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Spring 2021 MATH 76
Activity 11

CONVERGENCE TESTS
Strategies on alternating series

An infinite series $\sum_{k=M}^{\infty} a_k$ **is an alternating series if it has the form**

- $a_k = (-1)^k b_k$
- **or** $a_k = (-1)^{k+1} b_k$
- **or** $a_k = (-1)^{k-1} b_k$

where b_k **is positive. To decide the convergence use the Alternating series test.**

1. **If** $\lim_{k\to\infty} b_k \neq 0$ **then stop here and say that the series** $\sum_{k=M}^{\infty} a_k$ **diverges (Divergence Test.)**
2. **If** $\lim_{k\to\infty} b_k = 0$**, then:**
   - **If** b_{k+1} **is ALWAYS smaller or equal than** b_k **meaning** $b_{k+1} \leq b_k$**. Use the fact that dividing by a larger number results in a smaller outcome, use your calculator to check when convenient, graph** $f(k) = b_k$ **and see whether it is decreasing on some half line interval** $[M, \infty)$**, etc.. If yes then the series** $\sum_{k=M}^{\infty} a_k$ **converges by the Alternating Series Test.**
   - **Otherwise the Alternating Series Test is not applicable. Recheck if** $\lim_{k\to\infty} b_k = 0$**.**
3. **In case the series** $\sum_{k=M}^{\infty} a_k$ **converges either**
   - **it converges conditionally**
   - **or converges absolutely.**

   For that consider the NEW series $\sum_{k=M}^{\infty} |a_k| = \sum_{k=i}^{\infty} b_k$**. Use ANY other applicable test to determine if** $\sum_{k=M}^{\infty} b_k$ **converges or diverges.**

   (a) **If** $\sum_{k=M}^{\infty} b_k$ **converges then the original series** $\sum_{k=M}^{\infty} a_k$ **converges absolutely.**

   (b) **If** $\sum_{k=M}^{\infty} b_k$ **diverges then the original series** $\sum_{k=M}^{\infty} a_k$ **converges conditionally (note that you assumed that the series** $\sum_{k=M}^{\infty} a_k$ **converges.)**

Determine whether the following alternating series converges or diverges.

1. $\sum_{k=2}^{\infty} \frac{(-1)^k}{k^2 - 1}$

$a_k = \frac{(-1)^k}{k^2 - 1}$

$b_k = \frac{(-1)^k}{k^2}$

$b_{k+1} = \frac{(-1)^{k+1}}{(k+1)^2}$

$\lim_{k \to \infty} \frac{(-1)^k}{k^2} = \emptyset$ ✓

$b_{k+1} \leq b_k$ ✓

By the Alternating Series test this Converges.

2. $\sum_{k=1}^{\infty} (-1)^k k e^{-k}$

$a_k = (-1)^k k e^{-k}$

$b_k = \frac{k}{e^k}$

$\lim_{k \to \infty} (-1)^k k e^{-k} = \emptyset$

$\lim_{k \to \infty} \frac{(-1)^k k}{e^k} = \emptyset$

Seris Converges by the Alternating series test.

3. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k^2 + 4)}{2k^2 + 1}$

$a_k = \frac{(-1)^{k+1}(k^2+4)}{2k^2+1}$

$b_k = \frac{k^2}{2k^2}$

$\lim_{k = \infty} \frac{k}{2k^2} = \frac{1}{2}$

diverges based on the Alternating Series test.

4. $\sum_{k=1}^{\infty} \frac{(-2)^{k+1}}{k^2}$

$a_k = \frac{(-1)^{k+1}(2)^{k+1}}{k^2}$

$b_k = \frac{(2)^{k+1}}{k^2}$

$\lim_{k = \infty} \frac{2^{k+1}}{k^2} = \infty$

diverges based on the Alternating Series test.

The following series are all convergent (you can check by the Alternating Series Test). State whether the convergence is conditional or absolute.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k^4+1}}$

$\lim_{k=\infty} \left| \frac{1}{\sqrt{k^4+1}} \right| = 0$ Converges absolutely

$a_k = \frac{(-1)^k}{\sqrt{k^4+1}}$

$b_k = \frac{(1)^k}{k^{\frac{1}{2}}}$

$a_k \cdot a_{k+1}$ $\frac{a_k}{a_{k+1}}$

2. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}10^k}{k!}$

Ratio test

$a_k = \frac{(-1)^{k+1} 10^k}{k!}$

$\frac{(-1)^{k+1} 10^k}{k!} \cdot \frac{(k+1)!}{(-1)^{k+2} 10^{k+1}} = \frac{1}{(-1)^1 10^1} \boxed{= -\frac{1}{10}}$

$b_k = \frac{10^k}{k!}$

Converges absolutely

Does the following infinite series converge **absolutely**, converge **conditionally**, or diverge?

1. $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{1}{\sqrt{k^2+1}}$

$\lim_{k\to\infty} \left| \frac{1}{\sqrt{k^2+1}} \right| = 0$ $\boxed{\text{Converges Conditionally}}$

$a_k = (-1)^k \frac{1}{\sqrt{k^2+1}}$

$b_k = \frac{1}{\sqrt{k^2}}$

2. $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{k!}{\pi^k}$ $b_k = \frac{k!}{\pi^k}$

$\lim_{k\to\infty} \frac{k!}{\pi^k} = \infty$

$\frac{100 \cdot 99 \cdot 98}{\pi^{100}} = \infty$ diverge

Series diverges

Test	Series	Convergent if	Divergent if	Comments
Geometric series	$\sum_{k=M}^{\infty} ar^k,\ a \neq 0$	$\|r\| < 1$	$\|r\| \geq 1$	$\sum_{k=M}^{\infty} ar^k = \frac{ar^M}{1-r}$, if $\|r\| < 1$
Divergence test	$\sum_{k=M}^{\infty} x_k$	N/A	$\lim_{k\to\infty} x_k \neq 0$	Incon. if $\lim_{k\to\infty} x_k = 0$
Integral test	$\sum_{k=M}^{\infty} x_k,\ x_k = f(k)$ f is continuous, positive, decreasing	$\int_M^{\infty} f(x)dx$ conv.	$\int_M^{\infty} f(x)dx$ div.	
$p-$series	$\sum_{k=M}^{\infty} \frac{a}{k^p}$	$p > 1$	$p \leq 1$	
Ratio test	$\sum_{k=M}^{\infty} x_k,\ x_k > 0$	$0 \leq \lim_{k\to\infty} \frac{x_{k+1}}{x_k} < 1$	$\lim_{k\to\infty} \frac{x_{k+1}}{x_k} > 1$	Incon. if $\lim_{k\to\infty} \frac{x_{k+1}}{x_k} = 1$
Root test	$\sum_{k=M}^{\infty} x_k,\ x_k \geq 0$	$0 \leq \lim_{k\to\infty} x_k^{1/k} < 1$	$\lim_{k\to\infty} x_k^{1/k} > 1$	Incon. if $\lim_{k\to\infty} x_k^{1/k} = 1$
Comparison	$\sum_{k=M}^{\infty} x_k,\ x_k > 0$	$0 < x_k \leq y_k$ and $\sum_{k=M}^{\infty} y_k$ conv.	$0 < y_k \leq x_k$ and $\sum_{k=M}^{\infty} y_k$ div.	$\sum_{k=M}^{\infty} x_k$ is given you supply $\sum_{k=M}^{\infty} y_k$
Limit Comparison	$\sum_{k=M}^{\infty} x_k$, $x_k > 0, y_k > 0$	$0 \leq \lim_{k\to\infty} \frac{x_k}{y_k} < \infty$ and $\sum_{k=M}^{\infty} y_k$ conv.	$\lim_{k\to\infty} \frac{x_k}{y_k} > 0$ $\left(\text{or } \lim_{k\to\infty} \frac{x_k}{y_k} = \infty\right)$ and $\sum_{k=M}^{\infty} y_k$ div.	$\sum_{k=M}^{\infty} x_k$ is given you supply $\sum_{k=M}^{\infty} y_k$
Altern. series test	$\sum_{k=M}^{\infty} (-1)^k x_k$ $x_k > 0$	$\lim_{k\to\infty} x_k = 0$ and, $0 < x_{k+1} \leq x_k$	$\lim_{k\to\infty} x_k \neq 0$	Combined with divergence test.
Absolute Conv.	$\sum_{k=M}^{\infty} x_k$	$\sum_{k=M}^{\infty} \|x_k\|$ conv.	N/A	
Conditional Conv.	$\sum_{k=M}^{\infty} x_k$	$\sum_{k=M}^{\infty} x_k$ conv. and $\sum_{k=M}^{\infty} \|x_k\|$ div.	N/A	