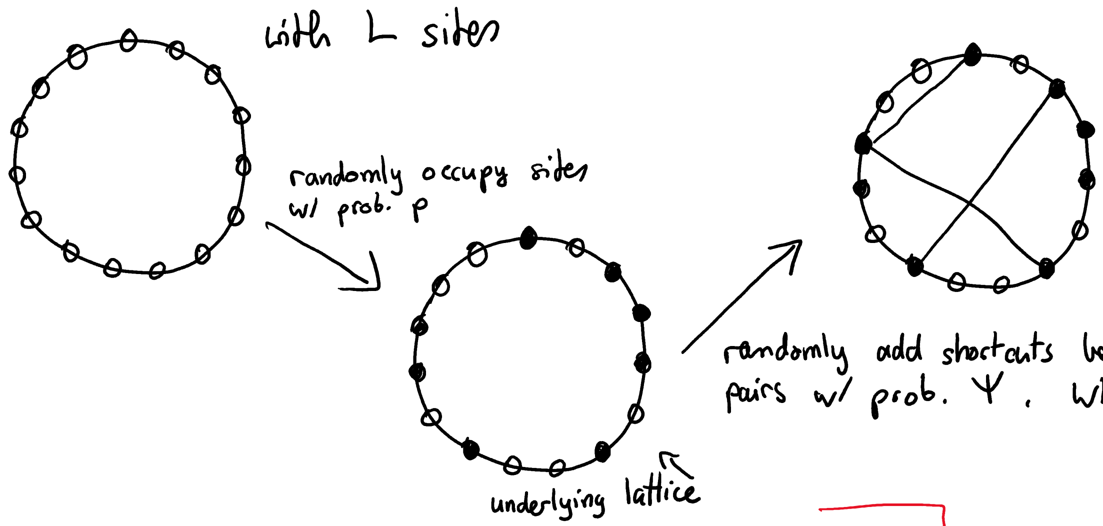
Percolation in Epidemics

Mittwoch, 6. Mai 2020 11:10

Site percolation

take a periodic 1D lattice with L sites



randomly add shortcuts between pairs w/ prob. Ψ. What is Ψ?

$$\boxed{\phi = \frac{\text{average number of shortcuts}}{\text{number of bonds in underlying lattice}} = \frac{\langle n_s \rangle}{n_b} = \frac{\langle n_s \rangle}{kL}}$$

Ψ = prob. of a randomly chosen pair having a shortcut

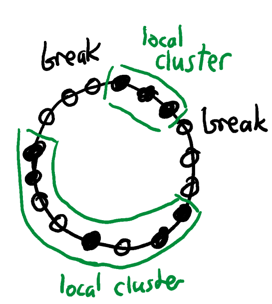
$$\Psi = 1 - \bar{\Psi} = 1 - \left(1 - \frac{2!}{L^2}\right)^{\langle n_s \rangle}$$

$$\bar{\Psi} = \text{prob. of not picking this pair when adding the shortcuts} = \Big(1 - \underbrace{\frac{2!}{L^2}}_{\text{prob. of picking a given pair}}\Big)^{\langle n_s \rangle}$$

We assume that $L \gg 1$, then

$$\boxed{\Psi = 1 - (1-x)^{\langle n_s \rangle}} \overset{x \to 0}{\underset{\text{Taylor Expansion}}{\approx}} \cancel{1} - \cancel{1} + \langle n_s \rangle x = \langle n_s \rangle \frac{2}{L^2} = \boxed{\frac{2\phi k}{L}}$$

What is the average number of local clusters of size i



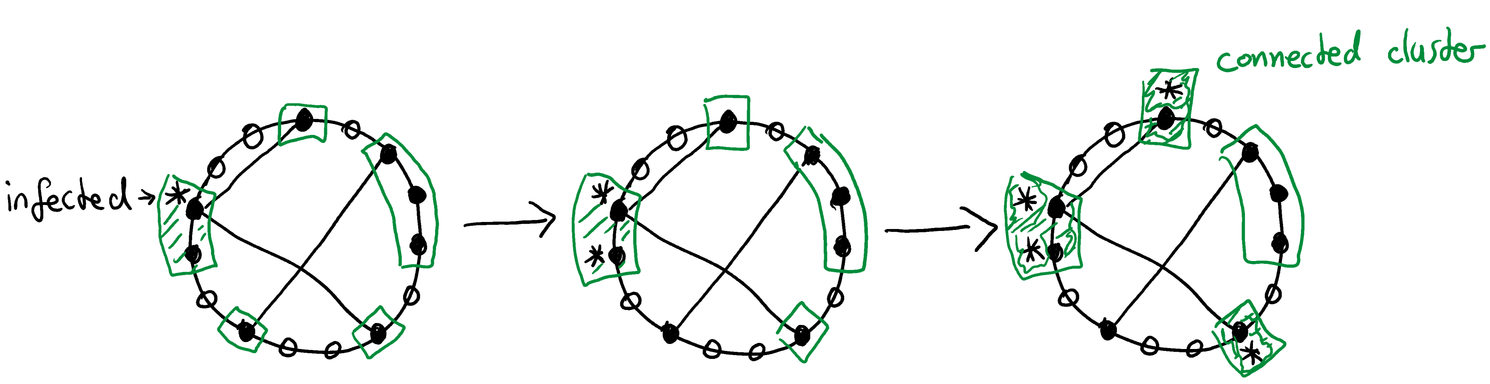
k = 3

Example

$$\boxed{\langle N_i \rangle = \text{prob. of local cluster having size } i \cdot L}$$

$$\boxed{= \underbrace{(1-p)^{2k}}_{\text{prob. of having breaks on the two edges}} \cdot \underbrace{p}_{\text{prob. of having one occupied}} \cdot \underbrace{\left(1-(1-p)^k\right)^{i-1}}_{\text{prob. of not having a break } i-1 \text{ times}} \cdot L}$$

Spreading of the disease



Adding a local cluster of size i to the connected cluster at step n.

$$\vec{v}_n = \begin{pmatrix} v_{n1} \\ v_{n2} \\ \vdots \\ v_{nL} \end{pmatrix} = \begin{pmatrix} \text{prob. of adding a local cluster of size 1 to the connected cluster} \\ \text{" " " " " " " " 2 " " " "} \\ \vdots \end{pmatrix}$$

for $p \leq p_c$ the prob. increases linearly with a transition Matrix M

$$\boxed{\vec{v}_{n+1} = M \vec{v}_n}$$ where M_{ij} = number of local clusters of size i connected to local clusters of size j

$$\Rightarrow \boxed{M_{ij}} = N_i \cdot (\text{prob. of a size } i \text{ cluster having a shortcut to a } j \text{ size cluster})$$

$$\boxed{= N_i \cdot (1 - \underbrace{(1-\Psi)^{ij}}_{\substack{\text{prob. of having no shortcut} \\ \text{between a pair } i \cdot j \text{ times}}})}$$

$\vec{v}_{n+1} = M \vec{v}_n$ is an iterative eq.

$$= M M \vec{v}_{n-1} = M^2 \vec{v}_{n-1}$$

$$\vdots$$

$$= M^n \vec{v}_0$$

express $\vec{v}_0$ in terms of Eigenvectors $\vec{\omega}_j$ of M

$$\vec{v}_{n+1} = M^n \sum_{j=1}^{L} a_j \vec{\omega}_j = \sum_{j=1}^{L} a_j \lambda^n \vec{\omega}_j$$

Perron-Frobenius theorem $\rightarrow$ M has a unique largest Eigenvalue λ_{max} (w/ positive Eigenvector)
and for large n, we have

$$\boxed{\vec{v}_{n+1} \approx \lambda_{max}^n \cdot a_{max} \vec{\omega}_{max}}$$

if $\lambda_{max} < 1$ $\Rightarrow$ $\vec{v}_{n+1} \rightarrow 0$ no spreading of desease

if $\lambda_{max} > 1$ $\Rightarrow$ $\vec{v}_{n+1}$ grows exponentially in λ_{max} $\rightarrow$ Epidemic

therefore

$\lambda_{max} = 1$ corresponds to the critical value at the separation between an epidemic and no epidemic.

Remember $\Psi = \dfrac{2\phi k}{L}$ for $L \gg 1$

If we fix ϕ and let $L \rightarrow \infty$, then $\Psi \rightarrow 0$ and we obtain via Taylor expansion

$$\boxed{M_{ij}} = N_i (1 - (1-\Psi)^{ij}) \approx N_i (1 - 1 + ij\Psi) = \boxed{N_i \, ij \, \Psi}$$

$$\Rightarrow M_{ij} \text{ can be written as } M = \vec{a} \underset{\text{outer product}}{\otimes} \vec{b}^T \text{ with } \vec{a} = \begin{pmatrix} N_1 \, 1 \\ N_2 \, 2 \\ \vdots \end{pmatrix} \Psi \text{ and } \vec{b} = \begin{pmatrix} 1 \\ 2 \\ \vdots \end{pmatrix}$$

$\Rightarrow M$ has rank 1 $\Rightarrow$ one non-zero Eigenvalue

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since $\vec{a} \otimes \vec{b}^T = \vec{a}\,\vec{b}^T = \begin{pmatrix} b_1 \vec{a} & b_2 \vec{a} & \cdots & b_L \vec{a} \end{pmatrix} \Rightarrow$ all columns are linearly dependent

The eigenvector corresponding to the non-zero eigenvalue is $\vec{a}$ since

$$\begin{pmatrix} b_1 \vec{a} & b_2 \vec{a} & \cdots & b_L \vec{a} \end{pmatrix} \cdot \vec{a} = b_1 a_1 \vec{a} + b_2 a_2 \vec{a} + \ldots = \underbrace{(\vec{a}\cdot\vec{b})}_{=\lambda}\,\vec{a}$$

So we have the Eigenvector $\vec{a}$ with $a_i = \Psi N_i\, i$ to the only non-zero Eigenvalue $\boxed{\lambda = \vec{a}\cdot\vec{b} = \Psi \sum_{j=1}^{L} N_j\, j^2}$

we insert $N_j = (1-p)^{2k}\, p\, (1-(1-p)^k)^{i-1}\, L$

and $\Psi = \frac{2\phi k}{L}$ for large L

$$\Rightarrow \lambda = \frac{2\phi k}{\cancel{L}} (1-p)^{2k}\, p\, \cancel{L} \sum_{j=1}^{L} j^2 q^{i-1}$$ where we set $q = 1-(1-p)^k$

$$= 2\phi k\,(1-q)^2 p \sum_{j=1}^{L} j^2 q^{i-1}$$ using geometric series (and „Mathematica")

$$\overset{L\to\infty}{\approx} 2\phi k\, \cancel{(1-q)^2}\, p\, \frac{1+q}{(1-q)^{\cancel{3}}}$$

$$= 2\phi k\, p\, \frac{1+1-(1-p)^k}{1-1+(1-p)^k} = 2\phi k\, p\, \frac{2-(1-p)^k}{(1-p)^k}$$

$$\Rightarrow \lambda = 2\phi k\, p\, \frac{2-(1-p)^k}{(1-p)^k}$$

$\lambda = 1$ corresponds to the critical probability p_c

$$\Rightarrow \boxed{1 = 2\phi k\, p_c\, \frac{2-(1-p_c)^k}{(1-p_c)^k}}$$