## The Falling Slinky

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The slinky, released from rest hanging under its own weight, falls in a peculiar manner. The bottom stays at rest until a wave hits it from above. Two cases—one unphysical one where the slinky is able to pass through itself, and the other where the coils of the slinky collide creating a shock wave travelling down the slinky are analysed. In the former case, the bottom begins to move much later than in the latter.

Hang a slinky up so that it is supported only on top, and let it go. The bottom of the slinky stays at rest (does not move) for a lengthy time. Let us analyse the behaviour in two cases, one where the slinky coils can interpenetrate each other (a physically unrealistic situation), and one where the coils inelastically collide. For the former case see the earlier work by Calkin [1] who examines the case of a spring whose equilibrium length is non-zero so there is no inversion of the spring.

Set up a labeling of the slinky with coordinate y uniformly along the slinky. The density in this coordinate is given by  $\rho$  which is a constant. Let x be a vertical real space coordinate. Then the location of the point y along the slinky is given by x(t,y). The stretching of the slinky will be given by  $\frac{\partial x}{\partial y}$  and the force due to this stretching is  $k\frac{\partial x}{\partial y}$ . We can write the Lagrangian by

$$L = \int \left[ \frac{1}{2} \rho \left( \frac{\partial x}{\partial t} \right)^2 - \frac{1}{2} k \left( \frac{\partial}{\partial x} \right)^2 - \rho g x \right] dy \tag{1}$$

giving us the equation of motion

$$\rho \frac{\partial^2 x}{\partial t^2} - k \frac{\partial^2 x}{\partial y^2} + \rho g = 0 \tag{2}$$

Intially the slinky is supported at its top end and stationary. The solution is

$$x(y) = \frac{1}{2} \frac{\rho g}{k} y^2. \tag{3}$$

where y=0 is taken to be the bottom of the slinky and y=L the top.

After it is released, the boundary conditions at the two ends must be that  $\frac{\partial x}{\partial y} = 0$  at the two ends so that there are no forces due to the stretched spring at the ends. The solution to the equations of motion are of the form, such that the velocity of any point on the spring is 0 at t=0 is

$$x(y,t) = -\frac{1}{2}gt^2 + f(y+Vt) + g(y-Vt)$$
(4)

where  $V = \sqrt{\frac{k}{\rho}}$ . In order to have the correct boundary conditions at t = 0 we require

$$f'(y) = g'(y) \tag{5}$$

or

$$f(y) = g(y) \tag{6}$$

In order that at all times at y=0 we have  $\frac{\partial x}{\partial y} = 0$  we require

$$f'(vt) + f'(-vt) = 0 (7)$$

or

$$f(y) = f(-y) \tag{8}$$

Finally,

$$f'(L - vt) = f'(L - vt) \tag{9}$$

or

$$f(2L+y) = f(y) \tag{10}$$

or that f be periodic with period 2L.

Since at t=0, we have

$$x(0,y) = \frac{1}{2} \frac{\rho g}{k} y^2 \tag{11}$$

we have

$$f(y) = \frac{1}{4} \frac{\rho g}{k} y^2 \tag{12}$$

for -L < y < L and periodic with period 2L determining the value at other values of y.

Thus, let us consider the complete solution. We can determine it for 0 < t < L/V by dividing the interval 0 < y < L into two parts, 0 < y < L - Vt and L - Vt < y < L. For the former, we have

$$x(t,y) = -\frac{1}{2}gt^2 + \frac{1}{2}\frac{\rho g}{k}\left((y - Vt)^2 + (y + Vt)^2\right)$$
(13)

$$= -\frac{1}{2}gt^2 + \frac{1}{2}\frac{\rho g}{k}V^2t^2 + \frac{1}{2}\frac{\rho g}{k}y^2 \tag{14}$$

$$= \frac{1}{2} \frac{\rho g}{k} y^2 \tag{15}$$

Ie, the bottom of the slinky remains static. The top y > L - Vt is given by

$$x(t,y) = -\frac{1}{2}gt^2 + \frac{1}{4}\frac{\rho g}{k}\left((2L - y + Vt)^2 + (y + vt)^2\right)$$
(16)

$$= \frac{\rho g}{k} (L^2 - Ly + \frac{1}{2}y^2 + LVt) \tag{17}$$

Ie, for any point y above the junction, the part of the slinky moves with constant velocity  $\frac{gL}{V}$ . The junction point, y = L - Vt occurs at

$$X(t) = \frac{1}{2} \frac{g}{V^2} (L^2 - 2LVt) \tag{18}$$

The problem with this solution is that at the point y = L - vt, the slope  $\frac{\partial x}{\partial y}$  changes from positive to negative. For y less than L - Vt the slope is  $gy/v^2$  while for larger values of y, it is  $\frac{g}{V^2}(y - L) < 0$ . But a slope of less than zero means that the slinky has passed through itself. Before that happens the coils will crash together, and the above solution becomes invalid.

Let us assume that when the coils come together the collision is a perfectly inelastic collision. Let us assume that above some point Y(t) the coils are all together, while below that point, the slinky, as above, remains motionless. Then we have a mass  $M = \rho(L - Y)$  above that transition point, while below it, we assume as above that the slinky remains motionless. Note that if  $\frac{dY}{dt} < V$  this will clearly not be a valid solution.

The velocity in x space of that point Y(t) is

$$U = \frac{dY}{dt} \frac{\partial x}{\partial y}|_{y=Y} \tag{19}$$

Thus the Momentum equation for that mass above the transition point is

$$\frac{dMU}{dt} = -Mg - k\frac{\partial x}{\partial y}|_{y=Y} \tag{20}$$

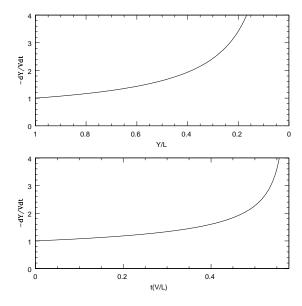


FIG. 1: Figure 1.  $-\frac{dY}{dt}$  as a function of Y and t for the boundary of the shock wave in the falling slinky.

or

$$\frac{d\left(\rho(L-Y)\frac{dY}{dt}\frac{\rho g}{k}Y\right)}{dt} = -\rho g(L-Y) - k\left(\frac{\rho g}{k}Y\right) = -\rho gL \tag{21}$$

The solution is

$$\frac{1}{V^2}(L-Y)^2(\frac{Y}{3L} + \frac{1}{6}) = \frac{1}{2}t^2 + ct \tag{22}$$

If  $c \neq 0$ , then for small t we have  $y = L - V\sqrt{ct}$ , and the velocity  $\frac{dY}{dt}$  goes to infinity as  $t \to 0$ . Thus we have

$$(L-Y)^2(\frac{2Y}{3L} + \frac{1}{3}) = V^2t^2 \tag{23}$$

As we can see from figure 1,  $-\frac{dY}{dt}$  is always greater than V. and approaches infinity as  $Y \to 0$ . This falling slinky has a shock wave where the top collapsed part of the slinky crashes into the lower stationary part, and as is usual for shock waves, they travel faster than the velocity of sound (in this case V) in the medium. In physical space,

$$\frac{dX}{dt} = \frac{dY}{dt} \frac{\partial y}{\partial x}|_{y=Y} = \frac{V^2 L t}{L - Y} \frac{g}{v^2} = gt \frac{L}{L - Y}$$
(24)

which goes to the finite velocity  $gt = g(\frac{L}{\sqrt{3}V})$  as  $Y \to 0$ .

In figure 2 I have plotted  $-\frac{V^2}{gL}\frac{dX}{dt}$  as a function of  $\frac{Y}{L}$  and Vt/L (which is  $1/\sqrt{3}$  when Y(t)=0.)

Note that initially the physical velocity of the shock front is finite  $(\frac{gL}{V})$  and is not just proportional to gt, the free fall velocity. It then falls to  $gt = \frac{g}{\sqrt{3}} \frac{L}{V}$  as the shock approaches the bottom end of the slinky. The total energy of the spring, potential plus kinetic is

$$E = M \frac{dX(t)^{2}}{dt} + \int_{0}^{Y(t)} \rho gx(y) + \frac{1}{2}k \left(\frac{\partial x}{\partial y}\right)^{2} dy = \rho \frac{g^{2}}{V^{2}} \left((L - Y)L^{2}(2\frac{Y}{3L} + \frac{1}{3}) + \frac{Y^{3}}{3}\right)$$
(25)

The total energy drops by a factor of 2, but not linearly in Y or t. The energy loss of course occurs in the collision between the coils of the slinky. Figure 3 are plots of the total energy as a function of Y and of t. The dotted curve is the potential energy in the stretched spring remaining as the shock reaches the point Y plus the gravitational potential energy at time t=0 plus the initial gravitational potential energy of the center of mass of the spring. The

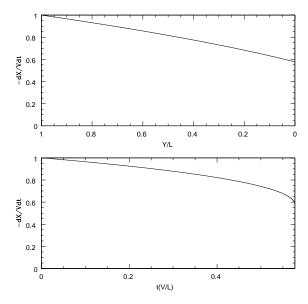


FIG. 2: Figure 2. The physical velocity,  $-\frac{dX}{dt}$ , as a function of Y and t for the boundary of the shock wave in the falling slinky.

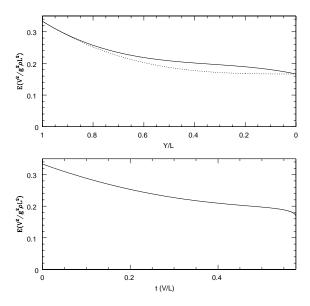


FIG. 3: Figure 3. The total energy (Kinetic plus potential energy—both gravitational with the bottom of the slinky as potential energy reference point and potential energy in the spring) as a function of Y and of t. Note that in neither case is the loss of energy linear. In fact, it is at the end of the trajectory of the shock that the largest rate of loss of energy occurs, where both the graviational and spring potential energies are the least.

center of mass energy we expect to be conserved. The conversion of the potential energy in the spring extention into heat during the collisions occurs intially, and again just at the end of the travel of the shock. That it is this internal potential energy of the stretched spring that is converted is clear if one goes into the center of mass frame of the spring which falls with acceleration g. There the only potential energy is the stretched spring, and at the end, the stretched spring is at rest, with no internal motion.

This problem illustrates a number of features of shock fronts[2]. They do not conserve energy in their gross motion (of course the lost energy goes into heating the system behind the shock). The motion of the shock front is also larger

than the velocity of sound in the medium (in this case it goes to infinity in the y coordinates, while the velocity of 'sound', V, is a constant). This problem does differ from many other shocks in that the heating due to the energy dissipation in the shock does not affect the equations of motion of the shock itself.

If the equilibrium of the spring is not with  $\frac{dx}{dy} = 0$ , but has some equilibrium value a, then the static solution for the hanging spring is

$$y = \frac{g\rho}{2k}(y+a)^2 \tag{26}$$

Time dependent solution is again

$$x(t,y) = -\frac{1}{2}gt^2 + f(y - Vt) + g(y + Vt)$$
(27)

with initial condition again

$$x(0,y) = \frac{g\rho}{2k}(y+a)^2$$
 (28)

$$\partial_t x(0,y) = 0 \tag{29}$$

giving f(y) = g(y) The boundary conditions at y = 0 and y = L are

$$\partial_u x(t,0) = a \tag{30}$$

$$\partial_y x(t, L) = a \tag{31}$$

The solution is

$$f(y) = \begin{cases} \frac{\rho g}{4k} (y+a)^2 & -L < y < L\\ \frac{\rho g}{4k} ((y-2L) + a)^2 + 4aL & L < y < 3L \end{cases}$$
 (32)

Thus the solution for  $0 < t < \frac{L}{v}$  is

$$x(t,y) = \begin{cases} \frac{\rho g}{2k} (y+a)^2 & 0 < y < vt \\ \frac{\rho g}{2k} (y-L+a)^2 + L^2 - 2LVt & vt < y < L \end{cases}$$
 (33)

The condition that the spring elements not collide is that  $\frac{dx}{dy}$  not be zero anywhere for 0 < y < L which is that y - L + a not be zero anywhere. This implies that a > L is the condition that a freely suspended spring not have a zero slope anywhere. The stretch at the top of the freely suspended spring is  $\frac{\rho g}{k}(L+a)$  while at the bottom it is  $\frac{\rho g}{k}a$  which has a ratio of  $\frac{L}{a} + 1$ . Thus the condition that the coils of the spring not collide (as in ref [1] when the freely suspended spring is dropped is that the ratio of the stretch at the top and the bottom must be less than a factor of 2 (a > L) when the spring is suspended by its end.

<sup>[1]</sup> M. G. Calkin "Motion of a falling spring" Am. J. Phys. 63 261 (1993)

<sup>[2]</sup> See for example http://en.wikipedia.org/wiki/Shock\_wave (retrieved Oct 8,2010). For a history of the study of shocks, see Krehl, Peter O. K. (2011), "Shock wave physics and detonation physics—a stimulus for the emergence of numerous new branches in science and engineering", The European Physical Journal H 36 85 (2011)