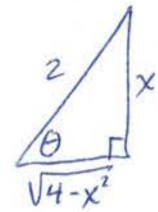
1. Evaluate each integral.

(a) (*) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

$x = 2\sin\theta \qquad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$

$dx = 2\cos\theta\, d\theta$



$$= \int \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}} \cdot 2\cos\theta\, d\theta$$

$$= \int \frac{8\sin^2\theta\cos\theta}{\sqrt{4(1-\sin^2\theta)}}\, d\theta$$

$$= \frac{8}{2}\int \frac{\sin^2\theta\,\cancel{\cos\theta}}{\cancel{\sqrt{\cos^2\theta}}}\, d\theta$$

$$= 4\int \sin^2\theta\, d\theta$$

power of $\sin\theta$ is even...

$$= 4\int \frac{1}{2}(1-\cos(2\theta))\,d\theta$$

$$= 2\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C$$

$$= 2(\theta - \sin\theta\cos\theta) + C$$

$$= 2\left(\sin^{-1}\left(\frac{x}{2}\right) - \frac{x}{2}\,\frac{\sqrt{4-x^2}}{2}\right) + C$$

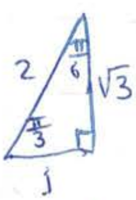
$$= \boxed{2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + C}$$

(b) (*) $\displaystyle\int_1^2 \frac{\sqrt{x^2-1}}{x^3}\,dx$

$x = \sec\theta \qquad \left(0 \le \theta < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < \theta \le \pi\right)$

$dx = \sec\theta\tan\theta\, d\theta$

$x=2:\ \sec\theta = 2$
$\theta = \frac{\pi}{3}$



$x=1:\ \sec\theta = 1$
$\theta = 0$

$$\int_1^2 \frac{\sqrt{x^2-1}}{x^3}\,dx = \int_0^{\frac{\pi}{3}} \frac{\sqrt{\sec^2\theta - 1}}{\sec^{\cancel{3}2}\theta}\,\cancel{\sec\theta}\tan\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{\tan^2\theta}{\sec^2\theta}\,d\theta = \int_0^{\frac{\pi}{3}} \frac{\sin^2\theta}{\cancel{\cos^2\theta}}\cdot\frac{\cancel{\cos^2\theta}}{1}\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} \sin^2\theta\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\Big|_0^{\frac{\pi}{3}}$$

part (a)

$$= \frac{1}{2}\cdot\frac{\pi}{3} - \frac{1}{4}\sin\left(\frac{2\pi}{3}\right) - (0-0)$$

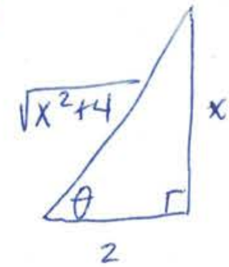
$$= \frac{\pi}{6} - \frac{1}{4}\cdot\frac{\sqrt{3}}{2}$$

$$= \boxed{\frac{\pi}{6} - \frac{\sqrt{3}}{8}}$$

(c) (**) $\int \sqrt{4x^2+16}\,dx$

$x = 2\tan\theta \qquad \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$

$dx = 2\sec^2\theta$



$$= 2\int \sqrt{x^2+4}\,dx$$

$$= 2\int \sqrt{4\tan^2\theta+4}\cdot 2\sec^2\theta\,d\theta$$

$$= 8\int \sqrt{\tan^2\theta+1}\,\sec^2\theta\,d\theta$$

$$= 8\int \sec^3\theta\,d\theta$$

$$= 8\cdot\frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right)+C$$

(from 3.2B #1(b))

$$= \cancel{4}\cdot\frac{\sqrt{x^2+4}}{\cancel{2}}\cdot\frac{x}{\cancel{2}} + 4\ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right|+C \overset{*}{``="} \boxed{x\sqrt{x^2+4}+4\ln|\sqrt{x^2+4}+x|+C}$$

different "C"

* Note that

$$\ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right| = \ln\left|\frac{\sqrt{x^2+4}+x}{2}\right|$$

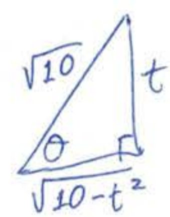
$$= \ln|\sqrt{x^2+4}+x| - \ln(2)$$

constant!

(d) (**) $\int_1^3 \frac{1}{t\sqrt{10-t^2}}\,dt$

$t = \sqrt{10}\sin\theta$

$dt = \sqrt{10}\cos\theta\,d\theta$



$$= \int_{\sin^{-1}\left(\frac{1}{\sqrt{10}}\right)}^{\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)} \frac{\cancel{\sqrt{10}}\cos\theta}{\cancel{\sqrt{10}}\sin\theta\sqrt{10-10\sin^2\theta}}\,d\theta$$

$$= \frac{1}{\sqrt{10}}\int_{\sin^{-1}\left(\frac{1}{\sqrt{10}}\right)}^{\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)} \frac{\cancel{\cos\theta}}{\sin\theta\cancel{\cos\theta}}\,d\theta$$

$$= \frac{1}{\sqrt{10}}\int_{\sin^{-1}\left(\frac{1}{\sqrt{10}}\right)}^{\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)} \csc\theta\,d\theta$$

$$= -\frac{1}{\sqrt{10}}\ln|\csc\theta+\cot\theta|\Bigg|_{\sin^{-1}\left(\frac{1}{\sqrt{10}}\right)}^{\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)}$$

$$= -\frac{1}{\sqrt{10}}\ln\left|\csc\left(\sin^{-1}\left(\tfrac{3}{\sqrt{10}}\right)\right)+\cot\left(\sin^{-1}\left(\tfrac{3}{\sqrt{10}}\right)\right)\right|$$

$$+\frac{1}{\sqrt{10}}\ln\left|\csc\left(\sin^{-1}\left(\tfrac{1}{\sqrt{10}}\right)\right)+\cot\left(\sin^{-1}\left(\tfrac{1}{\sqrt{10}}\right)\right)\right|$$

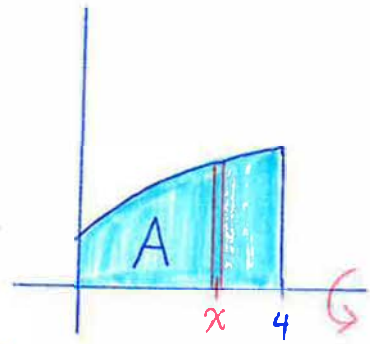
$$= \boxed{-\frac{1}{\sqrt{10}}\left(\ln\left(\frac{\sqrt{10}}{3}+\frac{1}{3}\right)-\ln\left(\sqrt{10}+3\right)\right)}$$

2. Consider the region A bounded by $y = \sqrt[4]{x^2+9}$, $y = 0$, $x = 0$, and $x = 4$.

(a) Find the volume of the solid formed by rotating A about the x-axis.

Using the disk method, we have

$R = \sqrt[4]{x^2+9}$, $r = 0$, so



$$V = \pi \int_0^4 (R^2 - r^2)\,dx$$

$$= \pi \int_0^4 \left(\sqrt[4]{x^2+9}\right)^2 dx = \pi \int_0^4 \sqrt{x^2+9}\,dx.$$

$x = 3\tan\theta$

$dx = 3\sec^2\theta\,d\theta$

$\sqrt{x^2+9} = 3\sec\theta$

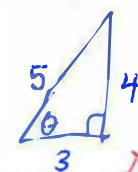
(check!)

$$= \pi \int_0^{\tan^{-1}(\frac{4}{3})} 3\sec\theta\,d\theta$$

$$= 3\pi \ln|\sec\theta + \tan\theta| \Big|_0^{\tan^{-1}(\frac{4}{3})}$$

$$= 3\pi\left(\ln\left|\tfrac{5}{3} + \tfrac{4}{3}\right| - \ln(1+0)\right) = \boxed{3\pi \ln(3)}$$

$\sec(\tan^{-1}(\frac{4}{3})) = \frac{5}{3}$:



(b) Find the volume of the solid formed by rotating A about the y-axis.

Using the shell method, we have

$r = x$, $h = \sqrt[4]{x^2+9}$, so

$$V = \frac{2\pi}{2} \int_0^4 2x \sqrt[4]{x^2+9}\,dx$$

$u = x^2 + 9$

$du = 2x\,dx$

$x = 4 : u = 25$

$x = 0 : u = 9$

$$= \pi \int_9^{25} u^{\frac{1}{4}}\,du$$

$$= \pi \cdot \frac{4}{5} u^{5/4} \Big|_9^{25}$$

$$= \frac{4\pi}{5}\left(25^{5/4} - 9^{5/4}\right)$$

$$= \frac{4\pi}{5}\left((5^2)^{5/4} - (3^2)^{5/4}\right) = \boxed{\frac{4\pi}{5}\left(5^{5/2} - 3^{5/2}\right)}$$