10.7.75

Puni Aditya - EE25BTECH11046

Question: Find the equations of tangents drawn from origin to the circle $x^2+y^2-2rx-2hy+h^2=0$,are

1) $x=0$
2) $y=0$
3) $\left(h^2-r^2\right)x-2rhy=0$
4) $\left(h^2-r^2\right)x+2rhy=0$

Solution:

A general conic section is described by the equation

$$\mathbf{x}^\top\mathbf{V}\mathbf{x}+2\mathbf{u}^\top\mathbf{x}+f=0 \tag{1}$$

where $\mathbf{V}$ is a symmetric matrix. A line passing through a point $\mathbf{h}$ and having unit direction vector $\mathbf{m}$ is

$$\mathbf{x}=\mathbf{h}+k\mathbf{m} \tag{2}$$

Substitute the line equation into the conic to find points of intersection.

$$k^2\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)+2k\left(\mathbf{m}^\top\mathbf{V}\mathbf{h}+\mathbf{u}^\top\mathbf{m}\right)+\left(\mathbf{h}^\top\mathbf{V}\mathbf{h}+2\mathbf{u}^\top\mathbf{h}+f\right)=0 \tag{3}$$

For the line to be tangent to the conic, the discriminant of quadratic in k must be zero. Let $g(\mathbf{h})=\mathbf{h}^\top\mathbf{V}\mathbf{h}+2\mathbf{u}^\top\mathbf{h}+f$ be the value of the conic expression at the point $\mathbf{h}$.

$$\left(\mathbf{m}^\top\mathbf{V}\mathbf{h}+\mathbf{u}^\top\mathbf{m}\right)^2-\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)(g(\mathbf{h}))=0 \tag{4}$$

$$\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\right)^2-g(\mathbf{h})\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)=0 \tag{5}$$

$$\mathbf{m}^\top\left((\mathbf{V}\mathbf{h}+\mathbf{u})(\mathbf{V}\mathbf{h}+\mathbf{u})^\top\right)\mathbf{m}-\mathbf{m}^\top(g(\mathbf{h})\mathbf{V})\mathbf{m}=0 \tag{6}$$

$$\mathbf{m}^\top\left[(\mathbf{V}\mathbf{h}+\mathbf{u})(\mathbf{V}\mathbf{h}+\mathbf{u})^\top-\left(\mathbf{h}^\top\mathbf{V}\mathbf{h}+2\mathbf{u}^\top\mathbf{h}+f\right)\mathbf{V}\right]\mathbf{m}=0 \tag{7}$$

$$\text{Let } \boldsymbol{\Sigma}=(\mathbf{V}\mathbf{h}+\mathbf{u})(\mathbf{V}\mathbf{h}+\mathbf{u})^\top-\left(\mathbf{h}^\top\mathbf{V}\mathbf{h}+2\mathbf{u}^\top\mathbf{h}+f\right)\mathbf{V} \tag{8}$$

$$\mathbf{m}^\top\boldsymbol{\Sigma}\mathbf{m}=0 \tag{9}$$

This is the general equation for the directions of tangents from an arbitrary point $\mathbf{h}$.

For $\mathbf{h}=\mathbf{0}$,

$$\boldsymbol{\Sigma}=(\mathbf{V}\mathbf{0}+\mathbf{u})(\mathbf{V}\mathbf{0}+\mathbf{u})^\top-\left(\mathbf{0}^\top\mathbf{V}\mathbf{0}+2\mathbf{u}^\top\mathbf{0}+f\right)\mathbf{V}=\mathbf{u}\mathbf{u}^\top-f\mathbf{V} \tag{10}$$

To solve this, the symmetric matrix $\mathbf{\Sigma}$ is diagonalized. The eigendecomposition of $\mathbf{\Sigma}$ is $\mathbf{\Sigma} = \mathbf{PDP}^\top$, where:
$\mathbf{D}$ is a diagonal matrix with the eigenvalues of $\mathbf{\Sigma}$ on its diagonal

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{11}$$

$\mathbf{P}$ is an orthogonal matrix whose columns are the corresponding orthonormal eigenvectors. So, $\mathbf{P}^\top\mathbf{P} = \mathbf{PP}^\top = 1$.

$$\mathbf{P} = \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} \end{pmatrix} \tag{12}$$

Substitute (11) and (12) in (9)

$$\mathbf{m}^\top(\mathbf{PDP}^\top)\mathbf{m} = 0 \tag{13}$$

$$\mathbf{m} = \mathbf{P}\begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm\sqrt{|\lambda_1|} \end{pmatrix} \tag{14}$$

For the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} -r \\ -h \end{pmatrix}, \ f = h^2 \tag{15}$$

From (8),

$$\mathbf{\Sigma} = \begin{pmatrix} -r \\ -h \end{pmatrix}\begin{pmatrix} -r & -h \end{pmatrix} - h^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r^2 - h^2 & rh \\ rh & 0 \end{pmatrix} \tag{16}$$

The characteristic equation is $\left|\mathbf{\Sigma} - \lambda\mathbf{I}\right| = 0$.
Let $\lambda^2 + a_1\lambda + a_2 = 0$. Using Faddeev-Leverrier Method,

$$\mathbf{B_1} = \mathbf{A} \tag{17}$$

$$a_1 = -tr\left(\mathbf{B_1}\right) = -\left(r^2 - h^2\right) \tag{18}$$

$$\mathbf{B_2} = \mathbf{A}\left(\mathbf{B_1} + a_1\mathbf{I}\right) \tag{19}$$

$$\mathbf{B_2} = \begin{pmatrix} r^2h^2 & 2rh\left(r^2 - h^2\right) \\ 0 & r^2h^2 \end{pmatrix} \tag{20}$$

$$a_2 = -\frac{1}{2}tr\left(\mathbf{B_2}\right) \tag{21}$$

$$a_2 = -r^2h^2 \tag{22}$$

So, $\lambda^2 - \left(r^2 - h^2\right)\lambda - r^2h^2 = 0$, giving the eigenvalues $\lambda_1 = r^2$ and $\lambda_2 = -h^2$.

$$\mathbf{P} = \frac{1}{\sqrt{r^2 + h^2}}\begin{pmatrix} r & h \\ h & -r \end{pmatrix} \tag{23}$$

Using (14),

$$\mathbf{m_1} = \frac{1}{\sqrt{r^2+h^2}} \begin{pmatrix} r & h \\ h & -r \end{pmatrix} \begin{pmatrix} \sqrt{\left|-h^2\right|} \\ \sqrt{\left|r^2\right|} \end{pmatrix} \tag{24}$$

$$= \frac{1}{\sqrt{r^2+h^2}} \begin{pmatrix} r & h \\ h & -r \end{pmatrix} \begin{pmatrix} h \\ r \end{pmatrix} \tag{25}$$

$$= \frac{1}{\sqrt{r^2+h^2}} \begin{pmatrix} 2rh \\ h^2 - r^2 \end{pmatrix} \tag{26}$$

$$\mathbf{m_1} = \frac{1}{\sqrt{r^2+h^2}} \begin{pmatrix} r & h \\ h & -r \end{pmatrix} \begin{pmatrix} \sqrt{\left|-h^2\right|} \\ -\sqrt{\left|r^2\right|} \end{pmatrix} \tag{27}$$

$$= \frac{1}{\sqrt{r^2+h^2}} \begin{pmatrix} r & h \\ h & -r \end{pmatrix} \begin{pmatrix} h \\ -r \end{pmatrix} \tag{28}$$

$$= \frac{1}{\sqrt{r^2+h^2}} \begin{pmatrix} 0 \\ h^2 + r^2 \end{pmatrix} \tag{29}$$

For $\mathbf{m_1}$, $\left(h^2 - r^2\right) x - (2rh)\, y = 0$. This is option **3**.
For $\mathbf{m_2}$, $\left(r^2 + h^2\right) \times x - 0 \times y = 0 \implies x = 0$. This is option **1**.
So, options (1) and (3) are true.

Example:
Let r = 3, h = 2.
For the circle $x^2 + y^2 - 6x - 4y + 4 = 0$,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, \ f = 4 \tag{30}$$

From (8),

$$\boldsymbol{\Sigma} = \begin{pmatrix} -3 \\ -2 \end{pmatrix} \begin{pmatrix} -3 & -2 \end{pmatrix} - 4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 6 & 0 \end{pmatrix} \tag{31}$$

The characteristic equation is $\left|\boldsymbol{\Sigma} - \lambda \mathbf{I}\right| = 0$.
Let $\lambda^2 + a_1 \lambda + a_2 = 0$. Using Faddeev-Leverrier Method,

$$\mathbf{B_1} = \mathbf{A} \tag{32}$$

$$a_1 = -tr\left(\mathbf{B_1}\right) = -(5) \tag{33}$$

$$\mathbf{B_2} = \mathbf{A}\left(\mathbf{B_1} + a_1 \mathbf{I}\right) \tag{34}$$

$$\mathbf{B_2} = \begin{pmatrix} 36 & 60 \\ 0 & 36 \end{pmatrix} \tag{35}$$

$$a_2 = -\frac{1}{2} tr\left(\mathbf{B_2}\right) \tag{36}$$

$$a_2 = -36 \tag{37}$$

So, $\lambda^2 - 5\lambda - 36 = 0$, giving the eigenvalues $\lambda_1 = 9$ and $\lambda_2 = -4$.

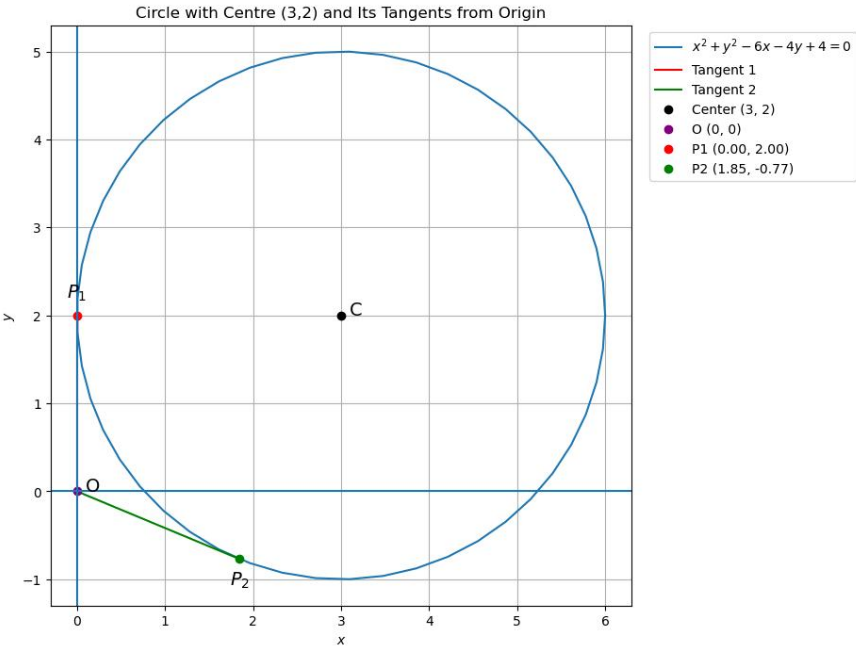
$$\mathbf{P} = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix} \tag{38}$$

Using (26) and (29),

$$\mathbf{m_1} = \frac{1}{\sqrt{13}} \begin{pmatrix} 12 \\ -5 \end{pmatrix} \tag{39}$$

$$\mathbf{m_2} = \frac{1}{\sqrt{13}} \begin{pmatrix} 0 \\ 13 \end{pmatrix} \tag{40}$$

For $\mathbf{m_1}$, $(-5)\,x - (12)\,y = 0$. This is option **3**.
For $\mathbf{m_2}$, $13 \times x - 0 \times y = 0 \implies x = 0$. This is option **1**.
So, options (1) and (3) are true.



Example