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**Pseudofinite structures**

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Dedication.

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# Introduction

# 1. The limit

## 1.1 Basic examples

### 1.1.1 $\mathcal{G}_k = \text{EDGE}_k$

We consider the classes of graphs

$$\text{EDGE}_k := \{(k, E); |E| = 1\},$$

and we let  $\mathcal{G}_k = \text{EDGE}_k$  and  $F = F_{\text{tree}}$ .

Intuitively one should not be able to find the edge on a significant i.e. non-infinitesimal fraction of samples with a tree that is allowed to explore only an infinitesimal fraction of edges.

**Theorem 1.1.1.** We will prove that

$$\llbracket (\exists x)(\exists y)\Gamma(x, y) \rrbracket = \mathbf{0}.$$

*Proof.* Let  $T \in F_{\text{tree}}$  be a tree of depth  $n^{1/t}$ , for some  $t > \mathbb{N}$  that outputs a pair of numbers less than  $n$ .

Start from the root of  $T$  and always choose the path that corresponds to an edge not existing. At the end we obtain some answer, that gives us a set of at most  $2 \cdot n^{1/t} + 2$  vertices. Now we can find at least:

$$\binom{n - 2n^{1/t} - 2}{2} = \frac{(n - 2n^{1/t} - 2)(n - 2n^{1/t} - 3)}{2} \quad (1.1)$$

different  $\omega \in \Omega$  such that  $T(\omega)$  is not an edge in  $\omega$ .

The probability that any of those graphs is sampled is

$$\frac{\binom{n - 2n^{1/t} - 2}{2}}{|\mathcal{G}_k|} = \frac{(n - 2n^{1/t} - 2)(n - 2n^{1/t} - 3)}{n(n - 1)} \quad (1.2)$$

$$= \left(1 - \frac{2n^{1/t} - 2}{n}\right) \cdot \left(1 - \frac{2n^{1/t} - 3}{n - 1}\right) \quad (1.3)$$

$$\geq \left(1 - \frac{2n^{1/t} - 2}{n}\right)^2 \quad (1.4)$$

$$\geq 1 - \frac{4n^{1/t} - 4}{n}. \quad (1.5)$$

And one can clearly see that  $\text{st}(1 - \frac{4n^{1/t} - 4}{n}) = 1$ . This proves that the boolean value we are considering is  $\mathbf{0}$  since we can the two witnesses for  $x$  and  $y$  into a tree that could find an edge with depth  $n^{1/t}$  for some  $t > \mathbb{N}$ .  $\square$

## 2. $F = F_{tree}$

### 2.1 Basic observations

**Example 2.1.1.** Let

$$\mathcal{G}_k = \{([k], E); E \text{ consists of exactly one } n/2\text{-clique}\},$$

and  $F = F_{tree}$ .

We will prove that for every  $t > \mathbb{N}$ :

$$\llbracket \Gamma \text{ has an } n^{1/t}\text{-clique} \rrbracket \quad (2.1)$$

$$= \llbracket (\exists \Lambda)(\forall u)(\forall v)((u, v \leq n^{1/t}) \rightarrow \Gamma(\Lambda(u), \Lambda(v))) \wedge (\Lambda : [n^{1/t}] \hookrightarrow \mathcal{M}) \rrbracket \quad (2.2)$$

$$= \bigvee_{\Lambda} \bigwedge_u \bigwedge_v \llbracket (u \neq v < n^{1/t}) \rightarrow (\Gamma(\Lambda(u), \Lambda(v)) \wedge \Lambda(u) \neq \Lambda(v)) \rrbracket \quad (2.3)$$

$$= \mathbf{1}. \quad (2.4)$$

For  $j \in [n^{1/t}]$  let  $\Lambda_j$  to be a tree of depth  $j \cdot (n^{1/t})^2$  which first tries to find an edge  $1 \leftrightarrow k$  for  $k \in [n^{1/t}]$  if it fails than it tries to find  $2 \leftrightarrow k$  and so on. Once it finds some edge  $(i, k)$ , then it starts again but from  $i + 1$  until it finds the first  $j$  elements of  $\Delta_\omega$  and responds with the  $j$ -th element. Since  $j$  is always bounded by  $n^{1/t}$ ,  $\Lambda$  really sends  $F$  to  $F$ .

**Example 2.1.2.** Let

$$\mathcal{G}_k = \{([k], E); E \text{ consists of exactly one edge}\}$$

and  $F = F_{tree}$ .

We will prove that

$$\llbracket (\exists x)(\exists y)\Gamma(x, y) \rrbracket = \mathbf{0}.$$

Let  $T$  be any binary tree of depth  $n^{1/t}$ ,  $t > \mathbb{N}$ , whose leaves are labeled by unordered pairs of edges.

Start from the root of  $T$  and always choose the path that corresponds to an edge not existing. At the end we obtain some answer, that gives us a set of at most  $2 \cdot n^{1/t} + 2$  vertices. Now we can find at least:

$$\binom{n - 2n^{1/t} - 2}{2} = \frac{(n - 2n^{1/t} - 2)(n - 2n^{1/t} - 3)}{2} \quad (2.5)$$

$$=: m \quad (2.6)$$

different  $\omega \in \Omega$  such that  $T(\omega)$  is not an edge in  $\omega$ . The standard part of the ratio the number of these counterexamples to  $\text{st}(\frac{m}{|\mathcal{G}_n|}) = 1$ .

This proves that the boolean value we are considering is  $\mathbf{0}$  since we can combine the two witnesses for  $x$  and  $y$  into a tree that could find an edge with depth  $n^{1/t}$  for some  $t > \mathbb{N}$ .  $\square$



**Theorem 2.1.3.** Let  $\varphi = (\forall \bar{x})\varphi_0(\bar{x})$  be a universal  $\{E\}$ -sentence, such that

$$\lim_{k \rightarrow \infty} \Pr_{G \in \mathcal{G}_k} (G \models \varphi) = 1.$$

Then  $\varphi$  is valid in the b.v. structure.

*Proof.* From  $\aleph_1$ -saturation of  $\mathcal{M}$  and our assumption, we know that for each  $m \in \mathbb{N}$  there exists a  $k_0 \in \mathbb{N}$  such that

$$\mathcal{M} \models (\forall k > k_0) \left( \Pr_{G \in \mathcal{G}_k} (G \models \varphi) > 1 - 1/m \right).$$

Therefore, since  $n > \aleph_1$ , we have that  $\text{st}(\Pr_{G \in \mathcal{G}_n} (G \models \varphi)) = 1$  and therefore  $\llbracket \varphi_0(\bar{\alpha}) \rrbracket = \mathbf{1}$  for each tuple  $\bar{\alpha}$  in  $F$ .

Therefore

$$\llbracket \varphi \rrbracket = \bigwedge_{\bar{\alpha}} \llbracket \varphi_0(\bar{\alpha}) \rrbracket \quad (2.7)$$

$$= \bigwedge_{\bar{\alpha}} \mathbf{1} \quad (2.8)$$

$$= \mathbf{1}. \quad (2.9)$$

□

**Theorem 2.1.4.** Let  $F = F_{tree}$ . Let  $\varphi_0(x_0, \dots, x_{l-1})$  be a q.f.  $\{E\}$ -formula. Let  $0 < p \leq 1$ , consider subset  $A \subseteq [g_k]^l$  such that for all  $\bar{a} \in A$

$$\Pr_{G \in \mathcal{G}_k} (G \models \varphi_0(\bar{a})) \geq p$$

and

$$\{\{G \models \varphi_0(\bar{a})\} \subseteq \mathcal{G}_k; \bar{a} \in A\} \text{ are mutually independent.}$$

moreover let  $A_k$  be the set with the largest cardinality that has this property.

If  $\lim_{k \rightarrow \infty} |A_k| = \infty$ , then  $\llbracket (\exists \bar{x})\varphi_0(\bar{x}) \rrbracket = \mathbf{1}$ .

*Proof.* Let  $\bar{x} = (x_0, \dots, x_{l-1})$ . Let  $T_{\bar{a}}$  be a tree of some standard depth  $d$ , that tests whether  $G \models \varphi_0(\bar{a})$ .

From  $\aleph_1$ -saturation of  $\mathcal{M}$  we have  $n' > \aleph_1$  many tuples  $\bar{a}_0, \dots, \bar{a}_{n'-1} \in A_n$ , such that  $\Pr_{G \in \mathcal{G}_k} (G \models \varphi_0(\bar{a}_i)) \geq p$ , we can assume  $n' < n^{1/t_0}$  for some  $t_0 > \aleph_1$ .

For  $j \in [l]$  construct a tree  $T_j$  inductively as follows: Start with  $T_{\bar{a}_0}$ . Replace the label of every accepting leaf by  $(\bar{a}_0)_j$  and remove the label of every rejecting leaf. Call this tree  $T_j^0$ . Assume we have already constructed  $T_j^m$ . Construct  $T_j^{i+1}$  by appending  $T_{\bar{a}_{m+1}}$  to every undefined leaf, relabeling every satisfied leaf to  $(\bar{a}_{i+1})_j$  and removing labels from every rejecting leaf. We will define  $T_j$  as  $T_j^{n'}$  with undefined leafs labeled by 0. (This can be done, because all instances of induction are in  $\text{Th}(\mathbb{N})$ .) Note that  $\text{dp}(T_j) = d \cdot n' < n^{1/t}$  for some  $t > \aleph_1$ .

Call  $\bar{\alpha}$  the tuple computed by  $T_0, \dots, T_{l-1}$ . We will prove that probability of  $\bar{\alpha}$  being a witness to  $\varphi_0(\bar{x})$  is 1. For each  $\bar{a}_i$  we have, that the probability of  $G \models \varphi_0(\bar{a}_i)$  is at least  $p$ . The mutual independence of  $\{G \models \varphi_0(\bar{a}_i); i \in [n']\}$  and the construction of  $T_j$  implies that  $T_j$  has a probability of  $(1 - p)^{n'}$  of failing, which is obviously almost 0. □

**Example 2.1.5.** Let

$\mathcal{G}_k = \{([k], E); E \text{ has at least one edge, and can have a second one } 0E1\},$

and let  $F = F_{tree}$ . Then  $\mu(\llbracket(\exists x)(\exists y)\Gamma(x, y)\rrbracket) = \frac{1}{2}$ .

*Proof.* Let  $T_0$  be a tree that always outputs 0 and  $T_1$  be a tree that always outputs 1. We can prove that  $\llbracket\Gamma(0, 1)\rrbracket \geq \llbracket\Gamma(\alpha, \beta)\rrbracket$  for any  $\alpha, \beta$ .  $\square$

**Example 2.1.6.** Let

$\mathcal{G}_k = \{([k], E); E \text{ at least one edge, and may have exactly } k/2 \text{ more from start}\}$

and let  $F = F_{tree}$ . Then  $\mu(\llbracket(\exists x)(\exists y)\Gamma(x, y)\rrbracket) = \frac{1}{2}$ .

**Definition 2.1.7.** We say that  $\{\mathcal{G}_k\}_{k=0}^\infty$  is **isomorphism closed**, if there is  $k_0$  such that for every  $k > k_0$  if we have that  $G_1 \in \mathcal{G}_k$ ,  $V_{G_2} = [g_k]$  and  $G_1 \cong G_2$  then  $G_2 \in \mathcal{G}_k$ .

We say that  $\{\mathcal{G}_k\}_{k=0}^\infty$  is **categorical** if there is  $k_0$  such that for every  $k > k_0$  if we have  $G_1, G_2 \in \mathcal{G}_k$  then  $G_1 \cong G_2$ . For a categorical sequence  $\{\mathcal{G}_k\}_{k=0}^\infty$  we denote  $G_k$  the lexicographically minimal element of  $\mathcal{G}_k$ .

**Lemma 2.1.8.** Let  $\{\mathcal{G}_k\}_{k=0}^\infty$  be categorical and isomorphism closed, then for large enough  $k$

$$|\mathcal{G}_k| = \frac{g_k!}{|\text{Aut}(G_k)|}.$$

*Proof.* Every  $\rho \in S_{g_k}$  defines an isomorphism  $\rho : G_k \rightarrow \rho(G_k)$ , where  $\rho(G_k)$  is a graph obtained from  $G_k$  by renaming every vertex  $v$  to  $\rho(v)$ .

**Claim:** For any  $\rho, \pi \in S_{g_k}$ :

$$\rho(G_k) = \pi(G_k) \iff \exists \tau \in \text{Aut}(G_k) : \rho \circ \tau = \pi.$$

*Proof of claim.* “ $\Rightarrow$ ” Let  $\rho(G_k) = \pi(G_k)$ , therefore  $\tau := \rho^{-1} \circ \pi \in \text{Aut}(G_k)$  and  $\rho \circ \tau = \rho \circ \rho^{-1} \circ \pi = \pi$ .

“ $\Leftarrow$ ” Let  $\rho \circ \tau = \pi$ . Then  $\pi(G_k) = \rho(\tau(G_k)) = \rho(G_k)$ .  $\square$

Notice that the  $\tau$  in the statement of the claim is uniquely determined by  $\rho^{-1} \circ \pi$ . Therefore if we defined a quotient set  $S_{g_k} / \sim$  with  $\rho \sim \pi \iff \rho(G_k) = \pi(G_k)$  then  $|S_{g_k} / \sim| = \frac{g_k!}{|\text{Aut}(G_k)|}$ .

The Lemma follows from noticing that if we start with  $\{G_k\}$  and then we build  $\mathcal{G}_k$  by finding isomorphic graphs on the vertex set  $[g_k]$  we can only do so by trying different permutation from  $S_{g_k}$  and these permutations find the same graph if and only if they are in the same  $\sim$ -class. Therefore there is a bijection between  $S_{g_k} / \sim$  and  $\mathcal{G}_k$ .  $\square$

**Lemma 2.1.9** (Candidate for optimal search trees). Let  $\{\mathcal{G}_k\}_{k=0}^\infty$  be categorical and isomorphism closed, let  $\varphi(x_0, \dots, x_{l-1})$  be an open  $\{E\}$ -formula, let  $\models \varphi(\bar{x}) \rightarrow \bigwedge_{i,j=0}^{l-1, l-1} x_i =^{b_{ij}} x_j$  for some  $b_{ij} \in \{0, 1\}$ , let  $k_0 \geq 0$  and define  $\{q_k\}_{k=k_0}^\infty$  as follows

$$q_k := \frac{g_k!}{|\text{Aut}(G_k)|} \cdot \frac{|\varphi(G_k)|}{|\bigcup_{G \in \mathcal{G}} \varphi(G_k)|}.$$

Then there is  $c \in \mathbb{N}$  and trees  $T_0, \dots, T_{l-1}$  of depth  $n^{(r)} \cdot c$ , (with  $n^{(r)}$  being defined in the proof) such that for the  $\bar{\alpha}$  computed by  $\bar{T}$  we have  $\llbracket\varphi(\bar{\alpha})\rrbracket = 1$ .

*Proof.* We will use the identity from the statement to construct a search tree (iterated  $T_{\bar{a}}$ ) which almost always finds a witness to  $\varphi$ .

We will analyze the problem in the finite case for big enough  $k > 0$ . We should only check those tuples included in  $\bigcup_{G \in \mathcal{G}_k} \varphi(G)$ . For example, if we are trying to find an edge then we need not check the constant tuples  $(a, a)$ . Moreover, to succeed we only need to check one specific tuple in each  $\varphi(G)$ ,  $G \in \mathcal{G}_k$ .

Consider the set  $S = \{(G, \bar{a}); G \in \mathcal{G}_k, G \models \varphi(\bar{a})\}$  and a projection to the second coordinate  $p_2 : S \rightarrow \bigcup_{G \in \mathcal{G}_k} \varphi(G)$ . Since  $|S| = \frac{q_k!}{|\text{Aut}(G)|} \cdot |\varphi(G_k)|$  we have that  $q_k$  is the average size of a  $p_2$  preimage of any  $\bar{a} \in \bigcup_{G \in \mathcal{G}_k} \varphi(G)$ .

**Claim:** For all  $\bar{a}, \bar{b} \in \bigcup_{G \in \mathcal{G}_k} \varphi(G)$  we have  $|p_2^{-1}[\bar{a}]| = |p_2^{-1}[\bar{b}]| = q_k$ .

*Proof of claim.* We will prove that for any  $\bar{a}, \bar{b} \in \bigcup_{G \in \mathcal{G}_k} \varphi(G)$  we have  $|p_2^{-1}[\bar{a}]| \leq |p_2^{-1}[\bar{b}]|$ , by symmetry, they must be equal and also equal to  $q_k$  which is the average size of any singleton preimage.

Let  $p_2^{-1}[\bar{a}] = \{G_0, \dots, G_{s-1}\} \times \{\bar{a}\}$  and let  $\rho = (b_0 a_0) \dots (b_{l-1} a_{l-1})$ , this is a permutation from the condition on  $\varphi$ . Then

$$p_2^{-1}[\bar{b}] \supseteq \{\rho(G_0), \dots, \rho(G_{s-1})\} \times \{\rho(\bar{a}) = \bar{b}\}. \quad \square$$

Now consider the multiset  $M = (\bigcup_{G \in \mathcal{G}_k} \varphi(G), \text{count} : \bar{a} \mapsto |p_2^{-1}[\bar{a}]|)$ , we will construct the searching tree by plucking elements from this multiset in the following way.

Let  $M^{(0)} := M$ ,  $\mathcal{G}_k^{(0)} = \mathcal{G}_k$ . For  $i \geq 0$  and  $M^{(i)}, \mathcal{G}_k^{(i)}$  built, take some  $\bar{a} \in M^{(i)}$  with maximal count( $\bar{a}$ ), put  $\mathcal{G}_k^{(i+1)} = \mathcal{G}_k^{(i)} \setminus p_2^{-1}[\bar{a}]$  and form  $M^{(i+1)}$  by removing  $\bar{a}$ , and for every  $\bar{b} \in p_1[p_2^{-1}[\bar{a}]] \setminus \{\bar{a}\}$  setting  $\text{count}_{M^{(i+1)}}(\bar{b}) = \max\{0, \text{count}_{M^{(i)}}(\bar{b}) - (\varphi(G_k))\}$ . We also add  $T_{\bar{a}}$  to the leaves of the tree we are constructing  $T_i$  and call it  $T_{i+1}$ .

For each  $i \geq 0$  we have that  $T_i$  finds a witness in  $G \in \mathcal{G}_k$  iff  $G \notin \mathcal{G}_k^{(i)}$ . So to calculate the probability of success of  $T_i$  we just need to find upper bounds on the cardinality of  $\mathcal{G}_k^{(i)}$ .

Define  $m_i := \max\{\text{count}(\bar{a}); \bar{a} \in M^{(i)}\}$ . Let  $k^{(0)} \geq 0$  be the greatest number such that for all  $i < k^{(0)}$ :  $m_i = q_k$ .

Define a set  $M_m^{(i)} = \{\bar{a}; \text{count}_{M_i}(\bar{a}) = m_i\}$ . We can see, that  $k^{(0)} \geq 1$  and  $M^{(0)} = \bigcup_{G \in \mathcal{G}_k} \varphi(G)$ . At each step  $i < k^{(0)}$  we construct  $T_{i+1}$  by searching for some  $\bar{a} \in M_m^{(i)}$ , this results in  $|\mathcal{G}_k^{(i+1)}| = |\mathcal{G}_k^{(i)}| - q_k$ . We also remove one instance of every  $\bar{b} \in p_1[p_2^{-1}(\bar{a})] \setminus \{\bar{a}\}$  from  $M^{(i)}$  to form  $M^{(i+1)}$ , this results in  $|M_m^{(i+1)}| \geq |M_m^{(i)}| - 1 - q_k \cdot (|\varphi(G_k)| - 1)$ .

Therefore

$$k^{(0)} \geq \left\lfloor \frac{|M_m^{(0)}|}{q_k \cdot |\varphi(G_k)|} \right\rfloor \quad (2.10)$$

$$= \left\lfloor \frac{|\bigcup_{G \in \mathcal{G}_k} \varphi(G)|}{q_k \cdot |\varphi(G_k)|} \right\rfloor, \quad (2.11)$$

$$\text{and } |\mathcal{G}_k^{(k^{(0)})}| = |\mathcal{G}_k| - k^{(0)} \cdot q_k = \frac{|\text{Aut}(G_k)|}{g_k!} - \left\lfloor \frac{|\bigcup_{G \in \mathcal{G}_k} \varphi(G)|}{q_k \cdot |\varphi(G_k)|} \right\rfloor \cdot q_k \leq \frac{|\text{Aut}(G_k)|}{g_k!} - \left\lfloor \frac{|\bigcup_{G \in \mathcal{G}_k} \varphi(G)|}{|\varphi(G_k)|} \right\rfloor.$$

However the right hand side of the last inequality is rarely  $\leq 0$ , so generally one has to continue with plucking even after  $k^{(0)}$ -many steps. We define  $k^{(j)}$  as the greatest number such that for all  $i < k^{(j)} : m_i \geq q_k - j$  and continue for  $k^{(r)}$  steps, where  $r$  is the smallest number such that

$$|\mathcal{G}_k^{k^{(r)}}| = |\mathcal{G}_k| - k^{(0)} \cdot q_k - \sum_{j=1}^r (k^{(j)} - k^{(j-1)}) \cdot (q_k - j) \quad (2.12)$$

$$= 0. \quad (2.13)$$

However, this requires a general analysis of  $k^{(j)}$  and I haven't manage to compute that.

For  $k = n$  in  $\mathcal{M}$  we put  $n^{(r)} := k^{(r)}$ .  $\square$

## 2.2 $\mathcal{G}_k = \text{ALL}_k$

**Theorem 2.2.1** (Everything exists). Let  $\varphi(\bar{x}, \bar{y}) = \bigwedge_{i=0}^{m-1} \psi_i(\bar{x}, \bar{y}) \wedge \bigwedge_{i=0}^{l-1} \vartheta_i(\bar{y})$ , where  $\psi_i, \vartheta_i$  are literals and  $\psi_i$  are not of the form  $(y_i = y_j)^b$ ,  $E(y_i, y_j)^b$ ,  $x_i \neq x_i$ ,  $E(x_i, x_i)$ ,  $b \in \{0, 1\}$ .

Let  $\bar{\beta}$  be a tuple of vertices computed by  $F_{tree}$  of the same length as  $\bar{y}$ . Then  $\llbracket (\exists \bar{x}) \varphi^\Gamma(\bar{x}, \bar{\beta}) \rrbracket = \llbracket \bigwedge_{i=0}^{l-1} \vartheta_i(\bar{\beta}) \rrbracket$ , specifically if  $l = 0$  then  $(\exists \bar{x}) \phi_0(\bar{x}, \bar{\beta})$  is valid in the b.v. graph.

*Proof.* We will construct one tree  $T$  computing the whole tuple of witnesses  $\bar{\alpha}$ , such a construction can be straightforwardly split into a tuple of tree each computing the specific element.

First we concatenate all the trees used to compute  $\bar{\beta}$ . At each leave we can now proceed knowing the value of  $\bar{\beta}$  at the specific  $\omega \in \Omega$ . Now we just construct a tree as in Theorem 2.1.4 but searching only over edges not checked previously and only to fulfill each  $\psi_i$ . Luckily we have so far searched only an infinitesimal part of the edges and since we assume  $\mathcal{G}_k = \text{ALL}_k$  both of the conditions of the theorem are satisfied, so by analogous argument, we have a tree that finds a witness all of the  $\psi_i(\bar{x}, \bar{\beta})$  with probability infinitesimally close to 1.

Therefore

$$\llbracket (\exists \bar{x}) \varphi^\Gamma(\bar{x}, \bar{\beta}) \rrbracket = \llbracket (\exists \bar{x}) \bigwedge_{i=0}^{m-1} \psi_i(\bar{x}, \bar{\beta}) \rrbracket \wedge \llbracket \bigwedge_{i=0}^{l-1} \vartheta_i(\bar{\beta}) \rrbracket \quad (2.14)$$

$$= \llbracket \bigwedge_{i=0}^{l-1} \vartheta_i(\bar{\beta}) \rrbracket. \quad (2.15)$$

$\square$

**Corollary 2.2.2.** For each  $\varphi(\bar{x})$  that is not a tautology in the theory of graphs we have that  $\llbracket (\forall \bar{x}) \varphi^\Gamma(\bar{x}) \rrbracket = \mathbf{0}$ .

**Corollary 2.2.3.** For each  $\varphi(\bar{x}, \bar{y})$  that is not falsifiable by  $\bar{y}$  in the theory of graphs we have that  $\llbracket (\forall \bar{y}) (\exists \bar{x}) \varphi^\Gamma(\bar{x}) \rrbracket = \mathbf{1}$ .

### 3. $F = F_{nbtree}$

#### 3.1 $\mathcal{G}_k = *PATH_k$

**Definition 3.1.1.** We define  $*PATH_k$  (the pointed paths on  $k$  vertices) as the set of all (undirected) graphs  $G$  on the vertex set  $[k]$ , where  $G$  is isomorphic to the path on  $n$  vertices and  $\deg_G(0) = 1$ .

**Definition 3.1.2.** After we fix  $n$ , we define  $F_{nbtree}$  as the set of all functions computed by some some labeled tree with the following shape:

- Each non-leaf node is labeled by some  $v \in [n]$ .
- For each  $\{u, v\} \subseteq [n]$  and a node  $N$  there is an outgoing edge from  $N$  labeled  $A$ .
- Each leaf is labeled by some  $m \in \mathcal{M}_n$ .
- The depth of the tree is at most  $n^{1/t}$  for some  $t > \mathbb{N}$ .

Computation of such a tree on a undirected graph  $G$  goes as follows. We interpret the non-leaf nodes as questions "what is the neighbour set of  $v$ ?" and the edges as answers from our graph  $G$ , and thus we follow a path determined by  $G$  until we find a vertex for which the answer is not an edge (in which case the computation returns 0) or until we find a leaf, in which case the computation returns the label of the leaf.

We now shift out focus to analysing the ability of trees from  $F_{nbtree}$  to find the non-zero degree 1 vertex in  $G \in *PATH_n$ . We say a tree  $T \in F_{nbtree}$  fails at a graph  $G$  if  $T(G)$  is not a non-zero vertex of degree one in  $G$ .

**Definition 3.1.3.** Let  $m \leq n$  and  $v \in [w]$  and  $U \subseteq [w]$  with  $|U| \leq 2$ , then we define

$$\mathcal{G}_m^{v?=U} := \{G \in \mathcal{G}_m; N_G(v) = U\},$$

where  $N_G$  is the neighbour-set function of  $G$ .

**Lemma 3.1.4.** There are bijections for all nonstandard  $m \leq n$  and distinct  $u, v, w \in [m] \setminus \{0\}$ :

$$\mathcal{G}_m^{v?=\{u,w\}} \cong \mathcal{G}_{m-2} \times [2] \tag{3.1}$$

$$\mathcal{G}_m^{v?=\{u,0\}} \cong \mathcal{G}_{m-2} \tag{3.2}$$

$$\mathcal{G}_m^{0?=\{u\}} \cong \mathcal{G}_{m-1}. \tag{3.3}$$

*Proof.* (sketch) For (3.1) we can just contract all of  $u, v, w$  into one vertex and relabel the rest of the graph, leaving the orientation as a one remaining bit of information. This is obviously reversible and a bijection.

For (3.2) we can do the same, but the orientation is given by 0.  $\square$

**Lemma 3.1.5.** Let  $T \in F_{nbtree}$ , with root labeled  $v \in [m] \setminus 0$ , we have for each  $T_{v?=\{u,w\}}$  a tree  $\tilde{T}_{v?=\{u,w\}}$  of the same depth, such that

$$P_m(T_{v?=\{u,w\}} \text{ fails} | v? = \{u, w\}) = P_{m-2}(\tilde{T}_{v?=\{u,w\}}). \quad (3.4)$$

For a tree  $T$  with the root labeled 0, we have a tree  $\tilde{T}_{v?=\{u,w\}}$  of the same depth, such that

$$P_m(T_{v?=\{u\}} \text{ fails} | v? = \{u\}) = P_{m-1}(\tilde{T}_{v?=\{u\}}). \quad (3.5)$$

*Proof.* (sketch) To construct the tree, we just replace all vertices in labels of  $T_{v?=\{u,w\}}$  by there renumbering from the bijection in (3.1).

(TODO: Elaborate) One can then check that the trees  $T_{v?=\{u,w\}}$  and  $\tilde{T}_{v?=\{u,w\}}$  are isomorphic in a sense that their computation of a graph  $G$  and  $\tilde{G}$  respectively,  $\tilde{G}$  being the corresponding  $(m-2)$ -vertex graph, agree with the structure of the path and that correctness of leaves is preserved under the renumbering. Essentially they emulate the same computation but on a smaller graph.  $\square$

**Lemma 3.1.6.** For all nonstandard  $t > \mathbb{N}$ ,  $m \geq n - 2n^{1/t}$  and  $k \in [n^{1/t} + 1]$  for all trees  $T \in F_{nbtree}$  of depth  $k$  we have

$$P_m(T \text{ fails}) \geq \prod_{i=0}^k \left(1 - \frac{2}{m - 2i - 2}\right).$$

*Proof.* We proceed by induction on  $k$ .

$k = 0$  : We have that the probability of success of a straight guess is at most  $\frac{1}{m-1}$ . Therefore

$$P(T \text{ fails}) \geq \left(1 - \frac{1}{m-1}\right) \geq \left(1 - \frac{2}{m-2}\right). \quad (3.6)$$

$(k-1) \Rightarrow k$  : First we assume that the root is labeled 0. Then we have

$$P(T \text{ fails}) = \sum_{u \in V \setminus \{0\}} P_{m-1}(0Eu) P_{m-1}(T_{0?=\{u\}} \text{ fails} | 0Eu) \quad (3.7)$$

$$\geq P_{m-1}(T_{0?=\{u\}} \text{ fails} | 0Eu) \quad (3.8)$$

$$= P_{m-1}(\tilde{T}_{0?=\{u\}} \text{ fails}) \quad (3.9)$$

$$\geq \prod_{i=0}^{k-1} \left(1 - \frac{2}{m - 2i - 2}\right) \quad (3.10)$$

$$\geq \prod_{i=0}^k \left(1 - \frac{2}{m - 2i - 2}\right). \quad (3.11)$$

Now we assume that the root is labeled  $v \neq 0$ . First we notice that

$$P_m(vE0) = \frac{1}{m-1} \quad (3.12)$$

$$P_m(N(V) = 1) = \frac{1}{m-1} \quad (3.13)$$

$$P_m(|N(V) \setminus \{0\}| = 2) = 1 - \frac{2}{m-1}, \quad (3.14)$$

the first two probabilities are obviously  $\frac{1}{m-1}$  because they correspond to  $v$  being positioned on one of the ends of the non-zero segment which has length  $m-1$ . The event in (3.14) is the complement of the union of the first two events, which have empty interseption, giving us that stated probability.

Then we have

$$P_m(T \text{ fails}) = P_m(vE0)P_m(T \text{ fails}|vE0) \quad (3.15)$$

$$+ P_m(|N(v) \setminus \{0\}| = 2)P_m(T \text{ fails}| |N(v) \setminus \{0\}| = 2) \quad (3.16)$$

$$+ P_m(|N(v)| = 1)P_m(vE0)P_m(T \text{ fails}| |N(v)| = 1) \quad (3.17)$$

$$\geq P_m(|N(v) \setminus \{0\}| = 2)P_m(T \text{ fails}| |N(v) \setminus \{0\}| = 2) \quad (3.18)$$

$$\geq (1 - \frac{2}{m-1}) \quad (3.19)$$

$$\cdot \sum_{\substack{u, w \in V \setminus \{0\} \\ u \neq w}} P_m(v? = \{u, w\})P_m(T_{v?=\{u, w\}} \text{ fails}|v? = \{u, w\}) \quad (3.20)$$

$$\geq (1 - \frac{2}{m-1})P_m(T_{v?=\{u_0, w_0\}} \text{ fails}|v? = \{u_0, w_0\}) \quad (3.21)$$

$$\geq (1 - \frac{2}{m-1})P_{m-2}(\tilde{T}_{v?=\{u_0, w_0\}} \text{ fails}) \quad (3.22)$$

$$\geq (1 - \frac{2}{m-1}) \prod_{i=0}^{k-1} (1 - \frac{2}{m-2i-4}) \quad (3.23)$$

$$\geq (1 - \frac{2}{m-2}) \prod_{i=1}^k (1 - \frac{2}{m-2i-2}) \quad (3.24)$$

$$\geq \prod_{i=0}^k (1 - \frac{2}{m-2i-2}). \quad (3.25)$$

where in (3.21) we choose  $u_0, w_0$  with the lowest value of

$$P_m(T_{v?=\{u_0, w_0\}}|v? = \{u_0, w_0\}),$$

the bound follows the fact that all  $P_m(v? = \{u, w\})$  are the same for distinct non-zero  $u, w$ . In (3.22) we use the lemma 3.1.5 and in (3.23) we use the induction hypothesis.  $\square$

**Corollary 3.1.7.** For a tree  $T \in F_{nbtree}$  we have that

$$P_n(T \text{ fails}) \approx 1.$$

*Proof.* Since  $T$  has depth at most  $n^{1/t}$  for some  $t > \mathbb{N}$  we by the previous lemma that

$$P_n(T \text{ fails}) \geq \prod_{i=0}^{n^{1/t}} \left(1 - \frac{2}{n-2i-2}\right) \quad (3.26)$$

$$\geq \left(1 - \frac{2n^{1/t}}{n-2n^{1/t}-2}\right) \quad (3.27)$$

$$\approx 1. \quad (3.28)$$

$\square$

Finally we can prove the following theorem.

**Theorem 3.1.8.**

$$\llbracket (\exists v)(\exists u)(\forall w)((v \neq 0) \wedge (\Gamma(v, u)) \wedge (\Gamma(v, w) \rightarrow u = w)) \rrbracket = \mathbf{0}$$

*Proof.* Expanding the value of the formula in the statement we get

$$\bigvee_{\alpha} \bigvee_{\beta} \bigwedge_{\gamma} \llbracket (\alpha \neq 0) \wedge (\Gamma(\alpha, \beta)) \wedge (\Gamma(\alpha, \gamma) \rightarrow \beta = \gamma) \rrbracket,$$

to prove it evaluates to  $\mathbf{0}$  we need to find for every  $\alpha, \beta$  some  $\gamma$  such that

$$\llbracket (\alpha \neq 0) \wedge (\Gamma(\alpha, \beta)) \wedge (\Gamma(\alpha, \gamma) \rightarrow \beta = \gamma) \rrbracket = \mathbf{0}.$$

For any  $\alpha, \beta$  we define

$$\gamma(\omega) := \begin{cases} v & N(\alpha(\omega)) = \{v\} \\ w & w \in N(\alpha(\omega)) \setminus \{\beta(\omega)\}, \end{cases}$$

such a function can be computed by a tree in  $F_{nbtree}$  which we can construct by concatenation of trees computing  $\alpha$  and  $\beta$ .

Let  $T$  be the tree computing  $\alpha$ . Now we proceed by contradiction, let

$$\epsilon := \mu(\llbracket (\alpha \neq 0) \wedge (\Gamma(\alpha, \beta)) \wedge (\Gamma(\alpha, \gamma) \rightarrow \beta = \gamma) \rrbracket) > 0,$$

by definition this means that

$$\epsilon = \text{st}(P_n[(\alpha \neq 0) \wedge (\Gamma(\alpha, \beta)) \wedge (\Gamma(\alpha, \gamma) \rightarrow \beta = \gamma)]) > 0.$$

But by definition of  $\gamma$  and Corollary 3.1.7 we have

$$\begin{aligned} 0 &< \epsilon \\ &= \text{st}(P_n[(\alpha \neq 0) \wedge (\Gamma(\alpha, \beta)) \wedge (\Gamma(\alpha, \gamma) \rightarrow \beta = \gamma)]) \\ &\leq \text{st}(P_n[(\alpha \neq 0) \wedge (\Gamma(\alpha, \beta)) \wedge |N(\alpha)| = 1]) \\ &\leq \text{st}(P_n[(\alpha \neq 0) \wedge |N(\alpha)| = 1]) \\ &= \text{st}(P_n[T \text{ does not fail}]) \\ &= 0. \end{aligned}$$

A contradiction. □



# Conclusion

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# A. Attachments

## A.1 First Attachment