Math 76 Exercises - 6.4C Sums of Series; Series Estimation of Definite Integrals

1. Find the sum of each convergent series.

(a). $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{n!}$

$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ for all x, so let $x=4$.

Then $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{n!} = \boxed{e^4}$

(b). $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n \cdot n!} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \cdot \frac{1}{n!}$ Similar to above, if we let $x = \frac{1}{2}$ we get $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cdot \frac{1}{n!} = e^{1/2} = \sqrt{e}$. Thus

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \cdot \frac{1}{n!} = \sqrt{e} - \underbrace{\left(\frac{1}{2}\right)^0 \cdot \frac{1}{0!}}_{n=0 \text{ term}} = \boxed{\sqrt{e} - 1}$$

(c). $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!}$

$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)\, x^{2n}}{(2n)!} = \cos x$ for all x, so

letting $x = \pi$ we get $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} = \cos(\pi) = \boxed{-1}$

(d). $\displaystyle\sum_{n=2}^{\infty} \frac{(-4)^n}{5^{2n+1}\,(2n+1)!} = \sum_{n=2}^{\infty} \frac{(-1)^n\, 4^n}{5^{2n+1}\,(2n+1)!}$

$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin x$ for all x

$$= \frac{1}{2} \sum_{n=2}^{\infty} \frac{(-1)^n\, 2^{2n} \cdot 2}{5^{2n+1}\,(2n+1)!}$$

$$= \frac{1}{2} \sum_{n=2}^{\infty} \frac{(-1)^n \left(\frac{2}{5}\right)^{2n+1}}{(2n+1)!} = \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{2}{5}\right)^{2n+1}}{(2n+1)!} - \underbrace{\frac{2}{5}}_{n=0 \text{ term}} - \underbrace{\frac{-\left(\frac{2}{5}\right)^3}{3!}}_{n=1 \text{ term}}\right]$$

$$= \frac{1}{2}\left(\sin\left(\frac{2}{5}\right) - \frac{2}{5} + \frac{\cancel{8}^{\,4}}{125 \cdot \cancel{6}_{\,3}}\right) = \boxed{\frac{1}{2} \sin\left(\frac{2}{5}\right) - \frac{1}{5} + \frac{2}{375}}$$

(e) $\displaystyle\sum_{n=3}^{\infty} \frac{2^{n+1}}{(n-1)!} = \sum_{n=2}^{\infty} \frac{2^{n+2}}{n!} = 4\sum_{n=2}^{\infty} \frac{2^n}{n!}$

$$= 4\left[\sum_{n=0}^{\infty} \frac{2^n}{n!} - \frac{2^0}{0!} - \frac{2^1}{1!}\right] = 4\left(e^2 - 1 - 2\right)$$

$$= \boxed{4e^2 - 12}$$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)\, 5^{2n+1}}$ $\qquad \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \tan^{-1} x$ for $-1 \le x \le 1$,

so letting $x = \frac{1}{5}$ we have $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \left(\frac{1}{5}\right)^{2n+1}}{2n+1}$

$$= \tan^{-1}\left(\tfrac{1}{5}\right) - \frac{(-1)^0 \left(\frac{1}{5}\right)^{2\cdot 0+1}}{2\cdot 0+1}$$

$$= \boxed{\tan^{-1}\left(\tfrac{1}{5}\right) - \tfrac{1}{5}}$$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{(-9)^{n+2}}{(2n-1)\, 2^{2n+1}}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+2}\, 9^{n+2}}{(2n-1)\, 2^{2n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n\, 3^{2(n+2)}}{(2n-1)\, 2^{2n+1}}$$

$2(n+1) - 1 = 2n+1$;
$2(n+1) + 1 = 2n+3$,

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\, 3^{2(n+3)}}{(2n+1)\, 2^{2n+3}} = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n\, 3^{2n+1}}{(2n+1)\, 2^{2n+1}}}_{\text{diverges since } \frac{3}{2} > 1} \cdot \frac{(-1)\cdot 3^5}{2^2}$$

$\boxed{\text{SERIES DIVERGES}}$

(h) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, 16^n}{(2n)!}$

$$= \sum_{n=2}^{\infty} \frac{(-1)^{n+1}\, 4^{2n}}{(2n)!} = -\sum_{n=2}^{\infty} \frac{(-1)^n\, 4^{2n}}{(2n)!}$$

$$= -\left[\sum_{n=0}^{\infty} \frac{(-1)^n\, 4^{2n}}{(2n)!} - \frac{(-1)^0\, 4^0}{0!} - \frac{(-1)^1\, 4^2}{2!}\right]$$

$$= -\left[\cos(4) - 1 + 8\right]$$

$$= \boxed{-\cos(4) - 7}$$

2. Estimate the following definite integrals to 3 decimal places using series.

(a) $\int_0^{\pi/3} \cos(x^2)\,dx$ $= \int_0^{\frac{\pi}{3}} \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!}\,dx$

$$= \int_0^{\pi/3} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}\Big|_0^{\frac{\pi}{3}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{4n+1}}{(4n+1)(2n)!} - 0.$$

$$\approx \underbrace{\frac{\left(\frac{\pi}{3}\right)^1}{1\cdot 0!}}_{\frac{\pi}{3}\approx 1.0471975\ldots} - \underbrace{\frac{\left(\frac{\pi}{3}\right)^5}{5\cdot 2!}}_{\approx 0.125934\ldots} + \underbrace{\frac{\left(\frac{\pi}{3}\right)^9}{9\cdot 4!}}_{\approx 0.007011\ldots} \left[- \underbrace{\cancel{\frac{\left(\frac{\pi}{3}\right)^{13}}{13\cdot 6!}}}_{\approx 0.00019 < 0.0005} + \cdots \right] \text{(don't need)}$$

$$\approx 1.0471975\ldots - 0.125934\ldots + 0.007011\ldots \approx \boxed{0.928}$$

(b) $\int_0^{\pi/4} x^4 \sin(x^2)\,dx$ $= \int_0^{\pi/4} x^4 \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}\,dx$

$$= \int_0^{\pi/4} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+6}}{(2n+1)!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+7}}{(4n+7)(2n+1)!}\Big|_0^{\pi/4}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right)^{4n+7}}{(4n+7)(2n+1)!} - 0$$

$$\approx \underbrace{\frac{\left(\frac{\pi}{4}\right)^7}{(7)(1!)}}_{0.026334867} - \underbrace{\frac{\left(\frac{\pi}{4}\right)^{11}}{11\cdot 3!}}_{0.001062783} \left[+ \underbrace{\cancel{\frac{\left(\frac{\pi}{4}\right)^{15}}{15\cdot 5!}}}_{0.000014828\ldots < 0.0005} \right] \text{(don't need)}$$

$$\approx \boxed{0.025}$$

(c) $\int_0^{0.4} x^2 \ln(1-x)\, dx$ $= -\int_0^{0.4} x^2 \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}\, dx$

$$= -\int_0^{0.4} \sum_{n=0}^{\infty} \frac{x^{n+3}}{n+1}\, dx$$

$$= -\int_0^{0.4} \sum_{n=1}^{\infty} \frac{x^{n+2}}{n}\, dx$$

$$= -\sum_{n=1}^{\infty} \frac{x^{n+3}}{n(n+3)} \Big|_0^{0.4}$$

$$= -\sum_{n=1}^{\infty} \frac{(0.4)^{n+3}}{n(n+3)} - 0$$

$$\approx -\underbrace{\frac{(0.4)^4}{1\cdot 4}}_{0.0064} - \underbrace{\frac{(0.4)^5}{2\cdot 5}}_{0.001024} \left[- \cancel{\frac{(0.4)^6}{3\cdot 6}} \right]$$ don't need

$\cancel{0.0002275}$ < 0.0005

Recall: $\frac{d}{dx}(\ln(1-x)) = \frac{-1}{1-x}$

$$= -\sum_{n=0}^{\infty} x^n \text{, so}$$

$$\ln(1-x) = -\int \sum_{n=0}^{\infty} x^n\, dx$$

$$= C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

When $x=0$ we get $0 = C - 0$

So $C = 0$. So

$$\ln(1-x) = \boxed{-\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}}.$$

$$\approx -0.0064 - 0.001024 \approx \boxed{0.007}$$

(d) $\int_{-0.3}^{0} \tan^{-1}(x^2)\, dx$

$$= \int_{-0.3}^{0} \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{2n+1}\, dx = \int_{-0.3}^{0} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1}\, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)(4n+3)} \Big|_{-0.3}^{0} = 0 - \sum_{n=0}^{\infty} \frac{(-1)^n (-0.3)^{4n+3}}{(2n+1)(4n+3)}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n \overbrace{(-1)^{4n+3}}^{=-1} (0.3)^{4n+3}}{(2n+1)(4n+3)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (0.3)^{4n+3}}{(2n+1)(4n+3)}$$

$$\approx \underbrace{\frac{(0.3)^3}{1\cdot 3}}_{0.009} \left[\cancel{- \underbrace{\frac{(0.3)^7}{3\cdot 7}}_{0.0000104\ldots < 0.0005} + \frac{(0.3)^{11}}{5\cdot 11}} \right]$$ (don't need)

$$\approx \boxed{0.009}$$