Q.1 $$Q(\theta|\theta^{old}) = \sum_{Z} p(Z|X, \theta^{old}) \ln p(X, Z|\theta) \quad \text{—(1)}$$

Let $\gamma(Z_n)$: marginal posterior distribution of latent variable Z_n

& $\xi(Z_{n-1}, Z_n)$: Joint posterior of two successive latent variables

$$\Rightarrow \quad \gamma(Z_n) = p(Z_n|X, \theta^{old})$$

$$\xi(Z_{n-1}, Z_n) = p(Z_{n-1}, Z_n|X, \theta^{old})$$

~~for each value of n, we~~ can

$\gamma(Z_{nk})$ denotes the conditional probability of $Z_{nk} = 1$

similar for $\xi(Z_{n-1,j}, Z_{nk})$

$$\gamma(Z_{nk}) = E[Z_{nk}] = \sum_{Z} \gamma(Z) Z_{nk}$$

$$\xi(Z_{n-1,j}, Z_{nk}) = \sum_{Z} \gamma(Z) Z_{n-1,j} Z_{nk}.$$

Also for HMM, Joint probability of X, Z is given by

$$p(X, Z|\theta) = p(Z_1|\pi)\left[\prod_{n=2}^{N} p(Z_n|Z_{n-1}, A)\right] \prod_{m=1}^{N} p(X_m|Z_{m}, \quad \text{—(2)}$$

where ϕ is emission probabilities.

Putting Eq (2) in (1) and using definitions of $\gamma(Z_{nk})$ and $\xi(Z_{n-1,j}, Z_{nk})$ we get.

$$\log P(X, Z|\theta) = \log P(Z_1|\pi) + \sum_{n=2}^{N} \log P(Z_n|Z_{n-1}, A) + \sum_{m=1}^{N} \log p(X_m|Z_m, \phi)$$

$$Q(\theta, \theta^{old}) = \sum_{Z} \log P(Z|X, \theta^{old}) \left[\log P(Z_1|\pi) + \qquad \right]$$

$$P(X,Z|\theta) = \pi_{z_1} \prod_{n=1}^{N} A_{z_{n-1} z_n} \phi_{z_n}(x_n)$$

The Q function now becomes.

$$Q(\theta,\theta^{old}) = \sum_{Z} P(Z|X,\theta^{old}) \log\left(\pi_{z_1} \prod_{n=1}^{N} A_{z_{n-1} z_n} \phi_{z_n}(x_n)\right)$$

$$= \sum_{Z} \log \pi_{z_1} P(Z|X,\theta^{old}) + \sum_{Z}\left(\sum_{n=1}^{N} \log A_{z_{n-1} z_n}\right) P(Z|X,\theta^{old})$$

$$+ \sum_{Z}\left(\sum_{n+1}^{N} \log \phi_{z_n}(x_n)\right) P(Z|X,\theta^{old})$$

First term:

$$\sum_{Z} \log \pi_{z_1} P(Z|X,\theta^{old}) = \sum_{k=1}^{K} \log \pi_k P(z_{1k}|X,\theta^{old})$$

$$= \sum_{k=1}^{K} \log \pi_k \gamma(z_{1k})$$

Second term:

$$\sum_{Z}\left(\sum_{n=1}^{N} \log A_{z_{n-1} z_n}\right) P(Z|X,\theta^{old}) = \cancel{\sum_{i=1}^{K}} \sum_{n=2}^{N} \sum_{k=j}$$

$$= \sum_{n=2}^{N} \sum_{j=1}^{K} \sum_{k=1}^{K} \xi(z_{n-1,j}, z_{nk}) \ln A_{jk}$$

Last term:

$$\sum_{Z}\left(\sum_{n+1}^{N} \log \phi_{z_n}(x_n)\right) P(Z|X,\theta^{old})$$

$$= \sum_{k=1}^{K} \sum_{n=1}^{N} \log \phi_k(x_n) \underbrace{P(z_{nk}|X,\theta^{old})}_{\gamma(z_{nk})}$$

$$Q(\theta,\theta^{old}) = \sum_{k=1}^{N} \gamma(z_{1k}) \ln \pi_k + \sum_{n=2}^{N} \sum_{j=1}^{K} \sum_{k=1}^{K} \xi(z_{n-1,j}, z_{nk}) \ln A_{jk}$$

$$+ \sum_{n=1}^{N} \sum_{k=1}^{K} \gamma(z_{nk}) \ln p(x_n|\phi_k)$$