Math 76 Exercises – 5.5 Alternating Series; Absolute and Conditional Convergence

1. Determine whether each of the following alternating series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

$$b_n = \left| \frac{(-1)^n}{\sqrt{n}} \right| = \frac{1}{\sqrt{n}}.$$

$\{b_n\}$ is decreasing and $\lim_{n\to\infty} b_n = 0$.

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-2)^{n+1}\, n}{8n^2-5}$

$$b_n = \left| \frac{(-2)^{n+1}\, n}{8n^2-5} \right| = \frac{2^{n+1}\, n}{8n^2-5}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2^{n+1}}{\frac{8n^2-5}{n}} = \lim_{n\to\infty} \frac{2^{n+1}}{8n - \cancel{5/n}} \quad \frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{\ln 2 \cdot 2^{n+1}}{8} = \infty \neq 0.$$

Thus $\displaystyle\sum_{n=2}^{\infty} \frac{(-2)^{n+1}\, n}{8n^2-5}$ diverges

(c) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n \ln n}{n}$

$$b_n = \left| \frac{(-1)^n \ln n}{n} \right| = \frac{\ln n}{n}.$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln n}{n} \quad \frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{1/n}{1} = 0.$$

If we let $f(x) = \frac{\ln x}{x}$ we see that $f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2}$ $= \frac{1-\ln x}{x^2}$, which is negative for $x > e$. So $\{b_n\}$ is decreasing. Thus $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n \ln n}{n}$ converges

(d) $\displaystyle\sum_{n=1}^{\infty}\left(-\frac{2}{3}\right)^n \cdot \frac{1}{n}$

$$b_n = \left|\left(-\tfrac{2}{3}\right)^n \cdot \tfrac{1}{n}\right| = \left(\tfrac{2}{3}\right)^n \cdot \tfrac{1}{n}$$

$a_n = \left(-\frac{2}{3}\right)^n \cdot \frac{1}{n}$

$$\lim_{n\to\infty} b_n = 0 \cdot 0 = 0 \quad ✓$$

$$b_{n+1} = \left(\tfrac{2}{3}\right)^{n+1} \cdot \tfrac{1}{n+1} = \left(\tfrac{2}{3}\right)\left(\tfrac{2}{3}\right)^n \cdot \tfrac{1}{n+1}$$
$$< \tfrac{2}{3} \cdot \left(\tfrac{2}{3}\right)^n \cdot \tfrac{1}{n}$$
$$= \tfrac{2}{3} b_n < b_n.$$

So $\{b_n\}$ is decreasing. ✓

Therefore $\displaystyle\sum_{n=1}^{\infty}\left(-\tfrac{2}{3}\right)^n \cdot \tfrac{1}{n}$ converges

2. For each convergent series above, determine whether the series is absolutely convergent or conditionally convergent.

(a) The series $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a p-series with $p = \frac{1}{2} \le 1$, so $\sum b_n = \sum |a_n|$ diverges. But $\sum a_n$ converges (where $a_n = \frac{(-1)^n}{\sqrt{n}}$), so $\sum a_n$ converges conditionally

(b) (The series diverges)

(c) By the Integral Test (see class exercises 5.3 #1(c)), the series $\sum |a_n| = \sum b_n = \sum \frac{\ln(n)}{n}$ diverges.

[Or use direct comparison test : $\frac{\ln(n)}{n} \ge \frac{1}{n}$, and $\sum \frac{1}{n}$ diverges.]

But $\sum a_n$ converges (where $a_n = \frac{(-1)^n \ln(n)}{n}$).

Therefore $\sum a_n$ converges conditionally.

(d) By direct comparison with $\sum\left(\frac{2}{3}\right)^n$: $\left(\frac{2}{3}\right)^n \cdot \frac{1}{n} < \left(\frac{2}{3}\right)^n$, and $\sum\left(\frac{2}{3}\right)^n$ is a convergent geometric series ($|r| = \frac{2}{3} < 1$).

Thus $\sum |a_n| = \sum b_n = \sum \left(\frac{2}{3}\right)^n \cdot \frac{1}{n}$ converges (where $a_n = \left(-\frac{2}{3}\right)^n \cdot \frac{1}{n}$).

So $\sum a_n$ converges absolutely

3. Consider the conditionally convergent series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,6}{n}$. $= 6 - \frac{6}{2} + \frac{6}{3} - \frac{6}{4} + \frac{6}{5} - \frac{6}{6} + \cdots$

(a) Write a rearrangement of the terms so that the sum of this series is 5.

$6 - \frac{6}{2} = 3$

$3 + \frac{6}{3} = 5$

$5 + \frac{6}{5} = 6.2$

$6.2 - \frac{6}{4} = 4.7$

$4.7 + \frac{6}{7} \simeq 5.6$

$5.6 - \frac{6}{6} \simeq 4.6$

$4.6 + \frac{6}{9} \simeq 5.2$

$5.2 - \frac{6}{8} \approx 4.5$

$4.5 + \frac{6}{11} \simeq 5.02$

$5.02 - \frac{6}{10} \approx 4.4$

$4.4 + \frac{6}{13} \simeq 4.88$

$4.88 + \frac{6}{15} \approx 5.28$, etc.

(b) Write a rearrangement of the terms so that the sum of this series is 1.

$6 - \frac{6}{2} = 3$

$3 - \frac{6}{4} = 1.5$

$1.5 - \frac{6}{6} = 0.5$

$0.5 + \frac{6}{3} = 2.5$

$2.5 - \frac{6}{8} = 1.75$

$1.75 - \frac{6}{10} = 1.15$

$1.15 - \frac{6}{12} = 0.65$

$0.65 + \frac{6}{5} = 1.85$

$1.85 - \frac{6}{14} \simeq 1.4$

$1.4 - \frac{6}{16} \approx 1.05$

$1.05 - \frac{6}{18} \simeq 0.71$

$0.71 + \frac{6}{7} \simeq 1.57$

$1.57 - \frac{6}{20} \approx 1.27$

$1.27 - \frac{6}{22} \approx 0.998$, etc.

4. Give an example of a_n for which the sequence $\{a_n\}$ converges, but the series $\sum a_n$ diverges.

Consider $a_n = \frac{1}{n}$.

The sequence $\{a_n\}$ converges to 0 since $\lim_{n\to\infty} a_n = 0$, but $\sum a_n$ diverges since $\sum \frac{1}{n}$ is a p-series with $p = 1$.

5. Is there an example of a_n for which the series $\sum a_n$ converges, but the sequence $\{a_n\}$ diverges? If so, find one. If not, explain why not.

It is not possible. By the Divergence Test, if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.