Introduction to Decision Theory

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The Big Picture

financial markets $\rightarrow$ data $\rightarrow$ statistics $\rightarrow$ inferences

As statisticians, we are tasked with turning the large amount of data generated by experiments and observations into inferences about the world.

This gives rise to a number of core statistical questions:

1. Modeling: How do we capture the uncertainty in our data and the world that produced it?
2. Methodology: What are the right mathematical and computational tools that allow us to draw these statistical inferences?
3. Analysis: How do we compare and evaluate the statistical inferences we make and the procedures we use to make them? In particular, how do we do optimal inference?

Decision Theory Framework

Decision theory provides a framework to answer all of the core questions. In particular, it allows us to formalize the notion of inference as a **decision problem** consisting of three key ingredients:

1. A **statistical model** is a family of distributions P, indexed by a parameter θ. We write

$$P = \{\mathbb{P}_\theta : \theta \in \Omega\}$$

   Here θ is the parameter, Ω is the parameter space, and each $\mathbb{P}_\theta$ is a distribution.

   P is the class of distributions to which we believe our random sample X belongs. In other words, we assume that the data X come from some $\mathbb{P}_\theta \in P$ but that the true θ is unknown.

   The fact that we don't know θ captures our uncertainty about the problem.

Decision Theory Framework

Example 1 (Weighted coin flips)

Observe a sequence of coin flips $X_1, \dots, X_n \in \{0, 1\}$ where 0 encodes tailes and 1 encodes heads.

It's a weighted coin, so I don't know how often I expect heads to arrive. The goal is to estimate the probability of heads given the observations.

To do this we model this process as independent draws from a Bernoulli distribution:

$$P = \{\mathrm{Ber}(\theta) : \theta \in [0, 1] = \Omega\}$$

In this case, $\mathbb{P}_\theta(X_i = 1) = \theta$.

Decision Theory Framework

2. A **decision procedure** δ is a map from $\mathcal{X}$ (the sample space) to the decision space $\mathcal{D}$

   Example 2 (Weighted coin flips)

   Taking $P = \{\text{Ber}(\theta) : \theta \in [0,1] = \Omega\}$ as before, we may be interested in estimating θ or testing hypotheses based on θ.

   (a) Estimating θ: the decision space is $\mathcal{D} = [0,1]$, and the decision procedure might be $\delta(X) = \frac{1}{n}\sum_{i=1}^{n} X_i$. This procedure is an example of an **estimator**.

   (b) Accept or rejecting the hypothesis $\theta > 1/2$: the decision space is $\mathcal{D} = \{\text{accept, reject}\}$, and one possible decision procedure is $\delta(X) =$ "reject if $\frac{1}{n}\sum_{i=1}^{n} X_i \leq 1/2$, accept otherwise" This procedure is an example of a **hypothesis test**.