Tyler Gillette

Spring 2021 MATH 76
Activity 9

SEQUENCES AND SERIES
You will need a calculator to complete this activity. You can use your phone if needed.

1. **Sequences**

(a) Consider the sequence defined by $a_1 = 1$ and $a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right)$ for $n = 2, 3, \ldots$.

i. Complete the following table.

n	a_n
1	$a_1 = 1$
2	$a_2 = \ldots\ldots$
3	$a_3 = \ldots\ldots$
4	$a_4 = \ldots\ldots$
5	$a_5 = \ldots\ldots$
6	$a_6 = \ldots\ldots$

ii. Does the sequence seem to converge? If yes, to what value?

(b) Consider the Fibonacci sequence defined by $f_1 = 1, f_2 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for $n = 2, 3, \ldots$.

i. Complete the following table.

n	f_n	$\frac{f_{n+1}}{f_n}$
1	1	$\frac{1}{1} = 1$
2	1	$\frac{\ldots}{1} = \ldots\ldots$
3	$\ldots$	$\frac{\ldots}{\ldots} = \ldots\ldots$
4	$\ldots$	$\frac{\ldots}{\ldots} = \ldots\ldots$
5	$\ldots$	$\frac{\ldots}{\ldots} = \ldots\ldots$
6	$\ldots$	$\frac{\ldots}{\ldots} = \ldots\ldots$

ii. Does the sequence $\{f_n\}$ seem to converge or diverge? If yes, to what value? How about the sequence $\left\{\frac{f_{n+1}}{f_n}\right\}$?

1. **Sequences**

(a) Consider the sequence defined by $a_1 = 1$ and $a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right)$ for $n = 2, 3, \ldots$.

i. Complete the following table.

n	a_n
1	$a_1 = 1$
2	$a_2 = \ldots\ldots$
3	$a_3 = \ldots\ldots$
4	$a_4 = \ldots\ldots$
5	$a_5 = \ldots\ldots$
6	$a_6 = \ldots\ldots$

ii. Does the sequence seem to converge? If yes, to what value?

I. 2 = 1.5 6 = 1.414

3 = 1.416 7 = 1.414

4 = 1.414 8 = 1.414

5 = 1.414

II. $\boxed{\text{Yes, it Converges to 1.414}}$

(b) Consider the Fibonacci sequence defined by $f_1 = 1$, $f_2 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for $n = 2, 3, \ldots$.

i. Complete the following table.

n	f_n	$\frac{f_{n+1}}{f_n}$
1	1	$\frac{1}{1} = 1$
2	1	$\frac{\cdots}{1} = \ldots\ldots$
3	$\ldots$	$\frac{\cdots}{\cdots} = \ldots\ldots$
4	$\ldots$	$\frac{\cdots}{\cdots} = \ldots\ldots$
5	$\ldots$	$\frac{\cdots}{\cdots} = \ldots\ldots$
6	$\ldots$	$\frac{\cdots}{\cdots} = \ldots\ldots$

ii. Does the sequence $\{f_n\}$ seem to converge or diverge? If yes, to what value? How about the sequence $\left\{\frac{f_{n+1}}{f_n}\right\}$?

I.

3 = 2, 2

4 = 3, $\frac{3}{2}$

5 = 5, $\frac{5}{3}$

6 = 8, $\frac{8}{5}$

7 = 13, $\frac{13}{8}$

II. The Sequence diverges and goes to ∞

No the Series $\frac{f_{n+1}}{f_n}$ will Converge.

(c) In addition to the techniques of computing limits that you have previously learned, there are results like the **Squeeze theorem** that are useful to find limits.

Compute the limit (if it exists) of the following sequences as n approaches ∞.

i. $a_n = \dfrac{(-1)^n}{2^n}$

ii. $a_n = \dfrac{5^n}{5^n + 1}$

iii. $a_n = \left(1 + \dfrac{1}{n}\right)^n$

iv. $a_n = \dfrac{\ln n}{n}$

v. $a_n = \sqrt{n+1} - \sqrt{n}$

vi. $a_n = \dfrac{\sin n}{n^2}$

(c) In addition to the techniques of computing limits that you have previously learned, there are results like the **Squeeze theorem** that are useful to find limits.

Compute the limit (if it exists) of the following sequences as n approaches ∞.

i. $a_n = \frac{(-1)^n}{2^n}$

$$\lim_{n\to\infty} \frac{(-1)^n}{2^n} = \boxed{0} \checkmark$$

+ − + −

b s b s

ii. $a_n = \dfrac{5^n}{5^n+1}$

$$\lim_{n\to\infty} \frac{5^n}{5^n+1} = \boxed{1} \checkmark$$

iii. $a_n = \left(1+\dfrac{1}{n}\right)^n$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \boxed{e} \checkmark$$

iv. $a_n = \dfrac{\ln n}{n}$

$$\lim_{n\to\infty} \frac{\ln n}{n} = \boxed{0} \checkmark$$

v. $a_n = \sqrt{n+1} - \sqrt{n}$

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = \boxed{0} \checkmark$$

vi. $a_n = \dfrac{\sin n}{n^2}$

$$\lim_{n\to\infty} \frac{\sin(n)}{n^2} = \boxed{0} \checkmark$$

2. **Series**

Understanding the infinite series "process"

Start the "process" with a sequence $\{x_k\} = \{x_1, x_2, x_3, \ldots\}$, and compute the partial sums

$$\begin{aligned} S_1 &= x_1 \\ S_2 &= x_1 + x_2 \\ \vdots &= \vdots \\ S_n &= x_1 + x_2 + \ldots + x_n = \sum_{k=1}^{n} x_k \end{aligned}$$

The numbers $S_1, S_2, \ldots, S_n$ form a sequence and the limit of that sequence, $\lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} x_k$, is the infinite series $\sum_{k=1}^{\infty} x_k$. If $\lim_{n\to\infty} S_n$ exists and is a real number then the infinite series $\sum_{k=1}^{\infty} x_k$ **converges**. Otherwise it **diverges**.

Now finding the limit $\lim_{n\to\infty} S_n$ is not trivial if S_n is cannot be expressed as a function of n. There are a few cases when this is possible. For example

- $\sum_{k=0}^{n} r^k = \dfrac{1 - r^{n+1}}{1 - r}$ when $r \neq 0, 1$
- when the sum telescopes $\sum_{k=1}^{n} \dfrac{1}{k(k+1)} = 1 - \dfrac{1}{n+1}$

When it is not trivial to express S_n as a function of n, which is often the case, **a convergence test** will be used to determine if the infinite series $\sum_{k=1}^{\infty} x_k$ converges or diverges.

(a) Consider the geometric series $\sum_{k=M}^{\infty} c \cdot r^k$ where M is a positive integer, and c is a nonzero constant. The goal of this problem is to come up with a set of conditions that will help decide whether the series $\sum_{k=M}^{\infty} c \cdot r^k$ converges or diverges.

i. Let

$$\begin{aligned} S_n &= \sum_{k=M}^{n} c \cdot r^k \\ &= c \cdot r^M + c \cdot r^{M+1} + \ldots + c \cdot r^n. \end{aligned}$$

Write rS_n as a sum

$$rS_n =$$

(a) Consider the geometric series $\sum_{k=M}^{\infty} c \cdot r^k$ where M is a positive integer, and c is a nonzero constant. The goal of this problem is to come up with a set of conditions that will help decide whether the series $\sum_{k=M}^{\infty} c \cdot r^k$ converges or diverges.

i. Let

$$S_n = \sum_{k=M}^{n} c \cdot r^k$$
$$= c \cdot r^M + c \cdot r^{M+1} + \ldots + c \cdot r^n.$$

Write rS_n as a sum

$rS_n =$

$$\boxed{r \cdot (S_n) = C \cdot r^{m+1} + C \cdot r^{m+2} + C \cdot r^{m+3} + C \cdot r^{m+n}}$$

ii. Write $S_n - rS_n$ as a sum

$$S_n - rS_n =$$

iii. Using your result in part ii., what is a formula for S_n?

iv. Next we want to evaluate $\lim_{n\to\infty} S_n$. Compute

A. $\lim_{n\to\infty} r^{n+1}$ when $|r| < 1$. Choose a value of r that satisfies the condition $|r| < 1$ and test the limit.

B. $\lim_{n\to\infty} r^{n+1}$ when $|r| > 1$. Choose a value of r that satisfies the condition $|r| > 1$ and test the limit. Note that if $r = 1$, the formula found in part iii. does not hold and if $r = -1$ the limit $\lim_{n\to\infty} r^{n+1}$ does not exist.

C. $\lim_{n\to\infty} S_n$

v. Write the conditions under which the infinite series $\sum_{k=M}^{\infty} c \cdot r^k$ converges. To what limit does it converge to?

vi. Write the conditions under which the infinite series $\sum_{k=M}^{\infty} c \cdot r^k$ diverges.

ii. Write $S_n - rS_n$ as a sum

$$S_n - rS_n = S_n(1-r)$$

iii. Using your result in part ii., what is a formula for S_n?

II $S_n - r(S_n) = \boxed{cr^m - cr^{n+1}}$ ✓

$cr^m - crm$

$cr^m + cr^{m+1} + cr^{m+2}$

$cr^{m+1} + cr^{m+2} + \dots cr^{m+n}$

III $S_n = cr^m - cr^{m+n}$

$$\boxed{S_n = \frac{cr^m - cr^{n+1}}{1-r}}$$ ✓

iv. Next we want to evaluate $\lim_{n\to\infty} S_n$. Compute

A. $\lim_{n\to\infty} r^{n+1}$ when $|r| < 1$. Choose a value of r that satisfies the condition $|r| < 1$ and test the limit.

B. $\lim_{n\to\infty} r^{n+1}$ when $|r| > 1$. Choose a value of r that satisfies the condition $|r| > 1$ and test the limit. Note that if $r = 1$, the formula found in part iii. does not hold and if $r = -1$ the limit $\lim_{n\to\infty} r^{n+1}$ does not exist.

C. $\lim_{n\to\infty} S_n$

IV $\boxed{\lim_{n\to\infty} cr^m = \infty}$

A $\lim_{n\to\infty} r^{n+1} = 0$

$\lim_{n\to\infty} 0.5^{n+1} = 0$ ✓

B $\lim_{n\to\infty} r^{n+1} = \infty$

$\lim_{n\to\infty} 5^{n+1} = \infty$

if $r \leq -1$ DNE

✓

C $$\boxed{\lim_{n\to\infty} S_n = \begin{cases} \frac{c \cdot r^m}{1-r} & \text{if } |r| < 1 \\ \infty \text{ or DNE} & \text{if } |r| \geq 1 \end{cases}}$$ ✓

v. Write the conditions under which the infinite series $\sum_{k=M}^{\infty} c \cdot r^k$ converges. To what limit does it converge to?

v) $\boxed{1) \ |r| < 1}$

vi. Write the conditions under which the infinite series $\sum_{k=M}^{\infty} c \cdot r^k$ diverges.

vi. $\boxed{|r| \geq 1}$