Math 76 Exercises – 7.1 Intro to Advanced Integration

Evaluate each integral.

1. $\displaystyle\int \frac{5}{2x-1}\,dx.$ $\displaystyle = \frac{5}{2}\int \frac{1\cdot 2}{2x-1}\,dx = \boxed{\frac{5}{2}\ln|2x-1|+C}$

Moral. You can integrate anything that looks like $\dfrac{\text{constant}}{\text{linear}}$!

2. $\displaystyle\int \frac{2x-5}{(x^2-5x)^3}\,dx.$

$u = x^2-5x$

$du = 2x-5\,dx$

$\displaystyle = \int \frac{1}{u^3}\,du$

$\displaystyle = \int u^{-3}\,du = -\frac{1}{2}u^{-2}+C = \boxed{-\frac{1}{2(x^2-5x)^2}+C}$

Moral. Always check to see if you can use u-substitution (or guess and check) before trying anything fancy!

3. $\displaystyle\int \frac{4x-1}{x^2+5}\,dx.$ $\displaystyle = \int \frac{4x\,dx}{x^2+5} - \int \frac{1}{x^2+5}\,dx$

$\displaystyle = 2\int \frac{2x}{x^2+5}\,dx - \boxed{\int \frac{1}{5\left(\frac{x^2}{5}+1\right)}\,dx}$

$\displaystyle = \boxed{2\ln|x^2+5| - \frac{\sqrt{5}}{5}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)+C}$

$\displaystyle = \frac{1\sqrt{5}}{5}\int \frac{1}{\left(\frac{x}{\sqrt{5}}\right)^2+1}\,\frac{1}{\sqrt{5}}\,dx$

$u = \dfrac{x}{\sqrt{5}}$

$du = \dfrac{1}{\sqrt{5}}\,dx$

$\displaystyle = \frac{\sqrt{5}}{5}\int \frac{1}{u^2+1}\,du$

$\displaystyle = \frac{\sqrt{5}}{5}\tan^{-1}(u)+C$

$\displaystyle = \frac{\sqrt{5}}{5}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)+C$

Moral: You can integrate anything that looks like $\dfrac{\text{linear}}{x^2+a^2}$!

In general we have $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

4. $\displaystyle\int \frac{3x-1}{x^2+6x+11}\,dx.$

$$x^2+6x+11 = x^2+6x+9+2 = (x+3)^2+2.$$

$$= \int \frac{3x-1}{(x+3)^2+2}\,dx$$

$u = x+3 \longleftrightarrow x = u-3.$

$du = dx$

$$= \int \frac{3(u-3)-1}{u^2+2}\,du = \int \frac{3u-10}{u^2+2}\,du$$

$$= \int \frac{3u}{u^2+2}\,du - \int \frac{10}{u^2+2}\,du \text{ , Use techniques from \#3}$$

to get $\frac{3}{2}\ln(u^2+2) - \frac{10}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$

$$= \boxed{\frac{3}{2}\ln((x+3)^2+2) - \frac{10}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right) + C}$$

Moral: You can integrate anything that looks like $\dfrac{\text{linear}}{\text{quadratic}}$!

5. $\displaystyle\int \frac{x^3-3x^2+1}{x^2+1}\,dx.$

$$\begin{array}{r}
x - 3 \\
x^2+0x+1 \overline{\smash{)}\,x^3-3x^2+0x+1} \\
-(x^3+0x^2+x) \\
\hline
-3x^2-x+1 \\
-(-3x^2-0x-3) \\
\hline
-x+4
\end{array}$$

These place-holders are helpful!

$$= \int x-3+\boxed{\frac{-x+4}{x^2+1}}\,dx$$

Notice the $\frac{\text{linear}}{\text{quadratic}}$ pattern again...

$$= \frac{1}{2}x^2 - 3x + \int \frac{-x\,dx}{x^2+1} + \int \frac{4}{x^2+1}\,dx$$

$$= \boxed{\frac{1}{2}x^2 - 3x - \frac{1}{2}\ln(x^2+1) + 4\tan^{-1}(x) + C}$$

Moral. When the integrand is an *improper* rational function, perform polynomial division to rewrite the quotient as a polynomial plus a *proper* rational function, then apply the previous techniques.

6. $\int \frac{5x}{\frac{2}{x} - 3}\,dx$ $= \int \frac{5x}{\frac{2-3x}{x}}\,dx = \int \boxed{\frac{5x^2}{2-3x}}\,dx$

An improper rational function...

$$\begin{array}{r} -\frac{5}{3}x - \frac{10}{9} \\ -3x+2 \overline{\big) 5x^2 + 0x + 0} \\ -(5x^2 - \frac{10}{3}x) \\ \hline \frac{10}{3}x + 0 \\ -(\frac{10}{3}x - \frac{20}{9}) \\ \hline \frac{20}{9} \end{array}$$

$= \int -\frac{5}{3}x - \frac{10}{9} + \frac{20}{9(-3x+2)}\,dx$

$= -\frac{5}{6}x^2 - \frac{10}{9}x + \frac{20}{9}\left(-\frac{1}{3}\right)\ln|-3x+2| + C$

$= \boxed{-\frac{5}{6}x^2 - \frac{10}{9}x - \frac{20}{27}\ln|-3x+2| + C}$

Moral. You can use algebra to simplify fractions before deciding on a technique.

7. $\int \frac{3}{x^{1/2} + 6x^{2/3}}\,dx$

Let $u^6 = x$. Then $u^3 = x^{\frac{1}{2}}$ and $u^4 = x^{2/3}$. $6u^5\,du = dx$.

$= \int \frac{3}{u^3 + 6u^4} \cdot 6u^5\,du = 18\int \frac{u^5}{u^3 + 6u^4}\,du$

$= 18\int \frac{u^5}{u^3(1+6u)}\,du = 18\int \boxed{\frac{u^2}{1+6u}}\,du$ Use polynomial division...

$= 18\int \frac{1}{6}u - \frac{1}{36} + \frac{1}{36(6u+1)}\,du = \int 3u - \frac{1}{2} + \frac{1}{2(6u+1)}\,du$

$= \frac{3}{2}u^2 - \frac{1}{2}u + \frac{1}{12}\ln|6u+1| + C = \boxed{\frac{3}{2}x^{1/3} - \frac{1}{2}x^{1/6} + \frac{1}{12}\ln|6x^{1/6}+1| + C}$

Moral. Radicals or fractional powers of x can sometimes be converted to whole powers by rationalizing.

8. $\displaystyle\int \frac{e^{12x}}{e^{6x} - 4e^{-6x}}\,dx = \int \frac{e^{12x}}{e^{-6x}(e^{12x}-4)}\,dx = \frac{1}{6}\int \frac{e^{12x}\,6e^{6x}}{e^{12x}-4}\,dx$

$u = e^{6x}$

$du = 6e^{6x}\,dx$

$$= \frac{1}{6}\int \boxed{\frac{u^2}{u^2-4}}\,du = \frac{1}{6}\int \boxed{1 + \frac{4}{u^2-4}}\,du$$

(polynomial division)

Preview of §7.5:

Note that $\dfrac{4}{u^2-4}$

$= \dfrac{4}{(u-2)(u+2)}$

$= \dfrac{1}{u-2} - \dfrac{1}{u+2}$

(check!)

$$= \frac{1}{6}\int 1 + \frac{1}{u-2} - \frac{1}{u+2}\,du$$

$$= \frac{1}{6}\left(u + \ln|u-2| - \ln|u+2|\right) + C$$

$$= \frac{1}{6}\left(u + \ln\left|\frac{u-2}{u+2}\right|\right) + C$$

$$= \boxed{\frac{1}{6}\left(e^{6x} + \ln\left|\frac{e^{6x}-2}{e^{6x}+2}\right|\right) + C}$$

9. $\displaystyle\int \frac{\sin(2x)}{\sin^2 x + 3}\,dx$

$$= \int \frac{2\sin x\cos x}{\sin^2 x + 3}\,dx$$

$u = \sin x$

$du = \cos x\,dx$

$$= \int \frac{2u}{u^2+3}\,du$$

$$= \ln(u^2+3) + C$$

$$= \boxed{\ln(\sin^2 x + 3) + C}$$

Moral. You can sometimes convert an integrand into a rational function using substitution.

10. Jamie says that

$$\int -\frac{1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + C,$$

and Alex says that

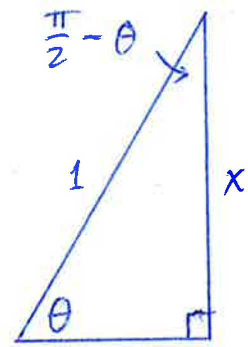
$$\int -\frac{1}{\sqrt{1-x^2}}\,dx = -\sin^{-1} x + C.$$

Who is right?

They are both right! We know that

$\frac{d}{dx}(\sin^{-1}x + C) = \frac{1}{\sqrt{1-x^2}}$, so $\frac{d}{dx}(-\sin^{-1}x + C) = -\frac{1}{\sqrt{1-x^2}}$,

and $\frac{d}{dx}(\cos^{-1}x + C) = -\frac{1}{\sqrt{1-x^2}}$.



Sure enough, if we let $\theta = \sin^{-1}x$

then $\sin\theta = x$. Thus* $\cos(\frac{\pi}{2}-\theta) = x$.

Therefore $\cos^{-1}x = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \sin^{-1}x$,

hence $\cos^{-1}x$ and $-\sin^{-1}x$ differ by a constant.

* (The rest of this explanation is for $0 \le x \le 1$, but it turns out that $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ for all x with $-1 \le x \le 1$.)

Moral. All antiderivatives of a function differ by a constant.