Q.1 $X \in R^{d \times N}$ {Data is zero mean}
Let P be the linear transform on X
$P \in R^{d \times d}$
Transformed Data will be $(x') = PX$
Co-relation Matrix $C_{XX} = \frac{X'(X')^T}{N} = \frac{PXX^TP^T}{N}$
$C_{XX} = XX^T$ is symmetric
$\Rightarrow$ C_{XX} is diagonalizable $\Rightarrow$ $C_{XX} = EDE^{-1}$ — ①
where E is Matrix who columns are eigenvectors of C_{XX}.

$C_{x'x'} = \frac{PXX^TP^T}{N}$ should be diagonalizable {optimal decorrelation}
$C_{x'x'} = PC_{XX}P^T = PEDE^{-1}P^T$ — using ①
putting $P = E^{-1} = E^T$ { $E^TE = I$ as E is orthogonal matrix }
$C_{x'x'} = D$ so $\boxed{P = E^T}$

$\rightarrow$ E is also a orthogonal matrix {as C_{XX} is symmetric}
Let e_1 and e_2 be two eigen vectors corresponding to λ_1 and λ_2 $(\lambda_1 \neq \lambda_2)$

now
$$
\begin{aligned}
\lambda_1 \langle e_1, e_2 \rangle &= (\lambda_1 e_1)^T e_2 \\
&= \cancel{(Ae_1)^T e_2}\ (C_{XX} e_1)^T e_2 \\
&= \cancel{e_1^T A^T e_2}\ e_1^T C_{XX}^T e_2 \\
&= e_1^T C_{XX} e_2 \quad (C_{XX} \text{ is sym.}) \\
&= e_1^T \lambda_2 e_2 \\
&= \lambda_2 \langle e_1, e_2 \rangle
\end{aligned}
$$

$\Rightarrow (\lambda_1 - \lambda_2) \cdot \langle e_1 e_2 \rangle = 0$
since $\lambda_1 \neq \lambda_2$ therefore $\langle e_1, e_2 \rangle = 0$
$C_{XX} = E \cdot D \cdot E^T = E \cdot D \cdot E^{-1}$