# Pattern Formation in a system with competing wavelengths

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We propose a modified version of the Swift-Hohenberg equation in an attempt to study a situation in which competition between two patterns of different wavelength exist.

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## I. INTRODUCTION

The Swift-Hohenberg equation serves as a model for pattern formation in a broad range of physical systems. This equation, which takes the form

$$u_t = ru - \left(1 + \partial_x^2\right)^2 u + N[u], \tag{1}$$

describes the dynamics of a real field $u$ over one spatial dimension in time, where $N$ is some nonlinear function of $u$. We have rescaled the equation so that the critical wavenumber that defines the natural wavelength of the patterned state is unity. We will be interested in two possible choices of $N$, namely $N_{23}[u] = bu^2 - u^3$ and $N_{35}[u] = bu^3 - u^5$. The strength of the linear forcing term $r$ and the strength of the quadratic nonlinearity $b$ are left as parameters of the system.

There may be physical systems in which multiple patterned states with different characteristic wavelengths can exist simultaneously. We would like to develop a model for such a system and study the interaction and competition between these patterned states. We propose a modification to the Swift-Hohenberg equation in which such competition may be possible:

$$u_t = ru - \left(1 + \partial_x^2\right)^2 \left[\left(q^2 + \partial_x^2\right)^2 + \delta\right] u + N[u]. \tag{2}$$

Linear stability analysis of this equation results in a marginal stability curve (Fig. 1) where two wavelengths can compete with each other. The first wavelength is $k = 1$, just as in the original SHE (Eq. 1), and the other occurs at

$$q_* = \frac{1}{2}\sqrt{1 + 3q^2 \pm \Delta}. \tag{3}$$

where $\Delta = \sqrt{(q^2-1)^2 - 8\delta}$ and $+/-$ is used when $q > 1$ and $q < 1$ respectively. We note that such solutions exist only when $(q^2 - 1)^2 > 8\delta$, which is the condition that there are local extrema other than $k = 1$ in the marginal stability curve. We must further make the restriction that $\delta > -(q^2 - 1)^2$ in order that $k = 1$ is in fact a local minimum. If both these conditions are met, there will be a minimum at $q^*$ in addition to 1. The curve that defines the region of interest to us is also proportional to the marginal stability curve of the original Swift-Hohenberg operator with characteristic wavenumber $q$ and forcing $\delta$.

Why is this the case, does it have some physical interpretation?

This is not a constraining restriction since we will generally be interested in small values of $\delta$. When $\delta = 0$, we have that $q_* = q$ and both wavelengths become unstable at $r = 0$.

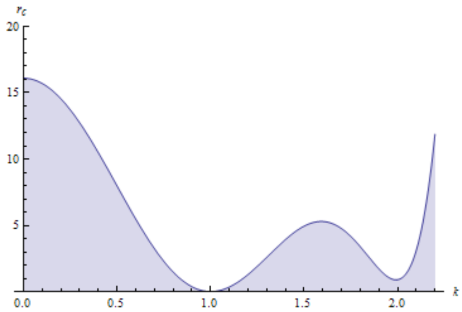


FIG. 1: Linear stability of the modified Swift-Hohenberg equation with $q = 2$ and $\delta = .1$. The shaded region indicates linearly stable parameter regimes of the homogeneous solution with a forcing $r$ for a given wavenumber $k$.

The instability of $q_*$ is shifted for nonzero values of $\delta$ by

$$r_* = \frac{1}{128}\left(3(q^2-1)+\Delta\right)^2\left((q^2-1)^2-(q^2-1)\Delta+4\delta\right). \tag{4}$$

We will be interested in the case that $\frac{8\delta}{(q^2-1)^2} << 1$, so that the two competing wavelengths will have similar marginal stabilities and can thus compete. This assumption allows us to approximate $q_*$ and $r_*$ as:

$$q_* \approx q\left[1-\frac{\delta}{2q^2(q^2-1)^2}+\frac{(9q^2-1)\,\delta^2}{8q^4\,(q^2-1)^3}\right] \tag{5}$$

$$r_* \approx (q^2-1)^2\delta-\delta^2 \tag{6}$$

We note that the $q = 1$ is a degenerate case that may need to be handled separately. In terms of the our analysis here, $q = 1$ requires that $\delta = 0$ and in this case, $q_* = q$ and $r_* = 0$. Another interesting regime could be when $q$ is close to one so that we have nearby wavelengths competing. We might also want to consider the case when $q >> 1$ (or $q << 1$) so that the wavelengths of the patterns occur on different lengthscales. This regime is of interest for quasicrystals?

This equation has been studied by Bentley (2012), though with a slightly different parametrization. The advantage of this parametrization over the one used by Bentley becomes apparent in the small $\delta$ limit. We see that $q$ is approximately the wavenumber of the second competing pattern, and $\delta$ is proportional to the relative shift between the onsets of the two patterns. We might also consider a slightly different parametrization in which the $\delta(1-k^2)^2$ is instead $\delta(1-k^2)$, the so-called "Proctor term" (I think). The advantage of our parametrization is two-fold: (1) the characteristic wavenumber of one of the patterns is exactly one. (2) The relations between $q_*$, $r_*$ and $q$ and $\delta$ are much simpler in our case. Bentley has looked at this equation as a model for magnetorotational Taylor-Couette flows. His focus is in the supercritical regime where the patterned states bifurcate from the homogeneous state with a supercritical pitchfork bifurcation. He is currently working to publish his work. We don't know what additional work he has done beyond his thesis that his advisor has shared with us.

## II. VARIATIONAL STRUCTURE

Just as in the case of the original Swift-Hohenberg equation (Eq. 1), this modified equation can be expressed in terms of a Lyaponuv functional as where

$$F[u] = \tag{7}$$

This implies that the system will always approach a steady state in time, and we can focus on time-independent solutions to give some insight into the dynamics of the system.

## III. WEAKLY NONLINEAR ANALYSIS

We would like to look at small amplitude solutions in the neighborhood of the $r = 0$ bifurcation where the periodic state branches off of the homogeneous state in the space of steady-state solutions. We will take a multi-scale approach, defining a slow timescale $T = \epsilon^2 t$, and long spatial scale $X = \epsilon x$ so that the derivatives become $\partial_t \to \partial_t + \epsilon^2 \partial_T$ and $\partial_x \to \partial_x + \epsilon \partial_X$. We will assume that the system will not change on the fast timescale, so we can neglect the $\partial_t$ term. With some trial and error, it can be seen that the appropriate scaling of forcing strength to probe the dynamics we are interested in will be $r = \epsilon^2 \mu$. In addition, we will choose to scale the shift parameter $\delta \to \epsilon^2 \delta$ so that the difference in onset of instability for the two wavelengths will be of the same order of the forcing that we consider.

There are a few issues to consider here, especially after looking at the work of Bentley. I picked the above scaling to match what is standard in the original Swift-Hohenberg equation. I had mostly completed this calculation before looking at Bentley's thesis, and noticed that he picked a slightly different scaling. His large spacial scale goes like $X = \epsilon^{1/2} x$ instead of

$X = \epsilon x$ . Considering that we have upped the spatial order of the equation from 4 to 8, his scaling makes sense because it "keeps the balance between the spatial and time orders the same." The other thing he does differently that what I have done is to look near the codimension 3 point where $\delta = 0$ and $q = 1$, whereas I have set up my equations to be valid for arbitrary (rational) values of q that do not necessarily need to be close together. I do, however, in this analysis assume that q is of order 1. Choosing q either very large or small is of interest, and should be done as well. The cost of my approach is that I require two coupled amplitudes that must be solved whereas Bentley can get a single amplitude equation in one variable. Since his amplitude varies over length scales of order $\epsilon^{-1/2}$ while mine two amplitudes vary over length scales order $\epsilon^{-1}$, I am guessing the physical explanation is that he is somehow incorporating the beat frequencies from my two amplitudes as a slowly varying modulation? Since I have gotten so far into my calculation before realizing that I might even consider a different scaling, I decided to go ahead and finish it to see what I get.

One other issue that arises, comes due to my choice of how to write down the equation. As was discussed in the Introduction, $q$ is not actually the wavenumber of the second pattern and yet it is the wavenumber of the leading order problem in this scaling. The shift in wavenumber happens at a very high order, so maybe it doesn't appear at the order of calculation I'm working at. I'm guessing that when it does appear, it appears as a correction to the slowly varying amplitude. The other option is that I should change the form of my solution to include corrections directly to the wavenumber, $u\ Be^{i(k_0+\epsilon k_1+\ldots)x}$. I'm not sure if this is a problem or not yet, but I need to check this a little more carefully.

The linear part of the modified Swift-Hohenberg equation (Eq. 2),

$$L = r - \left(1 + \partial_x^2\right)^2 \left[\left(q^2 + \partial_x^2\right)^2 + \delta\right], \tag{8}$$

can be expanded as $L = L_0 + \epsilon L_1 + \epsilon^2 L_2 + ...$ where:

$$L_0 = -\left(1+\partial_x^2\right)^2\left(q^2+\partial_x^2\right)^2 \tag{9a}$$

$$L_1 = -4\left(1+\partial_x^2\right)\left(q^2+\partial_x^2\right)\left(q^2+1+2\partial_x^2\right)\partial_x\partial_X \tag{9b}$$

$$L_2 = -2\left[14\partial_x^6+15\left(q^2+1\right)\partial_x^4+3\left(q^4+4q^2+1\right)\partial_x^2+q^2\left(q^2+1\right)\right]\partial_X^2$$
$$-\delta\left[1+\left(2+\partial_x^2\right)\partial_x^2\right]+\mu-\partial_T \tag{9c}$$

$$L_3 = -4\left[\left(14\partial_x^4+10\left(q^2+1\right)\partial_x^2+q^4+4q^2+1\right)\partial_X^2+\delta\left(1+\partial_x^2\right)\right]\partial_x\partial_X \tag{9d}$$

$$L_4 = -\left[\partial_X^2\left(70\partial_x^4+30\left(q^2+1\right)\partial_x^2+q^4+4q^2+1\right)+2\delta\left(1+3\partial_x^2\right)\right]\partial_X^2 \tag{9e}$$

$$L_5 = -4\left[\left(3(q^2+1)+14\partial_x^2\right)\partial_X^2+\delta\right]\partial_x\partial_X^3 \tag{9f}$$

$$L_6 = -\left[2\left(q^2+1+14\partial_x^2\right)\partial_X^2+\delta\right]\partial_X^4 \tag{9g}$$

### A. The quadratic-cubic nonlinearity

We will first consider the case when $N = N_{23}$ so that the modified Swift-Hohenberg equation can be written as $L[u] - N_{23}[u] = 0$. We will assume that the solution can be written as an asymptotic series with the leading term of order $\epsilon$, namely $u = \epsilon u_1 + \epsilon^2 u_2 + \epsilon^2 u_3 + ...$

We can then write out the resulting equation at each order of $\epsilon$ by matching terms at the proper order.

$$\mathcal{O}(\epsilon):\ -L_0 u_1 = 0 \tag{10a}$$

$$\mathcal{O}(\epsilon^2):\ -L_0 u_2 = L_1 u_1 + b u_1^2 \tag{10b}$$

$$\mathcal{O}(\epsilon^3):\ -L_0 u_3 = L_1 u_2 + L_2 u_1 + 2b u_1 u_2 - u_1^3 \tag{10c}$$

The solution to the $\mathcal{O}(\epsilon)$ equation can be expressed in terms of the yet to be determined complex amplitudes $A_1, B_1$ as:

$$u_1(x, X, T) = A_{11}(X, T)e^{ix} + B_{11}(X, T)e^{iqx} + c.c. \tag{11}$$

where *c.c.* denotes the complex conjugate of the expression written. The $\mathcal{O}(\epsilon^2)$ equation has solutions that can be written in the form:

$$u_2(x, X, T) = C_{20}(X, T) \tag{12}$$
$$+\left[A_{21}(X,T)e^{ix}+A_{22}(X,T)e^{2ix}+B_{21}(X,T)e^{iqx}+B_{22}(X,T)e^{2iqx}+c.c.\right]$$

Noting that $L_1 u_1 = 0$, we see that substituting Eqs. 44 and 46 into Eq. 10b results in the condition

$$\begin{aligned} 0 = &\left(2b(|A_{11}|^2 + |B_{11}|^2) - q^4 C_{20}\right) \\ &+ \left[\left(bA_{11}^2 - 9(q^2-4)^2 A_{22}\right) e^{2ix} + \left(bB_{11}^2 - 9q^4(1-4q^2)^2 B_{22}\right) e^{2iqx} \right. \\ &\left. + 2bB_{11}\left(A_{11}e^{i(q+1)x} + \bar{A}_{11}e^{i(q-1)x}\right) + c.c.\right] \end{aligned} \tag{13}$$

In order to proceed with the analysis, we will eventually need to make some assumptions about the choice of $q$. We will assume that $q = m/n$ is a rational number where $m$ and $n$ are relatively prime integers. In this case, we want to perform our Fourier analysis on a domain of size $2\pi n$. We can make use of orthagonality conditions to derive relations between the various amplitudes. Applying the integral operater $\frac{1}{2\pi n}\int_{-n\pi}^{n\pi}\,\mathrm{d}x$ to Eq. 46 gives:

$$\left[2b(|A_{11}|^2 + |B_{11}|^2) - q^4 C_{20}\right] + \delta_{\mathrm{d}}(q-1)\left[2bB_{11}\bar{A}_{11} + 2b\bar{B}_{11}A_{11}\right] = 0 \tag{14}$$

where $\delta_{\mathrm{d}}$ is the Dirac delta function. Assuming that $q \neq 1$, this gives a condition that

$$C_{20} = \frac{2b}{q^4}(|A_{11}|^2 + |B_{11}|^2) \tag{15}$$

and if $q = 1$, we get

$$C_{20} = \frac{2b}{q^4}(|A_{11} + B_{11}|^2) \tag{16}$$

Applying $\frac{1}{2\pi n}\int_{-n\pi}^{n\pi}\,\mathrm{d}x\; e^{-ix}$ and $\frac{1}{2\pi n}\int_{-n\pi}^{n\pi}\,\mathrm{d}x\; e^{-iqx}$ give:

$$\delta_{\mathrm{d}}(q-1/2)\left[bB_{11}^2 - 9q^4(1-4q^2)^2 B_{22}\right] + \delta_{\mathrm{d}}(q-2)\left[2bB_{11}\bar{A}_{11}\right] = 0 \tag{17}$$

and

$$\delta_{\mathrm{d}}(q-1/2)\left[2b\bar{B}_{11}A_{11}\right] + \delta_{\mathrm{d}}(q-2)\left[bA_{11}^2 - 9(q^2-4)^2 A_{22}\right] = 0 \tag{18}$$

respectively. These equations are trivially solved, unless $q = 2$ or $q = 1/2$.

Applying $\frac{1}{2\pi n}\int_{-n\pi}^{n\pi}\,\mathrm{d}x\; e^{2ix}$ and $\frac{q}{2\pi n}\int_{-n\pi}^{n\pi}\,\mathrm{d}x\; e^{2iqx}$ give:

$$\begin{aligned} 0 = &\left[bA_{11}^2 - 9(q^2-4)^2 A_{22}\right] \\ &+ \delta_{\mathrm{d}}(q-1)\left[2bB_{11}A_{11} + bB_{11}^2 - 9q^4(1-4q^2)^2 B_{22}\right] + \delta_{\mathrm{d}}(q-3)\left[2bB_{11}\bar{A}_{11}\right] \end{aligned} \tag{19}$$

and

$$\begin{aligned} 0 = &\left[bB_{11}^2 - 9q^4(1-4q^2)^2 B_{22}\right] \\ &+ \delta_{\mathrm{d}}(q-1)\left[2bB_{11}A_{11} + bA_{11}^2 - 9(q^2-4)^2 A_{22}\right] + \delta_{\mathrm{d}}(q-1/3)\left[2b\bar{B}_{11}A_{11}\right] \end{aligned} \tag{20}$$

respectively. In the case that $q = 1$, we again get the solution consistent with combining the two amplitudes into a single variable. The $q = 3, 1/3$ cases give a coupling between the two amplitudes in this equation, and all other cases give:

$$A_{22} = \frac{bA_{11}^2}{9(q^2-4)^2} \qquad B_{22} = \frac{bB_{11}^2}{9q^4(1-4q^2)^2} \tag{21}$$

Now, going on to the $\mathcal{O}(\epsilon^3)$, we can write down the solution in the form:

$$\begin{aligned} u_3(x,X,T) = C_{30}(X,T) + \Big[ & A_{31}(X,T)e^{ix} + A_{32}(X,T)e^{2ix} + A_{33}(X,T)e^{3ix} \\ & + B_{31}(X,T)e^{iqx} + B_{32}(X,T)e^{2iqx} + B_{33}(X,T)e^{3iqx} + c.c. \Big] \end{aligned} \tag{22}$$

In order to get the solvability conditions that determine the leading order amplitudes, we need only look at the following two projections of this equation: $\frac{1}{2\pi n}\int_{-n\pi}^{n\pi} \mathrm{d}x\; e^{-ix}$ and $\frac{q}{2\pi n}\int_{-n\pi}^{n\pi} \mathrm{d}x\; e^{-iqx}$. The conditions will come from forcing these projections that result in resonance terms to vanish. Leaving $q$ as an arbitrary rational number makes the calculation more difficult because we must now calculate more terms of the equation as they may have a component along these directions. We will calculate each term in the equation, and then look at the final equations resulting from the projections. An alternative way to look at this problem is to make use of the Fredhom alternative theorem, which will result in basically the same calculation.

We first not that $L_0 u_3$ and $L_1 u_2$ cannot have a component along the directions we are looking for as both directions lie completely in the kernels of these operators. We must therefore compute $L_2 u_1$ and the two nonlinear terms in order to find the solvability condition.

$$\begin{aligned} L_2 u_1 = & \left[4\left(q^2-1\right)^2 \partial_X^2 A_{11} + \mu A_{11} - \partial_T A_{11}\right] e^{ix} \\ & + \left[4q^2\left(q^2-1\right)^2 \partial_X^2 B_{11} + \left(\mu - \delta(q^2-1)^2\right) B_{11} - \partial_T B_{11}\right] e^{iqx} \\ & + c.c \end{aligned} \tag{23}$$

$$\begin{aligned} 2bu_1u_2 = 2b\Big[ & A_{11}A_{22}e^{3ix} + A_{11}A_{21}e^{2ix} + \left(A_{11}C_{20} + A_{22}\bar{A}_{11}\right)e^{ix} + A_{11}\bar{A}_{21} \\ & + B_{11}B_{22}e^{3iqx} + B_{11}B_{21}e^{2iqx} + \left(B_{11}C_{20} + B_{22}\bar{B}_{11}\right)e^{iqx} + B_{11}\bar{B}_{21} \\ & + \left(A_{11}B_{21} + B_{11}A_{21}\right)e^{i(q+1)x} + \left(\bar{A}_{11}B_{21} + B_{11}\bar{A}_{21}\right)e^{i(q-1)x} \\ & + A_{22}B_{11}e^{i(q+2)x} + \bar{A}_{22}B_{11}e^{i(q-2)x} + A_{11}B_{22}e^{i(2q+1)x} + \bar{A}_{11}B_{22}e^{i(2q-1)x} \\ & + c.c. \Big] \end{aligned} \tag{24}$$

$$
\begin{aligned}
-u_1^3 = -\Big[&A_{11}^3 e^{3ix} + \left(3|A_{11}|^2 A_{11} + 6|B_{11}|^2 A_{11}\right) e^{ix} \\
&+ B_{11}^3 e^{3iqx} + \left(3|B_{11}|^2 B_{11} + 6|A_{11}|^2 B_{11}\right) e^{iqx} \\
&+ 3A_{11}^2 B_{11} e^{i(q+2)x} + 3\bar{A}_{11}^2 B_{11} e^{i(q-2)x} + 3A_{11}B_{11}^2 e^{i(2q+1)x} + 3\bar{A}_{11}B_{11}^2 e^{i(2q-1)x} \\
&+ c.c.\Big]
\end{aligned} \tag{25}
$$

The final solvability conditions become:

$$
\begin{aligned}
&\Big[\Big(4\left(q^2-1\right)^2 \partial_X^2 A_{11} + \mu A_{11} - \partial_T A_{11}\Big) + 2b\left(A_{11}C_{20} + A_{22}\bar{A}_{11}\right) - \big(3|A_{11}|^2 A \\
&+\delta_{\mathrm{d}}(q-1)\Big[\Big(4q^2\left(q^2-1\right)^2 \partial_X^2 B_{11} + \left(\mu - \delta(q^2-1)^2\right) B_{11} - \partial_T B_{11}\Big) + 2b\left(B_{11}C_{20} + B_{22}\bar{B}_{11}\right) - \big(3|B_{11}|^2 B
\end{aligned} \tag{26}
$$

If $q \neq 1, 2, 1/2, 3, 1/3$, then the equations become

$$\dot{A} = \mu A + aA'' - \alpha|A|^2 A - \gamma|B|^2 A \tag{27}$$

$$\dot{B} = \left(\mu - \frac{\delta a}{4}\right) B + q^2 a B'' - \beta|B|^2 B - \gamma|A|^2 B \tag{28}$$

where $a = 4(q^2-1)^2$, $\alpha = 3 - \frac{4b^2}{q^4} - \frac{2b^2}{9(q^2-4)^2}$, $\beta = 3 - \frac{4b^2}{q^4} - \frac{2b^2}{9q^2(1-4q^2)^2}$, and $\gamma = 6 - \frac{4b^2}{q^4}$. We see from this equation that in the $q = 1$ case the spatial derivative term vanishes, and this would have been prevented if we used the spatial scaling of Bentley.

We can gain some insight into these coupled equations by rewriting the amplitudes in terms of polar coordinates as $A = re^{i\theta}$ and $b = se^{i\phi}$. Using these variables, the two complex equations can be written as the following 4 real equations:

$$
\begin{aligned}
\dot{r} &= \mu r + a(r'' - r\theta'^2) - \alpha r^3 - \gamma s^2 r && \text{(29a)} \\
r^2\dot{\theta} &= a(r^2\theta')' && \text{(29b)} \\
\dot{s} &= (\mu - \delta a/4)s + q^2 a(s'' - s\phi'^2) - \beta s^3 - \gamma r^2 s && \text{(29c)} \\
s^2\dot{\phi} &= q^2 a(s^2\phi')' && \text{(29d)}
\end{aligned}
$$

If we focus on time-independent solutions, we can now use the standard trick of mapping the above equations onto a problem of particles in a central potential. In this case there are two particles in the potential that are coupled together. The angular momentum of each particle $l_A = r^2\theta'$ and $l_B = s^2\phi'$ are constants of the motion along with the total energy of the system $h = h_A + h_B - h_c$. Here the energy of the two particles are

$$h_A = \frac{1}{2}ar'^2 + \frac{1}{2}\mu r^2 + \frac{al_A^2}{2r^2} - \frac{\alpha}{4}r^4 \tag{30a}$$

$$h_B = \frac{1}{2}q^2 as'^2 + \frac{1}{2}(\mu - \delta a/4)s^2 + \frac{q^2 al_B^2}{2s^2} - \frac{\beta}{4}s^4 \tag{30b}$$

and the coupling term is $h_c = \frac{1}{2}\gamma r^2 s^2$. In these new variables, the equations of motion become

$$h'_A = \gamma s^2 r r' \tag{31a}$$

$$h'_B = \gamma r^2 s s' \tag{31b}$$

$$l'_A = 0 \tag{31c}$$

$$l'_B = 0 \tag{31d}$$

## IV. WEAKLY NONLINEAR ANALYSIS FOR Q=0

In the $q = 0$ case, we have a degenerate minimum($k^4$) to the marginal stability curve at $k = 0$ in adition to the quadratic minimum at $k = 1$. This indicates that three lengthscales are required for the analysis. In addition to the original lengthscale of the problem $x$ corresponding to the wavelength of the preferred spatial pattern, there is also the long lengthscale $x = \epsilon X$ over which the envelope is modulated. In addition there is an intermediate lengthscale $x = \epsilon^{1/2}\chi$ associated with modulation of the zero wavenumber mode. Thus, the spatial derivative becomes $\partial_x \to \partial_x + \epsilon^{1/2}\partial\chi + \epsilon\partial_X$. With this new scaling, we find that it is necessary to control the value of the forcing parameter on a more course scale: $r \to \epsilon\eta + \epsilon^2\mu$ in order to reach a consistent set of equations. This physically corresponds to looking in a neighborhood of size $\mathcal{O}(\epsilon^2)$ that is $\mathcal{O}(\epsilon)$ away from $r = 0$. We'll ignore the timescales for now, but it seems like the interesting scale will be more course than the previous case as well (i.e. $\mathcal{O}(\epsilon)$).

The linear part of the modified Swift-Hohenberg equation (Eq. 2),

$$L = r - \partial_x^4 \left(1 + \partial_x^2\right)^2, \tag{32}$$

can be expanded as $L = L_0 + \epsilon^{1/2} L_{1/2} + \epsilon L_1 + \epsilon^{3/2} L_{3/2} + \epsilon^2 L_2 + ...$ where:

$$L_0 = -\,\partial_x^4 \left(1 + \partial_x^2\right)^2 \tag{33a}$$

$$L_{1/2} = -\,4 \left(1 + \partial_x^2\right) \left(1 + 2\partial_x^2\right) \partial_x^3 \partial_\chi \tag{33b}$$

$$L_1 = \eta - 2 \left(3 + 15\partial_x^2 + 14\partial_x^4\right) \partial_x^3 \partial_\chi - 4 \left(1 + \partial_x^2\right) \left(1 + 2\partial_x^2\right) \partial_x^3 \partial_X \tag{33c}$$

$$L_{3/2} = -\,4 \left(1 + 10\partial_x^2 + 14\partial_x^4\right) \partial_x \partial_\chi^3 - 4 \left(3 + 15\partial_x^2 + 14\partial_x^4\right) \partial_x^2 \partial_\chi \partial_X \tag{33d}$$

$$\begin{aligned} L_2 = \mu &- \left(1 + 30\partial_x^2 + 70\partial_x^4\right) \partial_\chi^4 - 12 \left(1 + 10\partial_x^2 + 14\partial_x^4\right) \partial_x \partial_\chi^2 \partial_X \\ &- 2 \left(3 + 15\partial_x^2 + 14\partial_x^4\right) \partial_x^2 \partial_X^2 \end{aligned} \tag{33e}$$

$$\begin{aligned} L_{5/2} = &-\,4 \left(1 + 30\partial_x^2 + 70\partial_x^4\right) \partial_\chi^3 \partial_X - 12 \left(1 + 10\partial_x^2 + 14\partial_x^4\right) \partial_x \partial_\chi \partial_X^2 \\ &- 4 \left(3 + 14\partial_x^2\right) \partial_x \partial_\chi^5 \end{aligned} \tag{33f}$$

$$\begin{aligned} L_3 = &-\,6 \left(1 + 30\partial_x^2 + 70\partial_x^4\right) \partial_\chi^2 \partial_X^2 - 4 \left(1 + 10\partial_x^2 + 14\partial_x^4\right) \partial_x \partial_X^3 \\ &- 20 \left(3 + 14\partial_x^2\right) \partial_x \partial_\chi^4 \partial_X - 2 \left(1 + 14\partial_x^2\right) \partial_\chi^6 \end{aligned} \tag{33g}$$

### A. The quadratic-cubic nonlinearity

We will first consider the case when $N = N_{23}$ so that the modified Swift-Hohenberg equation can be written as $L[u] + N_{23}[u] = 0$. We will assume that the solution can be written as an asymptotic series with the leading term of order $\epsilon^{1/2}$, namely $u = \epsilon^{1/2} u_{1/2} + \epsilon u_1 + \epsilon^{3/2} u_{3/2} + \epsilon^2 u_2 + ...$

We can then write out the resulting equation at each order of $\epsilon$ by matching terms at the proper order. At leading order we have $\mathcal{O}(\epsilon^{1/2}) : \; L_0 u_{1/2} = 0$, telling us that the solution is of the form $u_{1/2} = \Theta_{1/2} + (A_{1/2} e^{ix} + C.C.)$.

The next order is $\mathcal{O}(\epsilon) : \; L_0 u_1 + L_{1/2} u_{1/2} + b u_{1/2}^2 + b u_{1/2}^2 = 0$. Noting that $L_{1/2} u_{1/2} = 0$, the solvability condition implies that $A_{1/2} = \Theta_{1/2} = 0$ because $\Theta$ is assumed to be real. We finally get the equation $L_0 u_1 = 0$, telling us that the solution must be of the form $u_{1/2} = \Theta + (Ae^{ix} + C.C.)$ again with undetermined coefficients.

The order $\mathcal{O}(\epsilon^{3/2})$ produces an the identical equation as above for $u_{3/2}$, namely $L_0 u_{3/2} = 0$. We can thus redefine $u_1$ to include the solution at this order. This effectively allows us to set $u_{3/2} = 0$ for the rest of our calculations.

Moving on to order $\mathcal{O}(\epsilon^2)$, we find consistent nontrivial equation if we assume that $\eta \neq 0$. The solvability conditions from the equation, $L_0 u_2 + L_1 u_1 + b u_1^2 = 0$ produce the following set of equations for the undertermined coefficients of $u_1$.

$$\eta\Theta + b\Theta^2 + 2b|A|^2 = 0 \tag{34a}$$

$$4A_{\chi\chi} + \eta A + 2b\Theta A = 0 \tag{34b}$$

It is clear that a solution is only possible in the case that $\eta^2 \geq 8b^2|A|^2$, which provides maximum limit on the magnitude of $A$ in terms of the parameters of the problem. Note that we would probably want to include the time derivative at this order. Assuming that $A$ satisfies this inequality, we can write an expression the following expression for $\Theta$ in terms of $A$

$$\Theta = \frac{1}{2b}\left(-\eta \pm \sqrt{\eta^2 - 8b^2|A|^2}\right) \tag{35}$$

As well as a spatial evolution equation for $A$ decoupled from $\Theta$.

$$A_{\chi\chi} \pm \left(\frac{\eta^2}{2} - 4b^2|A|^2\right)^{1/2} A = 0 \tag{36}$$

We can now rescale this equation by $|A|$'s maximum value $A_* = \eta/\sqrt{8}b$ and rescale the intermediate length by $\chi^2 \to (\sqrt{2}/\eta)\chi^2$. We will furthermore select the plus sign from the root in order that it is possible to have a local minumum in the potential. The equation now becomes

$$a'' + \left(1 - |a|^2\right)^{1/2} a = 0 \tag{37}$$

where $a = A/A_*$ has a maximumum magnitude of 1 and the primes indicate derivatives with respect to the rescaled intermediate length.

We can re-express this equation into a form similar to those of a particle traveling in a central potential by making the substitution $A = re^{i\phi}$. In this case we get the two equations

$$r'' - \frac{l^2}{r^3} + \sqrt{1 - r^2}r = 0 \tag{38a}$$

$$l' = (r^2\phi')' = 0 \tag{38b}$$

where, $r < 1$. We can find the spatial hamiltonian for this system to be

$$h = \frac{1}{2}r'^2 + \frac{l^2}{2r^2} - \frac{1}{3}(1 - r^2)^{3/2} \tag{39}$$

$$\mathcal{O}(\epsilon^{1/2}) : L_0 u_{1/2} = 0 \tag{40a}$$

$$\mathcal{O}(\epsilon) : L_0 u_1 + L_{1/2} u_{1/2} + b u_{1/2}^2 \tag{40b}$$

$$\mathcal{O}(\epsilon^{3/2}) : L_0 u_{3/2} + L_{1/2} 1 u_1 + L_{1/2} u_1 = \tag{40c}$$

$$\mathcal{O}(\epsilon^{2}) : L_0 u_2 + L_{1/2} u_{3/2} + L_1 u_1 + b u_1^2 \tag{40d}$$

$$\mathcal{O}(\epsilon^{5/2}) : -L_0 u_{5/2} = L_{1/2} u_2 + L_1 u_{3/2} + L_{3/2} u_1 + 2b u_1 u_{3/2} \tag{40e}$$

$$\mathcal{O}(\epsilon^{3}) : -L_0 u_3 = L_{1/2} u_{5/2} + L_1 u_2 + L_{3/2} u_{3/2} + L_2 u_1 + b u_{3/2}^2 + 2b u_1 u_2 - u_1^3 \tag{40f}$$

## V. WEAKLY NONLINEAR ANALYSIS REDO-BENTLY

In this section, we consider the scaling analogous to what Bentley did in which the long spatial scale is defined by $x = \epsilon^{1/2} X$ instead of $x = \epsilon X$ as in the previous section. All other scalings are identical to the previous section. The linear part of the modified Swift-Hohenberg equation (Eq. 2),

$$L = r - \left(1 + \partial_x^2\right)^2 \left[\left(q^2 + \partial_x^2\right)^2 + \delta\right], \tag{41}$$

can be expanded as $L = L_0 + \epsilon^{1/2} L_{1/2} + \epsilon L_1 + \epsilon^{3/2} L_{3/2} + \epsilon^2 L_2 + ...$ where:

$$L_0 = -\left(1 + \partial_x^2\right)^2 \left(q^2 + \partial_x^2\right)^2 \tag{42a}$$

$$L_{1/2} = -4\left(1 + \partial_x^2\right)\left(q^2 + \partial_x^2\right)\left(q^2 + 1 + 2\partial_x^2\right)\partial_x \partial_X \tag{42b}$$

$$L_1 = -2\left[14\partial_x^6 + 15\left(q^2+1\right)\partial_x^4 + 3\left(q^4 + 4q^2 + 1\right)\partial_x^2 + q^2\left(q^2+1\right)\right]\partial_X^2 \tag{42c}$$

$$L_{3/2} = -4\left[14\partial_x^4 + 10\left(q^2+1\right)\partial_x^2 + q^4 + 4q^2 + 1\right]\partial_x \partial_X^3 \tag{42d}$$

$$L_2 = -\left[70\partial_x^4 + 30\left(q^2+1\right)\partial_x^2 + q^4 + 4q^2 + 1\right]\partial_X^4 - \delta\left(1 + \partial_x^2\right) + \mu - \partial_T \tag{42e}$$

$$L_{5/2} = -4\left[\left(3(q^2+1) + 14\partial_x^2\right)\partial_X^4 + \delta(1 + \partial_x^2)\right]\partial_x \partial_X \tag{42f}$$

$$L_3 = -2\left[\left(q^2 + 1 + 14\partial_x^2\right)\partial_X^4 + \delta(1 + 3\partial_x^2)\right]\partial_X^2 \tag{42g}$$

### A. The quadratic-cubic nonlinearity

We will first consider the case when $N = N_{23}$ so that the modified Swift-Hohenberg equation can be written as $L[u] + N_{23}[u] = 0$. We will assume that the solution can be written as an asymptotic series with the leading term of order $\epsilon$, namely $u = \epsilon u_1 + \epsilon^{3/2} u_{3/2} + \epsilon^2 u_2 + ...$

We can then write out the resulting equation at each order of $\epsilon$ by matching terms at the proper order.

$$\mathcal{O}(\epsilon): \; -L_0 u_1 = 0 \tag{43a}$$

$$\mathcal{O}(\epsilon^{3/2}): \; -L_0 u_{3/2} = L_{1/2} u_1 \tag{43b}$$

$$\mathcal{O}(\epsilon^2): \; -L_0 u_2 = L_{1/2} u_{3/2} + L_1 u_1 + b u_1^2 \tag{43c}$$

$$\mathcal{O}(\epsilon^{5/2}): \; -L_0 u_{5/2} = L_{1/2} u_2 + L_1 u_{3/2} + L_{3/2} u_1 + 2b u_1 u_{3/2} \tag{43d}$$

$$\mathcal{O}(\epsilon^3): \; -L_0 u_3 = L_{1/2} u_{5/2} + L_1 u_2 + L_{3/2} u_{3/2} + L_2 u_1 + b u_{3/2}^2 + 2b u_1 u_2 - u_1^3 \tag{43e}$$

The solution to the $\mathcal{O}(\epsilon)$ equation can be expressed in terms of the yet to be determined complex amplitudes $A_{11}, B_{11}$ as:

$$u_1(x, X, T) = A_{11}(X,T) e^{ix} + B_{11}(X,T) e^{iqx} + c.c. \tag{44}$$

Furthermore, since $L_{1/2} u_1 = 0$, we can absorb $u_{3/2}$ into $u_1$ as a correction.

For the next order in $\epsilon$,($\epsilon^2$) , we see that $L_{1/2} u_{3/2}$ vanishes, and will assume the following form for $u_2$:

$$\begin{aligned} u_2(x, X, T) &= C_{20}(X,T) \\ &\quad + \left[A_{21}(X,T) e^{ix} + A_{22}(X,T) e^{2ix} + B_{21}(X,T) e^{iqx} + B_{22}(X,T) e^{2iqx} + c.c.\right] \end{aligned} \tag{45}$$

The resulting condition becomes:

$$\begin{aligned} 0 &= \left(2b(|A_{11}|^2 + |B_{11}|^2) - q^4 C_{20}\right) \\ &\quad + \Bigg[4(q^2-1)^2 \left(A_{11}'' e^{ix} + q^2 B_{11}'' e^{iqx}\right) \\ &\quad + \left(b A_{11}^2 - 9(q^2-4)^2 A_{22}\right) e^{2ix} + \left(b B_{11}^2 - 9q^4(1-4q^2)^2 B_{22}\right) e^{2iqx} \\ &\quad + 2b B_{11} \left(A_{11} e^{i(q+1)x} + \bar{A}_{11} e^{i(q-1)x}\right) + c.c.\Bigg] \end{aligned} \tag{46}$$

In the case that $q \neq 1$, we see that this requires the leading order amplitudes to satisfy $A_{11}'' = B_{11}'' = 0$, which implies that they must be constants if we assume a finite value at $\pm\infty$ for the boundary conditions. This choice of scaling works well when $q = 1$ as is demonstrated in Bentley's thesis, but does not seem to produce useful results in the more general case.

Can I modify this scaling slightly or add an additional parameter that appears at this order to cancel out the problematic terms here? Would an additional time or length scale help, and if so, what would it represent physically?