Math 76 Exercises – 5.1B Sequences

1. Determine whether each sequence converges or diverges. If a sequence converges, find the limit.

(a) $\left\{\dfrac{n^2+5}{3-4n^2}\right\}$

$$\lim_{n\to\infty} \frac{n^2+5}{3-4n^2} = \lim_{n\to\infty} \frac{n^2}{-4n^2} = -\frac{1}{4}.$$

So the sequence converges to $-\frac{1}{4}$.

(b) $\left\{\dfrac{2n}{\sqrt{n}-107}\right\}$

$$\lim_{n\to\infty} \frac{2n}{\sqrt{n}-107} = \lim_{n\to\infty} \frac{2n}{\sqrt{n}} = \lim_{n\to\infty} 2\sqrt{n} = \infty.$$

So the sequence diverges.

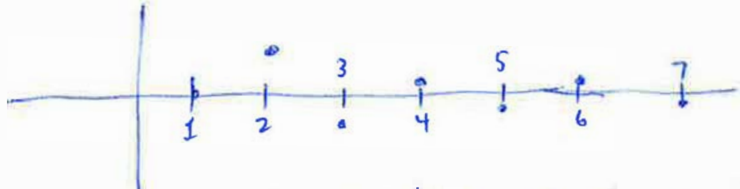
(c) $\left\{\dfrac{\ln n}{n}\right\}$

$$\lim_{n\to\infty} \frac{\ln n}{n} \qquad \text{“}\frac{\infty}{\infty}\text{”} \sim \text{use L'Hôpital's Rule} \ldots$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{1/n}{1} = 0.$$

So the sequence converges to 0.

(d) $\left\{\frac{(-1)^n \ln n}{n}\right\}$



When n is even, the terms are positive and go to 0.
When n is odd, the terms are negative and go to 0. (see part (c)).
Since all terms go to 0, the sequence converges to 0.

2. Determine whether each sequence above is monotonic or not.

(a) Every rational function has finitely many critical numbers and vertical asymptotes. So if we let N be the largest x-value that has a critical point or vertical asymptote for $f(x) = \frac{x^2+5}{3-4x^2}$, then the sequence $\left\{\frac{n^2+5}{3-4n^2}\right\}$ is monotonic for all $n \geq N$.

(b) For $a, b > 0$ we know that if $a < b$ then $a^2 < b^2$. So we can note that $\left(\frac{2n}{\sqrt{n-107}}\right)^2 = \frac{4n^2}{n-107}$ is a rational function and hence monotonic from some point on; thus $\frac{2n}{\sqrt{n-107}}$ is monotonic as well.

(c) Let $f(x) = \frac{\ln x}{x}$. Then $f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1-\ln x}{x^2}$ which is negative when $1 - \ln x < 0$, i.e. $\ln x > 1$, i.e. $x > e$. So $f(x)$ is decreasing for all $x > e$; hence the sequence $\left\{\frac{\ln n}{n}\right\}$ is monotonic for $n \geq 3$.

(d) The sequence alternates $(-,+,-,+,\ldots)$ so is <u>not</u> monotonic.

3. Determine whether each sequence above is bounded or not. For each bounded sequence $\{a_n\}$, find a number M such that $|a_n| \le M$ for all $n \ge 1$.

First of all, every convergent sequence is bounded (why?), so the sequences in parts (a), (c), and (d) are bounded.

Since the sequence in (b) diverges to ∞, that sequence is not bounded.

(a) For $n=1$ we get $\dfrac{1^2+5}{3-4\cdot 1^2} = -6$.

For $n=2$ we get $\dfrac{2^2+5}{3-4\cdot 2^2} = -\dfrac{9}{13}$

For $n=3$ we get $\dfrac{3^2+5}{3-4\cdot 3^2} = -\dfrac{14}{33}$, etc. (closer and closer to $-\frac{1}{4}$)

So all terms are bounded by $M=6$.

(c) For $n=1$ we get $\dfrac{\ln 1}{1} = 0$.

For $n=2$ we get $\dfrac{\ln 2}{2} \approx 0.347$.

For $n=3$ we get $\dfrac{\ln 3}{3} \approx 0.366$.

Since the sequence approaches 0 monotonically for $n \geq 3$, we can use $M = \dfrac{\ln 3}{3}$.

(d) This sequence is the alternating version of the sequence in part (c), so the absolute values are bounded by the same numbers.

So we can use $M = \dfrac{\ln 3}{3}$ again.

4. Determine whether each sequence converges or diverges. If a sequence converges, find the limit.

(a) $\left\{\frac{50\sin^2 n}{n^3}\right\}$ Using the Squeeze Theorem, we have

$0 \le \sin^2 n \le 1$ for all n, so

$$0 \le \lim_{n\to\infty} \frac{50\sin^2 n}{n^3} \le \lim_{n\to\infty} \frac{50}{n^3} = 0.$$

Therefore $\lim_{n\to\infty} \frac{50\sin^2 n}{n^3} = \boxed{0}$

(b) $\left\{\frac{3^n}{n!}\right\}_{n=1}^{\infty}$ We have $\frac{3^n}{n!} = \frac{3}{1}\cdot\frac{3}{2}\cdot\underbrace{\frac{3\cdot 3\cdot 3\cdots 3}{3\cdot 4\cdot 5\cdots(n-1)}}_{\le 1}\cdot\frac{3}{n}$

So $0 \le \frac{3^n}{n!} \le \frac{9}{2}\cdot 1\cdot\frac{3}{n} = \frac{27}{2}\cdot\frac{1}{n}.$

So $0 \le \lim_{n\to\infty}\frac{3^n}{n!} \le \lim_{n\to\infty}\frac{27}{2}\cdot\frac{1}{n} = 0.$ Thus $\lim_{n\to\infty}\frac{3^n}{n!} = \boxed{0}$

(c) $\left\{\left(1+\frac{1}{n}\right)^n\right\}$

Let $L = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. Indeterminate form of type 1^∞

Then $\ln L = \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)$ Indeterminate form $\infty\cdot 0$

$= \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{1/n}$ "$\frac{0}{0}$", so l'Hôpital's Rule applies

$\overset{H}{=} \lim_{n\to\infty}\frac{\frac{-1/n^2}{1+1/n}}{-1/n^2} = \lim_{n\to\infty}\frac{1}{1+1/n} = 1.$ So $\ln L = 1$.

Therefore $L = \boxed{e}$

(d) $\{a_n\}$ defined recursively by $a_1 = 4$ and $a_{n+1} = \frac{a_n}{3} + 10$.

Let $L = \lim_{n\to\infty} a_n$. Then $L = \lim_{n\to\infty} a_{n+1}$, so

$\lim_{n\to\infty}(a_{n+1}) = \lim_{n\to\infty}\left(\frac{a_n}{3} + 10\right)$ becomes $L = \frac{1}{3}L + 10$.

Thus $\frac{2}{3}L = 10$, so $L = \frac{3}{2}\cdot 10 = \boxed{15}$

5. Recall that the formula for the value of an annuity after k months (where P dollars is invested initially and at the end of every month, at a monthly interest rate of r) is

$$V(k) = \frac{P((1+r)^{k+1} - 1)}{r}.$$

If \$100 is invested monthly in an annuity paying 6% annually (compounded monthly),

(a) What will be the value of the annuity after 3 years?

$r = \frac{0.06}{12} = \frac{0.01}{2} = 0.005$.

$$V(36) = \frac{100((1.005)^{37} - 1)}{0.005}$$

$\approx \$4053.28$

(b) How many years will it take to reach a value of \$25,000?

$$V(k) = \frac{100((1.005)^{k+1} - 1)}{0.005} \overset{\text{set}}{=} 25{,}000$$

$$(1.005)^{k+1} - 1 = \frac{25000}{100}(0.005) = 1.25$$

$$(1.005)^{k+1} = 2.25$$

$$(k+1)\ln(1.005) = \ln(2.25)$$

$$k+1 = \frac{\ln(2.25)}{\ln(1.005)}$$

$k \approx 161.6$ months

$\approx \boxed{13 \text{ years, } 6 \text{ months}}$