A Flexible Multi Factor Affine State Switching Asset Pricing Model

Tanathorn Chanwangsa

March 2025

1 Introduction

1.1 Model setup

Let $(S_t)_{t\geq 0}$ be the underlying asset price in $\mathbb{Q}$ measure. Suppose that the stochastic discount factor is D_t, we define the discounted price process as $F_t \equiv D_t S_t$. Where we see that F_t is a $\mathbb{Q}-$martingale. Now consider a collection of n state processes $(Y_{i,t})_{t\geq 0}$ which are assumed to be non-negative orthogonal factors, hence lives in independent probability spaces. We suppose that the state process follows a Feller square root Jump-Diffusion model

$$dY_{i,t} = \sigma_i \sqrt{Y_{i,t}} dW_{i,t} + \int_0^\infty z\tilde{N}_i(dt, dz) \tag{1.1}$$

where the jump term is a compound Poisson jump process with jump rate λ_i, with positive jump intensity measure $\nu_i(dz)$. For the rest of this paper we will assume an exponential jump intensity with rate parameter of γ_i due to its analytical tractability in deriving closed form affine characteristic functions. Hence

$$\nu_i(dz) = \lambda_i \gamma_i e^{-\gamma_i z} dz$$

Now, let $X_t = \log(F_t)$, we model X_t as a mixture of the orthogonal factors plus a compensating drift term:

$$dX_t = \sum_{i=1}^n \alpha_i (dY_{i,t} + c_{i,t} dt) \tag{1.2}$$

Where $\{\alpha\}_{1\leq i\leq n}$ are the weighting coefficients and $\{c_t\}_{1\leq i\leq n}$ are the compensating drift terms to ensure that F_t is a $\mathbb{Q}$-martingale.

1.2 Compensating to obtain $\mathbb{Q}$-Martingale

To ensure F_t is a $\mathbb{Q}$-martingale, we first use Ito Lemma to write the SDE for F_t. We see that if $F_t = e^{X_t}$ then,

$$dF_t = F_{t^-}\left(dX_t + \frac{1}{2} d\langle X_t\rangle_t^{cont}\right) + F_{t^-}\sum_{i=1}^n (e^{\Delta_i X_t} - 1) \tag{1.3}$$

The derivation for the continuous portion of the Ito Lemma is straightforward, on a side note one can see that the jump contribution can be derived as follows:

Recall that for some function $f(X)$, the generator of the compound Poisson process is:

$$Qf(x) = \int_0^{\infty} [f(x+z) - f(x)]\nu(dz)$$

Appropriately choosing $f = e^X$ and noting that a jump in Y_i contributes to a change of $\alpha_i z_i$ in X we have that

$$Q_i f(x) = \int_0^{\infty} e^X [e^{\alpha_i z} - 1]\nu_i(dz) = F_{t^-} \int_0^{\infty} [e^{\alpha_i z} - 1]\nu_i(dz)$$

Hence the total SDE for F_t becomes

$$dF_t = F_{t^-}\left(\sum_{i=1}^{n} \alpha_i (dY_{i,t} + c_{i,t}dt + \frac{1}{2}\alpha_i \sigma_i^2 Y_{i,t}dt)\right) + F_{t^-}\sum_{i=1}^{n}\int_0^{\infty}(e^{\alpha_i z} - 1)\nu_i(dz) \tag{1.4}$$

To ensure that F_t is a martingale, we must have that $\mathbb{E}[dF_t|\mathscr{F}_t] = 0$ or equivalently

$$\sum_{i=1}^{n} \alpha_i (c_{i,t} + \frac{1}{2}\sigma_i^2 \alpha_i Y_{i,t})dt + \int_0^{\infty} \mathbb{E}[(e^{\alpha_i z} - 1)\nu_i(dz)] = 0 \tag{1.5}$$

we compute

$$\int_0^{\infty} \mathbb{E}[(e^{\alpha_i z} - 1)\nu_i(dz)] = \lambda_i \mathbb{E}[e^{\alpha_i Z_i} - 1] = \frac{\lambda_i \alpha_i}{\gamma_i - \alpha_i}$$

If $c_{i,t} = \beta_i + \delta_i Y_{i,t}$ we see that

$$\beta_i = -\frac{\lambda_i}{\gamma_i - \alpha_i} \tag{1.6}$$

$$\delta_i = -\frac{1}{2}\sigma_i^2 \alpha_i \tag{1.7}$$

Which completely determines the dynamics of the compensated process. Note that to ensure the well defined-ness of F_t specifically $\sup_t \mathbb{E}[||F_t||] < \infty$ then $\gamma_i > \alpha_i$.

2 Exponential Affine Characteristic Function

In this section we will derive the characteristic function of the X_t process. Consider the forward PDE for $\phi(u;t)$ where $\phi(u;t,x,\mathbf{y}) = \mathbb{E}[e^{iuX_t}|X_0 = x, \mathbf{Y}_0 = \mathbf{y}]$. We find that

$$\partial_t \phi = \sum_{i=1}^{n} \alpha_i (\beta_i + \delta_i y_i)\phi_x + \frac{1}{2}\sigma_i^2 y_i (\alpha_i^2 \phi_{xx} + 2\alpha_i \phi_{xy} + \phi_{yy}) + \mathbb{E}[\phi(x + \alpha_i Z_i, y_i + Z_i) - \phi(x,y)] \tag{2.1}$$

Noticing the affine structure we produce an ansatz

$$\phi(u;t) = \exp(iux + A(t) + \sum_{i=1}^{n} B_i(t) y_i)$$

We find that this gives us

$$\begin{aligned}(A'(t) + \sum_{i=1}^{n} B_i'(t) y_i) = \sum_{i=1}^{n} iu\alpha_i(\beta_i + \delta_i y_i) + \frac{1}{2}\sigma_i^2 y_i(-u^2\alpha_i^2 + 2\alpha_i iuB(t) + B(t)^2) \\ + \mathbb{E}[e^{iu\alpha_i Z_i + B_i(t)Z_i} - 1]\end{aligned} \tag{2.2}$$

Matching the terms we see that

$$A'(t) = \sum_{i=1}^{n} iu\alpha_i\beta_i + \mathbb{E}[e^{iu\alpha_i Z_i + B_i(t)Z_i} - 1]; \quad A(0) = 0 \tag{2.3}$$

$$B_i'(t) = iu\alpha_i\delta_i - \frac{1}{2}\sigma_i^2 u^2\alpha_i^2 + i\sigma_i^2\alpha_i uB(t) + \frac{1}{2}\sigma_i^2 B(t)^2; \quad B(0) = 0 \tag{2.4}$$

We can substitute in the expectations of the exponential jump intensity and the compensator term for δ_i to simplify

$$A'(t) = \sum_{i=1}^{n}\left(iu\alpha_i\beta_i + \frac{\lambda_i\gamma_i}{\gamma_i - (B_i(t) + iu\alpha_i)} - 1\right); \quad A(0) = 0 \tag{2.5}$$

$$B_i'(t) = -\frac{1}{2}iu\alpha_i^2\sigma_i^2 - \frac{1}{2}\sigma_i^2 u^2\alpha_i^2 + i\sigma_i^2\alpha_i uB(t) + \frac{1}{2}\sigma_i^2 B(t)^2; \quad B(0) = 0 \tag{2.6}$$

The ODE for $B_i(t)$ can be solved with the given the initial conditions. The solution after some algebra is

$$B_i(t) = -(-1)^{3/4}\sqrt{u}\alpha_i \tan\left(\tan^{-1}\left((-1)^{3/4}\sqrt{u}\right) - \frac{1}{2}(-1)^{3/4} t\sqrt{u}\alpha_i\sigma_i^2\right) - iu\alpha_i \tag{2.7}$$

Moving forward we use the express for $B_i(t)$ to find $A'(t)$, such a solution exists. We define

$$F_i(t) = \int\left[\frac{\gamma_i}{\gamma_i - (B_i(t) + iu\alpha_i)} - 1\right] dt$$

We find that

$$\begin{aligned}F_i(t) = -t - \frac{2\gamma_i \log\left(\gamma_i + (-1)^{3/4}\sqrt{u}\alpha_i \tan\left(\tan^{-1}\left((-1)^{3/4}\sqrt{u}\right) - \frac{1}{2}(-1)^{3/4} t\sqrt{u}\alpha_i\sigma_i^2\right)\right)}{\sigma_i^2\left(\gamma_i^2 - iu\alpha_i^2\right)} + \\ \frac{\gamma_i\left((-1)^{3/4}\gamma_i + \sqrt{u}\alpha_i\right)\log\left(\tan\left(\tan^{-1}\left((-1)^{3/4}\sqrt{u}\right) - \frac{1}{2}(-1)^{3/4} t\sqrt{u}\alpha_i\sigma_i^2\right) + i\right)}{\sqrt{u}\alpha_i\sigma_i^2\left(\gamma_i^2 - iu\alpha_i^2\right)} + \\ \frac{\sqrt[4]{-1}\gamma_i\left(\gamma_i + \sqrt[4]{-1}\sqrt{u}\alpha_i\right)\log\left(-\tan\left(\tan^{-1}\left((-1)^{3/4}\sqrt{u}\right) - \frac{1}{2}(-1)^{3/4} t\sqrt{u}\alpha_i\sigma_i^2\right) + i\right)}{\sqrt{u}\sigma_i^2\left(u\alpha_i^3 + i\gamma_i^2\alpha_i\right)}\end{aligned}$$

The solution for A is then

$$A(t) = \sum_{i=1}^{n} iu\alpha_i\beta_i t + \lambda_i(F_i(t) - F_i(0)) \tag{2.8}$$