Math 76 Exercises – 6.3A Approximating Functions with Polynomials

1. For each function, find the polynomial approximation centered at the given c for the degree n specified.

   (a) $f(x) = \tan x$ centered at $c = 0$; $n = 2$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tan x$	$\tan 0 = 0$
1	$\sec^2 x$	$\sec^2 0 = 1$
2	$2\sec^2 x \tan x$	0

$$\tan x \approx 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 = \boxed{x}$$

   (b) $g(x) = e^{-x}$ centered at $c = 0$; $n = 3$

n	$g^{(n)}(x)$	$g^{(n)}(0)$
0	e^{-x}	1
1	$-e^{-x}$	-1
2	e^{-x}	1
3	$-e^{-x}$	-1

$$g(x) \approx 1 - \frac{1}{1!}x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 = \boxed{1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3}$$

   (c) $h(x) = \cos(x^2)$ centered at $c = 0$; $n = 4$

n	$h^{(n)}(x)$	$h^{(n)}(0)$
0	$\cos(x^2)$	1
1	$-2x\sin(x^2)$	0
2	$-4x^2\cos(x^2) - 2\sin(x^2)$	0
3	$8x^3\sin(x^2) - (8x^3+4x)\cos(x^2)$	0
4	$(16x^4 + 8x^2)\sin(x^2) - (24x^2+4)\cos(x^2)$	-4

$$h(x) \approx 1 + 0x + 0x^2 + 0x^3 - \frac{4}{4!}x^4 = \boxed{1 - \frac{1}{6}x^4}$$

(d) $j(x) = \sqrt{x}$ centered at $c = 4$; $n = 3$

n	$j^{(n)}(x)$	$j^{(n)}(4)$
0	$\sqrt{x}$	2
1	$\frac{1}{2\sqrt{x}}$	$\frac{1}{4}$
2	$-\frac{1}{4x^{3/2}}$	$-\frac{1}{32}$
3	$\frac{3}{8x^{5/2}}$	$\frac{3}{256}$

$$\sqrt{x} \approx 2 + \tfrac{1}{4}(x-4) - \frac{1/32}{2!}(x-4)^2 + \frac{3/256}{3!}(x-4)^3$$

$$= \boxed{2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3}$$

(e) $k(x) = \tan^{-1}(4x)$ centered at $c = 0$; $n = 3$

n	$k^{(n)}(x)$	$k^{(n)}(0)$
0	$\tan^{-1}(4x)$	0
1	$\frac{4}{1+16x^2}$	4
2	$-\frac{128x}{(1+16x^2)^2}$	0
3	$\frac{6144x^2-128}{(1+16x^2)^3}$	-128

$$k(x) \approx 0 + 4x + 0x^2 - \frac{128}{3!}x^3$$

$$= \boxed{4x - \frac{64}{3}x^3}$$

(f) $m(x) = \ln(3x-5)$ centered at $c = 2$; $n = 3$

n	$m^{(n)}(x)$	$m^{(n)}(2)$
0	$\ln(3x-5)$	0
1	$\frac{3}{3x-5}$	3
2	$-\frac{9}{(3x-5)^2}$	-9
3	$\frac{27}{(3x-5)^3}$	27

$$m(x) \approx 0 + 3(x-2) - \frac{9}{2!}(x-2)^2 + \frac{27}{3!}(x-2)^3$$

$$= \boxed{3(x-2) - \frac{9}{2}(x-2)^2 + \frac{9}{2}(x-2)^3}$$

2. Use the results above to approximate the following numbers. Using a calculator, determine the absolute error of each approximation.

(a) tan(0.5)

$f(0.5) \approx 0.5$

Calculator: $\tan(0.5) \approx 0.5463$

Error: $|0.5463 - 0.5| = \boxed{0.0463}$

(b) tan(−1.3)

$f(-1.3) \approx -1.3$

Calculator: $\tan(-1.3) \approx -3.6021$

Error: $|-3.6021 - (-1.3)| = \boxed{2.3021}$

Input further from the center = bigger error!

(c) (*) $\frac{1}{e}$

$g(1) = e^{-1} = \frac{1}{e} \approx 1 - 1 + \frac{1}{2} - \frac{1}{3!} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3} = 0.333\ldots$

Calculator: $e^{-1} \approx 0.367879$

Error $\approx |0.367879 - 0.\overline{3}| \approx \boxed{0.034546}$

(d) (*) $\frac{1}{e^4}$

$g(4) = e^{-4} = \frac{1}{e^4} \approx 1 - 4 + \frac{1}{2} \cdot 4^2 - \frac{1}{3!} 4^3 = -5.\overline{6}$

Calculator: $e^{-4} \approx 0.0183$

Error: $|0.0183 - (-5.\overline{6})| \approx \boxed{5.68498}$

Again, big error!

(e) $\cos(0.64)$

$$h(0.8) = \cos(0.64) \approx 1 - \frac{1}{6}(0.8)^4 \approx 0.9317$$

Calculator: $\cos(0.64) \approx 0.8021$

Error: $|0.8021 - 0.9317| \approx \boxed{0.1296}$

(f) $\sqrt{3.7}$

$$j(3.7) \approx 2 + \frac{1}{4}(3.7-4) - \frac{1}{64}(3.7-4)^2 + \frac{1}{512}(3.7-4)^3$$

$$\approx 1.923541$$

Calculator: $\sqrt{3.7} \approx 1.923538$

Error: $|1.923538 - 1.923541| = \boxed{0.000003}$

(g) $\sqrt{8.2}$

$$j(8.2) \approx 2 + \frac{1}{4}(8.2-4) - \frac{1}{64}(8.2-4)^2 + \frac{1}{512}(8.2-4)^3$$

$$\approx 2.919078$$

Calculator: $\sqrt{8.2} \approx 2.863564$

Error: $|2.863564 - 2.919078| = \boxed{0.055514}$

(h) $\ln(1.5)$

$$3x - 5 \overset{set}{=} 1.5$$
$$3x = 6.5 = \frac{13}{2}$$
$$x = \frac{13}{6}$$

$$m\left(\frac{13}{6}\right) = \ln(1.5) \approx 3\left(\frac{13}{6} - 2\right) - \frac{9}{2}\left(\frac{13}{6} - 2\right)^2 + \frac{9}{2}\left(\frac{13}{6} - 2\right)^3$$

$$= 3 \cdot \frac{1}{6} - \frac{9}{2} \cdot \left(\frac{1}{6}\right)^2 + \frac{9}{2}\left(\frac{1}{6}\right)^3 \approx 0.6458\overline{3}$$

Calculator: $\ln(1.5) \approx 0.405465$

Error: $|0.405465 - 0.6458\overline{3}| \approx \boxed{0.240368}$