Math 76 Exercises – 5.2A Geometric Series

1. Do the following series converge or diverge? If a series converges, find its sum.

(a) $\sum_{n=0}^{\infty} \frac{2}{3^n} = \sum_{n=0}^{\infty} 2 \cdot \left(\frac{1}{3}\right)^n$. Since $r = \frac{1}{3}$ and $\left|\frac{1}{3}\right| < 1$,

the series converges. The first term is $F = 2$,

so the sum is $\frac{F}{1-r} = \frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} = \frac{6}{2} = \boxed{3}$

(b) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \left(= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right) \cdot \left(\frac{1}{2}\right)^{n-1}, \text{ etc.}\right)$

We have $r = \frac{1}{2}$ and $F = \frac{1}{2}$, (There are infinitely many ways to express the sum!)

So the sum is $\frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = \boxed{1}$ (converges)

(c) $\sum_{n=1}^{\infty} 5e^{-2n} = \sum_{n=1}^{\infty} 5 \cdot \left(\frac{1}{e^2}\right)^n$.

We have $r = \frac{1}{e^2}$ and $F = \frac{5}{e^2}$, so the sum is

$\frac{5/e^2}{1 - \frac{1}{e^2}} = \frac{5/e^2}{\frac{e^2-1}{e^2}} = \boxed{\frac{5}{e^2-1}}$ (converges)

(d) $\sum_{n=2}^{\infty} \left(-\frac{1}{4}\right)^{3n} = \sum_{n=2}^{\infty} \left(-\frac{1}{64}\right)^n$.

We have $r = -\frac{1}{64}$ and $F = \left(-\frac{1}{4}\right)^6$, so the sum

is $\frac{\left(-\frac{1}{4}\right)^6}{1-\left(-\frac{1}{64}\right)} = \frac{\frac{1}{4^6}}{1+\frac{1}{4^3}} = \frac{\frac{1}{4^6}}{\frac{4^3+1}{4^3}} = \frac{\cancel{4^3}}{\cancel{4^6}^{\,4^3}(4^3+1)} = \frac{1}{64 \cdot 65}$

$= \boxed{\frac{1}{4160}}$ (converges)

(e) $\sum_{n=3}^{\infty} \left(\frac{1}{5}\right)^{n+1} \left(= \sum_{n=3}^{\infty} \left(\frac{1}{5}\right) \cdot \left(\frac{1}{5}\right)^n \right)$

We have $r = \frac{1}{5}$ and $F = \left(\frac{1}{5}\right)^{3+1} = \frac{1}{5^4}$, so

the series converges to $\dfrac{\frac{1}{5^4}}{1 - \frac{1}{5}} = \dfrac{1/5^4}{4/5} = \dfrac{\cancel{5}}{4 \cdot 5^{\cancel{4}3}} = \boxed{\frac{1}{500}}$

(f) $\sum_{n=0}^{\infty} \frac{(1.2)^n}{7} = \sum_{n=0}^{\infty} \left(\frac{1}{7}\right)(1.2)^n$.

This is a geometric series with $r = 1.2 \geq 1$, so the series diverges

(g) $\sum_{n=1}^{\infty} \frac{3^n}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{3}{2}\right)^n$.

This is a geometric series with $r = \frac{3}{2} \geq 1$,

So the series diverges.

(h) $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{25 \cdot 4^{n-1}} = \sum_{n=2}^{\infty} \frac{(-1)^2 (-1)^{n-1}}{25 \cdot 4^{n-1}} = \sum_{n=2}^{\infty} \frac{1}{25}\left(-\frac{1}{4}\right)^{n-1}$.

We have $r = -\frac{1}{4}$ and $F = \left(\frac{1}{25}\right)\left(-\frac{1}{4}\right)^1 = -\frac{1}{100}$,

So the series converges to $\dfrac{-1/100}{1 - (-\frac{1}{4})} = \dfrac{-1/100}{5/4}$

$= -\dfrac{4}{500} = \boxed{-\frac{1}{125}}$

2. Express each repeating decimal as a fraction (ratio of two integers).

Hint: write each decimal as a geometric series first. For example,

$$0.\overline{3} = 0.3 + 0.03 + 0.003 + \cdots = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \ldots$$

(a) $1.\overline{35}$

$$= 1 + 0.35 + 0.0035 + 0.000035 + \ldots$$

$$= 1 + \frac{35}{100} + \frac{35}{100^2} + \frac{35}{100^3} + \ldots$$

$$= 1 + \sum_{n=1}^{\infty} 35 \cdot \left(\frac{1}{100}\right)^n = 1 + \frac{35/100}{1 - 1/100} = 1 + \frac{35/100}{99/100}$$

$$= 1 + 35/99 = \boxed{\frac{134}{99}}$$

(b) $0.\overline{142857}$

$$= .142857 + .000000142857 + .000000000000142857 + \ldots$$

$$= \frac{142857}{1000000} + \frac{142857}{(1000000)^2} + \frac{142857}{(1000000)^3} + \ldots$$

$$= \sum_{n=1}^{\infty} 142857\left(\frac{1}{1000000}\right)^n = \frac{\frac{142857}{1000000}}{\frac{999999}{1000000}} = \frac{142857}{999999} = \boxed{\frac{1}{7}}$$

(c) $4.301\overline{2}$

$$= 4.301 + 0.0002 + 0.00002 + 0.000002 + \ldots$$

$$= 4 + \frac{301}{1000} + \frac{2}{10^4} + \frac{2}{10^5} + \frac{2}{10^6} + \ldots$$

$$= \frac{4301}{1000} + \sum_{n=4}^{\infty} 2 \cdot \left(\frac{1}{10}\right)^n = \frac{4301}{1000} + \frac{2/10^4}{1 - 1/10}$$

$$= \frac{4301}{1000} + \frac{2/10^4}{9/10} = \frac{4301}{1000} + \frac{2 \cdot \cancel{10}}{9 \cdot 10^{\cancel{4}\,3}} = \frac{4301}{1000} + \frac{2}{9000}$$

$$= \frac{9 \cdot 4301 + 2}{9000} = \boxed{\frac{38711}{9000}}$$