Rotation Matrix

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Rotation Matrix

Proof

A rotation is a linear transformation that preserves the length of a vector **v**.

$$\|\mathbf{R}\mathbf{v}\| = \mathbf{v} \tag{1}$$

Let the rotation matrix be **R** with order n and there be two vector **u** and **v** in the 3D space.
The inner product of the two vectors must remain same.

Proof

$$\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v})^\top (\mathbf{u} + \mathbf{v}) \tag{2}$$

$$\|\mathbf{u} + \mathbf{v}\|^2 = \mathbf{u}^\top \mathbf{u} + \mathbf{v}^\top \mathbf{v} + 2\mathbf{u}^\top \mathbf{v} \tag{3}$$

$$\mathbf{u}^\top \mathbf{v} = \frac{1}{2} \left(\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 \right) \tag{4}$$

$$\text{Let } \mathbf{u_1} = \mathbf{R}\mathbf{u} \text{ and } \mathbf{v_1} = \mathbf{R}\mathbf{v} \tag{5}$$

Proof

Using (1) and (4),

$$\mathbf{u_1}^\top \mathbf{v_1} = \frac{1}{2} \left(\|\mathbf{u_1} + \mathbf{v_1}\|^2 - \|\mathbf{u_1}\|^2 - \|\mathbf{v_1}\|^2 \right) \tag{6}$$

$$= \frac{1}{2} \left(\|\mathbf{R}\left(\mathbf{u} + \mathbf{v}\right)\|^2 - \|\mathbf{Ru}\|^2 - \|\mathbf{Rv}\|^2 \right) \tag{7}$$

$$= \frac{1}{2} \left(\|(\mathbf{u} + \mathbf{v})\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 \right) \tag{8}$$

$$= \mathbf{u}^\top \mathbf{v} \tag{9}$$

Proof

$$\implies (\mathbf{Ru})^\top (\mathbf{Rv}) = \mathbf{u}^\top \mathbf{v} \qquad (10)$$

$$\mathbf{u}^\top \mathbf{R}^\top \mathbf{Rv} = \mathbf{u}^\top \mathbf{v} \qquad (11)$$

$$\mathbf{u}\mathbf{R}^\top \mathbf{Rv} = \mathbf{u}^\top \mathbf{Iv} \qquad (12)$$

$$\mathbf{u}^\top \left(\mathbf{R}^\top \mathbf{R} - \mathbf{I} \right) \mathbf{v} = 0 \qquad (13)$$

Let

$$\mathbf{R}^\top \mathbf{R} - \mathbf{I} = \mathbf{A}$$

Proof

The (13) becomes

$$\mathbf{u}^\top \mathbf{A} \mathbf{v} = 0 \tag{14}$$

(14) is true for all $\mathbf{u}$, $\mathbf{v}$. Let

$$\mathbf{u} = \mathbf{e_i} \text{ and } \mathbf{v} = \mathbf{e_j}, \text{ where } 0 \leq i,j \leq n \tag{15}$$

Substituting (15) in (14),

$$\mathbf{e_i}^\top \mathbf{A} \mathbf{e_j} = 0 \tag{16}$$

Proof

$\implies \mathbf{A_{ij}} = 0$ where $\mathbf{A_{ij}}$ is an element in $i - th$ row and $j - th$ column of $\mathbf{A}$ (17)

From (17), every element

$$\mathbf{A_{ij}} = 0 \tag{18}$$

$$\implies \mathbf{A} = \mathbf{O} \tag{19}$$

$$\mathbf{R}^{\top}\mathbf{R} - \mathbf{I} = \mathbf{O} \tag{20}$$

$$\implies \mathbf{R}^{\top}\mathbf{R} = \mathbf{I} \tag{21}$$

From (21), it can be concluded that the rotation matrix $\mathbf{R}$ is orthogonal and its determinant is 1.