Math 76 Exercises – 7.4A Derivatives and Areas in Polar Coordinates

1. Find the slope of the tangent line to the polar curve $r = 3 - 2\sin\theta$ at $\theta = -\dfrac{\pi}{4}$.

$f(\theta) = 3 - 2\sin\theta$. Parametric equations are

$x = r\cos\theta = (3 - 2\sin\theta)\cos\theta = 3\cos\theta - 2\sin\theta\cos\theta = 3\cos\theta - \sin 2\theta$

$y = r\sin\theta = (3 - 2\sin\theta)\sin\theta = 3\sin\theta - 2\sin^2\theta$.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos\theta - 4\sin\theta\cos\theta}{-3\sin\theta - 2\cos(2\theta)} = \frac{3\cos\theta - 2\sin 2\theta}{-3\sin\theta - 2\cos 2\theta}$$

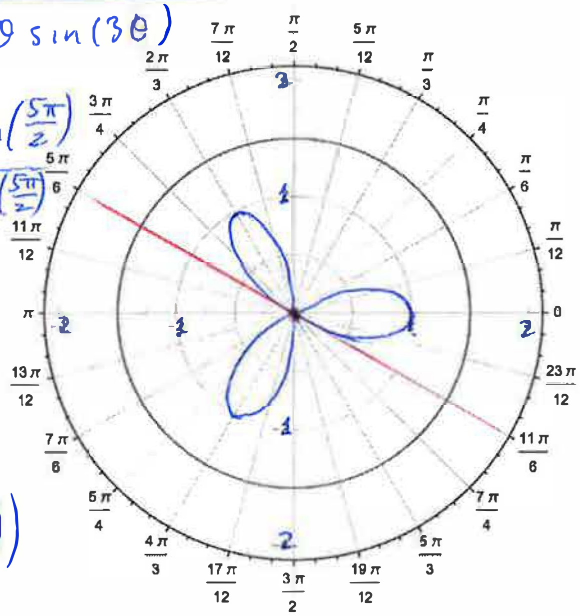
$$\left.\frac{dy}{dx}\right|_{\theta = -\frac{\pi}{4}} = \frac{3\cos(-\frac{\pi}{4}) - 2\sin(-\frac{\pi}{2})}{-3\sin(-\frac{\pi}{4}) - 2\cos(-\frac{\pi}{2})} = \frac{3(\frac{\sqrt{2}}{2}) - 2(-1)}{-3(-\frac{\sqrt{2}}{2}) - 2\cdot 0} = \boxed{\frac{\frac{3\sqrt{2}}{2} + 2}{\frac{3\sqrt{2}}{2}}}$$

2. Find the equation of the tangent line to the rose $r = \cos(3\theta)$ at the point where $\theta = \dfrac{5\pi}{6}$. Verify that your answer is plausible by graphing.

Parametric equations are $x = \cos(3\theta)\cos\theta$

$y = \cos(3\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos(3\theta)\cos\theta - 3\sin\theta\sin(3\theta)}{-\cos(3\theta)\sin\theta - 3\cos\theta\sin(3\theta)}$$



$$\left.\frac{dy}{dx}\right|_{\theta = \frac{5\pi}{6}} = \frac{\overset{0}{\cancel{\cos(\frac{5\pi}{2})}}\cos(\frac{5\pi}{6}) - 3\sin(\frac{5\pi}{6})\sin(\frac{5\pi}{2})}{\overset{0}{\cancel{-\cos(\frac{5\pi}{2})}}\sin(\frac{5\pi}{6}) - 3\cos(\frac{5\pi}{6})\sin(\frac{5\pi}{2})}$$

$$= \frac{-3\cdot\frac{1}{2}\cdot 1}{-3\cdot\frac{-\sqrt{3}}{2}\cdot 1} = -\frac{1}{\sqrt{3}} = m$$

Point of tangency is

$\left(\cos(\frac{5\pi}{2})\cos(\frac{5\pi}{6}), \cos(\frac{5\pi}{2})\sin(\frac{5\pi}{6})\right)$

$= (0, 0)$.

So the equation is $\boxed{y = \frac{-1}{\sqrt{3}}x}$

3. Find the points at which the tangent line to the graph of the cardioid $r = 1 - \sin\theta$ is

   (i) horizontal;

   (ii) vertical.

   Verify that your answer is plausible by graphing.

$x = (1-\sin\theta)\cos\theta = \cos\theta - \sin\theta\cos\theta = \cos\theta - \frac{1}{2}\sin(2\theta)$

$y = (1-\sin\theta)\sin\theta = \sin\theta - \sin^2\theta$

$\frac{dy}{d\theta} = \cos\theta - 2\sin\theta\cos\theta \overset{set}{=} 0$

$\cos\theta(1-2\sin\theta) = 0$

$\cos\theta = 0$ or $\sin\theta = \frac{1}{2}$

$\theta = \cancel{\frac{\pi}{2}}, \frac{3\pi}{2}, \ldots$ or $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$

$\frac{dx}{d\theta} = -\sin\theta - \cos(2\theta) \overset{set}{=} 0$

$-\sin\theta - (1-2\sin^2\theta) = 0$

$2\sin^2\theta - \sin\theta - 1 = 0$

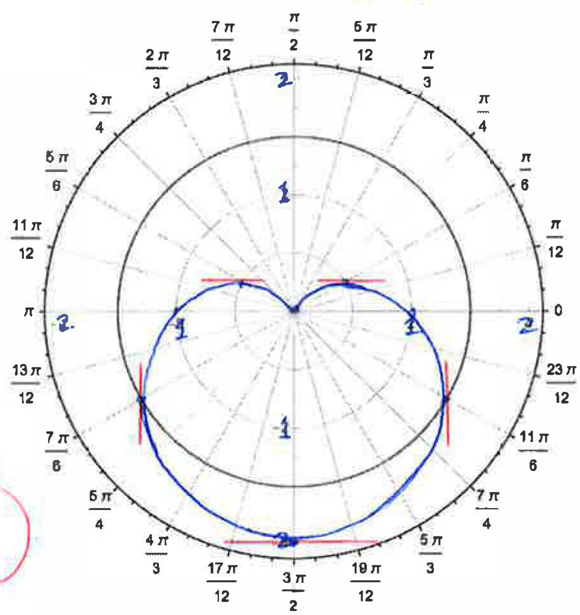
$(2\sin\theta + 1)(\sin\theta - 1) = 0$

$\sin\theta = -\frac{1}{2}$ or $\sin\theta = 1$

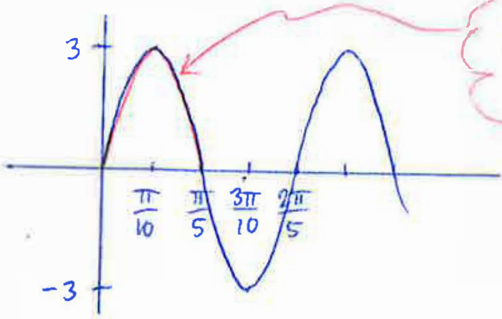
$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \ldots$ or $\theta = \cancel{\frac{\pi}{2}}, \ldots$

Cross out values of θ that make both $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ equal to 0.

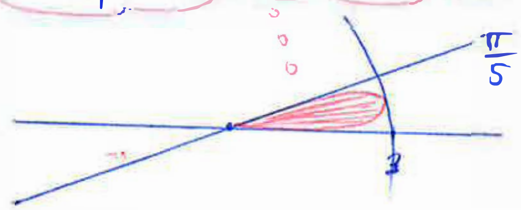
r	θ
1	0
$\frac{1}{2}$	$\frac{\pi}{6}$
0	$\frac{\pi}{2}$
$\frac{1}{2}$	$\frac{5\pi}{6}$
1	π
$\frac{3}{2}$	$\frac{7\pi}{6}$
2	$\frac{3\pi}{2}$
$\frac{3}{2}$	$\frac{11\pi}{6}$



4. Find the area of one leaf of the rose $r = 3\sin(5\theta)$.



One leaf of the rose corresponds to this "bump"



$$\text{Area} = \frac{1}{2}\int_0^{\frac{\pi}{5}} (3\sin(5\theta))^2\, d\theta$$

$$= \frac{1}{2}\cdot 9 \int_0^{\frac{\pi}{5}} \sin^2(5\theta)\, d\theta$$

$$= \frac{9}{2}\cdot\frac{1}{2}\int_0^{\frac{\pi}{5}} (1-\cos(10\theta))\, d\theta$$

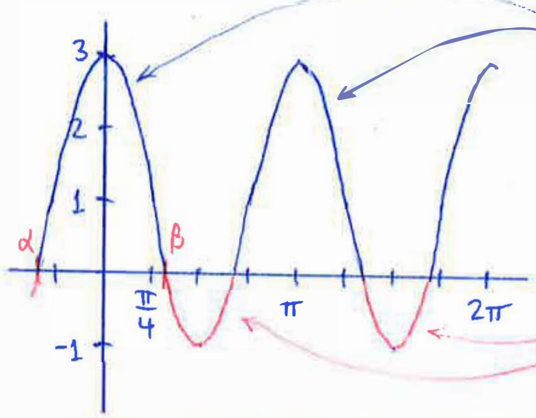
$$= \frac{9}{4}\left(\theta - \frac{1}{10}\sin(10\theta)\right)\Big|_0^{\frac{\pi}{5}}$$

$$= \frac{9}{4}\left(\frac{\pi}{5} - \frac{1}{10}\cancel{\sin(2\pi)}^{\,0} - (0-0)\right)$$

$$= \boxed{\frac{9\pi}{20}}$$

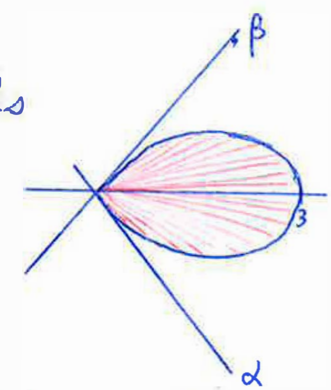
5. Set up an integral for the area enclosed by one of the larger loops of the polar curve

$$r = 1 + 2\cos(2\theta).$$



A larger loop corresponds to one of these "bumps".

(The smaller loops come from these "bumps".)



To find α and β, set $r = 0$:

$$1 + 2\cos(2\theta) \overset{\text{set}}{=} 0$$

$$\cos(2\theta) = -\frac{1}{2}$$

$$2\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \ldots \quad \text{Also } -\frac{2\pi}{3}$$

$$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \ldots \quad \text{also } -\frac{\pi}{3}.$$

So we can see $\alpha = -\frac{\pi}{3}$, $\beta = \frac{\pi}{3}$.

The "bump" is symmetric about the y-axis, so we can get the area from 0 to $\frac{\pi}{3}$ and then double it:

$$A = 2\cdot\frac{1}{2}\int_0^{\frac{\pi}{3}} (1+2\cos(2\theta))^2\, d\theta$$

$$= \boxed{\int_0^{\frac{\pi}{3}} (1+2\cos(2\theta))^2\, d\theta}$$