10.7.75

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Question: Find the equations of tangents drawn from origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$,are

1) $x = 0$
2) $y = 0$
3) $\left(h^2 - r^2\right) x - 2rhy = 0$
4) $\left(h^2 - r^2\right) x + 2rhy = 0$

Solution:

A general conic section is described by the equation

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

where $\mathbf{V}$ is a symmetric matrix. A line passing through a point $\mathbf{h}$ and having unit direction vector $\mathbf{m}$ is

$$\mathbf{x} = \mathbf{h} + k\mathbf{m} \tag{2}$$

Substitute the line equation into the conic to find points of intersection.

$$k^2 \left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right) + 2k\left(\mathbf{m}^\top \mathbf{V}\mathbf{h} + \mathbf{u}^\top \mathbf{m}\right) + \left(\mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f\right) = 0 \tag{3}$$

For the line to be tangent to the conic, the discriminant of quadratic in k must be zero. Let $g(\mathbf{h}) = \mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f$ be the value of the conic expression at the point $\mathbf{h}$.

$$\left(\mathbf{m}^\top \mathbf{V}\mathbf{h} + \mathbf{u}^\top \mathbf{m}\right)^2 - \left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right)(g(\mathbf{h})) = 0 \tag{4}$$

$$\left(\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})\right)^2 - g(\mathbf{h})\left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right) = 0 \tag{5}$$

$$\mathbf{m}^\top \left((\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top\right)\mathbf{m} - \mathbf{m}^\top (g(\mathbf{h})\mathbf{V})\mathbf{m} = 0 \tag{6}$$

$$\mathbf{m}^\top \left[(\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - \left(\mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f\right)\mathbf{V}\right]\mathbf{m} = 0 \tag{7}$$

$$\text{Let } \mathbf{\Sigma} = (\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - \left(\mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f\right)\mathbf{V} \tag{8}$$

$$\mathbf{m}^\top \mathbf{\Sigma}\mathbf{m} = 0 \tag{9}$$

This is the general equation for the directions of tangents from an arbitrary point $\mathbf{h}$.

For $\mathbf{h} = \mathbf{0}$,

$$\mathbf{\Sigma} = (\mathbf{V}\mathbf{0} + \mathbf{u})(\mathbf{V}\mathbf{0} + \mathbf{u})^\top - \left(\mathbf{0}^\top \mathbf{V}\mathbf{0} + 2\mathbf{u}^\top \mathbf{0} + f\right)\mathbf{V} = \mathbf{u}\mathbf{u}^\top - f\mathbf{V} \tag{10}$$

To solve this, the symmetric matrix $\mathbf{\Sigma}$ is diagonalized. The eigendecomposition of $\mathbf{\Sigma}$ is $\mathbf{\Sigma} = \mathbf{PDP}^\top$, where:
$\mathbf{D}$ is a diagonal matrix with the eigenvalues of $\mathbf{\Sigma}$ on its diagonal

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{11}$$

$\mathbf{P}$ is an orthogonal matrix whose columns are the corresponding orthonormal eigenvectors. So, $\mathbf{P}^\top\mathbf{P} = \mathbf{PP}^\top = 1$.

$$\mathbf{P} = \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} \end{pmatrix} \tag{12}$$

Substitute (11) and (12) in (9)

$$\mathbf{m}^\top(\mathbf{PDP}^\top)\mathbf{m} = 0 \tag{13}$$

$$(\mathbf{m}^\top\mathbf{P})\mathbf{D}(\mathbf{P}^\top\mathbf{m}) = 0 \tag{14}$$

Let $\mathbf{y} = \mathbf{P}^\top\mathbf{m} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

$$\mathbf{y}^\top\mathbf{D}\mathbf{y} = 0 \tag{15}$$

$$\begin{pmatrix} y_1 & y_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \tag{16}$$

$$\begin{pmatrix} y_1 & y_2 \end{pmatrix}\begin{pmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \end{pmatrix} = 0 \tag{17}$$

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = 0 \tag{18}$$

Since $\mathbf{P}$ is orthogonal,

$$\mathbf{y}^\top\mathbf{y} = \mathbf{m}^\top\mathbf{PP}^\top\mathbf{m} \tag{19}$$

$$= \mathbf{m}^\top\mathbf{I}\mathbf{m} \tag{20}$$

$$= \mathbf{m}^\top\mathbf{m} \tag{21}$$

$$= 1 \tag{22}$$

$$\implies y_1^2 + y_2^2 = 1 \tag{23}$$

From the pair of equations (18) and (23),

$$y_1^2 = \frac{-\lambda_2}{\lambda_1 - \lambda_2},\ y_2^2 = \frac{\lambda_1}{\lambda_1 - \lambda_2} \tag{24}$$

$$\mathbf{y} = \mathbf{P}^\top\mathbf{m} \tag{25}$$

$$\mathbf{PP}^\top\mathbf{m} = \mathbf{Py} \tag{26}$$

$$\mathbf{m} = \mathbf{Py} \tag{27}$$

For the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} -r \\ -h \end{pmatrix}, \ f = h^2 \tag{28}$$

From (8),

$$\mathbf{\Sigma} = \begin{pmatrix} -r \\ -h \end{pmatrix} \begin{pmatrix} -r & -h \end{pmatrix} - h^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r^2 - h^2 & rh \\ rh & 0 \end{pmatrix} \tag{29}$$

The characteristic equation is $\left| \mathbf{\Sigma} - \lambda \mathbf{I} \right| = 0$.
Let $\lambda^2 + a_1 \lambda + a_2 = 0$. Using Faddeev-Leverrier Method,

$$\mathbf{B_1} = \mathbf{A} \tag{30}$$

$$a_1 = -tr\left(\mathbf{B_1}\right) = -\left(r^2 - h^2\right) \tag{31}$$

$$\mathbf{B_2} = \mathbf{A}\left(\mathbf{B_1} + a_1 \mathbf{I}\right) \tag{32}$$

$$\mathbf{B_2} = \begin{pmatrix} r^2 h^2 & 2rh\left(r^2 - h^2\right) \\ 0 & r^2 h^2 \end{pmatrix} \tag{33}$$

$$a_2 = -\frac{1}{2} tr\left(\mathbf{B_2}\right) \tag{34}$$

$$a_2 = -r^2 h^2 \tag{35}$$

So, $\lambda^2 - \left(r^2 - h^2\right)\lambda - r^2 h^2 = 0$, giving the eigenvalues $\lambda_1 = r^2$ and $\lambda_2 = -h^2$.

$$\mathbf{P} = \frac{1}{\sqrt{r^2 + h^2}} \begin{pmatrix} r & h \\ h & -r \end{pmatrix} \tag{36}$$

$$y_1^2 = \frac{-\left(-h^2\right)}{r^2 - \left(-h^2\right)} = \frac{h^2}{r^2 + h^2} \implies y_1 = \pm \frac{h}{\sqrt{r^2 + h^2}} \tag{37}$$

$$y_2^2 = \frac{r^2}{r^2 - \left(-h^2\right)} = \frac{r^2}{r^2 + h^2} \implies y_2 = \pm \frac{r}{\sqrt{r^2 + h^2}} \tag{38}$$

Using (27), $\mathbf{m} = y_1 \mathbf{v_1} + y_2 \mathbf{v_2}$.

$$\mathbf{m_1} \propto h\mathbf{v_1} - r\mathbf{v_2} \propto h \begin{pmatrix} r \\ h \end{pmatrix} - r \begin{pmatrix} h \\ -r \end{pmatrix} = \begin{pmatrix} 0 \\ h^2 + r^2 \end{pmatrix} \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{39}$$

$$\mathbf{m_2} \propto h\mathbf{v_1} + r\mathbf{v_2} \propto h \begin{pmatrix} r \\ h \end{pmatrix} + r \begin{pmatrix} h \\ -r \end{pmatrix} = \begin{pmatrix} 2rh \\ h^2 - r^2 \end{pmatrix} \tag{40}$$

For $\mathbf{m_1} \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$: $1 \times x - 0 \times y = 0 \implies x = 0$. This is option **1**.
For $\mathbf{m_2} \propto \begin{pmatrix} 2rh \\ h^2 - r^2 \end{pmatrix}$: $\left(h^2 - r^2\right) x - (2rh)\, y = 0$. This is option **3**.
So, options (1) and (3) are true.

Example:
Let r = 3, h = 2.

For the circle $x^2 + y^2 - 6x - 4y + 4 = 0$,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, \ f = 4 \tag{41}$$

From (8),

$$\mathbf{\Sigma} = \begin{pmatrix} -3 \\ -2 \end{pmatrix} \begin{pmatrix} -3 & -2 \end{pmatrix} - 4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 6 & 0 \end{pmatrix} \tag{42}$$

The characteristic equation is $\left|\mathbf{\Sigma} - \lambda \mathbf{I}\right| = 0$.
Let $\lambda^2 + a_1\lambda + a_2 = 0$. Using Faddeev-Leverrier Method,

$$\mathbf{B_1} = \mathbf{A} \tag{43}$$
$$a_1 = -tr\left(\mathbf{B_1}\right) = -(5) \tag{44}$$

$$\mathbf{B_2} = \mathbf{A}\left(\mathbf{B_1} + a_1\mathbf{I}\right) \tag{45}$$
$$\mathbf{B_2} = \begin{pmatrix} 36 & 60 \\ 0 & 36 \end{pmatrix} \tag{46}$$
$$a_2 = -\frac{1}{2} tr\left(\mathbf{B_2}\right) \tag{47}$$
$$a_2 = -36 \tag{48}$$

So, $\lambda^2 - 5\lambda - 36 = 0$, giving the eigenvalues $\lambda_1 = 9$ and $\lambda_2 = -4$.

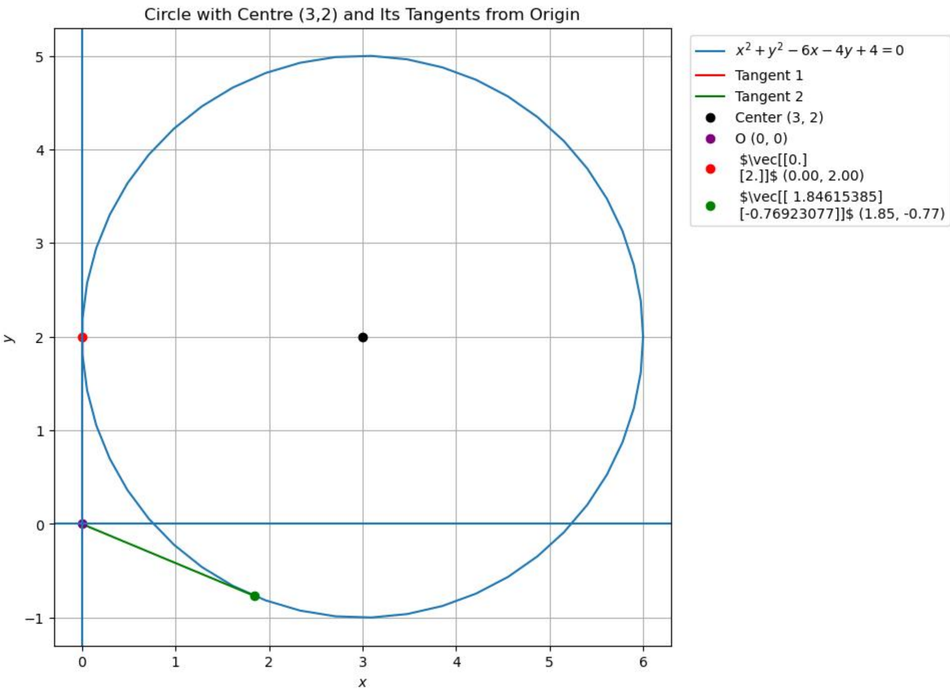
$$\mathbf{P} = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix} \tag{49}$$

$$y_1^2 = \frac{-(-4)}{9-(-4)} = \frac{4}{13} \implies y_1 = \pm\frac{2}{\sqrt{13}} \tag{50}$$
$$y_2^2 = \frac{9}{9-(-4)} = \frac{9}{13} \implies y_2 = \pm\frac{3}{\sqrt{13}} \tag{51}$$

Using (27), $\mathbf{m} = y_1\mathbf{v_1} + y_2\mathbf{v_2}$.

$$\mathbf{m_1} \propto 2\mathbf{v_1} - 3\mathbf{v_2} \propto 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 13 \end{pmatrix} \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{52}$$
$$\mathbf{m_2} \propto 2\mathbf{v_1} + 3\mathbf{v_2} \propto 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} + 3\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 12 \\ -5 \end{pmatrix} \tag{53}$$

For $\mathbf{m_1}$: $\begin{pmatrix} x \\ y \end{pmatrix} = k\begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies x = 0$. This is option **1**.
For $\mathbf{m_2}$: $\begin{pmatrix} x \\ y \end{pmatrix} = k\begin{pmatrix} 12 \\ -5 \end{pmatrix} \implies (-5)\,x - (12)\,y = 0$. This is option **3**.
So, options (1) and (3) are true.


Circle with Centre (3,2) and Its Tangents from Origin
$x^2 + y^2 - 6x - 4y + 4 = 0$
Tangent 1
Tangent 2
Center (3, 2)
O (0, 0)
$\vec[[0.]
[2.]]$ (0.00, 2.00)
$\vec[[1.84615385]
[-0.76923077]]$ (1.85, -0.77)
x
y

Example