2.10.85

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Question: Let **P** be the plane $3x+2y+3z = 16$ and let $S : \alpha\hat{i}+\beta\hat{j}+\gamma\hat{k},$ where $\alpha+\beta+\gamma = 7$ and the distance of (α, β, γ) from the plane is $\frac{2}{\sqrt{22}}$. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three distinct vectors in S such that $|\mathbf{u}-\mathbf{v}| = |\mathbf{v}-\mathbf{w}| = |\mathbf{w}-\mathbf{u}|$. Let V be the volume of the parallelepiped determined by vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Then the value of $\frac{80}{3}V$ is ______.

Solution:

Let the matrix of vectors be

$$\mathbf{A} = \begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix} \tag{1}$$

The volume

$$V = |det(\mathbf{A})| \tag{2}$$

Let

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The condition that the points lie on the plane

$$\alpha + \beta + \gamma = k_1 \tag{3}$$

gives

$$\mathbf{u}^\top \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = k_1 \tag{4}$$

$$\mathbf{v}^\top \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = k_1 \tag{5}$$

$$\mathbf{w}^\top \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = k_1 \tag{6}$$

From (4), (5) and (6), we get the system of equations

$$\mathbf{A}^\top \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{7}$$

$$\mathbf{A}^\top \mathbf{x} = k_1 \mathbf{x} \tag{8}$$

Let λ be the eigenvalue of the matrix $\mathbf{M}$.

$$\left|\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{9}$$

Taking transpose,

$$\left|\mathbf{M} - \lambda\mathbf{I}\right|^\top = 0 \tag{10}$$

$$\left|\mathbf{M}^\top - \lambda\mathbf{I}^\top\right| = 0 \tag{11}$$

$$\left|\mathbf{M}^\top - \lambda\mathbf{I}\right| = 0 \ \left(\because \mathbf{I}^\top = \mathbf{I}\right) \tag{12}$$

Hence, λ is also an eigenvalue of $\mathbf{M}^\top$.
$\implies$ $\mathbf{M}$ and $\mathbf{M}^\top$ have the same eigen values.
So, k_1 is an eigenvalue of $\mathbf{A}^\top$ and hence an eigenvalue of $\mathbf{A}$.
The vectors representing the sides of the equilateral triangle are

$$\mathbf{u} - \mathbf{v} = \mathbf{A}\mathbf{c_1} \tag{13}$$

$$\mathbf{v} - \mathbf{w} = \mathbf{A}\mathbf{c_2} \tag{14}$$

$$\mathbf{w} - \mathbf{u} = \mathbf{A}\mathbf{c_3} \tag{15}$$

where

$$\mathbf{c_1} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{c_2} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \mathbf{c_3} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{16}$$

The condition

$$\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{v} - \mathbf{w}\| = \|\mathbf{w} - \mathbf{u}\| = L \tag{17}$$

$$\implies (\mathbf{A}\mathbf{c_1})^\top (\mathbf{A}\mathbf{c_1}) = (\mathbf{A}\mathbf{c_2})^\top (\mathbf{A}\mathbf{c_2}) = (\mathbf{A}\mathbf{c_3})^\top (\mathbf{A}\mathbf{c_3}) = L^2 \tag{18}$$

$$V^2 = det\left(\mathbf{G}\right) \tag{19}$$

where the Gram matrix is $\mathbf{G} = \mathbf{A}^\top\mathbf{A}$. From (8), we find an eigenvector of $\mathbf{G}$:

$$\mathbf{G}\mathbf{x} = \mathbf{A}^\top\mathbf{A}\mathbf{x} \tag{20}$$

$$= \mathbf{A}^\top\left(k_1\mathbf{x}\right) \tag{21}$$

$$= k_1\left(\mathbf{A}^\top\mathbf{x}\right) \tag{22}$$

$$= k_1\left(k_1\mathbf{x}\right) \tag{23}$$

$$= k_1^2\mathbf{x} \tag{24}$$

So,

$$\lambda_1 = k_1^2 \tag{25}$$

is an eigenvalue of $\mathbf{G}$ with eigenvector $\mathbf{x}$.

(18) can be written in terms of $\mathbf{G}$ as

$$\mathbf{c_i}^\top \mathbf{G} \mathbf{c_i} = L^2 \text{ for } i = 1, 2, 3 \tag{26}$$

$$\because \mathbf{x}^\top \mathbf{c_i} = 0 \text{ where } i = 1, 2, 3 \tag{27}$$

The vectors $\mathbf{c_i}$ are orthogonal to $\mathbf{x}$ and lie in a 2D subspace W.
Since $\mathbf{G}$ is symmetric, its other two eigenvectors (forming an orthonormal basis), $\mathbf{v_2}, \mathbf{v_3}$, span W.
Let their eigenvalues be λ_2, λ_3.
From (26) and

$$\|\mathbf{c_i}\| = 2 \tag{28}$$

the quadratic form defined by $\mathbf{G}$ is constant on a circle in the subspace W. This is because more than two points have same distance from centre. Let $\mathbf{p}$ be a point in W.

$$\mathbf{p} = x_1 \mathbf{v_2} + x_2 \mathbf{v_3} \tag{29}$$

Let $q(\mathbf{p})$ be the quadratic form.

$$\begin{align}
q(\mathbf{p}) &= \mathbf{p}^\top \mathbf{G} \mathbf{p} \tag{30}\\
&= (x_1 \mathbf{v_2} + x_2 \mathbf{v_3})^\top \mathbf{G} (x_1 \mathbf{v_2} + x_2 \mathbf{v_3}) \tag{31}\\
&= \left(x_1 \mathbf{v_2}^\top + x_2 \mathbf{v_3}^\top\right)(x_1 \mathbf{G} \mathbf{v_2} + x_2 \mathbf{G} \mathbf{v_3}) \tag{32}\\
&= \left(x_1 \mathbf{v_2}^\top + x_2 \mathbf{v_3}^\top\right)(x_1 \lambda_2 \mathbf{v_2} + x_2 \lambda_3 \mathbf{v_3}) \tag{33}\\
&= \lambda_2 x_1^2 + \lambda_3 x_2^2 \tag{34}\\
& \tag{35}
\end{align}$$

$q(\mathbf{p})$ is an ellipse. Since, it is actually a circle,

$$\lambda_2 = \lambda_3 \tag{36}$$

Let

$$\lambda_2 = \lambda_3 = \lambda \tag{37}$$

Therefore, for any vector $\mathbf{p} \in W$,

$$\mathbf{p}^\top \mathbf{G} \mathbf{p} = \lambda \|\mathbf{p}\|^2 \tag{38}$$

From (38), using $\mathbf{c_1}$,

$$\begin{align}
\mathbf{c_1}^\top \mathbf{G} \mathbf{c_1} &= \lambda \|\mathbf{c_1}\|^2 \tag{39}\\
L^2 &= \lambda (2) \tag{40}\\
\lambda &= \frac{L^2}{2} \tag{41}
\end{align}$$

The eigenvalues of $\mathbf{G}$ are $k_1^2, \frac{L^2}{2}, \frac{L^2}{2}$.

For a matrix $\mathbf{M}$, if its eigenvalues are $\lambda_1, \lambda_2, \lambda_3$,

$$det(M) = \lambda_1 \lambda_2 \lambda_3 \tag{42}$$

Using (42),

$$det(\mathbf{G}) = k_1^2 \left(\frac{L^2}{2}\right)\left(\frac{L^2}{2}\right) = \frac{k_1^2 L^4}{4} \tag{43}$$

The volume is

$$V = \sqrt{det(\mathbf{G})} = \frac{k_1 L^2}{2} \tag{44}$$

$$\mathbf{P} : 3x + 2y + 3z = 16 \implies \mathbf{n} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}, k_2 = 16 \tag{45}$$

$$x + y + z = 7 \implies \mathbf{n_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, k_1 = 7 \tag{46}$$

$$\tag{47}$$

The distance condition gives two planes

$$\mathbf{P_1} : 3x + 2y + 3z = 18 \text{ and } \mathbf{P_2} : 3x + 2y + 3z = 14 \tag{48}$$

Let altitude of the triangle be h_T and θ be the angle between the plane normals.

$$h_T = \frac{d_{P_1 P_2}}{\sin\theta} \tag{49}$$

$$d_{P_1 P_2} = \frac{|18 - 14|}{\|\mathbf{n}\|} = \frac{4}{\sqrt{22}} \tag{50}$$

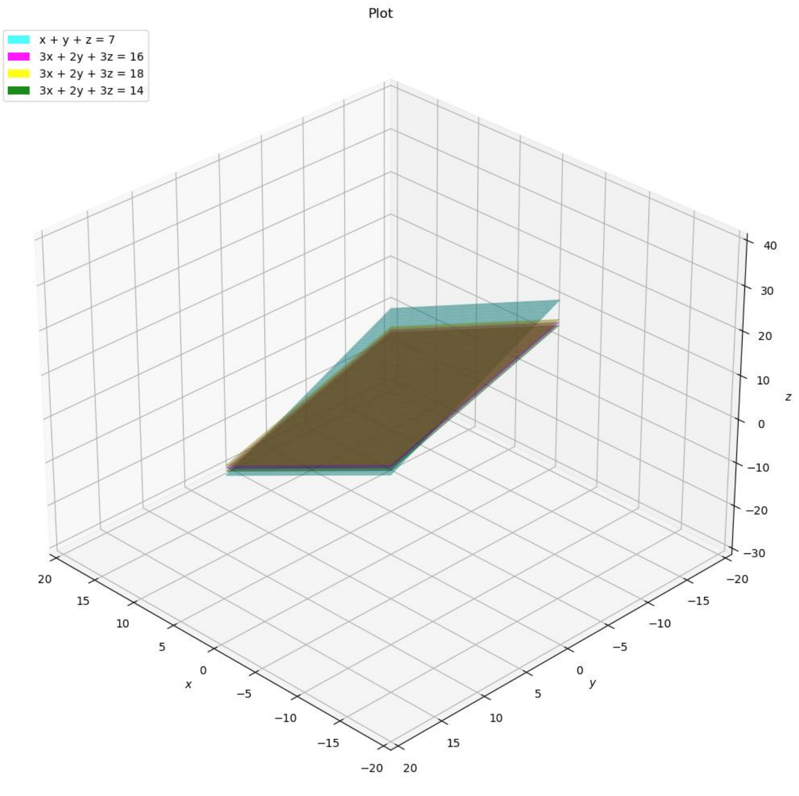
$$\sin\theta = \sqrt{1 - \left(\frac{\mathbf{n}^\top \mathbf{n_1}}{\|\mathbf{n}\| \|\mathbf{n_1}\|}\right)^2} = \sqrt{1 - \left(\frac{8}{\sqrt{22}\sqrt{3}}\right)^2} = \frac{1}{\sqrt{33}} \tag{51}$$

$$h_T = \frac{4/\sqrt{22}}{1/\sqrt{33}} = 4\sqrt{\frac{3}{2}} \tag{52}$$

$$L^2 = \left(\frac{2}{\sqrt{3}} h_T\right)^2 = \frac{4}{3}\left(16 \times \frac{3}{2}\right) = 32 \tag{53}$$

$$V = \frac{|7|(32)}{2} = 112 \tag{54}$$

$$\therefore \frac{80}{3} V = \frac{80}{3}(112) = \frac{8960}{3} \tag{55}$$


Plot
x + y + z = 7
3x + 2y + 3z = 16
3x + 2y + 3z = 18
3x + 2y + 3z = 14
x
y
z
20
15
10
5
0
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−10
−15
−20
40
30
20
10
0
−10
−20
−30

Plot