Rotation Matrix

Puni Aditya - EE25BTECH11046

A rotation is a linear transformation that preserves the length of a vector $\mathbf{v}$.

$$\|\mathbf{Rv}\| = \mathbf{v} \tag{1}$$

Let the rotation matrix be $\mathbf{R}$ with order n and there be two vector $\mathbf{u}$ and $\mathbf{v}$ in the 3D space.
The inner product of the two vectors must remain same.

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})^\top(\mathbf{u}+\mathbf{v}) \tag{2}$$

$$\|\mathbf{u}+\mathbf{v}\|^2 = \mathbf{u}^\top\mathbf{u} + \mathbf{v}^\top\mathbf{v} + 2\mathbf{u}^\top\mathbf{v} \tag{3}$$

$$\mathbf{u}^\top\mathbf{v} = \frac{1}{2}\left(\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2\right) \tag{4}$$

$$\text{Let } \mathbf{u_1} = \mathbf{Ru} \text{ and } \mathbf{v_1} = \mathbf{Rv} \tag{5}$$

Using (1) and (4),

$$\mathbf{u_1}^\top\mathbf{v_1} = \frac{1}{2}\left(\|\mathbf{u_1}+\mathbf{v_1}\|^2 - \|\mathbf{u_1}\|^2 - \|\mathbf{v_1}\|^2\right) \tag{6}$$

$$= \frac{1}{2}\left(\|\mathbf{R}(\mathbf{u}+\mathbf{v})\|^2 - \|\mathbf{Ru}\|^2 - \|\mathbf{Rv}\|^2\right) \tag{7}$$

$$= \frac{1}{2}\left(\|(\mathbf{u}+\mathbf{v})\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2\right) \tag{8}$$

$$= \mathbf{u}^\top\mathbf{v} \tag{9}$$

$$\implies (\mathbf{Ru})^\top(\mathbf{Rv}) = \mathbf{u}^\top\mathbf{v} \tag{10}$$

$$\mathbf{u}^\top\mathbf{R}^\top\mathbf{Rv} = \mathbf{u}^\top\mathbf{v} \tag{11}$$

$$\mathbf{u}\mathbf{R}^\top\mathbf{Rv} = \mathbf{u}^\top\mathbf{Iv} \tag{12}$$

$$\mathbf{u}^\top\left(\mathbf{R}^\top\mathbf{R} - \mathbf{I}\right)\mathbf{v} = 0 \tag{13}$$

Let

$$\mathbf{R}^\top\mathbf{R} - \mathbf{I} = \mathbf{A}$$

The (13) becomes

$$\mathbf{u}^\top\mathbf{Av} = 0 \tag{14}$$

(14) is true for all $\mathbf{u}$, $\mathbf{v}$. Let

$$\mathbf{u} = \mathbf{e_i} \text{ and } \mathbf{v} = \mathbf{e_j}, \text{ where } 0 \le \text{i,j} \le \text{n} \tag{15}$$

Substituting (15) in (14),

$$\mathbf{e_i}^\top \mathbf{A}\mathbf{e_j} = 0 \tag{16}$$

$$\implies \mathbf{A_{ij}} = 0 \text{ where } \mathbf{A_{ij}} \text{ is an element in } i-th \text{ row and } j-th \text{ column of } \mathbf{A} \tag{17}$$

From (17), every element

$$\mathbf{A_{ij}} = 0 \tag{18}$$

$$\implies \mathbf{A} = \mathbf{O} \tag{19}$$

$$\mathbf{R}^\top \mathbf{R} - \mathbf{I} = \mathbf{O} \tag{20}$$

$$\implies \mathbf{R}^\top \mathbf{R} = \mathbf{I} \tag{21}$$

$$det\left(\mathbf{R}^\top \mathbf{R}\right) = det\left(\mathbf{I}\right) \tag{22}$$

$$det\left(\mathbf{R}^\top\right) det\left(R\right) = det\left(\mathbf{I}\right) \tag{23}$$

$$det\left(\mathbf{R}\right)^2 = det\left(vecI\right) \tag{24}$$

$$det\left(\mathbf{R}\right)^2 = 1 \tag{25}$$

$$det\left(R\right) = 1 \tag{26}$$

From (21) and (26), it can be concluded that the rotation matrix **R** is orthogonal and its determinant is 1.