

2.10.85

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Question: Let \mathbf{P} be the plane $3x+2y+3z = 16$ and let $S : \alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$, where $\alpha+\beta+\gamma = 7$ and the distance of (α, β, γ) from the plane is $\frac{2}{\sqrt{22}}$. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three distinct vectors in S such that $|\mathbf{u} - \mathbf{v}| = |\mathbf{v} - \mathbf{w}| = |\mathbf{w} - \mathbf{u}|$. Let V be the volume of the parallelepiped determined by vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Then the value of $\frac{80}{3}V$ is _____.

Solution:

Let the matrix of vectors be $\mathbf{A} = \begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix}$. The volume $V = \|\mathbf{A}\|$. Let $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. The condition that the points lie on the plane $\alpha + \beta + \gamma = k_1$ where $k_1 = 7$ is:

$$\mathbf{A}^T \mathbf{x} = k_1 \mathbf{x} \quad (1)$$

So, k_1 is an eigenvalue of \mathbf{A}^T and hence an eigenvalue of \mathbf{A} . The vectors representing the sides of the equilateral triangle are

$$\mathbf{u} - \mathbf{v} = \mathbf{A}\mathbf{c}_1 \quad (2)$$

$$\mathbf{v} - \mathbf{w} = \mathbf{A}\mathbf{c}_2 \quad (3)$$

$$\mathbf{w} - \mathbf{u} = \mathbf{A}\mathbf{c}_3 \quad (4)$$

$$\text{where } \mathbf{c}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \mathbf{c}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

The condition $\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{v} - \mathbf{w}\| = \|\mathbf{w} - \mathbf{u}\| = L$ implies:

$$(\mathbf{A}\mathbf{c}_1)^T (\mathbf{A}\mathbf{c}_1) = (\mathbf{A}\mathbf{c}_2)^T (\mathbf{A}\mathbf{c}_2) = (\mathbf{A}\mathbf{c}_3)^T (\mathbf{A}\mathbf{c}_3) = L^2 \quad (5)$$

$$V^2 = |\mathbf{G}| \quad (6)$$

where the Gram matrix is $\mathbf{G} = \mathbf{A}^T \mathbf{A}$. From (1), we find an eigenvector of \mathbf{G} :

$$\mathbf{G}\mathbf{x} = \mathbf{A}^T \mathbf{A}\mathbf{x} \quad (7)$$

$$= \mathbf{A}^T (k_1 \mathbf{x}) \quad (8)$$

$$= k_1 (\mathbf{A}^T \mathbf{x}) \quad (9)$$

$$= k_1 (k_1 \mathbf{x}) \quad (10)$$

$$= k_1^2 \mathbf{x} \quad (11)$$

So, $\lambda_1 = k_1^2$ is an eigenvalue of \mathbf{G} with eigenvector \mathbf{x} .

(5) can be written in terms of \mathbf{G} as

$$\mathbf{c}_i^T \mathbf{G}\mathbf{c}_i = L^2 \text{ for } i = 1, 2, 3 \quad (12)$$

The vectors \mathbf{c}_i are orthogonal to \mathbf{x} because $\mathbf{x}^\top \mathbf{c}_i = 0$ and lie in a 2D subspace W .

Since \mathbf{G} is symmetric, its other two eigenvectors, $\mathbf{e}_2, \mathbf{e}_3$, span W .

Let their eigenvalues be λ_2, λ_3 .

$\because \mathbf{c}_1^\top \mathbf{G} \mathbf{c}_1 = L^2$ and $\mathbf{c}_2^\top \mathbf{G} \mathbf{c}_2 = L^2$; $\|\mathbf{c}_1\|^2 = 2$ and $\|\mathbf{c}_2\|^2 = 2$, the quadratic form defined by \mathbf{G} is constant on a circle (The ellipse becomes a circle due to equal distance of the points from centre) in the subspace W .

This requires the eigenvalues corresponding to this subspace to be equal: $\lambda_2 = \lambda_3 = \lambda$.

Therefore, for any vector $\mathbf{w} \in W$, $\mathbf{w}^\top \mathbf{G} \mathbf{w} = \lambda \|\mathbf{w}\|^2$. Using \mathbf{c}_1 :

$$\mathbf{c}_1^\top \mathbf{G} \mathbf{c}_1 = \lambda \|\mathbf{c}_1\|^2 \quad (13)$$

$$L^2 = \lambda (2) \quad (14)$$

$$\lambda = \frac{L^2}{2} \quad (15)$$

The eigenvalues of \mathbf{G} are $k_1^2, \frac{L^2}{2}, \frac{L^2}{2}$. The determinant of \mathbf{G} is the product of its eigenvalues:

$$|\mathbf{G}| = k_1^2 \left(\frac{L^2}{2} \right) \left(\frac{L^2}{2} \right) = \frac{k_1^2 L^4}{4} \quad (16)$$

The volume is

$$V = \sqrt{|\mathbf{G}|} = \frac{k_1 L^2}{2} \quad (17)$$

$$\mathbf{P} : 3x + 2y + 3z = 16 \implies \mathbf{n} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}, k_2 = 16 \quad (18)$$

$$x + y + z = 7 \implies \mathbf{n}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, k_1 = 7 \quad (19)$$

$$(20)$$

The distance condition gives two planes $\mathbf{P}_1 : 3x + 2y + 3z = 18$ and $\mathbf{P}_2 : 3x + 2y + 3z = 14$.

Let altitude of the triangle be h_T

$$h_T = \frac{d_{P_1 P_2}}{\sin \theta}, \text{ where } \theta \text{ is the angle between the plane normals.} \quad (21)$$

$$d_{P_1 P_2} = \frac{|18 - 14|}{\|\mathbf{n}\|} = \frac{4}{\sqrt{22}} \quad (22)$$

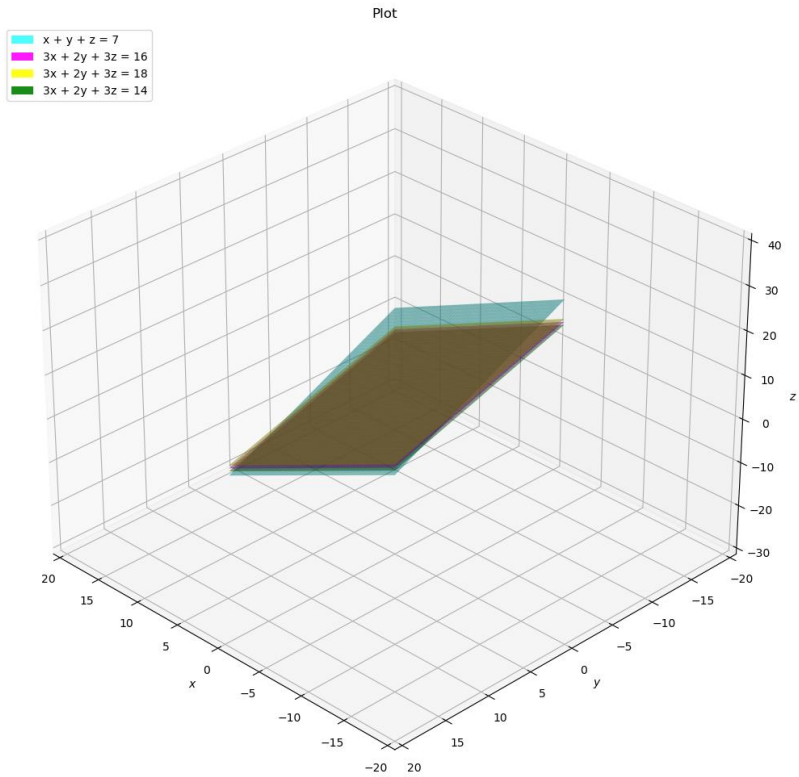
$$\sin \theta = \sqrt{1 - \left(\frac{\mathbf{n}^\top \mathbf{n}_1}{\|\mathbf{n}\| \|\mathbf{n}_1\|} \right)^2} = \sqrt{1 - \left(\frac{8}{\sqrt{22} \sqrt{3}} \right)^2} = \frac{1}{\sqrt{33}} \quad (23)$$

$$h_T = \frac{4/\sqrt{22}}{1/\sqrt{33}} = 4\sqrt{\frac{3}{2}} \quad (24)$$

$$L^2 = \left(\frac{2}{\sqrt{3}}h_T\right)^2 = \frac{4}{3}\left(16 \times \frac{3}{2}\right) = 32 \quad (25)$$

$$V = \frac{|7|(32)}{2} = 112 \quad (26)$$

$$\therefore \frac{80}{3}V = \frac{80}{3}(112) = \frac{8960}{3} \quad (27)$$



Plot