1. Find a power series centered at 0 for each function, if possible.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

(a) $f(x) = \dfrac{2}{1-3x}$

$$= 2 \cdot \frac{1}{1-3x}$$

$$= 2\sum_{n=0}^{\infty}(3x)^n$$

$$= \boxed{\sum_{n=0}^{\infty} 2 \cdot 3^n x^n}$$

(b) $g(x) = \dfrac{x}{1+x^2}$

$$= x \cdot \frac{1}{1-(-x^2)}$$

$$= x\sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty} x(-1)^n x^{2n} = \boxed{\sum_{n=0}^{\infty}(-1)^n x^{2n+1}}$$

(c) $h(x) = \dfrac{5x^3}{1-\frac{7}{x}}$ $h(x)$ is not defined at $x=0$, so there is no power series for $h(x)$ centered at 0.

(d) $j(x) = \dfrac{4x}{5x^2 - 8}$ $= \dfrac{-4x}{8-5x^2} = \dfrac{-4x}{8(1-\frac{5}{8}x^2)} = -\frac{1}{2}x \cdot \dfrac{1}{1-\frac{5}{8}x^2}$

$$= -\frac{1}{2}x \sum_{n=0}^{\infty} \left(\frac{5}{8}x^2\right)^n = \sum_{n=0}^{\infty} -\frac{1}{2}x \cdot \frac{5^n}{8^n} x^{2n}$$

$$= \boxed{\sum_{n=0}^{\infty} -\frac{5^n}{2 \cdot 8^n} x^{2n+1}}$$

2. Given that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, find a power series for $f(x) = e^{x^2}$.

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}}$$

3. Given that $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, find a power series for $g(x) = 5x\cos(x^3)$.

$$5x\cos(x^3) = 5x \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{5(-1)^n}{(2n)!} x^{6n+1}}$$

4. Given that $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$, find a power series for $h(x) = \ln(2+3x)$.

$$\ln(2+3x) = \ln\left(2\left(1+\tfrac{3}{2}x\right)\right) = \ln\left(2\left(1-\left(-\tfrac{3x}{2}\right)\right)\right)$$

$$= \ln 2 + \ln\left(1-\left(-\tfrac{3}{2}x\right)\right)$$

$$= \ln 2 - \sum_{n=1}^{\infty} \frac{\left(-\frac{3}{2}x\right)^n}{n}$$

$$= \ln 2 - \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 3^n}{2^n} x^n$$

$$= \boxed{\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^n}{2^n} \cdot x^n}$$

5. Find the first three non-zero terms of a power series for $e^x \cos x$.

$$e^x \cos x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)$$

$$= 1 + x - \cancel{\frac{x^2}{2}} + \cancel{\frac{x^2}{2}} - \frac{x^3}{2} + \frac{x^3}{6} + \cdots$$

$$= \boxed{1 + x - \frac{1}{3}x^3} + \cdots$$

6. Find the first three non-zero terms of a power series for $\dfrac{xe^x}{\ln(1-x)}$.

$$\frac{xe^x}{\ln(1-x)} = \frac{x\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}}{\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n}} = \frac{\displaystyle\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}}{\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n}}$$

$$= \frac{x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \cdots}{x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \cdots}$$

$$1 + \frac{1}{2}x - \frac{1}{12}x^2 + \cdots$$

$$x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \cdots \;\Big)\; x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \cdots$$

$$\begin{aligned}
& x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 + \cdots \\
\hline
& \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 - \tfrac{1}{12}x^4 - \tfrac{19}{120}x^5 + \cdots \\
& \tfrac{1}{2}x^2 + \tfrac{1}{4}x^3 + \tfrac{1}{6}x^4 + \tfrac{1}{8}x^5 + \cdots \\
\hline
& -\tfrac{1}{12}x^3 + \tfrac{1}{12}x^4 - \tfrac{1}{30}x^5 + \cdots \\
& -\tfrac{1}{12}x^3 - \tfrac{1}{24}x^4 - \tfrac{1}{36}x^5 + \cdots \\
\hline
& \tfrac{3}{24}x^4 - \tfrac{1}{180}x^5 + \cdots
\end{aligned}$$

$$= \boxed{1 + \frac{1}{2}x - \frac{1}{12}x^2} + \cdots$$