Introduction to Physics

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1 The Mathematics of Vectors

So far, we have learned that Determinism drives much of the approach we will use to understand and solve physics problems in this course. We know that there are two approaches to using the philosophy of Determinism, which we summarized as the following:

$$\text{analyze causes} \Longrightarrow \text{ acceleration } (\vec{\mathbf{a}}) \Longrightarrow \text{predict effect} \tag{1}$$

$$\text{infer causes} \Longleftarrow \text{ acceleration } (\vec{\mathbf{a}}) \Longleftarrow \text{measure effects} \tag{2}$$

We also learned that to find this acceleration we relied on a few mathematical tools depending on whether we were driving the process toward the right or left.

- Newton's Second Law, $\vec{a} = \Sigma\vec{F}/m$, where ΣF is the combination of all forces acting on the body and m is the mass of the body.
- Equations of Motion, $\vec{v}_{\mathrm{f}} = \vec{v}_{\mathrm{o}} + \vec{a}t$ and $\vec{d} = \vec{v}_{\mathrm{o}}t + (1/2)\vec{a}t^2$, where $\vec{v}_{\mathrm{o}}$ is the initial velocity of the body, $\vec{v}_{\mathrm{f}}$ is the final velocity of the body, and $\vec{d}$ is the displacement the body travels during the event.

So, we again updated our diagram:

$$\text{Newton's Second Law} \Longrightarrow \text{ acceleration } (\vec{a}) \Longrightarrow \text{Equations of Motion} \tag{3}$$

$$\text{Newton's Second Law} \Longleftarrow \text{ acceleration } \vec{a} \Longleftarrow \text{Equations of Motion} \tag{4}$$

We added more detail, describing mathematically what Newton's Second Law and the equations of motion entailed:

- Newton's Second Law, $\vec{a} = \Sigma\vec{F}/m$, where ΣF is the combination of all forces acting on the body and m is the mass of the body.
- Equations of Motion, $\vec{v}_{\mathrm{f}} = \vec{v}_{\mathrm{o}} + \vec{a}t$ and $\vec{d} = \vec{v}_{\mathrm{o}}t + (1/2)\vec{a}t^2$, where $\vec{v}_{\mathrm{o}}$ is the initial velocity of the body, $\vec{v}_{\mathrm{f}}$ is the final velocity of the body, and $\vec{d}$ is the displacement the body travels during the event.

To solve physics problems, we need to understand what these equations represent. Clearly, the physical meaning of each of these variables needs to be explored. Also, some of the variables in the equations have little arrows on top. We mentioned in the previous chapter that this meant the variables were vectors, and that this meant the variables not only had numerical values (quantities) but also direction as well.

We will discuss the physical meaning of the variables $\vec{d}$, $\vec{v}$, $\vec{a}$, $\vec{F}$, and $\vec{m}$. In the meantime, let us turn our attention toward vectors.

2 Vector Summation

A simple example clarifies why we need to distinguish between vectors and scalars. When a baseball player hits a double, he will pass first base, then turn towards second base, arriving there at some time. In some cases, the physical distance the baseball runner traveled to second base is important to know. For example, he may have a heart condition, for which the total distance traveled indicates the strain he will likely place on his heart. The total distance traveled is a scalar quantity, as it is a amount that has no direction associated with it.

On the other hand, if we are concerned about *where* the baseball player ends up after hitting the double (and not just how far he has traveled), then how far away the player is from home plate *and* in which direction become more important. This physical property, *displacement*, which combines information about how far away an object is located from a point *and* in which direction is an example of a *vector* quantity.

Therefore, right off the bat we see how one of the variables, displacement $\vec{d}$, is defined. To be more precise, displacement of an object after a time t has elapsed is defined as (1) how far from its original point the object is located and (2) in which direction.

We represent a vector with an arrow. The length of this arrow represents the *amount* associated with the variable; the direction of the arrow represents the direction associated with the variable. Therefore, Figure 1 the displacement vector pictured conveys that (1) the object is located 43 meters from its original starting point and (2) the object is located 48° above the horizontal from Point A. The fact that the object traveled 61 meters along the path shown in the figure is irrelevant as far as displacement is concerned.

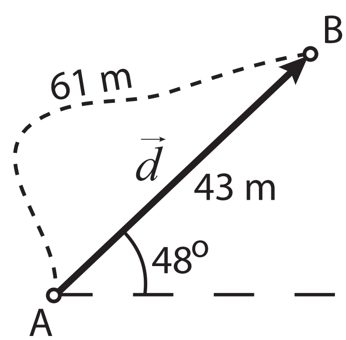

Figure 1: A typical displacement vector $\vec{d}$, pointing from Point A (where the object began its journey) to Point B (the destination of the object). The dashed line indicates the actual path of travel, which was 61 meters long.

2.1 Vector Algebra

Loosely speaking, an algebra comprises mathematical entities and rules for combining them. For example, real algebra comprises numbers and variables, along with the rules for adding, subtracting, multiplying, and dividing them.

Vector algebra, being an algebra, also comprises mathematical entities and the rules for combining them. With vector algebra, there are two entities that we must consider:

- Scalars
- Vectors

Vectors represent physical properties that not only correspond to an amount, but also a direction as well. The amount, by itself, is a scalar quantity. When combined with a direction, the variable becomes a vector.

A scalar quantity represents an *amount*, whereas a vector represents an *amount* and a *direction*. The following table illustrates some of the common vectors and scalars we will encounter over the next few weeks.

Variable	**Description**	
$\vec{d}$	Displacement	Vector points from start to end of path
$\vec{v}$	Velocity	How fast an object is traveling and in which direction
$\vec{a}$	Acceleration	The change in the velocity vector with respect to time
$\vec{F}$	Force	A push or pull on an object by its surroundings
d	Distance	The distance an object travels along a path
d	Distance	The length of the displacement vector $\vec{d}$
v	Speed	The speed at which an object is traveling

2.2 Head-to-tail Method of Vector Summation

One of the primary operations in vector algebra is summation, which comprises both addition and subtraction. The displacement vector provides a fairly easy means of understanding vector addition. In 2, a man travels along the path described by displacement $\vec{d_1}$, then travels along $\vec{d_2}$. Clearly, his net displacement points from Point A to Point B, which represent the beginning and end of his journey. However, we know that this displacement is the result of traveling along displacement $\vec{d_1}$ then along $\vec{d_2}$. Therefore, we can surmise that $\vec{d} = \vec{d_1} + \vec{d}2$ and that you can visually represent the addition of vectors by (1) placing them head to tail and (2) drawing a vector pointing from the beginning of the vector chain to the end. This method of summing vectors is called the *head to tail method*.

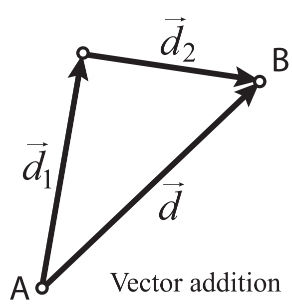

Figure 2: An object travels along two displacements, eventually ending up at Point B. The net displacement, which points from A to B, is simply the sum of the individual displacements, that is, $\vec{d} = \vec{d_1} + \vec{d_2}$.

To summarize: To add any number of vectors head to tail, place them in a chain where the tail of one vector touches the head of another vector. The sum of all vectors is the vector that points from the beginning of the chain to the end.

When discussing vector summation, we will call the sum of vectors the *resultant vector* and label it $\vec{R}$.

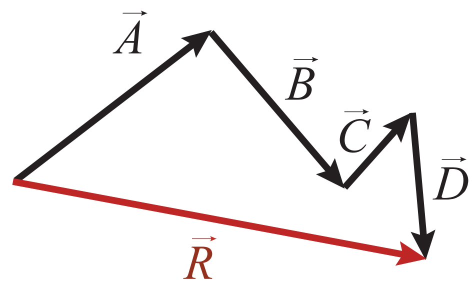

Figure 3: To add vectors head to tail, simply "hook" them in a chain; the resultant vector $\vec{R}$ points from the beginning of the chain to the end. In this example, $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$.

2.3 Parallelogram Method of Vector Summation

There is another way to sum vectors, and to visualize this process we turn to velocity. In Figure 4, we have a person wanting to ford a river such that he reaches Point X on the opposite bank. If the river is not flowing, which is depicted in part (a) of the figure, he can simply point his boat at X and row. The length of this velocity vector $\vec{v}_{\text{boat}}$ represents his rowing speed.

However, the river does flow at some velocity $\vec{v}_{\text{water}}$, as shown in part (b). Naturally, this flow will sweep the boat partially downstream such that the boat will end up traveling toward a different point on the shore, as shown in part (c). The net velocity vector, which describes the resulting speed of the boat and direction of its travel, is found simply by summing $\vec{v}_{\text{boat}}$ and $\vec{v}_{\text{water}}$, that is, $\vec{v}_{\text{net}} = \vec{v}_{\text{boat}} + \vec{v}_{\text{water}}$. To perform this summation, we hook the tails of the two velocity vectors together and form a parallelogram with the two vectors as sides. The resultant vector $\vec{v}_{\text{net}}$ points from the tails to the far corner of the parallelogram.

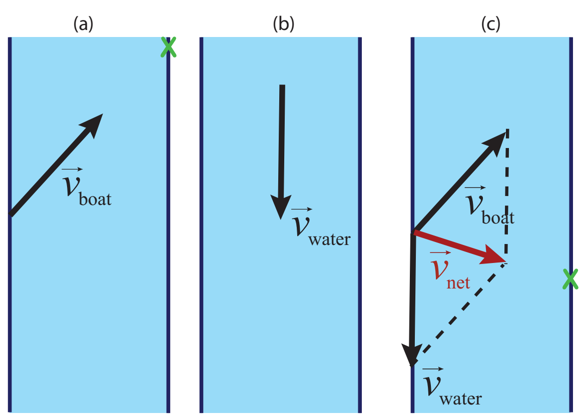

Figure 4: A boat attempting to ford a stream illustrates another classic example of vector addition. The easiest way to visualize this process is to (a) pretend that the stream is not moving in which case the boat will travels toward the X on the upper right, then (b) pretend that the stream is not moving but the boat is, and finally (c) sum the two velocity vectors. Note that the direction the arrow points in each case corresponds to the *direction of travel* while the length of the vector represents *speed*.

To summarize: To sum two vectors using the parallelogram method, hook their tails together, draw a parallelogram with the two vectors along the sides of the parallelogram, and draw a vector from the tails to the far corner of the parallelogram.

Adding multiple vectors using the parallelogram method is more clumsy than using the head-to-tail method. This requires adding two vectors, finding their resultant vector, then adding this resultant vector to the next vector in the problem.

We can return to our earlier problem where we added two displacements using the head-to-tail method and add them using the parallelogram method, as shown in Figure 5. Naturally, we get the same answer.

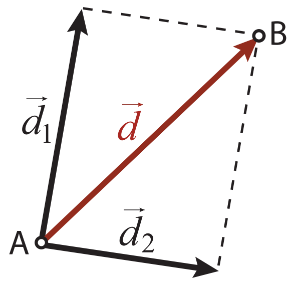

Figure 5: Another way to perform the summation $\vec{d} = \vec{d}_1 + \vec{d}_2$ using the same example as in Figure 2. This parallelogram method is usually more practical to use, albeit somewhat less intuitive. Because of the properties of the parallelogram, this method is in fact the same as the head-to-tail method.

2.3.1 Vector subtraction

What about subtraction? If we can find $\vec{R} = \vec{A} + \vec{B}$ then we can also find $\vec{R} = \vec{A} - \vec{B}$, since all we would have to do is flip B in the opposite direction and add. That is, $\vec{R} = \vec{A} + (-\vec{B})$. This process is illustrated in Figure 6.

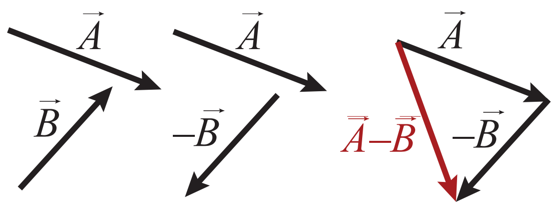

Figure 6: Vector subtraction is no harder than vector addition if we remember that $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$ and that $-\vec{B}$ is little more than the original $\vec{B}$ but flipped to point in the opposite direction.

2.3.2 Scalar-Vector Multiplication

Vectors are not the only entities found in vector algebra. We also have scalars, for which we know represent only amounts. That is, a scalar does not have any direction associated with it.

We already know how to add, subtract, multiply, and divide scalars – as mere amounts these follow the normal rules of arithmetic. But what about operations involving both scalars and vectors.

We cannot add a scalar to a vector. This would make no sense given that a scalar has no direction associated with it and a vector does. In which direction would the result point? The same conundrum applies when attempting to subtract a vector from scalar (or vice versa) – this operation is not allowed either.

However, multiplying a scalar and a vector *is* allowed. The scalar quantity, which we will denote as c, does little more than expand or shrink the vector, depending on whether $c > 1$ (expand) or $c < 1$ (shrink). Regardless, the direction of the vector is unaffected; only its length changes. What about division? The operation $\vec{A}/c$ is little more than $(1/c)\vec{A}$, so dividing a vector by a scalar poses little trouble. (However, one cannot divide a scalar by a vector.)

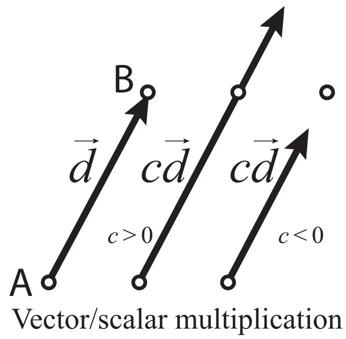

Figure 7: The effect of multiplying a vector by a scalar, which is to either expand or shrink the original vector depending on whether c is less than or greater than 1.

2.3.3 Combined operations

Now that we have defined the allowed operations in vector algebra and know to carry them out, we can examine an example where a number of these operations are involved. See 8 and be sure you understand it.

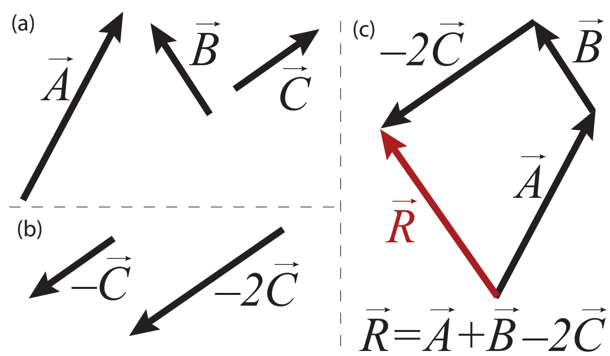

Figure 8: A typical example of vector algebra using the head-to-tail method. Here, the equation $\vec{R} = \vec{A} + \vec{B} - 2\vec{C}$. As depicted in part (b), the term $-2\vec{C}$ simply means (1) reverse the direction of $\vec{C}$ and (2) scale its length by 2.

2.4 Component Method of Vector Summation

So far, we used a visual approach to finding the sum of vectors. However, vector algebra would pose little benefit beyond estimation if the head-to-tail method was all we could do. We need a method that will generate exact, numerical answers. For that, we rely on the Component Method.

To sum vectors using the Component Method, we will follow the basic scheme shown in the accompanying table.

Step	What This Means
$\vec{R} = \vec{A} + \vec{B}$	The fundamental vector summation problem
$\vec{R} = (\vec{A}_x + \vec{A}_y) + (\vec{B}_x + \vec{B}_y)$	We transform $\vec{A}$, $\vec{B}$ into good vectors
$\vec{R} = (\vec{A}_x + \vec{B}_x) + (\vec{A}_y + \vec{B}_y)$	We rearrange the good vectors for easier summation.
$\vec{R} = \vec{R}_x + \vec{R}_y$	We recognize that $(\vec{A}_x + \vec{B}_x)$ is just $\vec{R}_x$, etc.
$\vec{R} = \vec{R}$	We build $\vec{R}$ using the Pythagorean theorem and arctangent.

2.4.1 Good vectors

We will demonstrate the summation of vectors using the Component Method using the example $\vec{R} = \vec{A} + \vec{B}$ shown in Figure 11.

If only vectors $\vec{A}$ and $\vec{B}$ had happened to point along the same dimension (i.e., in the same direction or in opposite directions), we could find $\vec{R}$ easily. We would just sum (i.e., add or subtract) their lengths.

Consider the example in Figure 9. Because $\vec{C}$ is longer than $\vec{D}$, adding vectors $\vec{C}$ and $\vec{D}$ simply produces a resultant vector of length 2 pointing in the same direction as $\vec{C}$. Easy!

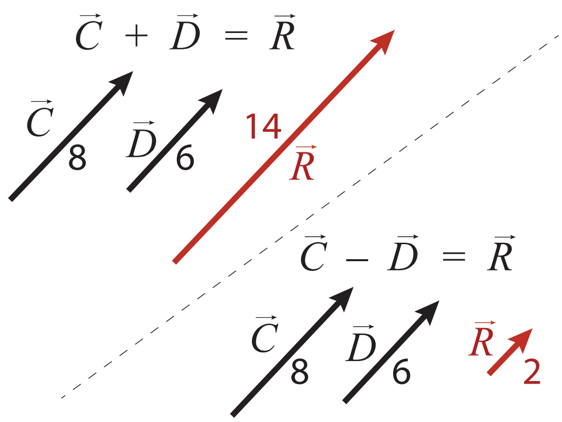

Figure 9: When vectors point along the same line, they are easy to add or subtract.

If our vectors happened to be perpendicular to each other, as in 10, we would find summing them easy too – our parallelogram becomes a simple rectangle divided into two right triangles. We could then use the Pythagorean theorem to find the length of the resultant vector $\vec{R}$ because $\vec{R}$ coincides with the hypotenuse. To determine its direction θ, we would just use the arctangent function. (We will discuss how to do this a bit more later on.) Although not as easy as the case where the two vectors lie along the same line, we at least know how to add vectors that are perpendicular.

When our vectors point along the same dimension or are perpendicular to each other, we will call them *good vectors* because they are easy to sum.

2.4.2 Bad vectors

Alas, vectors do not always point along the same directions nor are they always perpendicular to each other. We call these bad vectors! However, we can break the summation process down such that all we do is add vectors that point along the same directions or are perpendicular to each

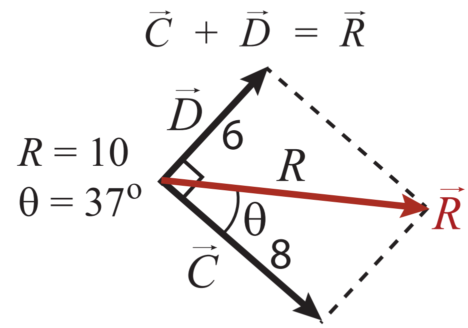

Figure 10: When vectors are perpendicular to each other, we can use the Pythagorean theorem and the arctangent to sum them. Although not as easy as the case shown in Figure 9, the mathematics involved is not complex.

other. That is, in essence, the Component Method: The component method replaces bad vectors that are hard to sum with good vectors that are easy to sum.

2.4.3 Sample Problem

Throughout this discussion, we will refer to the vectors once again shown in the following figure as a clarifying example. Note that these are bad vectors.

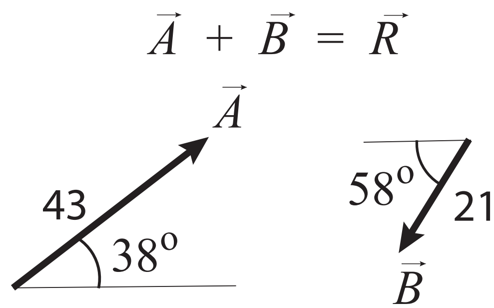

Figure 11: A sample vector addition problem, which we will use in the following subsections to learn the component method of vector addition.

Before even beginning, we should do a quick head-to-tail summation to see what R looks like. According to the following figure, our resultant vector should end up having a length of roughly 20 and point about 20° above the horizontal.

2.4.4 Vector Rehab

To reform our bad vectors and turn them into good vectors, we first place the vectors on a grid with the tails of vectors at the origin. See the following figure.

We next replace each bad vector with two good vectors that sum to it. We will call these two good vectors component vectors. They will be perpendicular to each other and will form a rectangle with the bad vector lined up on the diagonal, as shown. (I grayed out the original bad vectors in the above figure and represented the good vectors in green. I will remove the bad vectors completely in a bit.)

Why can we do this? The answer is simple: The two component vectors sum to equal the original bad vector; that is, $\vec{A}_x + \vec{A}_y = \vec{A}$. You should be able to verify this by looking at the figure

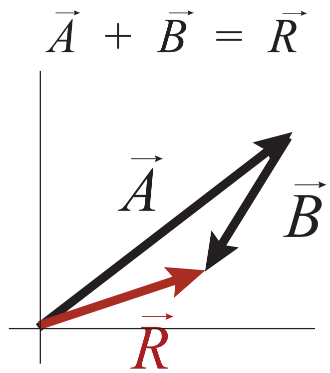

Figure 12: We can use the head-to-tail method to give us an idea of what the resultant vector $\vec{R}$ will look like with respect to its length and direction.

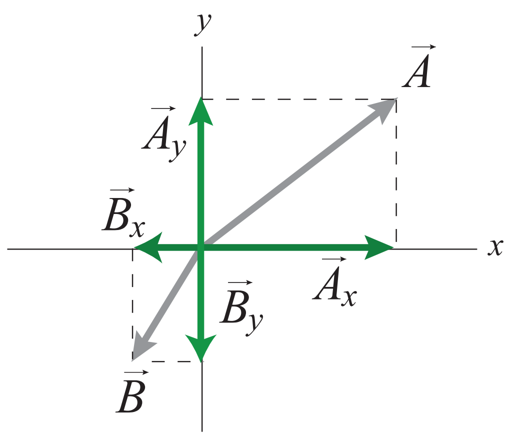

Figure 13: We replace the bad vectors $\vec{A}$ and $\vec{B}$ with good vectors (shown in green). Note that using the parallelogram method we can see that $\vec{A} = \vec{A}_{\text{x}} + \vec{A}_{\text{y}}$, which allows us to make this substitution. (The same reasoning applies for $\vec{B}$ and its good vector componenents.

and recalling how we sum vectors using the Parallelogram Method.

If you recall, we need to specify two things when we define a vector: Its length and its direction. We know the direction of each component vector, they are point straight up or down or they point left and right. But what about their lengths?

2.4.5 Trig Rears Its Ugly Head

One thing we notice immediately: These component vectors are always shorter than the original bad vector. But by how much? To find the (shrunken) lengths of the component vectors, we must find the shrink factor. This shrink factor, when applied to the length of the original bad vector, will tell us the length of each component. These shrink factors are always either the cosine or sine functions. But which one? You know the song. You love the song. Let's sing it!

> It's a touchy, touchy, touchy, touchy cosine.
> It's a touchy, touchy, touchy, touchy cosine.
> If no remember what do I sing?
> It's a touchy, touchy, touchy, touchy cosine.

In other words, if the angle sign touches the component vector, we use the cosine of that angle as our shrink factor. Always! And if we use the cosine shrink factor for one of the components, we use the sine shrink factor for the other component. Always! If you can remember these two simple rules, you will be handle half the trig that will ever appear in this course.

2.4.6 Reforming Vector $\vec{A}$

For bad vector $\vec{A}$, we notice that the angle symbol touches the $\vec{A}_{\mathrm{x}}$ component. Therefore, we would use the cosine shrink factor. We apply this shrink factor to the length of vector $\vec{A}$ to get the length of vector $\vec{A}_{\mathrm{x}}$. (Remember the song.)

$$A_x = \cos(38°)(43) = (0.78)(43) = 33.5$$

Remember, if we use cosine for one component vector, we must use sine for the other:

$$A_y = \sin(38°)(43) = (0.62)(43) = 26.7$$

Notice that we don't put arrows on top of A_x and A_y in our calculations. That's because we were only trying to find their lengths. We could also specify their directions, but we know their directions by looking at the figure.

Before moving on, I want to write the above equations a bit more formally. Be sure to recognize that the following express the exact same thing.

$$A_x = 43\cos(38°) = 33.5\,; \qquad A_y = 43\sin(38°) = 26.7$$

Unlike your textbook, we will not attach negative signs to our answers, since these are lengths. We don't need the negative signs because we can use the directions our vectors point in the figures to determine their direction.

Before worrying about vector $\vec{B}$, we should look at the results above and see if they look reasonably close to what we expected. For one, we certainly expect the length $\vec{A}_{\mathrm{x}}$ to be longer than $\vec{A}_{\mathrm{y}}$, and that is exactly what we get. So far, so good.

2.4.7 Reforming Vector $\vec{B}$

Now we turn our attention to bad vector B. According to our figure, the angle symbol touches $\vec{B}_{\mathrm{x}}$ so we use the cosine shrink factor:

$$B_x = 21\cos(58°) = 11.1$$

And if we use cosine for $\vec{B}_{\mathrm{x}}$, we must use sine to find the length of $\vec{B}_{\mathrm{y}}$:

$$B_y = 21\sin(58°) = 17.8$$

We check our work: According to our results, the length of $\vec{B}_{\mathrm{y}}$ should be slightly larger than $\vec{B}_{\mathrm{x}}$, which agrees with our figure. So we're good to go.

2.4.8 Summing the Good Vectors

Since we have substituted good vectors for our bad vectors, we no longer need the bad vectors. So we can remove them.

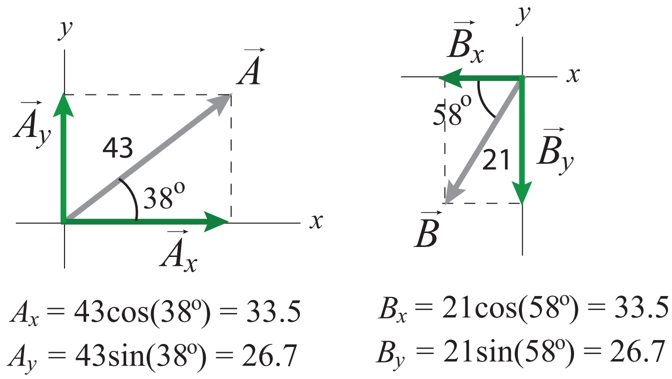

Figure 14: We use sine and cosine shrink factors to determine the lengths of $\vec{A}_{\mathrm{x}}$ and $\vec{A}_{\mathrm{y}}$.

Now all we have are good vectors. (Notice how clean our problem looks now!) To find our resultant vector $\vec{R}$, we just have to sum all the good vectors. That is,

$$\vec{R} = (\vec{A}_{\mathrm{x}} + \vec{A}_{\mathrm{y}}) + (\vec{B}_{\mathrm{x}} + \vec{B}_{\mathrm{y}}).$$

The easiest way (that is, the most non-boneheaded way) to sum our good vectors is to first sum those vectors that line up along the same axis. Here, we see that $\vec{A}_{\mathrm{x}}$ and $\vec{B}_{\mathrm{x}}$ are easy to sum because they point along the same axis, although in opposite directions. The same applies to $\vec{A}_{\mathrm{y}}$ and $\vec{B}_{\mathrm{y}}$.

Therefore, in essence, we are carrying out the following:

$$\vec{R} = (\vec{A}_{\mathrm{x}} + \vec{B}_{\mathrm{x}}) + (\vec{A}_{\mathrm{y}} + \vec{B}_{\mathrm{y}}).$$

Since the length of $\vec{A}_{\mathrm{x}} = 33.5$ and points to the right and the length of $\vec{B}_{\mathrm{x}} = 11.1$ and points to the left, then $\vec{A}_{\mathrm{x}} + \vec{B}_{\mathrm{x}}$ should produce a vector pointing toward the right with a length of 22.4. (Make sure you understand why.)

It is just as easy to compute $\vec{A}_{\mathrm{y}} + \vec{B}_{\mathrm{y}}$. These two vectors also point in opposite directions, with $\vec{A}_{\mathrm{y}}$ longer than $\vec{B}_{\mathrm{y}}$. So when we sum these two vectors, we produce a vector that points upwards; its length is 8.8. (Again, confirm this yourself.)

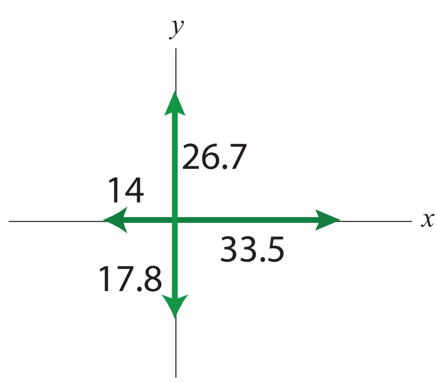

Figure 15: We have replaced the two vectors in the original problem with four vectors. However, these are good vectors and relatively easy to sum. Our next step is to sum the two vectors along the horizontal and then the two vectors along the vertical.

2.4.9 Building the Resultant Vector

Notice that two the remaining vectors are just the components of our resultant vector $\vec{R}$. (Use the Parallelogram Method to confirm this fact in your own mind.)

We're almost done. We haven't found $\vec{R}$ yet, but we found its components. We can label these components $\vec{R}_\mathrm{x}$ and $\vec{R}_\mathrm{y}$. All we have to do now is build vector $\vec{R}$ from these two components using the parallelogram method.

2.4.10 The Pythagorean Theorem

In the next figure, I show the resultant vector $\vec{R}$. Notice that it runs along the diagonal of a rectangle. In other words, its length is simply the hypotenuse of a right triangle, with one of the sides of the triangle $\vec{R}_\mathrm{x}$ and the other side $\vec{R}_\mathrm{y}$.

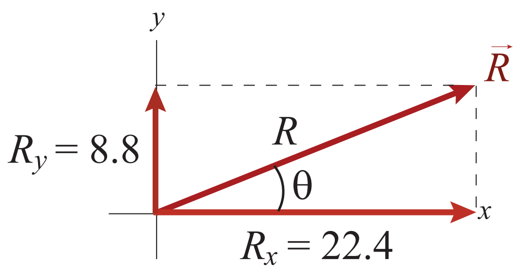

Figure 16: We see that the two vectors $\vec{R}_x$ and $\vec{R}_y$ sum to form the resultant $\vec{R}$. We can assign the angle θ to describe the direction of $\vec{R}$, so all that is left is to find R (the length of the resultant vector) and θ.

We know how to find the length of a hypotenuse using the Pythagorean theorem:

$$R^2 = R_\mathrm{x}^2 + R_\mathrm{y}^2$$

So to find the length of $\vec{R}$, we simply square the lengths of its two component vectors, add them, then take the square root.

$$R^2 = (22.4)^2 + (8.8)^2 = 501.8 + 77.4 = 579.2$$

Taking the square root of both sides we get the length $R = 24.1$. But we could have easily made a mistake when computing R. According to our figure, however, vector $\vec{R}$ should be a bit longer than the length of $\vec{R}_\mathrm{x}$, which is 22.4. So yeah, this looks about right. Okay, we found the length of $\vec{R}$. But $\vec{R}$ is a vector; we must also find its direction. How do we do that?

2.5 The Arctangent

Face it. I blew over the arctangent cavalierly in my lecture. Let me frame it with more care this time. We start with the tangent function. In trigonometry, the tangent of an angle is defined as the following: $\tan\theta = \text{opposite}/\text{adjacent}$, where "opposite" means the length of the side opposite the angle θ and "adjacent" means the length of the side adjacent to θ. (There is nothing to figure out here because that is just how the tangent is defined.)

In our problem (which I have shown again below), the tangent of θ is simply the ratio of $\vec{R}_\mathrm{y}$ over $\vec{R}_\mathrm{x}$, that is $\tan\theta = \vec{R}_\mathrm{y}/\vec{R}_\mathrm{x}$.

But we want θ, not $\tan\theta$! To peel θ away from its tan, we apply the arctangent function to both sides, producing

$$\theta = \arctan(\vec{R}_\mathrm{y}/\vec{R}_\mathrm{x}).$$

If this is all a bit confusing, just remember this: To find the direction θ, simply divide the length of the component opposite to θ by the length of the component adjacent to θ, then press the arctan button on your calculator. So let's do it:

$$\theta = \arctan(\vec{R}_{\mathrm{y}}/\vec{R}_{\mathrm{x}}) = \arctan(8.8/22.4) = \arctan(0.39) = 21.3°$$

Does this look right? According to our figure, 21.3° looks pretty reasonable. (Again, the online calculator at `bit.ly/1nco2aA` makes this calculation a snap.)

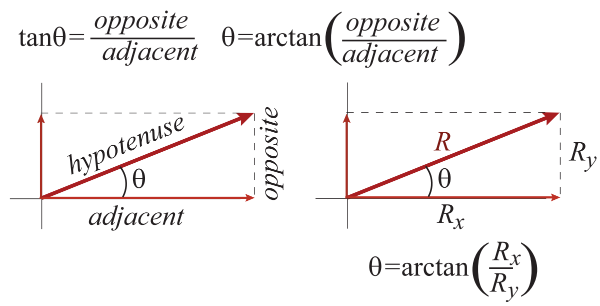

Figure 17: An important final step (and often forgotten) is to compute the direction of the resultant vector $\vec{R}$. For that, we employ the arctangent function.

2.5.1 Finally!

We found R! But to be exact, we need to express it completely. Since R is a vector, we need to state both its length and direction. Okay, we now can: $\vec{R}$ is 24.1 units at 21.3° above the x-axis.

One last thing: Earlier, we used the head-to-tail method to estimate our answer, which we said would have a length of roughly 20 units and point at roughly 20°. Our answers are reasonably close to this estimate. All seems well.