13th jan

## Vector Space

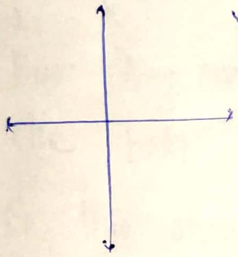
→ $S$ is a vector space.
$V$ is subset of $S$.
$V$ is basis of $S$, then it should satisfy :—

① Elements of $S$ is linear combinations of vectors of $V$.
② $V$ is linearly independent.

→ Number of elements in basis is cardinality of vector space.

→ A vector space can have multiple basis, but cardinality of basis will be same.

Ex:



← 2D

$$U_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$U_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Proof :-

$S$:
$$V = \{ v_1, v_2, \dots, v_m \}$$
$$U = \{ u_1, u_2, \dots, u_n \}$$

Proof by contradiction :

let $m \neq n$
let $m > n$

$U \subset V$ ← not possible (as both are basis)

If $U$ is basis and $U \subset V$ then $V$ will have some vectors that are not in $U$, so that vectors can be written as linear combination of vectors of $U$. So $V$ won't be linearly independent.

idea is :-

We will replace elements of $U$ (one by one) with elements of $V$ and increase cardinality of intersection of $U, V$.

so,

a) Which one element from U to remove.

b) Which element from V to insert in U.

c) New U should again be basis of S.

let $K$ elements are common in $U, V$.

$V = w_1, w_2, \ldots, w_K, v_{K+1}, \ldots, v_m$

$U = w_1, w_2, \ldots, w_K, u_{K+1}, \ldots, u_n$

$$v_{K+1} = \sum_{i=1}^{K} \alpha_i w_i + \sum_{i=K+1}^{n} \alpha_i u_i$$

There will be some $\alpha_i \ (K+1 \leq i \leq m) \neq 0$

let one of them is $\alpha_{K+1}$, i.e. $\alpha_{K+1} \neq 0$

Then replace $v_{K+1}$ with $u_{K+1}$

$U' = w_1, w_2, \ldots, w_K, v_{K+1}, u_{K+2}, \ldots, u_n$

We need to prove that $U'$ is basis of S..

$$v_{K+1} = \alpha_1 w_1 + \cdots + \alpha_K w_K + \underbrace{\alpha_{K+1}}_{\leftarrow \text{non zero}} u_{K+1} + \cdots + \alpha_n u_n$$

$$\Rightarrow u_{K+1} = \frac{-(\alpha_1 w_1 + \cdots + \alpha_K w_K) + v_{K+1} + (-1)(\alpha_{K+2} u_{K+2} + \cdots + \alpha_n u_n)}{\alpha_{K+1}}$$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{Any element} \\ \text{from } S}}{u_{K+1}} = \underbrace{\beta_1 w_1 + \cdots + \beta_K w_K + \beta_{K+1} v_{K+1} + \beta_{K+2} u_{K+2} + \cdots + \beta_n u_n}_{\text{Linear combination of } U'}$$

Now prove : elements of $U'$ are linearly independent.

Theorem

A vector can be expressed exactly in 1 manner as linear combination of basis.

Ex) $q$ is a vector space. $\{z_1, z_2, z_3\}$ + basis

$$q_1 = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 \quad \&$$
$$q_1 = \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3$$

$$\Rightarrow z_1(\alpha_1 - \beta_1) + z_2(\alpha_2 - \beta_2) + z_3(\alpha_3 - \beta_3) = 0$$

But Any one from $\alpha_1 - \beta_1$, $\alpha_2 - \beta_2$, $\alpha_3 - \beta_3$ are not 0. else both representation of $q_1$ will be same.

so this shows $z_1, z_2, z_3$ are not linearly dependent.
But they are basis.

Hence proved.

Using above theorem our proof :

let $U'$ is not linearly Independent.

$$U' = U - u_{K+1} + v_{K+1}$$

This is not Independent only if $v_{K+1}$ can be expressed as linear combination of $U - u_{K+1}$.

ie.

$$v_{K+1} = \gamma_1 w_1 + \gamma_2 w_2 + \cdots + \gamma_K w_K + \underset{\substack{\uparrow \\ \text{zero}}}{0} \cdot u_{K+1} + \gamma_{K+2} u_{K+2} + \cdots \gamma_n u_n$$

Here 2 representation of $v_{K+1}$ using basis.
so $U'$ is linearly independent.

So we increased cardinality of $U \cap V$ with 1 by replacing.

Now $|U \cap V| = k+1$

Like this we can replace $u_{k+2}, u_{k+3}, \cdots, u_n$

After that $|U \cap V| = n$

As $m > n$, now there will be some elements $v_{n+1} \cdots v_m$ in $V$.

So $U \subset V$.

But this is impossible.

So our assumption is wrong.

$m \ngtr n$

Like this $n \ngtr m$

$\therefore$ $m = n$ (proved)

by replacing.

$\dots v_n$ in V.

## Ortho Normal basis :-

In any vector space, there exist a basis where elements of basis is orthogonal to each other and magnitude of elements = 1.

**Theorem**

Every ~~vec~~ vector space have a ortho normal basis.

**Proof**

Orthonormal basis = orthogonal + unit vector

Goal:- Transfer the ~~b~~ vectors of basis to othogonal, unit vector.

- For unit vector divide each vector by it's magnitude.

let

Basis $v = v_1, v_2, \dots\dots, v_n$

$\uparrow$

after doing for unit modification

let $v_1 \cdot v_2 = \alpha_2$

$v_1 \cdot v_3 = \alpha_3$

$v_1 \cdot v_4 = \alpha_4$

$\vdots$

We can modify v to:

$$v' = v_1, \underbrace{(v_2 - \alpha_2 v_1), (v_3 - \alpha_3 v_1), \dots\dots}$$

Now these vectors are orthogonal to $v_1$

$$v_1 \cdot (v_2 - \alpha_2 v_1)$$

$$= v_1 \cdot v_2 - \alpha_2 \underbrace{v_1 \cdot v_1}_{\text{unit vector}}$$

$$= \alpha_2 - \alpha_2 \cdot 1 = 0$$

Now we have to prove that $v'$ is also basis.

① all vectors of $v'$ are linearly independent.

② We can get any vector of vector space from $v'$.

② Let P be a vector from vector space so my claim: P can be written as –

$$P = \beta_1 v_1 + \beta_2 (v_2 - \alpha_2 v_1) + \beta_3 (v_3 - \alpha_3 v_1) + \dots$$

$$= v_1(\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3 - \dots\dots) + \beta_2 v_2 + \beta_3 v_3 + \dots$$

$$= \gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3 + \dots\dots$$

old vectors of basis

That means if P can be written as linear combination of vectors of V then, it can be written with linear combination of $v'$.

① We know all the vector of V are linearly independent.

Assume all vectors in $v'$ are not linearly independent

let $v_k - \alpha_k v_1$ is linearly combination of some vectors in $v'$.

Then P can be written as:

$$P = \beta_1 v_1 + \beta_2 (v_2 - \alpha_2 v_1) + \dots + 0 \cdot (v_k - \alpha_k v_1) + (v_{k+1} - \alpha_{k+1} v_1) + \dots\dots$$

$$= v_1(\beta_1 - \alpha_2\beta_2 - \dots) + \beta_2 v_2 + \dots + 0 \cdot v_k + \beta_{k+1} v_{k+1} + \dots$$

$$= \gamma_1 v_1 + \gamma_2 v_2 + \dots + 0 \cdot v_k + \gamma_{k+1} v_{k+1} + \dots$$

This is 2 representation using basis. So all vectors in $v'$ is linearly independent.

- In next step we can again make vectors of $v'$ unit vectors and then all the vectors $v_3 \dots v_n$ orthogonal to $v_2$. And so on....

→ Number of linearly independent rows = number of linearly independent columns. (rank of a matrix)

Proof:-

Do some row operations, column operation in the matrix till you get only ~~a~~ none-zero elements present in diagonal, othere element of matrix will be zero. We may get some zeros in diagonal also.

Then we will get.

$$\underset{\substack{\uparrow \\ \text{Initial} \\ \text{matix}}}{A} \longrightarrow \underset{\substack{\uparrow \\ \text{matrix} \\ \text{after some operation.}}}{A_1}$$

We need to prove that:

Number of linearly independent rows, columns in $A, A_1$ is same.

~~Proof:- (Proof by contradiction)~~

~~(1) Number of linearly independent rows in $A, A_1$ is same.~~

~~Proof.~~

~~(1) Row operation ($R_i = R_i - \alpha R_j$) does not disturb~~

Case-2

If

Proof:-

(1) Num

m

By

row

proof

con

of

So

Case - 2

If ~~$R_i$ was~~ independent row.

Proof :- (by contradiction)

① Number of linearly independent rows in $A, A_1$ same.

$$A = \begin{bmatrix} \leftarrow n \rightarrow \\ R_1 \\ R_2 \\ R_3 \\ \vdots \end{bmatrix} \; (m \text{ rows}) \longrightarrow A' = \begin{bmatrix} R_1 \\ R_2 - \alpha R_1 \\ R_3 \\ \vdots \end{bmatrix}$$

By doing ~~do~~ Row operation only, the number of linearly independent rows remains same.

Proof :

consider a set $S$ containing $R_1, \dots, R_m$ and all linear combination of $R_1$ to $R_m$.

So '$S$' is a vector space & matrix $A$ contains its Basis.

Let us consider another set $S'$ containing all the elements (rows) of $A'$ and all linear combination of them. So $S'$ is also a vector space, basis present in $A'$.

Claim : $S = S'$

Because we can get $R_2$ ~~???~~ in $S'$ also

$$R_2 = \underbrace{\underbrace{(R_2 - \alpha R_1)}_{\text{in } A'} + \alpha R_1}_{\text{linear combination}}$$

As $S = S' \Rightarrow$ cardinality $(S)$ = cardinality $(S')$

$\Rightarrow$ # linearly independent rows in $S, S'$ are equal.

~~Prove: Inverse exists iff all the rows or columns are independent.~~
~~Proof:-~~

② Number of independent columns in $A, A'$ same.

Proof:-

We know, by doing row operation # linearly independent rows remains same.

let $\underbrace{Ax = 0}$ $\Rightarrow A'x = 0$ (solution remains same)

$\alpha_1 A_1 + \cdots + \alpha_m A_m = 0 \Rightarrow \alpha_1 A_1' + \cdots + \alpha_1 A_m' = 0$

common solution $= x$

Let $A = [C_1 \; C_2 \cdots C_n]$ f columns

$A' = [C_1' \; C_2' \cdots C_n']$

let $k$ = # linearly independent columns in $A$

$r$ = # linearly independent columns in $A'$

Proof by contradiction:

let $k > r$

Without loss of generality let $(C_1 \cdots C_k)$ are linearly independent.

As $k > r \Rightarrow (C_1' \cdots C_k')$ are not linearly independent.

$\therefore \alpha_1 C_1' + \alpha_2 C_2' + \cdots + \alpha_k C_k' = 0$

where $\alpha_1 \cdots \alpha_k$ are not all zero. (let $\alpha_z \neq 0$ $1 \leq z \leq k$)

let $\alpha_{k+1} = \alpha_{k+2} = \cdots = \alpha_n = 0$

so $\alpha_1 C_1' + \cdots + \alpha_n C_n' = 0$

$\Rightarrow A' \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = 0$

So $x' = [\alpha_1 \cdots \alpha_n]$ is a non trivial sol$^n$ for $A'x' = 0$

$\Rightarrow$ $x'$ is a sol$^n$ for $A$

$\Rightarrow$ $Ax' = 0$

$\Rightarrow$ $\alpha_1 c_1 + \cdots + \alpha_n c_n = 0$ — Here $\alpha_z \neq 0$

But for linearly independent all $\alpha$ have to 0.

$\Rightarrow$ $\alpha_1 c_1 + \cdots + \alpha_k c_k = 0$ ~~$K > K'$, upto $K'$ not all zero. after that $(K' \cdots K)$ will be linear combination~~

$\Rightarrow$ $\{c_1 \cdots c_k\}$ is linearly dependant ; which is a contradiction.

$\Rightarrow$ $K > r$ is not possible.

Similarly $K < r$ is not possible.

$\Rightarrow$ $K = r$.

$\therefore$ Number of independent row, columns same in $A, A'$.

By doing like we can modify $A, A' \cdots$ diagonal matrix.

Q) Prove inverse exist if all rows, or columns are independent.

Proof:- let matrix ~~$A_{m \times n}$~~ $A_{n \times m}$

All rows, columns are independent

$\Rightarrow$ rank$(A)$ = min$(m, n)$

let $m < n$

$\Rightarrow$ number of rows > number of columns

Here rank = $m$

$\Rightarrow$ $n - m$ rows will be linearly dependent

so $m \not< n$

likely $m \not> n$

$\therefore$ $m = n$ = rank$(A)$

$\Rightarrow$ determinant value $\neq 0$ $\Rightarrow$ inverse exist. (Proved)

## Null space

- Null space of any matrix A consists of all the vectors B such that $AB = 0$ and B is not zero.
- Null space is also a vector space

$S = \{x \mid Ax = 0\}$ where $A_{m \times n}$

↑ Null space . $x = n$ dimensional vector

S is vector space, because $u_1, u_2 \in S$ then
$Au_1 = 0$, $Au_2 = 0$, $A[\alpha_1 u_1 + \alpha_2 u_2] = 0$

let $K$ = rank of A

Ex]

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{Rank} = 1$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Q) What is the cardinality of basis of S ?

Ans :- $\boxed{n - K}$

**Nullity :-**

Number of linearly independent vectors present in null space of a matrix

= dimension of null space

= Nullity

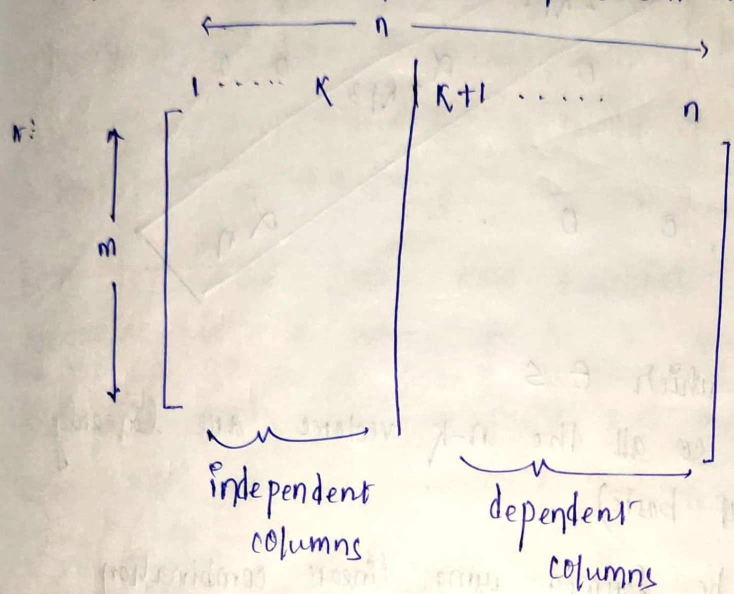
**Rank Nullity Theorem**

$\boxed{\text{Nullity of } A + \text{Rank of } A = \text{Total number of attributes/columns of } A}$

Proof:- Proof by construction

$S = \{x \mid Ax = 0\}$

construct a basis of $S$ with cardinality $= n-K$




we can make this possible without loss of generality

We know,

There exist a vector $V$ ~~and~~ belongs to $S$ $[\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1K}, \alpha_{1K+1}, \underbrace{0, \ldots, 0}_{n-(K+1)}]$.

Such that: ~~Also~~

$AV = 0$

because:

$\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1K}$ not all zero.

$\alpha_{1K+1}$ is non-zero such that

$A_{11} \cdot \alpha_{11} + A_{12} \cdot \alpha_{12} + \cdots + A_{1K} \cdot \alpha_{1K} + A_{1K+1} \cdot \alpha_{1K+1} = 0$

$\Rightarrow A_{11} \cdot \alpha_{11} + A_{12} \cdot \alpha_{12} + \cdots + A_{1K} \cdot \alpha_{1K} = -\alpha_{1K+1} A_{1K+1}$

As $A_{K+1}$ column is dependant column the above is possible.

So $v \in S$ can be:

① $\alpha_{11}\ \alpha_{12} \cdots \alpha_{1K},\ \boxed{\alpha_{1K+1}},\ 0\ 0\ 0 \cdots$ ← diagonal non-zero

② $\alpha_{21}\ \alpha_{22} \cdots \alpha_{2K},\ 0,\ \alpha_{2K+2},\ 0,\ 0 \cdots$

③ $\alpha_{31}\ \alpha_{32} \cdots \alpha_{3K},\ 0\ \ 0\ \ \alpha_{3K+3},\ 0,\ 0 \cdots$

⋮

(n-K) $\alpha_{n-K1}\ \alpha_{n-K2} \cdots \alpha_{n-KK},\ 0\ \ 0\ \cdots\cdots\ \alpha_n$

We constructed n-K vector which $\in S$

As diagonal only non-zero, so all the n-K vectors are linearly independent (1st property of basis)

2nd :- Any vector of S can be formed using linear combination of these n-1 vectors. (we need to prove this)

Proof :-

Take any element and find $\underbrace{\alpha_{1K+1}, \cdots, \alpha_n}_{\text{n-K vectors with only diagonal non-zero}}$ by doing row operation.

Then with combination of those vectors we can get our initial vector.

∴ Cardinality of $S$ = n-K (proved)

# Linear optimization

Example]

Let a pharma company make 2 types of tablets :- X, Y

In X it will get 10 rupees / tablet
In Y it will get 15 rupees / tablet } by selling

Let $x$ number of tablet X, $y$ number of tablet Y company made.

$\Rightarrow$ Total profit $= 10x + 15y$

Both X, Y uses some raw material A, B.

constraint is 'a' amont of A, 'b' amount of B is available.
For X : 2A, 4B needed. For Y 1A, 3B is needed.
Then How much maximum profit that pharma company can make.

Take $a = 10$
$b = 12$

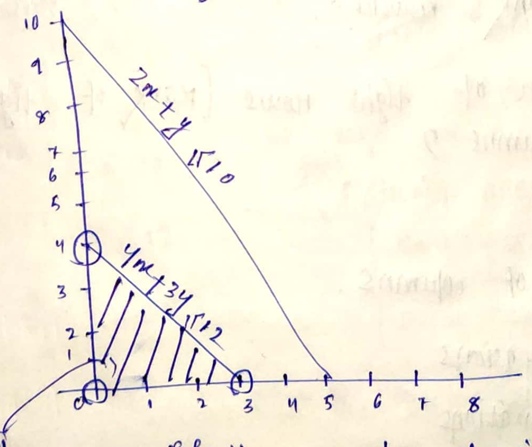
$\Rightarrow$ There are 4 constraints :

$$2x + y \leq a \Rightarrow 2x + y \leq 10 \quad \text{— (i)}$$

$$4x + 3y \leq b \Rightarrow 4x + 3y \leq 12 \quad \text{— (ii)}$$

(iii) —— $x \geq 0$
(iv) —— $y \geq 0$ } Here $x, y$ can't be -ve in this que.

maximize $10x + 15y$ :




There are infinite number of points to find the solution.

$\Rightarrow$ Then how to maximize ?

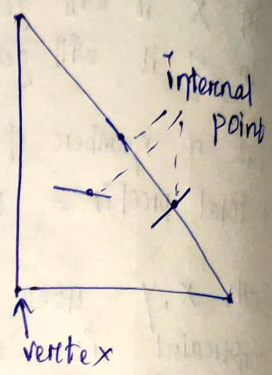
* Here in this case finite number of possible sol$^n$'s we need to identify. Then among them compare.

→ Here finite points will be corner points.
We need to prove the above statement.

Internal point

$v$ is a point called internal point if $u_1, u_2$ (different from $v$) exist such that there will be line joining $u_1, u_2$ and $v$ will be present on that line.

Else $v$ will be a vertex




→ Simplex algorithm
Theoritically exponential time
Practically (with some assumptions) finite time.

→ In the previous example, tight equations will be :

$4x + 3y \leq 12$

$y \geq 0 \Rightarrow 0.x + 1.y \geq 0$

matrix form : $\begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix}$

→ Tight equations give full rank matrix.

Q) Why at a vertex, number of tight rows (rank of tight matrix) = number of columns ?

Sol$^n$ Proof by contradiction.

Case-1

Assume rank < number of columns.

Here # columns = # dimensions

rank = # tight rows = # equations

let we have $x$ number of eq$^n$, $y$ dimensions ($y$ unknowns)

$n = y - x \Rightarrow n \geq 1$

So when we will try to solve these equations, we will have

Infinitely many solⁿ and solⁿ will be eqⁿ of n dimensions.
As $n \geq 1$, we can at least find an eqⁿ of line as solution.
So those points on line will be internal points.

case - 2

rank = # rows > # columns

But rank ≤ min (# rows, # columns)

⇒ rank $\not>$ # columns

∴ rank = # columns to get a perticular point (vertex as solution)

In previous example]

$$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$$

* what is a tight equation ? ( tight constraint )

Ans :- constraints are in-equalities . At a vertex those constraint's where inequality will be equality, those are tight constrains / eqⁿs.

Here :-

$2x + y \leq 10$

$4x + 3y \leq 12$

$x \geq 0$

$y \geq 0$

let take vertex (0,4)

⇒ tight constrains are :

$$\begin{cases} 4x + 3y \leq 12 & \{ 3 \times 4 = 12 \} \\ x \geq 0 & \{ 0 = 0 \} \end{cases}$$

Other : $2x + y \leq 10$ { $4 \leq 10$ }

$y \geq 0$ { $4 \geq 0$ } → Inequality

→ tight constraints depend on perticular point.

Ex:-

At (0,4) tight constraints :-

$$4x + 3y \leq 12$$
$$x \geq 0$$

At (3,0) tight constraints :-

$$4x + 3y \leq 12$$
$$y \geq 0$$

At (0,0) tight constraints :-

$$x \geq 0$$
$$y \geq 0$$

- At internal point may/may not get a tight constraint. If we will get # tight constraints will be less than dimension $(x, y, \cdots)$

## Theorem

~~Arrangian~~

Sufficient & necessity condition

A point is a vertex iff number of columns in tight constraint matrix equal to ~~number of columns~~ rank of matrix.

i.e.

rank = # tight constraints
# columns = # dimension

} if these two equal ⇒ vertex else internal point.

## Proof

Part-1 : If point is vertex, then # columns = rank

idea : Let rank < # columns

then we will show that point is internal point.

(Proof by contradiction)

i.e. For point to be vertex, # columns have to be equal with rank

Proof:

Using rank-nullity thm: rank(A) + nullity(A) = # columns (A)

As here rank < # columns, we will get at least 1 non-zero vector in Null space of tight constraint matrix A.

Let 'v' be a point (where we calculated tight constraints)

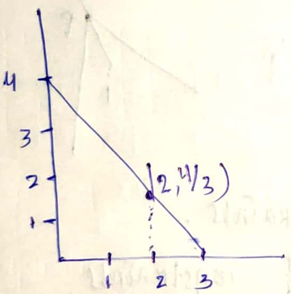
Let 'u' be the vector in Null space of A.

we will use 'u' to calculate 2 more points which will satisfy all ~~of~~ the constraints and 'v' will be present on line joining these 2 points.

Take $\alpha \approx 0$ but +ve. {nearly equal to zero}

'$v+\alpha u$', '$v-\alpha u$' are those 2 points.

<u>Example</u>




If point = (2,0) = v

Tight constraint : $y \geq 0$

Tight constraint matrix : $\begin{bmatrix} 0 & 1 \end{bmatrix}$

vector in Null space : $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = u$

$\left( As: \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = [0] \right)$

Take $\alpha = 1$

$\left.\begin{array}{l} v+\alpha u = (3,0) \\ v-\alpha u = (1,0) \end{array}\right\}$ satisfy all constraints

'v' is in the line of $v+\alpha u$, $v-\alpha u$

Ex-2

Point = $(0,3)$ = v

Tight constraint : $x \geq 0$

matrix : $\begin{bmatrix} 1 & 0 \end{bmatrix}$

$u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\alpha = 1$

$v+\alpha u = (0,4)$

$v-\alpha u = (0,2)$

point = v = $(2, 4/3)$

Tight constraint : $4x+3y \leq 12$

matrix : $\begin{bmatrix} 4 & 3 \end{bmatrix}$

vector in Null space = u = $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$

$\alpha = 0.01$ $\quad v+\alpha u = (2.03, 1.29)$ $\quad v-\alpha u = (1.97, 1.37)$

So, none of the constraints get violated.

So 'v' is an internal point.

$\therefore$ Our assumption is false.

$\therefore$ If a point is vertex, then # columns = rank.

Part - 2

If # columns = rank, then that point is a vertex.

Proof

If # columns = rank = n

$\Rightarrow$ we will get n linearly independent tight constraints.

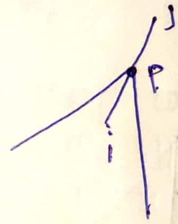
There will be n knowns $(x, y, z \cdots)$

If we will solve those eq$^n$ we will get a unique point 'p'

Our claim :- That point 'p' is a vertex

Assume that is not a vertex $\Rightarrow$ internal point.

$\Rightarrow$ There exist 2 points i, j such that, point 'p' will lie on the line joining i, j.




Assume 'i' is inside the boundary of constraints.

Then 'j' can't lie inside the boundary of constraints.

As point 'p' is the solution of tight constraints and $\exists$ i lies inside the boundary and j is opposite to i.

So 'p' is not an internal point.

$\therefore$ rank = # columns $\Leftrightarrow$ point is a vertex.

Constraint matrix of previous example :—

$$\begin{bmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

* Rank of this matrix = Number of columns

## Theorem

We have optimal $\underbrace{\text{solution}}_{\downarrow}$ at one of the vertices (corners)

maximum profit $10x + 15y$

→ We may have optimal solution at internal points, but we will definitely have solution at vertex.

**Proof**

idea:- Someone gave you solution 'v' (optimal point) as an internal point. You have to move the point towards some vertex. Finally point will be at the vertex.

By moving :

① We have to increase number of tight constraints
[At internal point # tight constraint < # columns.
At vertex # tight constraint = # columns. So we have to increase ~~#~~ number of tight constraints to match with # columns.

② profit ($10x + 15y$ value) should not decrease.
[At 'v' profit is maximum as v is an optimal point. So profit can't increase. And it should not decrease by moving. So it should remain constant)

Fact

* vectors in Null space of a matrix are orthogonal to vectors (rows) of matrix. (That's why we get dot product '0')

Proof of (1)

'v' is optimal point.

A = tight constraint matrix at 'v'

As 'v' is an internal point, # columns > rank (A)

$\Rightarrow$ There will be at least of vector 'u' at Null space of 'A'.

Total constraints

} Let these are tight constraints at v ← upper part

} Not tight constraints at v ← Lower part

As 'u' is a vector in Null space, 'u' is orthogonal with upper part.

We will move 'v' in the direction / opposite direction of 'u'.

claim

By moving 'v' we will reach at a vertex

i.e. we will able to increase number of tight constraints.

$\underset{\sim}{v}$ — Before moving

$\underset{\sim}{v + \alpha u}$ — After moving in the direction of u

'α' here can be +ve or -ve.

Assume by moving in the direction of 'u' we are not getting any extra tight constraints.

That means those extra constraints (lower part) are also orthogonal to 'u'.

$\Rightarrow$ 'u' belongs to Null space of all the constraints.

But we know, at constraint matrix rank = # columns

$\Rightarrow$ No Null space present for constraint matrix.

So our assumption is false.

$\therefore$ By moving 'v' in direction / opposite direction of 'u' we will increase number of tight constraints. And we will reach at a vertex.

Proof of ② >

$v$ $\quad\quad$ $v + \alpha u$

Assume profit decreased by moving in the direction / opposite of u.

i.e. $\text{Profit}(v) > \text{profit}(v + \alpha u)$

$\Rightarrow \text{Profit}(v) < \text{profit}(v - \alpha u)$

i.e. Profit will increase by moving in direction $-\alpha u$.

This is impossible (As 'v' is optimal) $\Rightarrow$ our assumption false

$\Rightarrow \text{profit}(v) = \text{profit}(v + \alpha u)$

Profit will not decrease.

$\therefore$ By moving 'v' in direction / opposite to u, we will reach at a vertex and profit will remain same.

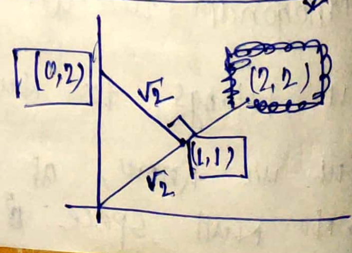
$\Rightarrow$ optimal solution will always present at vertices. (Proved)

Notes

* By moving in othogonal direction cost / value remain same.

→ Gradually we will even not look at all the vertices. We will start from 1 vertex and move with some direction to othere vertices.

Example >



## polyhedron

A polyhedron is a set that can be described in the form $\{x \in \mathbb{R}^n \mid Ax \geq b\}$. where $A = m \times n$ matrix
$b$ = vector in $\mathbb{R}^m$

→ A feasible set of any linear programming problem can be described by inequality constraints of Form $Ax \geq b$. Therefore set of constraint is a polyhedron.

→ Set of form $\{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ is also a polyhedron called as <u>polyhedron in standard form</u>.

→ A polyhedron can either "extend to infinity" or can be confined in a finite region.

→ A set $S \subset \mathbb{R}^n$ is <u>bounded</u> if there exists a constant $K$ such that the absolute value of every component of every element of $S$ is less than or equal to $K$.

→ Let '$a$' be a non zero vector in $\mathbb{R}^n$ and '$b$' be a scalar.

- The set $\{x \in \mathbb{R}^n \mid a'x = b\}$ is called a <u>hyperplane</u>.
- The set $\{x \in \mathbb{R}^n \mid a'x \geq b\}$ is called a <u>halfspace</u>.

$'$ → transpose

→ HyPerplane is boundary of corresponding half space.

→ vector '$a$' of hyperplane is perpendicular to hyperplane itself.

{let $x, y$ belongs to hyperplane $\Rightarrow a'x = a'y = b$
$\Rightarrow a'(x-y) = 0$

∴ '$a$' is orthogonal to any direction vector confined to hyperplane.

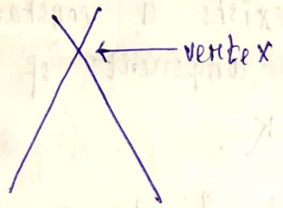
⇒ A polyhedron is intersection of finite number of half spaces.

10.02.2012

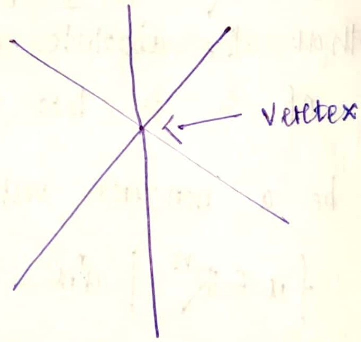
- Assume given a vertex. How to reach at optimal vertex where where cost (profit here) is maximum?
- At a vertex 'n' tight rows. (where $n = \#$ columns). We will find a direction vector with the given starting vertex such that moving in that direction will make some tight rows untight and some untight rows tight.

## Degenerate case

There is a chance that at a vertex, more than $n$ tight rows.
rows = equation of line.
We are assuming #rows = #columns (at vertex)
Ex] In 2 dimension, #columns = 2




our assumption
(2 tight rows)




Degenerate case
(more than 2 tight rows)

→ But we will now not consider degenerate case (at vertex #tight rows = # columns)

→ So now objective is ~~th~~ to find the direction vector such that some tight rows become untight and some untight become tight. cost should increase.

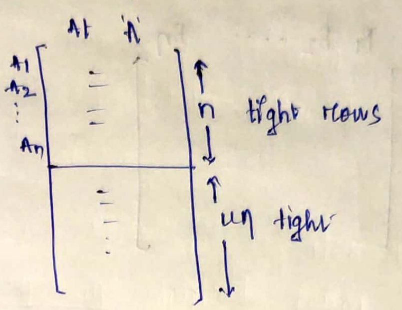
Assume $v$ = that direction vector; $c$ = cost vector (our Example: [10 15])

$c \cdot v > 0$

choose $v$ such that it does not go out of the polyhedra (Remain in feasible space).

$A$ = previous vertex (where to start moving).

Now: $A + \alpha v$




→ At 'A' $n$ linearly independents vectors (all $n$ dimensional) are present. So we can write any $n$-dimensional vector with linear combination of those $n$ vectors.

→ So 'C' can be written as linear combination of $A_1, A_2 \cdots A_n$

$$C = \alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n$$

Case-1

Some $\alpha < 0$ (at least one $\alpha < 0$)

Case-2

All $\alpha \geq 0$

For case-1

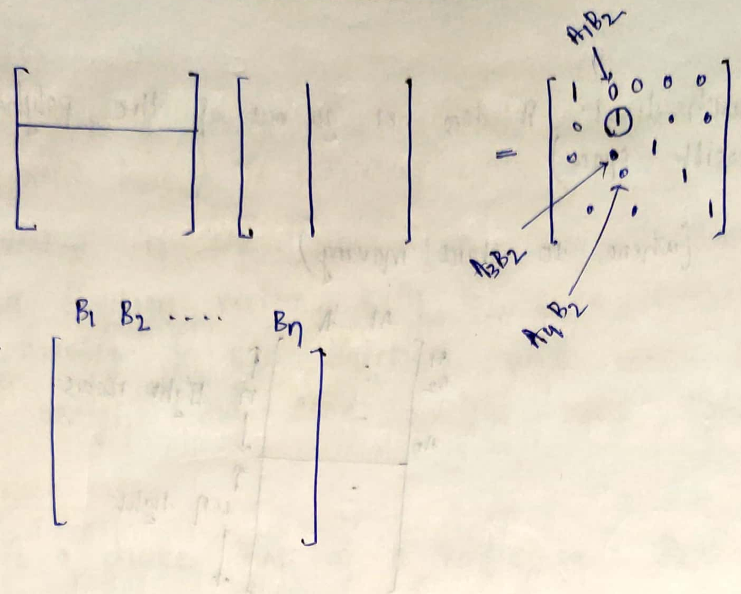
Without loss of generality assume $\alpha_2 < 0$

· $A$ in rank 'n' (full rank) matrix. So it's ~~~~ inverse ($A'$) exist.

$$A \cdot A^{-1} = A \cdot A' = I$$

notation change only

Identity matrix



Take direction vector $v = -(B_2)$

i.e. We will move $A$ in the direction $-B_2$.

<u>Prove two things</u>

① cost (objective function) increase

② It will not move out of polyhedra and reach at a new vertex.

① $$C(-B_2) = (\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n)(-B_2)$$

$$= \alpha_2 A_2 (-B_2)$$ ← Here $\alpha_2 < 0$, $-B_2 < 0$

$> 0$

⇑

cost increases

$B_2$ will orthogonal to $A_1, A_3, \ldots, A_n$

so $A_1 \cdot B_2 = 0$

$A_3 \cdot B_2 = 0$

$A_4 \cdot B_2 = 0$

$\vdots$

$A_n \cdot B_2 = 0$

② $$A = \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_n \end{array}\left[\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right]$$

← Now only $A_2$ will be untight

All other constraints will remain tight as we are moving orthogonal direction with them.

One of the untight constraint will be tight when we will reach at a different vertex, as at that vertex also #tight rows = # columns.

- As $-B_2$ is not orthogonal to $A_2$. $A_2$ will be untight.

## Case-2

When all $\alpha \geq 0$

- If we will move opposite direction of any $B_1$ or $B_2$ or $\cdots$ $B_n$, then cost will decrease.
  (As cost function is non -ve linear combination of tight rows)
- If we will move in same direction of any $B_1 \cdots B_n$, then constraint will violate.

How:-

~~Assume we will move~~ in ~~direction of $B_2$~~.

- If all $\alpha \geq 0$, then that point (A) is optimal point.

Proof by contradiction :-

Assume A is not optimal.

There is an optimal point z.

then:

① cost will increase from A to Z.

② Vertex Z will not violate any constraint.

cost function $C = \alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n$ $\quad\quad \alpha_i \geq 0$

let $U = Z - A$ $\quad\quad A \longrightarrow Z$

Then by moving in direction of U, cost will increase.

i.e. $C \cdot Z - C \cdot A = C \cdot U$

$C \cdot U > 0$

· As both point $A, Z$ are in polyhedra, so line joining $A-Z$ (All points on that line) will also inside polyhedra.

$A$ contains rows $A_1, A_2, \cdots, A_n$.

$$\left.\begin{array}{l} A_1 \cdot U \\ A_2 \cdot U \\ \vdots \\ A_n \cdot U \end{array}\right\} \text{All are } \leq 0$$

How ?

$$C \cdot U = (\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n) \cdot U$$

$$= \underbrace{\alpha_1 A_1 \cdot U}_{\leq 0} + \underbrace{\alpha_2 A_2 \cdot U}_{\leq 0} + \cdots + \underbrace{\alpha_n A_n \cdot U}_{\leq 0}$$

$$\leq 0$$

↑

This is contradicting with $C \cdot U > 0$

$\therefore$ $A$ is optimal point.

$\therefore$ ~~When~~

$$C = \alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n$$

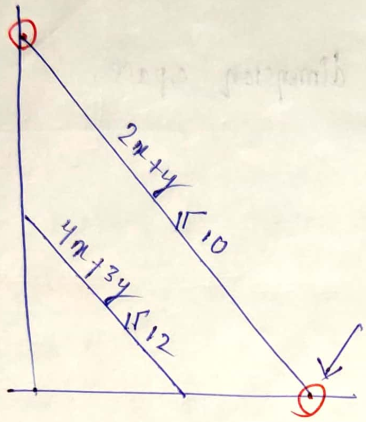
When all $\alpha_i \geq 0$; $1 \leq i \leq n$, then $A$ is optimal point.

14.02.2020

Q: How to get initial Feasible point ? (which lie inside polyhedra)

Wrong approach :—

Solve any n ($n$ = # dimensions) linearly independent constraints ; You will get a vertex .




You may end up with this type point by solving n linearly independent constraints. this point is not in polyhedra

Also getting n linearly independent constraints is exponential time .

Correct approach :—

$$
\begin{array}{llll}
a_{11} & a_{12} \cdots & a_{1n} & \leq b_1 \\
a_{21} & a_{22} \cdots & a_{2n} & \leq b_2 \\
\vdots & & & \vdots \\
a_{n1} & a_{n2} \cdots & a_{nn} & \leq b_n \\
\vdots & & & \\
a_{m1} & a_{m2} \cdots & a_{mn} & \leq b_n
\end{array}
$$

Case-1 : All b's are non-negative

Then we take 0 as starting point of algorithm. (All constraints will satisfy)

Case-2 : some b's are -ve.

We will not solve this problem directly. We create one more problem; from which we will get solution to original problem.

Suppose without loss of generality $b_1$ is -ve and is smallest among the $b_1, b_2, \dots, b_m$.

We are creating new problem in $(n+1)$ dimension space.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 1 \\ a_{21} & a_{22} & \cdots & a_{2n} & 1 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 1 \\ & & & & 1 \end{bmatrix} \begin{array}{l} \leq b_1 \\ \leq b_2 \\ \\ \leq b_m \\ \geq b_1 \end{array}$$

So our solution become $(x_1, x_2, \dots, x_n, z)$ instead of $(x_1, x_2, \dots, x_n)$

$b_1$ is smallest among $b_1 \cdots b_m$

$1 \cdot z \geq b_1$ $\quad (-z \leq -b_1)$

i.e. $z$ is at least $b_1$.

Set $x_1, x_2, \dots, x_n$ to 0 and $z = b_1$. Then every constraint will get satisfied.

~~Case-1~~ ~~N~~

Now our objective function: maximize $z$.

Case-1 : $z$ is -ve

our original problem

If $z$ is -ve then then Problem-1 don't have any solution.

How?

This case is possible when: Ex:

$5 \leq -3$ ← solution not possible

$5 + z \leq -3$

$\Rightarrow z \leq -8$ ← now we are getting negative $z$

Case-2 : if $z = 0$

Then problem ① have feasible point.

How to Find?

Drop value of $z$.

$(x_1 \cdots x_n)$ is feasible point.

Case-3 : if $z > 0$

Then problem ① has feasible point.

How to Find?

Drop $z$

$(x_1 \cdots x_n)$ are feasible point.

Note

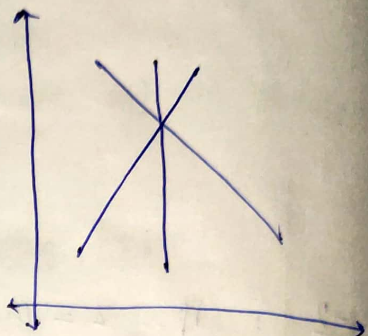
The above algorithm is simplex algorithm.

21 feb

## Degenerate algorithm

(When number of tight equation at a vertex is more than number of columns)

$$\begin{array}{c} \begin{array}{cccc} c_1 & c_2 & \cdots & c_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array} \begin{array}{l} \leq b_1 + \\ \leq b_2 + \\ \vdots \\ \leq b_m + \end{array} \boxed{\begin{array}{l} \varepsilon_1 \\ \varepsilon_2 \\ \\ \varepsilon_m \end{array}}$$



random very small (tend to zero) +ve numbers

Suppose in 2D, at a vertex 3 hyper planes are intersecting. If we move all the hyper planes here and there with negligible amount, then with high probability, he can solve the ~~den~~ degeneracy problem.

We will move the hyperplanes by adding $\varepsilon_i$ to equations.

But this process will not guarantee that it will remove degeneracy. If it is not removing, then ~~th~~ change the value of $\varepsilon_i$ and try again.

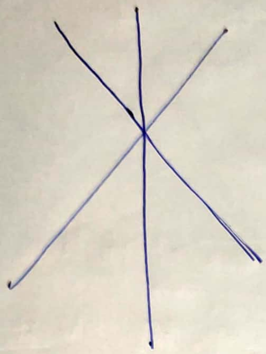
Let initial sol$^n$ space in $S_1$ and we added $\varepsilon$ in it, then it increased slightly and become $S_2$.

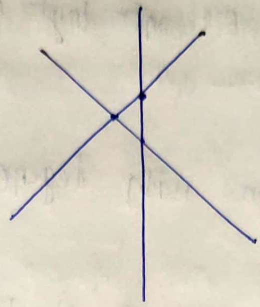
$S_1 \subseteq S_2$.

We want optimal sol$^n$ in $S_1$ (not in $S_2$)

Before moving hyper planes: ($S_1$)

After moving hyper planes
Possible vertex: ($S_2$)





For every vertex in $S_1$, there will be atleast one vertex in $S_2$.

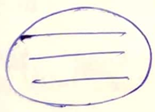
Now we will find optimum in $S_2$ and ~~th~~ from that we will find optimal solution for $S_1$.

→ $S_2$ might have a degenerate vertex, if it is the case select different $\epsilon$'s.

How?

Let $w \in S_2$ is optimal solution in $S_2$.

At $w$:



tight rows

Untight rows

· There exist a vertex in $S_1$ such that tight rows in $w$ are tight rows for that vertex also. That vertex will be optimal in $S_1$.

↑

This statement is valid if $\epsilon \to 0$

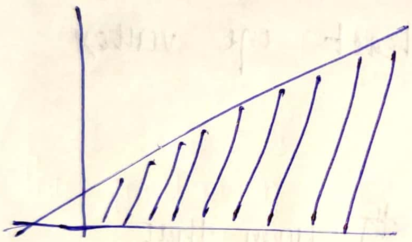
* If $S2$ is not degenerate, but tight equations of '$w$' $(w \in S2)$ don't give any vertex in $S_1$, that means $\epsilon$ is very large change the value of $\epsilon$ and try again.

## Assignment - 2

Implement simplex algo with degenerate case.

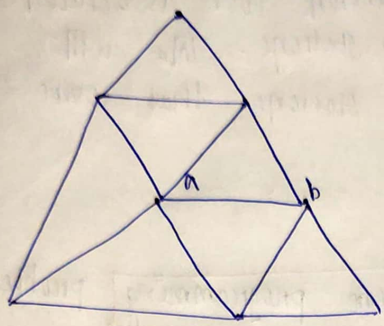
→ One more possible problem in simplex algorithm.

<u>Unbounded case</u>



In this case cost will increase, will not stop ever. So in this case stop the algorithm saying unbounded case.

# Vertex cover problem




edges are roads
internal vertex are junction.
↓
like a,b

**objective**

To monitor each road, at the same time minimize obserervation stations.

- This is a np complete problem. We don't have known polynomial time solution for this problem. But we have approximate solⁿ.

That solution can have number of observation station ≤ 2 * optimum algorithm number of obserrvation station

Algorithm :-

At any point take a road which is not covered and include both the end points (a,b) in solution. Mark all the roads which have any end point either a or b as covered.

Repeat this process till all roads covered.

At the end we will have solution set, but it may contain maximum 2 times the #stations we actually require. Because 2 end points are included every time.

## Modified problem

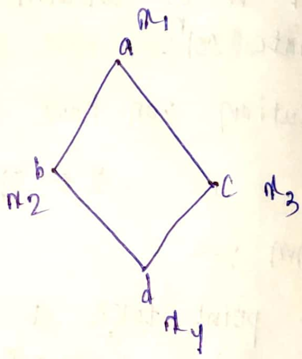
Each police station has some construction cost associated with it. Now we don't care # police stations. We will only care to minimize cost of police stations that covers all the roads.

## Approach

Convert this problem into $\boxed{\text{Integer linear programming}}$ problem. This is different from $\boxed{\text{linear programming problem}}$. In linear programming problem real numbers are allowed. But in this algo we require solution in integer.

This problem in integer linear programming

minimize cost at the given graph.




Take $x_1$ at a

$x_2$ at b

$x_3$ at c

$x_4$ at d

$\therefore$ constraints are $\Rightarrow$

$$x_1 + x_2 \geq 1$$
$$x_2 + x_4 \geq 1$$
$$x_3 + x_4 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$x_i \in \{0, 1\}$$

Because at least one of the vertex (to cover an edge) of an edge should get selected.

We converted the graph problem into Integer linear programming problem. But ILP problems belongs to NP ~~t~~ complete class problem. So no solution exist for ~~ILP~~ ILP problems.

So we have to convert this Integer linear programming (ILP) problem to Linear programming (LP) problem.

$x_i \in \{0,1\} \xrightarrow[\text{to}]{\text{changed}} 0 \leq x_i \leq 1$ ← LP problem

LP problem finally :-

$$\min\ (x_1 + x_2 + x_3 + x_4)$$

constraints : $x_1 + x_2 \geq 1$

$x_2 + x_4 \geq 1$

$x_3 + x_4 \geq 1$

$x_1 + x_3 \geq 1$

$0 \leq x_i \leq 1$

let solⁿ of this using simplex algorithm :

$x_1 = 0.6$

$x_2 = 0.4$

$x_3 = 0.4$

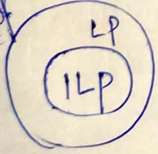
$x_4 = 0.6$

Total cost = 2

claim :-

Solution in LP problem is at, most optimal solution in ILP problem.

Proof




ILP problem has only integer solution allowed.
LP problem has real numbers are also allowed.
So in LP problem minimum cost can possible than ILP problem.

∴ Graph problem ⟶ ILP problem ⟶ LP problem

LP problem Solution ⟶ ILP problem Solution ⟶ Graph problem Solution

By using rounding technique we will change solution from LP to ILP.

(Take $\geq 0.5$ as 1 & $< 0.5$ as 0)

$\therefore x_1 = 1$

$x_2 = 0$

$x_3 = 0$

$x_4 = 1$

* Now optimal $sol^n$ for ILP can be at most twice (approximation factor = 2) than that of LP; because some $x_i$ can be become twice (if $x_i = 0.5$).

* From ILP $sol^n$ get solution of graph problem.

# Primal - Dual Theory

$S_1$ is primal

$S_2$ is dual of $S_1$

$S_3$ is dual of $S_2$

Termination condition in simplex algorithm :-

cost vector is linear combination of non-negative tight vectors at optimal point.

- Suppose we have a set of vector $S_1$. Let this is primal set. For this primal set we construct a dual set $S_2$. $S_3$ = dual of $S_2$.

Now we have to study the relation betn $S_1$ and $S_3$. This is called primal-dual theory.

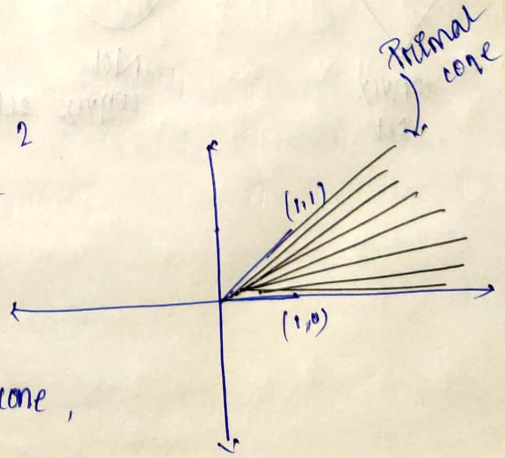
$$\begin{tikzcd} S_2 \arrow[r] & S_3 \arrow[dl, bend left, "equal"] \\ S_1 \arrow[u] \arrow[ur] & \end{tikzcd}$$

claim

Both $S_1$ and $S_3$ are equal. (dual of dual is primal)

2.3.20

## Primal & Dual cones

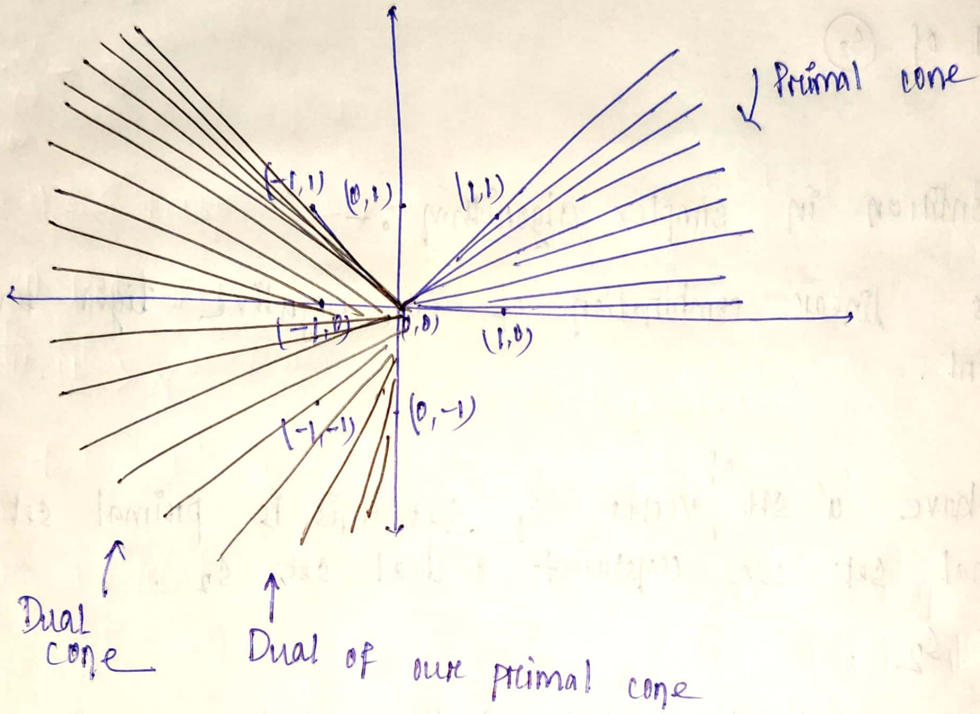



Let us take a 2D space. consider 2 vectors (1,0), (1,1). Let us look at all non -ve linear combination of these vectors.

This linear combination will form a cone, called primal space /cone.

→ This is not a vector space.

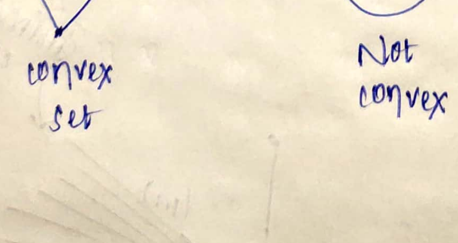
→ The dual of that cone is defined as the set of vectors whose dot product with any vector in the primal cone is at most '0' i.e. "≤ 0".




Claim :- Dual of Dual set = Primal

Convex set

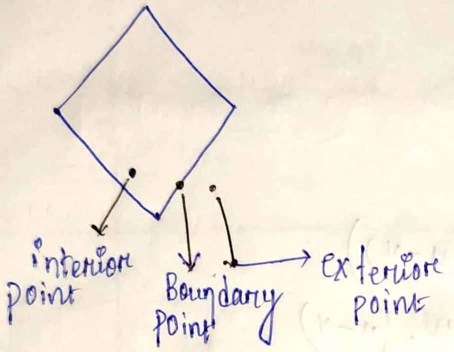
Let M be a set of points. Take 2 points on the line joining u,v. If all the points u,v belongs to set M, then M is a convex set.



## Interior point

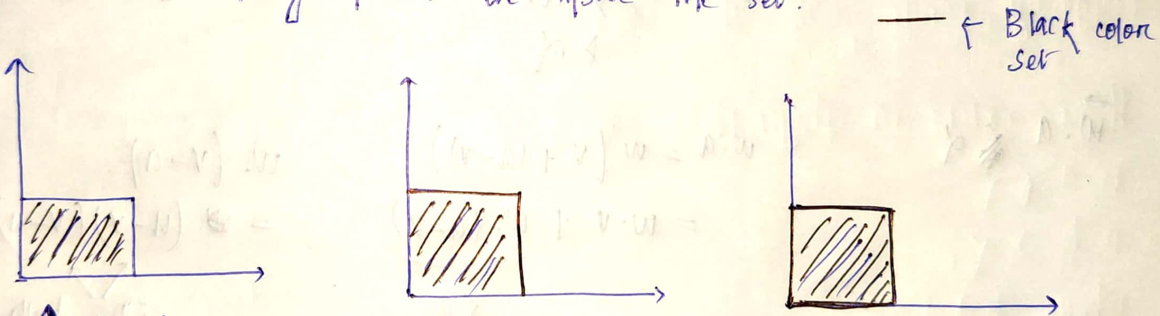
Take a point, draw a small ball around it. If the entire ball lies inside the set, then that point is an interior point.

- Exterior point
- Boundary point




## closed convex set

If all the boundary points lie inside the set.




Not closed convex set

Neither closed nor open convex set

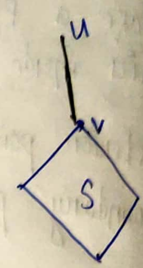
closed convex set

## Separating Hyperplane theorem

Suppose we have a closed convex set, and some point outside the set, then we will have a hyperplane which separates the closed convex set from the outside point.

## Proof

let $S$ be a convex set and $u$ be a point out side $S$.
Let 'v' be a boundary point of $S$ closest to $u$.




$\vec{w} = u - v$

$\vec{w} \cdot v = \alpha$

~~we~~ ~~or~~

let $a \in S$ and $a \neq v$

we can find that :

① $\vec{w} \cdot u > \alpha$

$$w \cdot u = w \cdot \left(v + (u - v)\right)$$
$$= w \cdot v + w \cdot (u - v)$$
$$= w \cdot v + \underbrace{w \cdot w}$$

$\Rightarrow w \cdot u > w \cdot v$ This will be always +ve

$> \alpha$

② $\vec{w} \cdot a \leq \alpha$

$$w \cdot a = w\left(v + (a - v)\right)$$
$$= w \cdot v + \underbrace{w \cdot (a - v)}_{\leq 0}$$

$\Rightarrow w \cdot a \leq w \cdot v$

$\leq \alpha$

$w \cdot (v - a)$

$= (u - v)(v - a)$

either both +ve or both −ve

$\Rightarrow w \cdot (v - a) \geq 0$

$\Rightarrow w \cdot (a - v) \leq 0$

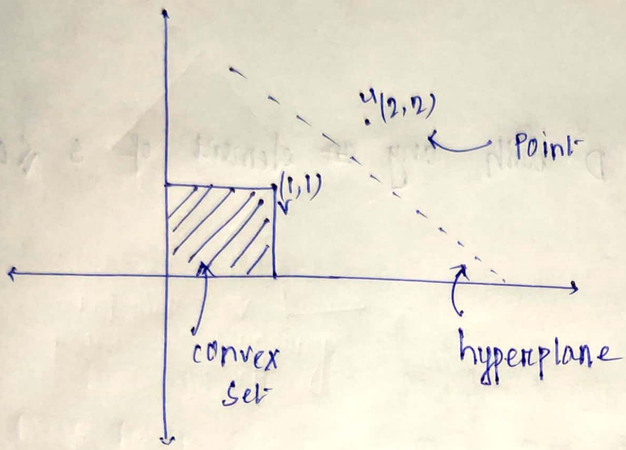
Separating hyperplane theorem says that :
Given $S, u, v$ we can find $\alpha, w$ such that $w \cdot a \leq \alpha$ & $w \cdot u > \alpha$

↑ hyperplane

~~Claim :- Dual of Dual is primal.~~

~~Proof- (Proof by contradiction)~~

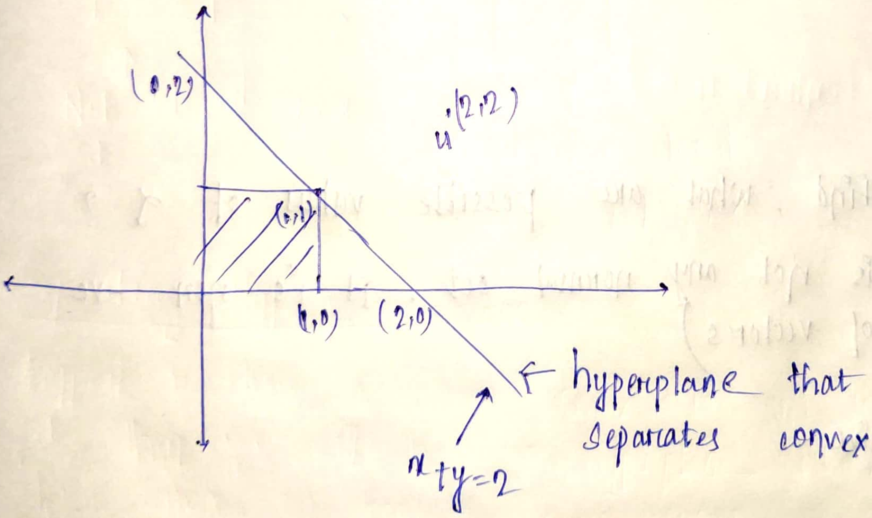
Ex] (Separating hyperplane thm)




$w = u - v = (2,2) - (1,1) = (1,1)$

$\underbrace{x + y = \boxed{2}}_{\text{hyperplane}}$

Substitute in the eqn




Here $\alpha = 2$

Claim :— Dual of dual is primal.

Proof

Let $S$ be primal set

$D$ is dual of $S$.

$D'$ is dual of $D$.

· Dot product of any element of $D$ with any ~~an~~ element of $S \leq 0$

(Proof by contradiction)

let $\exists\, b \in D'$ & $b \notin S$

→ $S$ is a closed convex set, because we have taken some vectors and $S$ is non -ve linear combinations of those vectors. (so boundary points will also included)

→ There exists a hyperplane $w, \alpha$ such that $\forall a \in S$

$w.a \leq \alpha$

$w.b > \alpha$

where $b \notin S$

· Now we want to find, what are possible values of $\alpha$ ?

(Because here 's' is not any normal set. It is non -ve linear combination of vectors)

① $\alpha$ can't be negative.

Proof

$S$ is non -ve linear combination of vectors. so $\vec{0} \in S$.

$w.0 = 0$.

if $\alpha$ -ve then $w.a < \alpha$ it don't include $\vec{0}$.

$\Rightarrow w.a$ is -ve (contradiction)

② $\alpha$ can't be +ve.

Proof

$$w.b > \alpha \quad \text{where } w \text{ is fixed, } b \text{ is fixed. Take } \overset{\text{constant}}{\overset{\downarrow}{w.b = \beta}}$$

if $\alpha$ +ve $\Rightarrow$ $w.b > 0$

$\beta > \alpha$

Take some $d \in S$ such that $w.d > \beta$

(This $d$ is possible in set $S$. Because $S$ is infinite set)

But above statement contradicts $\rightarrow$

$$\left.\begin{array}{l} w.d \not< \alpha \\ \alpha < \beta \end{array}\right\}$$

$\Rightarrow w.d < \beta$

$\therefore$ $\alpha = 0$

How? This also contradicts with $w.a' \not< \alpha$

$D' = S$

**Proof**

① $D' \subseteq S$

Prove by contradiction

Let take $D' \not\subseteq S$

$\Rightarrow \exists u \in D'$, $u \notin S$ Put $\alpha = 0$

$\exists a$ s.t $\forall a \in S$ $w.a \leq 0$

$w.u > 0$

Now we can say that $w \in D$ (dual of $s$). Because dot product of element of $w$ with any element of $s$ is at most 0.

Now $u \in D'$

$D'$ is dual of $D$

$\Rightarrow$ Every element of $D'$ has dot product $\leq 0$ with any element of $D$.

$\Rightarrow$ $w.u \leq 0$

But $w.u > 0$ (contradiction)

Therefore $D' \subseteq S$

② Similarly $S \subseteq D'$

$\therefore$ $D' = S$

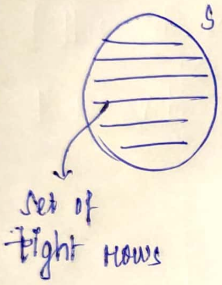
→ This theorem is basics for convex optimization.

Use of Primal dual theorem in simplex algorithm :—

~~In non-degenerate case~~, when is you at optimal point?

When cost vector is non -ve linear combination of tight rows.

⇒ cost function is in the cone of tight rows.




let $\underset{\substack{\uparrow \\ \text{cost fun}}}{c} \notin S$ $\Rightarrow$ $\forall a \in S$ $\quad w.a \leq 0$

$w.c > 0$

Then there exist a direction vector, when 'c' can be increase or decrease (depending on maximization or minimization problem) (moving in the direction of w)