

Rotation Matrix

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A rotation is a linear transformation that preserves the length of a vector \mathbf{v} .

$$\|\mathbf{R}\mathbf{v}\| = \|\mathbf{v}\| \quad (1)$$

Let the rotation matrix be \mathbf{R} with order n and there be two vector \mathbf{u} and \mathbf{v} in the 3D space.

The inner product of the two vectors must remain same.

$$\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v})^\top (\mathbf{u} + \mathbf{v}) \quad (2)$$

$$\|\mathbf{u} + \mathbf{v}\|^2 = \mathbf{u}^\top \mathbf{u} + \mathbf{v}^\top \mathbf{v} + 2\mathbf{u}^\top \mathbf{v} \quad (3)$$

$$\mathbf{u}^\top \mathbf{v} = \frac{1}{2} (\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2) \quad (4)$$

$$\text{Let } \mathbf{u}_1 = \mathbf{R}\mathbf{u} \text{ and } \mathbf{v}_1 = \mathbf{R}\mathbf{v} \quad (5)$$

Using (1) and (4),

$$\mathbf{u}_1^\top \mathbf{v}_1 = \frac{1}{2} (\|\mathbf{u}_1 + \mathbf{v}_1\|^2 - \|\mathbf{u}_1\|^2 - \|\mathbf{v}_1\|^2) \quad (6)$$

$$= \frac{1}{2} (\|\mathbf{R}(\mathbf{u} + \mathbf{v})\|^2 - \|\mathbf{R}\mathbf{u}\|^2 - \|\mathbf{R}\mathbf{v}\|^2) \quad (7)$$

$$= \frac{1}{2} (\|(\mathbf{u} + \mathbf{v})\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2) \quad (8)$$

$$= \mathbf{u}^\top \mathbf{v} \quad (9)$$

$$\implies (\mathbf{R}\mathbf{u})^\top (\mathbf{R}\mathbf{v}) = \mathbf{u}^\top \mathbf{v} \quad (10)$$

$$\mathbf{u}^\top \mathbf{R}^\top \mathbf{R} \mathbf{v} = \mathbf{u}^\top \mathbf{v} \quad (11)$$

$$\mathbf{u} \mathbf{R}^\top \mathbf{R} \mathbf{v} = \mathbf{u}^\top \mathbf{I} \mathbf{v} \quad (12)$$

$$\mathbf{u}^\top (\mathbf{R}^\top \mathbf{R} - \mathbf{I}) \mathbf{v} = 0 \quad (13)$$

Let

$$\mathbf{R}^\top \mathbf{R} - \mathbf{I} = \mathbf{A}$$

The (13) becomes

$$\mathbf{u}^\top \mathbf{A} \mathbf{v} = 0 \quad (14)$$

(14) is true for all \mathbf{u}, \mathbf{v} . Let

$$\mathbf{u} = \mathbf{e}_i \text{ and } \mathbf{v} = \mathbf{e}_j, \text{ where } 0 \leq i, j \leq n \quad (15)$$

Substituting (15) in (14),

$$\mathbf{e}_i^\top \mathbf{A} \mathbf{e}_j = 0 \quad (16)$$

$$\implies \mathbf{A}_{ij} = 0 \text{ where } \mathbf{A}_{ij} \text{ is an element in } i\text{-th row and } j\text{-th column of } \mathbf{A} \quad (17)$$

From (17), every element

$$\mathbf{A}_{ij} = 0 \quad (18)$$

$$\implies \mathbf{A} = \mathbf{O} \quad (19)$$

$$\mathbf{R}^\top \mathbf{R} - \mathbf{I} = \mathbf{O} \quad (20)$$

$$\implies \mathbf{R}^\top \mathbf{R} = \mathbf{I} \quad (21)$$

From (21), it can be concluded that the rotation matrix \mathbf{R} is orthogonal and its determinant is 1.