Spring 2021 MATH 76
Activity 8

SUMMATION

The $\sum$ notation.

Consider a list of real numbers named $x_1, x_2, x_3, \ldots,$ up to x_n where n is an integer. Note that x_1 is the first number on the list, x_2 the second number, and so on. The sum of these numbers $x_1 + x_2 + x_3 + \ldots + x_n$ is represented by $\displaystyle\sum_{k=1}^{n} x_k$. The symbol $\sum$, (greek letter "sigma") represents the sum, k is the index that takes values from 1 (the lower limit) to n (upper limit). The term x_k is the quantity (summand) to be added, and when

$$\begin{array}{ll} k=1 & x_k = x_1 \\ k=2 & x_k = x_2 \\ \vdots & \vdots \\ k=n & x_k = x_n. \end{array}$$

You should have already seen the $\sum$ notation in some topics such as:

- Riemann sum $R_n = \Delta x \displaystyle\sum_{k=1}^{n} f(x_k)$.
- Midpoint rule for numerical integration $\displaystyle\int_{x_0}^{x_n} f(x)dx \approx \Delta x \sum_{k=0}^{n-1} f\left(\frac{x_k + x_{k+1}}{2}\right)$.
- Trapezoid rule for numerical integration $\displaystyle\int_{x_0}^{x_n} f(x)dx \approx \frac{\Delta x}{2}\left(f(x_0) + f(x_n) + 2\sum_{k=1}^{n-1} f(x_k)\right)$.

1. Consider the following numbers $x_1 = 3, x_2 = -4, x_3 = 4, x_4 = 0, x_5 = 5, x_6 = -1, x_7 = 1, x_8 = 2, x_9 = -3$, and $x_{10} = -5$. Compute the following sums.

   (a) $\displaystyle\sum_{k=1}^{6} x_k$

   (b) $\displaystyle\sum_{k=1}^{5} x_{2k}$

   (c) $\displaystyle\sum_{k=1}^{5} x_{2k-1}$

   (d) $\displaystyle\sum_{k=0}^{9} x_{k+1}$

(a) $\sum_{k=1}^{6} x_k$ $= 3 + (-4) + 4, + 0 + 5 + -1 = \boxed{7}$ ✓

(b) $\sum_{k=1}^{5} x_{2k}$ $= -4 + 0 + (-1) + 2 + -5 = \boxed{-8}$ ✓

(c) $\sum_{k=1}^{5} x_{2k-1}$

Consider the following numbers $x_1 = 3, x_2 = -4, x_3 = 4, x_4 = 0, x_5 = 5, x_6 = -1, x_7 = 1, x_8 = 2, x_9 = -3$, and $x_{10} = -5$. Compute the following sums.

$1 \quad 3 \quad 5 \quad 7 \quad 9$

$= 3 + 4 + 5 + 1 + (-3) = \boxed{10}$ ✓

(d) $\sum_{k=0}^{9} x_{k+1}$

Consider the following numbers $x_1 = 3, x_2 = -4, x_3 = 4, x_4 = 0, x_5 = 5, x_6 = -1, x_7 = 1, x_8 = 2, x_9 = -3$, and $x_{10} = -5$. Compute the following sums.

$= 3 + (-4) + 4 + 0 + 5 + (-1) + 1 + 2, + (-3) + (-5) = \boxed{2}$ ✓

2. It is known that:

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

Find the following sums.

(a) $\displaystyle\sum_{k=1}^{10} k$

(b) $\displaystyle\sum_{k=0}^{10} k^2$

(c) $\displaystyle\sum_{k=3}^{10} k^3$

(d) $\displaystyle\sum_{k=1}^{10} (k^3 - 2k^2 - k)$

(e) $\displaystyle\sum_{k=0}^{4} \frac{k}{k+1}$

(a) $\sum_{k=1}^{10} k$ $= 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \boxed{55}$ ✓

(b) $\sum_{k=0}^{10} k^2$ $= 0 + 1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100$

$\boxed{= 385}$ ✓

(c) $\sum_{k=3}^{10} k^3$ $= 27 + 64 + 125 + 216 + 343 + 512 + 729 + 1000$

$\boxed{= 3016}$ ✓

(d) $\sum_{k=1}^{10} (k^3 - 2k^2 - k)$

$= (-2) + (-2) + 6 + 28 + 70 + 138 + 238 + 376 + 558 + 790$

$\boxed{= 2200}$ ✓

(e) $\sum_{k=0}^{4} \frac{k}{k+1}$ $= 0 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} \boxed{= \frac{163}{60}}$ ✓

3. (a) Consider the integral $\int_0^1 e^{-x^2}dx$.

i. With $n = 8, \Delta x = \dfrac{1-0}{8} = 0.125$, compute:

$$\begin{array}{lll} x_0 = 0 & x_1 = \dots & x_2 = \dots \\ x_3 = \dots & x_4 = \dots & x_5 = \dots \\ x_6 = \dots & x_7 = \dots & x_8 = 1. \end{array}$$

ii. Use the midpoint rule $\displaystyle\int_{x_0}^{x_8} f(x)dx \approx \Delta x \sum_{k=0}^{7} f\left(\frac{x_k + x_{k+1}}{2}\right)$ to approximate the integral $\int_0^1 e^{-x^2}dx$.

(b) Consider the integral $\displaystyle\int_0^1 \frac{1}{1+x^2}dx$. By using the same $x_0, \dots, x_8$ and Δx of part (a) apply the trapezoid method $\displaystyle\int_{x_0}^{x_8} f(x)dx \approx \frac{\Delta x}{2}\left(f(x_0) + f(x_8) + 2\sum_{k=1}^{7} f(x_k)\right)$ to approximate the integral $\displaystyle\int_0^1 \frac{1}{1+x^2}dx$.

3. (a) Consider the integral $\int_0^1 e^{-x^2}dx$.

i. With $n = 8, \Delta x = \frac{1-0}{8} = 0.125$, compute:

$\boxed{x_0 = 0}$	$x_1 = \ldots$	$x_2 = \ldots$
$x_3 = \ldots$	$x_4 = \ldots$	$x_5 = \ldots$
$x_6 = \ldots$	$x_7 = \ldots$	$x_8 = 1.$

[0, 0.125, 0.25, 0.375, 0.5, 0.625, 0.75, 0.875, 1]

ii. Use the midpoint rule $\int_{x_0}^{x_8} f(x)dx \approx \Delta x \sum_{k=0}^{7} f\left(\frac{x_k + x_{k+1}}{2}\right)$ to approximate the integral $\int_0^1 e^{-x^2}dx$.

[0, 0.125, 0.25, 0.375, 0.5, 0.625, 0.75, 0.875, 1]

0 1 2 3 4 5 6 7 8

$$(0.125)\sum_{k=0}^{7}\left(\frac{x_k + x_{k+1}}{2}\right) = \frac{1}{128} + \frac{3}{128} + \frac{5}{128} + \frac{7}{128} + \frac{9}{128} + \frac{11}{128} + \frac{13}{128} + \frac{15}{128}$$

0 1 2 3 4 5 6 7

$$\boxed{0.7473}$$ ✓

(b) Consider the integral $\int_0^1 \frac{1}{1+x^2}dx$. By using the same $x_0, \ldots, x_8$ and Δx of part (a) apply the trapezoid method $\int_{x_0}^{x_8} f(x)dx \approx \frac{\Delta x}{2}\left(f(x_0) + f(x_8) + 2\sum_{k=1}^{7} f(x_k)\right)$ to approximate the integral $\int_0^1 \frac{1}{1+x^2}dx$.

[0, 0.125, 0.25, 0.375, 0.5, 0.625, 0.75, 0.875, 1]

0 1 2 3 4 5 6 7 8

$$\frac{0.125}{2}\left(f(x_0) + 2(f(x_1)) + 2(f(x_2)) + \ldots f(x_8)\right)$$

$$= \frac{0.125}{2}$$

$$\frac{2}{1+(.125)^2}$$

$$= \frac{1}{16}\left(1 + \frac{128}{65} + \frac{32}{17} + \frac{128}{73} + \frac{8}{5} + \frac{128}{89} + \frac{32}{25} + \frac{128}{113} + \frac{1}{2}\right)$$

0 1 2 3 4 5 6 7 8

$$= \frac{1}{16} + \frac{8}{65} + \frac{2}{17} + \frac{8}{73} + \frac{1}{10} + \frac{8}{89} + \frac{2}{25} + \frac{8}{113} + \frac{1}{32}$$

0 1 2 3 4 5 6 7 8

$$\boxed{= 0.7847471236}$$ ✓

4. The goal of this problem is to derive a formula for the sum $\displaystyle\sum_{k=0}^{n} \frac{1}{2^k}$.

(a) Let $S = \displaystyle\sum_{k=0}^{n} \frac{1}{2^k}$. Write out $\frac{1}{2}S$ as a sum.

(b) Write $S - \frac{1}{2}S$ as a sum.

(c) Using your result in part (b), what is a formula for S?

(d) Compute $\displaystyle\sum_{k=0}^{100} \frac{1}{2^k}$.

(a) Let $S = \sum_{k=0}^{n} \frac{1}{2^k}$. Write out $\frac{1}{2}S$ as a sum. $\frac{1}{2}S =$

$$S = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$$

$$\boxed{\frac{1}{2}S = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \frac{1}{2^n} + \frac{1}{2^{n+1}}} \checkmark$$

(b) Write $S - \frac{1}{2}S$ as a sum.

$$S - \frac{1}{2}S = \left(\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}\right) - \left(\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \frac{1}{2^n} + \frac{1}{2^{n+1}}\right)$$

$$\boxed{S - \frac{1}{2}S = 1 + \frac{1}{2^{n+1}}} \checkmark$$

(c) Using your result in part (b), what is a formula for S?

$$S - \frac{1}{2}S = 1 + \frac{1}{2^{n+1}} \Rightarrow \frac{1}{2}S = 1 - \frac{1}{2^{n+1}}$$

$$2 \cdot \frac{1}{2}S = 2\left(1 - \frac{1}{2^{n+1}}\right)$$

$$\boxed{S = 2 - \frac{1}{2^n}} \checkmark$$

(d) Compute $\sum_{k=0}^{100} \frac{1}{2^k}$. $S = 2 - \frac{1}{2^n}$

$$\boxed{2 - \frac{1}{2^{100}}} \checkmark$$