Math 76 Exercises – 6.1 Power Series; Interval and Radius of Convergence

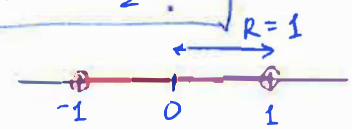
For each power series, find the interval and radius of convergence. Where is the power series centered?

1. $\sum_{n=1}^{\infty} 3x^n$

This is a geometric series centered at 0 with $r = x$. So the series will converge for $|x| < 1$ and diverge otherwise.

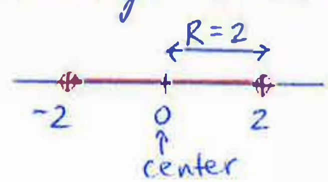
Interval of convergence (I.O.C.): $\boxed{(-1, 1)}$

Radius of convergence (R.O.C.): $\boxed{R = 1}$.



2. $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$ $= \sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n$. This is a geometric series centered at 0 with $r = \frac{x}{2}$. So the series will converge for $\left|\frac{x}{2}\right| < 1$, i.e. $-1 < \frac{x}{2} < 1$, i.e. $-2 < x < 2$, and diverge otherwise.

$$\boxed{\begin{array}{l} \text{I.O.C.}: (-2, 2) \\ R = 2 \end{array}}$$



3. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

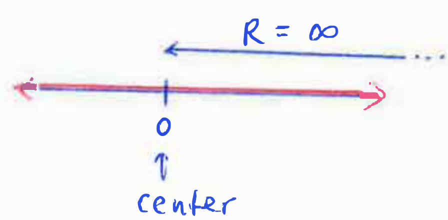
Power series centered at 0.

Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|$

$= \lim_{n\to\infty} |x| \cdot \frac{1}{(n+1)} = 0 < 1$ for all x.

So the series converges for all x.

$$\boxed{\begin{array}{l} \text{I.O.C.}: (-\infty, \infty) \\ R = \infty \end{array}}$$

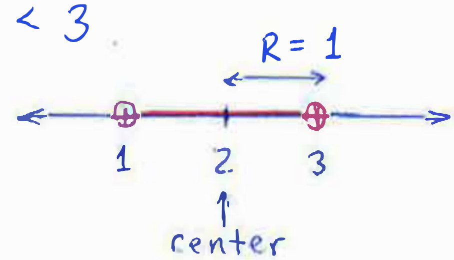


4. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{5}$ Geometric series centered at 2 with $r = x-2$.

Series converges for $|x-2| < 1$ and diverges otherwise.

$$-1 < x-2 < 1$$
$$1 < x < 3$$

$$\boxed{\begin{array}{l} \text{I.O.C.}: (1, 3) \\ R = 1 \end{array}}$$



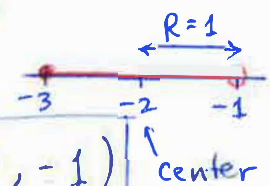
5. $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n}$ Power series centered at -2.

Ratio Test: $\lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{n+1} \cdot \frac{n}{(x+2)^n} \right| = \lim_{n\to\infty} |x+2| \cdot \frac{n}{n+1}$

$= |x+2| \cdot 1 \overset{\text{set}}{<} 1$. Series converges for $-1 < x+2 < 1$, i.e. $-3 < x < -1$.

Test $x = -3$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges (Alt. Series Test)

$x = -1$: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series test).



So $\boxed{\begin{array}{l} \text{I.O.C. is } [-3, -1) \\ R = 1 \end{array}}$

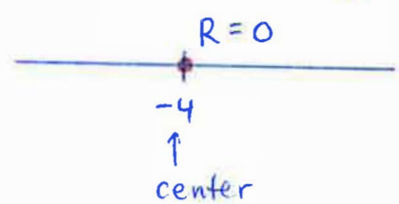
6. $\sum_{n=1}^{\infty} n!(x+4)^n$

Centered at $x = -4$

$\lim_{n\to\infty} \left| \frac{(n+1)!\,(x+4)^{n+1}}{n!\,(x+4)^n} \right| = \lim_{n\to\infty} (n+1)|x+4| = \begin{cases} \infty & \text{if } x \neq -4 \\ 0 & \text{if } x = 4. \end{cases}$

So the series converges at $x = -4$ and diverges otherwise. I.O.C.: $\{-4\}$

$R = 0$



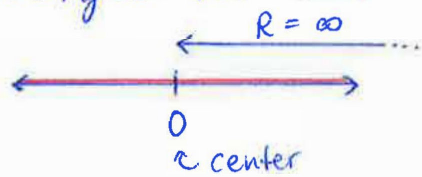
7. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ Centered at 0.

$$\lim_{n\to\infty} \left| \frac{x^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{x^{2n+2}}^{\,x^2}}{\cancel{x^{2n}}} \cdot \frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}} \right|$$

$$= \lim_{n\to\infty} x^2 \cdot \frac{1}{(2n+2)(2n+1)} = 0 < 1.$$

Series converges for all x.

$$\boxed{\begin{array}{c} \text{I.O.C.}: (-\infty, \infty) \\ R = \infty \end{array}}$$



8. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^{2n+1}}{2n+1}$ Centered at -1.

$$\lim_{n\to\infty} \left| \frac{(x+1)^{2n+3}}{2n+3} \cdot \frac{2n+1}{(x+1)^{2n+1}} \right|$$

$2(n+1)+1 = 2n+3$

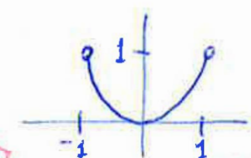
$$= \lim_{n\to\infty} (x+1)^2 \cdot \frac{2n+1}{2n+3} = (x+1)^2 \cdot 1 \overset{\text{set}}{<} 1$$

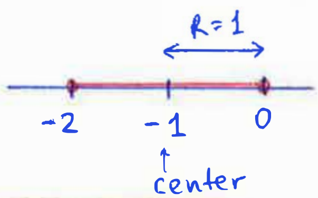
$$-1 < x+1 < 1$$

$$-2 < x < 0$$

$☺^2 < 1$

$\Rightarrow -1 < ☺ < 1$





$$\boxed{\begin{array}{c} \text{I.O.C.}: [-2, 0] \\ R = 1 \end{array}}$$

Check $x = -2$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

converges (why?)

Check $x = 0$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 1^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

converges (why?)

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-3)^n}{2^n}$

$$= \sum_{n=1}^{\infty} \frac{\left(2\left(x - \frac{3}{2}\right)\right)^n}{2^n} = \sum_{n=1}^{\infty} \left(x - \frac{3}{2}\right)^n$$

Geometric series centered at $\frac{3}{2}$ with $r = \left(x - \frac{3}{2}\right)$. Converges for $\left|x - \frac{3}{2}\right| < 1$ and diverges otherwise.

$$-1 < x - \frac{3}{2} < 1$$

$$\frac{1}{2} < x < \frac{5}{2}$$

$$\boxed{\begin{array}{c} \text{I.O.C.}: \left(\frac{1}{2}, \frac{5}{2}\right) \\ R = 1 \end{array}}$$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-4)^n}{\sqrt{n}}$ Centered at 4

$$\lim_{n\to\infty} \left| \frac{(x-4)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(x-4)^n} \right| = \lim_{n\to\infty} |x-4| \sqrt{\frac{n}{n+1}} = |x-4| \cdot 1 \overset{\text{set}}{<} 1.$$

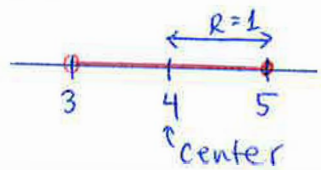
$-1 < x-4 < 1$

$3 < x < 5$

$x=3$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series test)

$x=5$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges (alt. series test)

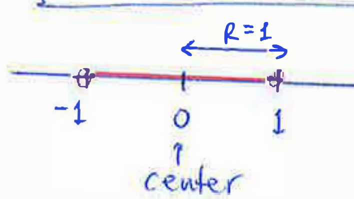
I.O.C.: $(3, 5]$
$R = 1$



11. $\displaystyle\sum_{n=1}^{\infty} (-1)^n x^{2n}$

$= \displaystyle\sum_{n=1}^{\infty} (-x^2)^n$ Geometric series centered at 0 with $r = -x^2$. Converges for $|-x^2| < 1$ and diverges otherwise. $-1 < x < 1$.

I.O.C.: $(-1, 1)$
$R = 1$



12. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+1)^n}{n^2+5}$ Centered at -1

$$\lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{(n+1)^2+5} \cdot \frac{n^2+5}{(x+1)^n} \right| = \lim_{n\to\infty} |x+1| \cdot \frac{n^2+5}{(n+1)^2+5} = |x+1| \cdot 1 \overset{\text{set}}{<} 1$$

$-1 < x+1 < 1$

$-2 < x < 0$

$x=-2$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+5}$ converges absolutely (compare with $\frac{1}{n^2}$)

$x=0$: $\displaystyle\sum_{n=1}^{\infty} \frac{1^n}{n^2+5}$ converges (compare with $\frac{1}{n^2}$)

I.O.C.: $[-2, 0]$
$R = 1$