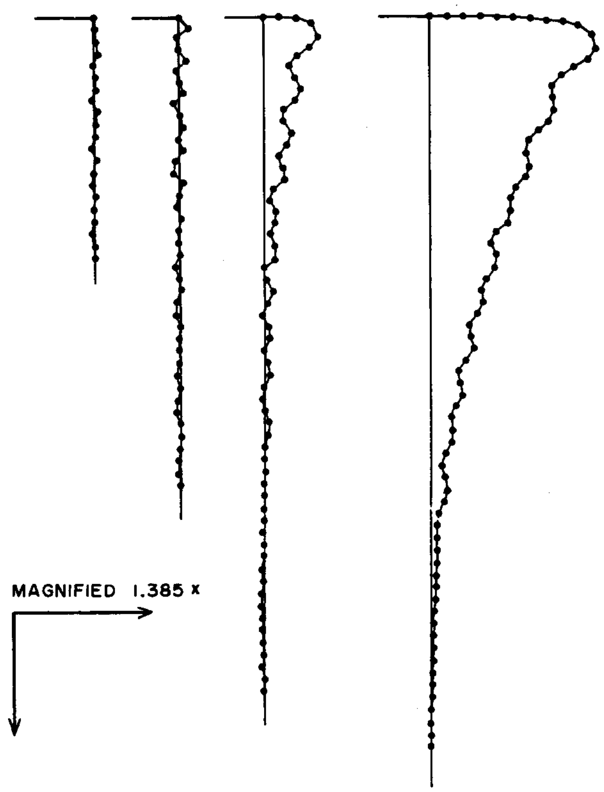

Fig. 5. Four snapshots of 80-link chain.

In Fig. 3 (Fig. 4) we display the resulting shape of a chain which started at rest with only its first point hanging over the table's edge, when approximated by 10 and 20 (40 and 80) segments. The chain's motion is captured at the moment when its last point just loses contact with the table. Note that the horizontal scale has been expanded to accentuate the corresponding displacement. It is interesting to observe (Fig. 5) that the solution (in agreement with Ref. 6) closely follows the inverted L shape until the first half of the chain has left the table.

[1]M. R. Spiegel, *Theoretical Mechanics* (Schaum's Outline Series, McGraw-Hill, New York, 1967).

[2]G. G. McLean and E. W. Nelson, *Theory and Problems of Engineering Mechanics* (Schaum's Outline Series, McGraw-Hill, New York, 1962).

[3]H. Lamb, *Dynamics* (Cambridge, U. P., Cambridge, 1961).

[4]D. Hestenes, *New Foundations for Classical Mechanics* (Reidel, Dordrecht, 1986).

[5]J. R. Sanmartin and M. A. Vallejo, "Widespread error in a standard problem in the dynamics of deformable bodies," Am. J. Phys. **46**, 949–950 (1978).

[6]D. Prato and R. J. Gleiser, "Another look at the uniform rope sliding over the edge of a smooth table," Am. J. Phys. **50**, 536–539 (1982).

[7]J. R. Sanmartin and M. A. Vallejo, "Comment on 'Another look at the uniform rope sliding over the edge of a smooth table,' " Am. J. Phys. **51**, 585 (1983).

Motion of a falling spring

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The motion of a spring, initially suspended from its top end and hanging in equilibrium in the earth's field, and then released and allowed to fall, is studied. The difference between the behaviors of loosely wound and tightly wound springs is emphasized.

I. INTRODUCTION

The motion of a mass suspended by a spring, a system often used in undergraduate labs to study simple harmonic motion, has been the subject of many articles in the *Journal*.[1–13] Much of the interest has focused on the effect that the mass of the spring has on the motion, and in particular on the period, which is the commonly measured parameter.

The motion is in fact not usually simple harmonic, something most easily seen by considering a free spring (no mass on the end). Bowen[9] and Cushing[10,11] have given simple solutions in this case for somewhat different initial conditions: for Bowen the spring is released from an unstretched state, and initially falls, the top end remaining fixed; for Cushing the bottom end of a freely hanging spring is pulled down a small distance below the equilibrium position and released.

We consider here a closely related problem, which apart from its intrinsic interest, forces us to consider the difference between loosely wound springs (the coils of the unstretched spring are separated from one another) and tightly wound springs (the coils of the unstretched spring touch one another). This does not appear to have been done before in a dynamic context, although Cushing[10] recognizes the distinction, and Mak[13] has considered the static case. In particular, we consider a spring initially suspended from its top end and hanging in equilibrium in the earth's gravitational field. The top end is released, and we wish to find the subsequent motion. We begin with the loosely wound spring.

II. LOOSELY WOUND SPRING

A loosely wound spring of mass m, spring constant k, and unstretched length l_0 is suspended from its top end and

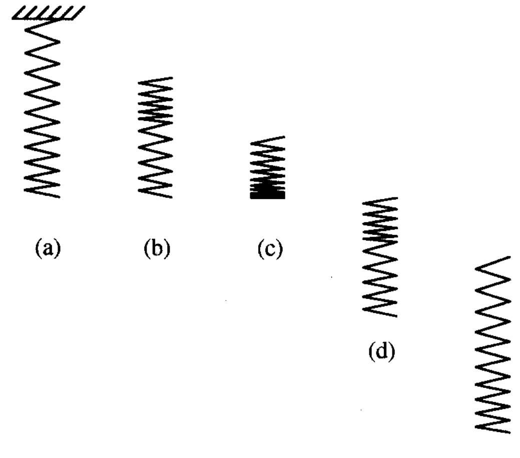

Fig. 1. Snapshots of the falling spring, (a) at $t=0$, (b) at $t=\tau/2$, (c) at $t=\tau$, (d) at $t=3\tau/2$, and (e) at $t=2\tau$.

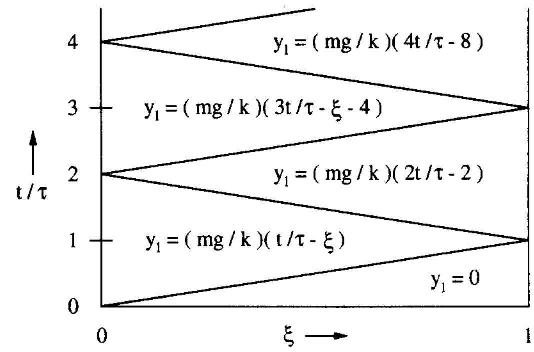

Fig. 2. Displacement y_1 of the falling spring as a function of position ξ and time t.

hangs vertically in equilibrium in the earth's gravitational field g. At time $t=0$ the top end is released and the spring falls. The problem is to find its subsequent motion. We denote points on the spring by a dimensionless variable ξ, with $\xi=0$ at the upper end, $\xi=1$ at the lower end, and the difference in ξ between any two points equal to the fraction of the mass of the spring between the two points. The motion of the spring is then described by giving the location $y(\xi,t)$ of all points ξ of the spring as functions of time t. The variable $y(\xi,t)$ is measured positive downward from the initial point of support. As is well known (see, for example, Refs. 4 and 6), $y(\xi,t)$ satisfies the inhomogeneous wave equation,

$$m\frac{\partial^2 y}{\partial t^2}=k\frac{\partial^2 y}{\partial \xi^2}+mg. \tag{1}$$

According to this, disturbances on the spring travel from one end to the other in a time $\tau = \sqrt{m/k}$. Thus for the falling spring the bottom remains stationary until at least this time after the top is released. The tension $f(\xi,t)$ in the spring at point ξ and time t is given by

$$f(\xi,t)=k\left(\frac{\partial y}{\partial \xi}-l_0\right), \tag{2}$$

so at a free end the boundary condition is

$$\left(\frac{\partial y}{\partial \xi}\right)_{\text{free end}}=l_0. \tag{3}$$

Initially, the spring is hanging in equilibrium ($\partial^2 y/\partial t^2=0$). Integration of Eq. (1) with the boundary conditions $y(\xi=0)=0$ at the top end and Eq. (3) at the bottom ($\xi=1$) end leads to the solution [see Fig. 1 (a)],

$$y_0(\xi)=\left(l_0+\frac{mg}{k}\right)\xi-\frac{mg}{2k}\xi^2. \tag{4}$$

As a check, note that according to Eq. (2) the tension at ξ is $mg(1-\xi)$, and equals the weight of the spring below ξ.

At $t=0$ the top end is released. The problem is then to find the solution $y(\xi,t)$ to the wave equation (1), subject to the boundary condition (3) at the top ($\xi=0$) and bottom ($\xi=1$) ends, and to the initial ($t=0$) conditions $y=y_0$, $\partial y/\partial t=0$. It is somewhat neater to work with the displacement $y_1(\xi,t)=y(\xi,t)-y_0(\xi)$ from the initial position. This satisfies the homogeneous wave equation, boundary conditions $\partial y_1/\partial\xi=-mg/k$ at the top end, $\partial y_1/\partial \xi=0$ at the bottom end, and initial conditions $y_1=0, \partial y_1/\partial t=0$. The solution can be obtained by using, for example, the method outlined in Ref. 11, with the result (see also Fig. 2),

$$y_1(\xi,t)=\begin{array}{ll} 0 & 0\leqslant t/\tau\leqslant\xi \\ (mg/k)(t/\tau-\xi) & \xi\leqslant t/\tau\leqslant 2-\xi \\ (mg/k)(2t/\tau-2) & 2-\xi\leqslant t/\tau\leqslant 2+\xi \\ (mg/k)(3t/\tau-\xi-4) & 2+\xi\leqslant t/\tau\leqslant 4-\xi \\ (mg/k)(4t/\tau-8) & 4-\xi\leqslant t/\tau\leqslant 4+\xi \\ \cdots & \cdots \end{array}, \tag{5}$$

where $\tau = \sqrt{m/k}$. One can easily check that this satisfies the wave equation (including the boundaries between the various regions, the δ function arising from the step in $\partial y/\partial\xi$ canceling the δ function arising from the step in $\partial y/\partial t$), and the boundary and initial conditions. A succession of snapshots of the spring at intervals $\tau/2$ is given in Fig. 1. For $0\leqslant t\leqslant\tau$ [Fig. 1 (b)] the information that the top end has been released propagates down the spring with a speed $1/\tau$ (in the material variable ξ), so the section of the spring below $\xi=t/\tau$ remains unaffected. The tension in the upper section ($0\leqslant\xi<t/\tau$) is $-mg\xi$, the negative sign indicating that this section of the spring is compressed. Indeed, the upward force on any piece of the upper section arising from this compression balances the downward force due to gravity, so the upper section moves downward with constant velocity. At $t=\tau$ [Fig. 1 (c)] the compression wave reaches the bottom and the entire compressed spring is then moving downward with a velocity $g\tau$. For $\tau<t\leqslant 2\tau$ [Fig. 1 (d)] the information that the bottom has been encountered now propagates up the spring with a speed $1/\tau$. The section of the spring above $\xi=2-t/\tau$ remains compressed and continues to move downward with velocity $g\tau$. The section of the spring below this point has the same coil spacing as in equilibrium, and it moves downward with a uniform velocity $2g\tau$. At $t=2\tau$ [Fig. 1 (e)] the coil spacing is back to what it was at $t=0$, but now the whole spring is moving downward with a velocity $2g\tau$. The cycle then repeats, with the additional downward velocity $2g\tau$.

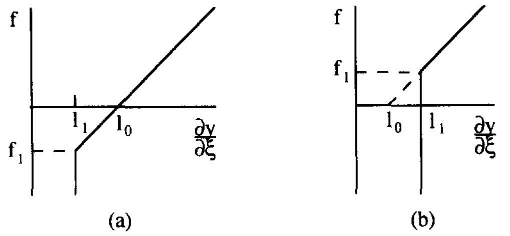

Fig. 3. Tension f as a function of stretch $\partial y/\partial \xi$, (a) for a loose spring, (b) for a pretensioned spring.

III. TIGHTLY WOUND SPRING

If the experiment is done using a Slinky, the qualitative behavior is similar to that already described for a loosely wound spring: it collapses down from the top, the bottom remaining stationary until the collapsed section reaches it. However, the time required for this is not the $\sqrt{m/k}$ obtained in the previous section, but approximately $0.6\sqrt{m/k}$. The reason for this difference is not hard to find. In the treatment of the previous section the upper section of the falling spring is compressed, with

$$\frac{\partial y}{\partial \xi}=l_0-\frac{mg}{k}\xi \tag{6}$$

However, for real springs the coils have finite thickness, so this cannot be less than $l_1(<l_0)$, the minimum compressed length of the spring with coils touching. The tension f in such a spring as a function of extension is shown in Fig. 3 (a) (see also Ref. 10). The theory of the previous section is thus consistent only for springs for which

$$mg\leqslant k(l_0-l_1). \tag{7}$$

As a test, note that if such a spring is placed on its end on a table, the coils at the bottom remain separated. An ordinary Slinky does not satisfy this condition. Indeed, a new Slinky is usually pretensioned: in the absence of external forces the coils are touching, the Slinky has a length l_1 and a finite force $f_1=k(l_1-l_0)$, where $l_0(<l_1)$ is the (unachievable) zero force length, is required to cause the coils to separate [see Fig. 3 (b)]. The theory of the preceding section, including the wave equation, does not apply, and we must start again.

The static hanging Slinky has been considered previously by Mak.[13] Because of pretensioning, there is a bottom section $\xi_1\leqslant\xi\leqslant 1$, with

$$mg(1-\xi_1)=f_1, \tag{8}$$

where the coils are in contact ($\partial y/\partial\xi=l_1$). In the top section $0\leqslant\xi\leqslant\xi_1$ the coils are separated and the Slinky behaves like a loosely wound spring, with

$$y_0(\xi)=\left(l_1+\frac{mg}{k}\xi_1\right)\xi-\frac{mg}{2k}\xi^2,\quad 0\leqslant\xi\leqslant\xi_1. \tag{9}$$

Note that the tension,

$$f_1+k\left(\frac{\partial y_0}{\partial\xi}-l_1\right)=f_1+mg(\xi_1-\xi)=mg(1-\xi), \tag{10}$$

at ξ is just sufficient to support the weight of the Slinky below ξ. The location of the bottom of the Slinky is obtained by setting $\xi=\xi_1$ in Eq. (9), and adding to it the length $l_1(1-\xi_1)$ of the bottom section, to give

$$y_0(\text{bottom})=l_1+\frac{mg}{2k}\xi_1^2. \tag{11}$$

This agrees with the result obtained by Mak.[13]

Now suppose that at $t=0$ the top of the Slinky is released. The Slinky begins to collapse down one coil after another successively from the top. Let us suppose that at time t the leading edge of the (collapsed) top section is at $\xi(t)$. Then the location of points in the top section of the Slinky is given by

$$y(\xi,t)=l_1\xi+\frac{mg}{k}\xi_1\xi(t)-\frac{mg}{2k}\xi(t)^2. \tag{12}$$

To write this down note that $\partial y/\partial\xi$ must be l_1, and $y(\xi,t)$ for the top section must match $y(\xi,t)$ for the middle section at $\xi=\xi(t)$. The middle section $[\xi(t)\leqslant\xi\leqslant\xi_1]$ of the Slinky is of course unaffected, and remains as given by Eq. (9).

The time dependence of $\xi(t)$ can be determined by applying Newton's second law to the system. At time t the mass of the top section is $m\xi(t)$ and its velocity is

$$\frac{\partial y(\xi,t)}{\partial t}=\frac{mg}{k}[\xi_1-\xi(t)]\frac{d\xi(t)}{dt}. \tag{13}$$

Since the bottom sections of the Slinky are stationary, the total momentum of the Slinky is $m\xi(t)(\partial y/\partial t)$. Equating this to the total impulse mgt which has acted on the Slinky, we find

$$\xi(t)[\xi_1-\xi(t)]\frac{d\xi(t)}{dt}=\frac{k}{m}t. \tag{14}$$

Integration then gives

$$\xi(t)^2\left(\xi_1-\frac{2}{3}\xi(t)\right)=\frac{k}{m}t^2. \tag{15}$$

In particular, the time for the Slinky to collapse fully and the bottom to begin to move is obtained by setting $\xi(t)=\xi_1$. It is given by

$$t(\text{bottom})=\sqrt{\frac{m}{3k}\xi_1^3}. \tag{16}$$

The factor $1/\sqrt{3}$ can be understood simply by considering the special case of a Slinky for which the coils just touch one another in the absence of external forces (so that $f_1=0$, $\xi_1=1$, and $l_1=l_0$). Hanging in equilibrium, this Slinky behaves like a loosely wound spring, and its center of mass lies $(1/2)l_0+1/3(mg/2k)$ above the bottom. When the Slinky is fully collapsed its center of mass lies $(1/2)l_0$ above the bottom. The center of mass thus falls a distance $(1/3)(mg/2k)$. Equating this to the well-known expression $(1/2)gt^2$ for distance fallen in time t in a uniform gravitational field, we recover Eq. (16). A similar argument can, of course, be given for the loosely wound springs of Sec. II. In this case, however, the center of mass of the compressed state of Fig. 1 (c) is located $(1/2)l_0-(2/3)(mg/2k)$ above the bottom, so the center of mass falls a distance $mg/2k$ before the bottom begins to move. Equating this to $(1/2)gt^2$ then gives the time $\sqrt{m/k}$ appropriate to the loosely wound spring case.

ACKNOWLEDGMENTS

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[1] H. L. Armstrong, "The oscillating spring and weight—An experiment often misinterpreted," Am. J. Phys. **37**, 447–449 (1969).
[2] Francis W. Sears, "A demonstration of the spring-mass correction," Am. J. Phys. **37**, 645–648 (1969).
[3] J. G. Fox and J. Mahanty, "The effective mass of an oscillating spring," Am. J. Phys. **38**, 98–100 (1970).
[4] T. W. Edwards and R. A. Hultsch, "Mass distribution and frequencies of a vertical spring," Am. J. Phys. **40**, 445–449 (1972).
[5] Paul Mazur and Frank A. Blood, Jr., "Comparison of normal modes of an end-weighted discrete and continuous linear chain," Am. J. Phys. **40**, 1694–1696 (1972).
[6] Thomas C. Heard and Neal D. Newby, Jr., "Behavior of a soft spring," Am. J. Phys. **45**, 1102–1106 (1977).
[7] Robert Weinstock, "Oscillations of a particle attached to a heavy spring: An application of the Stieltjes integral," Am. J. Phys. **47**, 508–514 (1979).
[8] Ernesto E. Galloni and Mario Kohen, "Influence of the mass of the spring on its static and dynamic effects," Am. J. Phys. **47**, 1076–1078 (1979).
[9] Jeffrey M. Bowen, "Slinky oscillations and the notation of effective mass," Am. J. Phys. **50**, 1145–1148 (1982).
[10] James T. Cushing, "The spring—mass system revisited," Am. J. Phys. **52**, 925–933 (1984).
[11] James T. Cushing, "The method of characteristics applied to the massive spring problem," Am. J. Phys. **52**, 933–937 (1984).
[12] Maurice Leclerc, "Effective elastic constant and effective mass of an oscillating spring: An energy approach," Am. J. Phys. **55**, 178–180 (1987).
[13] S. Y. Mak, "The static effectiveness mass of a slinky™," Am. J. Phys. **55**, 994–997 (1987).

A bifurcation problem in hydrostatics

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The equilibrium positions of a partially submerged rod supported at one end are considered. Simple examples of bifurcations occur, and the breaking of these bifurcations by an "imperfection" is discussed.

I. INTRODUCTION

As may be seen from the papers by Delbourgo,[1] by Gilbert,[2] and by Erdös, Schibler, and Herndon,[3] problems concerning the equilibrium and stability of floating objects can be less straightforward than they might at first seem. For example, Delbourgo, following the pioneering work of Huygens,[4] considered the stability of a long homogeneous plank, of rectangular cross section, freely floating at the surface of a static liquid. He showed that the appropriate (two-dimensional) parameter domain divides into nine regions in each of which one or more of six possible configurations of the plank is/are stable.

In this paper we consider a much simpler hydrostatic situation, namely the configurations of a "floating" rigid rod that is supported by a hinge at one end. We determine how the stability characteristics of these configurations vary as the point of support is raised or lowered quasistatically, and we discuss the effect of a small "imperfection"[5] in the system (namely a small couple applied to the rod at the hinge).

Although this problem is nothing like as rich—or as difficult—as Delbourgo's problem, it has the advantage that one of the parameters (the height of the support) can be varied continuously, so this problem could perhaps be used as a way of demonstrating bifurcation theory, hysteresis, and "jump" phenomena. In addition, it has the merit that the mathematical analysis is very simple, and that the only knowledge of physics required is an elementary understanding of buoyancy and of moments of forces.

Aspects of "constrained floating" problems of this sort are discussed by Besant and Ramsay,[6] by Ramsay,[7] and by Laws.[8] They derive equilibrium and stability conditions for a body of arbitrary shape that is freely pivoted about a fixed point. However, they do not consider in detail changes of stability as the pivot point is varied.

In their detailed studies of floating objects, Erdös *et al.*[3] include discussion of a prism that is free to rotate about a fixed horizontal longitudinal axis through its center of gravity; they determine the angular position of the prism in the stable state, as a function of the height of the axis. This is analogous to our first problem here.

II. THE PROBLEM

We consider a long thin uniform rigid rod PQ, of length L, cross-sectional area A, and density $\sigma\rho$, supported at P by means of a hinge that allows the rod to move only in a given vertical plane. The rod hangs (statically) from P, under gravity, but P is positioned such that some or all of the rod is immersed in a static liquid of uniform density ρ (see Fig. 1). We wish to determine the possible equilibrium positions of the rod (i.e., the angles α between PQ and the downward vertical), and the stability of these equilibria, for given height h of P above the liquid surface. Clearly α