

# 10.7.75

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**Question:** Find the equations of tangents drawn from origin to the circle  $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ , are

$$1) x = 0$$

$$2) y = 0$$

$$3) (h^2 - r^2)x - 2rhy = 0$$

$$4) (h^2 - r^2)x + 2rhy = 0$$

**Solution:**

A general conic section is described by the equation

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1)$$

where  $\mathbf{V}$  is a symmetric matrix. A line passing through a point  $\mathbf{h}$  and having unit direction vector  $\mathbf{m}$  is

$$\mathbf{x} = \mathbf{h} + k\mathbf{m} \quad (2)$$

Substitute the line equation into the conic to find points of intersection.

$$k^2 (\mathbf{m}^T \mathbf{V} \mathbf{m}) + 2k (\mathbf{m}^T \mathbf{V} \mathbf{h} + \mathbf{u}^T \mathbf{m}) + (\mathbf{h}^T \mathbf{V} \mathbf{h} + 2\mathbf{u}^T \mathbf{h} + f) = 0 \quad (3)$$

For the line to be tangent to the conic, the discriminant of quadratic in  $k$  must be zero.

Let  $g(\mathbf{h}) = \mathbf{h}^T \mathbf{V} \mathbf{h} + 2\mathbf{u}^T \mathbf{h} + f$  be the value of the conic expression at the point  $\mathbf{h}$ .

$$(\mathbf{m}^T \mathbf{V} \mathbf{h} + \mathbf{u}^T \mathbf{m})^2 - (\mathbf{m}^T \mathbf{V} \mathbf{m})(g(\mathbf{h})) = 0 \quad (4)$$

$$(\mathbf{m}^T (\mathbf{V} \mathbf{h} + \mathbf{u}))^2 - g(\mathbf{h})(\mathbf{m}^T \mathbf{V} \mathbf{m}) = 0 \quad (5)$$

$$\mathbf{m}^T ((\mathbf{V} \mathbf{h} + \mathbf{u})(\mathbf{V} \mathbf{h} + \mathbf{u})^T) \mathbf{m} - \mathbf{m}^T (g(\mathbf{h}) \mathbf{V}) \mathbf{m} = 0 \quad (6)$$

$$\mathbf{m}^T [(\mathbf{V} \mathbf{h} + \mathbf{u})(\mathbf{V} \mathbf{h} + \mathbf{u})^T - (\mathbf{h}^T \mathbf{V} \mathbf{h} + 2\mathbf{u}^T \mathbf{h} + f) \mathbf{V}] \mathbf{m} = 0 \quad (7)$$

$$\text{Let } \Sigma = (\mathbf{V} \mathbf{h} + \mathbf{u})(\mathbf{V} \mathbf{h} + \mathbf{u})^T - (\mathbf{h}^T \mathbf{V} \mathbf{h} + 2\mathbf{u}^T \mathbf{h} + f) \mathbf{V} \quad (8)$$

$$\mathbf{m}^T \Sigma \mathbf{m} = 0 \quad (9)$$

This is the general equation for the directions of tangents from an arbitrary point  $\mathbf{h}$ .

For  $\mathbf{h} = \mathbf{0}$ ,

$$\Sigma = (\mathbf{V} \mathbf{0} + \mathbf{u})(\mathbf{V} \mathbf{0} + \mathbf{u})^T - (\mathbf{0}^T \mathbf{V} \mathbf{0} + 2\mathbf{u}^T \mathbf{0} + f) \mathbf{V} = \mathbf{u} \mathbf{u}^T - f \mathbf{V} \quad (10)$$

To solve this, the symmetric matrix  $\mathbf{\Sigma}$  is diagonalized. The eigendecomposition of  $\mathbf{\Sigma}$  is  $\mathbf{\Sigma} = \mathbf{PDP}^\top$ , where:

$\mathbf{D}$  is a diagonal matrix with the eigenvalues of  $\mathbf{\Sigma}$  on its diagonal

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (11)$$

$\mathbf{P}$  is an orthogonal matrix whose columns are the corresponding orthonormal eigenvectors. So,  $\mathbf{P}^\top \mathbf{P} = \mathbf{PP}^\top = \mathbf{1}$ .

$$\mathbf{P} = (\mathbf{v}_1 \quad \mathbf{v}_2) \quad (12)$$

Substitute (11) and (12) in (9)

$$\mathbf{m}^\top (\mathbf{PDP}^\top) \mathbf{m} = 0 \quad (13)$$

$$(\mathbf{m}^\top \mathbf{P}) \mathbf{D} (\mathbf{P}^\top \mathbf{m}) = 0 \quad (14)$$

$$\text{Let } \mathbf{y} = \mathbf{P}^\top \mathbf{m} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = 0 \quad (15)$$

$$\begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \quad (16)$$

$$\begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \end{pmatrix} = 0 \quad (17)$$

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = 0 \quad (18)$$

Since  $\mathbf{P}$  is orthogonal,

$$\mathbf{y}^\top \mathbf{y} = \mathbf{m}^\top \mathbf{PP}^\top \mathbf{m} \quad (19)$$

$$= \mathbf{m}^\top \mathbf{Im} \quad (20)$$

$$= \mathbf{m}^\top \mathbf{m} \quad (21)$$

$$= 1 \quad (22)$$

$$\implies y_1^2 + y_2^2 = 1 \quad (23)$$

From the pair of equations (18) and (23),

$$y_1^2 = \frac{-\lambda_2}{\lambda_1 - \lambda_2}, \quad y_2^2 = \frac{\lambda_1}{\lambda_1 - \lambda_2} \quad (24)$$

$$\mathbf{y} = \mathbf{P}^\top \mathbf{m} \quad (25)$$

$$\mathbf{PP}^\top \mathbf{m} = \mathbf{Py} \quad (26)$$

$$\mathbf{m} = \mathbf{Py} \quad (27)$$

For the circle  $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ ,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -r \\ -h \end{pmatrix}, \quad f = h^2 \quad (28)$$

From (8),

$$\Sigma = \begin{pmatrix} -r \\ -h \end{pmatrix} \begin{pmatrix} -r & -h \end{pmatrix} - h^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r^2 - h^2 & rh \\ rh & 0 \end{pmatrix} \quad (29)$$

The characteristic equation is  $|\Sigma - \lambda \mathbf{I}| = 0$ .

let  $\lambda^2 + a_1\lambda + a_2 = 0$ . Using Faddev-Leverrier Method,

$$\mathbf{B}_1 = \mathbf{A} \quad (30)$$

$$a_1 = -tr(\mathbf{B}_1) = -(r^2 - h^2) \quad (31)$$

$$\mathbf{B}_2 = \mathbf{A}(\mathbf{B}_1 + a_1\mathbf{I}) \quad (32)$$

$$\mathbf{B}_2 = \begin{pmatrix} r^2h^2 & 2rh(r^2 - h^2) \\ 0 & r^2h^2 \end{pmatrix} \quad (33)$$

$$a_2 = -\frac{1}{2}tr(\mathbf{B}_2) \quad (34)$$

$$a_2 = -r^2h^2 \quad (35)$$

So,  $\lambda^2 - (r^2 - h^2)\lambda - r^2h^2 = 0$ , giving the eigenvalues  $\lambda_1 = r^2$  and  $\lambda_2 = -h^2$ .

$$\mathbf{P} = \frac{1}{\sqrt{r^2 + h^2}} \begin{pmatrix} r & h \\ h & -r \end{pmatrix} \quad (36)$$

$$y_1^2 = \frac{-(-h^2)}{r^2 - (-h^2)} = \frac{h^2}{r^2 + h^2} \implies y_1 = \pm \frac{h}{\sqrt{r^2 + h^2}} \quad (37)$$

$$y_2^2 = \frac{r^2}{r^2 - (-h^2)} = \frac{r^2}{r^2 + h^2} \implies y_2 = \pm \frac{r}{\sqrt{r^2 + h^2}} \quad (38)$$

Using (27),  $\mathbf{m} = y_1\mathbf{v}_1 + y_2\mathbf{v}_2$ .

$$\mathbf{m}_1 \propto h\mathbf{v}_1 - r\mathbf{v}_2 \propto h \begin{pmatrix} r \\ h \end{pmatrix} - r \begin{pmatrix} h \\ -r \end{pmatrix} = \begin{pmatrix} 0 \\ h^2 + r^2 \end{pmatrix} \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (39)$$

$$\mathbf{m}_2 \propto h\mathbf{v}_1 + r\mathbf{v}_2 \propto h \begin{pmatrix} r \\ h \end{pmatrix} + r \begin{pmatrix} h \\ -r \end{pmatrix} = \begin{pmatrix} 2rh \\ h^2 - r^2 \end{pmatrix} \quad (40)$$

For  $\mathbf{m}_1 \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ :  $1 \times x - 0 \times y = 0 \implies x = 0$ . This is option **1**.

For  $\mathbf{m}_2 \propto \begin{pmatrix} 2rh \\ h^2 - r^2 \end{pmatrix}$ :  $(h^2 - r^2)x - (2rh)y = 0$ . This is option **3**.