Math 76 Exercises – 6.4A Taylor and Maclaurin Series

1. Write the Maclaurin series for e^x, $\sin x$, $\cos x$, and $\tan^{-1}(x)$. What is the radius of convergence of each?

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad ; \quad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad ; \quad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad ; \quad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad ; \quad R = 1$$

(interval of convergence is $[-1, 1]$)

2. Write the Maclaurin series for each function. What is the radius of convergence?

(a) $f(x) = 4e^{8x}$

$$= 4 \sum_{n=0}^{\infty} \frac{(8x)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{4 \cdot 8^n x^n}{n!}}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\cancel{4} \cdot \cancel{8^{n+1}}\, 8}{\cancel{(n+1)!}\,(n+1)} \frac{\cancel{n!}}{\cancel{4} \cdot \cancel{8^n}} |x| = \lim_{n\to\infty} \frac{8}{n+1} |x| = 0 < 1$$

for all x. So $\boxed{R = \infty}$

(b) $g(x) = 5x^2 \cos x$

$$= 5x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \boxed{\sum_{n=0}^{\infty} \frac{5(-1)^n}{(2n)!} x^{2n+2}}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\cancel{5}}{(2(n+1))!} \cdot \frac{(2n)!}{\cancel{5}} \left| \frac{x^{2(n+1)+2}}{x^{2n+2}} \right| = \lim_{n\to\infty} \frac{\cancel{2n!}\, |x^2|}{(2n+2)(2n+1)\cancel{(2n)!}}$$

$$= 0 < 1 \text{ for all } x.$$

So $\boxed{R = \infty}$

(c) $h(x) = 2\sin(3x)$

$$= 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(3x)^{2n+1} = \boxed{\sum_{n=0}^{\infty} \frac{2(-1)^n 3^{2n+1}}{(2n+1)!} x^{2n+1}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\cancel{2}\cdot 3^{2(n+1)+1}}{(2(n+1)+1)!}\cdot\frac{(2n+1)!}{\cancel{2}\cdot 3^{2n+1}}\left|\frac{x^{2(n+1)+1}}{x^{2n+1}}\right|$$

$2(n+1)+1 = 2n+3$

$$= \lim_{n\to\infty} \frac{3^2}{(2n+3)(2n+2)}|x^2| = 0 < 1$$

for all x. So $\boxed{R = \infty}$

(d) $k(x) = x\tan^{-1}(x^3)$

$$= x\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(x^3)^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{3(2n+1)+1} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{6n+4}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2n+1}{2(n+1)+1}\cdot\left|\frac{x^{6(n+1)+4}}{x^{6n+4}}\right| = \lim_{n\to\infty} 1\cdot|x^6| = x^6 \overset{\text{set}}{<} 1.$$

So $|x|<1$. $\boxed{R = 1}$

3. For the function $k(x)$ above, find $k^{(22)}(0)$.

Hint: Consider the Maclaurin series for $k(x)$. The coefficient of the x^{22} term is $\dfrac{k^{(22)}(0)}{n!}$ (where n is what you have to plug in to get x^{22}).

We reach the x^{22} term when $6n+4 = 22$, i.e. $n=3$. So we have

$$\frac{k^{(22)}(0)}{3!}x^{22} = \frac{(-1)^3}{2\cdot 3+1}x^{6\cdot 3+4}$$

$$k^{(22)}(0) = \frac{(-1)^3}{2\cdot 3+1}\cdot 3! = \frac{-1}{7}\cdot 6 = \boxed{-\frac{6}{7}}$$

4. Find a series solution to each of the following indefinite integrals:

(a) $\int e^{x^4}\,dx$

$$e^{x^4} = \sum_{n=0}^{\infty} \frac{(x^4)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{4n}}{n!}$$

So $$\int e^{x^4}\,dx = \sum_{n=0}^{\infty} \frac{1}{n!}\cdot\frac{x^{4n+1}}{4n+1} + C = \boxed{\sum_{n=0}^{\infty} \frac{1}{n!\,(4n+1)}\cdot x^{4n+1} + C}$$

(b) $\int \sin(x^2)\,dx$

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{4n+2}$$

So $$\int \sin(x^2)\,dx = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\cdot\frac{1}{4n+3}\, x^{4n+3} + C}$$

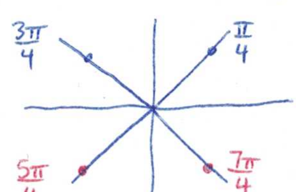
5. Write the Taylor series for each function centered at the given a.

(a) $f(x) = \sin x;\ a = \dfrac{\pi}{4}$

n	$f^{(n)}(x)$	$f^{(n)}(\frac{\pi}{4})$
0	$\sin x$	$\frac{\sqrt{2}}{2}$
1	$\cos x$	$\frac{\sqrt{2}}{2}$
2	$-\sin x$	$-\frac{\sqrt{2}}{2}$
3	$-\cos x$	$-\frac{\sqrt{2}}{2}$
4	$\sin x$	$\frac{\sqrt{2}}{2}$
5	$\cos x$	$\frac{\sqrt{2}}{2}$
	$\vdots$	$\vdots$

Hmm... $\left\{\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \right.$

$$= \left\{\sin\left(\tfrac{\pi}{4}(2n+1)\right)\right\}_{n=0}^{\infty}$$



From the work at right, we get

$$f^{(n)}\left(\tfrac{\pi}{4}\right) = \sin\left(\tfrac{\pi}{4}(2n+1)\right),$$

So

$$\sin x = \boxed{\sum_{n=0}^{\infty} \frac{\sin\left(\frac{\pi}{4}(2n+1)\right)}{n!}\left(x - \frac{\pi}{4}\right)^n}$$

(b) $g(x) = \frac{1}{x}$; $a = 5$

n	$g^{(n)}(x)$	$g^{(n)}(5)$
0	$\frac{1}{x}$	$\frac{1}{5}$
1	$-\frac{1}{x^2}$	$-\frac{1}{5^2}$
2	$\frac{2}{x^3}$	$\frac{2}{5^3}$
3	$\frac{-3\cdot 2}{x^4}$	$\frac{-3\cdot 2}{5^4}$
4	$\frac{4\cdot 3\cdot 2}{x^5}$	$\frac{4\cdot 3\cdot 2}{5^5}$
5	$\frac{-5\cdot 4\cdot 3\cdot 2}{x^6}$	$-\frac{5!}{5^6}$

From the work at right, we have

$$g^{(n)}(5) = \frac{(-1)^n n!}{5^{n+1}}, \text{ so}$$

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{n!}}{5^{n+1}} \cdot \frac{1}{\cancel{n!}}(x-5)^n$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} (x-5)^n}$$

(c) $h(x) = \ln x$; $a = 2$

n	$h^{(n)}(x)$	$h^{(n)}(2)$
0	$\ln x$	$\ln 2$
1	$\frac{1}{x}$	$\frac{1}{2}$
2	$-\frac{1}{x^2}$	$-\frac{1}{2^2}$
3	$\frac{2}{x^3}$	$\frac{2!}{x^3}$
4	$-\frac{3\cdot 2}{x^4}$	$-\frac{3!}{x^4}$

For $n \geq 1$ we have $h^{(n)}(2) = \frac{(-1)^{n-1}(n-1)!}{n!}$.

So

$$\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cancel{(n-1)!}}{2^n} \cdot \frac{1}{\underset{n}{\cancel{n!}}} (x-2)^n$$

$$= \boxed{\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 2^n} (x-2)^n}$$

(d) $k(x) = \frac{x-1}{e^x}$; $a = 1$

Following the hint, let $m(x) = e^{-x}$...

n	$m^{(n)}(x)$	$m^{(n)}(1)$
0	e^{-x}	$\frac{1}{e}$
1	$-e^{-x}$	$-\frac{1}{e}$
2	e^{-x}	$\frac{1}{e}$
3	$-e^{-x}$	$-\frac{1}{e}$
4	e^{-x}	$\frac{1}{e}$
⋮	⋮	⋮

We get $m^{(n)}(1) = (-1)^n \cdot \frac{1}{e}$. So

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{e} \cdot \frac{1}{n!}(x-1)^n.$$

Therefore $k(x) = (x-1)e^{-x}$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{e \cdot n!} (x-1)^{n+1}}$$