FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

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Pseudofinite structures

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Study programme: Matemathics
Study branch: Mathemathics for Information technologies

Prague 2022

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Dedication.

Title: Pseudofinite structures

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Department: Department of Algebra

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Abstract: Abstract.

Keywords: key words

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Introduction

1. The limit

1.1 Basic examples

1.1.1 $\mathcal{G}_k = \mathbf{EDGE}_k$

We consider the classes of graphs

$$\mathrm{EDGE}_k := \{(k, E); |E| = 1\},$$

and we let $\mathcal{G}_k = \mathrm{EDGE}_k$ and $F = F_{tree}$.

Intuitivelly one should not be able to find the edge on a significant i.e. non-infinitesimal fraction of samples with a tree that is allowed to explore only an infinitesimal fraction of edges.

Theorem 1.1.1. We will prove that

$$[\![(\exists x)(\exists y)\Gamma(x, y)]\!] = \mathbf{0}.$$

Proof. Let $T \in F_{tree}$ be a tree of depth $n^{1/t}$, for some $t > \mathbb{N}$ that outputs a pair of numbers less than n.

Start from the root of T and always choose the path that corresponds to an edge not existing. At the end we obtain some answer, that gives us a set of at most $2 \cdot n^{1/t} + 2$ vertices. Now we can find at least:

$$\binom{n - 2n^{1/t} - 2}{2} = \frac{(n - 2n^{1/t} - 2)(n - 2n^{1/t} - 3)}{2} \tag{1.1}$$

different $\omega \in \Omega$ such that $T(\omega)$ is not an edge in ω.

The probability that any of those graphs is sampled is

$$\begin{aligned}
\frac{\binom{n-2n^{1/t}-2}{2}}{|\mathcal{G}_k|} &= \frac{(n - 2n^{1/t} - 2)(n - 2n^{1/t} - 3)}{n(n-1)} &(1.2)\\
&= \left(1 - \frac{2n^{1/t} - 2}{n}\right) \cdot \left(1 - \frac{2n^{1/t} - 3}{n-1}\right) &(1.3)\\
&\geq \left(1 - \frac{2n^{1/t} - 2}{n}\right)^2 &(1.4)\\
&\geq 1 - \frac{4n^{1/t} - 4}{n}. &(1.5)
\end{aligned}$$

And one can clearly see that $\mathrm{st}(1 - \frac{4n^{1/t}-4}{n}) = 1$. This proves that the boolean value we are considering is $\mathbf{0}$ since we can the two witnesses for x and y into a tree that could find an edge with depth $n^{1/t}$ for some $t > \mathbb{N}$. □

1.1.2 Sparse $\mathcal{G}_k$

One can see that in Theorem 1.1.1 the exact properties of graphs in $\mathcal{G}_k$ do not play a crucial role. If $\mathcal{G}_k$ consisted of all graphs on k verticies containing say exactly one triangle, or any other fixed subgraph of constant size, and no other edges, we would still find that the non-existence is valid in the limit graph.

A more general case would be to consider a family of graphs in which there is an infinitesimally small chance that two independent uniformly random verticies have an edge between. However, this is not sufficient.

Example 1.1.2. Let $\mathcal{G}_k$ consist of those graphs on the vertex-set k which contain the edge $E(0,1)$ and then has exactly one other edge.

As k increases, the number of edges get smaller than any standard positive fraction. But

$$[\![(\exists x)(\exists y)E(x,y)]\!] = \mathbf{1},$$

as witnessed by x being the constant 0 and y the constant 1 both of which are computable by a tree of depth 0.

One can see that having distinguished verticies can ruin the sparseness implying the non-existence of edges in the limit graph. We want to distiguish from this situation by considering the sequences $\mathcal{G}_k$ satisfying the following natural definition.

Definition 1.1.3. We say that $\{\mathcal{G}_k\}_{k=0}^{\infty}$ is **isomorphism closed**, if there is k_0 such that for every $k > k_0$ if we have that $G_1 \in \mathcal{G}_k$, $V_{G_2} = g_k$ and $G_1 \cong G_2$ then $G_2 \in \mathcal{G}_k$.

Theorem 1.1.4. Let an isomorphism closed $\mathcal{G}_k$ have the following property. There is a sequence $\{b_k\}_k$ and for big enough k, a uniformly sampled 2-element $\{u,v\} \subseteq g_k$ and every $G \in \mathcal{G}_k$ we have

$$\Pr[E_G(u,v)] \le b_k,$$

and some k_0 such that $\lim_{k\to\infty} k^{1/k_0} b_k = 0$. Then

$$[\![(\exists x)(\exists y)\Gamma(x,y)]\!] = \mathbf{0}.$$

Proof. Let us define the number $c_{u,v} := |\{G \in \mathcal{G}_k; E_G(u,v)\}|$, which is the number of graphs G in $\mathcal{G}_k$ satisfying $E_G(u,v)$. Of course $c_{u,u} = 0$ for every u.

Claim: Let $u \ne v, u' \ne v'$ be vertices, then $c_{u,v} = c_{u',v'}$.
proof of claim: Let $\rho := (u\,u')(v\,v')$ be a permutation with cycles $(u\,u')$ and $(v\,v')$. We can let ρ act on $\mathcal{G}_k$ by sending G to a graph $\rho(G)$ which renames the edges coordinate-wise.

The fact that $\mathcal{G}_k$ is isomorphism closed implies that ρ restricts to a bijection:

$$\rho' : \{G \in \mathcal{G}_k; E_G(u,v), \neg E_G(u',v')\} \to \{G \in \mathcal{G}_k; E_G(u',v'), \neg E_G(u,v)\}$$

which proves the claim. □

Now we define a matrix with entries

$$a_{G,\{u,v\}} := \begin{cases} 1 & E_G(u,v) \\ 0 & \text{otherwise} \end{cases}$$

where the rows are indexed by one of $|\mathcal{G}_k|$ many graphs in $\mathcal{G}_k$ and the columns are indexed by the $\frac{k(k-1)}{2}$ many 2-element sets of numbers in k. We take any distinct vertices u, v and define $p := \Pr_{G \in \mathcal{G}_k}[E_G(u,v)] = \frac{c_{u,v}}{|\mathcal{G}_k|}$, by the claim the choice of u, v does not matter.

The assumption from the statement is equivalent to the equality

$$\sum_{\{u,v\}} a_{G,\{u,v\}} \leq \frac{k(k-1)}{2} b_k$$

for every G. We combine this with the claim and the definition of p to get

$$\frac{k(k-1)}{2} |\mathcal{G}_k| \, p = \sum_{\{u,v\}} \sum_{G \in \mathcal{G}_k} a_{G,\{u,v\}} \tag{1.6}$$

$$= \sum_{G \in \mathcal{G}_k} \sum_{\{u,v\}} a_{G,\{u,v\}} \tag{1.7}$$

$$\leq |\mathcal{G}_k| \frac{k(k-1)}{2} b_k \tag{1.8}$$

which implies

$$p \leq b_k.$$

Now let $k := n$ and let $T \in F_{tree}$ be a tree of depth $n^{1/t}$ for some $t > \mathbb{N}$, where every leaf of T is labeled by some edge. Walk down the tree T from the root by answering negatively to every edge, this gives us a set E_T of all edges T inspected or outputed and $|E_T| \leq n^{1/t} + 1$.

Now we just need to prove that the probability T find an edge is infinitesimally small. This is enough to prove the theorem, since the trees computing any two witnesses for x and y in the statement can be combined to construct T and if any tree T succeeds with only infinitesimally small probability, no random vertices can witness an edge on a set of non-zero measure.

For that we just use the fact that $p \leq b_n$ and derive

$$\Pr_{G \in \mathcal{G}_n}[T \text{ finds an edge}] \leq \sum_{\{u,v\} \in E_T} \Pr_{G \in \mathcal{G}_n}[E_G(u,v)] \tag{1.9}$$

$$= \sum_{\{u,v\} \in E_T} \frac{c_{u,v}}{|\mathcal{G}_n|} \tag{1.10}$$

$$\leq \sum_{\{u,v\} \in E_T} p \tag{1.11}$$

$$= (n^{1/t} + 1)p \tag{1.12}$$

$$\leq (n^{1/t} + 1) b_k \tag{1.13}$$

$$\leq n^{1/k_0} b_k \tag{1.14}$$

$$\approx 0. \tag{1.15}$$

□

2. $F = F_{tree}$

2.1 Basic observations

Example 2.1.1. Let

$$\mathcal{G}_k = \{([k], E); E \text{ consists of exactly one } n/2\text{-clique}\},$$

and $F = F_{tree}$.

We will prove that for every $t > \mathbb{N}$:

$$\begin{aligned}
&[\![\Gamma \text{ has an } n^{1/t}\text{-clique}]\!] && (2.1)\\
&= [\![(\exists \Lambda)(\forall u)(\forall v)(((u, v \leq n^{1/t}) \to \Gamma(\Lambda(u), \Lambda(v))) \wedge (\Lambda : [n^{1/t}] \hookrightarrow \mathcal{M}))]\!] && (2.2)\\
&= \bigvee_{\Lambda} \bigwedge_{u} \bigwedge_{v} [\![(u \neq v < n^{1/t}) \to (\Gamma(\Lambda(u), \Lambda(v)) \wedge \Lambda(u) \neq \Lambda(v))]\!] && (2.3)\\
&= \mathbf{1}. && (2.4)
\end{aligned}$$

For $j \in [n^{1/t}]$ let Λ_j to be a tree of depth $j \cdot (n^{1/t})^2$ which first tries to find an edge $1 \leftrightarrow k$ for $k \in [n^{1/t}]$ if it fails than it tries to find $2 \leftrightarrow k$ and so on. Once it finds some edge (i, k), then it starts again but from $i + 1$ until it finds the first j elelements of Δ_ω and responds with the j-th element. Since j is always bounded by $n^{1/t}$, Λ really sends F to F.

Example 2.1.2. Let

$$\mathcal{G}_k = \{([k], E); E \text{ consists of exactly one edge}\}$$

and $F = F_{tree}$.

We will prove that

$$[\![(\exists x)(\exists y)\Gamma(x, y)]\!] = \mathbf{0}.$$

Let T be any binary tree of depth $n^{1/t}$, $t > \mathbb{N}$, whose leaves are labeled by unordered pairs of edges.

Start from the root of T and always choose the path that corresponds to an edge not existing. At the end we obtain some answer, that gives us a set of at most $2 \cdot n^{1/t} + 2$ vertices. Now we can find at least:

$$\begin{aligned}
\binom{n - 2n^{1/t} - 2}{2} &= \frac{(n - 2n^{1/t} - 2)(n - 2n^{1/t} - 3)}{2} && (2.5)\\
&=: m && (2.6)
\end{aligned}$$

different $\omega \in \Omega$ such that $T(\omega)$ is not an edge in ω. The standard part of the ratio the number of these counterexamples to $\mathrm{st}(\frac{m}{|\mathcal{G}_n|}) = 1$.

This proves that the boolean value we are considering is $\mathbf{0}$ since we can combine the two witnesses for x and y into a tree that could find an edge with depth $n^{1/t}$ for some $t > \mathbb{N}$. □

Theorem 2.1.3. Let $\varphi = (\forall \overline{x})\varphi_0(\overline{x})$ be a universal $\{E\}$-sentence, such that

$$\lim_{k\to\infty} \Pr_{G\in\mathcal{G}_k}(G \models \varphi) = 1.$$

Then φ is valid in the b.v. structure.

Proof. From $\aleph_1$-saturation of $\mathcal{M}$ and our assumption, we know that for each $m \in \mathbb{N}$ there exists a $k_0 \in \mathbb{N}$ such that

$$\mathcal{M} \models (\forall k > k_0)\left(\Pr_{G\in\mathcal{G}_k}(G \models \varphi) > 1 - 1/m\right).$$

Therefore, since $n > \mathbb{N}$, we have that $\mathrm{st}(\Pr_{G\in\mathcal{G}_n}(G \models \varphi)) = 1$ and therefore $[\![\varphi_0(\overline{\alpha})]\!] = \mathbf{1}$ for each tuple $\overline{\alpha}$ in F.

Therefore

$$[\![\varphi]\!] = \bigwedge_{\overline{\alpha}} [\![\varphi_0(\overline{\alpha})]\!] \tag{2.7}$$
$$= \bigwedge_{\overline{\alpha}} \mathbf{1} \tag{2.8}$$
$$= \mathbf{1}. \tag{2.9}$$

□

Theorem 2.1.4. Let $F = F_{tree}$. Let $\varphi_0(x_0, \dots, x_{l-1})$ be a q.f. $\{E\}$-formula. Let $0 < p \leq 1$, consider subset $A \subseteq [g_k]^l$ such that for all $\overline{a} \in A$

$$\Pr_{G\in\mathcal{G}_k}(G \models \varphi_0(\overline{a})) \geq p$$

and

$$\{\{G \models \varphi_0(\overline{a})\} \subseteq \mathcal{G}_k; \overline{a} \in A\} \text{ are mutually independent.}$$

moreover let A_k be the set with the largest cardinality that has this property.

If $\lim_{k\to\infty} |A_k| = \infty$, then $[\![(\exists \overline{x})\varphi_0(\overline{x})]\!] = \mathbf{1}$.

Proof. Let $\overline{x} = (x_0, \dots, x_{l-1})$. Let $T_{\overline{a}}$ be a tree of some standard depth d, that tests whether $G \models \varphi_0(\overline{a})$.

From $\aleph_1$-saturation of $\mathcal{M}$ we have $n' > \mathbb{N}$ many tuples $\overline{a}_0, \dots, \overline{a}_{n'-1} \in A_n$, such that $\Pr_{G\in\mathcal{G}_k}(G \models \varphi_0(\overline{a}_i)) \geq p$, we can assume $n' < n^{1/t_0}$ for some $t_0 > \mathbb{N}$.

For $j \in [l]$ construct a tree T_j inductively as follows: Start with $T_{\overline{a}_0}$. Replace the label of every accepting leaf by $(\overline{a}_0)_j$ and remove the label of every rejecting leaf. Call this tree T_j^0. Assume we have already constructed T_j^m. Construct T_j^{i+1} by appending $T_{\overline{a}_{m+1}}$ to every undefined leaf, relabeling every satisfied leaf to $(\overline{a}_{i+1})_j$ and removing labels from every rejecting leaf. We will define T_j as $T_j^{n'}$ with undefined leafs labeled by 0. (This can be done, because all instances of induction are in $\mathrm{Th}(\mathbb{N})$.) Note that $\mathrm{dp}(T_j) = d \cdot n' < n^{1/t}$ for some $t > \mathbb{N}$.

Call $\overline{\alpha}$ the tuple computed by $T_0, \dots, T_{l-1}$. We will prove that probability of $\overline{\alpha}$ being a witness to $\varphi_0(\overline{x})$ is 1. For each $\overline{a}_i$ we have, that the probability of $G \models \varphi_0(\overline{a}_i)$ is at least p. The mutual independence of $\{G \models \varphi_0(\overline{a}_i); i \in [n']\}$ and the construction of T_j implies that T_j has a probability of $(1-p)^{n'}$ of failing, which is obviously almost 0. □

Example 2.1.5. Let

$$\mathcal{G}_k = \{([k], E); E \text{ has at least one edge, and can have a second one } 0E1\},$$

and let $F = F_{tree}$. Then $\mu([\![(\exists x)(\exists y)\Gamma(x, y)]\!]) = \frac{1}{2}$.

Proof. Let T_0 be a tree that always outputs 0 and T_1 be a tree that always outputs 1. We can prove that $[\![\Gamma(0, 1)]\!] \geq [\![\Gamma(\alpha, \beta)]\!]$ for any α, β. □

Example 2.1.6. Let

$$\mathcal{G}_k = \{([k], E); E \text{ at least one edge, and may have exactly } k/2 \text{ more from start}\}$$

and let $F = F_{tree}$. Then $\mu([\![(\exists x)(\exists y)\Gamma(x, y)]\!]) = \frac{1}{2}$.

Definition 2.1.7. We say that $\{\mathcal{G}_k\}_{k=0}^{\infty}$ is **isomorphism closed**, if there is k_0 such that for every $k > k_0$ if we have that $G_1 \in \mathcal{G}_k$, $V_{G_2} = [g_k]$ and $G_1 \cong G_2$ then $G_2 \in \mathcal{G}_k$.

We say that $\{\mathcal{G}_k\}_{k=0}^{\infty}$ is **categorical** if there is k_0 such that for every $k > k_0$ if we have $G_1, G_2 \in \mathcal{G}_k$ then $G_1 \cong G_2$. For a categorical sequence $\{G_k\}_{k=0}^{\infty}$ we denote G_k the lexicographically minimal element of $\mathcal{G}_k$.

Lemma 2.1.8. Let $\{\mathcal{G}_k\}_{k=0}^{\infty}$ be categorical and isomorphism closed, then for large enough k

$$|\mathcal{G}_k| = \frac{g_k!}{|\mathrm{Aut}(G_k)|}.$$

Proof. Every $\rho \in S_{g_k}$ defines an isomorphism $\rho : G_k \to \rho(G_k)$, where $\rho(G_k)$ is a graph obtained from G_k by renaming every vertex v to $\rho(v)$.

Claim: For any $\rho, \pi \in S_{g_k}$:

$$\rho(G_k) = \pi(G_k) \iff \exists \tau \in \mathrm{Aut}(G_k) : \rho \circ \tau = \pi.$$

Proof of claim. "$\Rightarrow$" Let $\rho(G_k) = \pi(G_k)$, therefore $\tau := \rho^{-1} \circ \pi \in \mathrm{Aut}(G_k)$ and $\rho \circ \tau = \rho \circ \rho^{-1} \circ \pi = \pi$.

"$\Leftarrow$" Let $\rho \circ \tau = \pi$. Then $\pi(G_k) = \rho(\tau(G_k)) = \rho(G_k)$. □

Notice that the τ in the statement of the claim is uniquely determinted by $\rho^{-1} \circ \pi$. Therefore if we defined a quotient set $S_{g_k}/\sim$ with $\rho \sim \pi \iff \rho(G_k) = \pi(G_k)$ then $|S_{g_k}/\sim| = \frac{g_k!}{|\mathrm{Aut}(G_k)|}$.

The Lemma follows from noticing that if we start with $\{G_k\}$ and then we build $\mathcal{G}_k$ by finding isomorphic graphs on the vertex set $[g_k]$ we can only do so by trying different permutation from S_{g_k} and these permutations find the same graph if and only if they are in the same $\sim$-class. Therefore there is a bijection between $S_{g_k}/\sim$ and $\mathcal{G}_k$. □

Lemma 2.1.9 (Candidate for optimal search trees)**.** Let $\{\mathcal{G}_k\}_{k=0}^{\infty}$ be categorical and isomorphism closed, let $\varphi(x_0, \ldots, x_{l-1})$ be an open $\{E\}$-formula, let $\models \varphi(\overline{x}) \to \bigwedge_{i,j=0}^{l-1,l-1} x_i =^{b_{ij}} x_j$ for some $b_{ij} \in \{0, 1\}$, let $k_0 \geq 0$ and define $\{q_k\}_{k=k_0}^{\infty}$ as follows

$$q_k := \frac{g_k!}{|\mathrm{Aut}(G_k)|} \cdot \frac{|\varphi(G_k)|}{|\bigcup_{G \in \mathcal{G}} \varphi(G_k)|}.$$

Then there is $c \in \mathbb{N}$ and trees $T_0, \ldots, T_{l-1}$ of depth $n^{(r)} \cdot c$, (with $n^{(r)}$ being defined in the proof) such that for the $\overline{\alpha}$ computed by $\overline{T}$ we have $[\![\varphi(\overline{\alpha})]\!] = \mathbf{1}$.

Proof. We will use the identity from the statement to construct a search tree (iterated $T_{\overline{a}}$) which almost always finds a witness to φ.

We will analyze the problem in the finite case for big enough $k > 0$. We should only check those tuples included in $\bigcup_{G \in \mathcal{G}_k} \varphi(G)$. For example, if we are trying to find an edge then we need not check the constant tuples (a, a). Moreover, to succeed we only need to check one specific tuple in each $\varphi(G), G \in \mathcal{G}_k$.

Consider the set $S = \{(G, \overline{a}); G \in \mathcal{G}_k, G \models \varphi(\overline{a})\}$ and a projection to the second coordinate $p_2 : S \to \bigcup_{G \in \mathcal{G}_k} \varphi(G)$. Since $|S| = \frac{g_k!}{|\mathrm{Aut}(G)|} \cdot |\varphi(G_k)|$ we have that q_k is the average size of a p_2 preimage of any $\overline{a} \in \bigcup_{G \in \mathcal{G}_k} \varphi(G)$.

Claim: For all $\overline{a}, \overline{b} \in \bigcup_{G \in \mathcal{G}_k} \varphi(G)$ we have $\left|p_2^{-1}[\overline{a}]\right| = \left|p_2^{-1}[\overline{b}]\right| = q_k$.

Proof of claim. We will prove that for any $\overline{a}, \overline{b} \in \bigcup_{G \in \mathcal{G}_k} \varphi(G)$ we have $\left|p_2^{-1}[\overline{a}]\right| \leq \left|p_2^{-1}[\overline{b}]\right|$, by symmetry, they must be equal and also equal to q_k which is the average size of any singleton preimage.

Let $p_2^{-1}[\overline{a}] = \{G_0, \ldots, G_{s-1}\} \times \{\overline{a}\}$ and let $\rho = (b_0\ a_0) \ldots (b_{l-1}\ a_{l-1})$, this is a permutation from the condition on φ. Then

$$p_2^{-1}[\overline{b}] \supseteq \{\rho(G_0), \ldots, \rho(G_{s-1})\} \times \{\rho(\overline{a}) = \overline{b}\}. \quad \square$$

Now consider the multiset $M = (\bigcup_{G \in \mathcal{G}_k} \varphi(G), \mathrm{count} : \overline{a} \mapsto \left|p_2^{-1}[\overline{a}]\right|)$, we will construct the searching tree by plucking elements from this multiset in the following way.

Let $M^{(0)} := M, \mathcal{G}_k^{(0)} = \mathcal{G}_k$. For $i \geq 0$ and $M^{(i)}, \mathcal{G}_k^{(i)}$ built, take some $\overline{a} \in M^{(i)}$ with maximal $\mathrm{count}(\overline{a})$, put $\mathcal{G}_k^{(i+1)} = \mathcal{G}_k^{(i)} \setminus p_2^{-1}[\overline{a}]$ and form $M^{(i+1)}$ by removing $\overline{a}$, and for every $\overline{b} \in p_1[p_2^{-1}[\overline{a}]] \setminus \{\overline{a}\}$ setting $\mathrm{count}_{M^{(i+1)}}(\overline{b}) = \max\{0, \mathrm{count}_{M^{(i)}}(\overline{b}) - (\varphi(G_k))\}$. We also add $T_{\overline{a}}$ to the leaves of the tree we are constructing T_i and call it T_{i+1}.

For each $i \geq 0$ we have that T_i finds a witness in $G \in \mathcal{G}_k$ iff $G \notin \mathcal{G}_k^{(i)}$. So to calculate the probability of success of T_i we just need to find upper bounds on the cardinality of $\mathcal{G}_k^{(i)}$.

Define $m_i := \max\{\mathrm{count}(\overline{a}); \overline{a} \in M^{(i)}\}$. Let $k^{(0)} \geq 0$ be the greatest number such that for all $i < k^{(0)}$: $m_i = q_k$.

Define a set $M_m^{(i)} = \{\overline{a}; \mathrm{count}_{M_i}(\overline{a}) = m_i\}$. We can see, that $k^{(0)} \geq 1$ and $M^{(0)} = \bigcup_{G \in \mathcal{G}_k} \varphi(G)$. At each step $i < k^{(0)}$ we construct T_{i+1} by searching for some $\overline{a} \in M_m^{(i)}$, this results in $\left|\mathcal{G}_k^{(i+1)}\right| = |\mathcal{G}_k^i| - q_k$. We also remove one instance of every $\overline{b} \in p_1[p_2^{-1}(\overline{a})] \setminus \{\overline{a}\}$ from $M^{(i)}$ to form $M^{(i+1)}$, this results in $\left|M_m^{(i+1)}\right| \geq \left|M_m^{(i)}\right| - 1 - q_k \cdot (|\varphi(G_k) - 1|)$.

Therefore

$$k^{(0)} \geq \left\lfloor \frac{\left|M_m^{(0)}\right|}{q_k \cdot |\varphi(G_k)|} \right\rfloor \tag{2.10}$$

$$= \left\lfloor \frac{\left|\bigcup_{G \in \mathcal{G}_k} \varphi(G)\right|}{q_k \cdot |\varphi(G_k)|} \right\rfloor, \tag{2.11}$$

and $\left|\mathcal{G}_k^{(k^{(0)})}\right| = |\mathcal{G}_k| - k^{(0)} \cdot q_k = \frac{|\mathrm{Aut}(G_k)|}{g_k!} - \left\lfloor \frac{\left|\bigcup_{G \in \mathcal{G}_k} \varphi(G)\right|}{q_k \cdot |\varphi(G_k)|} \right\rfloor \cdot q_k \leq \frac{|\mathrm{Aut}(G_k)|}{g_k!} - \left\lfloor \frac{\left|\bigcup_{G \in \mathcal{G}_k} \varphi(G)\right|}{|\varphi(G_k)|} \right\rfloor$.

However the right hand side of the last inequality is rarely ≤ 0, so generally one has to continue with plucking even after $k^{(0)}$-many steps. We define $k^{(j)}$ as the greatest number such that for all $i < k^{(j)} : m_i \geq q_k - j$ and continue for $k^{(r)}$ steps, where r is the smallest number such that

$$\left|\mathcal{G}_k^{k^{(r)}}\right| = |\mathcal{G}_k| - k^{(0)} \cdot q_k - \sum_{j=1}^{r}(k^{(j)} - k^{(j-1)}) \cdot (q_k - j) \tag{2.12}$$

$$= 0. \tag{2.13}$$

However, this requires a general analysis of $k^{(j)}$ and I haven't manage to compute that.

For $k = n$ in $\mathcal{M}$ we put $n^{(r)} := k^{(r)}$. □

2.2 $\mathcal{G}_k = \mathbf{ALL}_k$

Theorem 2.2.1 (Everything exists)**.** Let $\varphi(\overline{x}, \overline{y}) = \bigwedge_{i=0}^{m-1} \psi_i(\overline{x}, \overline{y}) \wedge \bigwedge_{i=0}^{l-1} \vartheta_i(\overline{y})$, where ψ_i, ϑ_i are literals and ψ_i are not of the form $(y_i = y_j)^b$, $E(y_i, y_j)^b$, $x_i \neq x_i$, $E(x_i, x_i)$, $b \in \{0, 1\}$.

Let $\overline{\beta}$ be a tuple of vertices computed by F_{tree} of the same length as $\overline{y}$. Then $[\![(\exists \overline{x})\varphi^\Gamma(\overline{x}, \overline{\beta})]\!] = [\![\bigwedge_{i=0}^{l-1} \vartheta_i(\overline{\beta})]\!]$, specifically if $l = 0$ then $(\exists \overline{x})\phi_0(\overline{x}, \overline{\beta})$ is valid in the b.v. graph.

Proof. We will construct one tree T computing the whole tuple of witnesses $\overline{\alpha}$, such a construction can be straightforwardly split into a tuple of tree each computing the specific element.

First we concatenate all the trees used to compute $\overline{\beta}$. At each leave we can now proceed knowing the value of $\overline{\beta}$ at the specific $\omega \in \Omega$. Now we just construct a tree as in Theorem 2.1.4 but searching only over edges not checked previously and only to fulfill each ψ_i. Luckily we have so far searched only an infinitesimal part of the edges and since we assume $\mathcal{G}_k = \mathrm{ALL}_k$ both of the conditions of the theorem are satisfied, so by analogous argument, we have a tree that finds a witness all of the $\psi_i(\overline{x}, \overline{\beta})$ with probability infinitesimally close to 1.

Therefore

$$[\![(\exists \overline{x})\varphi^\Gamma(\overline{x}, \overline{\beta})]\!] = [\![(\exists \overline{x}) \bigwedge_{i=0}^{m-1} \psi_i(\overline{x}, \overline{\beta})]\!] \wedge [\![\bigwedge_{i=0}^{l-1} \vartheta_i(\overline{\beta})]\!] \tag{2.14}$$

$$= [\![\bigwedge_{i=0}^{l-1} \vartheta_i(\overline{\beta})]\!]. \tag{2.15}$$

□

Corollary 2.2.2. For each $\varphi(\overline{x})$ that is not a tautology in the theory of graphs we have that $[\![(\forall \overline{x})\varphi^\Gamma(\overline{x})]\!] = \mathbf{0}$.

Corollary 2.2.3. For each $\varphi(\overline{x}, \overline{y})$ that is not falsifiable by $\overline{y}$ in the theory of graphs we have that $[\![(\forall \overline{y})(\exists \overline{x})\varphi^\Gamma(\overline{x})]\!] = \mathbf{1}$.

3. $F = F_{nbtree}$

3.1 $\mathcal{G}_k = *\mathbf{PATH}_k$

Definition 3.1.1. We define $*\text{PATH}_k$ (the pointed paths on k vertices) as the set of all (undirected) graphs G on the vertex set $[k]$, where G is isomorphic to the path on n vertices and $\deg_G(0) = 1$.

Definition 3.1.2. After we fix n, we define F_{nbtree} as the set of all functions computed by some some labeled tree with the following shape:

- Each non-leaf node is labeled by some $v \in [n]$.
- For each $\{u, v\} \subseteq [n]$ and a node N there is an outgoing edge from N labeled A.
- Each leaf is labeled by some $m \in \mathcal{M}_n$.
- The depth of the tree is at most $n^{1/t}$ for some $t > \mathbb{N}$.

Computation of such a tree on a undirected graph G goes as follows. We interpret the non-leaf nodes as questions "what is the neighbour set of v?" and the edges as answers from our graph G, and thus we follow a path determined by G until we find a vertex for which the answer is not an edge (in which case the computation returns 0) or until we find a leaf, in which case the computation returns the label of the leaf.

We now shift out focus to analysing the ability of trees from F_{nbtree} to find the non-zero degree 1 vertex in $G \in *\text{PATH}_n$. We say a tree $T \in F_{nbtree}$ fails at a graph G if $T(G)$ is not a non-zero vertex of degree one in G.

Definition 3.1.3. Let $m \leq n$ and $v \in [w]$ and $U \subseteq [w]$ with $|U| \leq 2$, then we define

$$\mathcal{G}_m^{v?=U} := \{G \in \mathcal{G}_m; N_G(v) = U\},$$

where N_G is the neighbour-set function of G.

Lemma 3.1.4. There are bijections for all nonstandard $m \leq n$ and distinct $u, v, w \in [m] \setminus \{0\}$:

$$\mathcal{G}_m^{v?=\{u,w\}} \cong \mathcal{G}_{m-2} \times [2] \tag{3.1}$$

$$\mathcal{G}_m^{v?=\{u,0\}} \cong \mathcal{G}_{m-2} \tag{3.2}$$

$$\mathcal{G}_m^{0?=\{u\}} \cong \mathcal{G}_{m-1}. \tag{3.3}$$

Proof. (sketch) For (3.1) we can just contract all of u,v,w into one vertex and relabel the rest of the graph, leaving the orientation as a one remaining bit of information. This is obviously reversible and a bijection.

For (3.2) we can do the same, but the orientation is given by 0. □

Lemma 3.1.5. Let $T \in F_{nbtree}$, with root labeled $v \in [m] \setminus 0$, we have for each $T_{v?=\{u,w\}}$ a tree $\tilde{T}_{v?=\{u,w\}}$ of the same depth, such that

$$P_m(T_{v?=\{u,w\}} \text{ fails}|v? = \{u,w\}) = P_{m-2}(\tilde{T}_{v?=\{u,w\}}). \tag{3.4}$$

For a tree T with the root labeled 0, we have a tree $\tilde{T}_{v?=\{u,w\}}$ of the same depth, such that

$$P_m(T_{v?=\{u\}} \text{ fails}|v? = \{u\}) = P_{m-1}(\tilde{T}_{v?=\{u\}}). \tag{3.5}$$

Proof. (sketch) To construct the tree, we just replace all vertices in labels of $T_{v?=\{u,w\}}$ by there renumbering from the bijection in (3.1).

(TODO: Elaborate) One can then check that the trees $T_{v?=\{u,w\}}$ and $\tilde{T}_{v?=\{u,w\}}$ are isomorphic in a sense that their computation of a graph G and $\tilde{G}$ respectively, $\tilde{G}$ being the corresponding $(m-2)$-vertex graph, agree with the structure of the path and that correctness of leaves is preserved under the renumbering. Essentially they emulate the same computation but on a smaller graph. □

Lemma 3.1.6. For all nonstandard $t > \mathbb{N}, m \geq n - 2n^{1/t}$ and $k \in [n^{1/t} + 1]$ for all trees $T \in F_{nbtree}$ of depth k we have

$$P_m(T \text{ fails}) \geq \prod_{i=0}^{k} \left(1 - \frac{2}{m-2i-2}\right).$$

Proof. We proceed by induction on k.

$k = 0$: We have that the probability of success of a straight guess is at most $\frac{1}{m-1}$. Therefore

$$P(T \text{ fails}) \geq \left(1 - \frac{1}{m-1}\right) \geq \left(1 - \frac{2}{m-2}\right). \tag{3.6}$$

$(k-1) \Rightarrow k$: First we assume that the root is labeled 0. Then we have

$$\begin{aligned}
P(T \text{ fails}) &= \sum_{u \in V \setminus \{0\}} P_{m-1}(0Eu) P_{m-1}(T_{0?=\{u\}} \text{ fails}|0Eu) && (3.7)\\
&\geq P_{m-1}(T_{0?=\{u\}} \text{ fails}|0Eu) && (3.8)\\
&= P_{m-1}(\tilde{T}_{0?=\{u\}} \text{ fails}) && (3.9)\\
&\geq \prod_{i=0}^{k-1} \left(1 - \frac{2}{m-2i-2}\right) && (3.10)\\
&\geq \prod_{i=0}^{k} \left(1 - \frac{2}{m-2i-2}\right). && (3.11)
\end{aligned}$$

Now we assume that the root is labeled $v \neq 0$. First we notice that

$$\begin{aligned}
P_m(vE0) &= \frac{1}{m-1} && (3.12)\\
P_m(N(V) = 1) &= \frac{1}{m-1} && (3.13)\\
P_m(|N(V) \setminus \{0\}| = 2) &= 1 - \frac{2}{m-1}, && (3.14)
\end{aligned}$$

the first two probabilities are obviously $\frac{1}{m-1}$ because they correspond to v being positioned on one of the ends of the non-zero segment which has length $m-1$. The event in (3.14) is the complement of the union of the first two events, which have empty intersetion, giving us that stated probability.

Then we have

$$\begin{aligned}
P_m(T \text{ fails}) &= P_m(vE0)P_m(T \text{ fails}|vE0) &(3.15)\\
&+ P_m(|N(v) \setminus \{0\}| = 2)P_m(T \text{ fails}|\,|N(v) \setminus \{0\}| = 2) &(3.16)\\
&+ P_m(|N(v)| = 1)P_m(vE0)P_m(T \text{ fails}|\,|N(v)| = 1) &(3.17)\\
&\geq P_m(|N(v) \setminus \{0\}| = 2)P_m(T \text{ fails}|\,|N(v) \setminus \{0\}| = 2) &(3.18)\\
&\geq (1 - \frac{2}{m-1}) &(3.19)\\
&\cdot \sum_{\substack{u,w \in V \setminus \{0\} \\ u \neq w}} P_m(v? = \{u,w\})P_m(T_{v?=\{u,w\}} \text{ fails}|v? = \{u,w\}) &(3.20)\\
&\geq (1 - \frac{2}{m-1})P_m(T_{v?=\{u_0,w_0\}} \text{ fails}|v? = \{u_0,w_0\}) &(3.21)\\
&\geq (1 - \frac{2}{m-1})P_{m-2}(\tilde{T}_{v?=\{u_0,w_0\}} \text{ fails}) &(3.22)\\
&\geq (1 - \frac{2}{m-1})\prod_{i=0}^{k-1}(1 - \frac{2}{m-2i-4}) &(3.23)\\
&\geq (1 - \frac{2}{m-2})\prod_{i=1}^{k}(1 - \frac{2}{m-2i-2}) &(3.24)\\
&\geq \prod_{i=0}^{k}(1 - \frac{2}{m-2i-2}). &(3.25)
\end{aligned}$$

where in (3.21) we choose u_0, w_0 with the lowest value of

$$P_m(T_{v?=\{u_0,w_0\}}|v? = \{u_0,w_0\}),$$

the bound follows the fact that all $P_m(v? = \{u,w\})$ are the same for distinct non-zero u, w. In (3.22) we use the lemma 3.1.5 and in (3.23) we use the induction hypothesis. □

Corollary 3.1.7. For a tree $T \in F_{nbtree}$ we have that

$$P_n(T \text{ fails}) \approx 1.$$

Proof. Since T has depth at most $n^{1/t}$ for some $t > \mathbb{N}$ we by the previous lemma that

$$\begin{aligned}
P_n(T \text{ fails}) &\geq \prod_{i=0}^{n^{1/t}} \left(1 - \frac{2}{n-2i-2}\right) &(3.26)\\
&\geq \left(1 - \frac{2n^{1/t}}{n - 2n^{1/t} - 2}\right) &(3.27)\\
&\approx 1. &(3.28)
\end{aligned}$$

□

Finally we can prove the following theorem.

Theorem 3.1.8.

$$[\![(\exists v)(\exists u)(\forall w)((v \neq 0) \wedge (\Gamma(v,u)) \wedge (\Gamma(v,w) \to u = w))]\!] = \mathbf{0}$$

Proof. Expanding the value of the formula in the statement we get

$$\bigvee_{\alpha} \bigvee_{\beta} \bigwedge_{\gamma} [\![(\alpha \neq 0) \wedge (\Gamma(\alpha,\beta)) \wedge (\Gamma(\alpha,\gamma) \to \beta = \gamma)]\!],$$

to prove it evalues to $\mathbf{0}$ we need to find for every α, β some γ such that

$$[\![(\alpha \neq 0) \wedge (\Gamma(\alpha,\beta)) \wedge (\Gamma(\alpha,\gamma) \to \beta = \gamma)]\!] = \mathbf{0}.$$

For any α, β we define

$$\gamma(\omega) := \begin{cases} v & N(\alpha(\omega)) = \{v\} \\ w & w \in N(\alpha(\omega)) \setminus \{\beta(\omega)\}, \end{cases}$$

such a function can be computed by a tree in F_{nbtree} which we can construct by concatenation of trees computing α and β.

Let T be the tree computing α. Now we proceed by contradiction, let

$$\epsilon := \mu([\![(\alpha \neq 0) \wedge (\Gamma(\alpha,\beta)) \wedge (\Gamma(\alpha,\gamma) \to \beta = \gamma)]\!]) > 0,$$

by definition this means that

$$\epsilon = \mathrm{st}(P_n[(\alpha \neq 0) \wedge (\Gamma(\alpha,\beta)) \wedge (\Gamma(\alpha,\gamma) \to \beta = \gamma)]) > 0.$$

But by definition of γ and Corollary 3.1.7 we have

$$\begin{aligned} 0 &< \epsilon \\ &= \mathrm{st}(P_n[(\alpha \neq 0) \wedge (\Gamma(\alpha,\beta)) \wedge (\Gamma(\alpha,\gamma) \to \beta = \gamma)]) \\ &\leq \mathrm{st}(P_n[(\alpha \neq 0) \wedge (\Gamma(\alpha,\beta)) \wedge |N(\alpha)| = 1]) \\ &\leq \mathrm{st}(P_n[(\alpha \neq 0) \wedge |N(\alpha)| = 1]) \\ &= \mathrm{st}(P_n[T \text{ does not fail}]) \\ &= 0. \end{aligned}$$

A contradiction. ∎

Conclusion

Bibliography