Math 76 Exercises – 6.2B Differentiation and Integration of Power Series

1. Use integration or differentiation to find a power series centered at 0 for each function.

   (a) $f(x) = \ln(1-x)$

$$f'(x) = -\frac{1}{1-x} = -\sum_{n=0}^{\infty} x^n \qquad (\text{for } -1 < x < 1)$$

So $f(x) = C - \int \sum_{n=0}^{\infty} x^n \, dx = C - \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$

for at least $-1<x<1$

Note that $f(0) = \ln 1 = 0 = C - 0 = C$.

So $C=0$. Thus $f(x) = \boxed{-\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}} = \boxed{-\sum_{n=1}^{\infty} \frac{1}{n} x^n}$

   (b) $g(x) = \tan^{-1}(x)$

$$g'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$(\text{for } -1 < x < 1)$

So $g(x) = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} \, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$

for at least $-1<x<1$

Note that $g(0) = \tan^{-1}(0) = 0 = C + 0 = C$,

So $C=0$. Thus $g(x) = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}}$

   (c) $h(x) = \dfrac{2}{(1+x)^2}$

$$\int h(x)\,dx = -2\frac{1}{1+x} + C = -2\sum_{n=0}^{\infty} (-x)^n + C = 2\sum_{n=0}^{\infty} (-1)^{n+1} x^n + C$$

$(\text{for } -1 < x < 1)$

So $h(x) = \boxed{2\sum_{n=1}^{\infty} (-1)^{n+1} n\, x^{n-1}}$

$n=0$ term is 0 and we don't want x^{-1}

for at least $-1 < x < 1$

(d) $j(x) = 5x^2 \tan^{-1}(2x)$

From part (b), $\tan^{-1}(2x) = -\sum_{n=1}^{\infty} \frac{1}{n}(2x)^n$, so

$$j(x) = -5x^2 \sum_{n=1}^{\infty} \frac{1}{n} 2^n x^n$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{-5 \cdot 2^n}{n} x^{n+2}}$$

2. For each power series above, find the interval of convergence.

(a) We know $f(x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n$ for at least $-1 < x < 1$.

Now check the endpoints: $x = 1$: $-\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$x = -1$: $-\sum_{n=1}^{\infty} \frac{1}{n}(-1)^n$ converges.

So the interval of convergence is $\boxed{[-1, 1)}$.

(b) We know $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ for at least $-1 < x < 1$.

Now check the endpoints: $x = 1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot 1$ converges

$x = -1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \underbrace{(-1)^{2n+1}}_{= -1 \text{ for all } n}$ converges

So the interval of convergence is $\boxed{[-1, 1]}$

(c) We know $h(x) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1}$ for at least $-1 < x < 1$.

Now check endpoints: $x = 1$: $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot n$ diverges

$x = -1$: $\sum_{n=1}^{\infty} (-1)^{n+1} n (-1)^{n-1} = \sum_{n=1}^{\infty} n$ diverge

So we get $\boxed{(-1, 1)}$

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(d) Using Ratio Test, we have $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{5}\cdot 2^{n+1}\, x^{n+3}}{n+1}\cdot\frac{n}{\cancel{5}\cdot 2^n\, x^{n+2}}\right|$

$= \lim_{n\to\infty}\frac{2n}{n+1}|x| = 2|x| \overset{\text{set}}{<} 1 \qquad |x| < \frac{1}{2}. \qquad -\frac{1}{2} < x < \frac{1}{2}.$

Check endpoints: $\underline{x=\frac{1}{2}}$: $\sum_{n=1}^{\infty}\frac{-5\cdot 2^n}{n}\left(\frac{1}{2}\right)^{n+2} = \sum_{n=1}^{\infty}\frac{-5}{4n}$ diverges

$\underline{x=-\frac{1}{2}}$: $\sum_{n=1}^{\infty}\frac{-5\cdot 2^n}{n}\left(-\frac{1}{2}\right)^{n+2} = \sum_{n=1}^{\infty}\frac{5}{4n}(-1)^{n+1}$ converges

So interval is $\boxed{\left[-\frac{1}{2},\frac{1}{2}\right)}$

3. Given that $e^{x^2} = \sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$ for all x, find a power series that equals $\int e^{x^2}\,dx$.

$$\int e^{x^2}dx = \boxed{C + \sum_{n=0}^{\infty}\frac{1}{n!}\frac{x^{2n+1}}{2n+1}}.$$

4. Given that $\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$ for all x, find a power series that equals $\cos x$.

$$\cos x = \frac{d}{dx}(\sin x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2n+1)x^{2n}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n\,\cancel{(2n+1)}}{\cancel{(2n+1)}\cdot(2n)!}x^{2n}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}.$$