Wide limits – nonstandard analysis meets computational complexity

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- Key method – Forcing

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- For any set S we can ask whether $S \models \varphi$. 'Is the sentence φ valid in S?'
- **The problem:** Is $\{|S|; S \models \varphi, S \text{ is finite}\}$ always finite or cofinite?

Solution

Theorem

Let $A := \{|S|; S \text{ is finite}, S \models \varphi\}$. Then either A is finite or $\mathbb{N} \setminus A$ is finite.

Proof.

Assume A is not finite, we will prove, $\mathbb{N} \setminus A$ is infinite.
For contradiction, assume $\mathbb{N} \setminus A$ is infinite. Compactness theorem from mathematical logic lets us then construct countable sets $S_1 \models \varphi$ and $S_2 \models \neg\varphi$. But such sets without no structure are isomorphic! A contradiction. □

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existential formulas in the limit $\leftrightarrow$ hardness of search problems

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Definition (Wide sequence)

A sequence $\{\mathcal{G}_k\}_{k=1}^{\infty}$ *of non-empty sets of finite graphs a* **wide sequence** *if*

- *there is a strictly increasing sequence of positive integers* $\{g_k\}_{k=1}^{\infty}$ *such that the vertex set of all* $G \in \mathcal{G}_k$ *is* $\{0, \dots, g_k - 1\}$,
- $\lim_{k\to\infty} |\mathcal{G}_k| = \infty$. *(Hence, wide!)*

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- And so on...

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- $\mathcal{M}$ contains an isomorphic copy of $\mathbb{N}$ and then some 'infinite numbers', we call those elements **nonstandard numbers**.
- From now on we fix one $\mathcal{M} \models \mathrm{Th}(\mathbb{N})$ which satisfies a technical condition. (It is $\aleph_1$-saturated.)

Pseudofinite structures

- Let $n \in \mathcal{M} \setminus \mathbb{N}$ be a nonstandard number. We can then use it as an index in $\mathcal{G}_k$ to obtain the n-th level of $\{\mathcal{G}_k\}_{k=0}^{\infty}$.

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- Example: In EDGE_n we get graphs on the infinite vertex set $\{0, \ldots, n-1\}$ such that there is exactly one edge.

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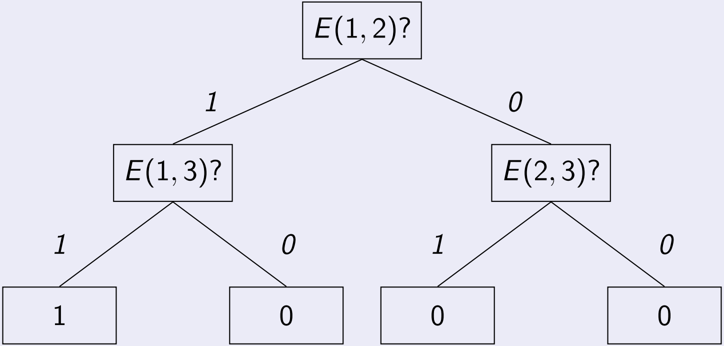
- The main vertex family we consider is F_{rud} the family of functions computed by decision trees of depth $n^{1/t}$ for t nonstandard.

The vertex family F_{rud}

Definition

The vertex family F_{rud} consists of all functions α with input $\mathcal{G}_n$ and output computed by a directed tree of depth at most $g_n^{1/t}$, for some nonstandard t, whose form we present by the following example.



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Theorem

There exists a surjective, order preserving ring homomorphism $\varphi : \mathbb{Q}^{\mathcal{M}}_{fin} \to \mathbb{R}$ whose kernel is the ideal of infinitesimal numbers. We call the imagine $st(q)$ the standard part of the $\mathcal{M}$-rational q.

Nonstandard analysis

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Theorem

Let $\{b_k\}_{k=1}^{\infty}$ *be a sequence of rational numbers then*

$$\lim_{k \to \infty} b_k = r \in \mathbb{R}$$

if and only if for all nonstandard $n \in \mathcal{M}$

$$st(b_n) = r \in \mathbb{R}.$$

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- The counting measure is defined as $A \mapsto |A|/|\mathcal{G}_n| \in \mathbb{Q}^{\mathcal{M}}$.
- The boolean algebra $\mathcal{B}$ is defined as the factor-algebra $\mathcal{A}/\mathcal{I}$ where $\mathcal{I}$ is the ideal of $\mathcal{A}$ of all elements with infinitesimal counting measure.

The Boolean algebra $\mathcal{B}$ – what we need

- **Key point**: The truth values of the first order sentences of the wide limit $\lim_F \mathcal{G}_n$ are the $\mathcal{M}$-definable subsets of $\mathcal{G}_n$ modulo sets of infinitesimal measure. We then have that $\mathcal{B}$ is a σ-algebra and that $\mu : A/\mathcal{I} \to \mathrm{st}(|A|/|\mathcal{G}_n|)$ is measure in the classical sense.

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- Moreover, we have that $\mathcal{B}$ is a complete Boolean algebra, it contains all infinite disjunctions and conjunctions.

The definition of a wide limit

Definition (Wide limit)

For a wide sequence $\mathcal{G}_k$, vertex family F and a nonstandard number n we define the wide limit

$$\lim_F \mathcal{G}_n$$

as a $\mathcal{B}$-valued graph whose vertex set is F and we interpret equality as

$$\lim_F \mathcal{G}_n[\![\alpha = \beta]\!] = \{\omega \in \mathcal{G}_n; \alpha(\omega) = \beta(\omega)\}/\mathcal{I},$$

and the edge relational symbol as

$$\lim_F \mathcal{G}_n[\![E(\alpha, \beta)]\!] = \{\omega \in \mathcal{G}_n; E_\omega(\alpha(\omega), \beta(\omega))\}/\mathcal{I},$$

and let it commute with $\neg$, $\wedge$, $\vee$ and interpret $\forall$ and $\exists$ respectively as infinite conjunctions and disjunctions.

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Theorem

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Proof.

The universe of the wide limit is F_{rud} the (random) vertices computed by decision trees of depth $n^{1/t} = g_n^{1/t}$ for some $t > \mathbb{N}$.
So we need to prove that any two such trees find the edge in uniformly smapled $\omega \in \mathrm{EDGE}_n$ with uniform probability. We can just compose the trees into one tree T which actually outputs the edge.

Theorem

$$\lim_{F_{rud}} EDGE_n[\![(\exists x)(\exists y)E(x,y)]\!] = \mathbf{0}$$

Proof cont.

So assume T is a possibly successful tree. We will prove it is definitely not successful.

Go down the tree and answer $\mathbf{0}$ to every query of the tree. Then the T cannot possibly succeed, what is the ratio of $\omega \in \mathrm{EDGE}_k$ which result in such answers?

Theorem

$$\lim_{F_{rud}} EDGE_n[\![(\exists x)(\exists y)E(x,y)]\!] = \mathbf{0}$$

Proof cont.

So we compute the ratio of those graphs in which the edge in not among $n^{1/t}$ many specified pairs of vertices.

$$\frac{\binom{n-n^{1/t}}{2}}{\binom{n}{2}} = \frac{(n-n^{1/t})(n-n^{1/t}-1)}{n(n-1)} \tag{1}$$

$$= \left(1-\frac{n^{1/t}}{n}\right)\left(1-\frac{n^{1/t}+1}{n-1}\right) \tag{2}$$

$$\geq 1-\frac{2n^{1/t}+2}{n-1} \tag{3}$$

but the standard part of (3) is 1! So the probability any tree succeeds is infinitesimally close to 0. □

Too sparse wide sequences

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Theorem

If the probability that a uniformly random pair of vertices on a uniformly random $G \in \mathcal{G}_k$ is $o(1/\sqrt[j]{k})$ for any $j \in \mathbb{N}$ then

$$\lim_{F_{rud}} \mathcal{G}_n[\![(\exists x)(\exists y)E(x,y)]\!] = \mathbf{0}.$$

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Theorem (Everything exists)

Let $\varphi(\overline{x})$ be a formula in the langauge of graphs (possibly with parameters), then

$$\lim_{F_{rud}} ALL_k[\![(\exists\overline{x})\varphi(x)]\!] = \mathbf{1}.$$

Proof. (sketch).

Iterate search on $n^{1/t}$ different tuples of ω and use mutual independence. $\square$

Large clique

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- Open problem: Is this the biggest clique here? What about in the wide sequence which contains a large clique and possibly other edges?

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- Analyze the F_{poly}, connection to **P** vs. **NP**.
- Easily generalizable to any relational and algebraic structure.
- One can then take a wide limit of $\mathbf{CSP}(\mathbb{A})$, connection to tractability.

What's next?

- Further analyze when we get the truth values strictly in the set $\{\mathbf{0}, \mathbf{1}\}$.
- Analyze the F_{poly}, connection to **P** vs. **NP**.
- Easily generalizable to any relational and algebraic structure.
- One can then take a wide limit of $\mathbf{CSP}(\mathbb{A})$, connection to tractability.
- The ultimate direction to pursue is to characterize some wide limit independently of the direct definition – this immediatelly results in (probabilistic) lower and upper bounds for F.

$$\lim_{F} \mathcal{G}_n [\![\textbf{Thank you!}]\!] = \mathbf{1}$$