Tyler Gillette
Kelm

Spring 2021 MATH 76
Activity 5

1. BY PARTS
   Note that the integration by parts formula is $\int udv = uv - \int vdu$.

   (a) $\int xe^x dx$
   (b) $\int \sin^{-1}(x) dx$
   (c) $\int x\cos(x) dx$
   (d) $\int \tan^{-1}(x) dx$
   (e) $\int xe^{-4x} dx$
   (f) $\int x^2 e^{-3x} dx$
   (g) $\int \ln(x) dx$
   (h) $\int e^x \sin(x) dx$
   (i) $\int \cos^2(x) dx$
   (j) $\int \frac{x^3}{(x^2+2)^2} dx$
   (k) $\int_1^9 \frac{\ln(x)}{\sqrt{x}} dx$
   (l) $\int \frac{\ln(x)}{x^7} dx$
   (m) $\int \frac{\ln(x)}{x^n} dx$ for any positive real number n.
   (n) $\int (\ln x)^2 dx$
   (o) $\int \ln(x^2+1) dx$
   (p) $\int \sin(\sqrt{x}) dx$
   Hint: Try a $u-$ sub first.
   (q) $\int x^3 e^{x^2} dx$.
   Hint: Try a $u-$ sub first.
   (r) $\int \cos(x)\ln(\sin x) dx$.
   Hint: Try a $u-$ sub first.
   (s) $\int x^n \ln(x) dx$ for a real number $n > 0$.
   (t) $\int x \cdot 2^x dx$

2. TRIG INTEGRALS
   Compute the following integrals.

   (a) $\int \sin^2(x)\cos^2(x)\ dx$
   (b) $\int \sec^4(x)\tan^2(x)\ dx$
   (c) $\int \cos^5(x)\ dx$
   (d) $\int \sin^3(2x)\ dx$
   (e) Evaluate the integral $\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2 x - 1} dx$.

(a) $\int xe^x dx$

$\int udv = uv - \int vdu.$ LIATE

$U = X \quad dv = e^x$

$du = 1 \quad V = e^x$

$= (X)(e^x) - \int e^x dx$

$\boxed{= xe^x - e^x + c}$ ✓

(b) $\int \sin^{-1}(x)dx$

$\int udv = uv - \int vdu.$

$U = \text{Arcsin}(x) \quad dV = 1$

$dU = \frac{1}{\sqrt{1-x^2}} \quad V = X$

$= \text{Arcsin}(x)X - \int \frac{x}{\sqrt{1-x^2}} dx$

$u = 1-x^2 \quad du = -2x\,dx$

$dx = \frac{1}{-2x} du$

$= X\text{Arcsin}(x) + \frac{1}{2}\int \frac{1}{\sqrt{U}} dU$

$= X\text{Arcsin}(x) + \frac{1}{2}(2\sqrt{U})$

$\boxed{= x\,\text{Arcsin}(x) + \sqrt{1-x^2} + c}$ ✓

(c) $\int x\cos(x)dx$

LIATE

$\int udv = uv - \int vdu.$

$U = X \quad dV = \text{Cos}(x)$

$dU = 1 \quad V = \sin(x)$

$= X\,\text{Sin}(x) - \int \sin(x)dx$

$\boxed{= x\sin(x) + \text{Cos}(x) + c}$ ✓

(d) $\int \tan^{-1}(x)dx$

$\int udv = uv - \int vdu.$

$U = \text{Arctan}(x) \quad dV = 1$

$dU = \frac{1}{x^2+1} \quad V = x$

$= X\text{ArcTan}(x) - \int \frac{x}{x^2+1} dx$

$U = x^2+1 \quad du = 2x\,dx$

$dx = \frac{1}{2x} dU$

$= X\,\text{Arctan}(x) - \frac{1}{2}\int \frac{1}{U} dx$

$= x\,\text{Artan}(x) - \frac{\ln(U)}{2}$

$$\boxed{= x\,\text{Arctan}(x) - \frac{\ln(x^2+1)}{2} + C}\ \checkmark$$

(e) $\int xe^{-4x}dx$ $\qquad\qquad \int udv = uv - \int vdu.$

$U = x \qquad dv = e^{-4x}$

$du = 1 \qquad V = -\frac{e^{-4x}}{4}$

$$= -\frac{xe^{-4x}}{4} + \int \frac{e^{-4x}}{4}dx$$

$$\boxed{= -\frac{xe^{-4x}}{4} - \frac{e^{-4x}}{16} + C}\ \checkmark$$

(f) $\int x^2e^{-3x}dx$ $\qquad\qquad \int udv = uv - \int vdu.$

$U = x^2 \qquad dv = e^{-3x}$

$du = 2x \qquad V = -\frac{e^{-3x}}{3}$

$$= -\frac{x^2e^{-3x}}{3} + \int \frac{2xe^{-3x}}{3}dx$$

$$= -\frac{x^2e^{-3x}}{3} + \frac{2}{3}\int xe^{-3x}dx$$

$U = x \qquad dv = e^{-3x}$

$du = 1 \qquad V = -\frac{e^{-3x}}{3}$

$$= -\frac{x^2e^{-3x}}{3} + \frac{2}{3}\left(-\frac{xe^{-3x}}{3} + \int \frac{e^{-3x}}{3}dx\right)$$

$$\boxed{= -\frac{x^2e^{-3x}}{3} - \frac{2xe^{-3x}}{9} - \frac{2e^{-3x}}{27} + C}\ \checkmark$$

(g) $\int \ln(x)dx$ $\qquad\qquad \int udv = uv - \int vdu.$

$U = \ln(x) \qquad dv = 1$

$du = \frac{1}{x} \qquad V = x$

$$= x\ln(x) - \int \frac{x}{x}dx$$

$$\boxed{= x\ln(x) - x + C}\ \checkmark$$

(h) $\int e^x \sin(x)dx$

$\int udv = uv - \int vdu.$

$U = \sin(x) \quad dv = e^x$

$du = \cos x \quad V = e^x$

$= e^x \sin x - \int e^x \cos x \, dx \quad U = \cos x \quad dv = e^x$

$du = -\sin x \quad V = e^x$

$e^x \sin x dx = e^x \sin x - (e^x \cos x - \int e^x \sin x \, dx)$

$+ e^x \sin x \, dx$

$\frac{2 e^x \sin x dx}{\cancel{2}} = \frac{e^x \sin x - e^x \cos x}{2}$

$\boxed{e^x \sin x \, dx = \frac{e^x \sin x - e^x \cos x}{2} + C}$ ✓

(i) $\int \cos^2(x)dx$

$\int udv = uv - \int vdu.$

$U = \cos^2(x) \quad dv = 1$

$du = -2\cos(x)\sin(x) \quad V = x$

$= x\cos^2(x) + \int 2x \cos(x)\sin(x)\, dx$

$= x\cos^2(x) + 2\int x \sin(2x)\, dx \quad U = x \quad dv = \sin(2x)$

$du = 1 \quad V = -\frac{\cos(2x)}{2}$

$= x\cos^2(x) + 2\left(-\frac{x\cos(2x)}{2} + \int \frac{\cos(2x)}{2} dx\right)$

$\boxed{U = 2x \quad du = 2\,dx \\ dx = \frac{1}{2} du}$

$= x\cos^2(x) + 2\left(-\frac{x\cos(2x)}{2} + \frac{1}{4}\int \cos(U)\, dU\right)$

$= x\cos^2(x) - \frac{2x\cos(2x)}{2} + \frac{2\sin(2x)}{4} + C$

$= x\cos^2(x) - \frac{2x\cos(2x)}{2} + \frac{\sin(2x)}{2} + C$

$\boxed{= x\cos^2(x) - x\cos(2x) + \frac{\sin(2x)}{2} + C}$ ✓

(j) $\int \frac{x^3}{(x^2+2)^2}dx$

$\int \frac{x^3}{(x^2+2)^2}dx$ $\quad U = x^2+2 \quad du = 2x\,dx$ $\quad \int u\,dv = uv - \int v\,du.$

$dx = \frac{1}{2x}du$

$= \frac{1}{2}\int \frac{x^3}{x(U)^2}dx$ $\quad U = x^2+2$

$U - 2 = x^2$

$= \frac{1}{2}\int \frac{x^2}{(U)^2}dU$

$= \frac{1}{2}\int \frac{U-2}{U^2}dU = \frac{1}{2}\left(\int \frac{U}{U^2}du - \int \frac{2}{U^2}du\right)$ $\quad \frac{1}{2U}$ $-$

$= \frac{1}{2}\ln(x^2+2) + \frac{1}{x^2+2} + C$ $\quad -\frac{1}{U}$

$\frac{1}{x^2+2} + \frac{\ln(x^2+2)}{2} + c$

$$\boxed{= \frac{\ln(x^2+2)}{2} + \frac{1}{x^2+2} + C}$$ ✓

(k) $\int_1^9 \frac{\ln(x)}{\sqrt{x}}dx$

$\int_1^9 \frac{\ln(x)}{\sqrt{x}}dx$ $\quad U = \ln(x) \quad dv = \frac{1}{\sqrt{x}}$ $\quad \int u\,dv = uv - \int v\,du.$

$du = \frac{1}{x} \quad V = 2\sqrt{x}$

$= \ln(x)2\sqrt{x} - \int \frac{2\sqrt{x}}{x}dx \quad U = x \quad du = 1\,dx$

$= \ln(x)2\sqrt{x} - \int \frac{2\sqrt{U}}{U}dU$

$= \ln(x)2\sqrt{x} - 2\int \frac{x^{\frac{1}{2}}}{x}dU$

$$= \ln(x)2\sqrt{x} - 4\sqrt{x} \Big|_1^9$$

$$= \left(\ln(9)2\sqrt{9} - 4\sqrt{9}\right) - \left(\ln(1)2\sqrt{1} - 4\sqrt{1}\right)$$

$2 \quad +4$

$$= 6\ln(9) - 12 \quad -6$$

$$\boxed{= 6\ln(9) - 18}$$ ✓

(l) $\int \frac{\ln(x)}{x^7} dx$

$$\int \frac{\ln(x)}{x^7} dx \qquad u = \ln(x) \quad dv = \frac{1}{x^7}$$

$$du = \frac{1}{x} \qquad v = -\frac{1}{6x^6}$$

$$= -\frac{\ln(x)}{6x^6} + \int \frac{1}{6x^7} dx$$

$$= -\frac{\ln(x)}{6x^6} + \frac{1}{6}\int \frac{1}{x^7} dx$$

$$= -\frac{\ln(x)}{6x^6} + \frac{1}{6}\left(-\frac{1}{6x^6}\right)$$

$$= -\frac{\ln(x)}{6x^6} - \frac{1}{36x^6} = \boxed{-\frac{6\ln(x)+1}{36x^6} + C}$$ ✓

(m) $\int \frac{\ln(x)}{x^n} dx$ for any positive real number n.

$$\int \frac{\ln(x)}{x^n} dx \qquad u = \ln(x) \quad dv = \frac{1}{x^n}$$

n

$$du = \frac{1}{x} \qquad v = \frac{x^{1-n}}{1-n}$$

$$= \frac{x^{1-n}\ln(x)}{1-n} - \int \frac{1}{(1-n)x^n}\,dx$$

$$= \frac{x^{1-n}\ln(x)}{1-n} - \frac{1}{1-n}\int \frac{1}{x^n}\,dx$$

$$= \frac{x^{1-n}\ln(x)}{1-n} - \frac{1}{1-n}\left(\frac{x^{1-n}}{1-n}\right) + C$$

$$\boxed{= \frac{x^{1-n}\ln(x)}{1-n} - \frac{x^{1-n}}{(1-n)^2} + C} \checkmark$$

(n) $\int (\ln x)^2 dx$

$$\int \ln^2 x\,dx \qquad u = \ln^2(x) \qquad dv = 1$$

$$du = \frac{2\ln(x)}{x} \qquad v = x$$

$$= x\ln^2(x) - \int \frac{x\,2\ln(x)}{x}\,dx$$

$$= x\ln^2(x) - 2\int \ln(x)\,dx \qquad u = \ln(x) \qquad dv = 1$$

$$du = \frac{1}{x} \qquad v = x$$

$$= x\ln^2(x) - 2\left(x\ln(x) - \int \frac{x}{x}\,dx\right)$$

$$\boxed{= x\ln^2(x) - 2x\ln(x) + 2x + C} \checkmark$$

(o) $\int \ln(x^2+1)dx$

$$\int \ln(x^2+1)\,dx \qquad u = \ln(x^2+1) \quad dv = 1$$

$$du = \frac{2x}{x^2+1} \qquad v = x$$

$$= x\ln(x^2+1) - \int \frac{2x^2}{x^2+1}\,dx$$

$= x\ln(x^2+1) - 2\int \frac{x}{x^2+1}\,dx$

$\downarrow$

$\int \frac{x^2+1}{x^2+1} - \int \frac{1}{x^2+1}$

$\int 1 - \int \frac{1}{x^2+1}$

$x - \text{Arctan}(x)$

$\boxed{= x\ln(x^2+1) - 2x + 2\text{Arctan}(x) + C}$ ✓

(p) $\int \sin(\sqrt{x})dx$

Hint: Try a $u-$ sub first.

$\int \sin(\sqrt{x})\,dx$ $\quad u = \sqrt{x} \quad du = \frac{1}{2\sqrt{x}}dx$

$dx = 2\sqrt{x}\,du$

$= 2\int u \sin u\,du \qquad f = u \qquad g' = \sin(u)$

$f' = 1 \qquad g = -\cos(u)$

$= 2\int -u\cos(u) - \int -\cos(u)\,du$

$= 2\left(-u\cos(u) + \sin(u)\right) + C$

$= 2\left(-\sqrt{x}\cos(\sqrt{x}) + \sin(\sqrt{x})\right) + C$

$\boxed{= -2\sqrt{x}\cos(\sqrt{x}) + 2\sin(\sqrt{x}) + C}$ ✓

(q) $\int x^3 e^{x^2} dx.$
Hint: Try a $u-$ sub first.

$$\int x^3 e^{x^2}$$

$u = x^2 \quad du = 2x\, dx$

$dx = \frac{1}{2x} du$

$$= \frac{1}{2}\int u e^u du$$

$f = u \quad g' = e^u$

$f' = 1 \quad g = e^u$

$$= \frac{1}{2}\left(ue^u - \int e^u du\right)$$

$$= \frac{ue^u}{2} - \frac{e^u}{2} + C$$

$$\boxed{= \frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + C}$$ ✓

(r) $\int \cos(x) \ln(\sin x) dx.$
Hint: Try a $u-$ sub first.

$$\int \cos(x) \ln(\sin x)\, dx$$

$u = \sin(x) \quad du = \cos(x) dx$

$dx = \frac{1}{\cos(x)} du$

$$= \int \ln u \, du$$

$f = \ln(u) \quad g' = 1$

$f' = \frac{1}{u} \quad g = u$

$$= u\ln(u) - \int \frac{u}{u} du$$

$$= u\ln(u) - u + C$$

$$\boxed{= \sin(x)\ln(\sin(x)) - \sin(x) + C}$$ ✓

(s) $\int x^n \ln(x)dx$ for a real number $n > 0$.

$$\int x^n \ln(x)\,dx \qquad u = \ln(x) \quad dv = x^n$$
$$du = \frac{1}{x} \qquad v = \frac{x^{n+1}}{n+1}$$

$$= \frac{x^{n+1}\ln(x)}{n+1} - \int \frac{x^n}{n+1}\,dx$$

$$= \frac{x^{n+1}\ln(x)}{n+1} - \frac{1}{n+1}\int x^n\,dx$$

$$= \frac{x^{n+1}\ln(x)}{n+1} - \frac{1}{n+1}\left(\frac{x^{n+1}}{n+1}\right) + C$$

$$\boxed{= \frac{x^{n+1}\ln(x)}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C} \quad \checkmark$$

(t) $\int x \cdot 2^x dx$

$$\int x 2^x\,dx$$

$$u = x \qquad dv = 2^x$$
$$du = 1 \qquad v = \frac{2^x}{\ln(2)}$$

$$= \frac{2x}{\ln(2)} - \int \frac{2^x}{\ln(2)}\,dx$$

$$= \frac{2x}{\ln(2)} - \frac{1}{\ln(2)}\int 2^x\,dx$$

$$= \frac{2x}{\ln(2)} - \frac{1}{\ln(2)}\left(\frac{2^x}{\ln(2)}\right) + C$$

$$\boxed{= \frac{2^x}{\ln(2)} - \frac{2^x}{\ln^2(2)} + C}$$ ✓

2. TRIG INTEGRALS
Compute the following integrals.

(a) $\int \sin^2(x)\cos^2(x)\,dx$

$$\int \sin^2(x)\cos^2(x)\,dx$$

$$\cos^2(x) = 1 - \sin^2(x)$$

$$= \int \sin^2(x)(1 - \sin^2(x))\,dx$$

$$= \int \sin^2(x) - \int \sin^4(x)\,dx$$

$$= -\frac{\cos(x)\sin(x)}{2} + \frac{1}{2}\int 1\,dx$$

$$= -\frac{\cos(x)\sin(x)}{2} + \frac{x}{2} + C$$

$$= -\frac{\cos(x)\sin^3(x)}{4} + \frac{3}{4}\int \sin^2(x)\,dx$$

$$= -\frac{\cos(x)\sin^3(x)}{4} + \frac{3}{4}\left(-\frac{\cos(x)\sin(x)}{2}\right) + C$$

$$\boxed{= \frac{\cos(x)\sin^3(x)}{4} + \frac{\cos(x)\sin(x)}{8} + \frac{x}{8} + C}$$ ✓

(b) $\int \sec^4(x)\tan^2(x)\,dx$

$$\int \sec^4(x)\tan^2(x)\,dx$$

$$\sec^2(x) = \tan^2(x) + 1$$

$$= \int \sec^2(x)\tan^2(x)(\tan^2(x)+1)\,dx$$

$$u = \tan(x) \qquad du = \sec^2(x)\,dx$$

$$= \int u^2(u^2+1)\,du$$

$$= \int u^4 + u^2\,du$$

$$= \frac{u^5}{5} + \frac{u^3}{3} + C$$

$$\boxed{= \frac{\tan^5(x)}{5} + \frac{\tan^3(x)}{3} + C} \quad \checkmark$$

(c) $\int \cos^5(x)\,dx$

$$\int \cos^5(x)\,dx$$

$$= \int \cos(x)(1-\sin^2(x))^2\,dx$$

$$u = \sin(x) \qquad du = \cos(x)\,dx$$

$$= \int (1-u^2)(1-u^2)\,du$$

$$= \int 1 - u^2 - u^2 + u^4\,du$$

$$= \int 1 - 2u^2 + u^4 \, du$$

$$= x - \frac{2u^3}{3} + \frac{u^5}{5} + C$$

$$\boxed{= x - \frac{2\sin^3(x)}{3} + \frac{\sin^5(x)}{5} + C}$$ ✓

(d) $\int \sin^3(2x)\ dx$

$$\int \sin^3(2x)\, dx \qquad u = 2x \qquad du = 2\, dx$$

$$dx = \frac{1}{2} du$$

$$= \frac{1}{2}\int \sin^3(u)\, du$$

$$= \frac{1}{2}\int (1 - \cos^2(u)) \sin(u)\, du$$

$$v = \cos(u) \qquad dv = -\sin(u)\, dx$$

$$dx = -\frac{1}{\sin(u)}\, dv$$

$$= \frac{1}{2}\int (v^2 - 1)\, dv$$

$$= \frac{1}{2}\left(\frac{v^3}{3} - v\right) + C$$

$$= \frac{v^3}{6} - \frac{v}{2} + C$$

$$\boxed{= \frac{\cos^3(2x)}{6} - \frac{\cos(2x)}{2} + C}$$ ✓

(e) Evaluate the integral $\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2 x - 1}dx$.

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sqrt{\sec^2 x - 1}\, dx$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \operatorname{Tan}(x)\, dx$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sin x}{\cos x} dx \qquad U = \cos(x) \quad dU = -\sin(x)\, dx$$

$$dx = -\frac{1}{\sin(x)} dU$$

$$= -\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{U} dU$$

$$= -\ln(U) \Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= -\ln(\cos(x)) \Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$$

$$= -\ln\left(\cos\left(\frac{\pi}{3}\right)\right) + \ln\left(\cos\left(-\frac{\pi}{3}\right)\right)$$

$$= -\ln\left(\frac{1}{2}\right) + \ln\left(\frac{1}{2}\right)$$

$$\boxed{= \ln\left(\frac{\frac{1}{2}}{\frac{1}{2}}\right) = \ln(1) = 0}$$

✓