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Pseudofinite structures

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Dedication.

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Introduction

1. $F = F_{tree}$

1.1 Basic observations

Example 1. Let

$$\mathcal{G}_k = \{([k], E); E \text{ consists of exactly one } n/2\text{-clique}\},$$

and $F = F_{tree}$.

We will prove that for every $t > \mathbb{N}$:

$$\llbracket \Gamma \text{ has an } n^{1/t}\text{-clique} \rrbracket \quad (1.1)$$

$$= \llbracket (\exists \Lambda)(\forall u)(\forall v)((u, v \leq n^{1/t}) \rightarrow \Gamma(\Lambda(u), \Lambda(v))) \wedge (\Lambda : [n^{1/t}] \hookrightarrow \mathcal{M}) \rrbracket \quad (1.2)$$

$$= \bigvee_{\Lambda} \bigwedge_u \bigwedge_v \llbracket (u \neq v < n^{1/t}) \rightarrow (\Gamma(\Lambda(u), \Lambda(v)) \wedge \Lambda(u) \neq \Lambda(v)) \rrbracket \quad (1.3)$$

$$= \mathbf{1}. \quad (1.4)$$

For $j \in [n^{1/t}]$ let Λ_j to be a tree of depth $j \cdot (n^{1/t})^2$ which first tries to find an edge $1 \leftrightarrow k$ for $k \in [n^{1/t}]$ if it fails than it tries to find $2 \leftrightarrow k$ and so on. Once it finds some edge (i, k) , then it starts again but from $i + 1$ until it finds the first j elements of Δ_ω and responds with the j -th element. Since j is always bounded by $n^{1/t}$, Λ really sends F to F .

Example 2. Let

$$\mathcal{G}_k = \{([k], E); E \text{ consists of exactly one edge}\}$$

and $F = F_{tree}$.

We will prove that

$$\llbracket (\exists x)(\exists y)\Gamma(x, y) \rrbracket = \mathbf{0}.$$

Let T be any binary tree of depth $n^{1/t}$, $t > \mathbb{N}$, whose leaves are labeled by unordered pairs of edges.

Start from the root of T and always choose the path that corresponds to an edge not existing. At the end we obtain some answer, that gives us a set of at most $2 \cdot n^{1/t} + 2$ vertices. Now we can find at least:

$$\binom{n - 2n^{1/t} - 2}{2} = \frac{(n - 2n^{1/t} - 2)(n - 2n^{1/t} - 3)}{2} \quad (1.5)$$

$$=: m \quad (1.6)$$

different $\omega \in \Omega$ such that $T(\omega)$ is not an edge in ω . The standard part of the ratio the number of these counterexamples to $\text{st}(\frac{m}{|\mathcal{G}_n|}) = 1$.

This proves that the boolean value we are considering is $\mathbf{0}$ since we can combine the two witnesses for x and y into a tree that could find an edge with depth $n^{1/t}$ for some $t > \mathbb{N}$. \square

Theorem 1. Let $\varphi = (\forall \bar{x})\varphi_0(\bar{x})$ be a universal $\{E\}$ -sentence, such that

$$\lim_{k \rightarrow \infty} \Pr_{G \in \mathcal{G}_k} (G \models \varphi) = 1.$$

Then φ is valid in the pseudofinite limit.

Proof. From \aleph_1 -saturation of \mathcal{M} and our assumption, we know that for each $m \in \mathbb{N}$ there exists a $k_0 \in \mathbb{N}$ such that

$$\mathcal{M} \models (\forall k > k_0) \left(\Pr_{G \in \mathcal{G}_k} (G \models \varphi) > 1 - 1/m \right).$$

Therefore, since $n > \mathbb{N}$, we have that $\text{st}(\Pr_{G \in \mathcal{G}_n} (G \models \varphi)) = 1$ and therefore $\llbracket \varphi_0(\bar{\alpha}) \rrbracket = \mathbf{1}$ for each tuple $\bar{\alpha}$ in F .

Therefore

$$\llbracket \varphi \rrbracket = \bigwedge_{\bar{\alpha}} \llbracket \varphi_0(\bar{\alpha}) \rrbracket \tag{1.7}$$

$$= \bigwedge_{\bar{\alpha}} \mathbf{1} \tag{1.8}$$

$$= \mathbf{1}. \tag{1.9}$$

□

Theorem 2. Let $F = F_{tree}$. Let $\varphi_0(x_0, \dots, x_{l-1})$ be a q.f. $\{E\}$ -formula. Let $0 < p \leq 1$, consider subset $A \subseteq [g_k]^l$ such that for all $\bar{a} \in A$

$$\Pr_{G \in \mathcal{G}_k} (G \models \varphi_0(\bar{a})) \geq p$$

and

$$\{\{G \models \varphi_0(\bar{a})\} \subseteq \mathcal{G}_k; \bar{a} \in A\} \text{ are mutually independent.}$$

moreover let A_k be the set with the largest cardinality that has this property.

If $\lim_{k \rightarrow \infty} |A_k| = \infty$, then $\llbracket (\exists \bar{x})\varphi_0(\bar{x}) \rrbracket = \mathbf{1}$.

Proof. Let $\bar{x} = (x_0, \dots, x_{l-1})$. Let $T_{\bar{a}}$ be a tree of some standard depth d , that tests whether $G \models \varphi_0(\bar{a})$.

From \aleph_1 -saturation of \mathcal{M} we have $n' > \mathbb{N}$ many tuples $\bar{a}_0, \dots, \bar{a}_{n'-1} \in A_n$, such that $\Pr_{G \in \mathcal{G}_k} (G \models \varphi_0(\bar{a}_i)) \geq p$, we can assume $n' < n^{1/t_0}$ for some $t_0 > \mathbb{N}$.

For $j \in [l]$ construct a tree T_j inductively as follows: Start with $T_{\bar{a}_0}$. Replace the label of every accepting leaf by $(\bar{a}_0)_j$ and remove the label of every rejecting leaf. Call this tree T_j^0 . Assume we have already constructed T_j^m . Construct T_j^{i+1} by appending $T_{\bar{a}_{m+1}}$ to every undefined leaf, relabeling every satisfied leaf to $(\bar{a}_{i+1})_j$ and removing labels from every rejecting leaf. We will define T_j as $T_j^{n'}$ with undefined leafs labeled by 0. (This can be done, because all instances of induction are in $\text{Th}(\mathbb{N})$.) Note that $\text{dp}(T_j) = d \cdot n' < n^{1/t}$ for some $t > \mathbb{N}$.

Call $\bar{\alpha}$ the tuple computed by T_0, \dots, T_{l-1} . We will prove that probability of $\bar{\alpha}$ being a witness to $\varphi_0(\bar{x})$ is 1. For each \bar{a}_i we have, that the probability of $G \models \varphi_0(\bar{a}_i)$ is at least p . The mutual independence of $\{G \models \varphi_0(\bar{a}_i); i \in [n']\}$ and the construction of T_j implies that T_j has a probability of $(1 - p)^{n'}$ of failing, which is obviously almost 0. □

Example 3. Let

$\mathcal{G}_k = \{([k], E); E \text{ has at least one edge, and can have a second one } 0E1\},$

and let $F = F_{tree}$. Then $\mu(\llbracket (\exists x)(\exists y)\Gamma(x, y) \rrbracket) = \frac{1}{2}.$

Proof. Let T_0 be a tree that always outputs 0 and T_1 be a tree that always outputs 1. We can prove that $\llbracket \Gamma(0, 1) \rrbracket \geq \llbracket \Gamma(\alpha, \beta) \rrbracket$ for any α, β . \square

Example 4. Let

$\mathcal{G}_k = \{([k], E); E \text{ at least one edge, and may have exactly } k/2 \text{ more from start}\}$

and let $F = F_{tree}$. Then $\mu(\llbracket (\exists x)(\exists y)\Gamma(x, y) \rrbracket) = \frac{1}{2}.$

Definition 1. We say that $\{\mathcal{G}_k\}_{k=0}^\infty$ is **isomorphism closed**, if there is k_0 such that for every $k > k_0$ if we have that $G_1 \in \mathcal{G}_k$, $V_{G_2} = [g_k]$ and $G_1 \cong G_2$ then $G_2 \in \mathcal{G}_k$.

We say that $\{\mathcal{G}_k\}_{k=0}^\infty$ is **categorical** if there is k_0 such that for every $k > k_0$ if we have $G_1, G_2 \in \mathcal{G}_k$ then $G_1 \cong G_2$. For a categorical sequence $\{\mathcal{G}_k\}_{k=0}^\infty$ we denote G_k the lexicographically minimal element of \mathcal{G}_k .

Lemma 1. Let $\{\mathcal{G}_k\}_{k=0}^\infty$ be categorical and isomorphism closed, then for large enough k

$$|\mathcal{G}_k| = \frac{g_k!}{|\text{Aut}(G_k)|}.$$

Proof. Every $\rho \in S_{g_k}$ defines an isomorphism $\rho : G_k \rightarrow \rho(G_k)$, where $\rho(G_k)$ is a graph obtained from G_k by renaming every vertex v to $\rho(v)$.

Claim: For any $\rho, \pi \in S_{g_k}$:

$$\rho(G_k) = \pi(G_k) \iff \exists \tau \in \text{Aut}(G_k) : \rho \circ \tau = \pi.$$

Proof of claim. “ \Rightarrow ” Let $\rho(G_k) = \pi(G_k)$, therefore $\tau := \rho^{-1} \circ \pi \in \text{Aut}(G_k)$ and $\rho \circ \tau = \rho \circ \rho^{-1} \circ \pi = \pi$.

“ \Leftarrow ” Let $\rho \circ \tau = \pi$. Then $\pi(G_k) = \rho(\tau(G_k)) = \rho(G_k)$. \square

Notice that the τ in the statement of the claim is uniquely determined by $\rho^{-1} \circ \pi$. Therefore if we defined a quotient set S_{g_k} / \sim with $\rho \sim \pi \iff \rho(G_k) = \pi(G_k)$ then $|S_{g_k} / \sim| = \frac{g_k!}{|\text{Aut}(G_k)|}$.

The Lemma follows from noticing that if we start with $\{G_k\}$ and then we build \mathcal{G}_k by finding isomorphic graphs on the vertex set $[g_k]$ we can only do so by trying different permutation from S_{g_k} and these permutations find the same graph if and only if they are in the same \sim -class. Therefore there is a bijection between S_{g_k} / \sim and \mathcal{G}_k . \square

Lemma 2 (Candidate for optimal search trees). Let $\{\mathcal{G}_k\}_{k=0}^\infty$ be categorical and isomorphism closed, let $\varphi(x_0, \dots, x_{l-1})$ be an open $\{E\}$ -formula, let $\models \varphi(\bar{x}) \rightarrow \bigwedge_{i,j=0}^{l-1, l-1} x_i =^{b_{ij}} x_j$ for some $b_{ij} \in \{0, 1\}$, let $k_0 \geq 0$ and define $\{q_k\}_{k=k_0}^\infty$ as follows

$$q_k := \frac{g_k!}{|\text{Aut}(G_k)|} \cdot \frac{|\varphi(G_k)|}{|\bigcup_{G \in \mathcal{G}} \varphi(G_k)|}.$$

Then there is $c \in \mathbb{N}$ and trees T_0, \dots, T_{l-1} of depth $n^{(r)} \cdot c$, (with $n^{(r)}$ being defined in the proof) such that for the $\bar{\alpha}$ computed by \bar{T} we have $\llbracket \varphi(\bar{\alpha}) \rrbracket = 1$.

Proof. We will use the identity from the statement to construct a search tree (iterated $T_{\bar{a}}$) which almost always finds a witness to φ .

We will analyze the problem in the finite case for big enough $k > 0$. We should only check those tuples included in $\bigcup_{G \in \mathcal{G}_k} \varphi(G)$. For example, if we are trying to find an edge then we need not check the constant tuples (a, a) . Moreover, to succeed we only need to check one specific tuple in each $\varphi(G)$, $G \in \mathcal{G}_k$.

Consider the set $S = \{(G, \bar{a}); G \in \mathcal{G}_k, G \models \varphi(\bar{a})\}$ and a projection to the second coordinate $p_2 : S \rightarrow \bigcup_{G \in \mathcal{G}_k} \varphi(G)$. Since $|S| = \frac{q_k!}{|\text{Aut}(G)|} \cdot |\varphi(G_k)|$ we have that q_k is the average size of a p_2 preimage of any $\bar{a} \in \bigcup_{G \in \mathcal{G}_k} \varphi(G)$.

Claim: For all $\bar{a}, \bar{b} \in \bigcup_{G \in \mathcal{G}_k} \varphi(G)$ we have $|p_2^{-1}[\bar{a}]| = |p_2^{-1}[\bar{b}]| = q_k$.

Proof of claim. We will prove that for any $\bar{a}, \bar{b} \in \bigcup_{G \in \mathcal{G}_k} \varphi(G)$ we have $|p_2^{-1}[\bar{a}]| \leq |p_2^{-1}[\bar{b}]|$, by symmetry, they must be equal and also equal to q_k which is the average size of any singleton preimage.

Let $p_2^{-1}[\bar{a}] = \{G_0, \dots, G_{s-1}\} \times \{\bar{a}\}$ and let $\rho = (b_0 a_0) \dots (b_{l-1} a_{l-1})$, this is a permutation from the condition on φ . Then

$$p_2^{-1}[\bar{b}] \supseteq \{\rho(G_0), \dots, \rho(G_{s-1})\} \times \{\rho(\bar{a}) = \bar{b}\}. \quad \square$$

Now consider the multiset $M = (\bigcup_{G \in \mathcal{G}_k} \varphi(G), \text{count} : \bar{a} \mapsto |p_2^{-1}[\bar{a}]|)$, we will construct the searching tree by plucking elements from this multiset in the following way.

Let $M^{(0)} := M$, $\mathcal{G}_k^{(0)} = \mathcal{G}_k$. For $i \geq 0$ and $M^{(i)}, \mathcal{G}_k^{(i)}$ built, take some $\bar{a} \in M^{(i)}$ with maximal count(\bar{a}), put $\mathcal{G}_k^{(i+1)} = \mathcal{G}_k^{(i)} \setminus p_2^{-1}[\bar{a}]$ and form $M^{(i+1)}$ by removing \bar{a} , and for every $\bar{b} \in p_1[p_2^{-1}[\bar{a}]] \setminus \{\bar{a}\}$ setting $\text{count}_{M^{(i+1)}}(\bar{b}) = \max\{0, \text{count}_{M^{(i)}}(\bar{b}) - (\varphi(G_k))\}$. We also add $T_{\bar{a}}$ to the leaves of the tree we are constructing T_i and call it T_{i+1} .

For each $i \geq 0$ we have that T_i finds a witness in $G \in \mathcal{G}_k$ iff $G \notin \mathcal{G}_k^{(i)}$. So to calculate the probability of success of T_i we just need to find upper bounds on the cardinality of $\mathcal{G}_k^{(i)}$.

Define $m_i := \max\{\text{count}(\bar{a}); \bar{a} \in M^{(i)}\}$. Let $k^{(0)} \geq 0$ be the greatest number such that for all $i < k^{(0)}$: $m_i = q_k$.

Define a set $M_m^{(i)} = \{\bar{a}; \text{count}_{M_i}(\bar{a}) = m_i\}$. We can see, that $k^{(0)} \geq 1$ and $M^{(0)} = \bigcup_{G \in \mathcal{G}_k} \varphi(G)$. At each step $i < k^{(0)}$ we construct T_{i+1} by searching for some $\bar{a} \in M_m^{(i)}$, this results in $|\mathcal{G}_k^{(i+1)}| = |\mathcal{G}_k^{(i)}| - q_k$. We also remove one instance of every $\bar{b} \in p_1[p_2^{-1}(\bar{a})] \setminus \{\bar{a}\}$ from $M^{(i)}$ to form $M^{(i+1)}$, this results in $|M_m^{(i+1)}| \geq |M_m^{(i)}| - 1 - q_k \cdot (|\varphi(G_k)| - 1)$.

Therefore

$$k^{(0)} \geq \left\lfloor \frac{|M_m^{(0)}|}{q_k \cdot |\varphi(G_k)|} \right\rfloor \tag{1.10}$$

$$= \left\lfloor \frac{|\bigcup_{G \in \mathcal{G}_k} \varphi(G)|}{q_k \cdot |\varphi(G_k)|} \right\rfloor, \tag{1.11}$$

$$\text{and } \left| \mathcal{G}_k^{(k^{(0)})} \right| = |\mathcal{G}_k| - k^{(0)} \cdot q_k = \frac{|\text{Aut}(G_k)|}{g_k!} - \left\lfloor \frac{|\bigcup_{G \in \mathcal{G}_k} \varphi(G)|}{q_k \cdot |\varphi(G_k)|} \right\rfloor \cdot q_k \leq \frac{|\text{Aut}(G_k)|}{g_k!} - \left\lfloor \frac{|\bigcup_{G \in \mathcal{G}_k} \varphi(G)|}{|\varphi(G_k)|} \right\rfloor.$$

However the right hand side of the last inequality is rarely ≤ 0 , so generally one has to continue with plucking even after $k^{(0)}$ -many steps. We define $k^{(j)}$ as the greatest number such that for all $i < k^{(j)} : m_i \geq q_k - j$ and continue for $k^{(r)}$ steps, where r is the smallest number such that

$$|\mathcal{G}_k^{k^{(r)}}| = |\mathcal{G}_k| - k^{(0)} \cdot q_k - \sum_{j=1}^r (k^{(j)} - k^{(j-1)}) \cdot (q_k - j) \quad (1.12)$$

$$= 0. \quad (1.13)$$

However, this requires a general analysis of $k^{(j)}$ and I haven't manage to compute that.

For $k = n$ in \mathcal{M} we put $n^{(r)} := k^{(r)}$. \square

1.2 $\mathcal{G}_k = \text{ALL}_k$

Theorem 3 (Everything exists). Let $\varphi(\bar{x}, \bar{y}) = \bigwedge_{i=0}^{m-1} \psi_i(\bar{x}, \bar{y}) \wedge \bigwedge_{i=0}^{l-1} \vartheta_i(\bar{y})$, where ψ_i, ϑ_i are literals and ψ_i are not of the form $(y_i = y_j)^b$, $E(y_i, y_j)^b$, $x_i \neq x_i$, $E(x_i, x_i)$, $b \in \{0, 1\}$.

Let $\bar{\beta}$ be a tuple of vertices computed by F_{tree} of the same length as \bar{y} . Then $\llbracket (\exists \bar{x}) \varphi^\Gamma(\bar{x}, \bar{\beta}) \rrbracket = \llbracket \bigwedge_{i=0}^{l-1} \vartheta_i(\bar{\beta}) \rrbracket$, specifically if $l = 0$ then $(\exists \bar{x}) \phi_0(\bar{x}, \bar{\beta})$ is valid in the b.v. graph.

Proof. We will construct one tree T computing the whole tuple of witnesses $\bar{\alpha}$, such a construction can be straightforwardly split into a tuple of tree each computing the specific element.

First we concatenate all the trees used to compute $\bar{\beta}$. At each leave we can now proceed knowing the value of $\bar{\beta}$ at the specific $\omega \in \Omega$. Now we just construct a tree as in Theorem 2 but searching only over edges not checked previously and only to fulfill each ψ_i . Luckily we have so far searched only an infinitesimal part of the edges and since we assume $\mathcal{G}_k = \text{ALL}_k$ both of the conditions of the theorem are satisfied, so by analogous argument, we have a tree that finds a witness all of the $\psi_i(\bar{x}, \bar{\beta})$ with probability infinitesimally close to 1.

Therefore

$$\llbracket (\exists \bar{x}) \varphi^\Gamma(\bar{x}, \bar{\beta}) \rrbracket = \llbracket (\exists \bar{x}) \bigwedge_{i=0}^{m-1} \psi_i(\bar{x}, \bar{\beta}) \rrbracket \wedge \llbracket \bigwedge_{i=0}^{l-1} \vartheta_i(\bar{\beta}) \rrbracket \quad (1.14)$$

$$= \llbracket \bigwedge_{i=0}^{l-1} \vartheta_i(\bar{\beta}) \rrbracket. \quad (1.15)$$

\square

Corollary 1. For each $\varphi(\bar{x})$ that is not a tautology in the theory of graphs we have that $\llbracket (\forall \bar{x}) \varphi^\Gamma(\bar{x}) \rrbracket = \mathbf{0}$.

Corollary 2. For each $\varphi(\bar{x}, \bar{y})$ that is not falsifiable by \bar{y} in the theory of graphs we have that $\llbracket (\forall \bar{y}) (\exists \bar{x}) \varphi^\Gamma(\bar{x}) \rrbracket = \mathbf{1}$.

Conclusion

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A. Attachments

A.1 First Attachment