Math 76 Exercises – 6.4B Binomial Series

1. Write the binomial expansion for $(x-2)^5$.

$$(x-2)^5 = \binom{5}{0}x^5 + \binom{5}{1}(-2)x^4 + \binom{5}{2}(-2)^2x^3 + \binom{5}{3}(-2)^3x^2 + \binom{5}{4}(-2)^4x + \binom{5}{5}(-2)^5$$

$$= x^5 + 5(-2)x^4 + 10(4)x^3 + 10(-8)x^2 + 5(16)x + (-32)$$

$$= \boxed{x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32}$$

2. Write the binomial series for each of the following functions. What is the radius of convergence of each?

(a) $f(x) = \dfrac{1}{(1+x)^{3/2}} = (1+x)^{-3/2}$

$$= \sum_{n=0}^{\infty}\binom{-3/2}{n}x^n = \sum_{n=0}^{\infty}\frac{(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})\cdots(-\frac{3}{2}-n+1)}{n!}x^n$$

$$= \boxed{1 + \sum_{n=1}^{\infty}\frac{(-1)^n\, 1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{2^n\cdot n!}x^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\cancel{1\cdot 3\cdot 5}\cdots(2(n+1)-1)}{\cancel{2^{n+1}}\,\cancel{(n+1)!}}\cdot\frac{\cancel{2^n}\cdot\cancel{n!}\,|x|}{\cancel{1\cdot 3}\cdots\cancel{(2n-1)}} = \lim_{n\to\infty}\frac{2n+3}{2(n+1)}|x| = |x|.$$

So $\underline{R = 1}$.

(b) $g(x) = \sqrt[3]{8-x}$

$$= \sqrt[3]{8\left(1-\frac{x}{8}\right)} = 2\left(1-\frac{x}{8}\right)^{1/3}$$

$$= 2\sum_{n=0}^{\infty}\binom{1/3}{n}\left(-\frac{x}{8}\right)^n = \boxed{2\sum_{n=0}^{\infty}\binom{1/3}{n}\frac{(-1)^n}{8^n}x^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{2}\binom{1/3}{n+1}x^{n+1}}{8^{n+1}}\cdot\frac{8^n}{\cancel{2}\binom{1/3}{n}x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{8}\,\frac{\cancel{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})\cdots(\frac{1}{3}-n+1)}(\frac{1}{3}-n)}{(n+1)!}\cdot\frac{n!}{\cancel{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})\cdots(\frac{1}{3}-n+1)}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{8}\,\frac{(\frac{1}{3}-n)}{n+1}\right| = \lim_{n\to\infty}\frac{|x|}{8}\,\frac{n-\frac{1}{3}}{n+1} = \frac{|x|}{8} \overset{\text{set}}{<} 1 \qquad |x| < 8.$$

$$\boxed{R = 8}$$

(c) $h(x) = \dfrac{2x}{(1-x)^4} = 2x(1-x)^{-4} = 2x \sum_{n=0}^{\infty} \binom{-4}{n}(-x)^n$

$$= \boxed{\sum_{n=0}^{\infty} 2\binom{-4}{n}(-1)^n x^{n+1}}$$

$$\frac{\binom{-4}{n+1}}{\binom{-4}{n}} = \frac{-4-n}{n+1} \quad \text{(check!)}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{2}\binom{-4}{n+1} x^{\cancel{n+2}}}{\cancel{2}\binom{-4}{n} x^{\cancel{n+1}}}\right|$$

$$= \lim_{n\to\infty} |x| \cdot \left|\frac{-4-n}{n+1}\right| = \lim_{n\to\infty} |x| \cdot \frac{n+4}{n+1} = |x|$$

So $\boxed{R = 1}$

(d) $k(x) = \sqrt{4x^2+9} = \sqrt{9\left(\frac{4}{9}x^2+1\right)}$

$$= 3\left(\frac{4}{9}x^2+1\right)^{1/2} = 3\sum_{n=0}^{\infty}\binom{1/2}{n}\left(\frac{4}{9}x^2\right)^n = \boxed{\sum_{n=0}^{\infty} \frac{3\cdot 4^n}{9^n}\binom{1/2}{n} x^{2n}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{3}\cdot \cancel{4^{n+1}}^{\,4}}{\cancel{9^{n+1}}_{\,9}} \frac{\binom{1/2}{n+1} \cancel{x^{2(n+1)}}^{\,x^2}}{\binom{1/2}{n} \cancel{x^{2n}}} \cdot \frac{\cancel{9^n}}{\cancel{3}\cdot\cancel{4^n}}\right|$$

$$\frac{\binom{1/2}{n+1}}{\binom{1/2}{n}} = \frac{\frac{1}{2}-n}{n+1}$$

$$= \lim_{n\to\infty} \frac{4}{9}x^2 \frac{n-\frac{1}{2}}{n+1} = \frac{4}{9}x^2 \overset{\text{set}}{<} 1.$$

$x^2 < \frac{9}{4}$ $\quad |x| < \frac{3}{2}$ $\quad$ So $\boxed{R = \frac{3}{2}}$

3. Use the series above to approximate the following to 3 decimal places.

(a) $(1.21)^{-3/2}$

Use $x = 0.21$ in #2(a); we get

$$1 + \underbrace{\frac{-3/2}{1!}(0.21)}_{-0.315} + \underbrace{\frac{(-3/2)(-5/2)}{2!}(0.21)^2}_{0.0826875}$$

$$+ \underbrace{\frac{(-3/2)(-5/2)(-7/2)}{3!}(0.21)^3}_{-0.020258438} + \underbrace{\frac{(-3/2)\cdots(-9/2)}{4!}(0.21)^4}_{0.004786056} + \underbrace{\frac{(-3/2)\cdots(-11/2)}{5!}(0.21)^5}_{-0.001105579}$$

$$\approx \boxed{0.751109539}$$

(Calculator: $(1.21)^{-3/2} \approx 0.7513$)

(b) $\sqrt[3]{7.9}$

Use $x = 0.1$ in # 2(b); we get

$$\sqrt[3]{7.9} \approx 2 + \underbrace{2 \cdot \frac{\frac{1}{3}(-1)}{1! \cdot 8}(0.1)}_{-0.008\overline{3}} + \underbrace{2 \cdot \frac{(\frac{1}{3})(-\frac{2}{3})}{2!\, 8^2}(0.1)^2}_{-.0000347\overline{2}} + 2 \cdot \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})(-1)}{3!\, 8^3}(0.1)^3$$

Already this is within 0.0001 of 0

... So we don't need this term

$$\approx \underline{1.9916319}44.$$

Calculator:

$$\sqrt[3]{7.9} \approx \underline{1.9916317}01.$$

4. For the function $f(x)$ above, find $f^{(17)}(0)$.

The coefficient of x^{17} in the binomial series of $f(x)$ is

$$\frac{f^{(17)}(0)}{17!} = \frac{(-1)^{17}\, 1 \cdot 3 \cdot 5 \cdots 33}{2^{17} \cdot 17!}, \text{ so}$$

$$f^{(17)}(0) = \boxed{-\frac{1 \cdot 3 \cdot 5 \cdots 33}{2^{17}}} \approx -4.83 \times 10^{13}$$

5. For the function $g(x)$ above, find $g^{(31)}(0)$.

The coefficient of the x^{31} term of the binomial series for $g(x)$ is

$$\frac{g^{(31)}(0)}{31!} = \binom{1/3}{31}\frac{(-1)^{31}}{8^{31}} = \frac{-\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\cdots\left(\frac{1}{3}-31+1\right)}{31!\,8^{31}}$$

So $$g^{(31)}(0) = \boxed{-\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\cdots\left(\frac{1}{3}-30\right)}{8^{31}}}$$

6. For the function $h(x)$ above, find $h^{(25)}(0)$.

Similar to above, we have

$$\frac{h^{(25)}(0)}{25!} = 2\binom{-4}{25}(-1) = \frac{-2\cdot(-4)(-5)(-6)\cdots(-4-25+1)}{25!}$$

So $$h^{(25)}(0) = -2(-4)(-5)(-6)\cdots(-28) = \boxed{\frac{28!}{3}}$$

7. For the function $k(x)$ above, find $k^{(12)}(0)$.

Similar to above, we have

$$\frac{k^{(12)}(0)}{12!} = \frac{3\cdot 4^6}{9^6}\binom{1/2}{6}$$

Note that the x^{12} term arises from $\underline{n=6}$!

$$= \frac{3\cdot 4^6}{9^6}\,\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\left(-\frac{9}{2}\right)}{6!}$$

So $$k^{(12)}(0) = -\frac{3\cdot 4^6}{9^6}\,\frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 12!}{6!} = -\frac{\cancel{3}\cdot 4^6\cdot \cancel{3}\cdot 5\cdot 7\cdot \cancel{9}\cdot 12\cdot 11\cdot 10\cdot \cancel{9}\cdot 8\cdot 7}{9^{\cancel{6}\,3}}$$

$$= \boxed{-\frac{3{,}532{,}390{,}400}{243}}$$