Q.2 $p(X) = \sum_{i=1}^{K} \pi_i \mathcal{N}(X; \mu_i, \Sigma_i)$

$x_1, x_2 \ldots x_n$ be N datapoints

$$\mathcal{L}(X;\theta) = \sum_{n=1}^{N} \log \left[\sum_{i=1}^{K} \pi_i \mathcal{N}(x_n; \mu_i, \Sigma_i) \right]$$

$\theta = \{\pi_i, \mu_i, \Sigma_i\}$ $i=1$ to K

$$\mathcal{N}(X, \mu_i, \Sigma_i) = \frac{1}{\sqrt{(2\pi)^d |\Sigma_i|}} e^{-\frac{1}{2}\sum_{j=1}^{d} \frac{(x_j - \mu_{ij})^2}{\sigma_{ij}^2}}$$

Assum Σ_i are diagonal.

(1) $$\frac{\partial \mathcal{L}(X,\theta)}{\partial \mu_{ij}} = \sum_{n=1}^{N} \frac{\pi_i N(x_n, \mu_i, \Sigma_i)}{\sum_{i=1}^{K} \pi_i N(x_n, \mu_i, \Sigma_i)} \cdot \frac{(x_{nj} - \mu_{ij})}{\sigma_{ij}^2}$$

$$\Rightarrow \frac{\partial \mathcal{L}(X,\theta)}{\partial \mu_i} = \sum_{n=1}^{N} \gamma(z_{ni}) \cdot \Sigma_i^{-1}(x_n - \mu_i) = 0$$

$$\Rightarrow \sum_{n=1}^{N} \gamma(z_{ni}) x_n = \left(\sum_{n=1}^{N} \gamma(z_{ni})\right) \mu_i$$

$$\mu_i = \frac{\sum_{n=1}^{N} \gamma(z_{ni}) x_n}{\sum_{n=1}^{N} \gamma(z_{ni})}$$

(2) $$\frac{\partial \mathcal{L}(X,\theta)}{\partial \sigma_{ij}} = \sum_{n=1}^{N} \frac{\pi_i N(x_n, \mu_i, \Sigma_i)}{\sum_{i=1}^{K} \pi_i N(x_n, \mu_i, \Sigma_i)} \left(\frac{(x_{nj} - \mu_{ij})^2}{\sigma_{ij}^3} - \frac{1}{\sigma_{ij}} \right)$$

$$= \sum_{n=1}^{N} \gamma(z_{ni}) \left(\frac{(x_{nj} - \mu_{ij})^2}{\sigma_{ij}^2} - 1 \right) = 0$$

$$\sigma_{ij}^2 = \frac{\sum_{n=1}^{N} \gamma(z_{ni}) (x_{nj} - \mu_{ij})^2}{\sum_{n=1}^{N} \gamma(z_{ni})}$$

$$\Sigma_i = \frac{1}{\sum_{n=1}^{N} \gamma(z_{ni})} \sum_{n=1}^{N} \gamma(z_{ni}) \cdot (x_n - \mu_i)(x_n - \mu_i)^T$$

π_i

(3) $$\frac{\partial \mathcal{L}(x,\theta)}{\partial \pi_i} = \sum_{n=1}^{N} \frac{\mathcal{N}(x_n, \mu_i, \Sigma_i)}{\sum_{i=1}^{K} \pi_i \mathcal{N}(x_n, \mu_i, \Sigma_i)} - \frac{\mathcal{N}(x_n, \mu_K, \Sigma_K)}{\sum_{i=1}^{K} \pi_i \mathcal{N}(x_n, \mu_i, \Sigma_i)}$$

$$\sum_i^K \pi_i = 1$$

$$\pi_K = 1 - \sum_{i=1}^{K-1} \pi_i$$

$$= \sum_{n=1}^{N} \frac{1}{\pi_i} \gamma(z_{ni}) - \frac{\gamma(z_{nK})}{\pi_K} = 0$$

$$= \cancel{\sum_{n=1}^{N} \frac{1}{\pi_i} \gamma(} \frac{N_i}{\pi_i} - \frac{N_K}{\pi_K} = 0$$

$$= \frac{N_i}{\pi_i} = \frac{N_K}{\pi_K}$$

similarly $\frac{N_i}{\pi_i} = c \ \forall \ i$

$\sum N_i = \sum c \pi_i = N = c$

$\Rightarrow \frac{N_i}{\pi_i} = N$ so $\pi_i = \frac{N_i}{N}$