Galileo Redux

or, How Do Nonrigid, Extended Bodies Fall?

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The mythology of physics includes three wonderful tales—that of Archimedes in the baths of Syracuse, Galileo's dropping objects from the tower of Pisa, and Newton's being hit by a falling apple in an English orchard. All three appeal to students and offer useful pegs onto which a teacher can hang ideas.

The Galileo experiment was restricted to solid, rigid objects. Newton's development of the science of mechanics in his *Principia Mathematica* focused on mass points or on solid, rigid bodies considered as mass points such as the Sun, the planets, and the Moon. This has led to an unconscious bias in beginning students towards a world consisting of mass points. The treatment of elasticity and deformable bodies appears only after elementary courses. This means, of course, that nonphysics majors have a simplistic picture of the physical world far removed from reality.

The Falling Slinky[1]

A Slinky offers a splendid opportunity for introducing nonrigid bodies within the Galilean and Newtonian framework. It provides a rather spectacular demonstration, all the more instructive since it is counterintuitive.

It is perhaps worth noting that although the fascinating properties of Slinkies as wave carriers have captured the imagination of many physicists, the literature does not indicate any exploitation of them as falling bodies. A search of the *American Journal of Physics* over the last 15 years shows several articles dealing with such subjects as wave cut off,[2] Slinky whistlers,[3] and the Slinky as a model for transverse waves in a plasma.[4] There is one exception. A paper by Buerger[5] does examine a Slinky falling under gravity as well as the compression shock wave running through the Slinky when it is released.

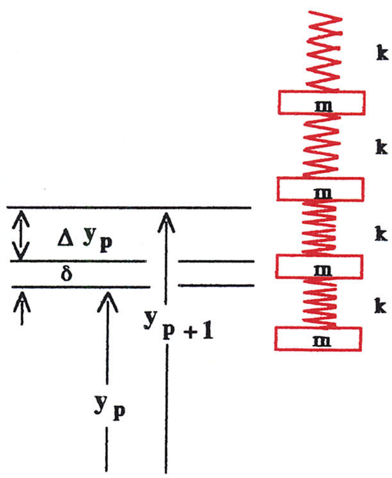

Fig. 1. Schematic of Slinky model extended under its own weight. Note the change in distance between the masses from top to bottom.

The demonstration that is the basis for this note was to suspend a Slinky so that it was extended vertically. We did this from a gallery about five meters above our gymnasium floor. On releasing the Slinky, we observed it falling, but not as might have been predicted; the top of the spring toy begins to fall, but the bottom remains absolutely stationary. Only when the entire Slinky is collapsed onto itself, does the bottom begin to move. We confirmed this by videotaping the demonstration. The extended Slinky is 189 cm long. The tape shows that it requires 10 frames (one-third second) for the Slinky to collapse to its unextended length of 6 cm. Only then does the bottom begin to move.

Clearly this is in apparent disagreement with Galileo. However, the disagreement is indeed apparent. If a small ball is placed next to the center of mass of the Slinky and released simultaneously with the Slinky, both strike the ground at the same time. This is so, even though the ball begins moving at the instant of release, while the bottom of the Slinky remains stationary and begins to move only after the top has collapsed onto itself.

The Hanging Slinky in Equilibrium

In order to analyze the force and energy relations for a hanging Slinky in equilibrium let us consider Fig. 1, which is a sketch of the model of the hanging Slinky. We see that the mass is not distributed uniformly. Since the separation between turns decreases from top to bottom, the nearer an element of the Slinky is to the point of suspension, the smaller is the mass per unit length. This is evident from the fact that the lower the point considered, the smaller is the weight of the mass beneath it. It is obvious from the diagram that the mass per unit length increases along the Slinky from top to bottom.

We wanted to construct a reasonable model for the Slinky, one that would enable us both to make calculations that agree with observed behavior and predictions that suggest further experiments. How do we choose a "reasonable" model? We do have a useful guideline. Apply Occam's razor and choose the simplest model or explanation that is in agreement with the actual results. Increase the complexity of the model only if necessary to arrive at agreement. Even using this guideline, we have an additional degree of freedom. There is an aesthetic element to our choice. Some models are simply more pleasing than others. All other factors being equal, we will, naturally, choose the more pleasing.

Details of the model we constructed and the associated calculations are available by mail. Send a stamped, self-addressed envelope to author Newburgh. One approach is to consider a Slinky of total mass M with N turns. The thickness of the wire is δ and the mass per turn, m, equals M/N. We number each turn with the running index i. The length of the unstretched Slinky is L_0, which equals $N\delta$; the length of the hanging Slinky (stretched by gravity) is L, which equals $L_0 + \Delta L$. Since the material from which the Slinky is made is uniform, the spring constant of each (single) turn (or coil) is k.

To simplify the model further, we can look at it as a set of discrete point masses (each of value m) with a diminishing separation between masses as one goes from top to bottom. These masses, m, are located at the midpoints of each turn, as shown schematically in Fig. 1.

We find that, under the influence of gravity, the Slinky's length increases by a value ΔL such that

$$\Delta L = N^2 mg/2k \qquad (1)$$

and the center of mass measured from the point of suspension, H,[6] is

$$y_{cm} = L_0/2 + 2\Delta L/3 \qquad (2)$$

and measured from the floor is

$$y_{cm} = H - (L_0/2 + 2\Delta L/3) \qquad (3)$$

Equation (1) is in essential agreement with the analysis of French,[7] which treated the Slinky as a continuous distribution of mass rather than as a set of discrete mass points connected by massless springs.

A simple energy argument gives one confidence in the validity of this model. The gravitational potential energy of the unstretched Slinky is

$$U = Mg\,(H - L_0/2) \qquad (4)$$

as measured from the floor, and that of the stretched Slinky is

$$U = Mg\,(H - L_0/2 - 2/3\,\Delta L) \qquad (5)$$

To position a Slinky so that it is hanging vertically and at rest, first take a compressed, stationary Slinky. Since it is at rest, there must be a force equal to the weight of the Slinky, Mg, applied to the bottom turn, and pushing in the upward direction. If the top is attached to a point of suspension and the upward force removed, the Slinky will fall and oscillate until finally damped. One can, however, lower the Slinky reversibly. A reversible process is quasi-stationary so that the Slinky will remain in equilibrium at all times. This can be achieved by an upward force equal to Mg minus an infinitesimal amount ε. In extending the Slinky, the bottom moves through a distance ΔL.

Let us now calculate the total work done in lowering the Slinky. There are two external forces acting as it moves. The first is that due to gravity and equals Mg. This force acts downward in the direction of motion. The second is the upward force F_{up}, equal to $Mg - \varepsilon$, applied at the Slinky bottom. This force is opposite to the direction of motion. There is no force acting at the point of suspension until the upward force is removed.

Although these two forces are equal in magnitude (apart from the infinitesimal ε), they do not do the same amount of work. That done by gravity is calculated from the motion of the center of mass and is positive. Since the center of mass moves a distance (2/3) ΔL, this work is positive and is

$$W_{gr} = Mg(2/3)\,\Delta L \qquad (6)$$

The upward force acts over a distance ΔL. Since the force and motion are in opposite directions, the work is negative in the amount

$$W_{up} = -Mg\Delta L - \varepsilon\Delta L \qquad (7)$$

Since $\varepsilon\Delta L$ is an infinitesimal quantity, the net work for the process is therefore

$$W_{net} = -Mg\,(1/3)\,\Delta L \qquad (8)$$

In a reversible process there is no dissipation. Therefore our analysis indicates a net loss in the system energy equal to $-Mg(1/3)\Delta L$. Examination of the changes in potential energy confirms this statement.

As the center of mass drops, the gravitational potential energy decreases by $-Mg(2/3)\Delta L$. Before the Slinky was dropped, it had no elastic potential energy. We calculate this gain and find it to be $Mg(1/3)\Delta L$. The net change in potential energy in the process is therefore

$$\Delta PE_{total} = \Delta PE_{gr} + \Delta PE_{el} = \qquad (9)$$

$$-Mg\,(1/3)\,\Delta L = -(2k/3N)\,(\Delta L)^2$$

Our model is therefore consonant with the energy calculations.

Forces When the Slinky Falls

What happens when the Slinky is released? At this instant the problem changes from one of equilibrium to one of dynamics. Consider the topmost turn, the N^{th} turn. Before the release three forces act on it: an upward spring force f_N, a downward spring force f_{N-1}, and a downward gravitational force, mg. These forces add up to zero. At the instant of release f_N goes quickly to zero. There is then a net unbalanced force on the topmost turn, and it begins to collapse onto the $(N-1)^{th}$ turn. This event is repeated sequentially. In short, the Slinky collapses with a finite velocity so that the onset of motion of the individual turns can be likened to a pulse propagation.

A simple way to visualize the process is to consider a queue of automobiles stopped at a red light. When the light turns green, all the cars do not move at once. The first to move is the car that is first in line. (This corresponds to the motion of the topmost turn of the Slinky.) Only after the first has started does the second go, and so on down the line. A pedestrian on the sidewalk sees it as a propagating pulse of motion. The analogy with the falling Slinky goes further than this. Just as the last car in the queue begins to move only when all in front have gone, so does the bottom turn of the Slinky begin to fall only after all those above it have fallen onto it.

As already mentioned, in equilibrium there are three forces acting on each turn, one up and two down. When the turn above has collapsed onto it, there are but two forces acting, both downward. This means that the acceleration downward is greater than that due to gravity alone. We shall return to this point when we discuss motion of the center of mass.

Data for the Slinky are as follows. The unextended Slinky is 6 cm long. When hanging, it is stretched to 189 cm. The measured time for complete collapse of the Slinky (at which time the bottom begins to move) is one-third second. The time is measured by counting video frames, which have a rate of 30 per second. Using the standard kinematic equations, we find that an acceleration of 9.8 m/s^2 would correspond to a distance of 54.4 cm only, confirming the assertion that the acceleration is greater than that of gravity.

Discussion

We can draw a number of useful lessons from this demonstration. We have found that students become excited as they realize the far-reaching implications of an apparently simple experiment with a familiar "toy."

Motion of the Center of Mass of a Nonrigid Body

The average acceleration of the top of the Slinky calculated from the 1/3 s needed to fall 1.83 m is 32.9 m/s^2. Taking the acceleration as decreasing uniformly from its initial value to a final one of 9.8 m/s^2 gives an initial acceleration of 56 m/s^2 or 5.7 g. However, the experiment demonstrates clearly that the center of mass falls with an acceleration of 9.8 m/s^2. We marked the center of mass with a piece of tape, placed a ball bearing next to it, and then released the Slinky and the bearing at the same instant. They reached the ground simultaneously.

From this we can generalize Galileo's conclusions from the Leaning Tower experiment. It is not true that all matter falls at the same rate in the absence of air resistance. Clearly different parts of the Slinky fall at different rates, in some cases astonishingly high ones. These different rates result from the internal forces within the extended body. There is one unique point in the Slinky—the center of mass—that is unaffected by the internal forces. This unique point moves according to the external force—gravity—as did the bearing. One is reminded of Gilbert's line[8] from the *Pirates of Penzance*, "a paradox, a paradox, a most ingenious paradox." This demonstration of the uniqueness of the center of mass helps students understand both the power and the limitations of Newton's Second Law, as well as showing them systems more typical of the real world.

Inclusion of the Mass of the Spring

The second point is that all or nearly all elementary mechanics problems treat springs as having either no mass or negligible mass. Interestingly enough, strings and ropes are introduced quite early with a discussion of the relation of wave velocity to mass per unit length, often as an introduction to dimensional analysis. In analyzing both the hanging and falling Slinky we cannot neglect the mass of the Slinky.

Since we cannot neglect the mass, we must account for it in our model. Our approach has been to replace the actual continuous mass distribution with a model of discrete masses having a nonuniform spatial distribution. By comparing the decrease in gravitational potential energy with the increase in elastic potential energy as the Slinky is extended vertically, we have obtained a measure of confidence in our model. The experiment therefore offers a useful demonstration of model building for beginning students, enabling the teacher to show the relations among models, idealizations, and real systems. In addition, the particular model used serves as a first introduction to the notion of discreteness. Some familiarity with discrete models in classical physics has obvious pedagogic advantages when quantum physics surfaces.

Comments

We hope that this note has shown the usefulness of the falling Slinky as a pedagogic tool for beginning students. As a demonstration it is dramatic and exciting, especially as the stationarity of the bottom of the Slinky during the collapse of the top is completely counterintuitive. As a real physical object, this popular spring toy illustrates the general applicability of mechanics to real situations. It amplifies and confirms the generality of Galileo's Leaning Tower experiment. It allows the teacher to examine theoretical model making in the analysis of real problems and introduces discrete models quite early in the curriculum. It helps the students appreciate the beauty and validity of Newton's Second Law. Finally, it has that element of playfulness—fun—that is so much a part of physics.

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References

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