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Spring 2021 MATH 76
Activity 7

PARTIAL FRACTIONS

Assume that $\dfrac{f(x)}{g(x)}$ is a rational integrand where the degree of $f(x)$ is **smaller** than the degree of $g(x)$.

- Simple linear factors $g(x) = (x - r_1)(x - r_2)\dots(x - r_n)$

  $$\frac{f(x)}{g(x)} = \frac{A_1}{x - r_1} + \frac{A_2}{x - r_2} + \dots + \frac{A_n}{x - r_n}$$

- Repeated linear factors $g(x) = (x - r)^n$

  $$\frac{f(x)}{g(x)} = \frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \dots + \frac{A_n}{(x - r)^n}$$

- Irreducible quadratic factor $g(x) = ax^2 + bx + c$ and cannot be factored $(b^2 - 4ac < 0)$

  $$\frac{f(x)}{g(x)} = \frac{Ax + B}{ax^2 + bx + c}$$

1. Write the following rational expressions in the appropriate partial fractions form. **Do not find the constants.**

   (a) $\dfrac{9x^3 + 30x - 20x^2 - 97}{(x - 2)(x - 3)(x^2 + 5)}$

   (b) $\dfrac{7x^2 + 75x - 150}{x^3 - 25x}$

   (c) $\dfrac{7x - 26}{x^2 - 6x - 16}$

   (d) $\dfrac{2 + x^4}{x^3 + 9x}$. Try a long division first.

2. Find the partial fraction decomposition of the following rational expressions. **Find the constants.**

   (a) $\dfrac{8x - 36}{(x-5)^2}$

   (b) $\dfrac{-3x - 23}{x^2 - x - 12}$

   (c) $\dfrac{3x^2 - 3x + 10}{(x-2)(x^2+4)}$

3. Evaluate the following integrals.

(a) $\displaystyle\int \frac{x-1}{x^2+x}dx$

(b) $\displaystyle\int \frac{2x-3}{x^3+x}dx$

(c) $\displaystyle\int \frac{4x^3-3x+5}{x^2-2x}dx.$

4. Here are some steps to evaluate $\displaystyle\int \frac{dx}{x^2+4x+13}$.

(a) Verify that $x^2+4x+13$ is irreducible.

(b) Write $x^2+4x+13=(x+\ldots)^2+\ldots$

(c) Use a $u-$ substitution $u=x+\ldots$ to simplify the integrand.

(d) Check that for $a>0$, $\displaystyle\int \frac{du}{u^2+a^2}=\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C$ and use it to compute the integral $\displaystyle\int \frac{dx}{x^2+4x+13}$.

5. The following integrals are given. Fill in the details to explain the answers.

(a) $\displaystyle\int \frac{2x+1}{x^2+4}dx = \ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

(b) $\displaystyle\int \frac{2x+2}{x^2+6x+10}dx = \ln(x^2+6x+10) - 4\tan^{-1}(x+3) + C$

(c) $\displaystyle\int \frac{2x+2}{(x-3)^2(x+1)}dx = -\frac{2}{x-3} + C$

(d) $\displaystyle\int \frac{3x}{x^3-x^2-2x}dx = \ln|x-2| - \ln|x+1| + C$

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(a) $\dfrac{9x^3 + 30x - 20x^2 - 97}{(x-2)(x-3)(x^2+5)}$ $= \dfrac{9x^3+30x-20x^2-97}{(x-2)(x-3)(x^2+5)}$

Sin^nx
quiz 90/10

$$\frac{9x^3+30x-20x^2-97}{(x-2)(x-3)(x^2+5)} = \frac{A}{(x-2)} + \frac{B}{(x-3)} + \frac{C}{(x^2+5)}$$

1

(b) $\dfrac{7x^2 + 75x - 150}{x^3 - 25x}$

$$\frac{7x^2+75x-150}{x^3-25x}$$

$$\frac{7x^2+75x-150}{x(x+5)(x-5)} = \frac{A}{x} + \frac{B}{(x+5)} + \frac{C}{(x-5)}$$

1

(c) $\dfrac{7x - 26}{x^2 - 6x - 16}$

$$\frac{7x-26}{x^2-6x-16}$$

$$\frac{7x-26}{(x+2)(x-8)} = \frac{A}{x+2} + \frac{B}{x-8}$$

1

(d) $\dfrac{2 + x^4}{x^3 + 9x}$. Try a long division first.

$$\frac{2+x^4}{x^3+9x}$$

$$\frac{2+x^4}{x(x^2+9)} = \frac{A}{x} + \frac{B}{x^2+9}$$

2 (a) $\frac{8x-36}{(x-5)^2}$

$$\frac{8x-36}{(x-5)^2}$$

$$\left[(x-5)^2\right]\frac{8x-36}{(x-5)^2} = \frac{A}{x-5} + \frac{B}{(x-5)^2}\left[(x-5)^2\right]$$

① $8x-36 = A(x-5) + B(x-5)$ $\qquad \frac{\quad}{x-5}(x-5)^{\cancel{2}}$

② $8x-36 = A(x-5) + B(x-5)$

③ $8x-36 = \underline{Ax} - A5 + \underline{Bx} - B5$

④ $8x-36 = (A+B)x - (A5 - B5)$

$8x-36 =$

$$\frac{8x-36}{(x-5)^2} = \frac{A(x-5)+B}{(x-5)^2}$$

$$8x-36 \cdot (x-5)^2 = A(x-5)+B\cdot(x-5)^2$$

$\boxed{\begin{matrix}A=8\\B=4\end{matrix}}$ ✓ $\qquad 8x-36 = A(x-5)+B$

$x=5 \rightarrow B = 40-36 = 4$

$x = \qquad A=8$

2 (b) $\frac{-3x-23}{x^2-x-12}$

$$\frac{-3x-23}{x^2-x-12}$$

$$\left[(x+3)(x-4)\right]\frac{-3x-23}{(x+3)(x-4)} = \frac{A}{(x+3)} + \frac{B}{x-4}\left[(x+3)(x-4)\right]$$

$$-3x-23 = A(x-4) + B(x+3)$$

$x = -3$

$-3(-3)-23 = A(-3-4) + \cancel{B(-3+3)}$

$\frac{-14}{-7} = A(-7)$

$\boxed{A=2}$ ✓

$x = 4$

$-3(4)-23 = 0 + B(4+3)$

$\frac{-35}{7} = B(7)$

$\boxed{B=-5}$ ✓

2

(c) $\dfrac{3x^2-3x+10}{(x-2)(x^2+4)}$

$$\frac{3x^2-3x+10}{(x-2)(x^2+4)}$$

$$\left[(x-2)(x^2+4)\right]\frac{3x^2-3x+10}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$$

$$3x^2-3x+10 = \frac{A(x-2)(x^2+4)}{x-2} + \frac{Bx+C\,(x-2)(x^2+4)}{x^2+4}$$

$A=2$

$$3x^2-3x+10 = 2(x^2+4) + Bx+C(x-2)$$

$$3x^2-3x+10 = 2x^2+8 + Bx^2 - 2Bx + Cx - 2C$$

$$3x^2-3x+10 = (2+B)x^2-(2B+C)x+8-2C$$

$$3=2+B \rightarrow \boxed{B=1}$$

$$-3=-2B+C \rightarrow -3=-2+C = \boxed{C=-1}$$

$$10=8-2C \quad 10=10 \checkmark$$

$$\boxed{A=2} \checkmark$$

$$\boxed{B=1} \checkmark$$

$$\boxed{C=-1} \checkmark$$

3

(a) $\int \frac{x-1}{x^2+x}dx$

$$\int \frac{x-1}{x^2+x}dx$$

$$= \int \frac{x-1}{x(x+1)} \rightarrow [x(x+1)]\frac{x-1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}[x(x+1)]$$

$$x-1 = A(x+1) + Bx$$

$\underline{x=-1}$

$$-1-1 = A(-1+1)+B(-1)$$

$$\frac{-2}{-1} = \frac{B(-1)}{}$$

$\underline{x=0}$

$$\frac{0-1}{1} = \frac{A(1)}{1}+0$$

$\boxed{B=2}$ $\boxed{A=-1}$

$$\int -\frac{1}{x}\,dx + \int \frac{2}{x+1}\,dx$$

$$\boxed{= -\ln(x) + 2\ln(x+1) + C} \checkmark$$

3

(b) $\int \frac{2x-3}{x^3+x}dx$

$$\int \frac{2x-3}{x^3+x}\,dx$$

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$$\left[x(x^2+1)\right]\frac{2x-3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}\left[x(x^2+1)\right]$$

$$0x^2 + 2x - 3 = A(x^2+1) + Bx^2 + Cx$$

$$Ax^2 + A + Bx^2 + Cx$$

$\boxed{\begin{matrix}-3=A\\ 2=C\end{matrix}}$

$0 = A+B$ $\quad 2(1)-3 = -6+B+2$

$$-1 = -6+B+2$$

$$\boxed{3=B}$$

$\boxed{\begin{matrix}A=-3\\ B=3\\ C=2\end{matrix}}$

$$\int \frac{-3}{x}\,dx + \int \frac{3x+2}{x^2+1}\,dx$$

$$= -3\ln|x| + \int \frac{3x+2}{x^2+1}\,dx$$

$$= -3\ln|x| + \int \frac{3x}{x^2+1}\,dx + \int \frac{2}{x^2+1}\,dx$$

$$= -3\ln|x| + 3\int \frac{x}{x^2+1}\,dx + 2\int \frac{1}{x^2+1}\,dx$$

$$u = x^2+1$$

$$du = 2x\,dx$$

$$dx = \tfrac{1}{2x}\,du$$

$$= -3\ln|x| + \frac{3}{2}\int \frac{1}{u}\,du +$$

$$\checkmark \quad \boxed{= -3\ln|x| + \frac{3}{2}\ln(x^2+1) + 2\,\mathrm{Arctan}(x) + C}$$

3

(c) $\int \frac{4x^3 - 3x + 5}{x^2 - 2x} dx.$

$$\int \frac{4x^3 - 3x + 5}{x^2 - 2x} dx$$

$$(x(x-2)) \frac{4x^3 - 3x + 5}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2} (x(x-2))$$

$$4x^3 - 3x + 5 = A(x-2) + Bx$$

$x = 0$

$$\frac{5}{-2} = \frac{A(-2)}{-2}$$

$$\boxed{A = -\frac{5}{2}} \checkmark$$

$x = 2$

$$\frac{31}{2} = \frac{B(2)}{2}$$

$$\boxed{B = \frac{31}{2}} \checkmark$$

$$= \int \frac{-\frac{5}{2}}{x} dx + \int \frac{\frac{31}{2}}{x-2} dx$$

$$\boxed{= -\frac{5}{2} \ln|x| + \frac{31}{2} \ln|x-2| + C} \checkmark$$

4. Here are some steps to evaluate $\int \frac{dx}{x^2+4x+13}$.

(a) Verify that $x^2+4x+13$ is irreducible.

Nothing Adds to 4 and multiply to 13

(b) Write $x^2+4x+13 = (x+\ldots)^2+\ldots$

$x^2+4x+2+11$

$\boxed{(x+2)^2+11}$ ✓

(c) Use a $u-$ substitution $u = x+\ldots$ to simplify the integrand.

$U = x+2$

$dU = 1\,dx$

$\boxed{\int \frac{dU}{U^2+11}}$ ✓

(d) Check that for $a>0$, $\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C$ and use it to compute the integral $\int \frac{dx}{x^2+4x+13}$.

$\boxed{\frac{1}{\sqrt{11}}\operatorname{Tan}^{-1}\left(\frac{x+2}{\sqrt{11}}\right)+C}$ ✓

5 (a) $\int \frac{2x+1}{x^2+4}dx = \ln(x^2+4)+\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$

$$\int \frac{2x+1}{x^2+4}\,dx$$

$$= \int \frac{2x}{x^2+4}dx + \int \frac{1}{x^2+4}\,dx$$

$U = x^2$

$dU = 2x\,dx$

$$= \ln(x^2+4) + \int \frac{dU}{\sqrt{U^2+2^2}}dx$$

$$\boxed{= \ln(x^2+4)+\frac{1}{2}\operatorname{Tan}^{-1}\left(\frac{x}{2}\right)+C}$$ ✓

5 (b) $\int \frac{2x+2}{x^2+6x+10}dx = \ln(x^2+6x+10) - 4\tan^{-1}(x+3) + C$

$$\int \frac{2x+2}{x^2+6x+10}\,dx \qquad x+1 = \frac{1}{2}(2x+6)-2$$

$$= 2\int \frac{x+1}{x^2+6x+10}\,dx$$

$$= 2\int \left(\frac{2x+6}{2(x^2+6x+10)} - \frac{2}{x^2+6x+10}\right)dx$$

$$= 2\left(\int \frac{x+3}{x^2+6x+10}\,dx - 2\int \frac{1}{x^2+6x+10}\,dx\right)$$

$$u = x^2+6x+10 \qquad -2\int \frac{1}{(x+3)^2+1}\,dx$$
$$du = 2x+6\,dx$$

$$= \frac{1}{2}\int \frac{1}{u}\,du \qquad u = x+3 \qquad du = 1dx$$

$$= \frac{\ln(u)}{2} \qquad 2\int \frac{1}{u^2+1}\,du$$

$$2\,\mathrm{Arctan}(u)$$

$$= 2\left(\frac{\ln(x^2+6x+10)}{2} - 2\,\mathrm{Arctan}(x+3)\right) + C$$

$$\boxed{= \ln(x^2+6x+10) - 4\,\mathrm{Arctan}(x+3) + C} \quad \checkmark$$

5

(c) $\displaystyle\int \frac{2x+2}{(x-3)^2(x+1)}dx = -\frac{2}{x-3} + C$

$$\int \frac{2x+2}{(x-3)^2(x+1)}dx$$

$$= \int \frac{2\cancel{(x+1)}}{(x-3)^2\cancel{(x+1)}}dx$$

$$= 2\int \frac{1}{(x-3)^2}dx \quad u = x-3 \quad du = 1\,dx$$

$$= 2\int \frac{1}{u^2}du$$

$$= 2\left(-\frac{1}{u}\right) + C$$

$$\boxed{= -\frac{2}{x-3} + C}$$

✓

(d) $\displaystyle\int \frac{3x}{x^3 - x^2 - 2x}dx = \ln|x-2| - \ln|x+1| + C$

$$\int \frac{3x}{x^3 - x^2 - 2x}dx$$

$$= \int \frac{3\cancel{x}}{\cancel{x}(x^2 - x - 2)}dx \quad = 3\int \frac{1}{x^2 - x - 2}dx$$

$$= 3\int \frac{1}{(x-2)(x+1)}\,dx$$

$$= 3\left(\left[(x-2)(x+1)\right]\frac{1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}\left[(x-2)(x+1)\right]\right)$$

$$\cancel{\frac{1}{(x-2)(x+1)}} = A(x+1) + B(x-2)$$

$x+2$ $\quad$ $x-1$

$$\boxed{A = \frac{1}{3}}$$

$$1 = B(-3)$$

$$\boxed{B = -\frac{1}{3}}$$

$$= 3\left(\int \frac{\frac{1}{3}}{x-2}\,dx - \int \frac{\frac{1}{3}}{x+1}\,dx\right)$$

$$= \int \frac{1}{x-2}\,dx - \int \frac{1}{x+1}\,dx$$

$$\boxed{= \ln|x-2| - \ln|x+1| + C}$$ ✓