Remembering that $\Delta E = \Delta \text{PE} + \Delta \text{KE}$, then this means $\Delta \text{KE} = 0$ and therefore the kinetic energy must not have changed at all.)

Here is why: From our first two work-energy theorems we can derive a third work-energy theorem: Because $E = \text{PE} + \text{KE}$ then $\Delta E = \Delta \text{PE} + \Delta \text{KE}$. Adding the total work we get

$$W_{\text{net}} = W_C + W_{\text{NC}} = -\Delta \text{PE} + \Delta E = F_{\text{net}} d \cos \theta_{\vec{F}\vec{d}} = \Delta \text{KE} \quad (12)$$

But when the object is raised at constant velocity, $\vec{F}_{\text{net}} = 0$. So whenever the forces acting on an object vector sum to 0 (that is, whenever they cancel) the kinetic energy of the system will not change. Note that this makes perfect sense — whenever the net force acting on a system is 0, it cannot accelerate and therefore cannot change its speed. Because the kinetic energy of an object is speed-dependent (remember that $\text{KE} = \frac{1}{2}mv^2$), then that means its kinetic energy cannot change. All is well.

5 Conservation of energy

From the work-energy theorem for non-conservative forces,

$$W_{\text{NC}} = F_{\text{NC}} d \cos \theta_{\vec{F}\vec{d}} = \Delta E, \quad (13)$$

we see that if no non-conservative forces do work on a system, then the total energy of the system is conserved, which also means $\Delta \text{PE} = -\Delta \text{KE}$ and therefore whatever potential energy is lost must go toward increasing the kinetic energy.

When energy is conserved, $E_o = E_f$, which means the original energy $E_o = \text{PE} + \text{KE}$ before an event is the same as the final energy $E_f = \text{PE} + \text{KE}$ after the event. Furthermore, when energy is conserved, the path an object takes from the point before the event to the point after the event does not matter.

Let's apply our understanding of the work-energy theorems and conservation to a couple of easy problems.

5.0.1 Example: Bobsled on a hill

In Figure 6, a man is initially sitting on a bobsled when it suddenly starts taking off down the hill. (The bobsled is represented by a green cube) If his initial velocity is 0 and we ignore friction, how fast is he traveling when he reaches the bottom?

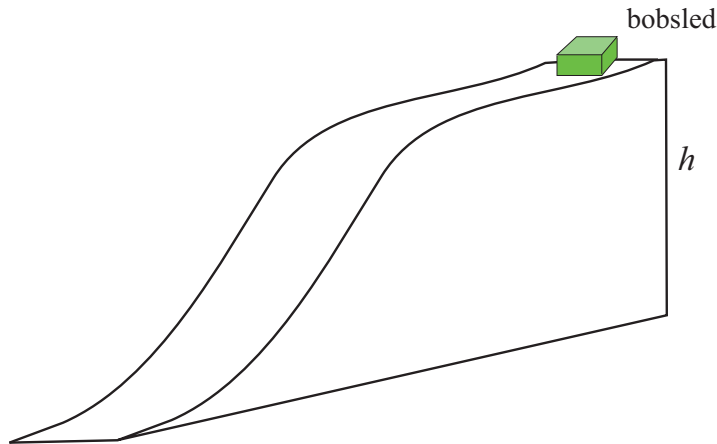


Figure 6: The bobsled problem, where a bobsled of unknown mass slides down a frictionless track. The track is more complex than an inclined plane, so the normal force changes direction constantly as the bobsled slides down. Newton's second law is worthless for solving this problem, but with conservation of energy the problem is a snap.

First off, notice that Newton's second law is useless for solving this problem. The normal force that acts on the man changes in direction and magnitude throughout his motion, so his acceleration is not constant. This means our equations of motion no longer apply. This problem seems hopeless.

However, let's look at the work-energy theorems one-by-one. The forces that act on the man throughout his motion include the gravitational force and the normal force. The normal force is non-conservative, so we apply the work-energy theorem for non-conservative forces:

$$W_C = F_C d \cos \theta_{\vec{F}\vec{d}} = \Delta E, \quad (14)$$

However, no matter where the bobsled is on the hill, the path it is taking will always be perpendicular to the surface. Therefore, the non-conservative forces (and there is only one here) do no work, and so energy is conserved. Because $\Delta E = 0$, then the energy of the bobsled when it is at rest at the top of the hill is the same as the energy it has at the bottom of the hill. Therefore,

$$E_o = E_f \quad \longrightarrow \quad PE_i = KE_f \quad \longrightarrow \quad mgh = \frac{1}{2}mv^2$$

Solving for speed we get

$$v = \sqrt{mg}$$

Notice how simple it was to solve for the final speed using conservation of energy. As we said before, energy offers in many cases numerous advantages over Newton's second law.

On a final note, notice that the mass of the bobsled cancels and, therefore, does not factor into the speed of the bobsled at the bottom. This is a consequence of a frictionless surface. If friction is factored into the equation, then we would need to find the energy lost due to this non-conservative force. But the force of friction, which depends on the normal force, would change in magnitude and direction and so in this case the problem would prove intractable.

6 A caveat

We need to clarify some of the language we have used above. Whenever non-conservative forces do work on a system, they tend to create thermal energy (that is, heat). This thermal energy, which is related to the motion of the molecules in the objects and surrounding air) is actually a combination of kinetic energy and potential energy. However, if the energy transforms into heat, it dissipates and is difficult to account for in our physical measurements.

For example, it is relatively easy to measure the location of an object, its mass, and how fast it is traveling. It is much harder to measure the location, mass, and speed of the molecules that make up the object, the surface it is resting on, and the surrounding air. Because the potential and kinetic energies of macroscopic systems are more easily measured, we call those *mechanical energies*. We should therefore revise our description of the work-energy theorems as the following:

- Whenever a conservative force does positive work on a system, it lowers the mechanical potential energy of the system.
- Whenever a non-conservative force does positive work on a system, it raises the total mechanical energy of the system.
- Whenever a net force acts on a system, it increases the mechanical kinetic energy of the system.

7 Power

There was a time when an unscrupulous car company (which we will not identify out of fairness) demonstrated the “power” of its new Chevette by having it tow a 747 jet airliner across the tarmac of an airfield. There was no doubt that the car was exerting a sizable force to the jet and was pulling it a fairly sizable distance. But was this a demonstration of power?

As my father used to say, “a mouse can pull an airplane if you gear it low enough.” So simply pulling the aircraft proves very little.

Suppose that we replace the Chevette (a real hunk of junk) with a red Ford F-150 King Ranch (a truck that only the coolest can possess). This truck can also pull the airliner, but at a much quicker speed because it is a truly badass vehicle.

When the Chevette (again, a real hunk of junk) pulled the airliner through a considerable distance, it performed a lot of work. After all, it applied a considerable force on the airliner and made it move through a large distance. But we all know that the engine inside a Chevette¹ is hardly powerful. So we can imagine that power P must involve not only how much work a force performs, but also how quickly it accomplishes this work. And it does:

$$P \equiv \frac{W}{t} \tag{15}$$

We can therefore say that the power exerted by a force is the rate at which this force can do work. The SI unit of work would seem to be the joule/second, but we give it a special name — the watt (W). One watt of power is the power exerted by gravity if it pushes a 1-newton object (like an apple) down a distance of 10 centimeters.

There are instances when the definition of power is re-cast into speed:

$$P \equiv \frac{W}{t} = \frac{Fd \cos \theta}{t}. \tag{16}$$

¹Have I mentioned that the Chevette is a real hunk of junk?

If we assume (a) the force applied to the object is constant and (b) the object is moving along the direction this force is applied (i.e., $\cos \theta_{\vec{F}\vec{d}} = 1$) then

$$P \equiv \frac{W}{t} = F \frac{d}{t} = Fv. \quad (17)$$

This states that the power exerted by the forces that propel an object at a speed v is given by

$$P \equiv \frac{W}{t} = Fv. \quad (18)$$

Which equation for power do we use? It depends on whether you know how fast the object is traveling or the time it takes for the object to travel from one point to another.

Let us return to our refrigerator problem in Figure 2. If it takes 2 minutes for us to push the refrigerator through the distance of 10 meters, then the power we exert is

$$P = \frac{W}{t} = \frac{5000 \text{ J}}{120 \text{ s}} = 41.6 \text{ W}. \quad (19)$$