## Math 76 Exercises – 5.3 The Divergence and Integral Tests

1. Determine whether each of the following series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (n^2-1)}{8-n-3n^2}$

We have $\displaystyle\lim_{n\to\infty} \frac{n^2-1}{8-n-3n^2} = -\frac{1}{3}$,

so $\displaystyle\lim_{n\to\infty} \frac{(-1)^n (n^2-1)}{8-n-3n^2}$ does not exist. (why?)

Therefore the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (n^2-1)}{8-n-3n^2}$ diverges by the Divergence Test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

This is a p-series with $p = 2 > 1$, so the series converges.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

$\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} \overset{H}{=} \lim_{n\to\infty} \frac{1/n}{1}$ (l'Hôpital's Rule)

$= 0$, so the Divergence Test does not apply.

Let $f(x) = \frac{\ln x}{x}$. Then $f'(x) = \dfrac{x \cdot \frac{1}{x} - \ln x}{x^2} = \dfrac{1-\ln x}{x^2}$,

which is negative for $x > e$. So $f(x)$ is continuous, positive, and decreasing for $x > e$. Thus the Integral Test applies.

$\longrightarrow$

$$\int_2^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\int_2^t \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \frac{1}{2}(\ln x)^2 \Big|_2^t$$

$$= \lim_{t\to\infty} \frac{1}{2}(\ln t)^2 - \frac{1}{2}(\ln(2))^2 = \infty \quad \text{(diverges)},$$

so by the Integral Test, the series $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ diverges

(d) $\sum_{n=5}^{\infty} \frac{42}{\sqrt[3]{n}} = 42 \sum_{n=5}^{\infty} \frac{1}{n^{1/3}}$. This is a p-series with $p = \frac{1}{3} \leq 1$, so the series diverges.

(e) $\sum_{n=1}^{\infty} \frac{3}{1+n^2}$ $\qquad \lim_{n\to\infty} \frac{3}{1+n^2} = 0$, so the Divergence Test does not apply.

The series seems "comparable" to $\sum \frac{3}{n^2}$, which converges, so we suspect $\sum \frac{3}{1+n^2}$ converges.

Let's use the Integral Test to make sure:

Let $f(x) = \frac{3}{1+x^2}$. $f(x)$ is continuous and positive everywhere. Moreover, $f'(x) = -\frac{6x}{(1+x^2)^2}$ which is negative for $x > 0$, so $f(x)$ is decreasing for $x > 0$.

Therefore the Integral Test applies.

$\longrightarrow$

$$\int_1^\infty \frac{3}{1+x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{3}{1+x^2}\,dx = \lim_{t\to\infty} 3\tan^{-1}x \Big|_1^t$$

$$= \lim_{t\to\infty}\left(3\tan^{-1}t - 3\tan^{-1}1\right) = 3\cdot\frac{\pi}{2} - 3\cdot\frac{\pi}{4} = \frac{3\pi}{4}$$

(converges).

Therefore the series $\sum_{n=1}^{\infty}\frac{3}{1+n^2}$ converges

(f) $\sum_{n=4}^{\infty}\frac{7n-15}{n(n-3)}$ $\lim_{n\to\infty}\frac{7n-15}{n(n-3)} = 0$, so the Divergence Test does not apply.

$f(x) = \frac{7x-15}{x(x-3)}$ is a rational function which is continuous and positive for $x>3$, and decreasing from some point on since $\lim_{x\to\infty} f(x) = 0$. So the Integral Test applies. We have

$$\int \frac{7x-15}{x(x-3)}\,dx = \int \frac{5}{x} + \frac{2}{x-3}\,dx$$ (Do the partial fractions to check!)

$$= 5\ln|x| + 2\ln|x-3| + C$$, so

$$\int_4^\infty \frac{7x-15}{x(x-3)}\,dx = \lim_{t\to\infty}\left(5\ln|t| + 2\ln|t-3| - (5\ln 4 - 2\ln 1)\right)$$

$$= \infty$$ (diverges).

So the series $\sum_{n=4}^{\infty}\frac{7n-15}{n(n-3)}$ diverges

2. Recall that the $k$th **remainder**, or *tail*, of a series $\sum_{n=?}^{\infty} a_n$ is

$$R_k = \sum_{n=k+1}^{\infty} a_n.$$

Note that if $s_k$ is the $k$th partial sum, then the sum of the series is

$$\sum_{n=?}^{\infty} a_n = s_k + R_k.$$

If the series $\sum a_n$ converges, and $f(x)$ is a continuous, positive, and decreasing function such that $f(n) = a_n$ for all $n$, then from the Integral Test we have that

$$\int_{n+1}^{\infty} f(x)\, dx \le R_n \le \int_{n}^{\infty} f(x)\, dx.$$

For each of the following convergent series, find how many terms must be used to estimate the sum with an error less than 0.001.

(a) (*) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

$f(x) = \frac{1}{x^2}$

$$\int_n^{\infty} \frac{1}{x^2}\, dx = \lim_{b\to\infty} \int_n^b \frac{1}{x^2}\, dx = \lim_{b\to\infty} -\frac{1}{x}\Big|_n^b = \lim_{b\to\infty}\left(-\underbrace{\frac{1}{b}}_{\to 0} + \frac{1}{n}\right) = \frac{1}{n}.$$

Want $n$ such that

$\frac{1}{n} < 0.001 = \frac{1}{1000}$.

$n > 1000$.

So we must use over 1000 terms!

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges very slowly!

(b) (*) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

$f(x) = \frac{1}{x(\ln x)^2}$

$$\int_n^{\infty} \frac{1}{x(\ln x)^2}\, dx = \lim_{b\to\infty} \int_n^b \frac{1}{x(\ln x)^2}\, dx$$

$u = \ln x$

$du = \frac{1}{x} dx$

$$= \lim_{b\to\infty} \int_{\ln n}^{\ln b} \frac{1}{u^2}\, du = \lim_{b\to\infty}\left(-\frac{1}{u}\right)\Big|_{\ln n}^{\ln b} = \lim_{b\to\infty}\left(-\underbrace{\frac{1}{\ln b}}_{\to 0} + \frac{1}{\ln n}\right)$$

$$= \frac{1}{\ln n}.$$

Want $n$ such that

$\frac{1}{\ln n} < \frac{1}{1000}$

$\ln n > 1000$

$n > e^{1000}$.



So we must use over $e^{1000}$ terms!

$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges very slowly!

3. For what values of $p$ does the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converge?

First note that if $p=0$, then we get $\sum_{n=2}^{\infty} \frac{1}{n}$, which is a p-series with "p" = 1 which diverges.

If $p<0$ then we get $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n}$ where $q=-p>0$, which also diverges since $\frac{(\ln n)^q}{n} > \frac{1}{n}$ for all $n>e$.

If $p>0$ then the Integral Test applies (check), so let $f(x) = \frac{1}{x(\ln x)^p}$.

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx \qquad \begin{matrix} u=\ln x \\ du=\frac{1}{x}dx \end{matrix} < \begin{matrix} x=t : u=\ln t \\ x=2 : u=\ln 2 \end{matrix}$$

$$= \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^p}\,dx$$

$$= \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u^p}\,du.$$

From the in-class exercises 7.8A #2, we see that this integral converges for $p>1$ and diverges for $p\le 1$.

Therefore $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges for $p>1$

and diverges for $p\le 1$.