# 2.10.85

Puni Aditya - EE25BTECH11046

**Question:** Let $\mathbf{P}$ be the plane $3x+2y+3z = 16$ and let $S : \alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$, where $\alpha+\beta+\gamma = 7$ and the distance of $(\alpha, \beta, \gamma)$ from the plane is $\frac{2}{\sqrt{22}}$. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three distinct vectors in $S$ such that $|\mathbf{u} - \mathbf{v}| = |\mathbf{v} - \mathbf{w}| = |\mathbf{w} - \mathbf{u}|$. Let $V$ be the volume of the parallelepiped determined by vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Then the value of $\frac{80}{3}V$ is ______.

**Solution:**

Let the matrix of vectors be $\mathbf{A} = \begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix}$. The volume $V = \left\|\mathbf{A}\right\|$. Let $\mathbf{x} = \begin{pmatrix}1\\1\\1\end{pmatrix}$. The condition that the points lie on the plane $\alpha + \beta + \gamma = k_1$ where $k_1 = 7$ is:

$$\mathbf{A}^\top \mathbf{x} = k_1 \mathbf{x} \tag{1}$$

So, $k_1$ is an eigenvalue of $\mathbf{A}^\top$ and hence an eigenvalue of $\mathbf{A}$.
The vectors representing the sides of the equilateral triangle are

$$\mathbf{u} - \mathbf{v} = \mathbf{A}\mathbf{c_1} \tag{2}$$
$$\mathbf{v} - \mathbf{w} = \mathbf{A}\mathbf{c_2} \tag{3}$$
$$\mathbf{w} - \mathbf{u} = \mathbf{A}\mathbf{c_3} \tag{4}$$

where $\mathbf{c_1} = \begin{pmatrix}-1\\1\\0\end{pmatrix}, \mathbf{c_2} = \begin{pmatrix}0\\1\\-1\end{pmatrix}, \mathbf{c_3} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$.
The condition $||\mathbf{u} - \mathbf{v}|| = ||\mathbf{v} - \mathbf{w}|| = ||\mathbf{w} - \mathbf{u}|| = L$ implies:

$$(\mathbf{A}\mathbf{c_1})^\top (\mathbf{A}\mathbf{c_1}) = (\mathbf{A}\mathbf{c_2})^\top (\mathbf{A}\mathbf{c_2}) = (\mathbf{A}\mathbf{c_3})^\top (\mathbf{A}\mathbf{c_3}) = L^2 \tag{5}$$

$$V^2 = \left|\mathbf{G}\right| \tag{6}$$

where the Gram matrix is $\mathbf{G} = \mathbf{A}^\top\mathbf{A}$. From (1), we find an eigenvector of $\mathbf{G}$:

$$\mathbf{G}\mathbf{x} = \mathbf{A}^\top\mathbf{A}\mathbf{x} \tag{7}$$
$$= \mathbf{A}^\top (k_1\mathbf{x}) \tag{8}$$
$$= k_1 \left(\mathbf{A}^\top\mathbf{x}\right) \tag{9}$$
$$= k_1 (k_1\mathbf{x}) \tag{10}$$
$$= k_1^2\mathbf{x} \tag{11}$$

**Here, the approach is wrong because it is inherently assumed that A and $\mathbf{A}^\top$ have the same eigenvector, which is only true in the case of symmetric matrices. Here, A need not be symmetric.**

So, $\lambda_1 = k_1^2$ is an eigenvalue of $\mathbf{G}$ with eigenvector $\mathbf{x}$.
(5) can be written in terms of $\mathbf{G}$ as

$$\mathbf{c_i}^\top \mathbf{G}\mathbf{c_i} = L^2 \text{ for } i = 1, 2, 3 \tag{12}$$

The vectors $\mathbf{c_i}$ are orthogonal to $\mathbf{x}$ because $\mathbf{x}^\top \mathbf{c_i} = 0$ and lie in a 2D subspace $W$.
Since $\mathbf{G}$ is symmetric, its other two eigenvectors, $\mathbf{e_2}, \mathbf{e_3}$, span $W$.
Let their eigenvalues be $\lambda_2, \lambda_3$.
$\because$ $\mathbf{c_1}^\top \mathbf{G}\mathbf{c_1} = L^2$ and $\mathbf{c_2}^\top \mathbf{G}\mathbf{c_2} = L^2$; $\|\mathbf{c_1}\|^2 = 2$ and $\|\mathbf{c_2}\|^2 = 2$, the quadratic form defined by $\mathbf{G}$ is constant on a circle (The ellipse becomes a circle due to equal distance of the points from centre) in the subspace $W$.
This requires the eigenvalues corresponding to this subspace to be equal: $\lambda_2 = \lambda_3 = \lambda$. Therefore, for any vector $\mathbf{w} \in W$, $\mathbf{w}^\top \mathbf{G}\mathbf{w} = \lambda \|\mathbf{w}\|^2$. Using $\mathbf{c_1}$:

$$\mathbf{c_1}^\top \mathbf{G}\mathbf{c_1} = \lambda \|\mathbf{c_1}\|^2 \tag{13}$$

$$L^2 = \lambda (2) \tag{14}$$

$$\lambda = \frac{L^2}{2} \tag{15}$$

The eigenvalues of $\mathbf{G}$ are $k_1^2, \frac{L^2}{2}, \frac{L^2}{2}$. The determinant of $\mathbf{G}$ is the product of its eigenvalues:

$$\left|\mathbf{G}\right| = k_1^2 \left(\frac{L^2}{2}\right)\left(\frac{L^2}{2}\right) = \frac{k_1^2 L^4}{4} \tag{16}$$

The volume is

$$V = \sqrt{\left|\mathbf{G}\right|} = \frac{k_1 L^2}{2} \tag{17}$$

$$\mathbf{P} : 3x + 2y + 3z = 16 \implies \mathbf{n} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}, k_2 = 16 \tag{18}$$

$$x + y + z = 7 \implies \mathbf{n_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, k_1 = 7 \tag{19}$$

$$\tag{20}$$

The distance condition gives two planes $\mathbf{P_1}$ : $3x + 2y + 3z = 18$ and $\mathbf{P_2}$ : $3x + 2y + 3z = 14$. Let altitude of the triangle be $h_T$

$$h_T = \frac{d_{P_1 P_2}}{\sin\theta}, \text{ where } \theta \text{ is the angle between the plane normals.} \tag{21}$$

$$d_{P_1 P_2} = \frac{|18 - 14|}{\|\mathbf{n}\|} = \frac{4}{\sqrt{22}} \tag{22}$$

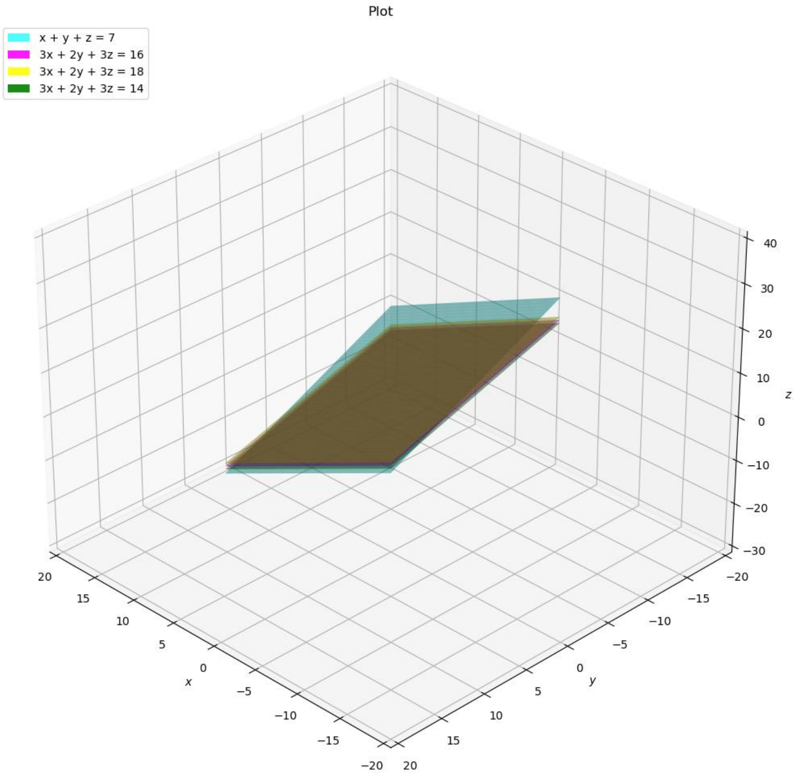
$$\sin\theta = \sqrt{1 - \left(\frac{\mathbf{n}^\top \mathbf{n_1}}{\|\mathbf{n}\|\,\|\mathbf{n_1}\|}\right)^2} = \sqrt{1 - \left(\frac{8}{\sqrt{22}\,\sqrt{3}}\right)^2} = \frac{1}{\sqrt{33}} \tag{23}$$

$$h_T = \frac{4/\sqrt{22}}{1/\sqrt{33}} = 4\sqrt{\frac{3}{2}} \tag{24}$$

$$L^2 = \left(\frac{2}{\sqrt{3}}h_T\right)^2 = \frac{4}{3}\left(16 \times \frac{3}{2}\right) = 32 \tag{25}$$

$$V = \frac{|7|\,(32)}{2} = 112 \tag{26}$$

$$\therefore \frac{80}{3}V = \frac{80}{3}(112) = \frac{8960}{3} \tag{27}$$




Plot