

2.10.85

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Question

Let \mathbf{P} be the plane $3x + 2y + 3z = 16$ and let $S : \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$, where $\alpha + \beta + \gamma = 7$ and the distance of (α, β, γ) from the plane is $\frac{2}{\sqrt{22}}$. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three distinct vectors in S such that $|\mathbf{u} - \mathbf{v}| = |\mathbf{v} - \mathbf{w}| = |\mathbf{w} - \mathbf{u}|$. Let V be the volume of the parallelepiped determined by vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Then the value of $\frac{80}{3} V$ is _____.

Theoretical Solution

Let the matrix of vectors be

$$\mathbf{A} = \begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix} \quad (1)$$

The volume

$$V = |\det(\mathbf{A})| \quad (2)$$

Let

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Theoretical Solution

The condition that the points lie on the plane

$$\alpha + \beta + \gamma = k_1 \quad (3)$$

gives

$$\mathbf{u}^\top \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = k_1 \quad (4)$$

$$\mathbf{v}^\top \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = k_1 \quad (5)$$

$$\mathbf{w}^\top \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = k_1 \quad (6)$$

From (4), (5) and (6), we get the system of equations

$$\mathbf{A}^\top \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (7)$$

$$\mathbf{A}^\top \mathbf{x} = k_1 \mathbf{x} \quad (8)$$

Theoretical Solution

Let λ be the eigenvalue of the matrix \mathbf{M} .

$$|\mathbf{M} - \lambda \mathbf{I}| = 0 \quad (9)$$

Taking transpose,

$$|\mathbf{M} - \lambda \mathbf{I}|^T = 0 \quad (10)$$

$$|\mathbf{M}^T - \lambda \mathbf{I}^T| = 0 \quad (11)$$

$$|\mathbf{M}^T - \lambda \mathbf{I}| = 0 \quad (\because \mathbf{I}^T = \mathbf{I}) \quad (12)$$

Hence, λ is also an eigenvalue of \mathbf{M}^T .

$\implies \mathbf{M}$ and \mathbf{M}^T have the same eigen values.

Theoretical Solution

So, k_1 is an eigenvalue of \mathbf{A}^\top and hence an eigenvalue of \mathbf{A} .

The vectors representing the sides of the equilateral triangle are

$$\mathbf{u} - \mathbf{v} = \mathbf{A}\mathbf{c}_1 \quad (13)$$

$$\mathbf{v} - \mathbf{w} = \mathbf{A}\mathbf{c}_2 \quad (14)$$

$$\mathbf{w} - \mathbf{u} = \mathbf{A}\mathbf{c}_3 \quad (15)$$

where

$$\mathbf{c}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \mathbf{c}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (16)$$

Theoretical Solution

The condition

$$\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{v} - \mathbf{w}\| = \|\mathbf{w} - \mathbf{u}\| = L \quad (17)$$

$$\implies (\mathbf{A}\mathbf{c}_1)^\top (\mathbf{A}\mathbf{c}_1) = (\mathbf{A}\mathbf{c}_2)^\top (\mathbf{A}\mathbf{c}_2) = (\mathbf{A}\mathbf{c}_3)^\top (\mathbf{A}\mathbf{c}_3) = L^2 \quad (18)$$

$$V^2 = \det(\mathbf{G}) \quad (19)$$

where the Gram matrix is $\mathbf{G} = \mathbf{A}^\top \mathbf{A}$.

Theoretical Solution

From (8), we find an eigenvector of \mathbf{G} :

$$\mathbf{G}\mathbf{x} = \mathbf{A}^\top \mathbf{A}\mathbf{x} \quad (20)$$

$$= \mathbf{A}^\top (k_1 \mathbf{x}) \quad (21)$$

$$= k_1 (\mathbf{A}^\top \mathbf{x}) \quad (22)$$

$$= k_1 (k_1 \mathbf{x}) \quad (23)$$

$$= k_1^2 \mathbf{x} \quad (24)$$

So,

$$\lambda_1 = k_1^2 \quad (25)$$

is an eigenvalue of \mathbf{G} with eigenvector \mathbf{x} .

(18) can be written in terms of \mathbf{G} as

$$\mathbf{c}_i^\top \mathbf{G} \mathbf{c}_i = L^2 \text{ for } i = 1, 2, 3 \quad (26)$$

$$\therefore \mathbf{x}^\top \mathbf{c}_i = 0 \text{ where } i = 1, 2, 3 \quad (27)$$

The vectors \mathbf{c}_i are orthogonal to \mathbf{x} and lie in a 2D subspace W .

Since \mathbf{G} is symmetric, its other two eigenvectors (forming an orthonormal basis), $\mathbf{v}_2, \mathbf{v}_3$, span W .

Let their eigenvalues be λ_2, λ_3 .

From (26) and

$$\|\mathbf{c}_i\| = 2 \quad (28)$$

the quadratic form defined by \mathbf{G} is constant on a circle in the subspace W . This is because more than two points have same distance from centre. Let \mathbf{p} be a point in W .

$$\mathbf{p} = x_1 \mathbf{v}_2 + x_2 \mathbf{v}_3 \quad (29)$$

$$q(\mathbf{p}) = \mathbf{p}^\top \mathbf{G} \mathbf{p} \quad (30)$$

$$= (x_1 \mathbf{v}_2 + x_2 \mathbf{v}_3)^\top \mathbf{G} (x_1 \mathbf{v}_2 + x_2 \mathbf{v}_3) \quad (31)$$

$$= (x_1 \mathbf{v}_2^\top + x_2 \mathbf{v}_3^\top) (x_1 \mathbf{G} \mathbf{v}_2 + x_2 \mathbf{G} \mathbf{v}_3) \quad (32)$$

$$= (x_1 \mathbf{v}_2^\top + x_2 \mathbf{v}_3^\top) (x_1 \lambda_2 \mathbf{v}_2 + x_2 \lambda_3 \mathbf{v}_3) \quad (33)$$

$$= \lambda_2 x_1^2 + \lambda_3 x_2^2 \quad (34)$$

$$(35)$$

$q(\mathbf{p})$ is an ellipse. Since, it is actually a circle,

$$\lambda_2 = \lambda_3 \quad (36)$$

Theoretical Solution

Let

$$\lambda_2 = \lambda_3 = \lambda \quad (37)$$

Therefore, for any vector $\mathbf{p} \in W$,

$$\mathbf{p}^\top \mathbf{G} \mathbf{p} = \lambda \|\mathbf{p}\|^2 \quad (38)$$

From (38), using \mathbf{c}_1 ,

$$\mathbf{c}_1^\top \mathbf{G} \mathbf{c}_1 = \lambda \|\mathbf{c}_1\|^2 \quad (39)$$

$$L^2 = \lambda (2) \quad (40)$$

$$\lambda = \frac{L^2}{2} \quad (41)$$

The eigenvalues of \mathbf{G} are $k_1^2, \frac{L^2}{2}, \frac{L^2}{2}$.

Theoretical Solution

For a matrix \mathbf{M} , if its eigenvalues are $\lambda_1, \lambda_2, \lambda_3$,

$$\det(\mathbf{M}) = \lambda_1 \lambda_2 \lambda_3 \quad (42)$$

Using (42),

$$\det(\mathbf{G}) = k_1^2 \left(\frac{L^2}{2} \right) \left(\frac{L^2}{2} \right) = \frac{k_1^2 L^4}{4} \quad (43)$$

The volume is

$$V = \sqrt{\det(\mathbf{G})} = \frac{k_1 L^2}{2} \quad (44)$$

Theoretical Solution

$$\mathbf{P} : 3x + 2y + 3z = 16 \implies \mathbf{n} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}, k_2 = 16 \quad (45)$$

$$x + y + z = 7 \implies \mathbf{n}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, k_1 = 7 \quad (46)$$

$$(47)$$

The distance condition gives two planes

$$\mathbf{P}_1 : 3x + 2y + 3z = 18 \text{ and } \mathbf{P}_2 : 3x + 2y + 3z = 14 \quad (48)$$

Let altitude of the triangle be h_T and θ be the angle between the plane normals.

Theoretical Solution

$$h_T = \frac{d_{P_1 P_2}}{\sin \theta} \quad (49)$$

$$d_{P_1 P_2} = \frac{|18 - 14|}{\|\mathbf{n}\|} = \frac{4}{\sqrt{22}} \quad (50)$$

$$\sin \theta = \sqrt{1 - \left(\frac{\mathbf{n}^\top \mathbf{n}_1}{\|\mathbf{n}\| \|\mathbf{n}_1\|} \right)^2} = \sqrt{1 - \left(\frac{8}{\sqrt{22} \sqrt{3}} \right)^2} = \frac{1}{\sqrt{33}} \quad (51)$$

$$h_T = \frac{4/\sqrt{22}}{1/\sqrt{33}} = 4\sqrt{\frac{3}{2}} \quad (52)$$

$$L^2 = \left(\frac{2}{\sqrt{3}}h_T\right)^2 = \frac{4}{3}\left(16 \times \frac{3}{2}\right) = 32 \quad (53)$$

$$V = \frac{|7|(32)}{2} = 112 \quad (54)$$

$$\therefore \frac{80}{3}V = \frac{80}{3}(112) = \frac{8960}{3} \quad (55)$$

Plot

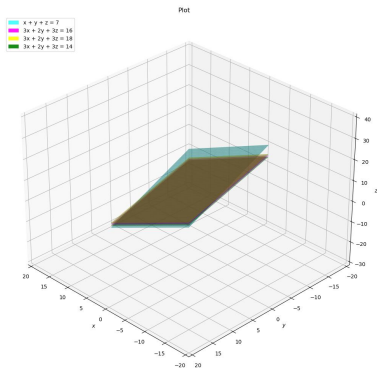


Figure: Plot