CS2040S
Data Structures and Algorithms

Augmented Trees!

Last week...

Dictionaries

Binary search trees

Tries

Balanced search trees

- AVL trees
- Scapegoat trees
- B-trees

Today: Dynamic Data Structures

1. Maintain a set of items

2. Modify the set of items

3. Answer queries.

Today: Dynamic Data Structures

1. Maintain a set of items

2. Modify the set of items

3. Answer queries.

Big picture idea:

Trees are a good way to store, summarize, and search dynamic data.

B-trees are at the heart of *every* database!

Dynamic Data Structures

- Operations that create a data structure
  - build (preprocess)
- Operations that modify the structure
  - insert
  - delete
- Query operations
  - search, select, etc.

“Why do we need to learn how an AVL tree works?”

Just use a Java TreeMap, amiright?

"Why do we need to learn how an AVL tree works?"

1. Learn how to think like a computer scientist.

"Why do we need to learn how an AVL tree works?"

1. Learn how to think like a computer scientist.
2. Learn to modify existing data structures to solve new problems.

Augmented Data Structures

Many problems require storing additional data in a standard data structure.

Augment more frequently than invent

Augmented Data Structures

Many problems require storing additional data in a standard data structure.

Augment more frequently than invent

Useful for summarizing and processing data

Today

Three examples of augmenting balanced BSTs

1. Order Statistics

2. Interval Queries

3. Orthogonal Range Searching

Augmenting data structures

Basic methodology:

1. Choose underlying data structure

    (tree, hash table, linked list, stack, etc.)

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3. Modify data structure to *maintain* additional info when the structure changes.

   (subject to insert/delete/etc.)

Augmenting data structures

Basic methodology:

1. Choose underlying data structure

   (tree, hash table, linked list, stack, etc.)

2. Determine additional info needed.

3. Modify data structure to *maintain* additional info when the structure changes.

   (subject to insert/delete/etc.)

4. Develop new operations.

Order Statistics

Input

A set of integers.

Output: select(k)

Item with rank k in the set.

52	**7**	**13**	**43**	**22**	**92**	**18**	**9**	**65**	**67**	**87**	**25**

select(4)

select(2) returns:

52	**7**	**13**	**43**	**22**	**92**	**18**	**9**	**65**	**67**	**87**	**25**

1. 52
2. 9
3. 13
4. 43
5. 25

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Order Statistics

Input

A set of integers.

Output: select(k)

Item with rank k in the set.

52	**7**	**13**	**43**	**22**	**92**	**18**	**9**	**65**	**67**	**87**	**25**

select(4)

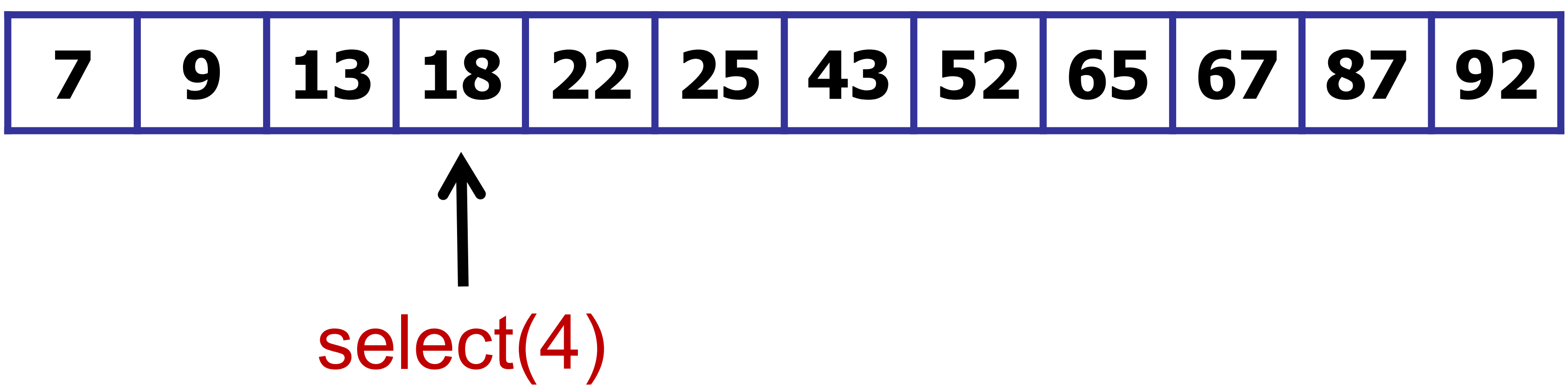
Order Statistics

Input

A set of integers.

Output: select(k)

Item with rank k in the set.



Order Statistics

Input

A set of integers.

Output: select(k) → Sort: O(n log n)

Item with rank k in the set.

7	**9**	**13**	**18**	**22**	**25**	**43**	**52**	**65**	**67**	**87**	**92**

select(4)

Order Statistics

Input

A set of integers.

Output: select(k) → QuickSelect: O(n)

Item with rank k in the set.

7	**9**	**13**	**18**	**22**	**25**	**43**	**52**	**65**	**67**	**87**	**92**

↑

select(4)

Order Statistics

Solution 1:

Sort: O(n log n)

Solution 2:

QuickSelect: O(n)

7	**9**	**13**	**18**	**22**	**25**	**43**	**52**	**65**	**67**	**87**	**92**

select(4)

Dynamic Order Statistics

Solution 1:

Preprocess: sort --- O(n log n)

Select: O(1)

Solution 2:

Preprocess: nothing --- O(1)

QuickSelect: O(n)

Dynamic Order Statistics

Solution 1:

Preprocess: sort --- O(n log n)

Select: O(1)

Solution 2:

Preprocess: nothing --- O(1)

QuickSelect: O(n)

Trade-off: how many items to select?

Dynamic Order Statistics

Implement a data structure that supports:

- insert(int key)
- delete(int key)

and also:

- select(int k)

7	**9**	**13**	**18**	**22**	**25**	**43**	**52**	**65**	**67**	**87**	**92**

select(4)

Dynamic Order Statistics

Solution 1:

Basic structure: sorted array A.

insert(int item): add item to sorted array A.

select(int k): return A[k]

7	**9**	**13**	**18**	**22**	**25**	**43**	**52**	**65**	**67**	**87**	**92**

Dynamic Order Statistics

Solution 2:

Basic structure: unsorted array A.

insert(int item): add item to end of array A.

select(int k): run QuickSelect(k)

7	**9**	**13**	**18**	**22**	**25**	**43**	**52**	**65**	**67**	**87**	**92**

When is it more efficient to maintain a sorted array (Solution 1)?

A. Always

B. When there are more inserts than selects.

C. When there are more selects than inserts.

D. Never

E. I’m confused.

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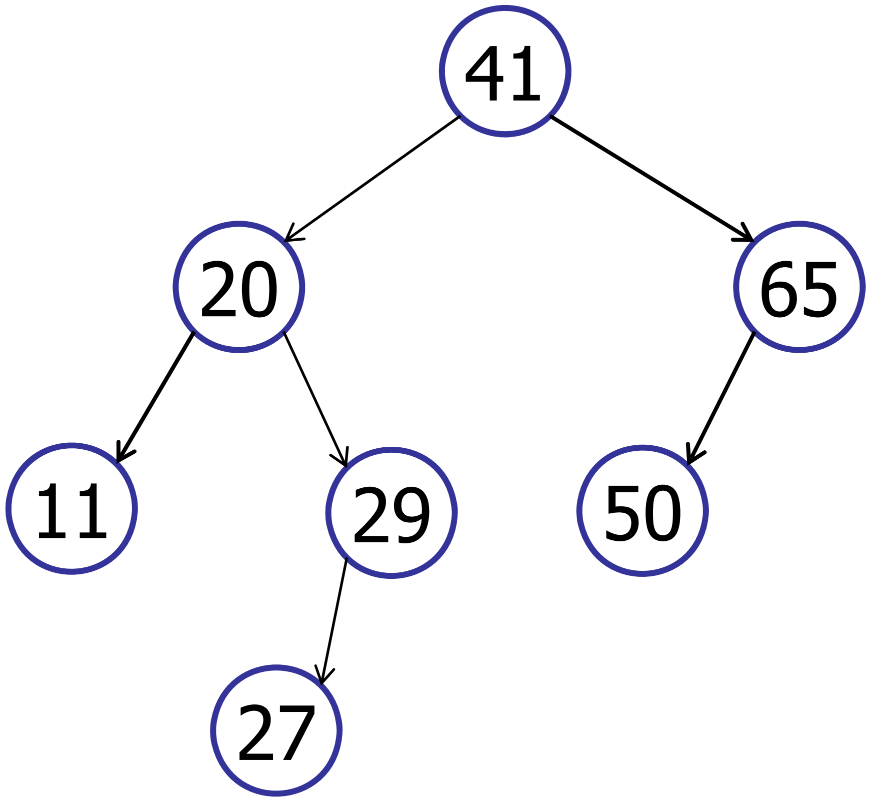
Dynamic Order Statistics

	Insert	Select
Solution 1: Sorted Array	O(n)	O(1)
Solution 2: Unsorted Array	O(1)	O(n)

7	9	13	18	22	25	43	52	65	67	87	92

Dynamic Order Statistics

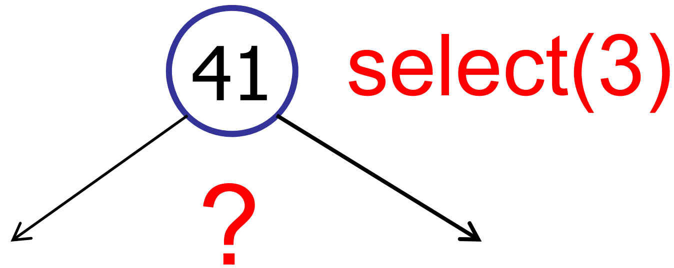
Today: use a (balanced) tree



11	**20**	**27**	**29**	**41**	**50**	**65**

Dynamic Order Statistics

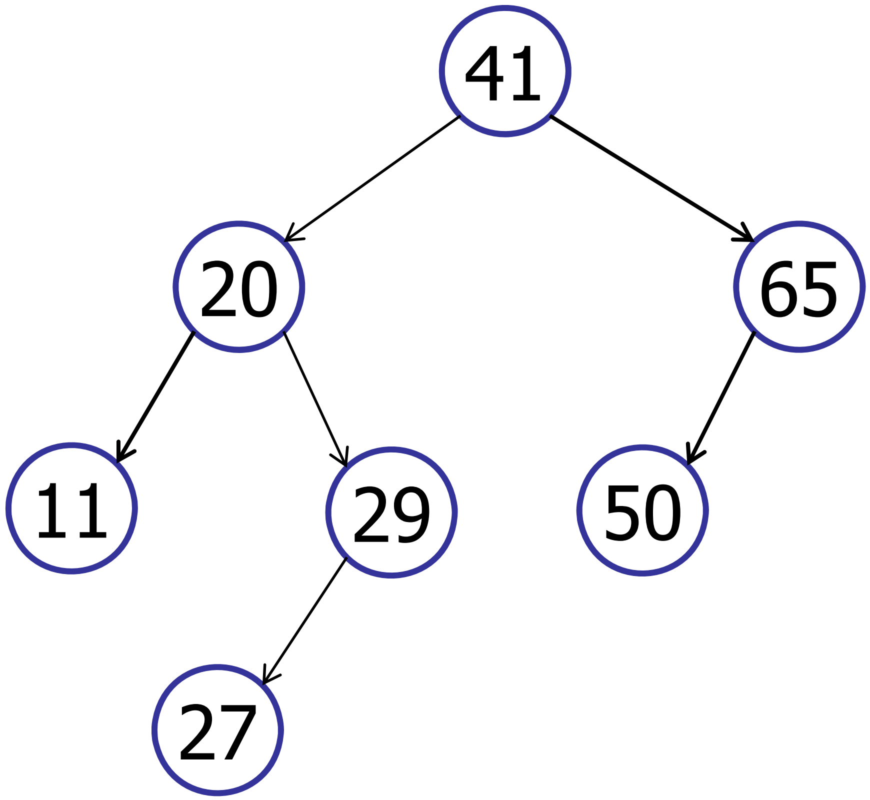
How to find the right item?



Dynamic Order Statistics

Simple solution: traversal

select(k): O(k)
in-order
traversal



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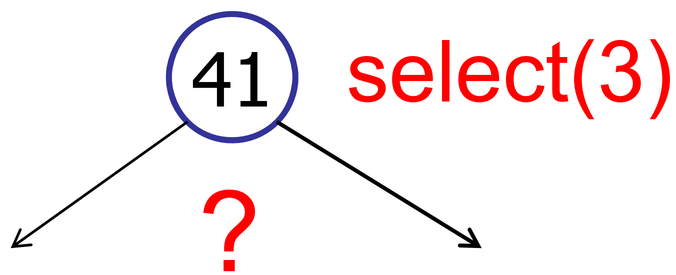
Dynamic Order Statistics

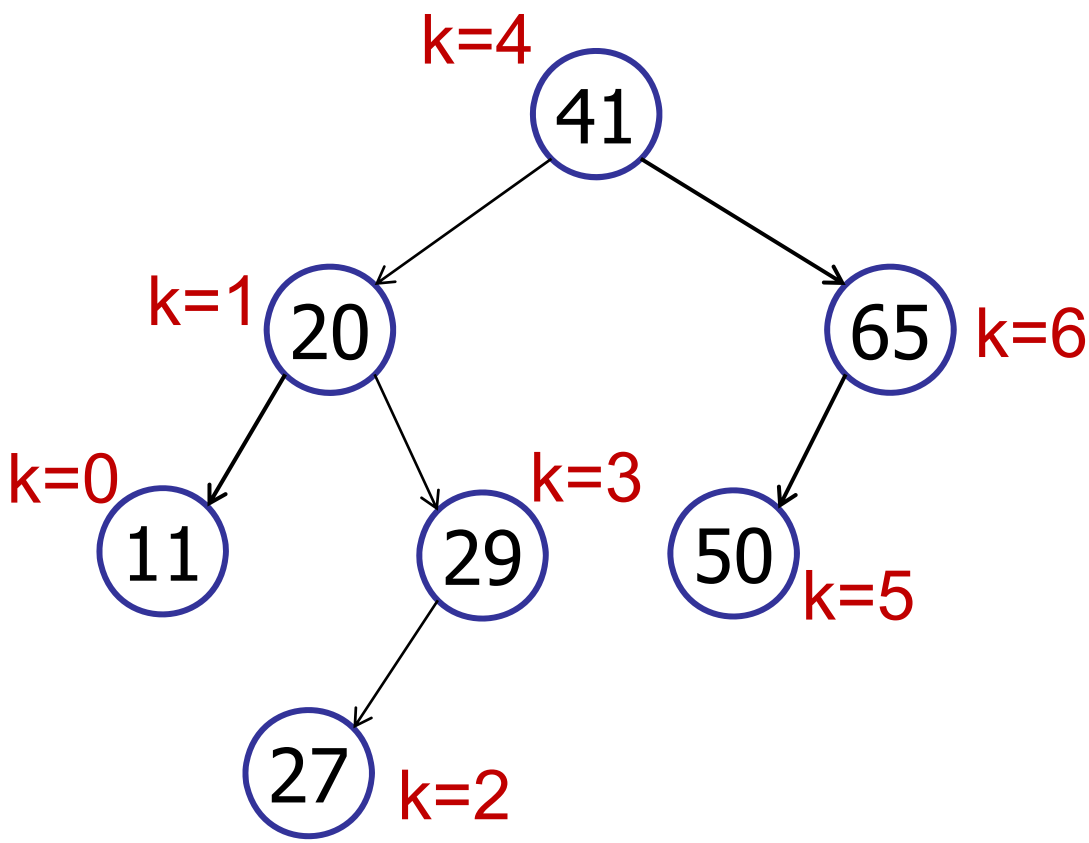
Augment!

What extra information would help?



Dynamic Order Statistics

Idea: store rank in every node



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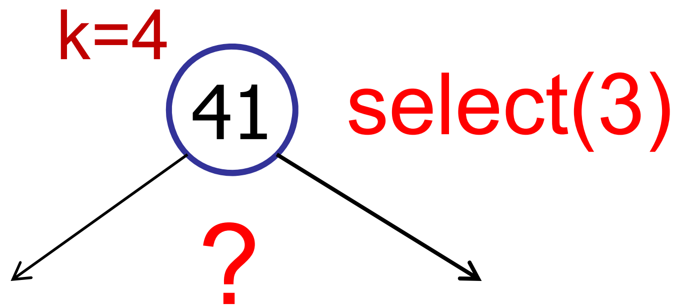

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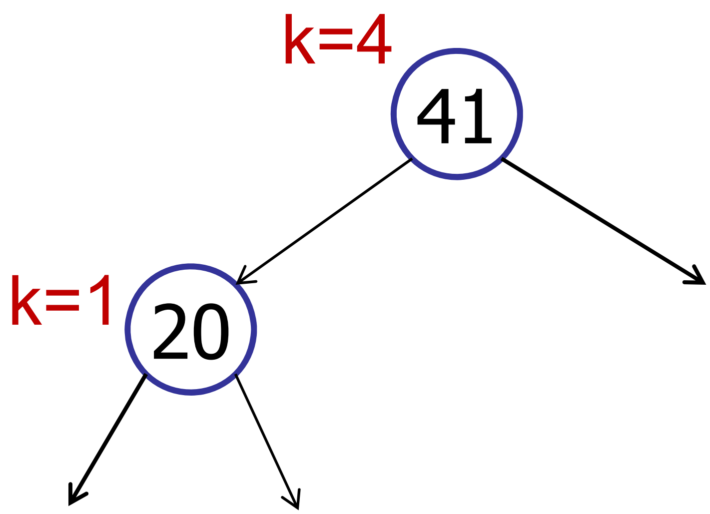
Dynamic Order Statistics

Idea: store rank in every node

select(3)



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Dynamic Order Statistics

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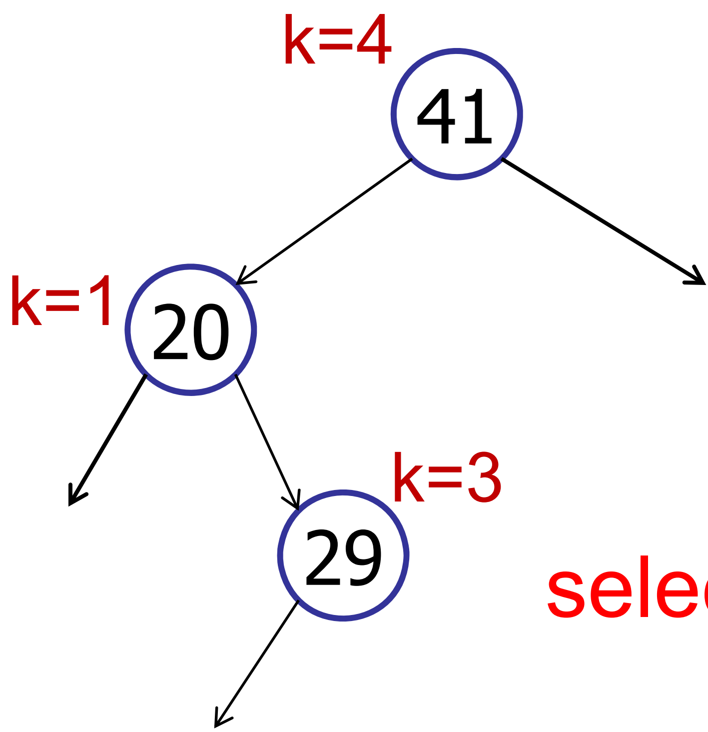
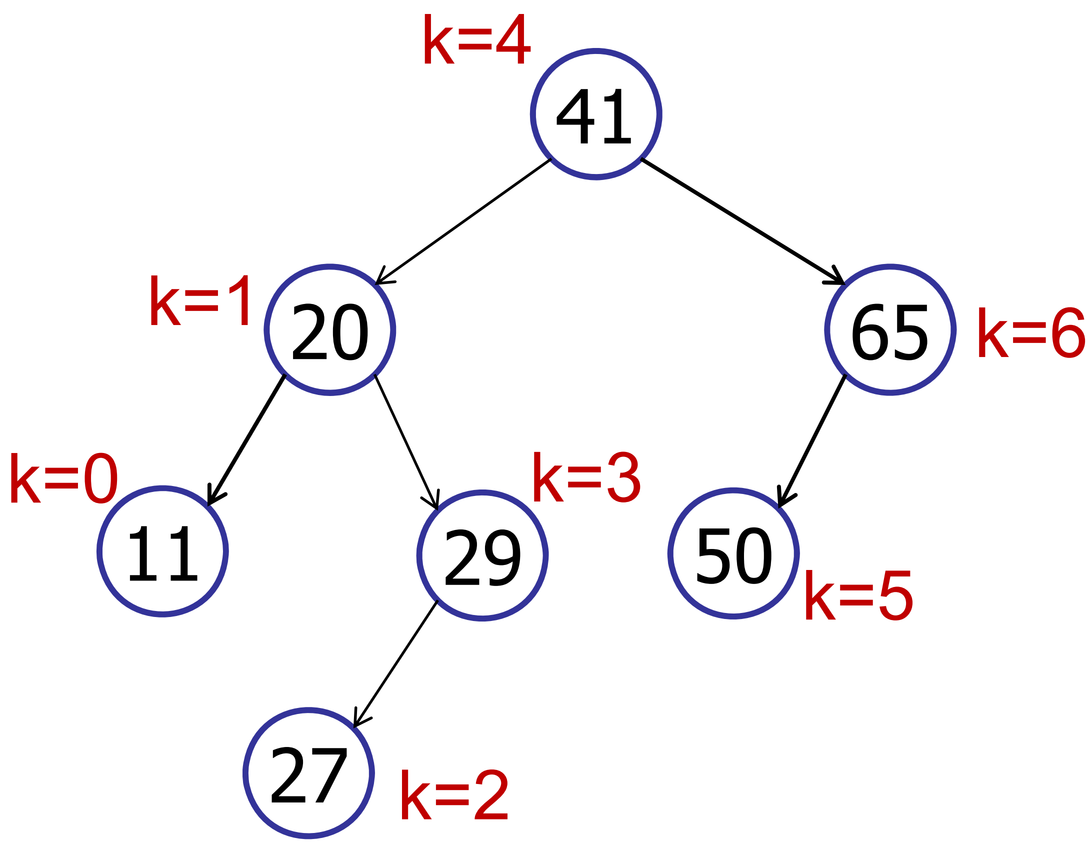


select(3)

11	**20**	**27**	**29**	**41**	**50**	**65**

Dynamic Order Statistics

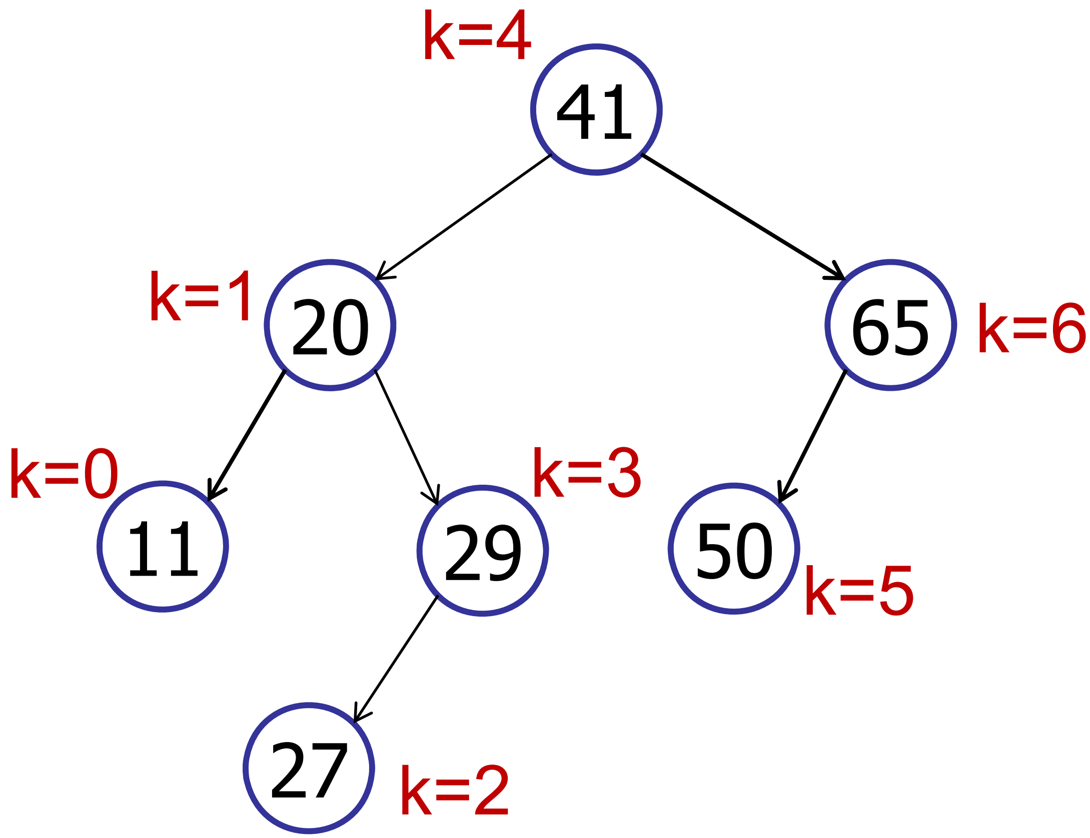
Idea: store rank in every node



11	**20**	**27**	**29**	**41**	**50**	**65**

Dynamic Order Statistics

What is the problem if we store rank in every node?



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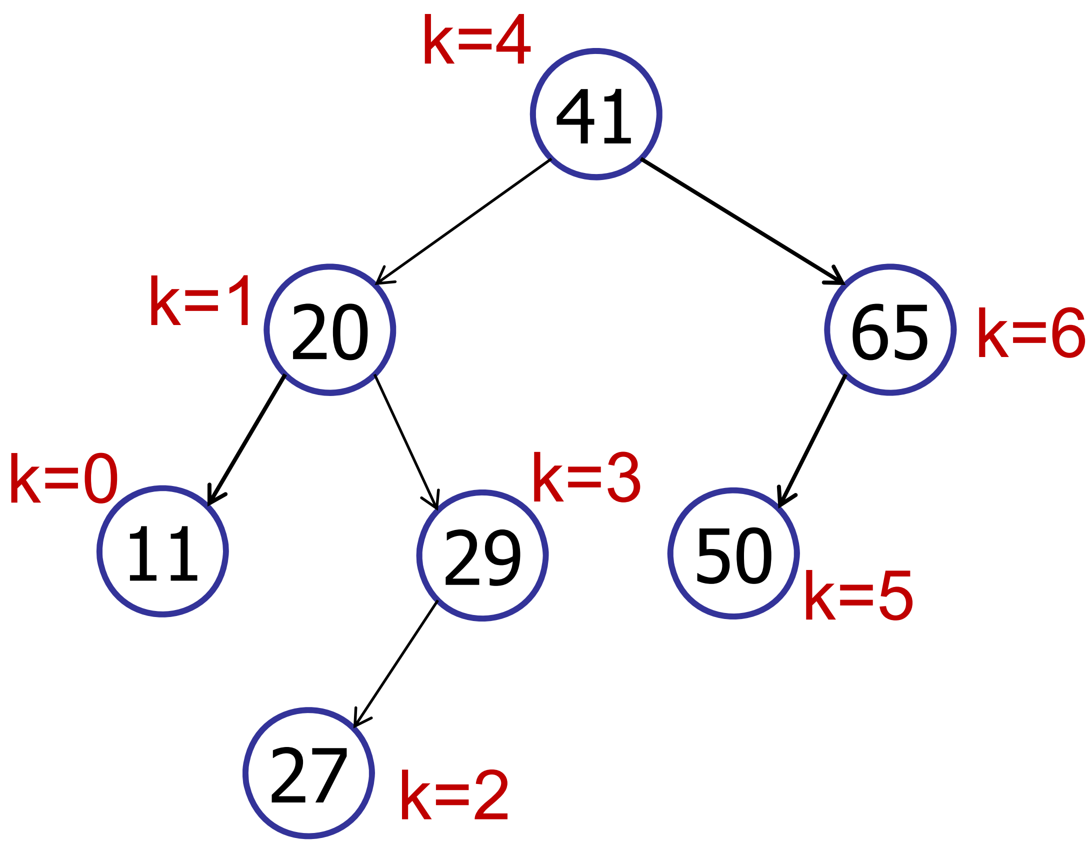
11	**20**	**27**	**29**	**41**	**50**	**65**

Dynamic Order Statistics

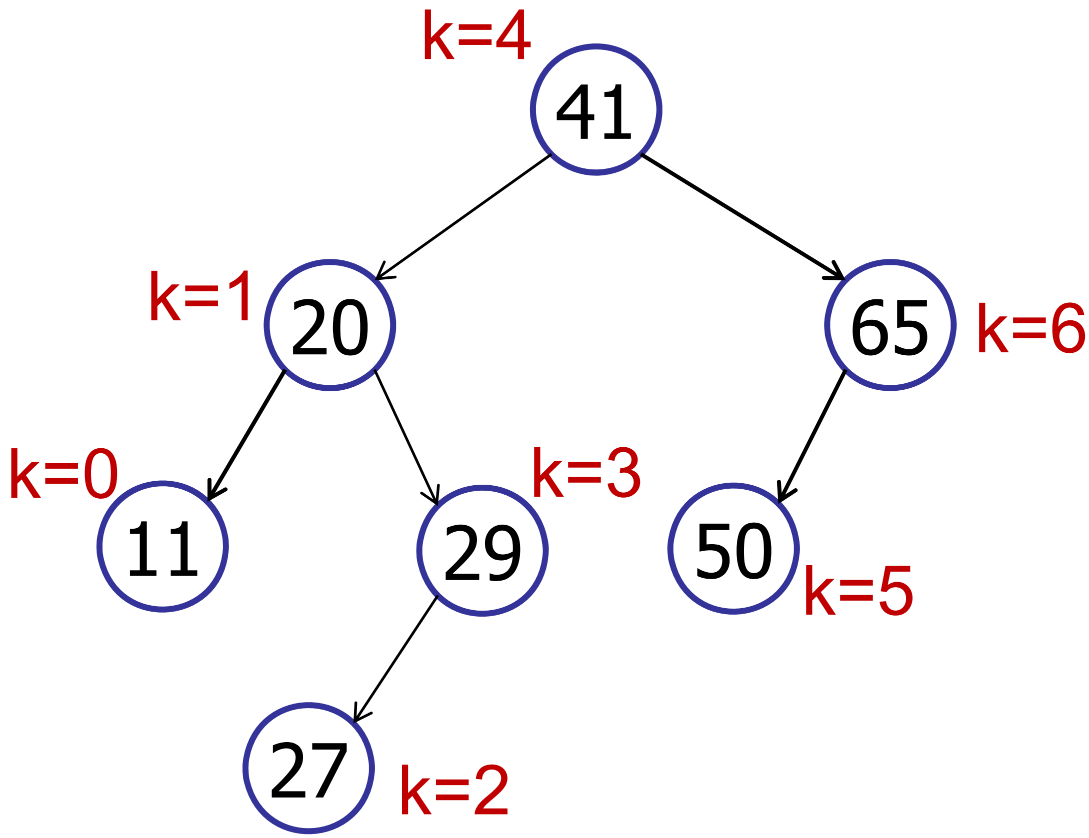
Idea: store rank in every node



Problem: insert(5)

Dynamic Order Statistics

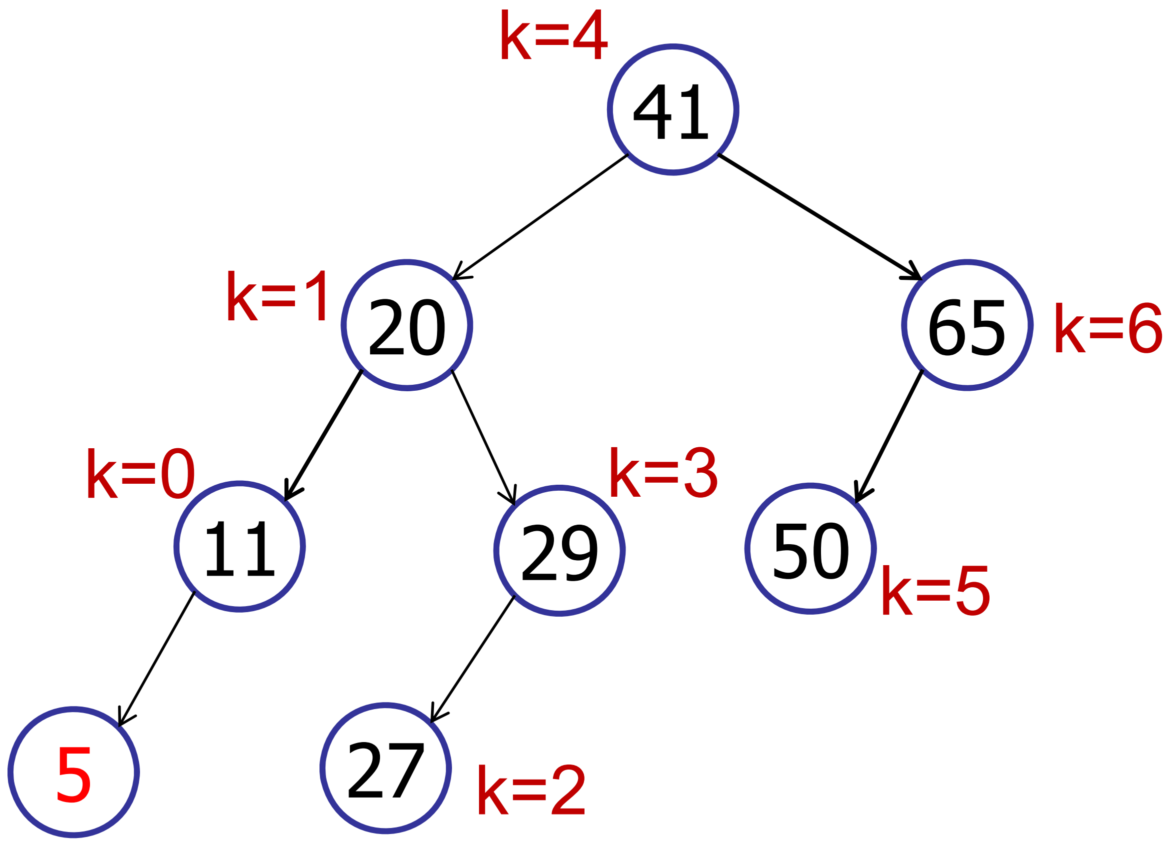
Idea: store rank in every node



Problem: insert(5) requires updating *all* the ranks!

Dynamic Order Statistics

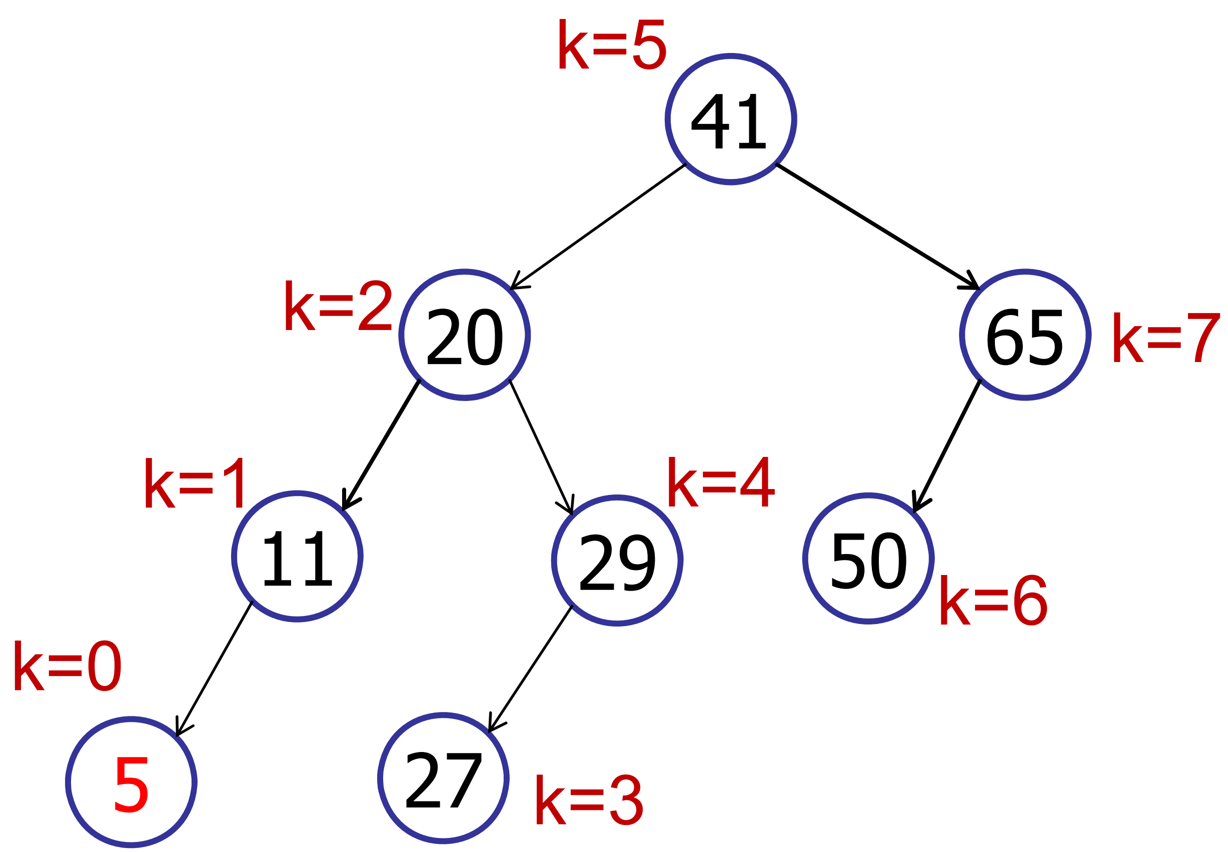
Idea: store rank in every node



5	**11**	**20**	**27**	**29**	**41**	**50**	**65**

Dynamic Order Statistics

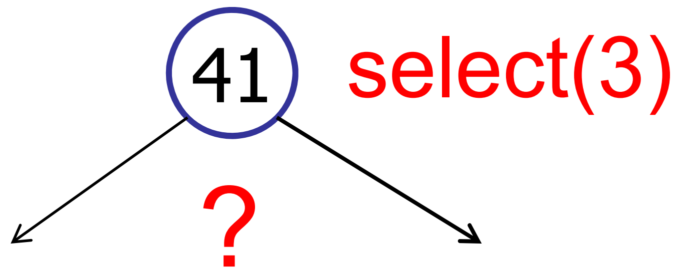
Conclusion: too expensive to store rank in every node!



5	**11**	**20**	**27**	**29**	**41**	**50**	**65**

Dynamic Order Statistics

What should we store in each node?



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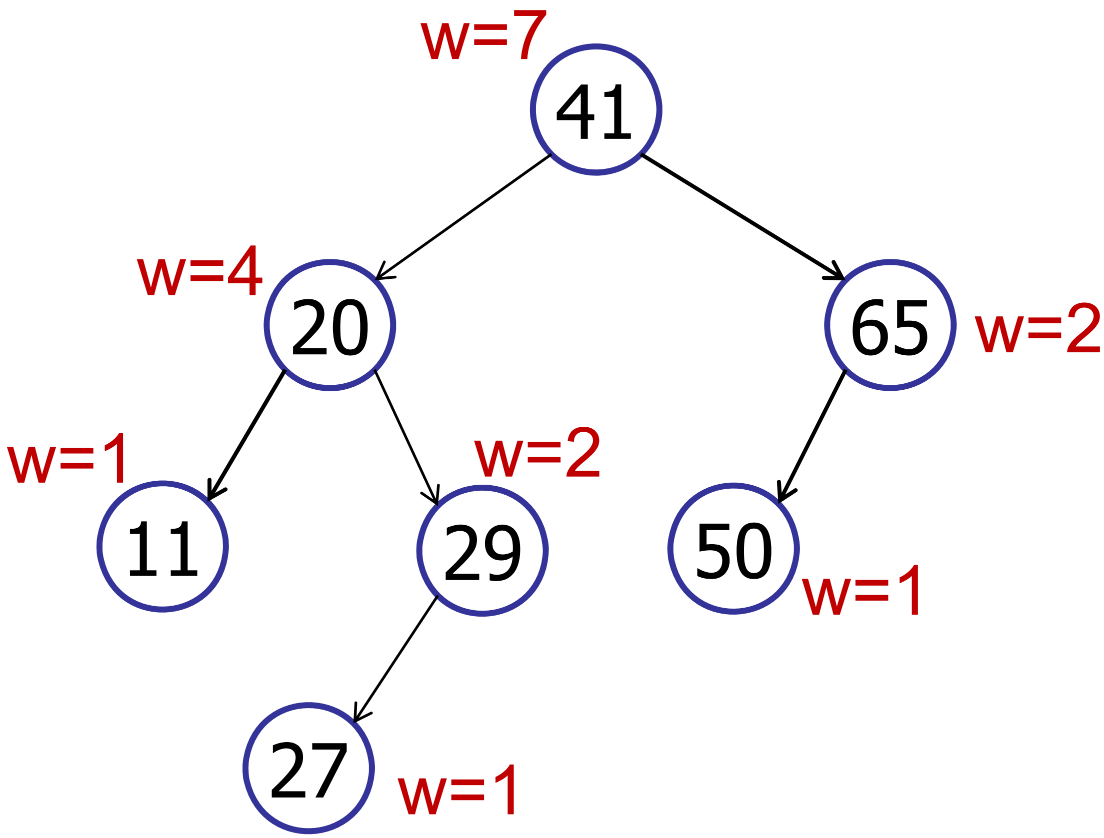
Dynamic Order Statistics

Idea: store *size* of sub-tree in every node



Dynamic Order Statistics

Idea: store size of sub-tree in every node

The <u>weight</u> of a node is the size of the tree rooted at that node.

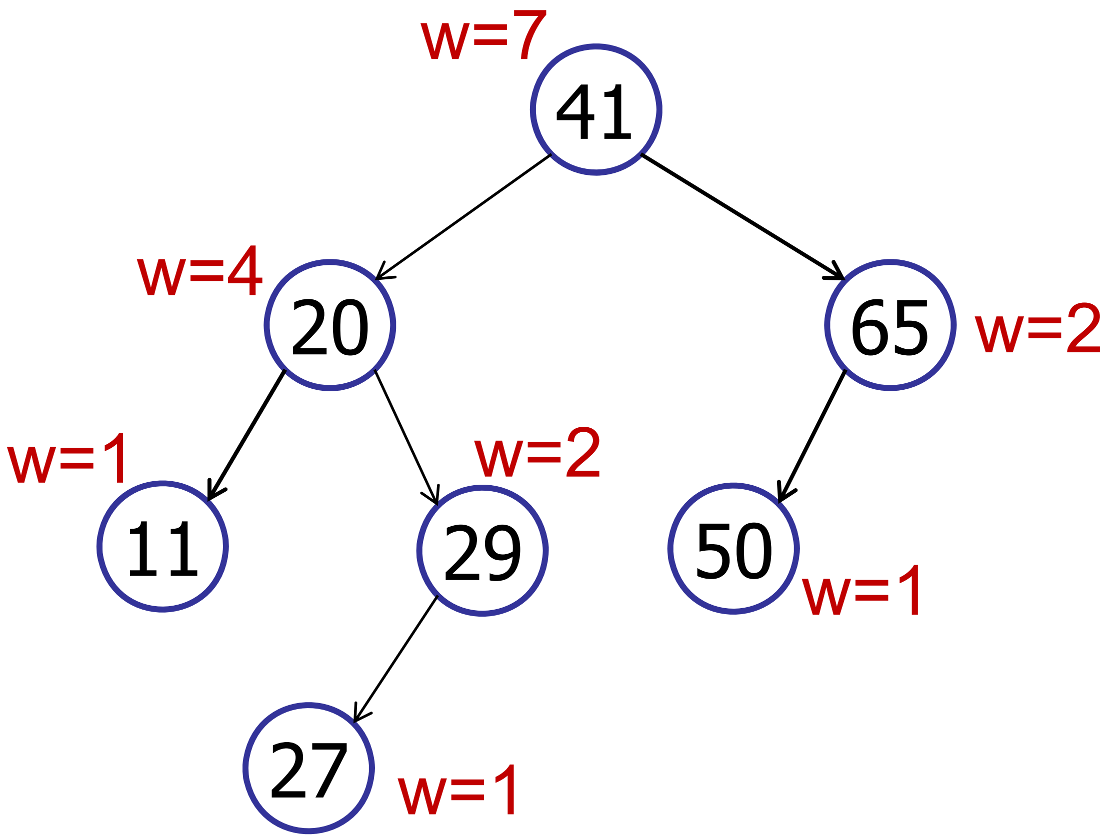
Define weight:

$w(\text{leaf}) = 1$

$w(v) = w(v.\text{left}) + w(v.\text{right}) + 1$

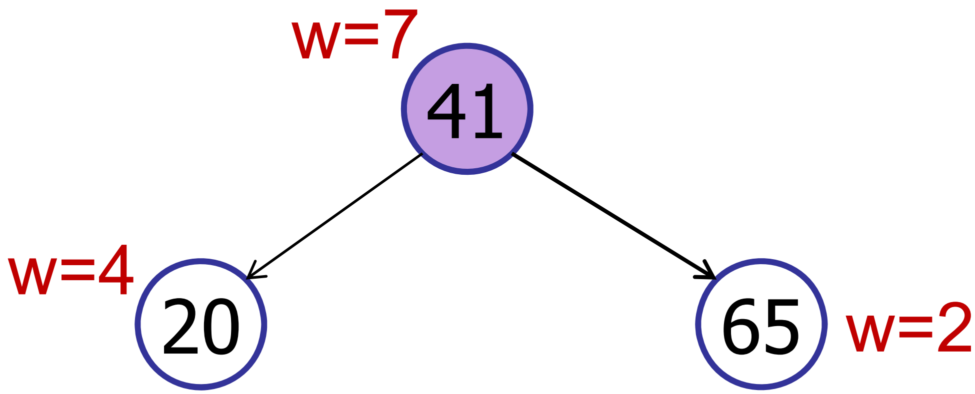
Dynamic Order Statistics

Idea: store *size* of sub-tree in every node

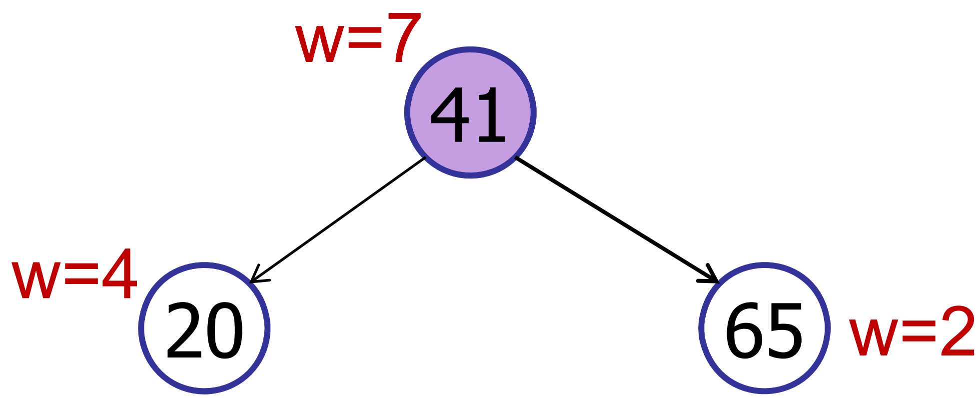


Dynamic Order Statistics

Example: select(3)



What is the rank of 41?



1. 1
2. 3
3. 5
4. 7
5. 9
6. Can’t tell.

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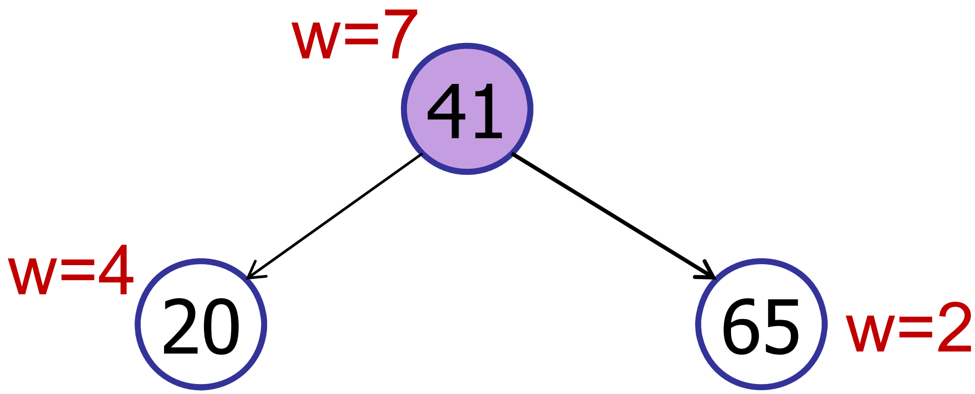
Dynamic Order Statistics

Example: select(3)

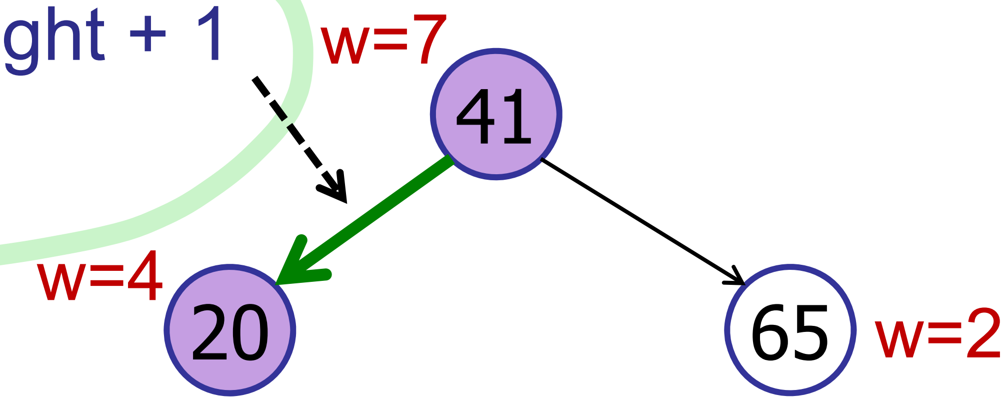


"rank in subtree" = left.weight + 1

Dynamic Order Statistics

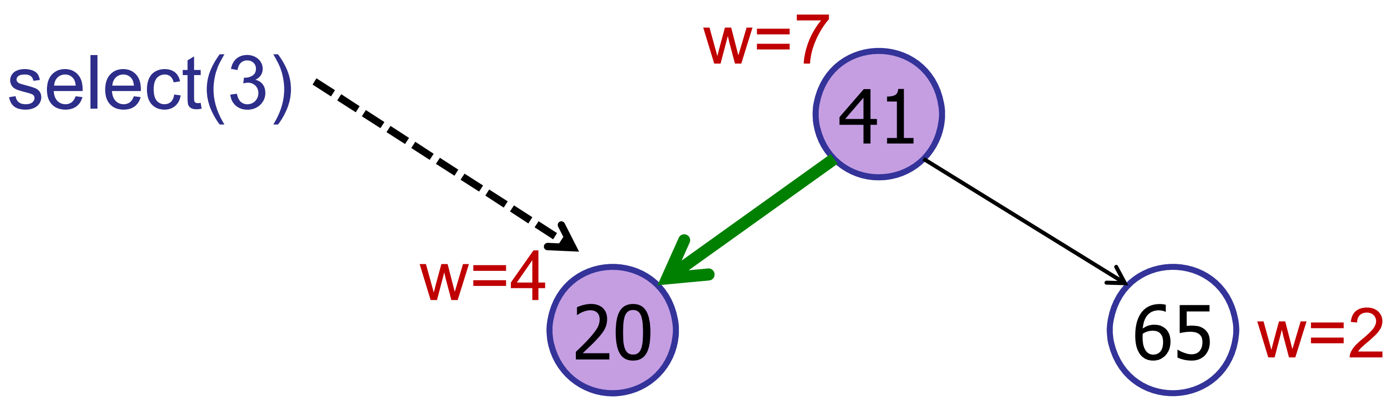
Example: select(3)

3 < left.weight + 1
Go left!



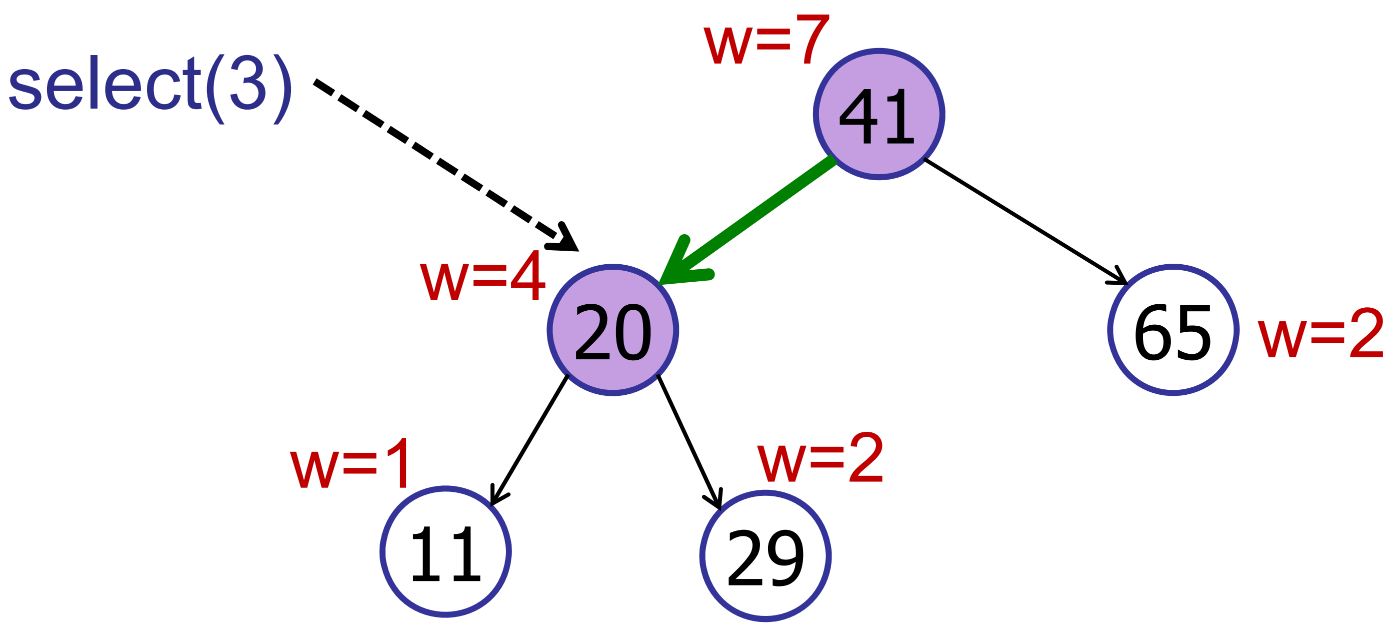
Dynamic Order Statistics

Example: select(3)



Dynamic Order Statistics

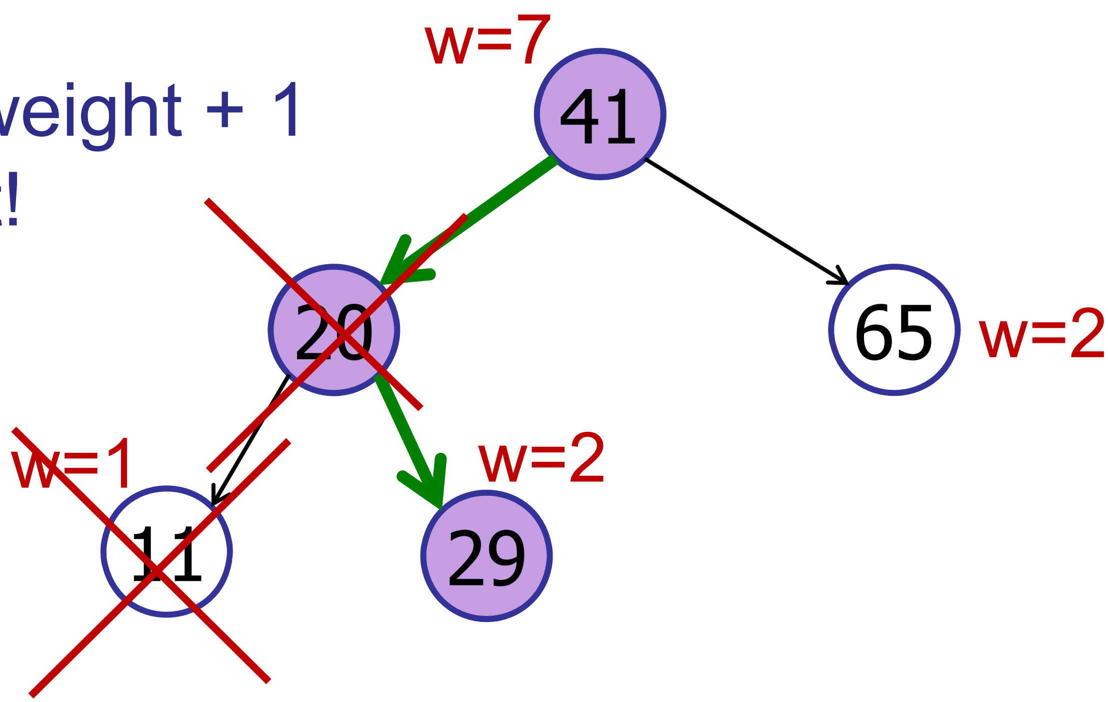
Example: select(3)



Dynamic Order Statistics

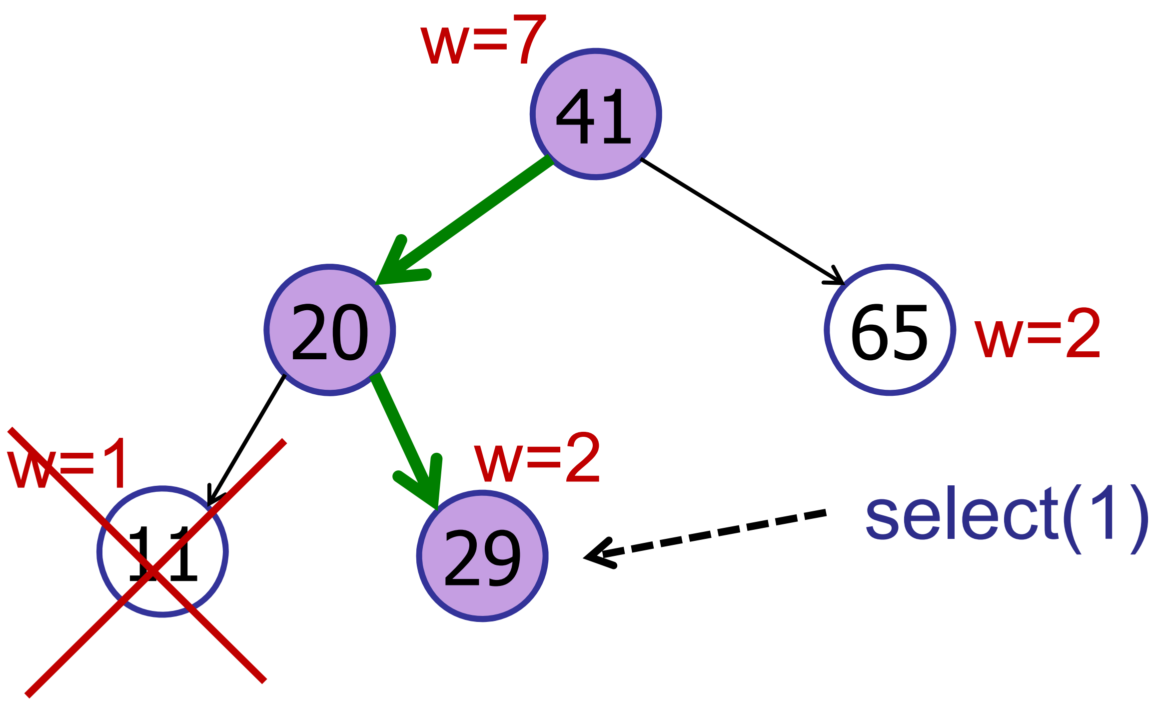
Example: select(3)

3 > left.weight + 1
Go right!



Dynamic Order Statistics

Example: select(3)



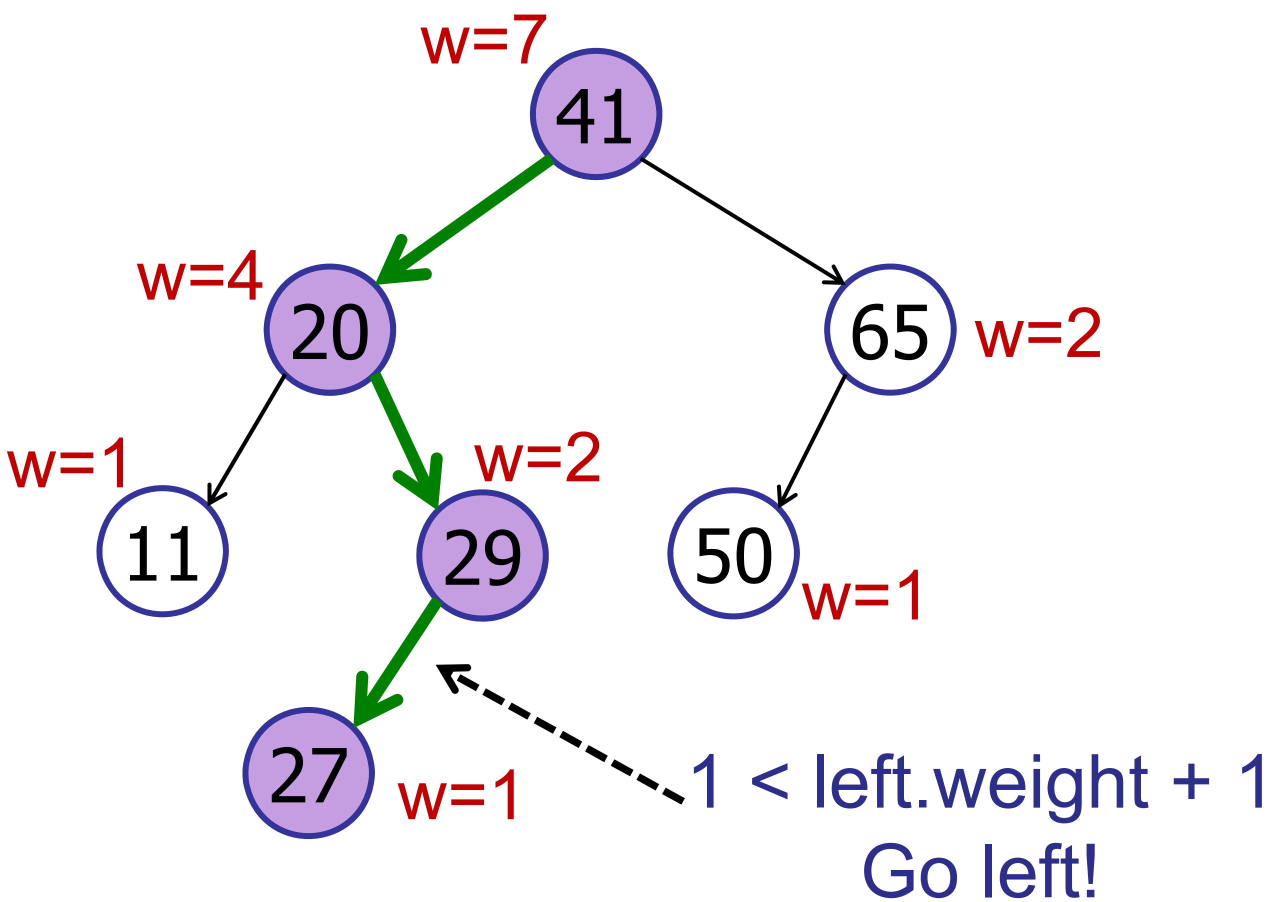
Item to select:

3 – (left.weight + 1) =

3 – (1 + 1) = 1

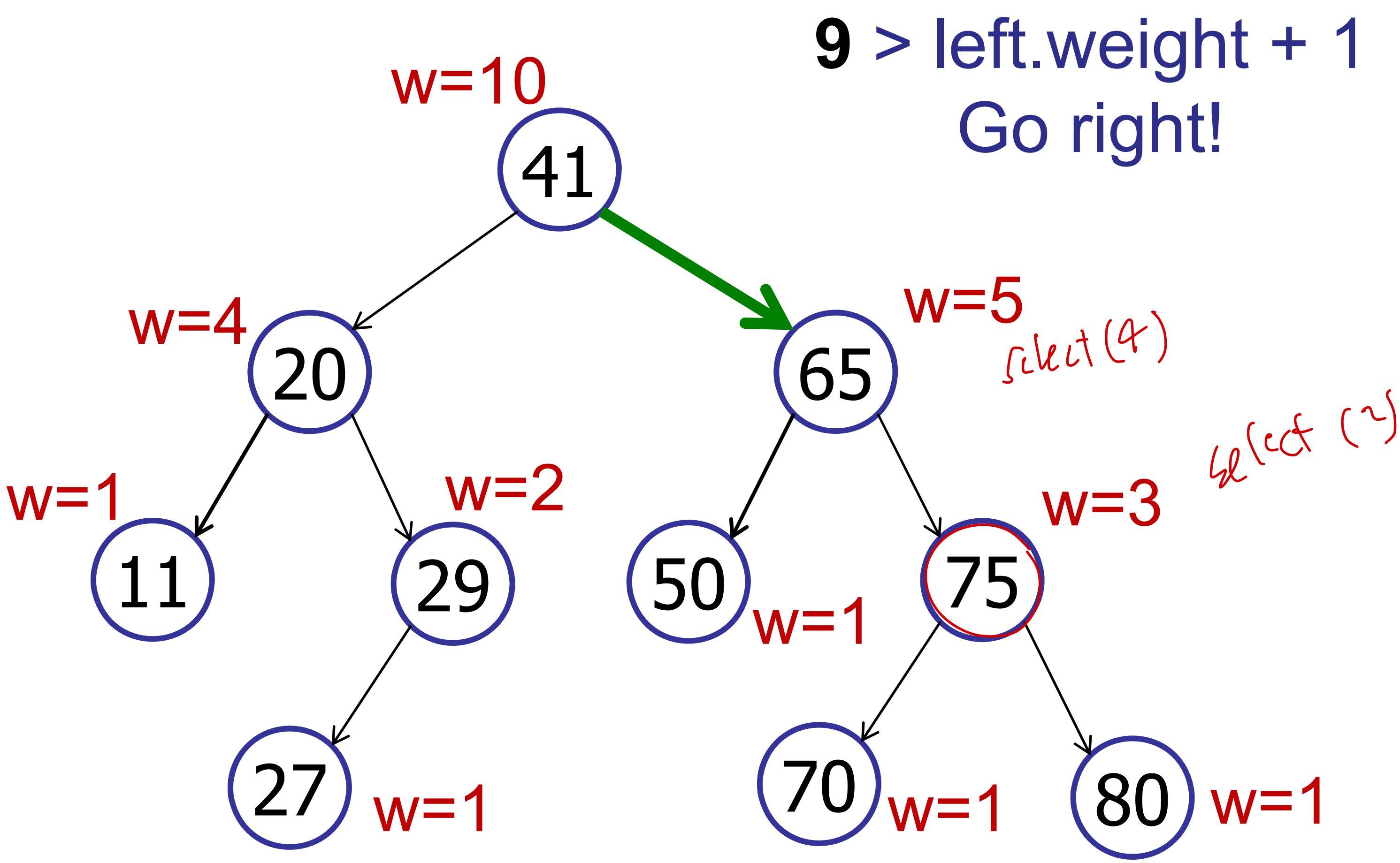
Dynamic Order Statistics

Example: select(3)



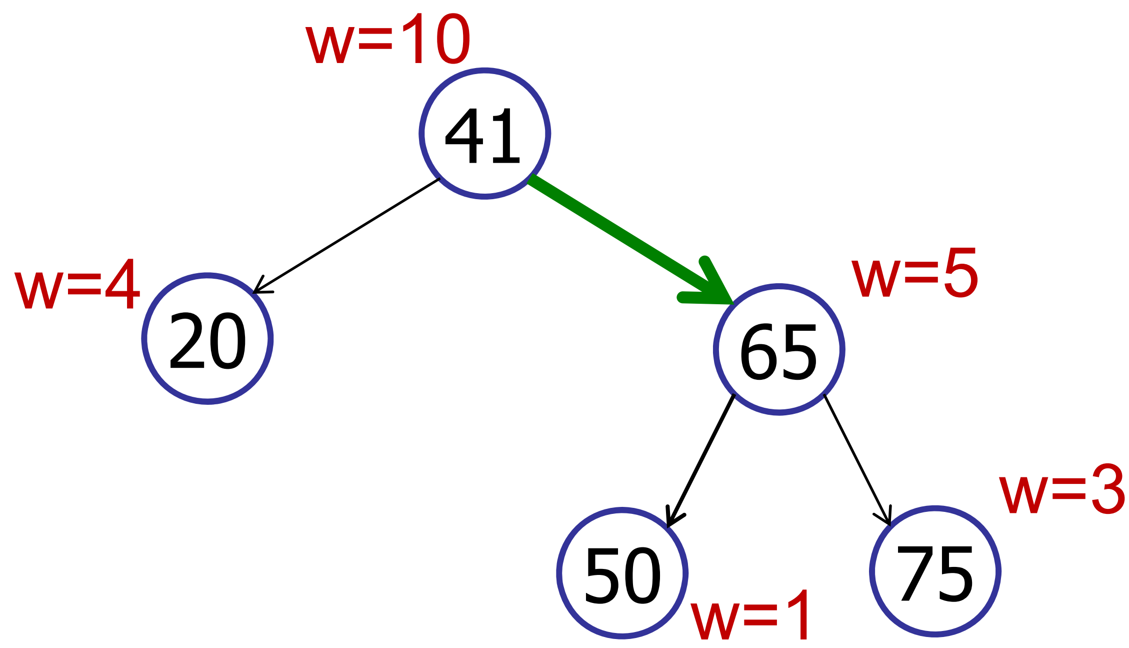
Dynamic Order Statistics

Example: select(9)



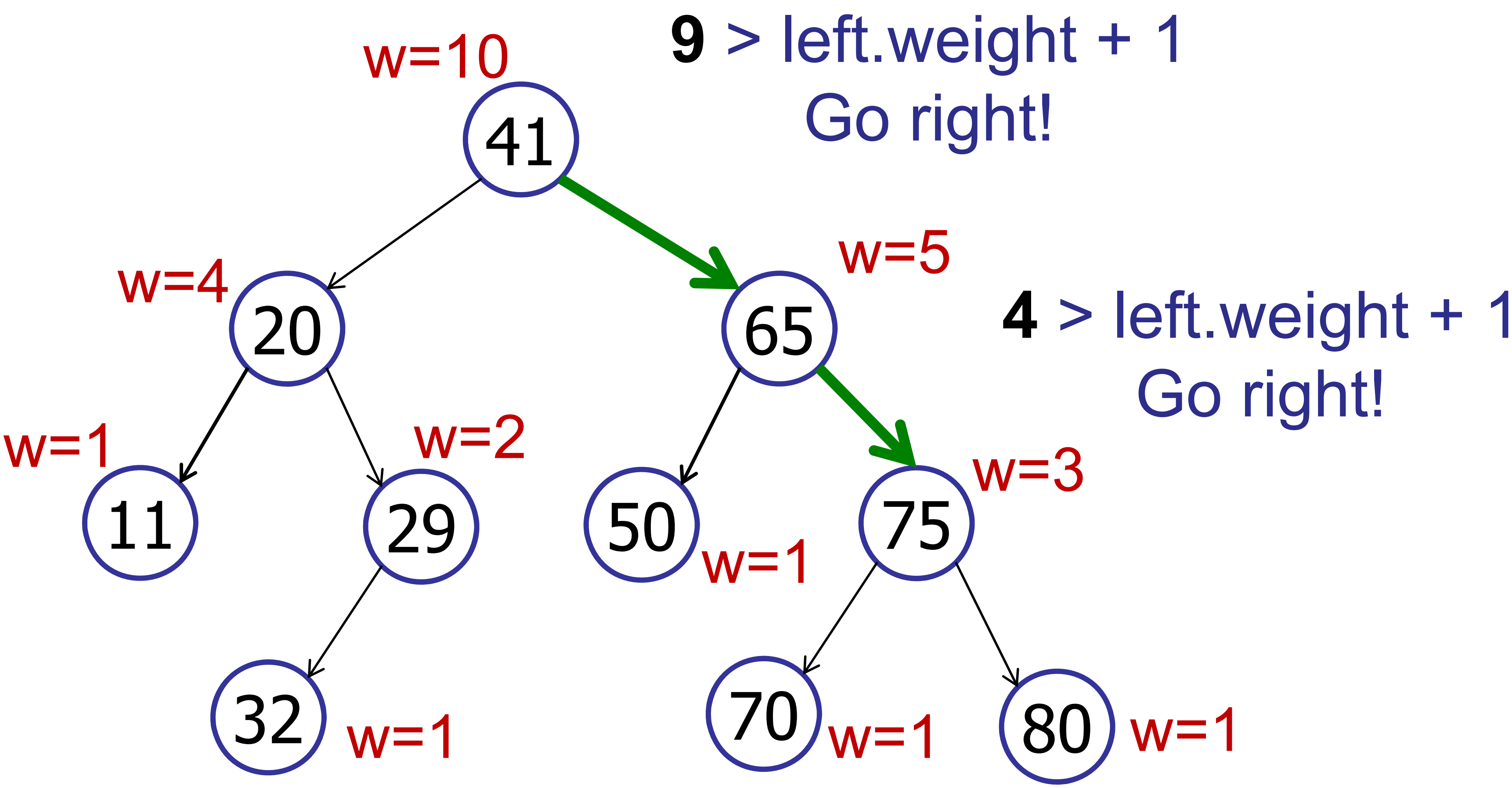
select(9)

1. Go left at 65
2. Go right at 65
3. Stop at 65
4. I'm confused



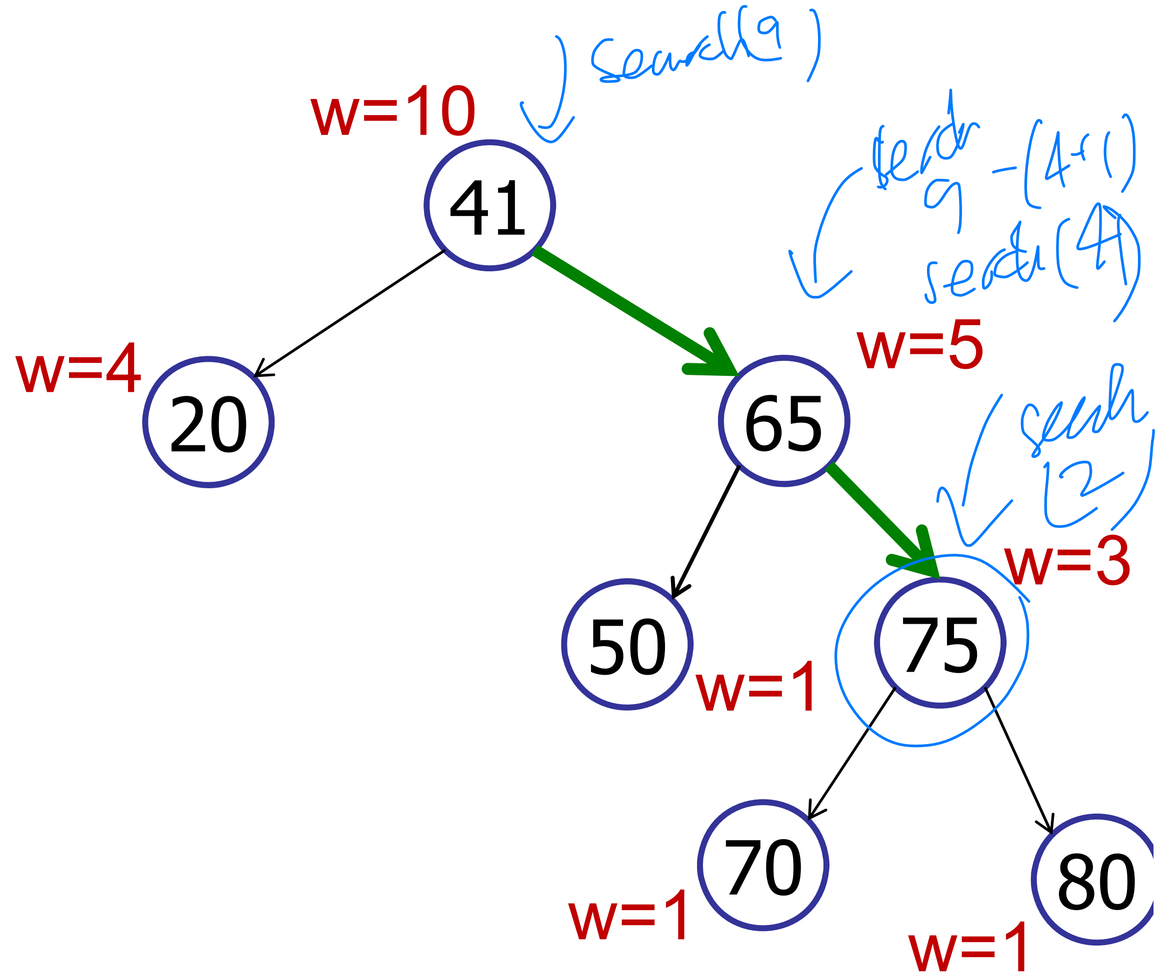
Dynamic Order Statistics

select(9)



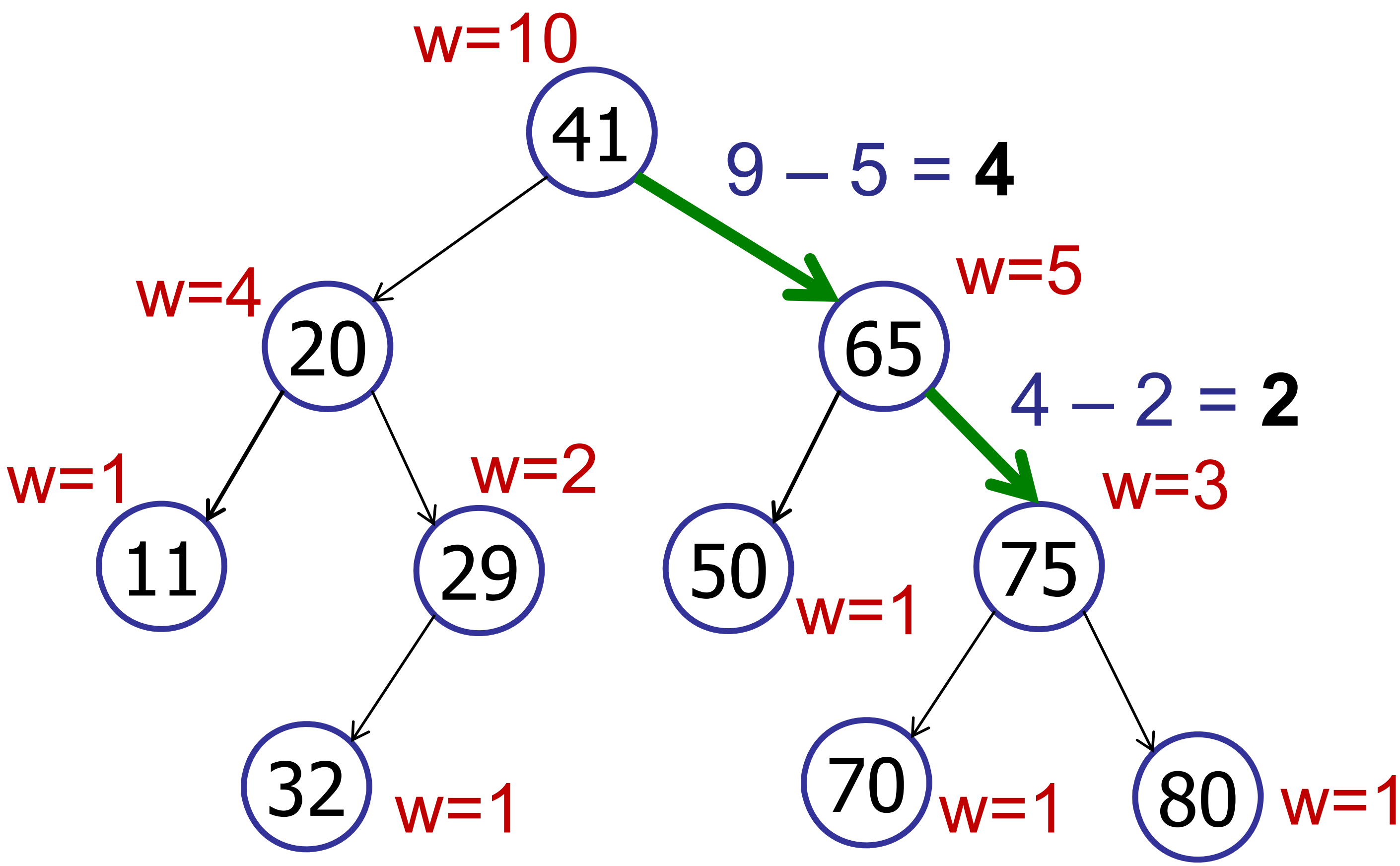
select(9)

1. Go left at 75
2. Go right at 75
3. Stop at 75
4. I’m confused



Dynamic Order Statistics

select(9)



Dynamic Order Statistics

select(k)

```
rank = left.weight + 1;
if (k == rank) then
        return v;
else if (k < rank) then
        return left.select(k);
else if (k > rank) then
        return right.select(k-rank);
```

Dynamic Order Statistics

select(k) : finds the node with rank k

Example: find the 10th tallest student in the class.

Dynamic Order Statistics

select(k) : finds the node with rank k

Example: find the 10th tallest student in the class.

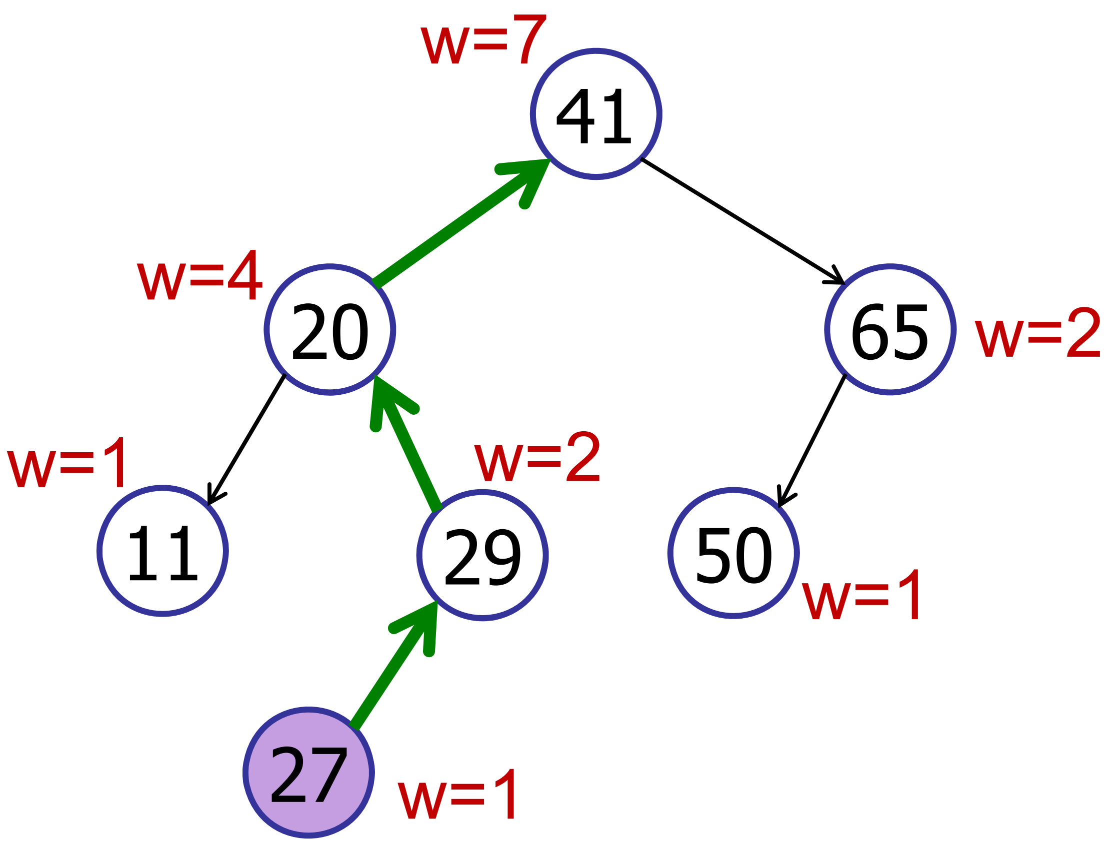
---

rank(v) : computes the rank of a node v

Example: determine the percentile of Johnny's height.
Is Johnny in the 10^{th} percentile or the 90^{th} percentile?

Dynamic Order Statistics

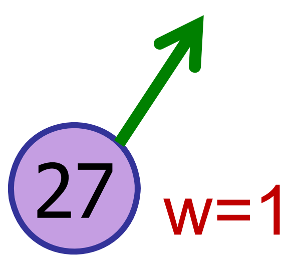
Example: rank(27)



rank = 1

Dynamic Order Statistics

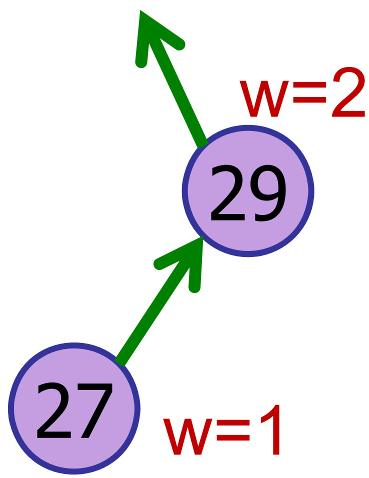
Example: rank(27)



rank = 1

Dynamic Order Statistics

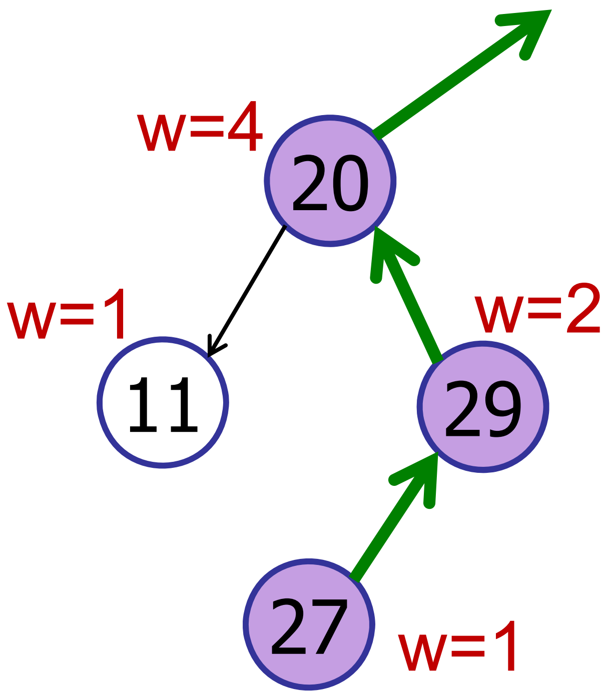
Example: rank(27)



rank = 1

Dynamic Order Statistics

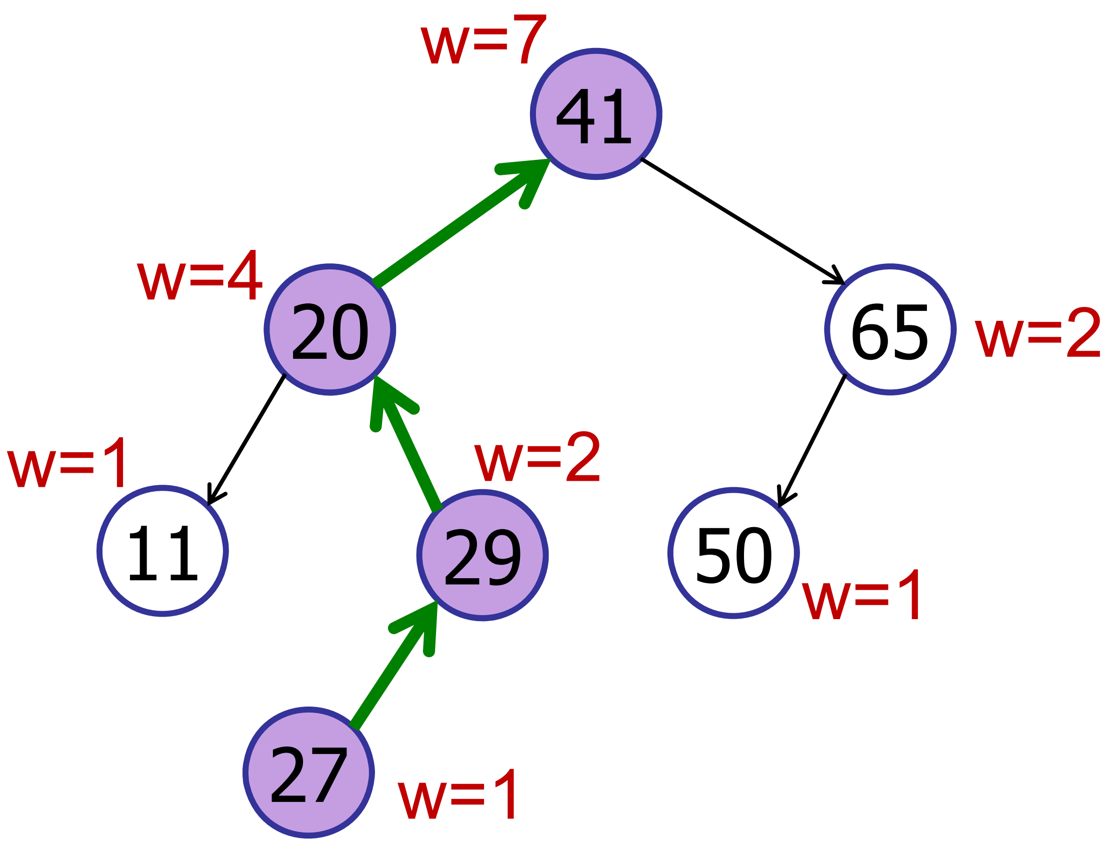
Example: rank(27)



rank = 1 + 2

Dynamic Order Statistics

Example: rank(27)



rank = 1 + 2 = 3

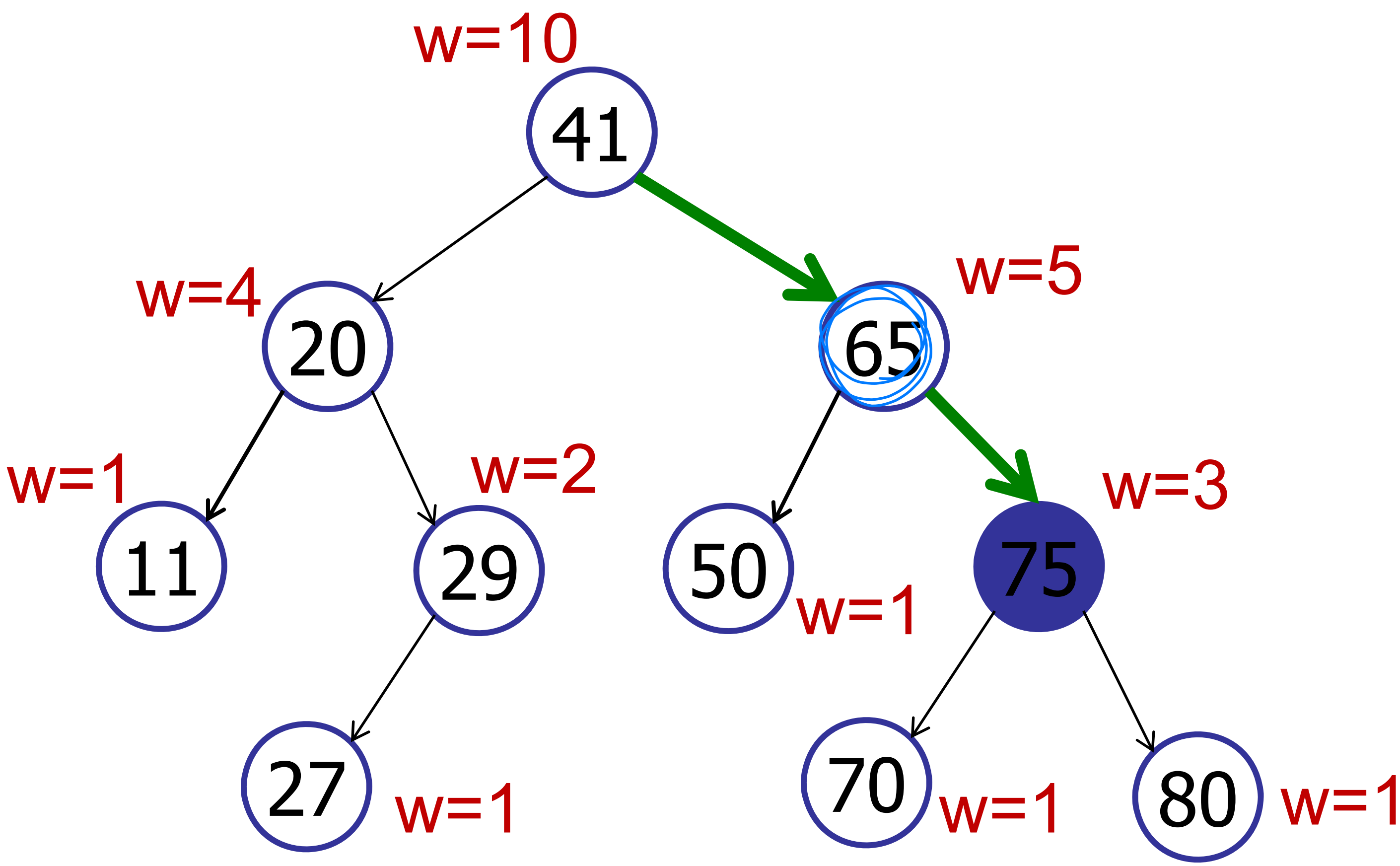
Dynamic Order Statistics

Rank(v) : computes the rank of a node v

```
rank(node)
    rank = node.left.weight + 1;
    while (node != null) do
        if node is left child then
            do nothing
        else if node is right child then
            rank += node.parent.left.weight + 1;
        node = node.parent;    → update current node
    return rank;
```

Dynamic Order Statistics

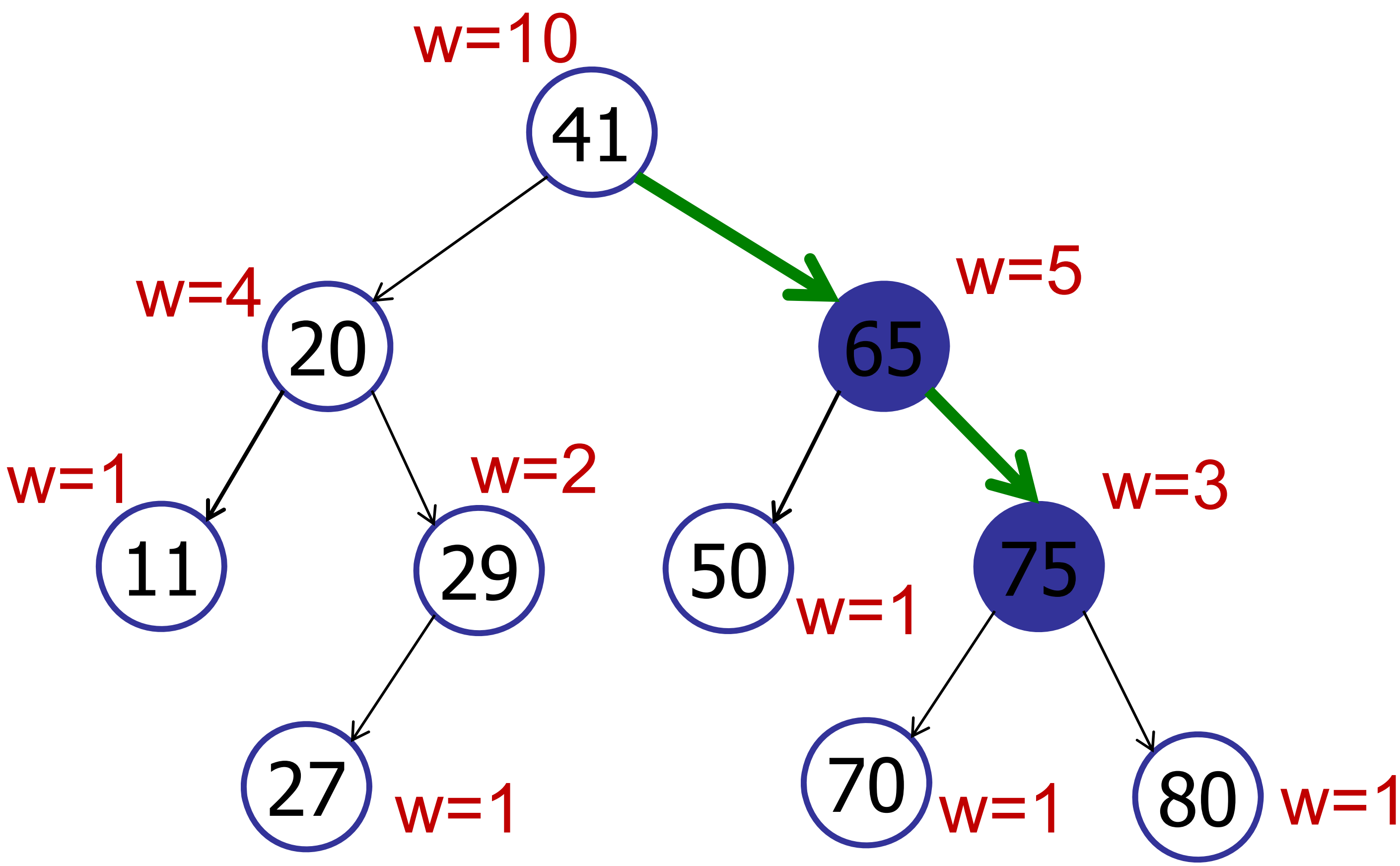
rank(75)



rank = 2

Dynamic Order Statistics

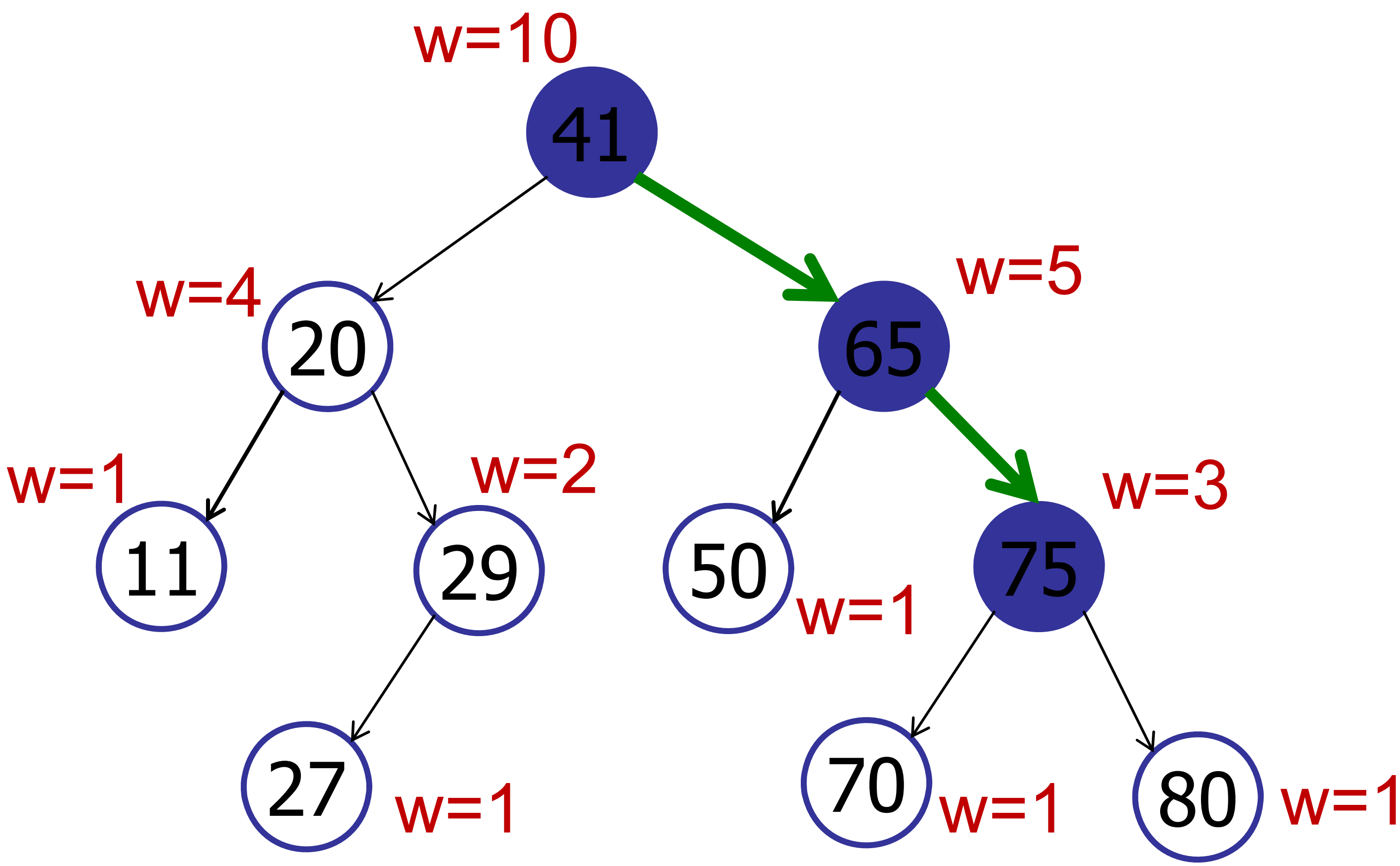
rank(75)



rank = 2 + 2

Dynamic Order Statistics

rank(75)



rank = 2 + 2 + 5 = 9

Dynamic Order Statistics

Rank(v) : computes the rank of a node v

```
rank(node)
    rank = node.left.weight + 1;
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        if node is left child then
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        else if node is right child then
            rank += node.parent.left.weight + 1;
        node = node.parent;
    return rank;
```

Augmenting data structures

Basic methodology:

1. Choose underlying data structure:

   AVL tree

2. Determine additional info needed:

   Weight of each node

3. Maintained info as data structure is modified.

   Update weights as needed

4. Develop new operations using the new info.

   Select and Rank

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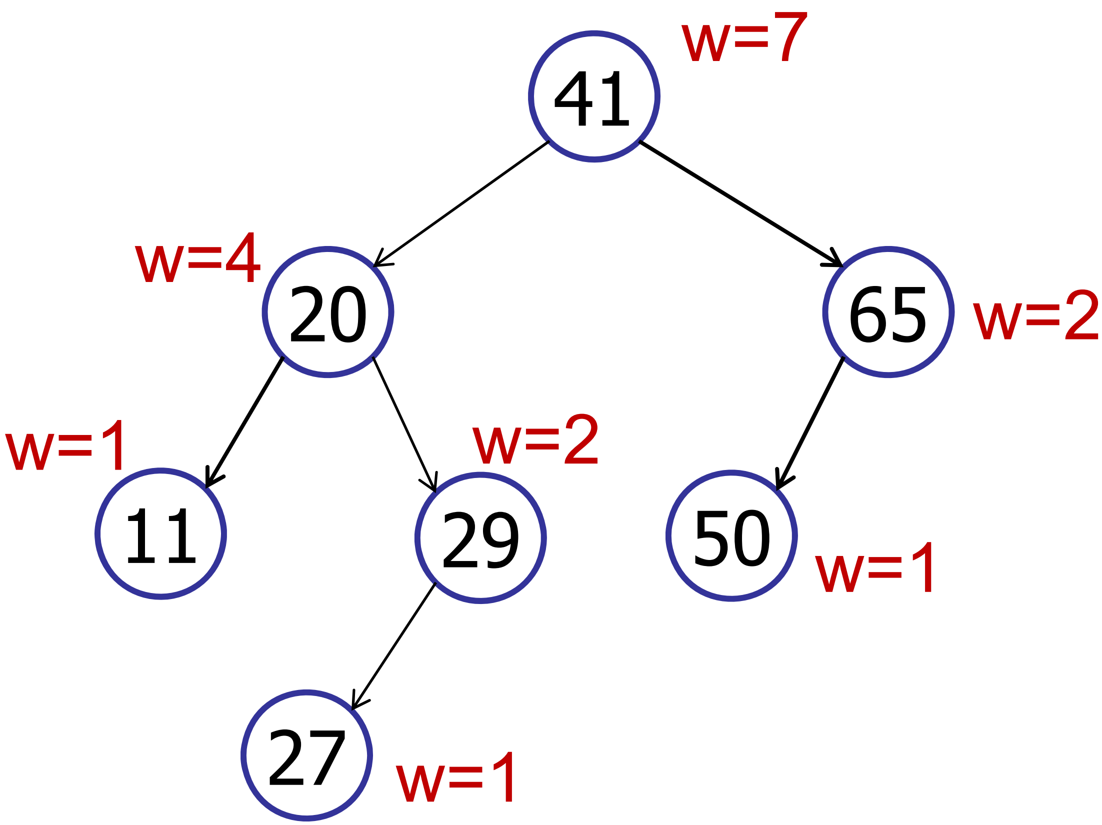
Augmented Trees

Maintain weight during insertions:



Augmented Trees

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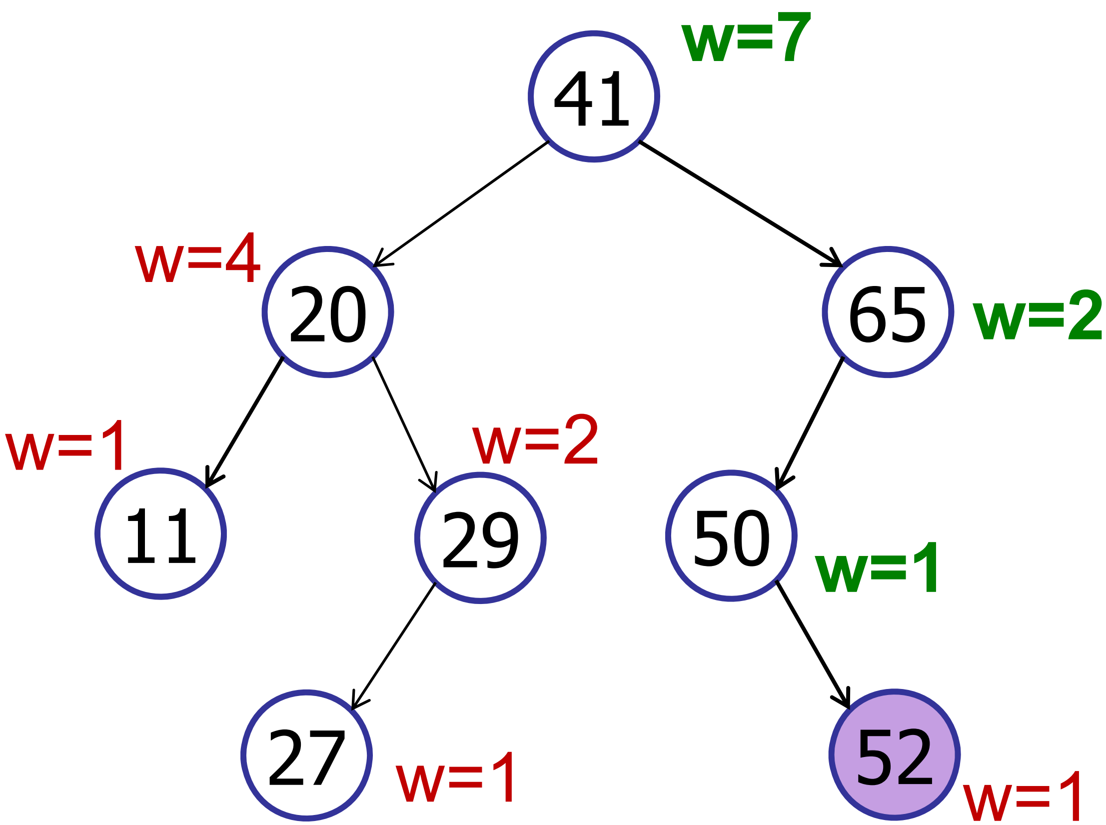

Augmented Trees

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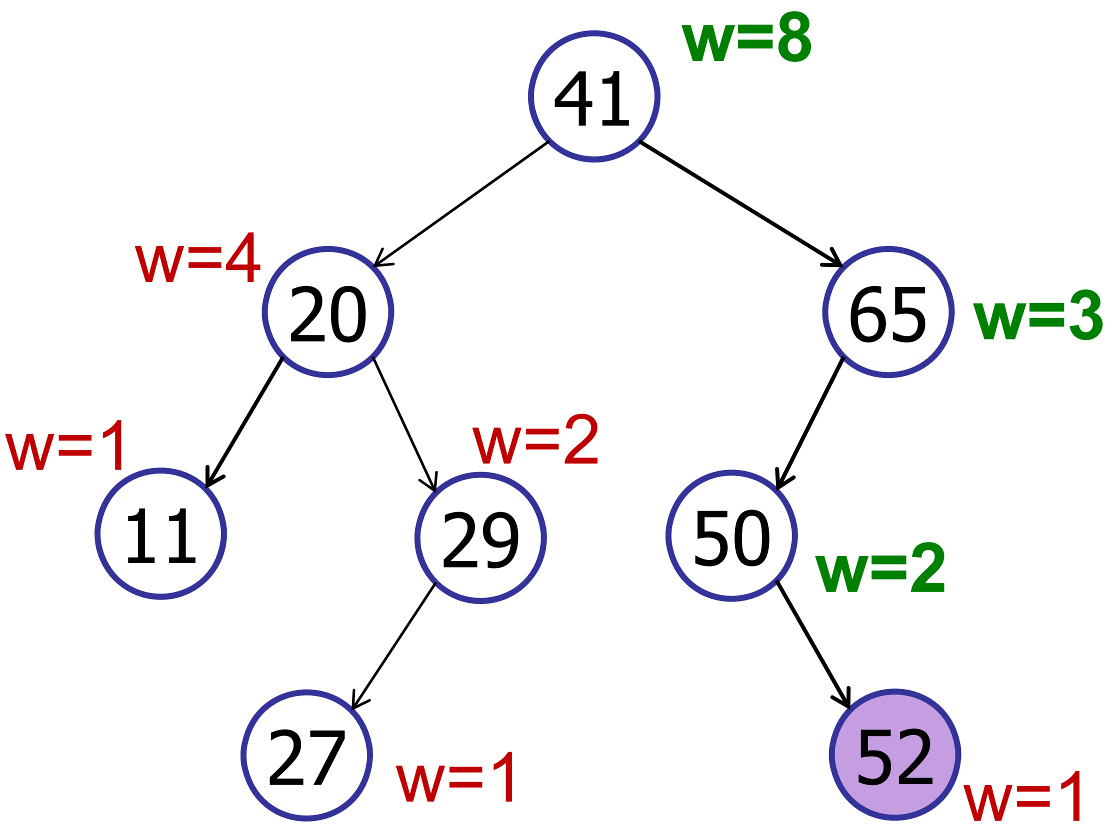
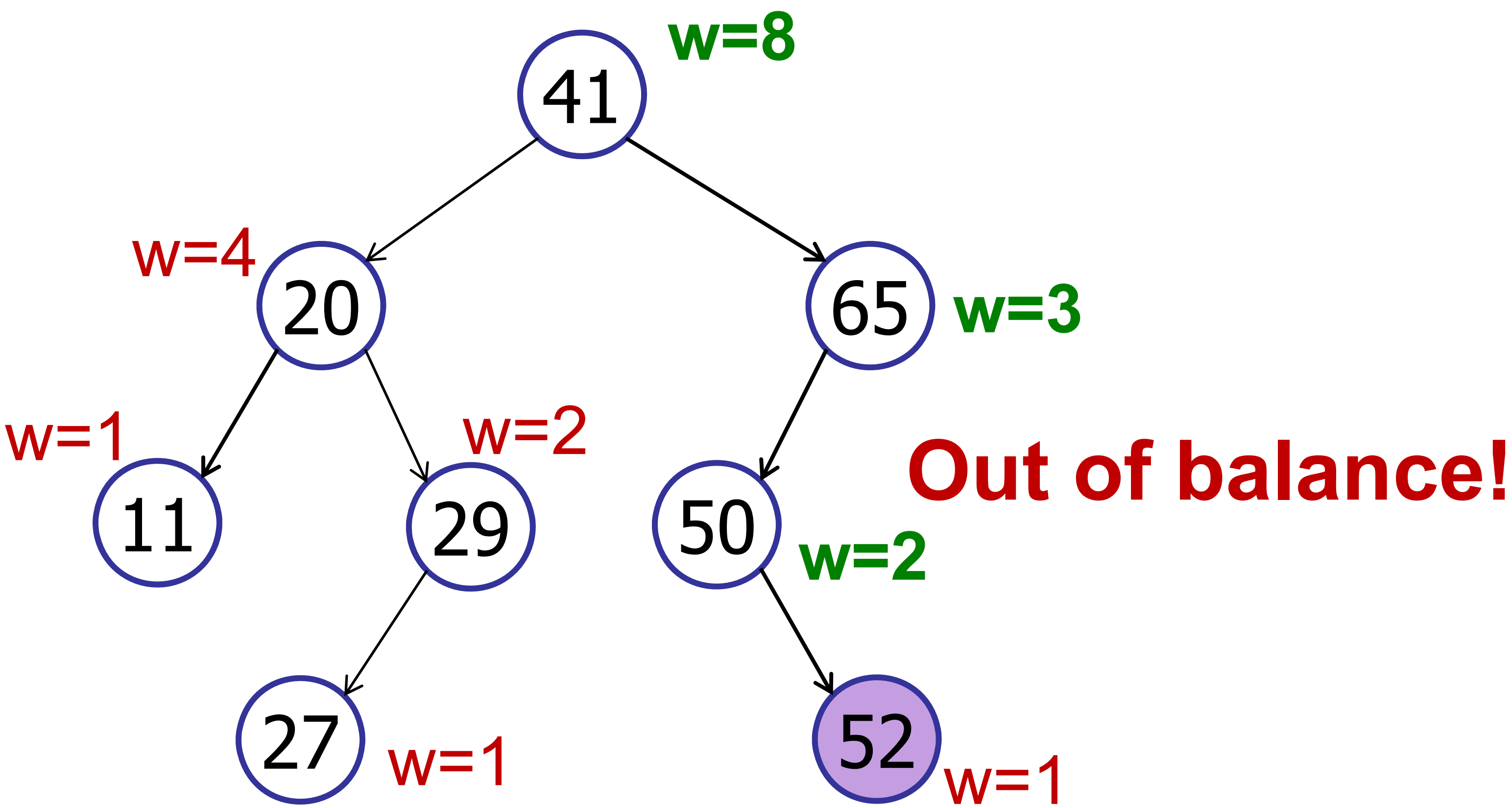


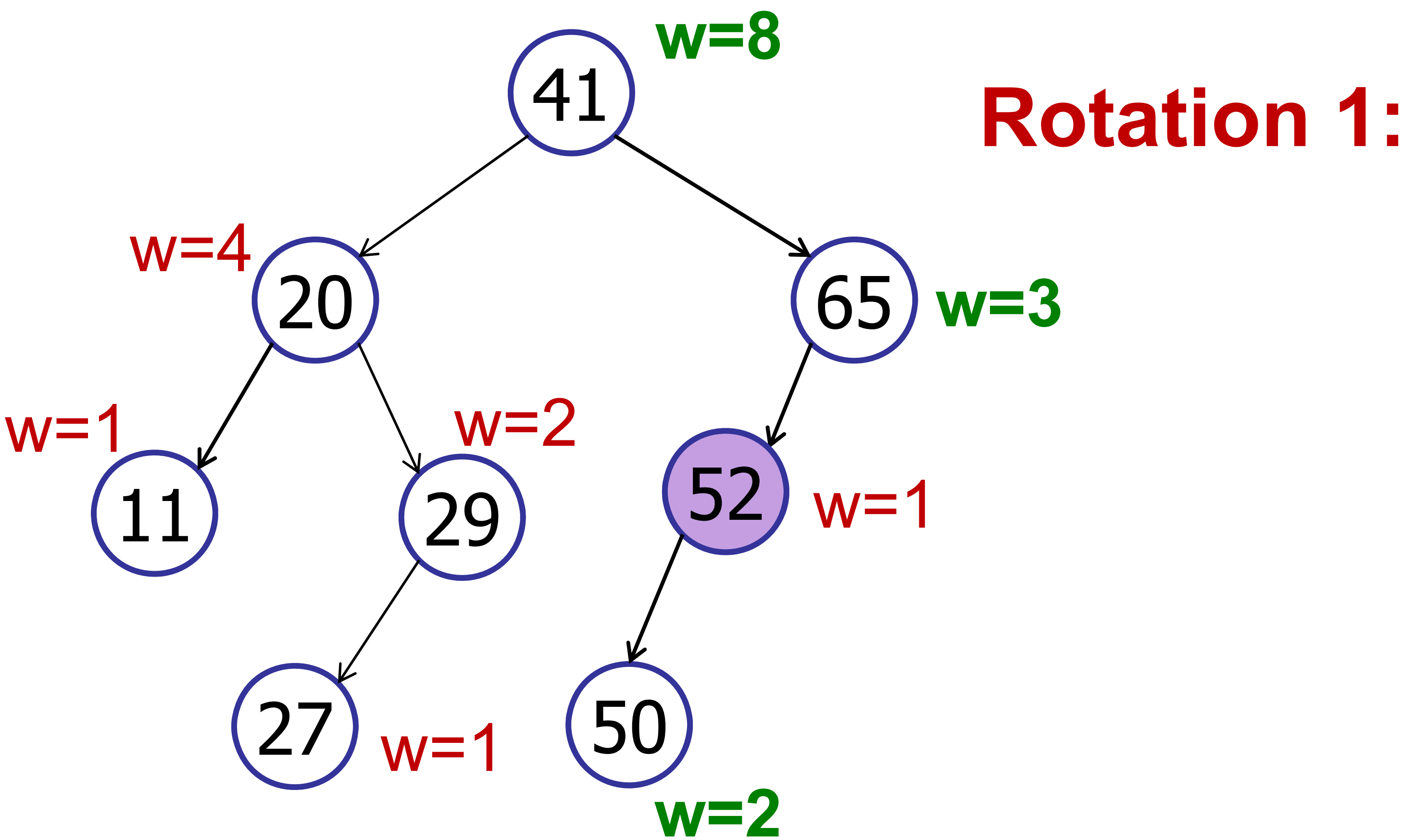
Augmented Trees

Maintain weight during insertions:



Augmented Trees

Maintain weight during insertions:



Augmented Trees

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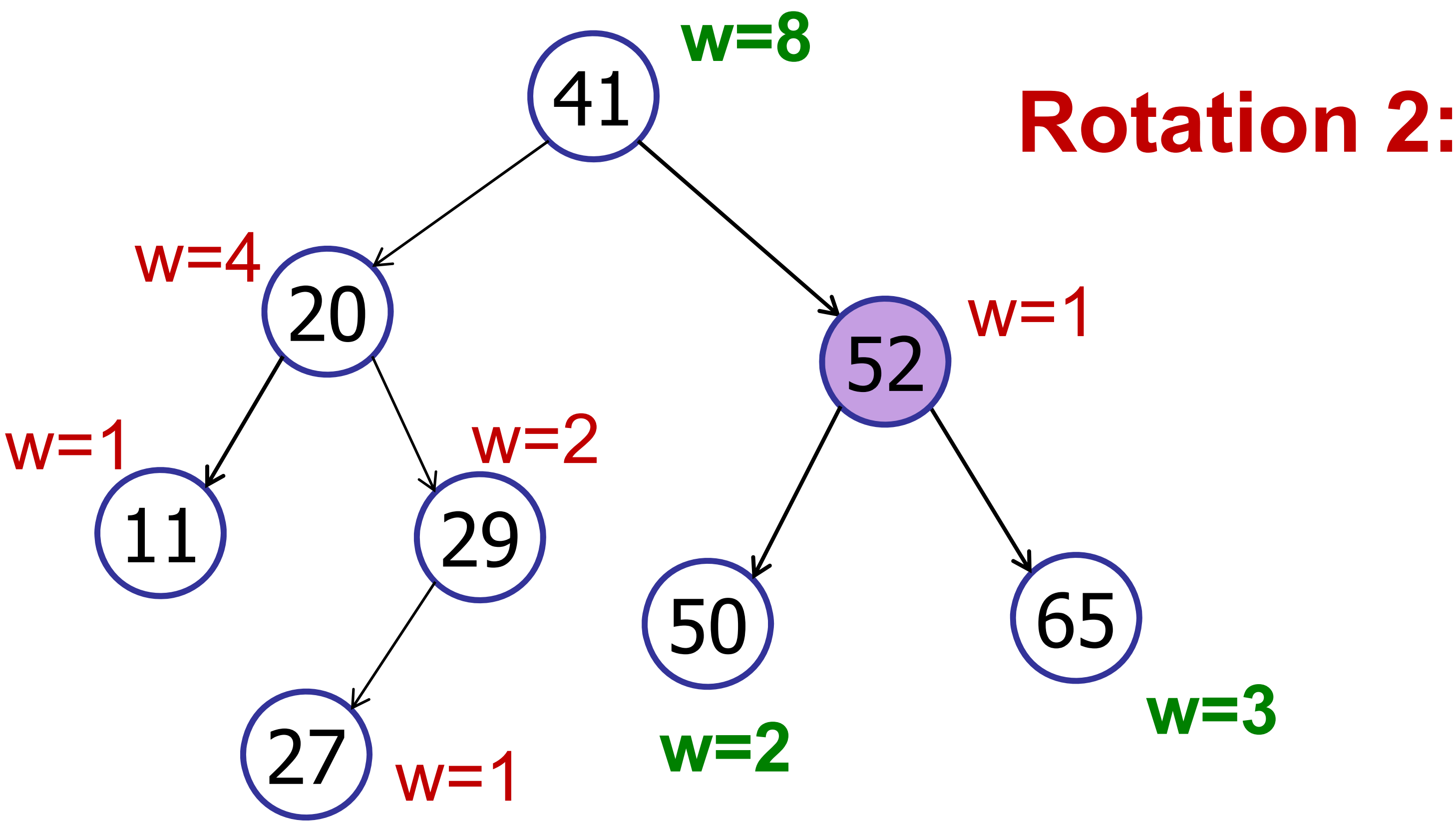
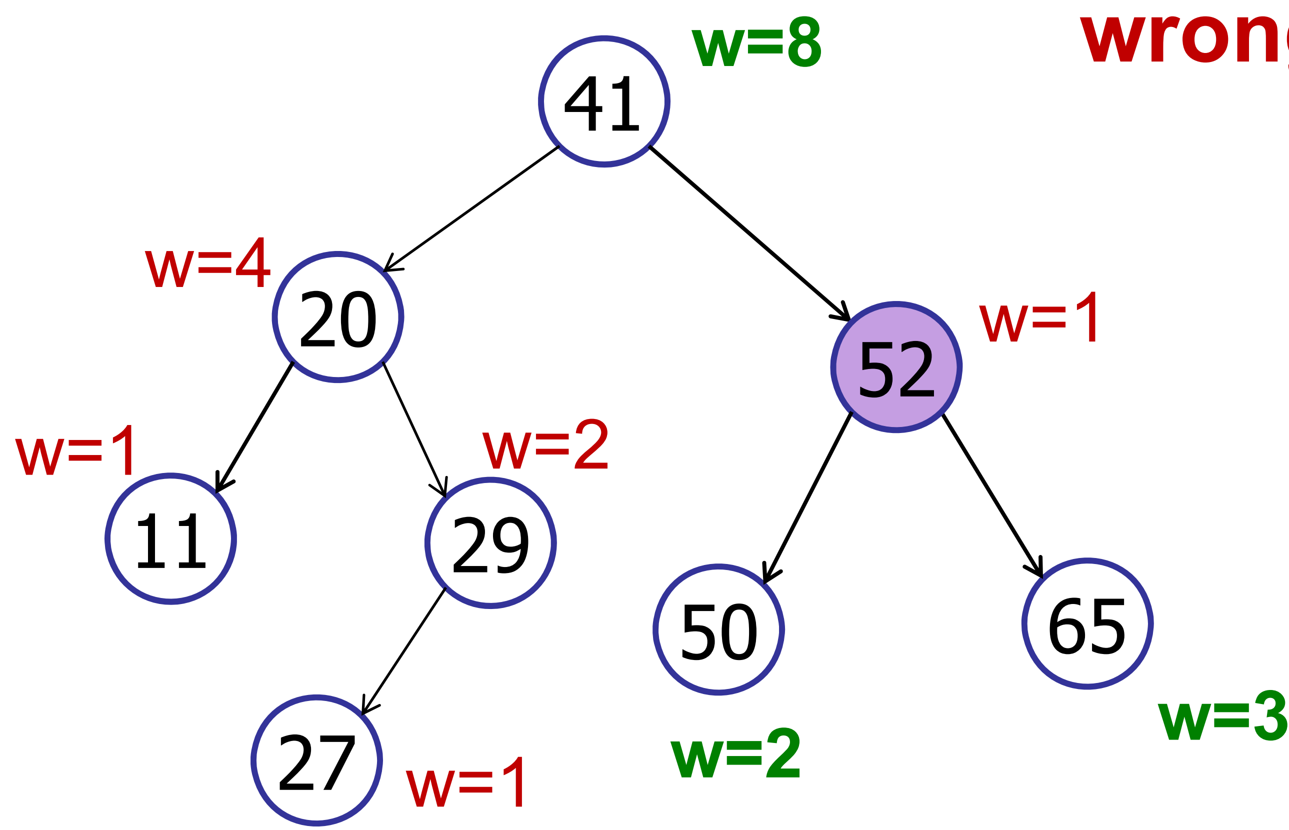


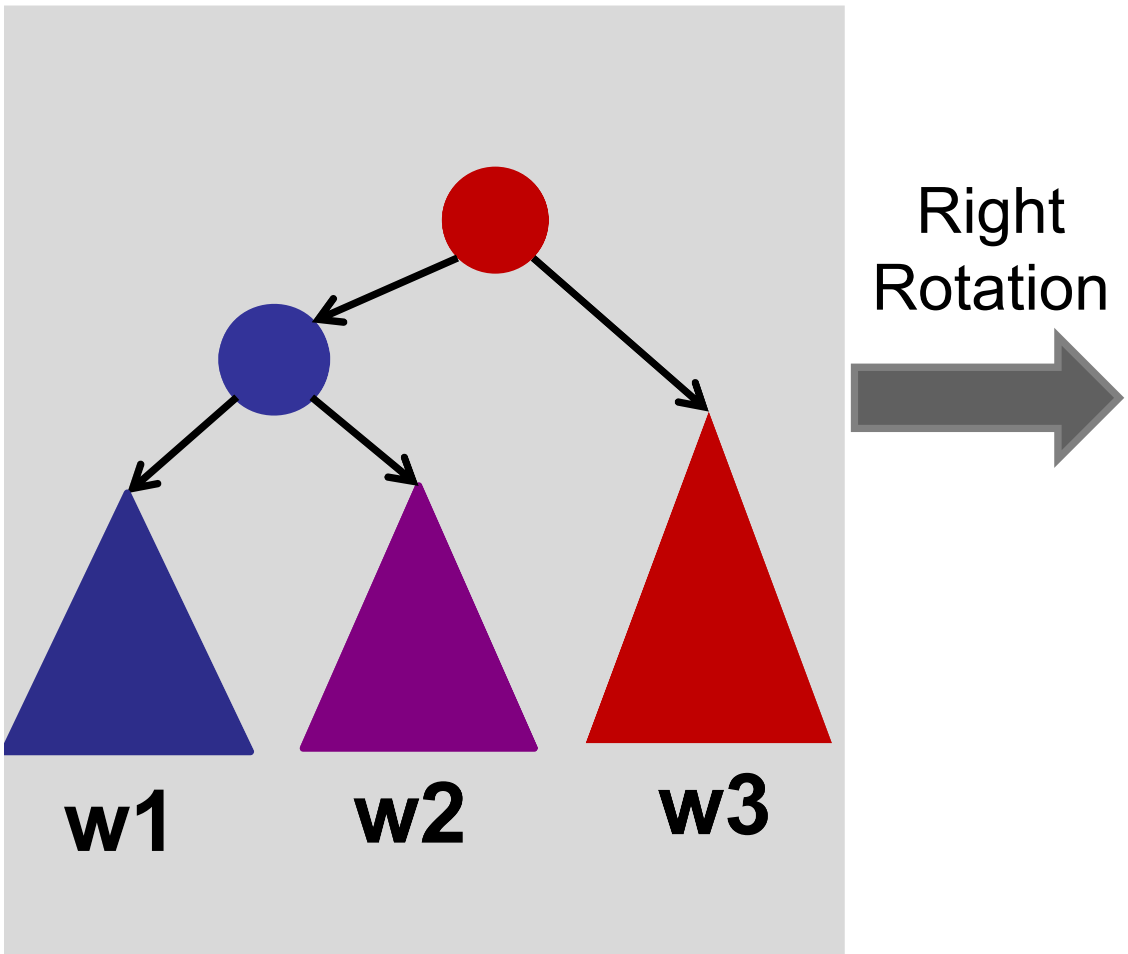
Augmented Trees

How to update weights on rotation?



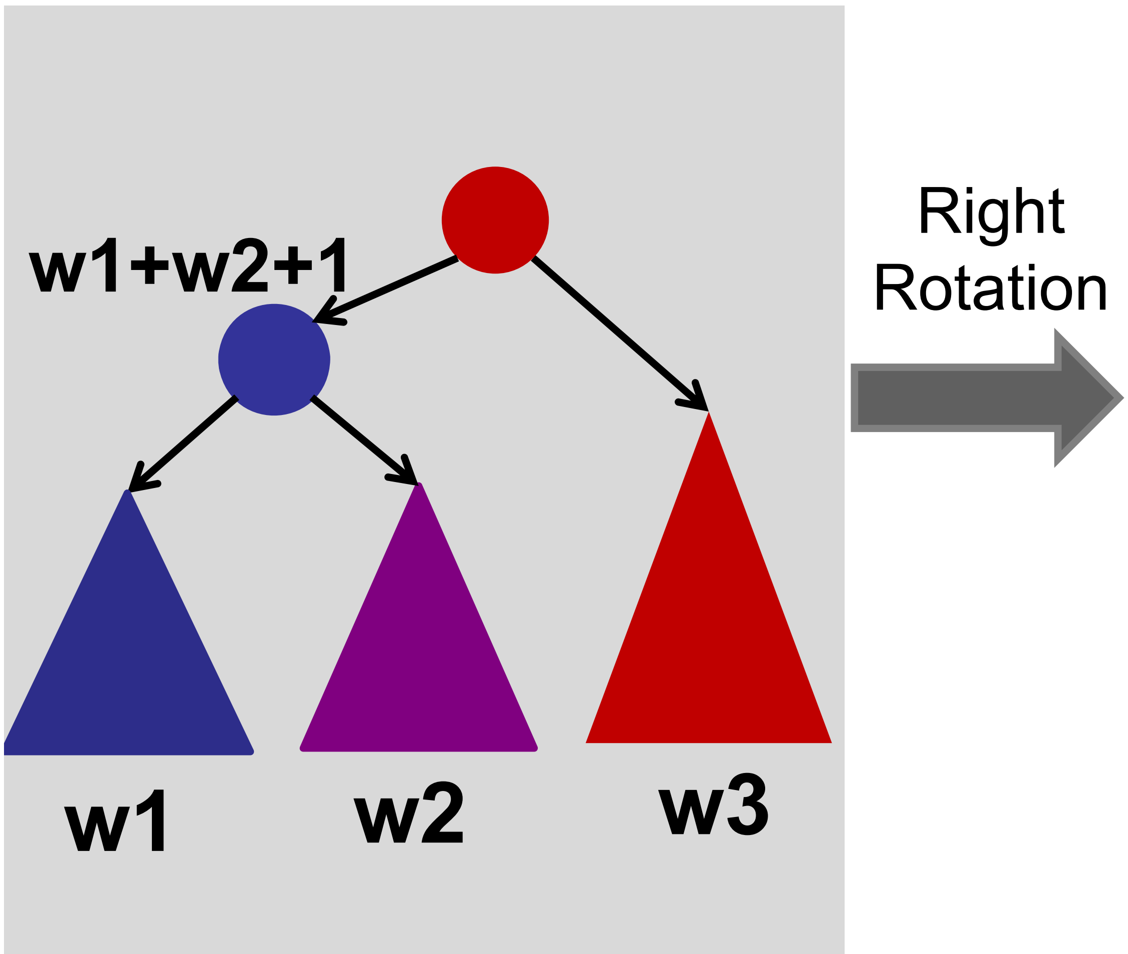
Augmented Trees

Maintain weight during rotations:



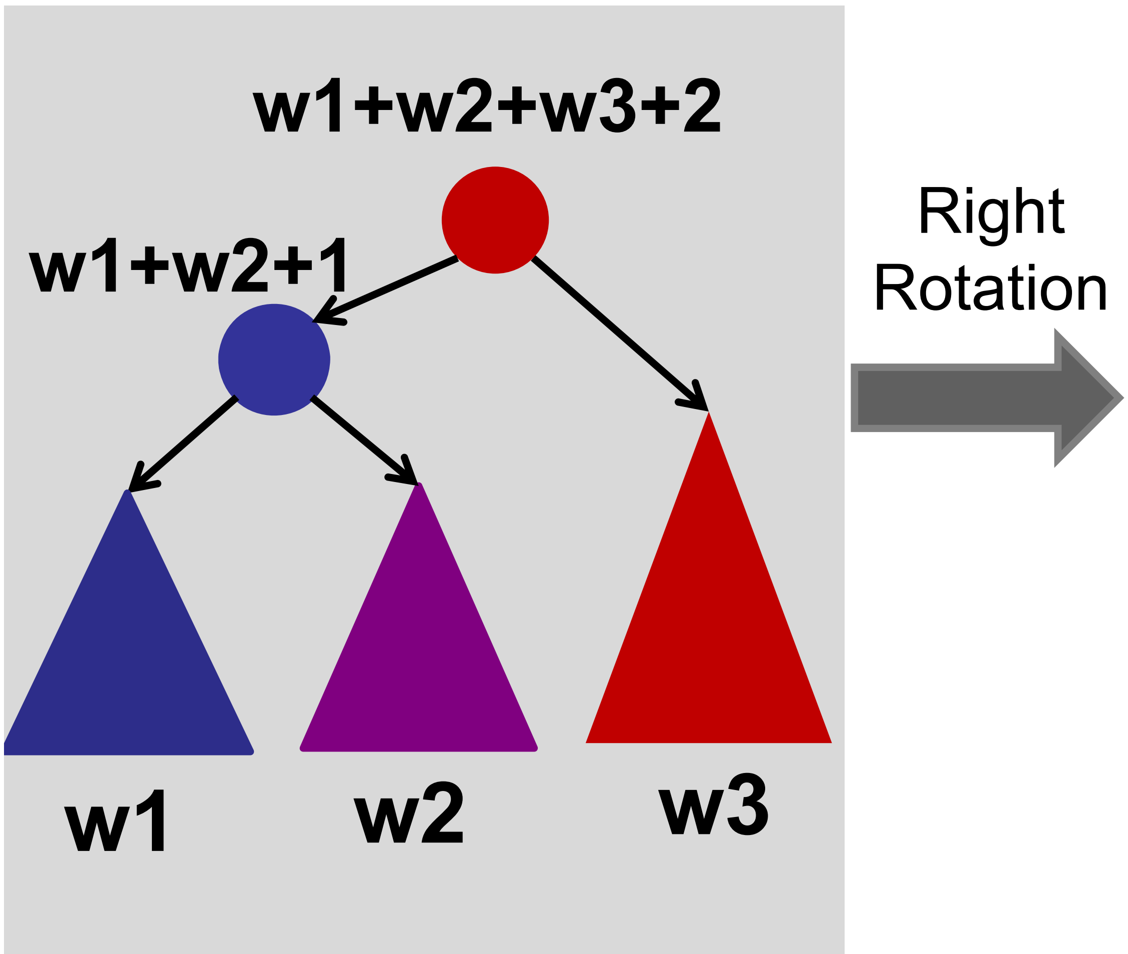
Augmented Trees

Maintain weight during rotations:



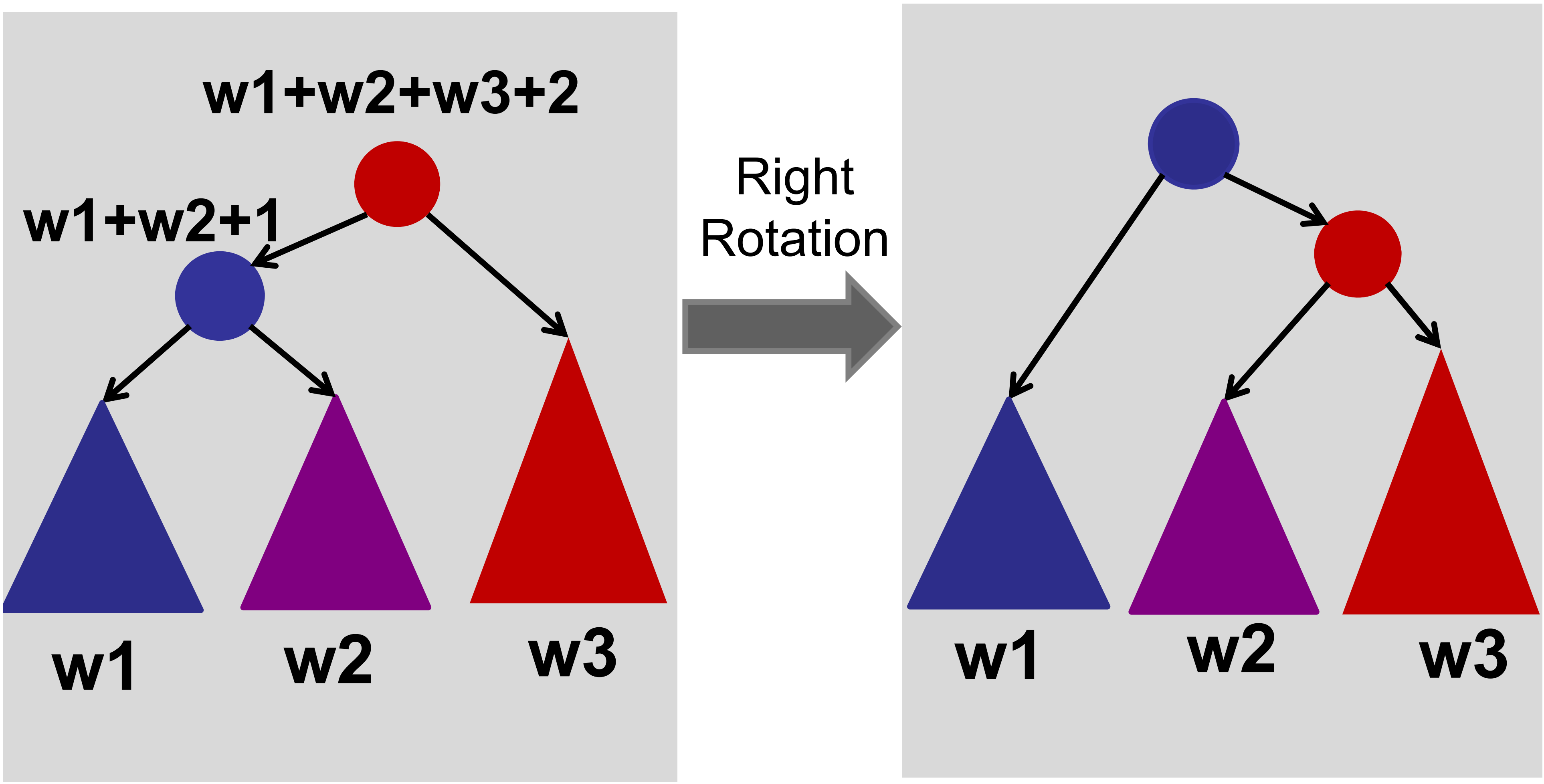
Augmented Trees

Maintain weight during rotations:



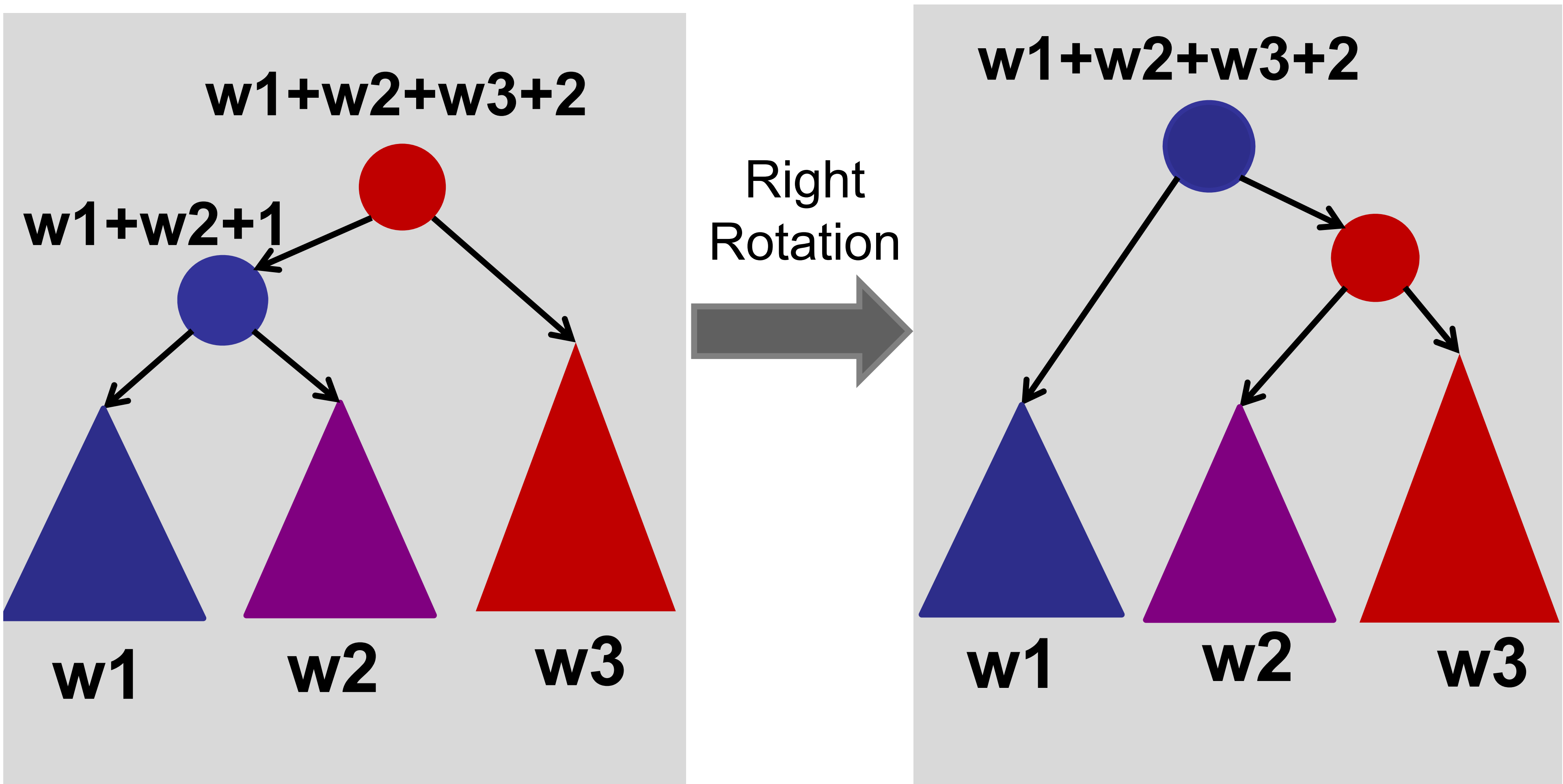
Augmented Trees

Maintain weight during rotations:



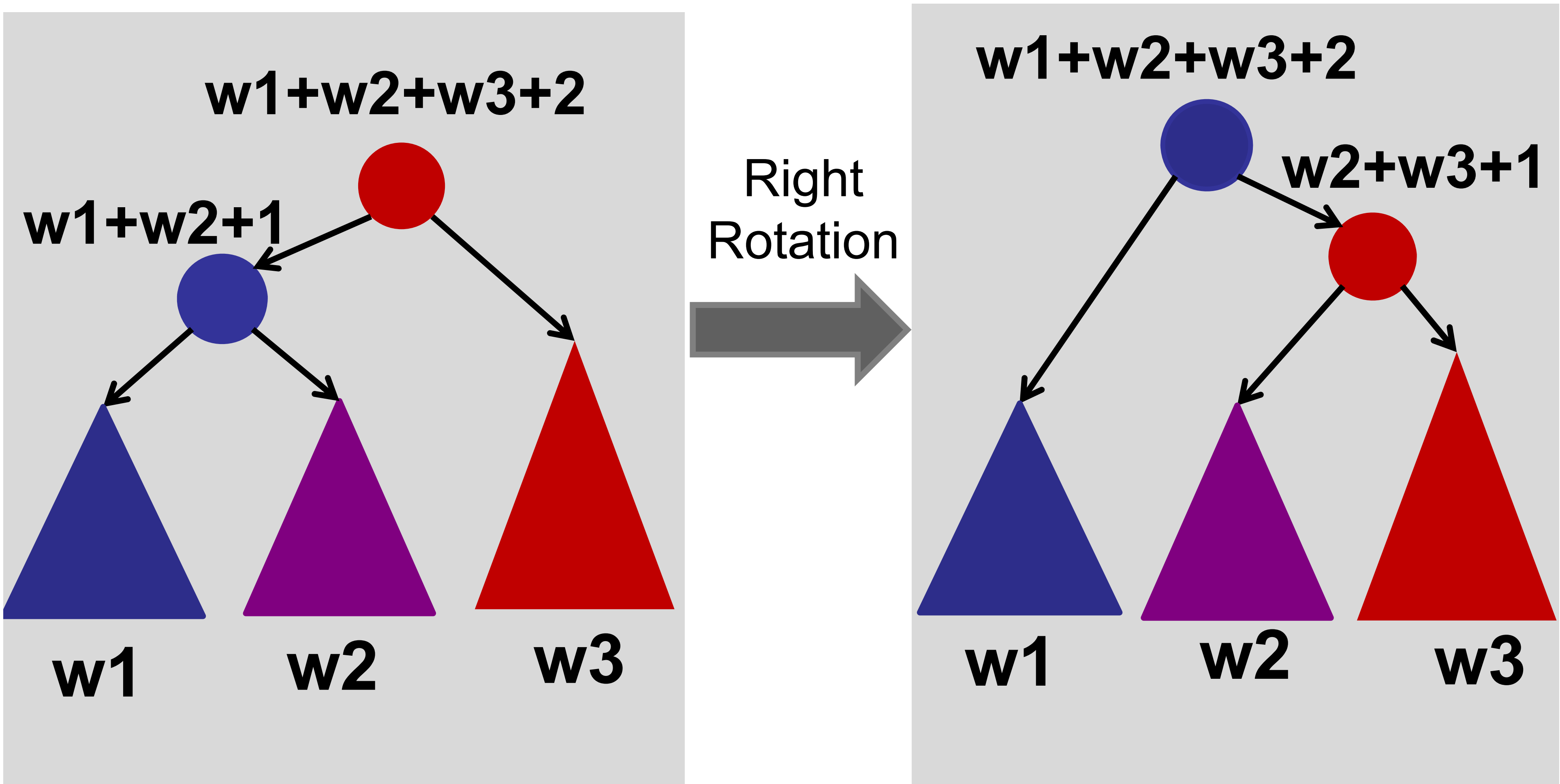
Augmented Trees

Maintain weight during rotations:



Augmented Trees

Maintain weight during rotations:



How long does it take to update the weights during a rotation?

1. O(1)
2. O(log n)
3. O(n)
4. O(n^2)
5. What is a rotation?

How long does it take to update the weights during a rotation?

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Augmented Trees

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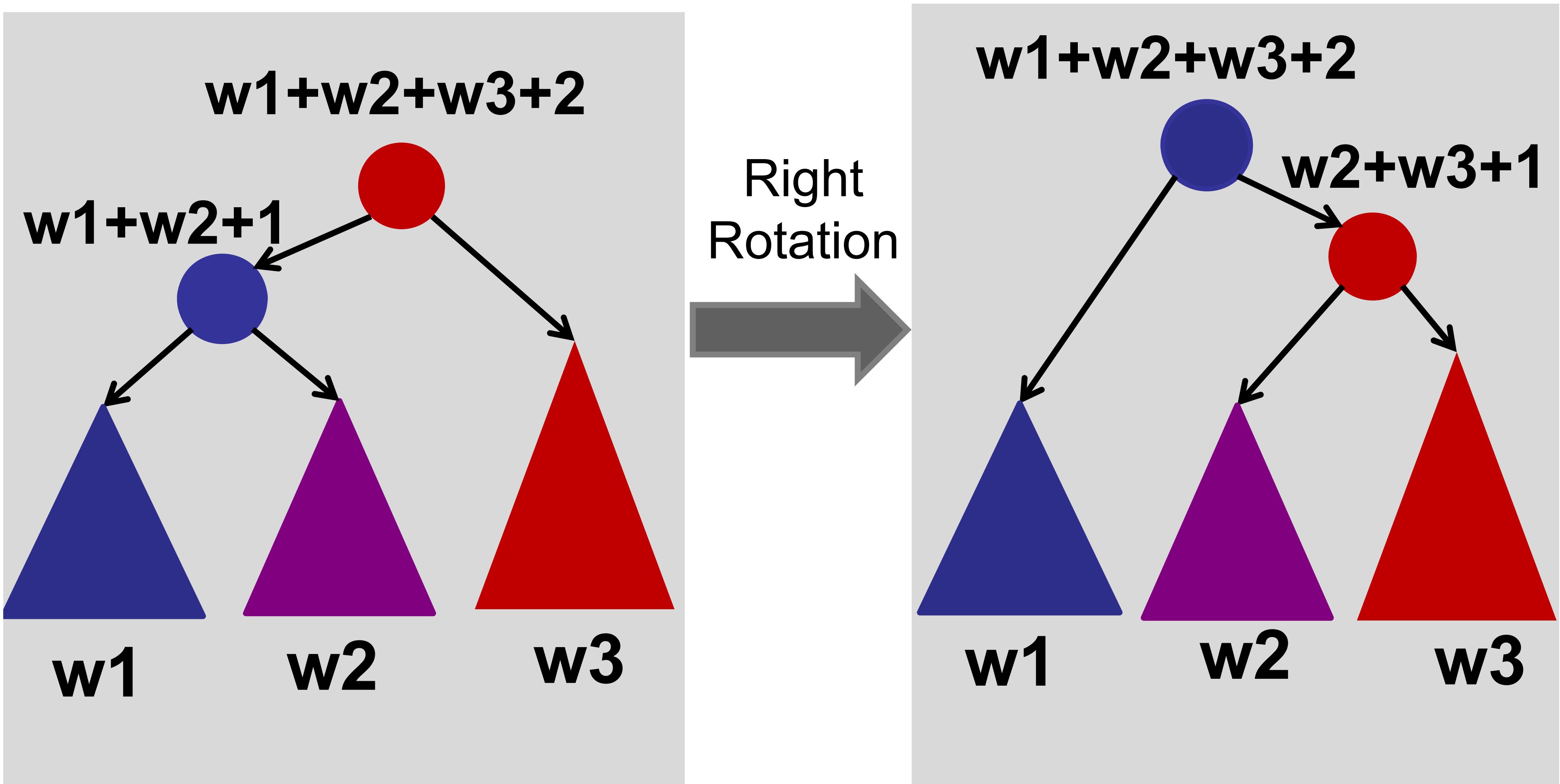


Augmenting data structures

Basic methodology:

1. Choose underlying data structure

   (tree, hash table, linked list, stack, etc.)

2. Determine additional info needed.

3. Verify that the additional info can be maintained as the data structure is modified.

   (subject to insert/delete/etc.)

4. Develop new operations using the new info.

Today

Three examples of augmenting balanced BSTs

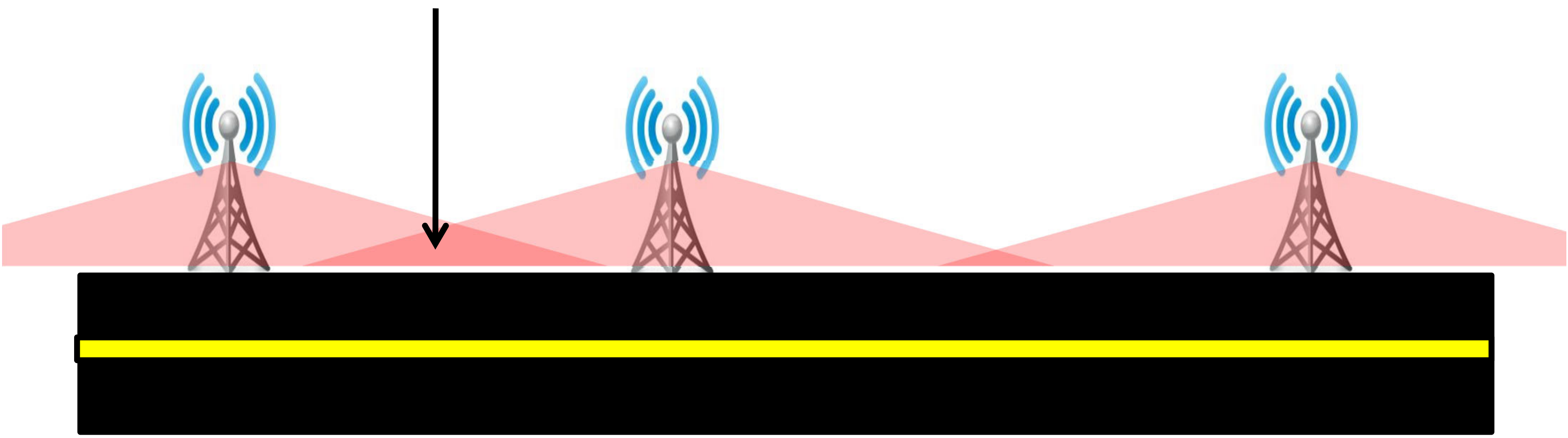
1. Order Statistics

2. Intervals

3. Orthogonal Range Searching

Cell Tower Coverage

Find a tower that covers my location.



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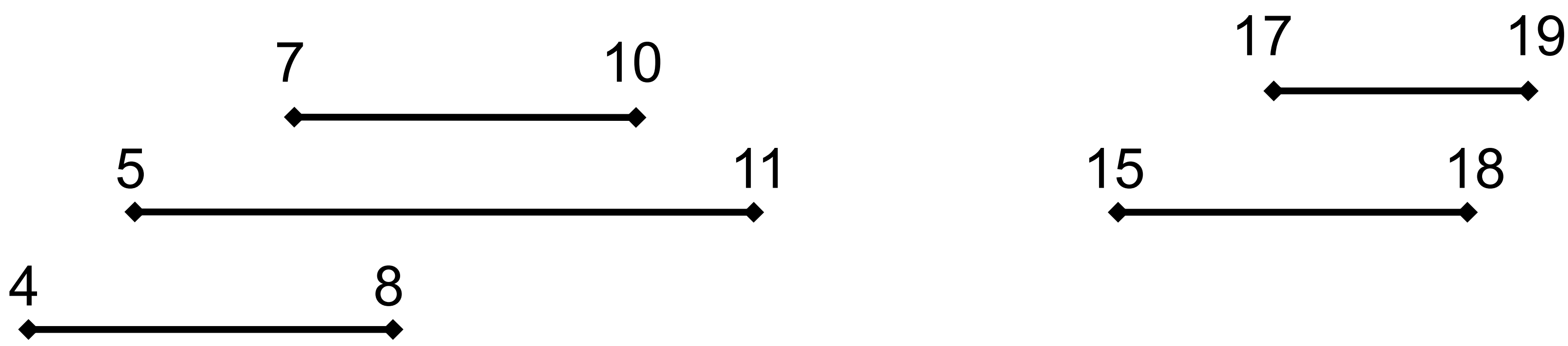

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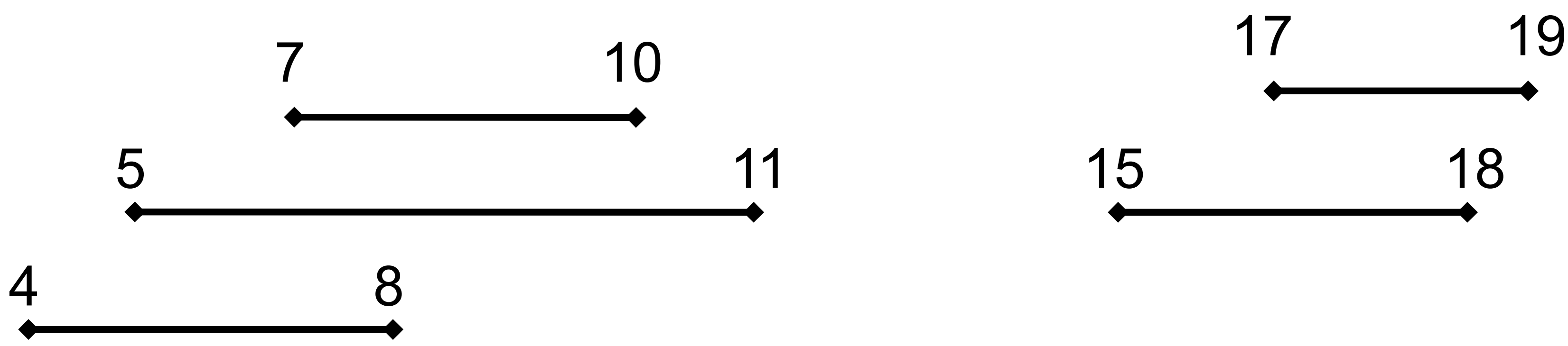
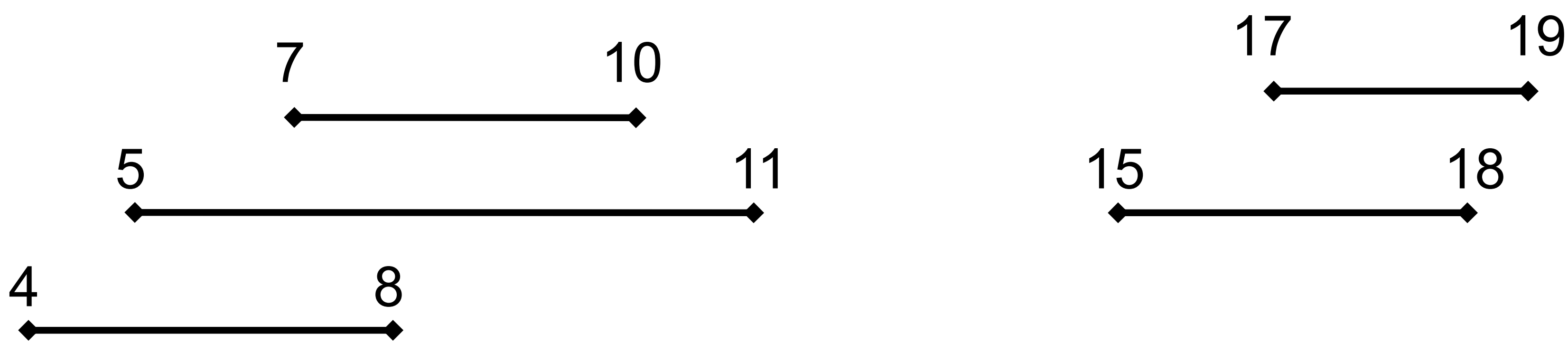


insert(begin, end)
delete(begin, end)

Cell Tower Coverage

Find a tower that covers my location.

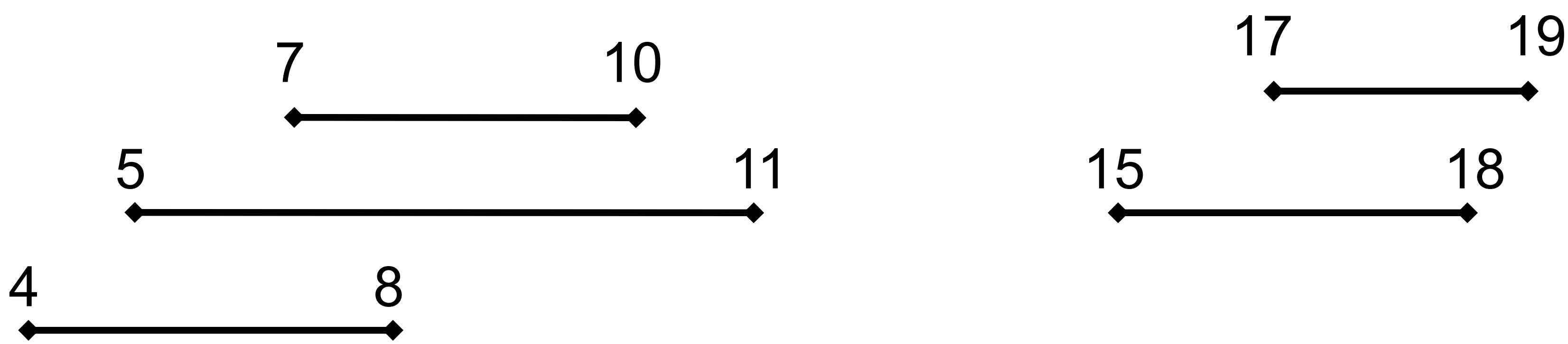


insert(begin, end)
delete(begin, end)

query(x): find an interval that overlaps x.

Cell Tower Coverage

Find a tower that covers my location.

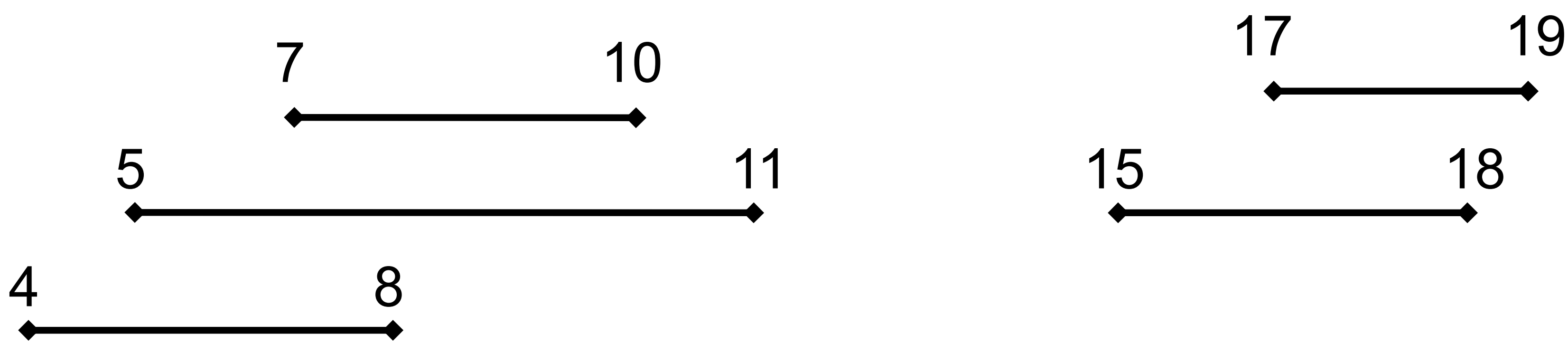


Idea 1: Keep intervals in a list.

Query: scan entire list.

Cell Tower Coverage

Find a tower that covers my location.



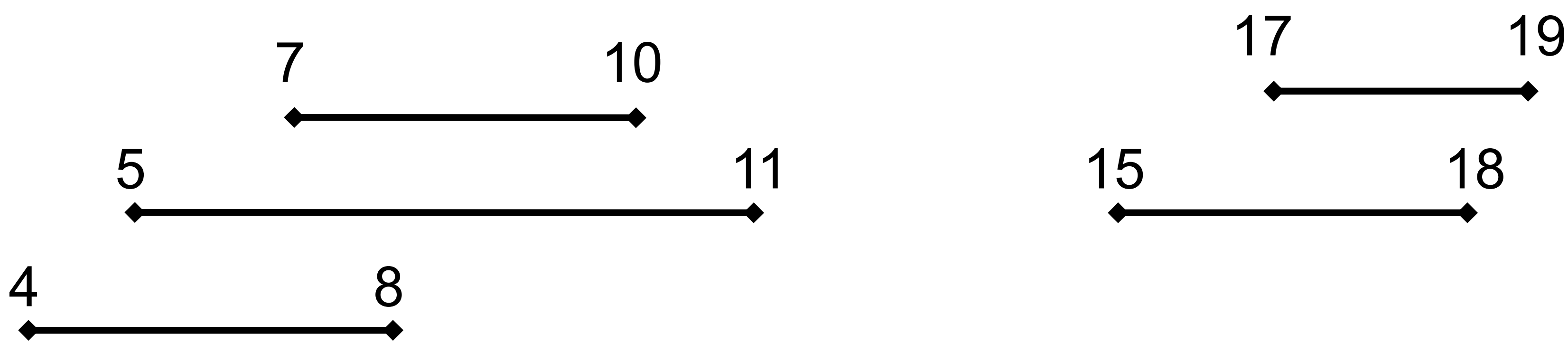
Idea 1: Keep intervals in a list.

Query: scan entire list.

Does sorting help?

Cell Tower Coverage

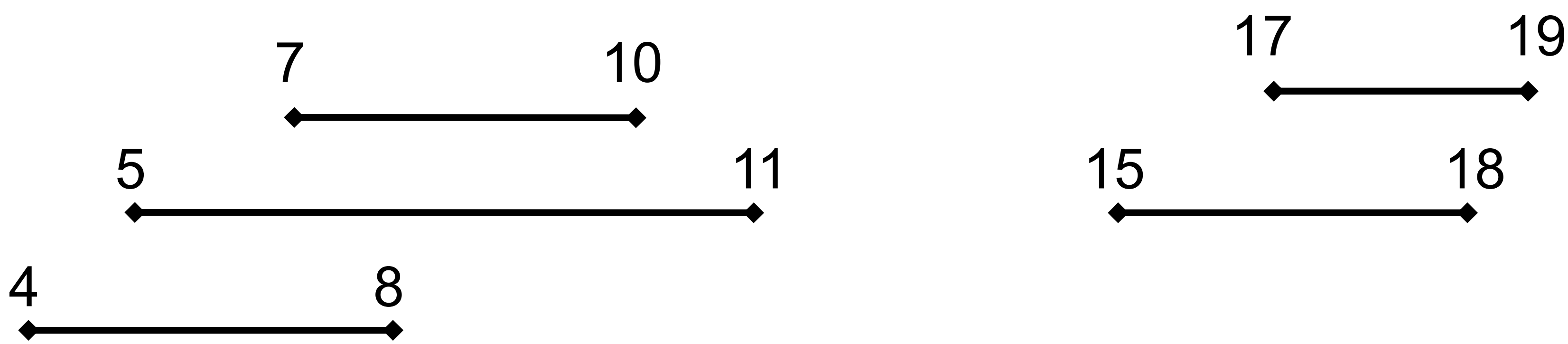
Find a tower that covers my location.



Idea 2: O(1) queries??

Cell Tower Coverage

Find a tower that covers my location.



Idea 2: O(1) queries

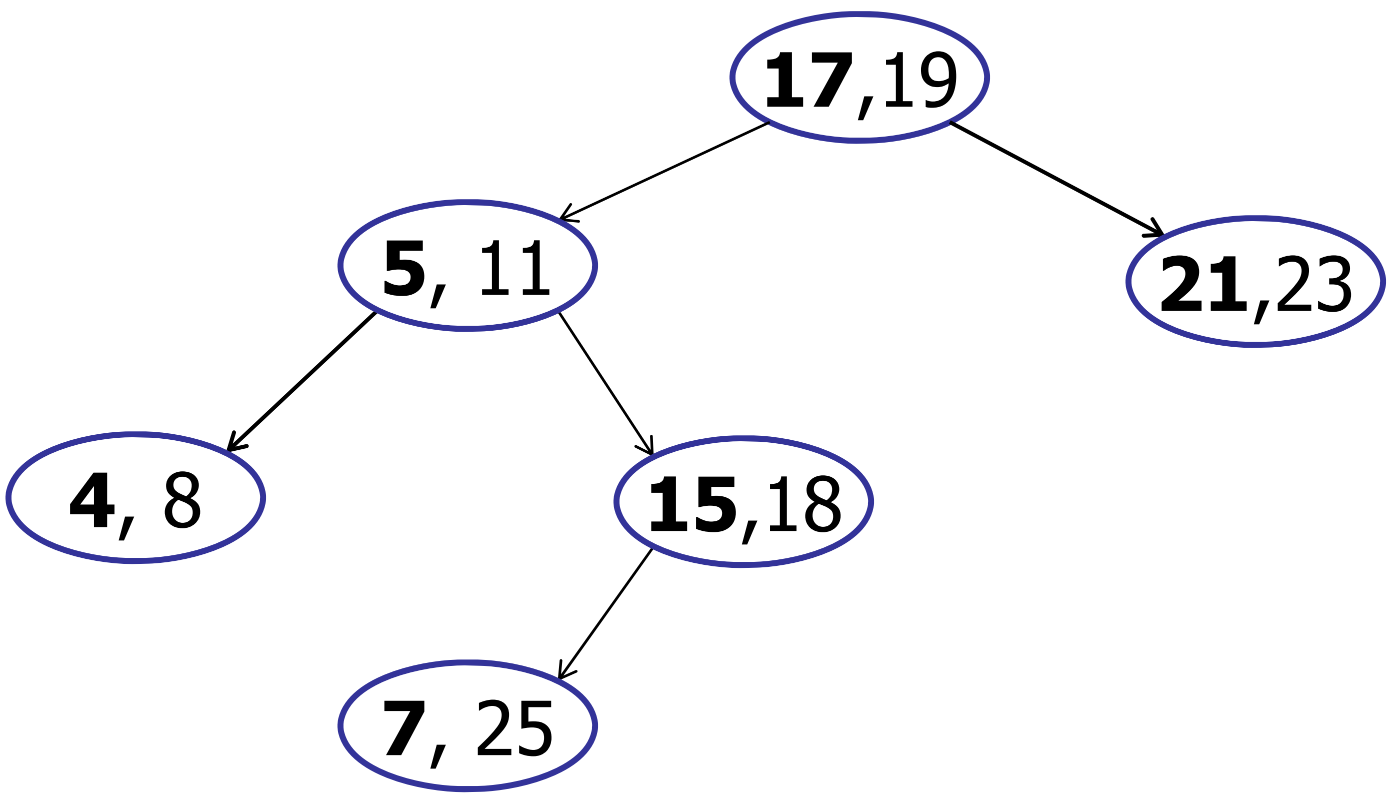
			A	**A**	**A**	**A**	**A**	**B**	**B**	**C**				**D**	**D**	**D**	**D**	**E**	
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Problems??

Idea 3: Interval Trees

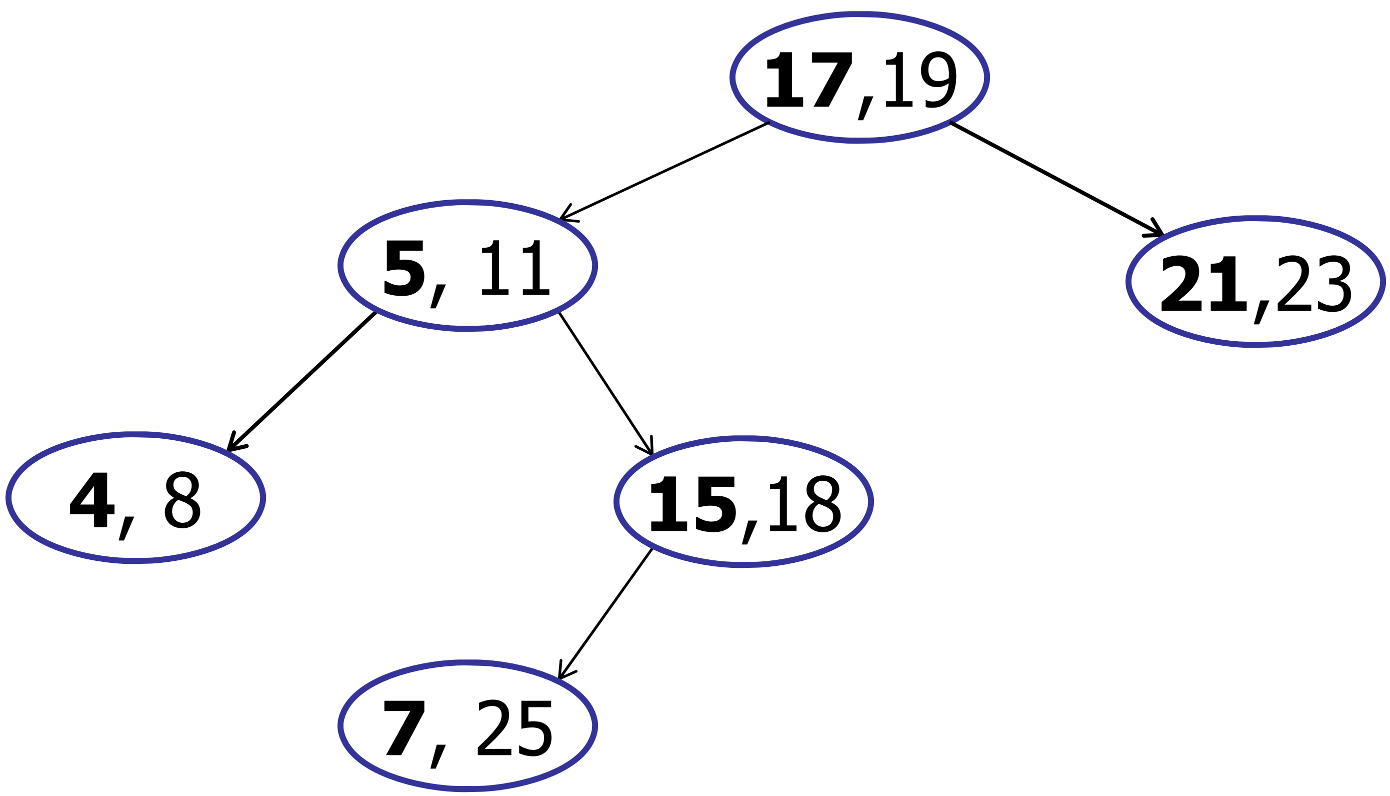
Interval Trees

Each node is an interval



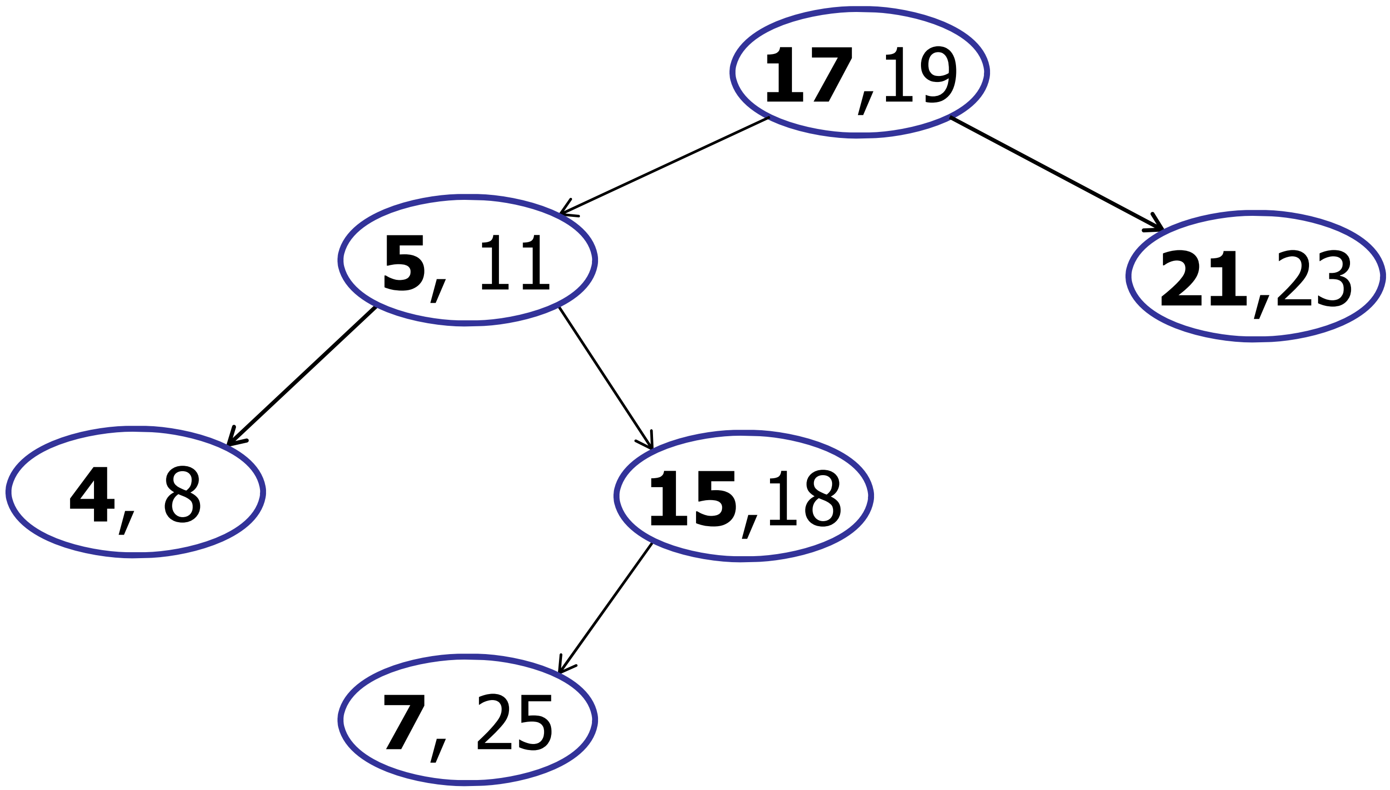
Interval Trees

Sorted by left endpoint



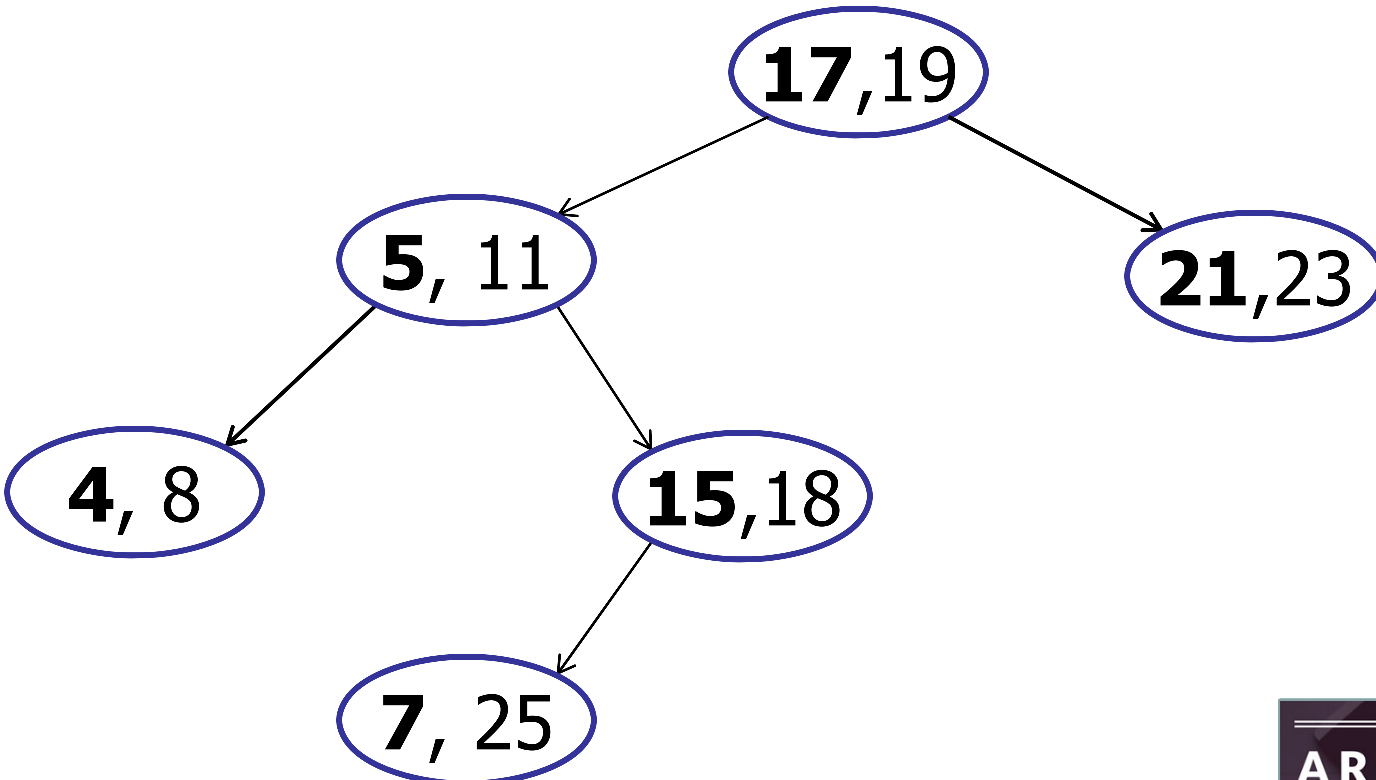
Interval Trees

search-interval(25) = ?



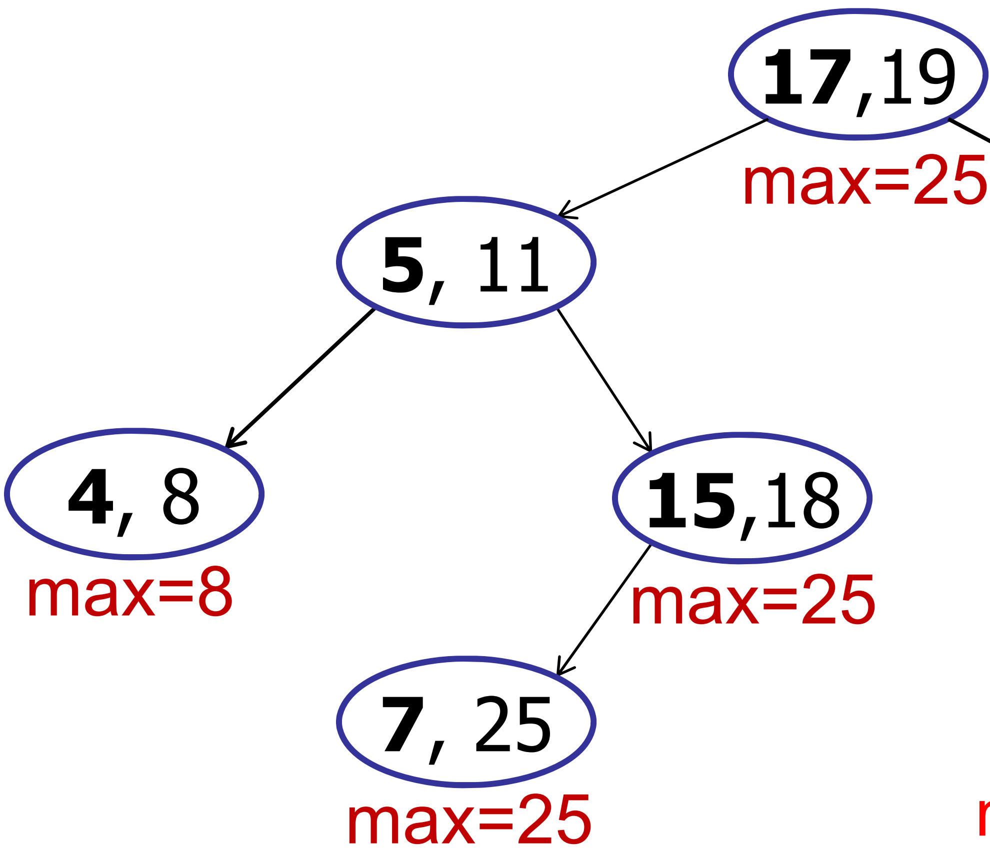
Interval Trees

Augment: ??



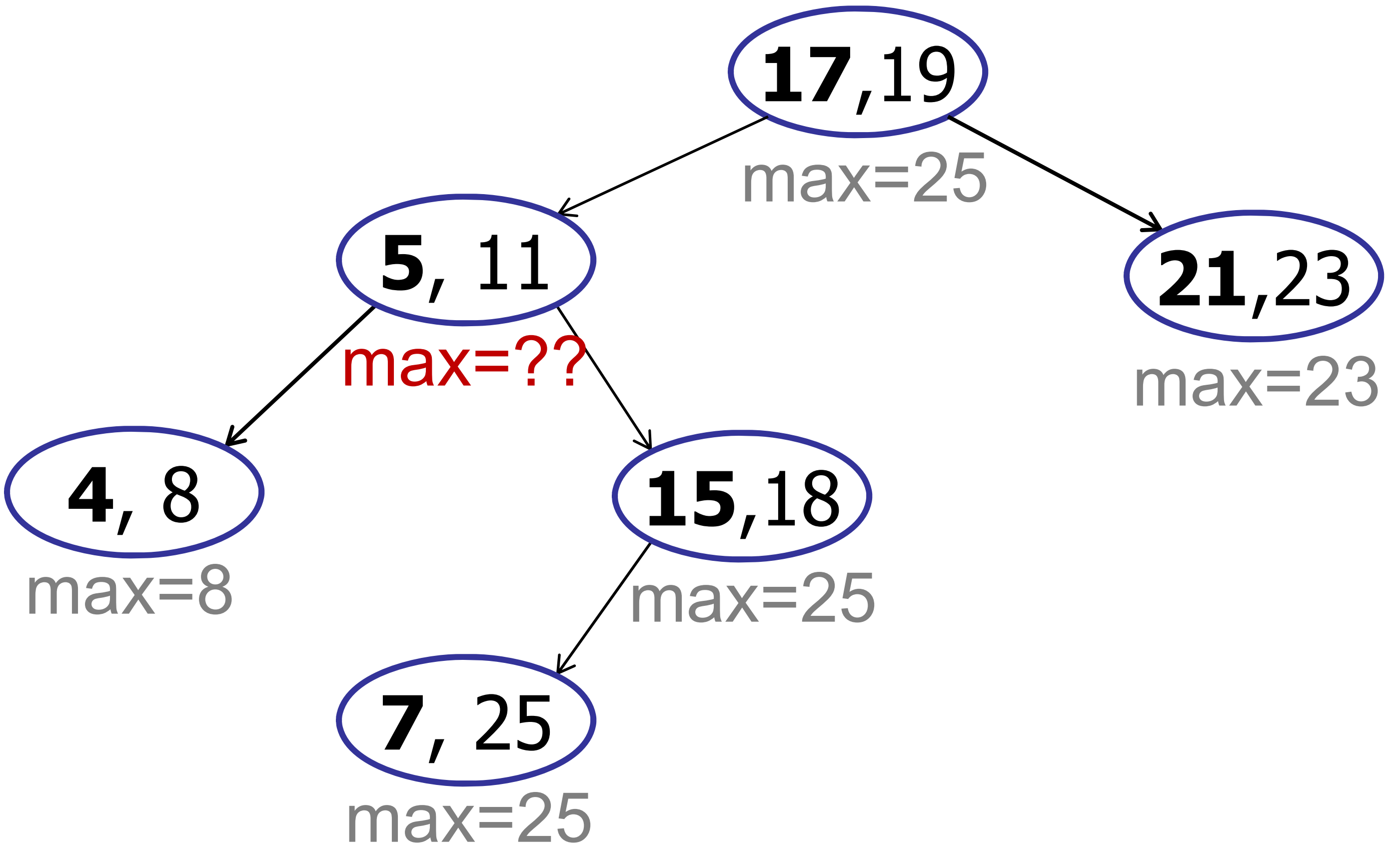
Interval Trees

Augment: maximum endpoint in subtree



$$\text{max} = \begin{cases} \text{node.right} \\ \text{left.max} \\ \text{right.max} \end{cases}$$

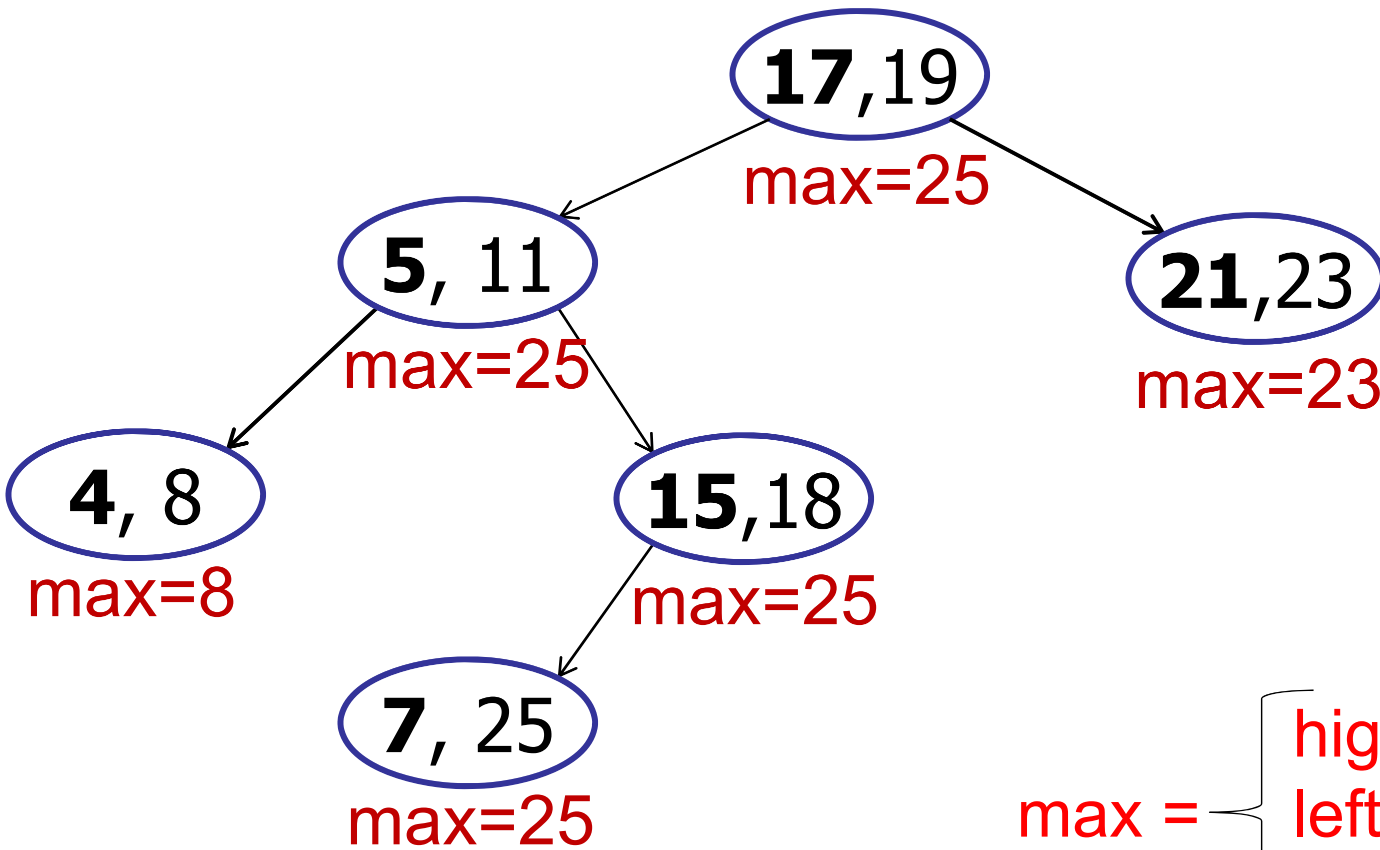
max=??



1. 5
2. 8
3. 11
4. 18
5. 25
6. 19

Interval Trees

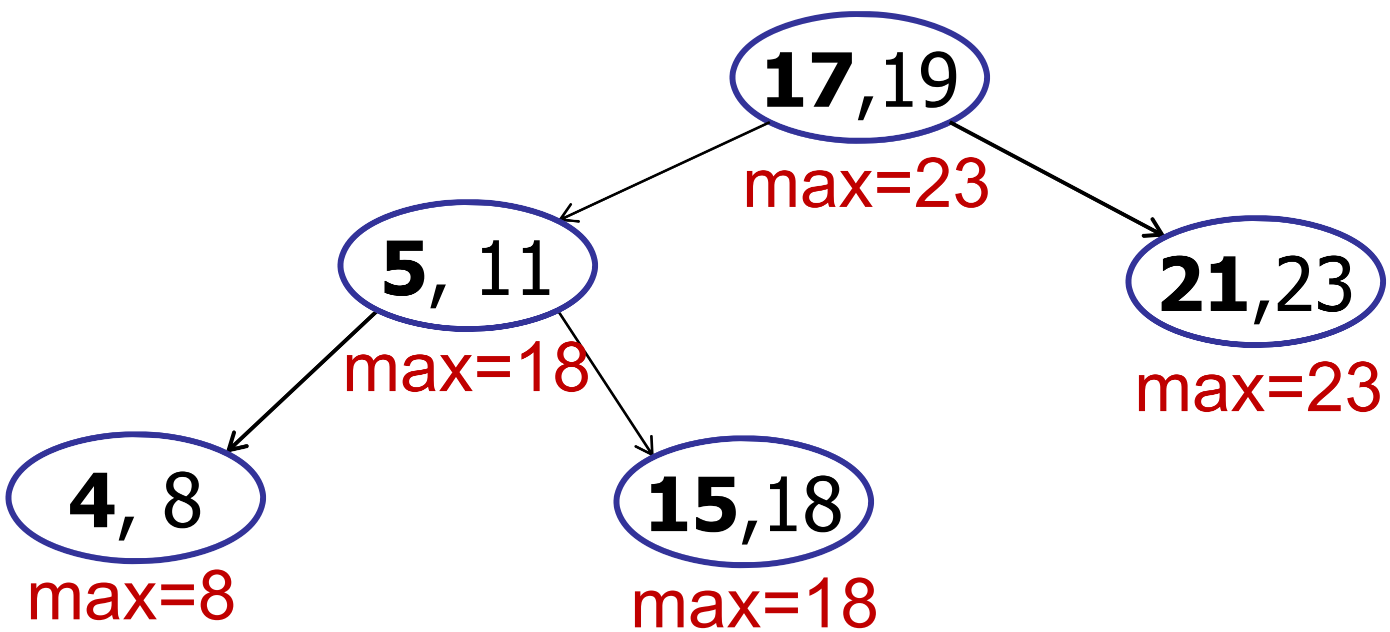
Augment: maximum endpoint in subtree



$$\max = \begin{cases} \text{high} \\ \text{left.max} \\ \text{right.max} \end{cases}$$

Interval Trees

Insertion: *example* – **insert(7, 25)**



Interval Trees

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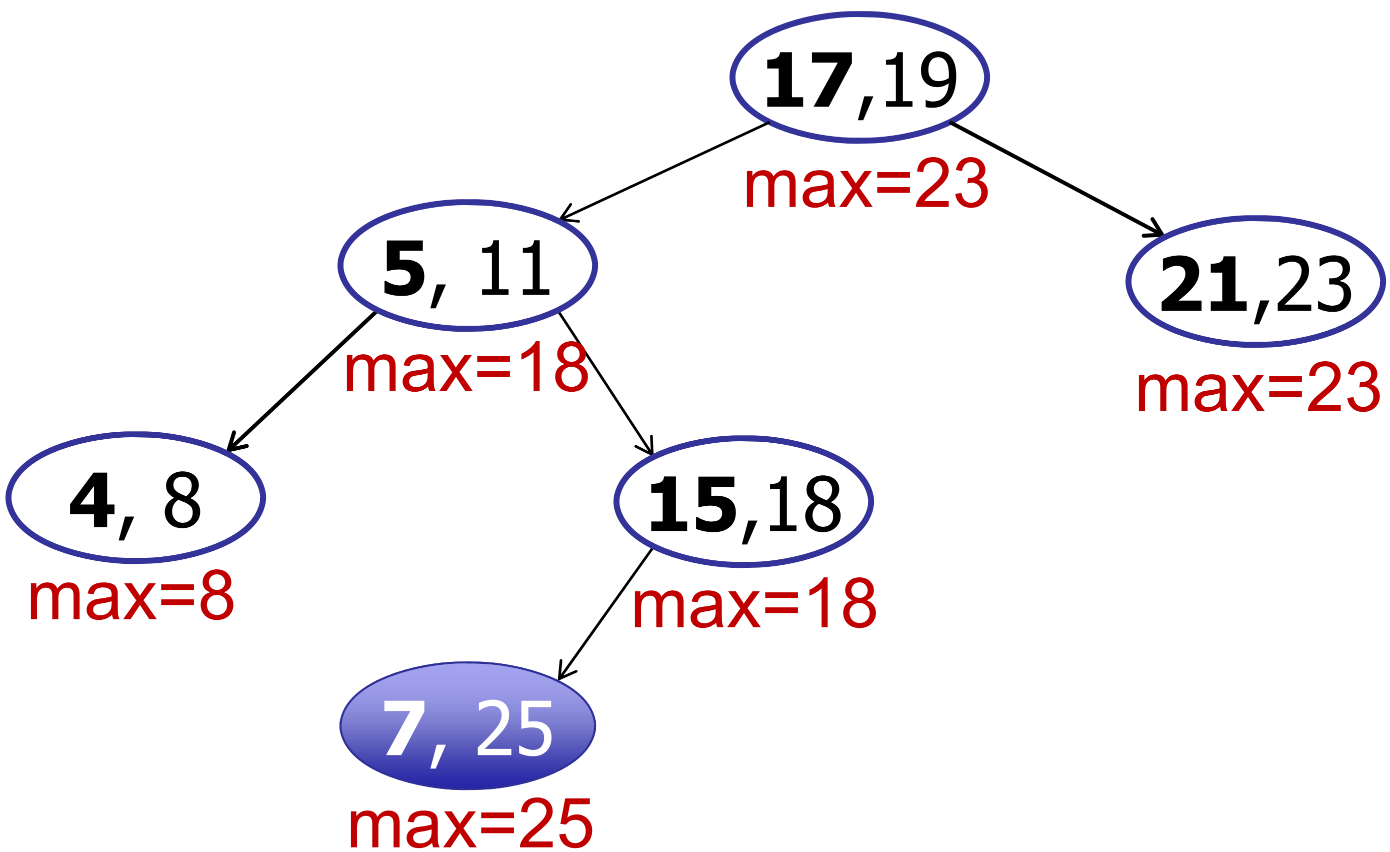
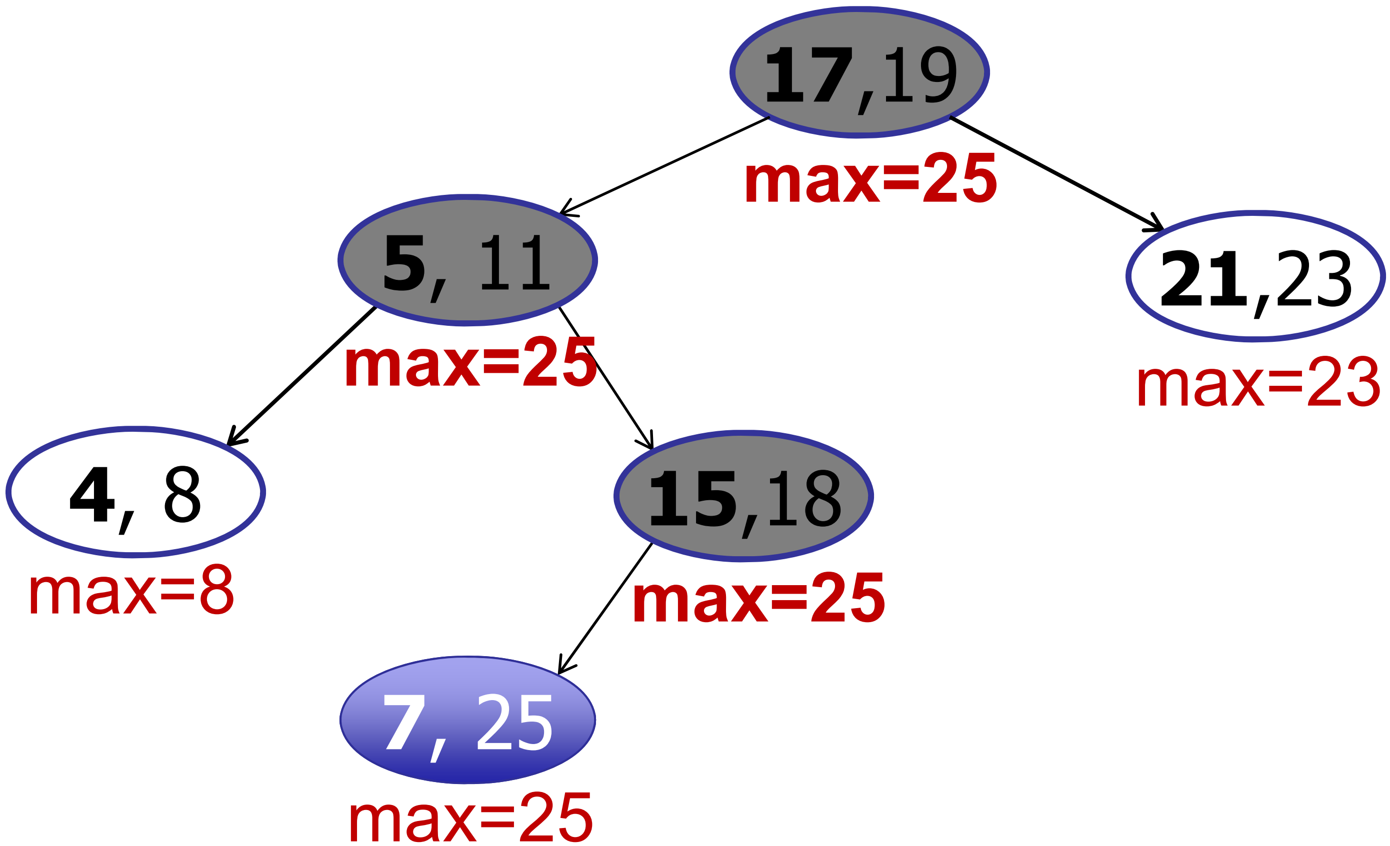


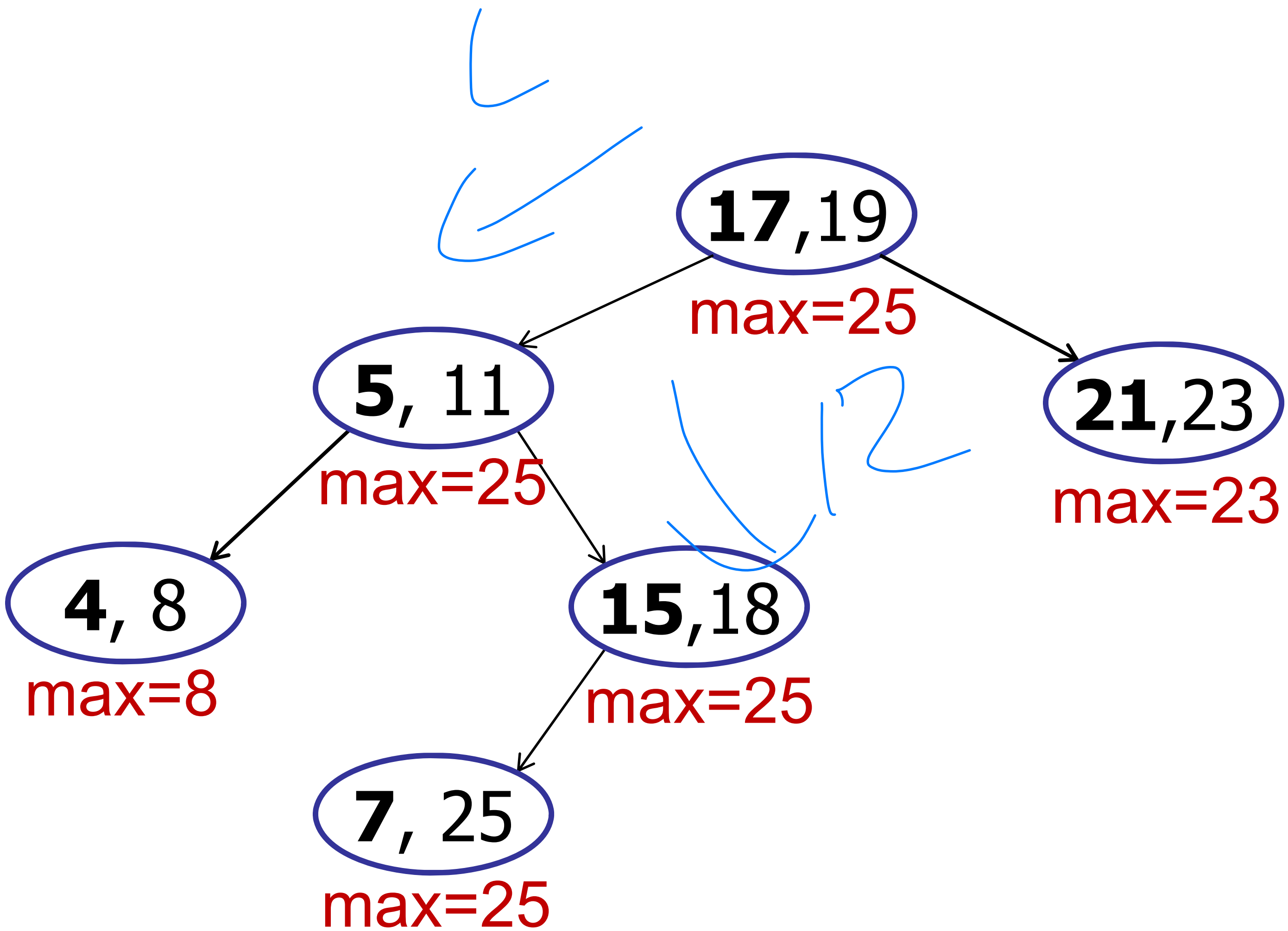
Interval Trees

Insertion: *example* – **insert(7, 25)**



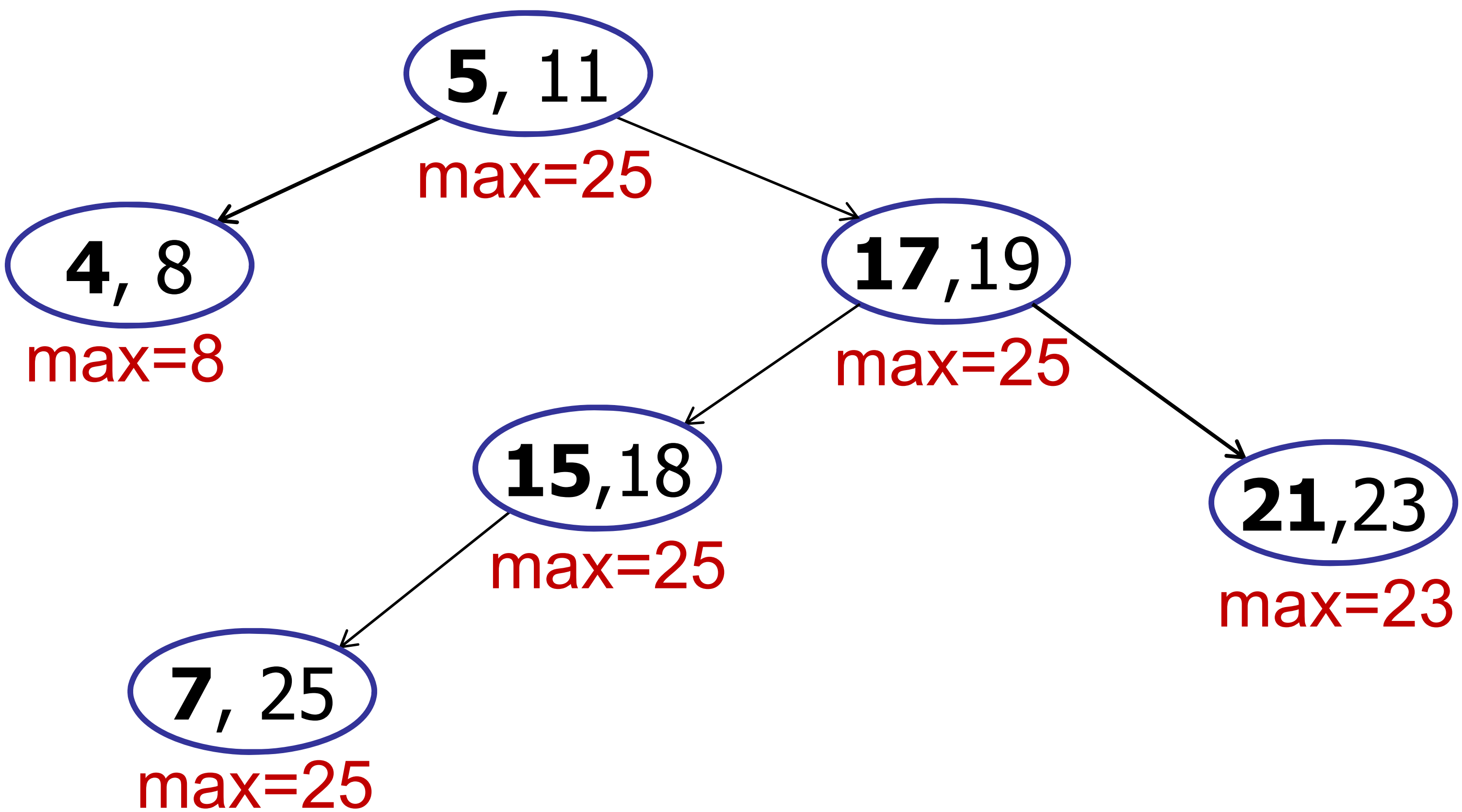
Interval Trees

Insertion: *out-of-balance*



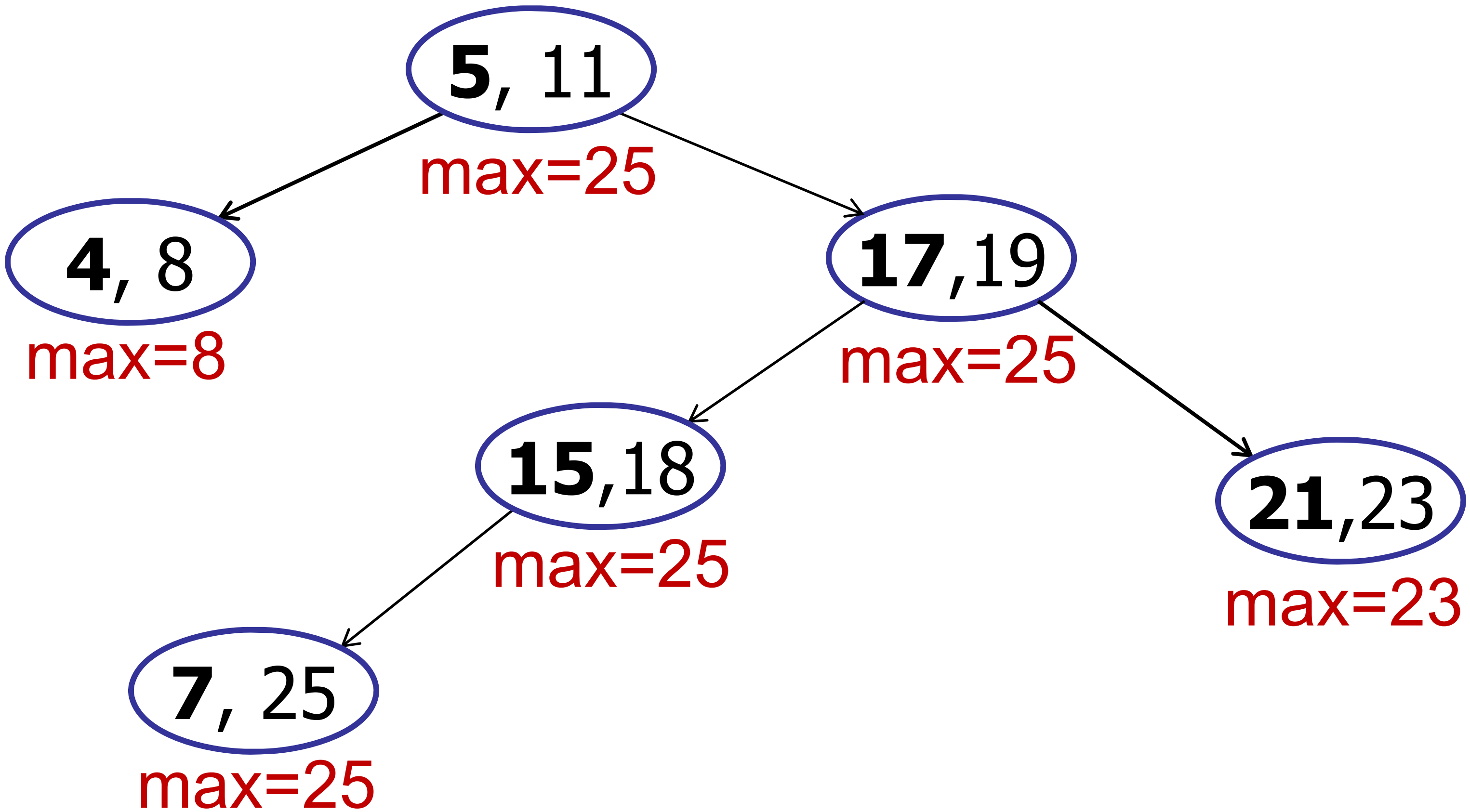
Interval Trees

Insertion: right-rotate (17, 19)



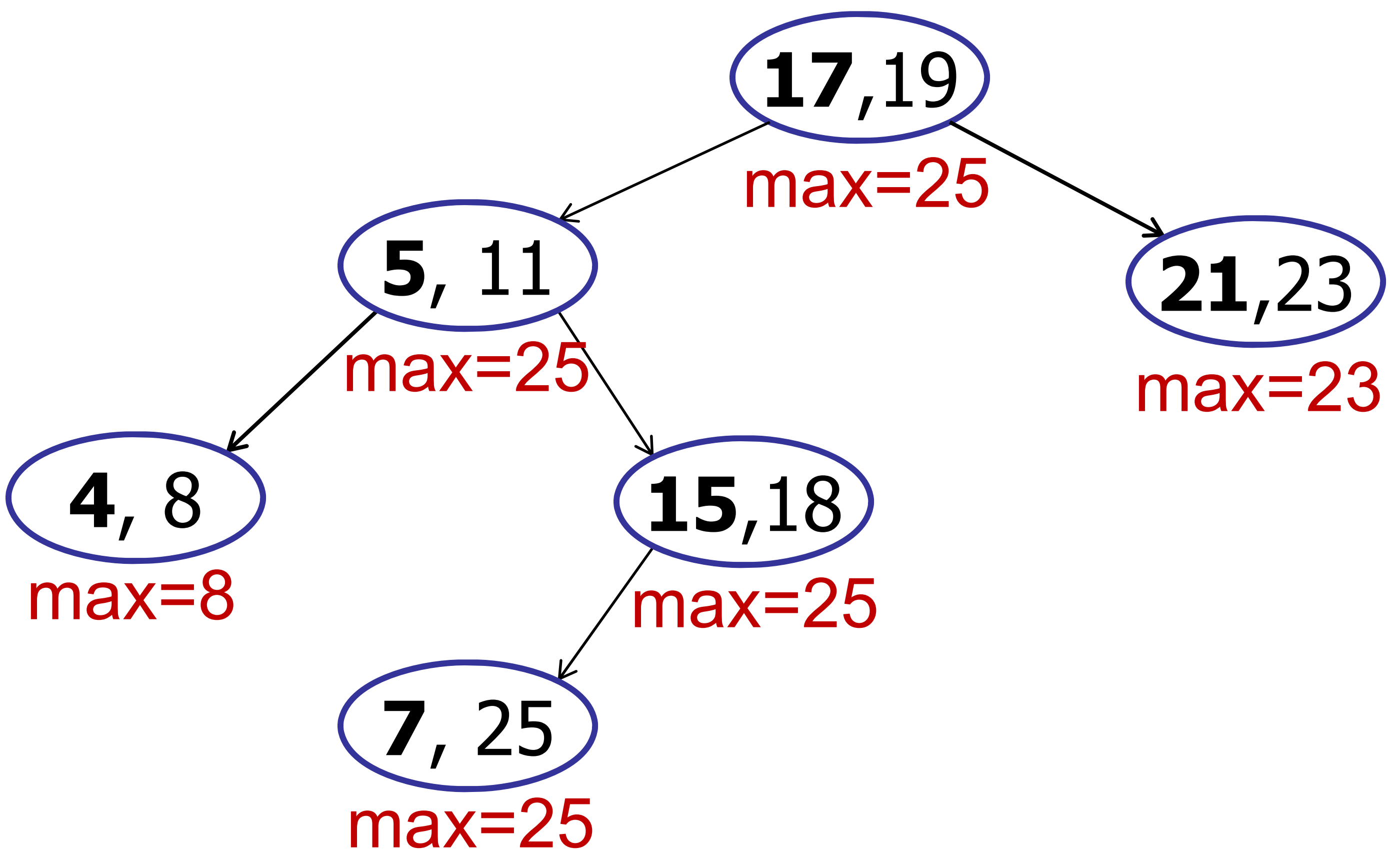
Interval Trees

Insertion: *right-rotate (17, 19),* OOPS!



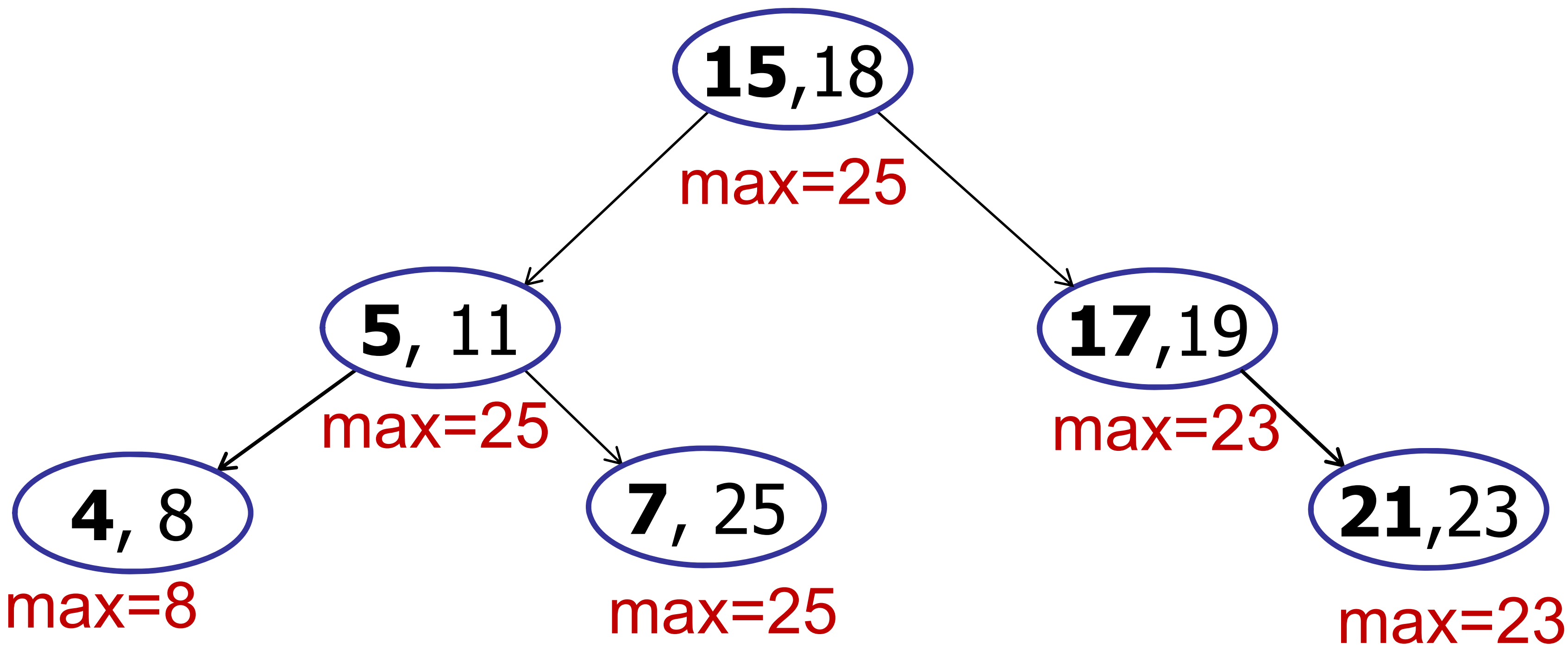
Interval Trees

Insertion: *out-of-balance*



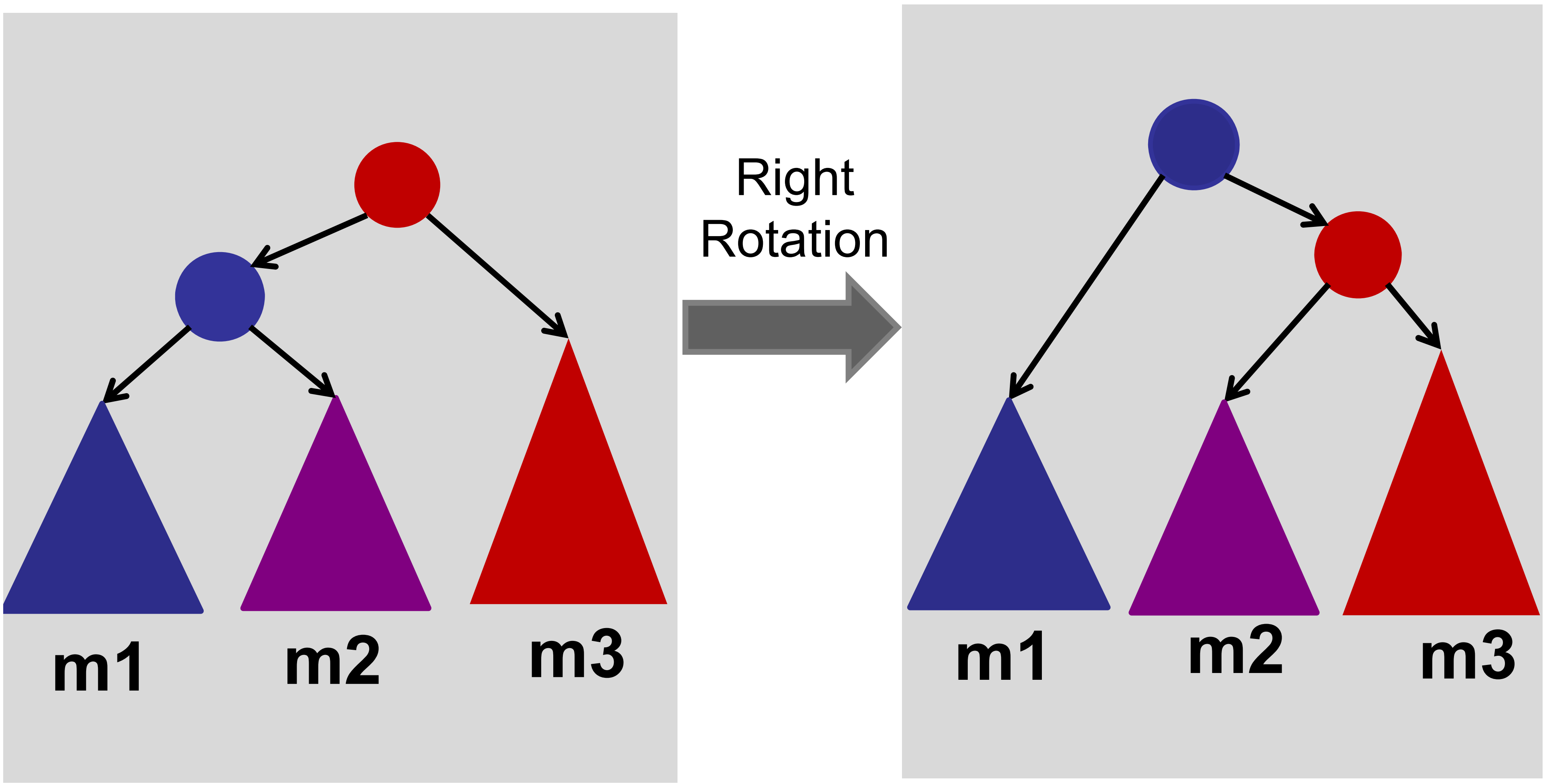
Interval Trees

Insertion: left-rotate, right-rotate



Interval Trees

Maintain MAX during rotations:



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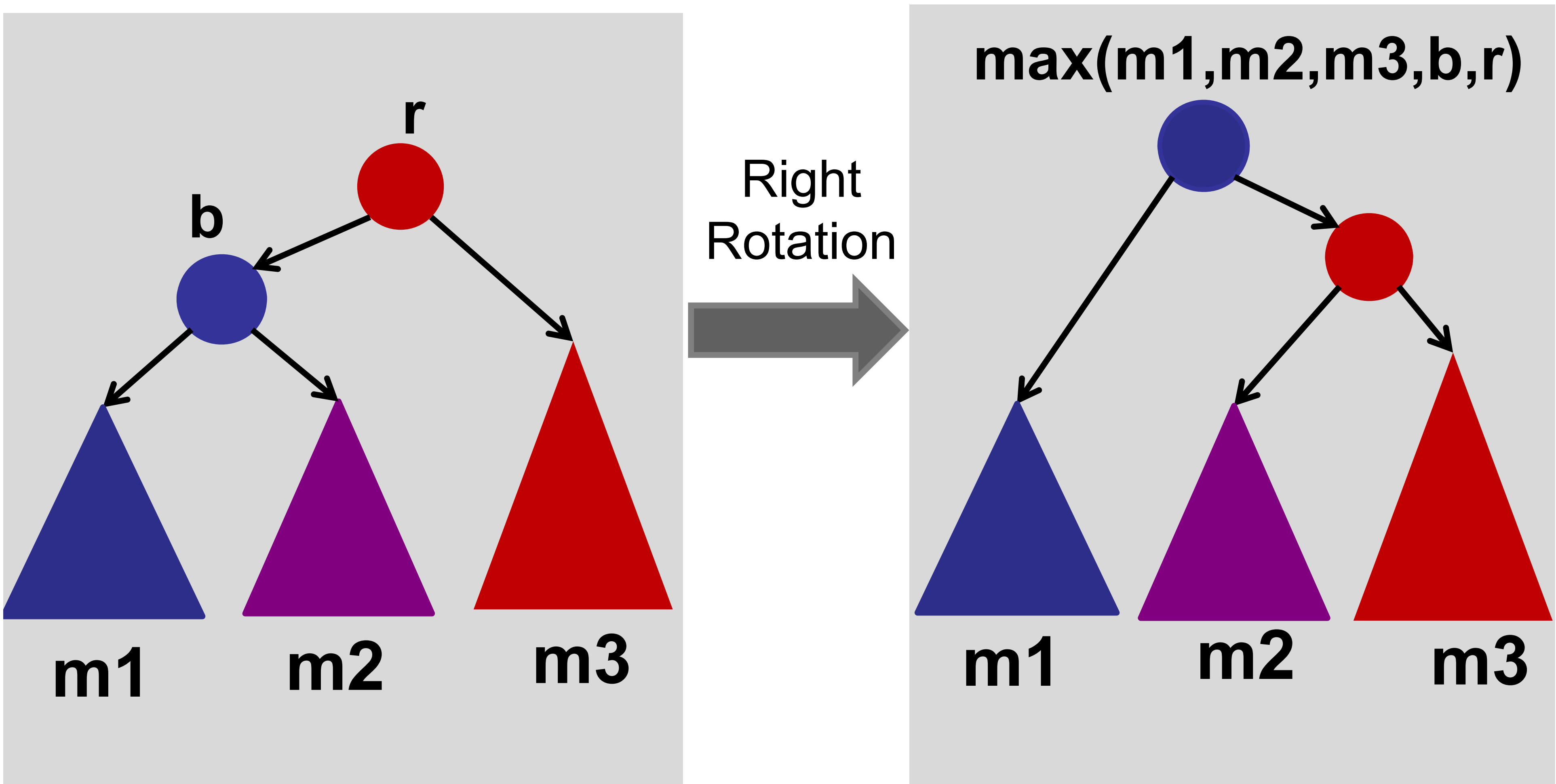

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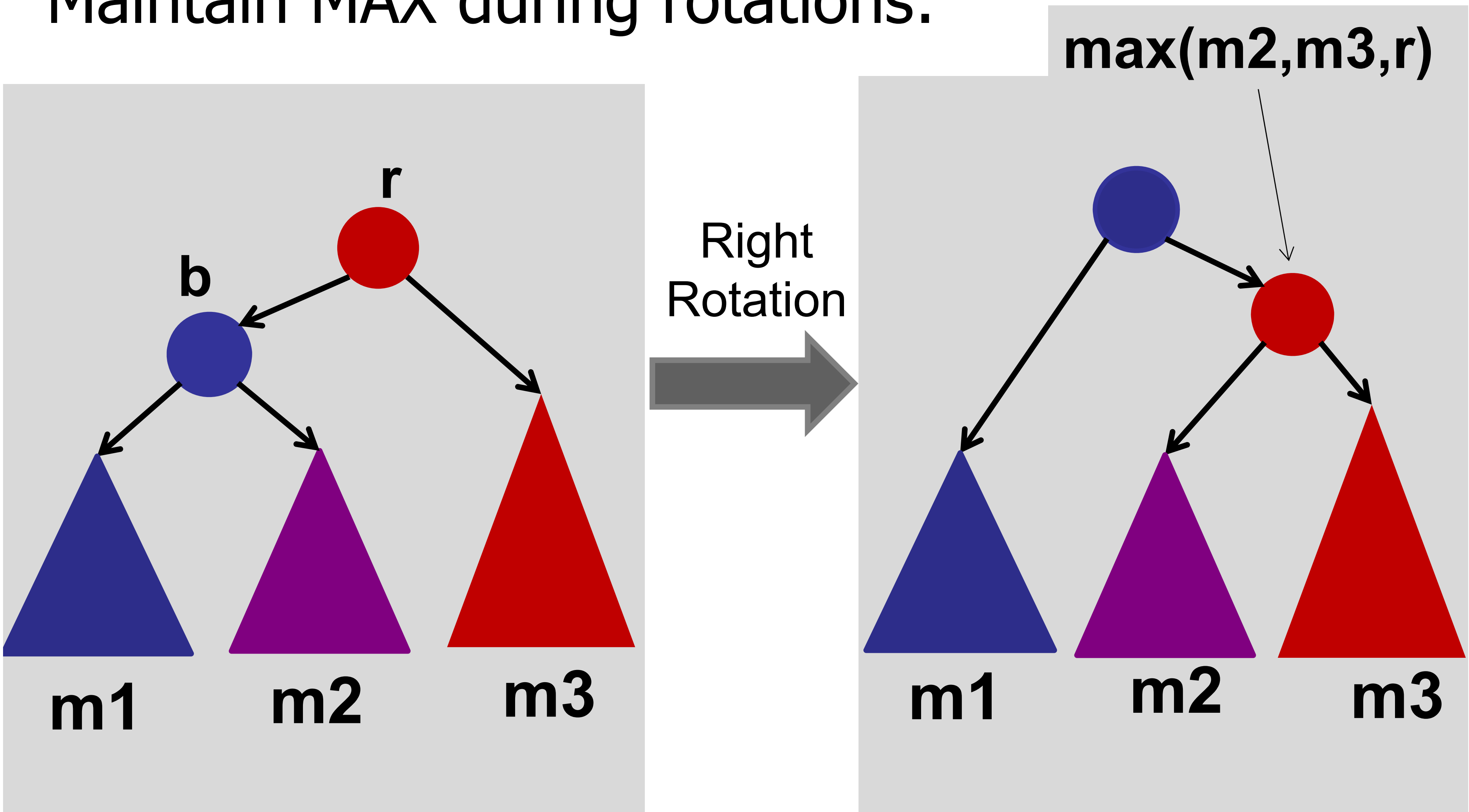
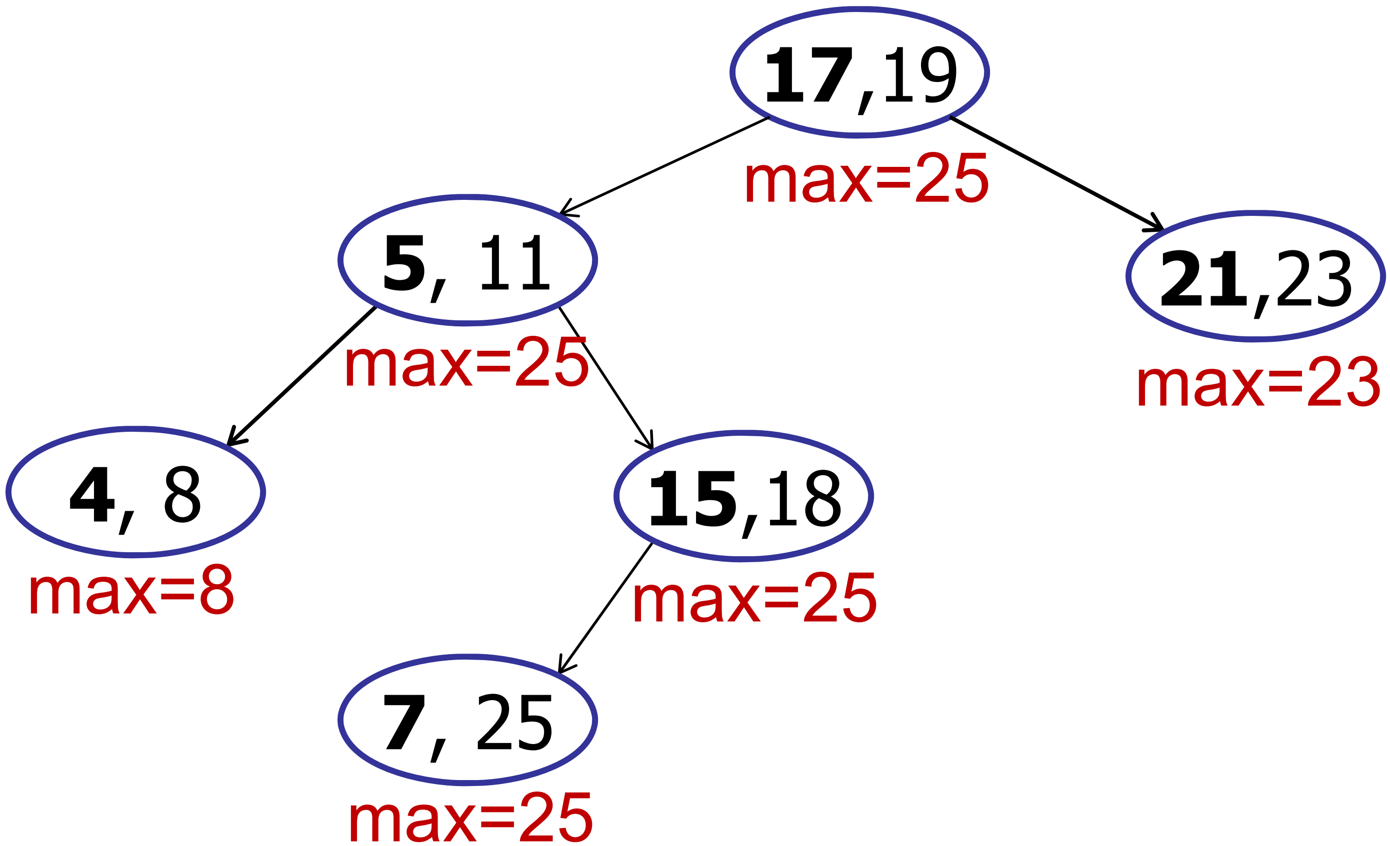


Interval Trees

Searching: interval-search(22)



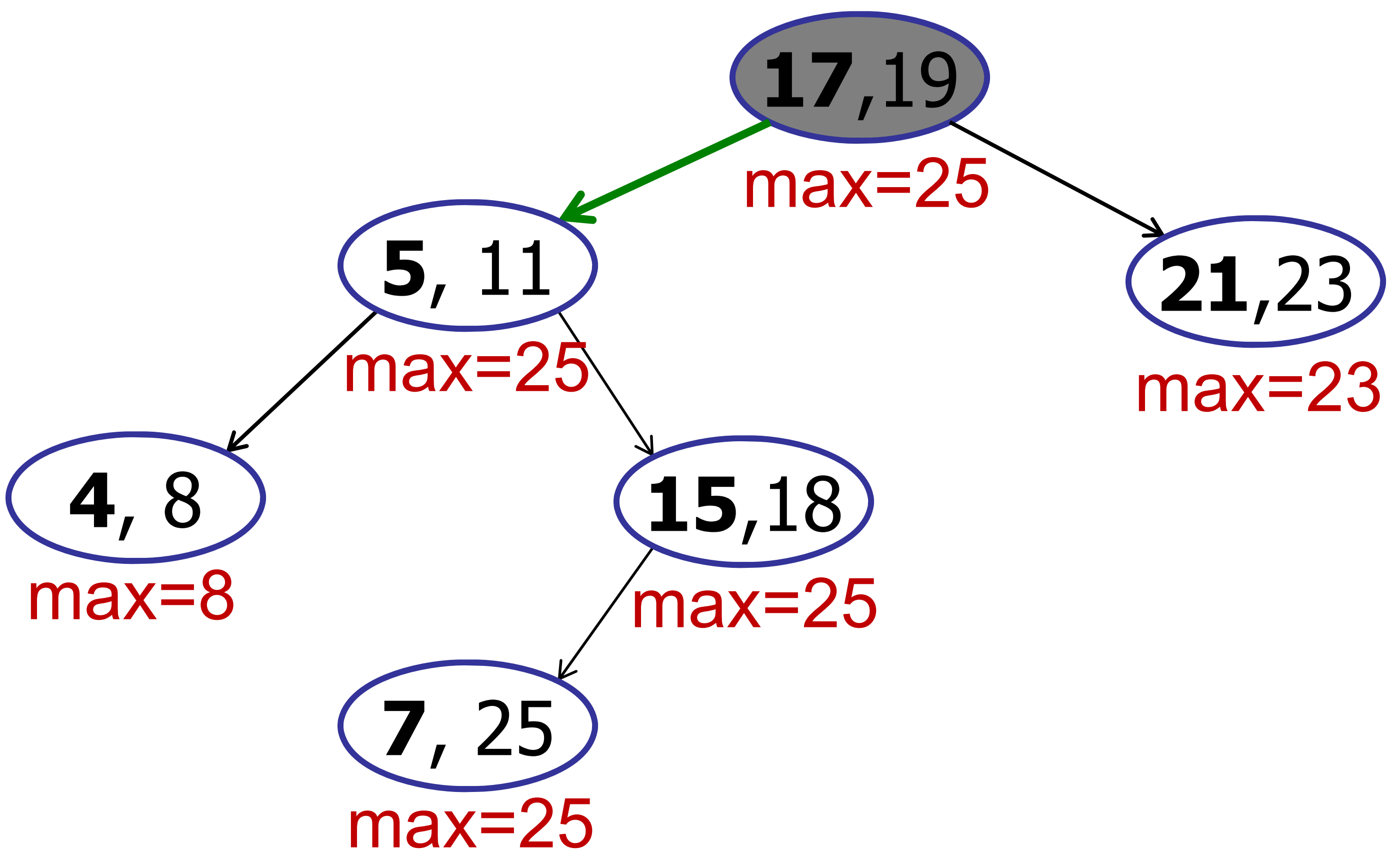
Interval Trees

interval-search(x) : find interval containing x

```
interval-search(x)
    c = root;
    while (c != null and x is not in c.interval) do
        if (c.left == null) then
            c = c.right;
        else if  (x > c.left.max)  then
            c = c.right;
        else c = c.left;
    return c.interval;
```

Interval Trees

Searching: interval-search(22)



Interval Trees

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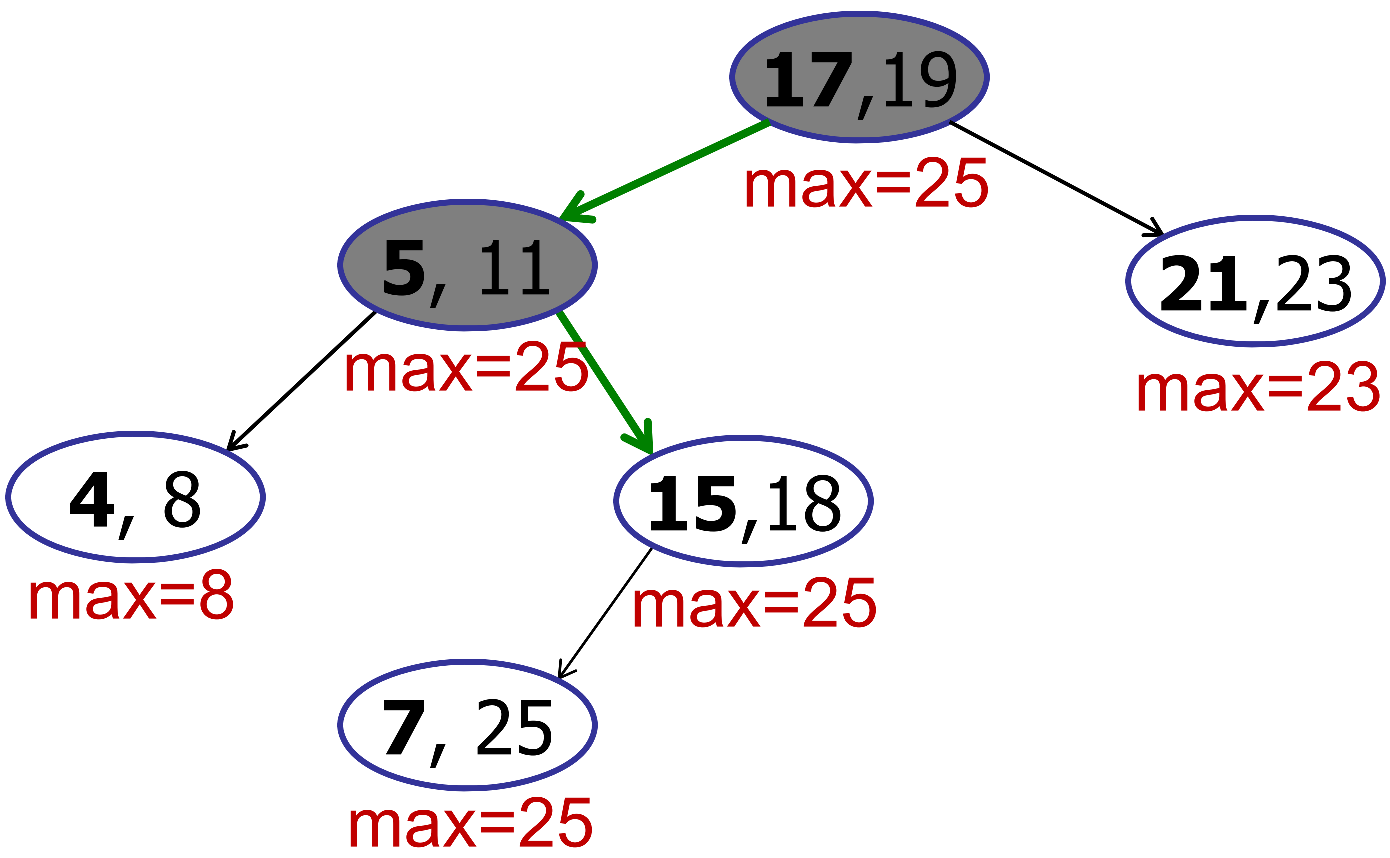
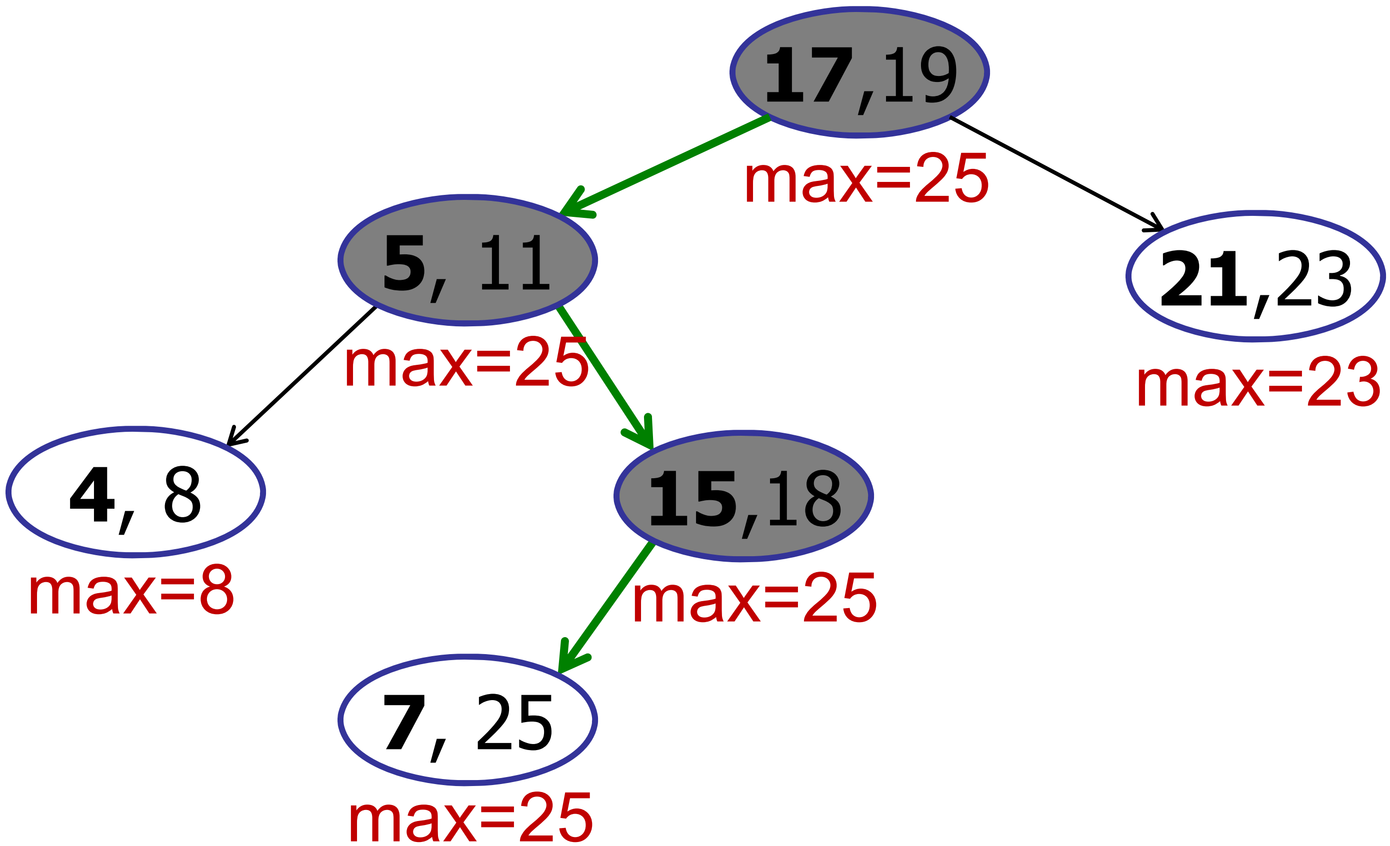


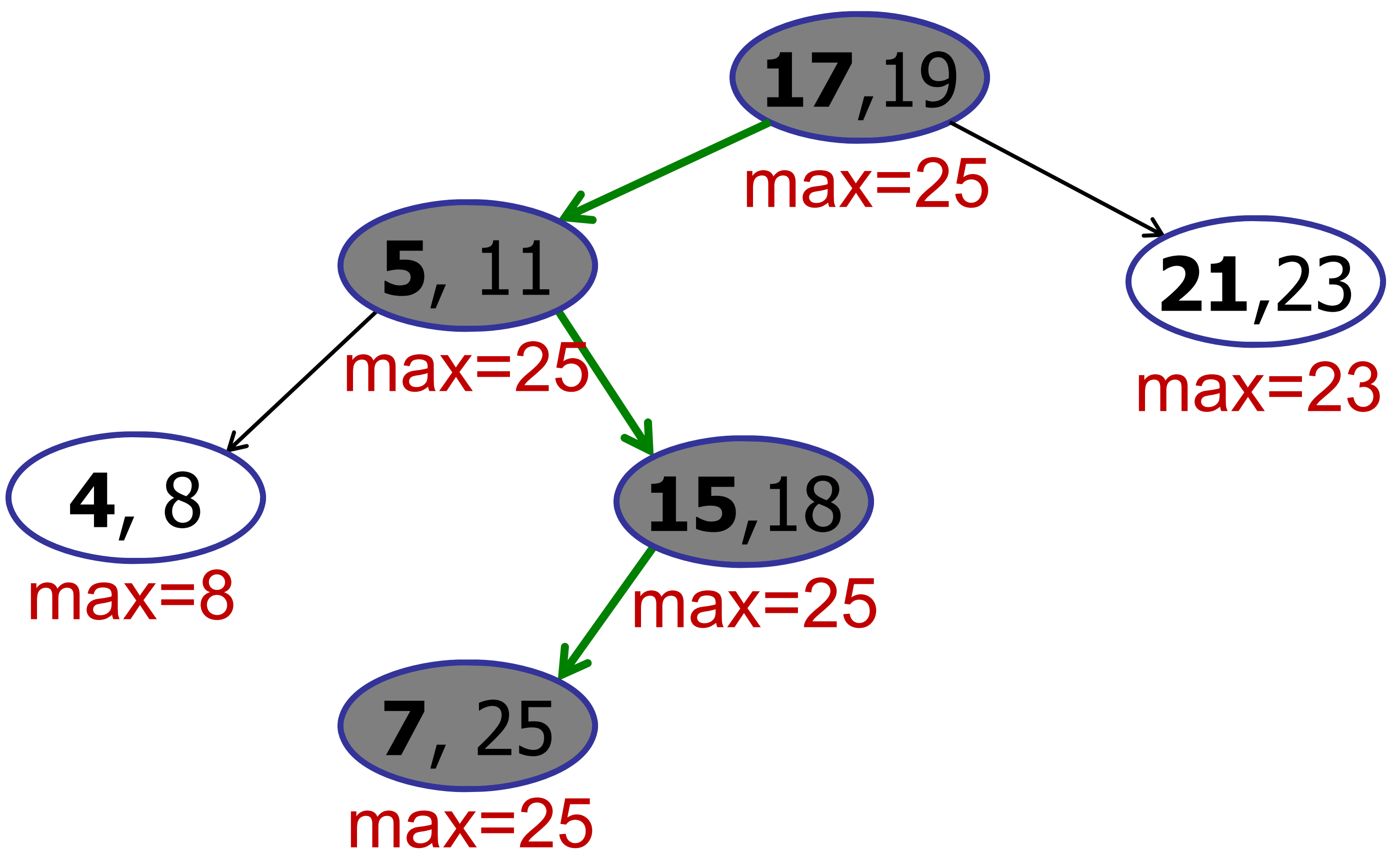
Interval Trees

Searching: interval-search(22)



Interval Trees

Searching: *interval-search(22)*



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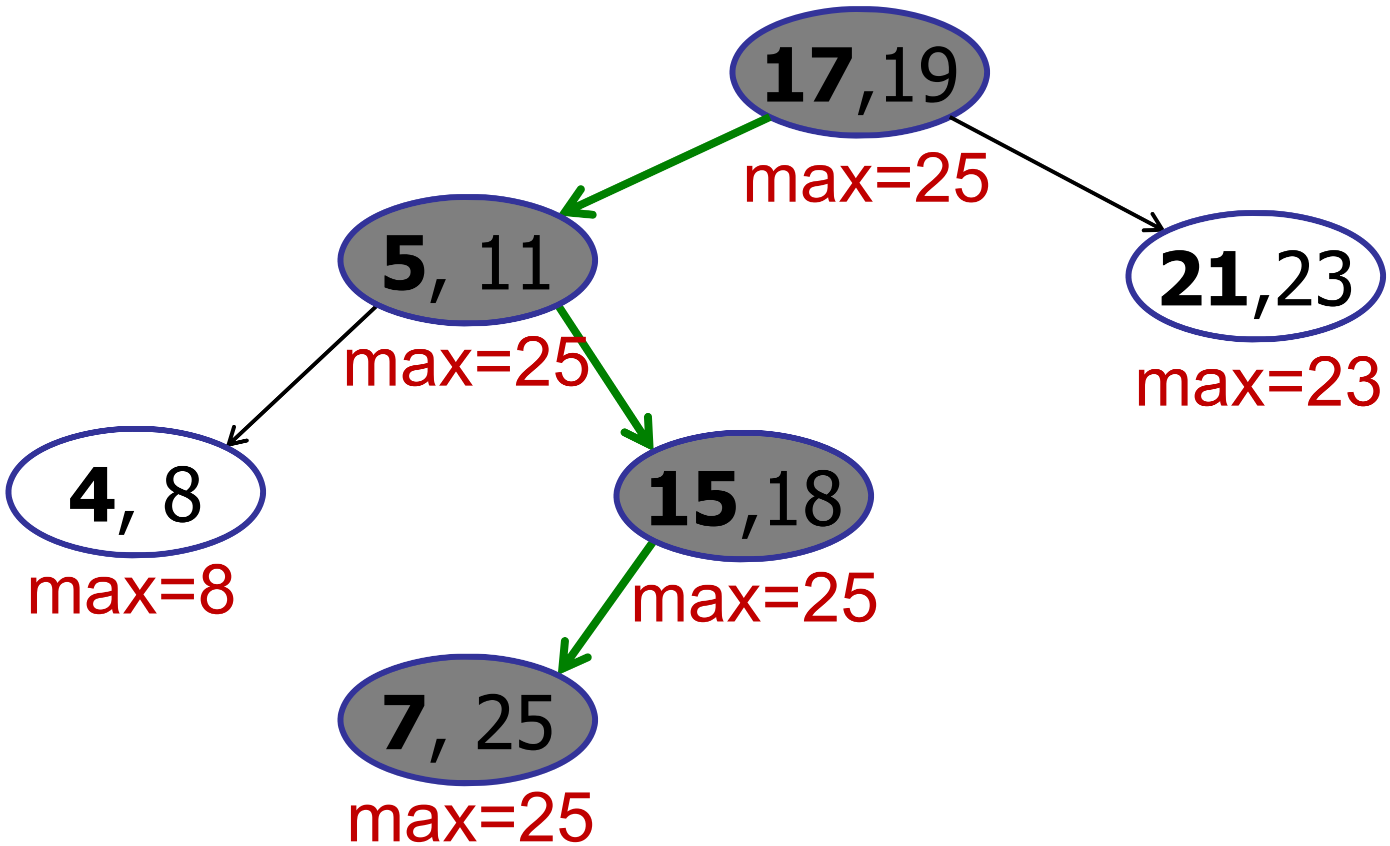
        else c = c.left;
    return c.interval;
```

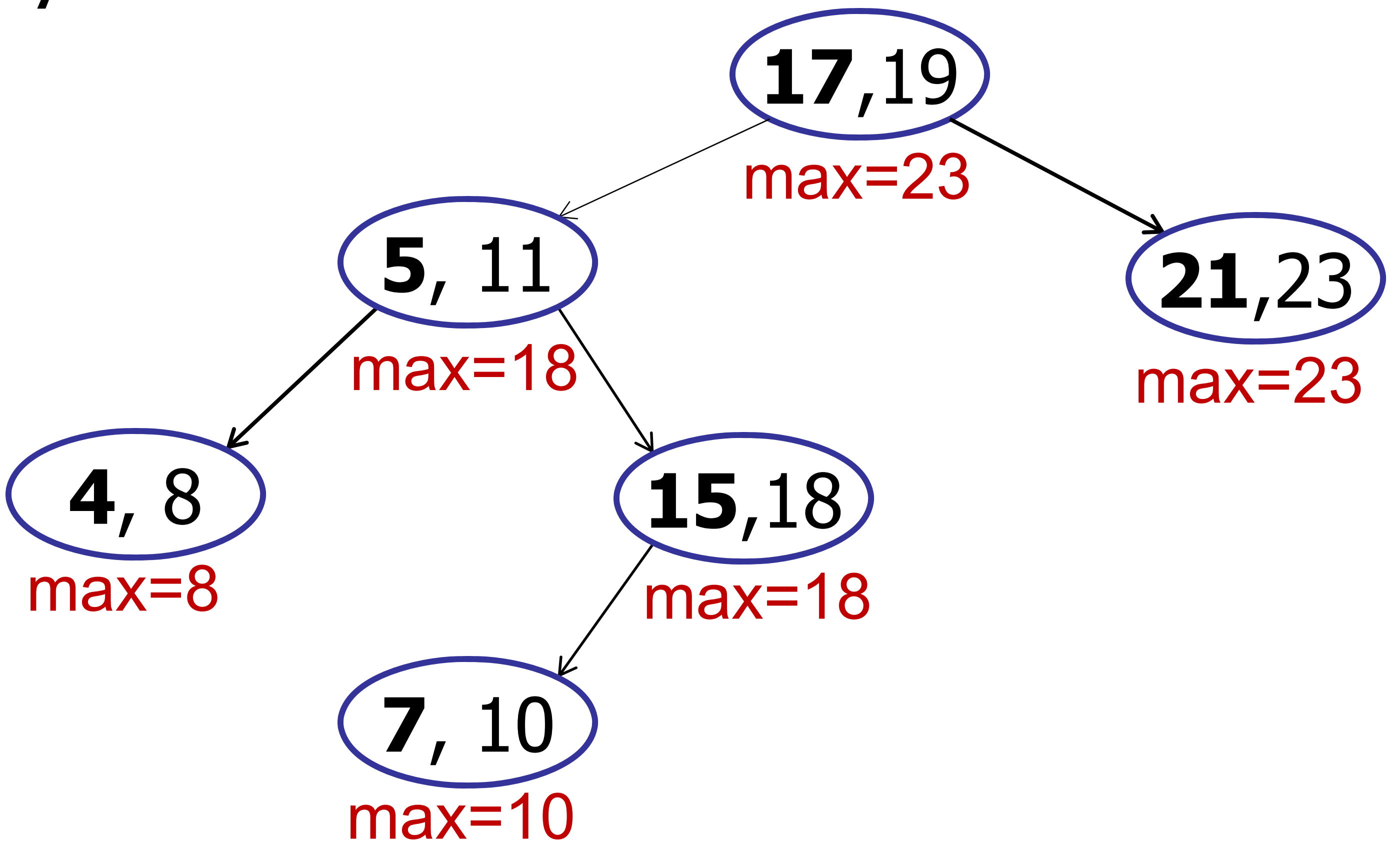
Interval Trees

Will any search find (21, 23)?



Interval Trees

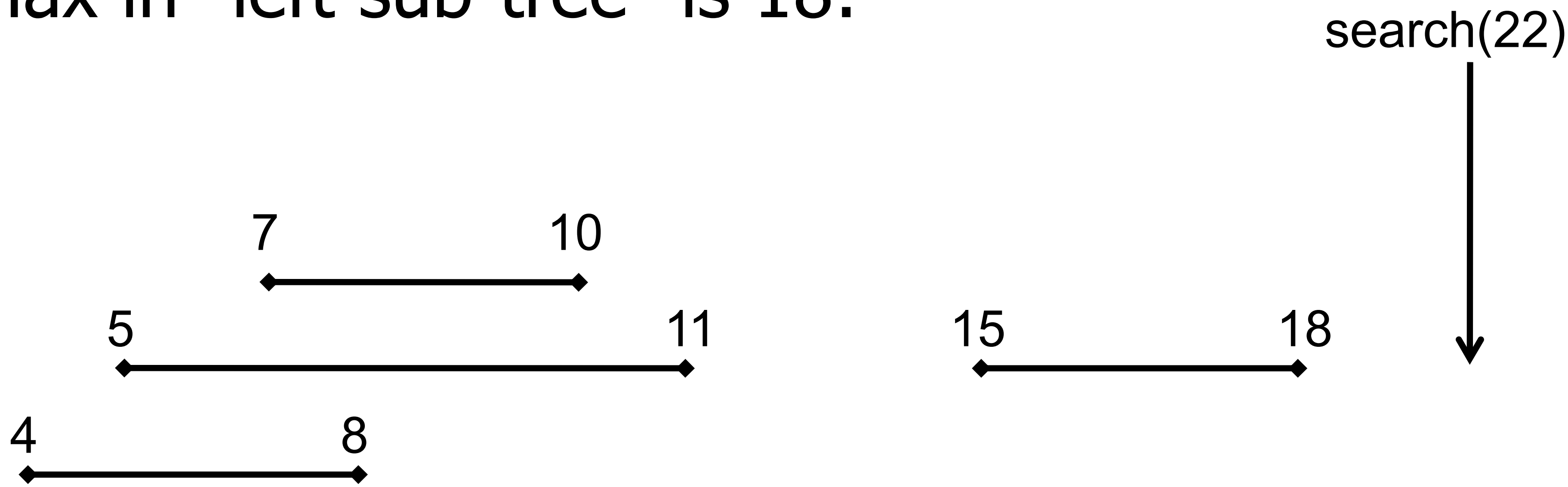
Why does it work?



Claim: If search goes right, then no overlap in left subtree.

Interval Trees

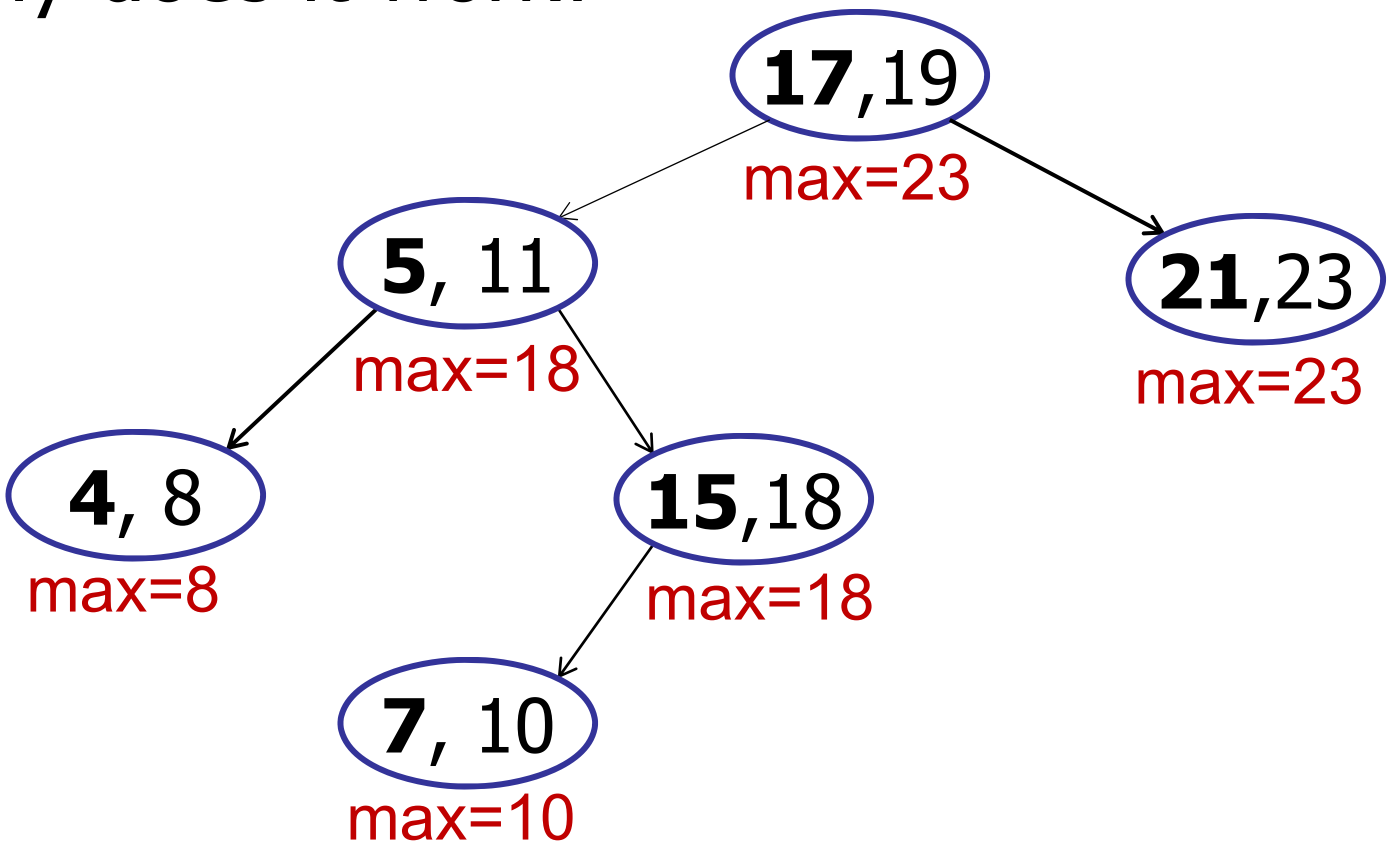
Max in “left sub-tree” is 18:



Safe to go right: 22 is not in the left sub-tree.

Interval Trees

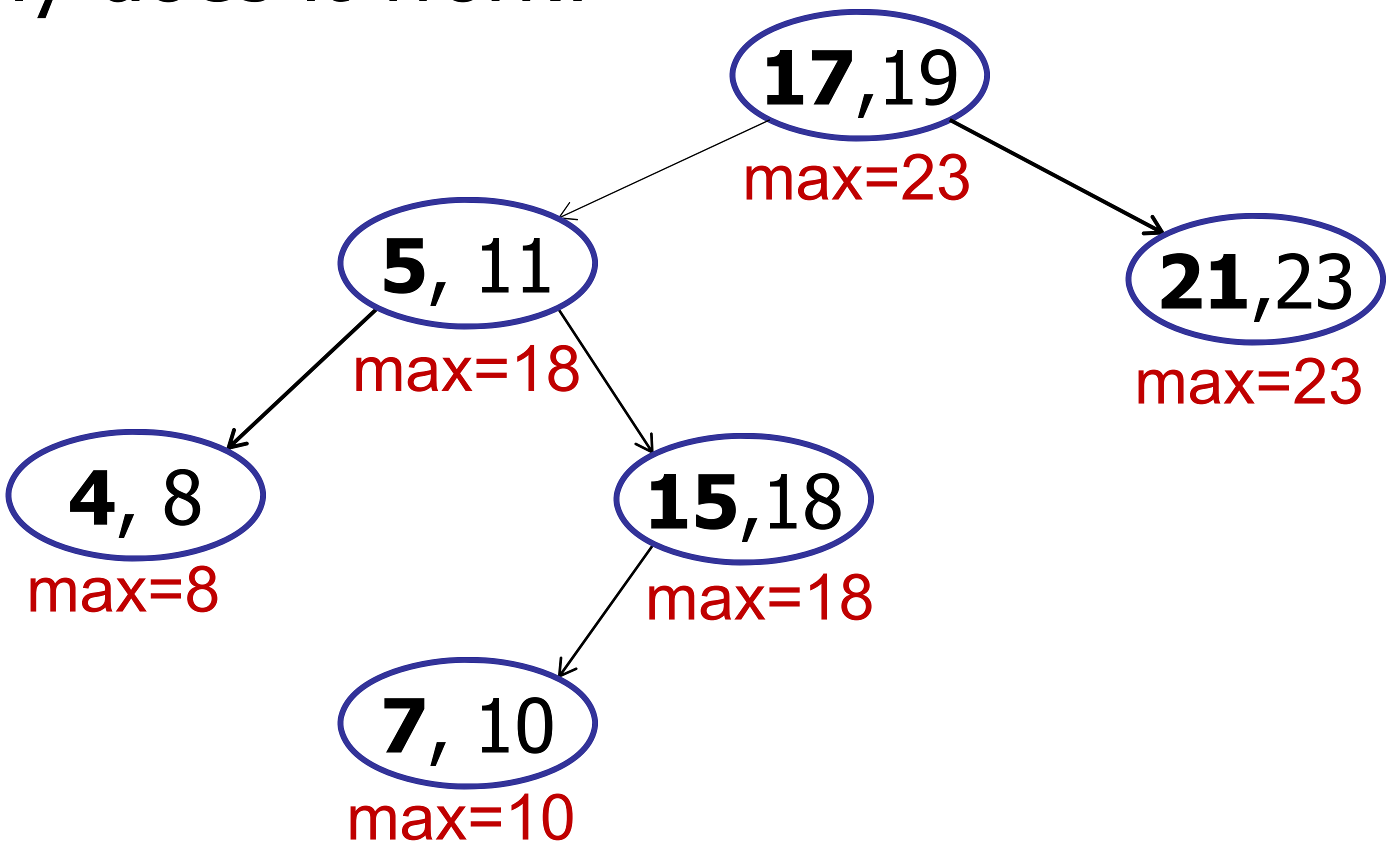
Why does it work?



Claim: If search goes left and there is no overlap in the left subtree…

Interval Trees

Why does it work?



Claim: If search goes left, then safe to go left.

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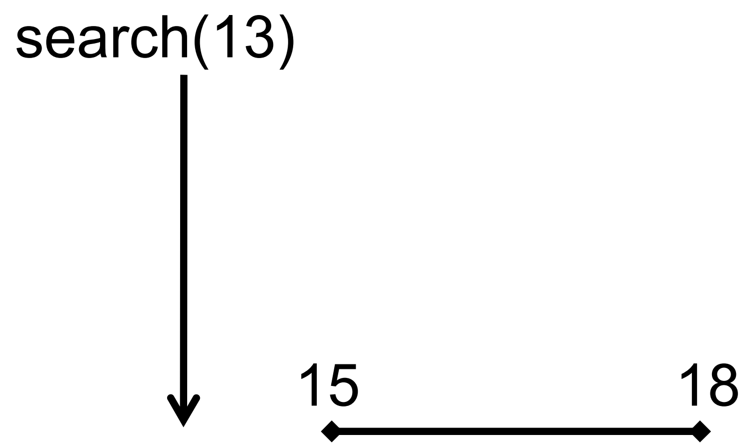
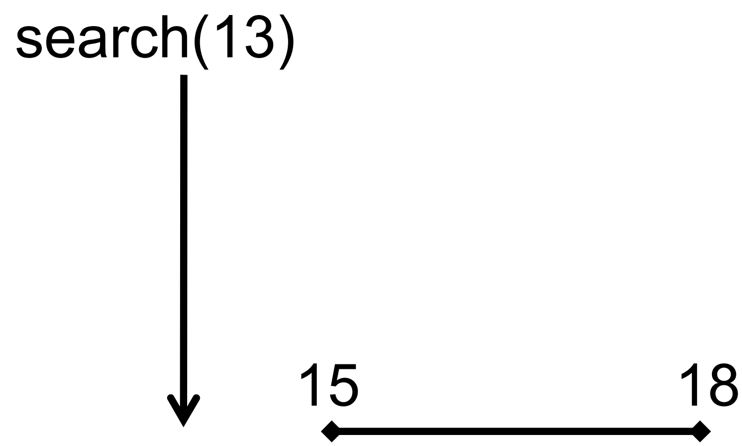
Max in “left sub-tree” is 18:



Go left: search(13) < 18

Interval Trees

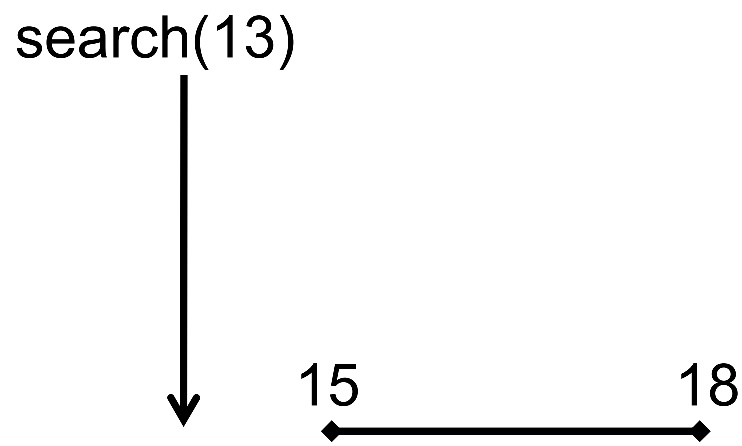
Max in “left sub-tree” is 18:



Go left: search(13) < 15 < 18

Interval Trees

Max in "left sub-tree" is 18:

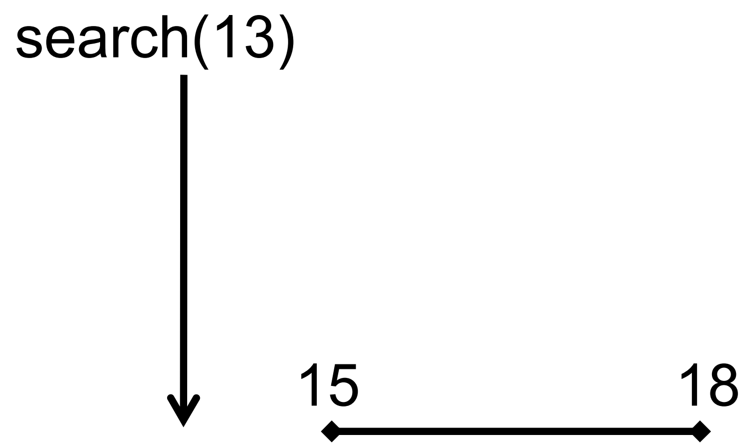


Go left: search(13) < 15 < 18

Tree sorted by left endpoint.

Interval Trees

Max in "left sub-tree" is 18:



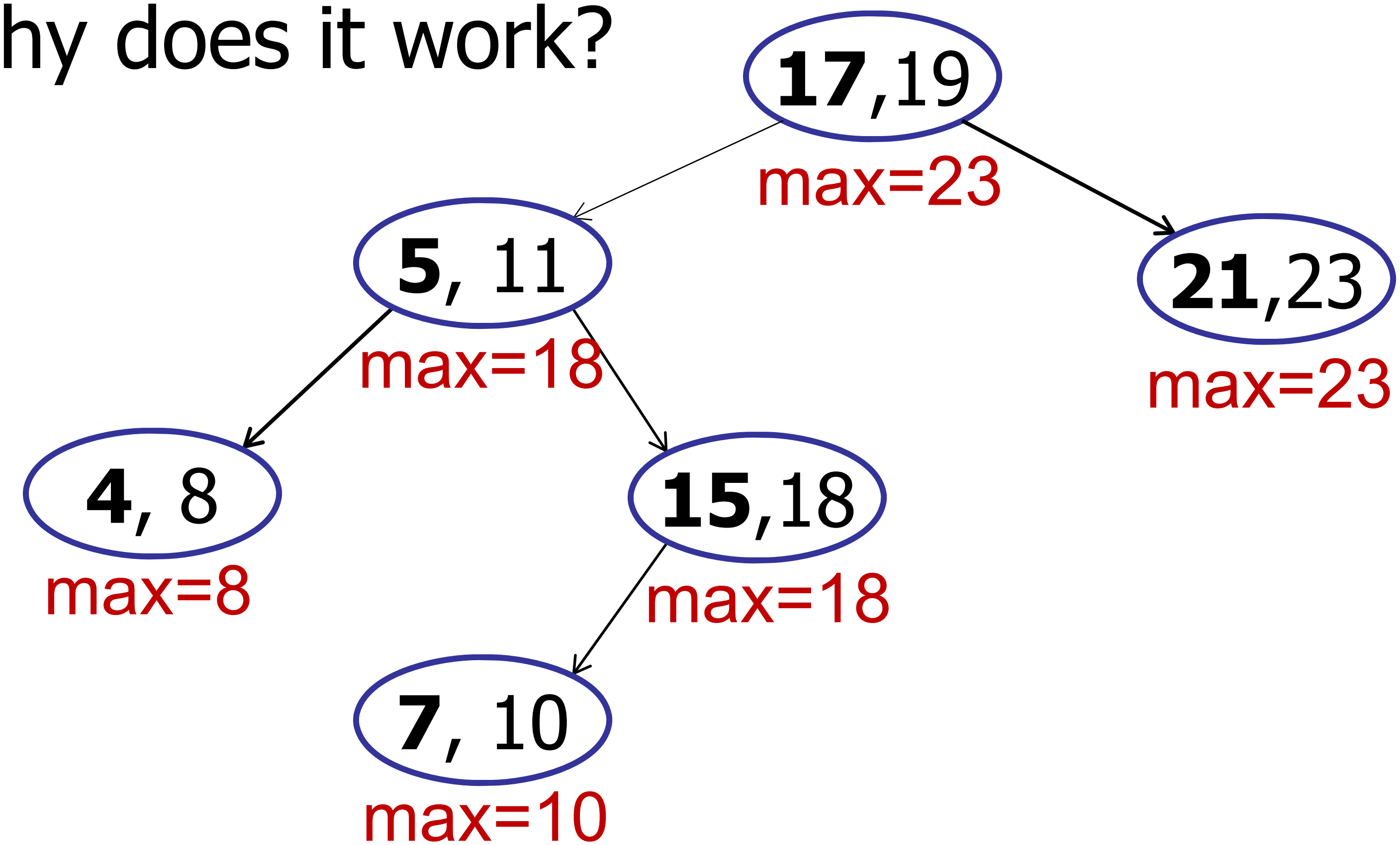
Go left: search(13) < 15 < 18

Tree sorted by left endpoint.

13 < every interval in right subtree

Interval Trees

Why does it work?



Claim: If search goes left and no overlap, then key < every interval in right sub-tree.

Interval Trees

If search goes right: then no overlap in left subtree.

➔ Either search finds key in right subtree or it is not in the tree.

If search goes left: if there is no overlap in left subtree, then there is no overlap in right subtree either.

➔ Either search finds key in left subtree or it is not in the tree.

Conclusion: search finds an overlapping interval, if it exists.

The running time of interval-search is:

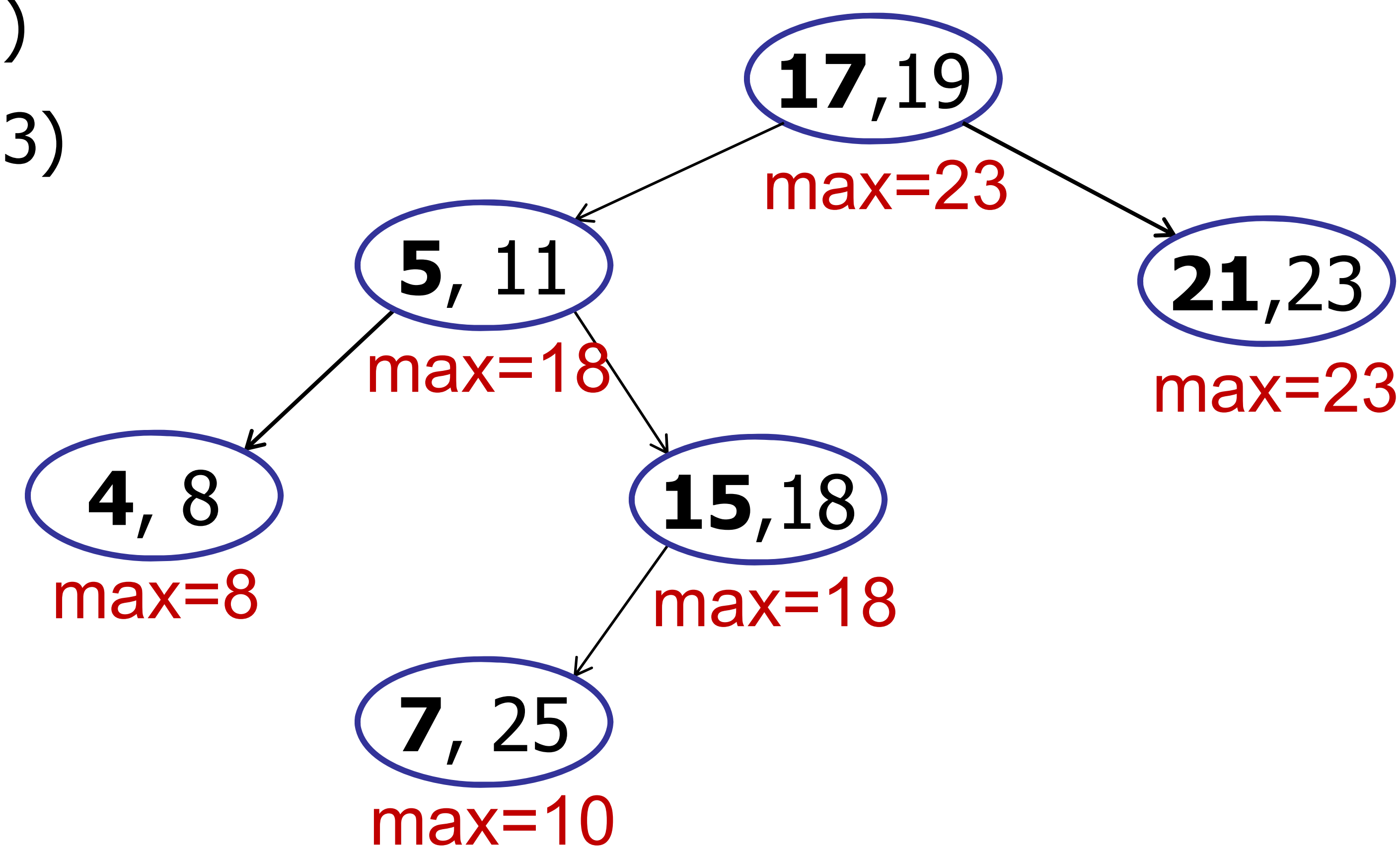
1. $O(1)$
2. $\boxed{O(\log n)}$
3. $O(n)$
4. $O(n \log n)$
5. $O(n^2)$
6. Can't say.

Interval Trees

Extension: List all intervals that overlap with point?

E.g.: search(22) returns:

- (7,25)
- (21,23)



Interval Trees

Extension: List all intervals that overlap with point?

All-Overlaps Algorithm:

Repeat until no more intervals:

– Search for interval.

– Add to list.

– Delete interval.

Repeat for all intervals on list:

– Add interval back to tree.

The running time of All-Overlaps, if there are k overlapping intervals?

1. O(1)
2. O(k)
3. $\boxed{\text{O(k log n)}}$
4. O(k + log n)
5. O(kn)
6. O(kn log n)

Interval Trees

Extension: List all intervals that overlap with point?

All-Overlaps Algorithm: O(k log n)

Repeat until no more intervals:

- Search for interval.
- Add to list.
- Delete interval.

Repeat for all intervals on list:

- Add interval back to tree.

Best known solution: O(k + log n)

Today

Three examples of augmenting BSTs

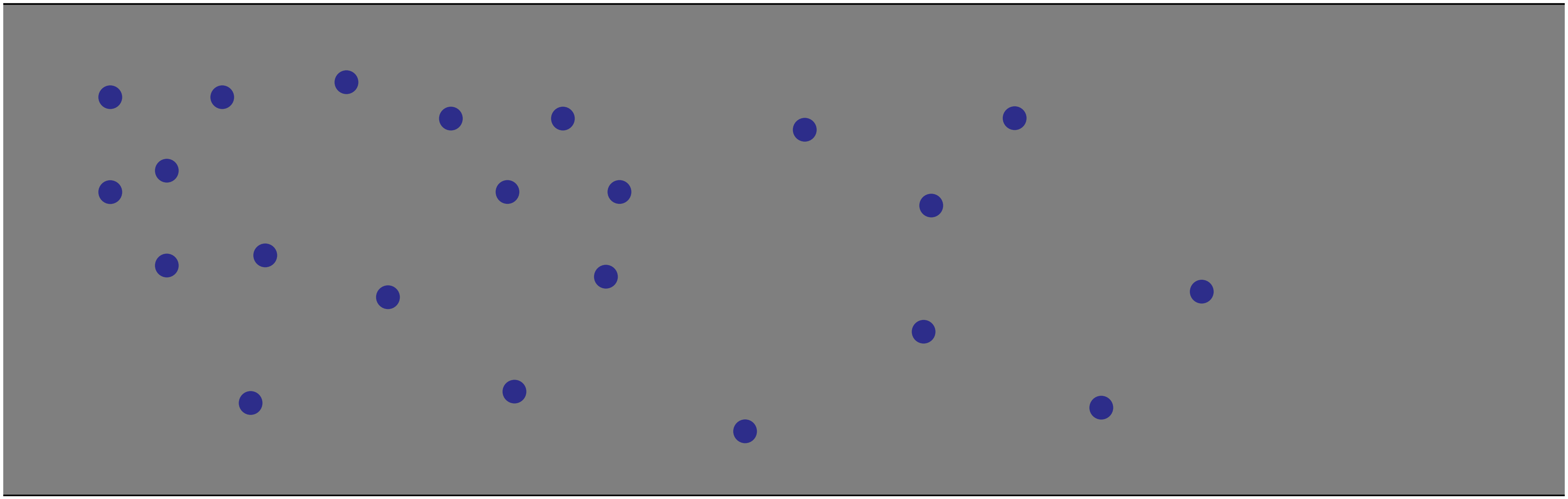
1. Order Statistics

2. Intervals

3. Orthogonal Range Searching

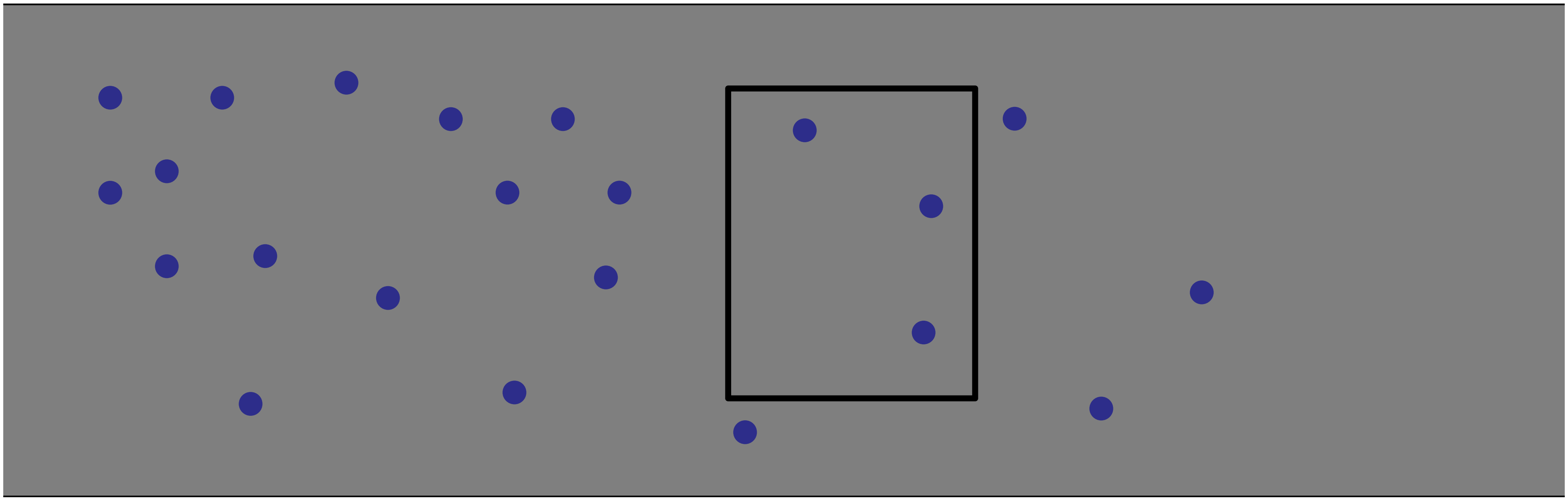
Orthogonal Range Searching

Input: n points in a 2d plane



Orthogonal Range Searching

Input: n points in a 2d plane

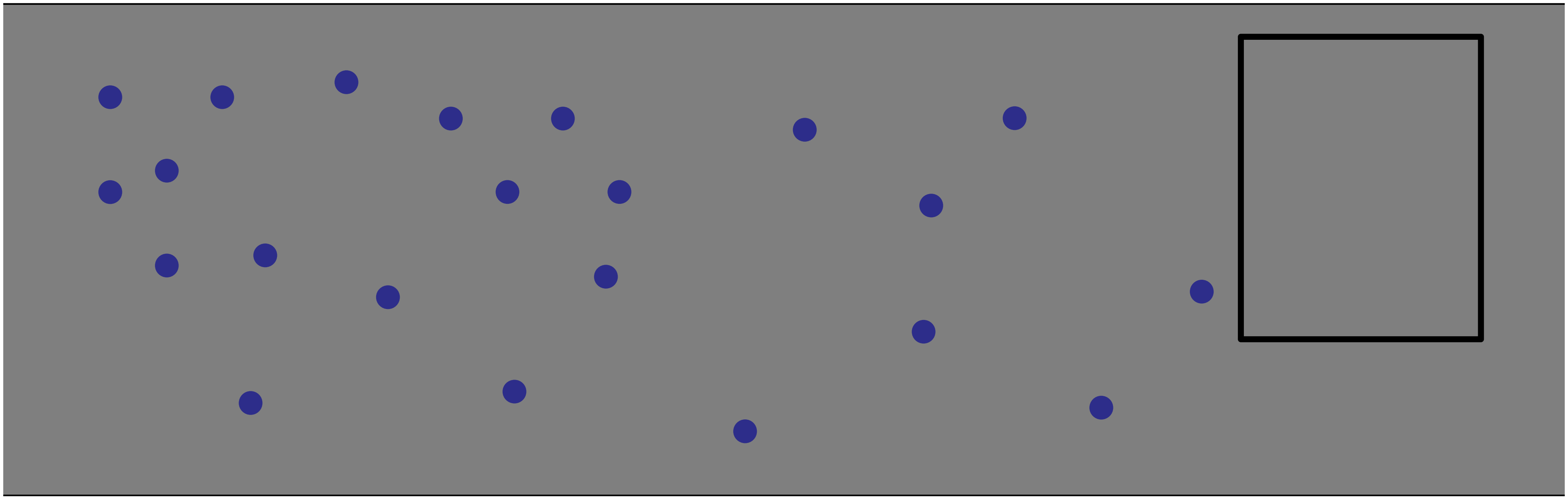


Query: Box

- Contains at least one point?
- How many?

Orthogonal Range Searching

Input: n points in a 2d plane

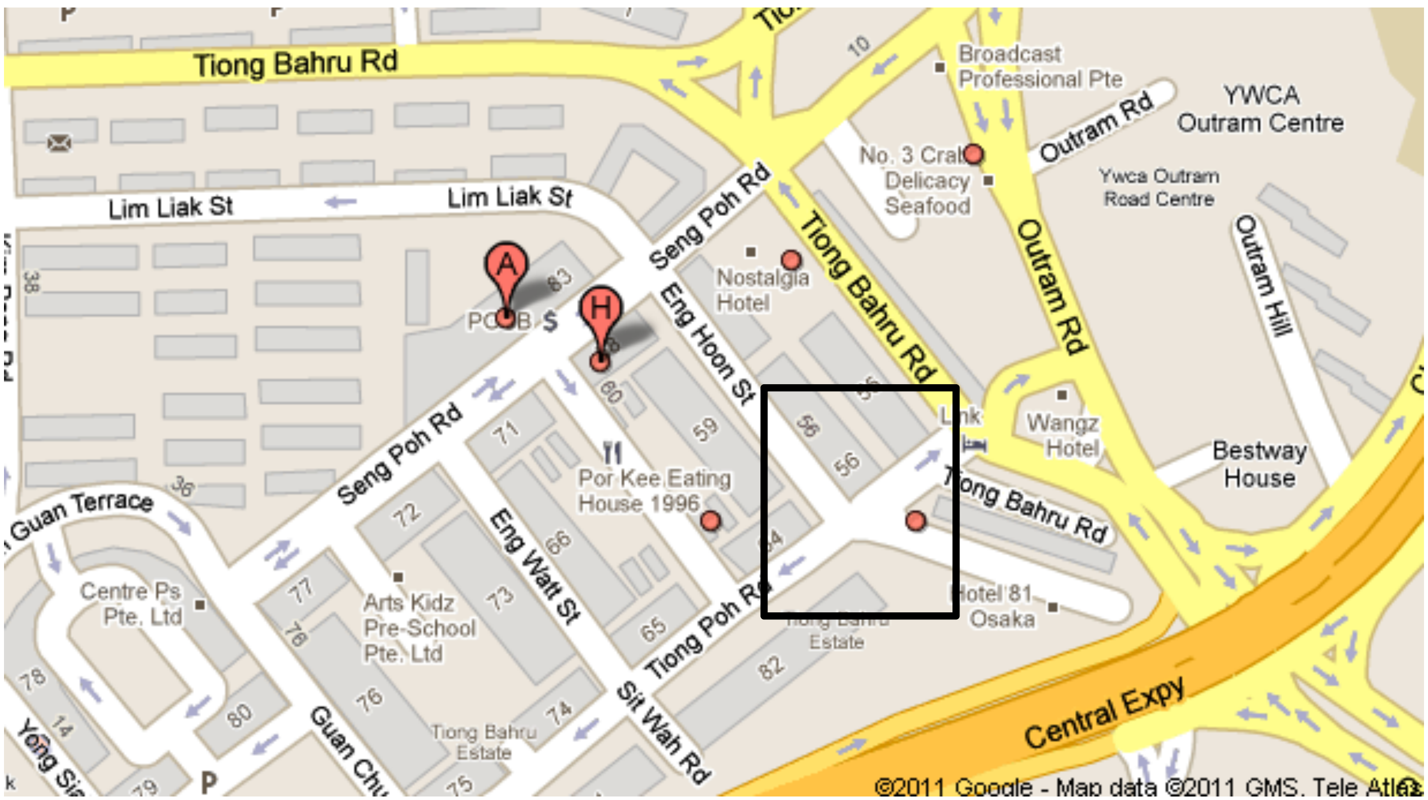


Query: Box

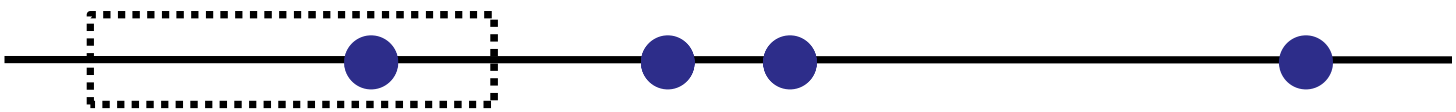
- Contains at least one point?
- How many?

Practical Example

Are there any good restaurants within one block of me?



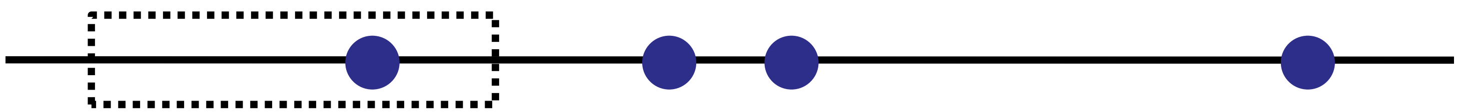
One Dimension



One Dimension

Range Queries

- Important in databases
- "Find me everyone between ages 22 and 27."



One Dimension

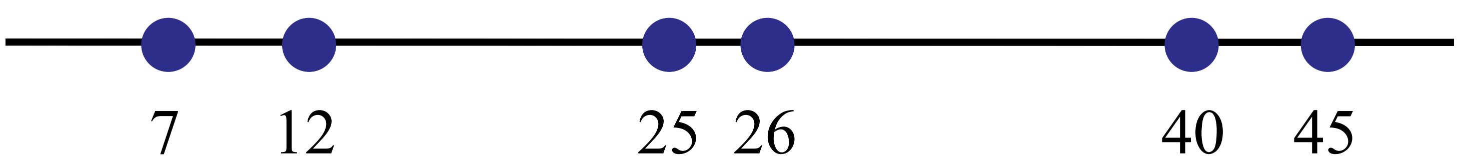
Strategy:

1. Use a binary search tree.

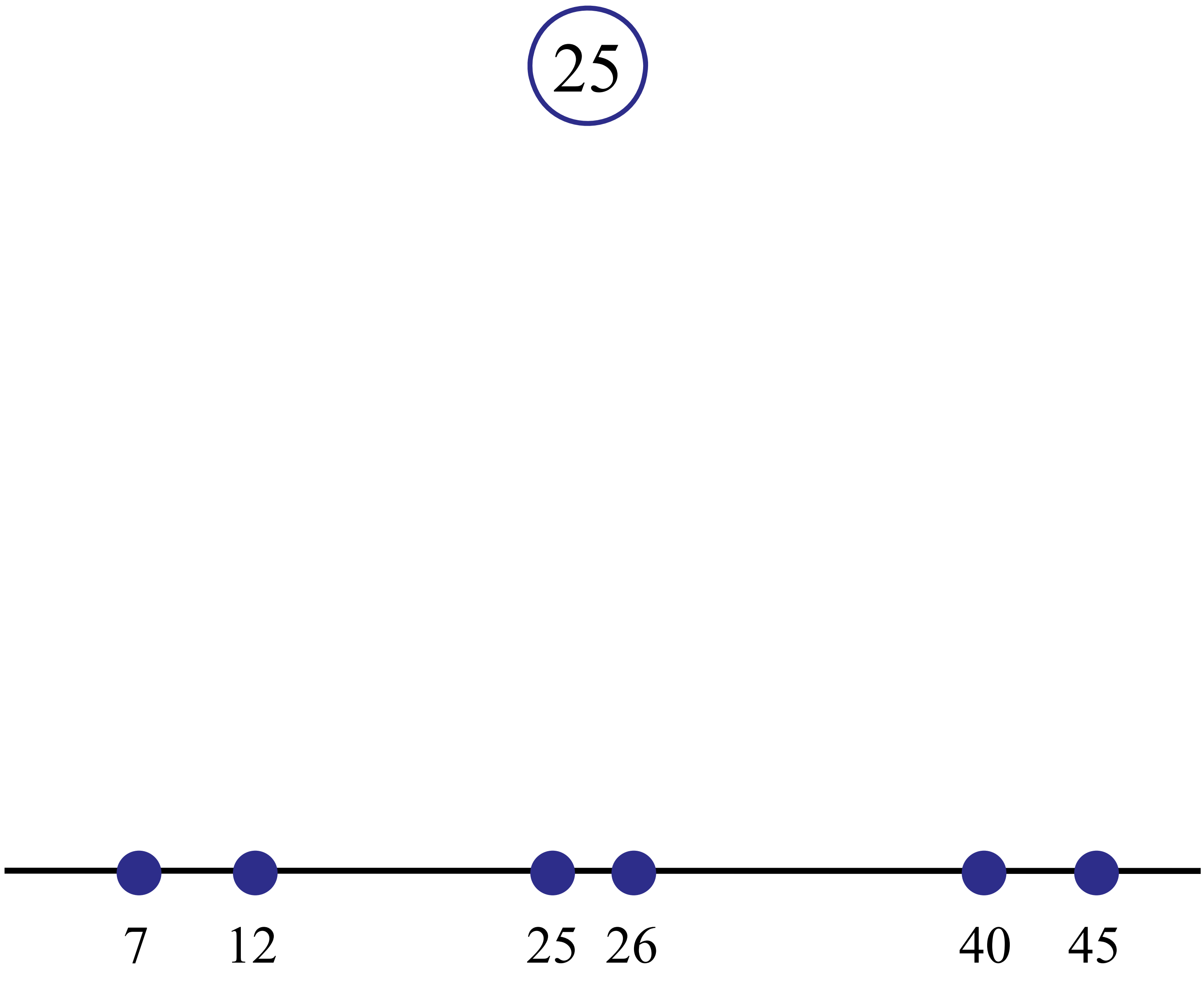
2. Store all points in the leaves of the tree. (Internal nodes store only copies.)

3. Each internal node v stores the MAX of any leaf in the left sub-tree.

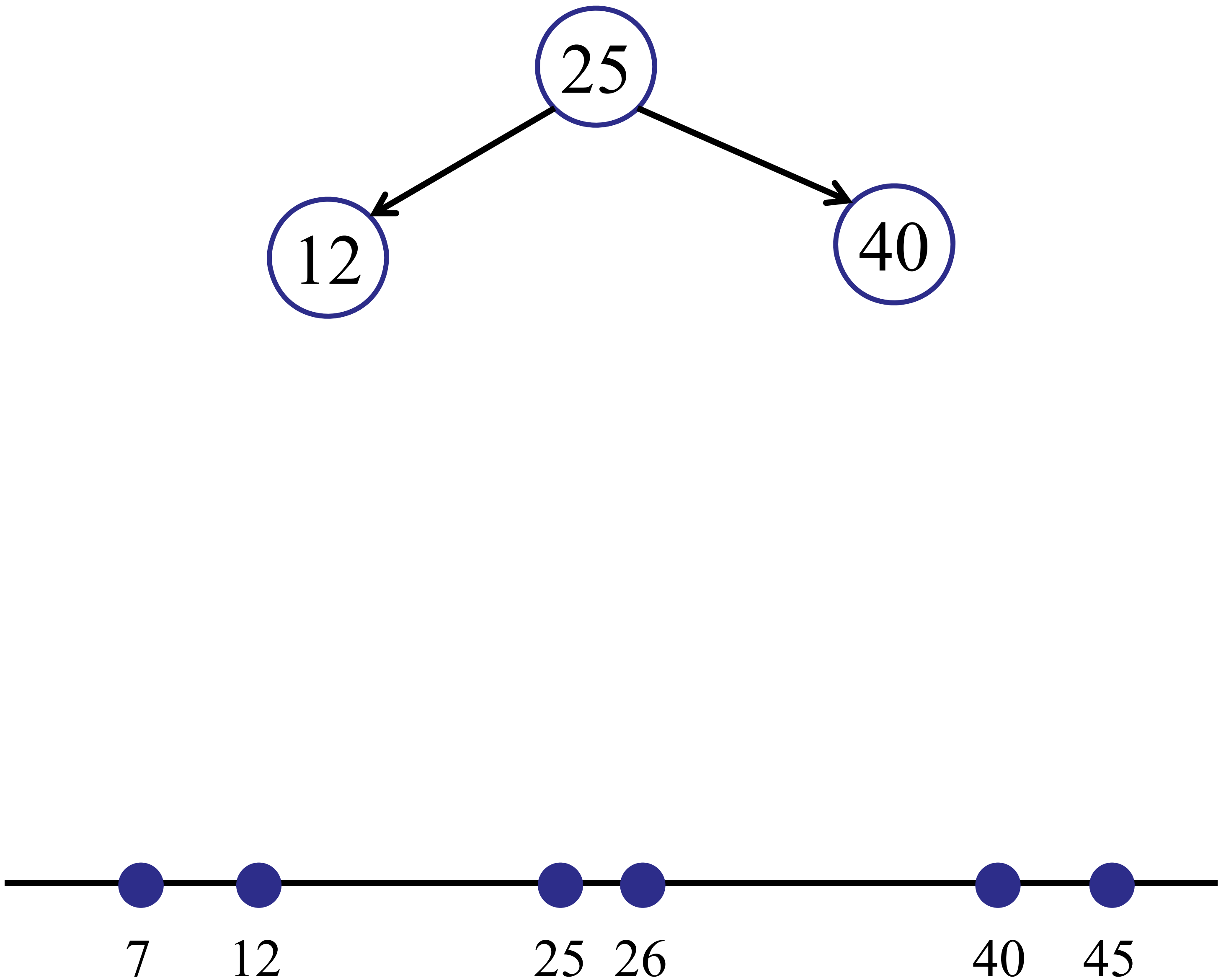
Example



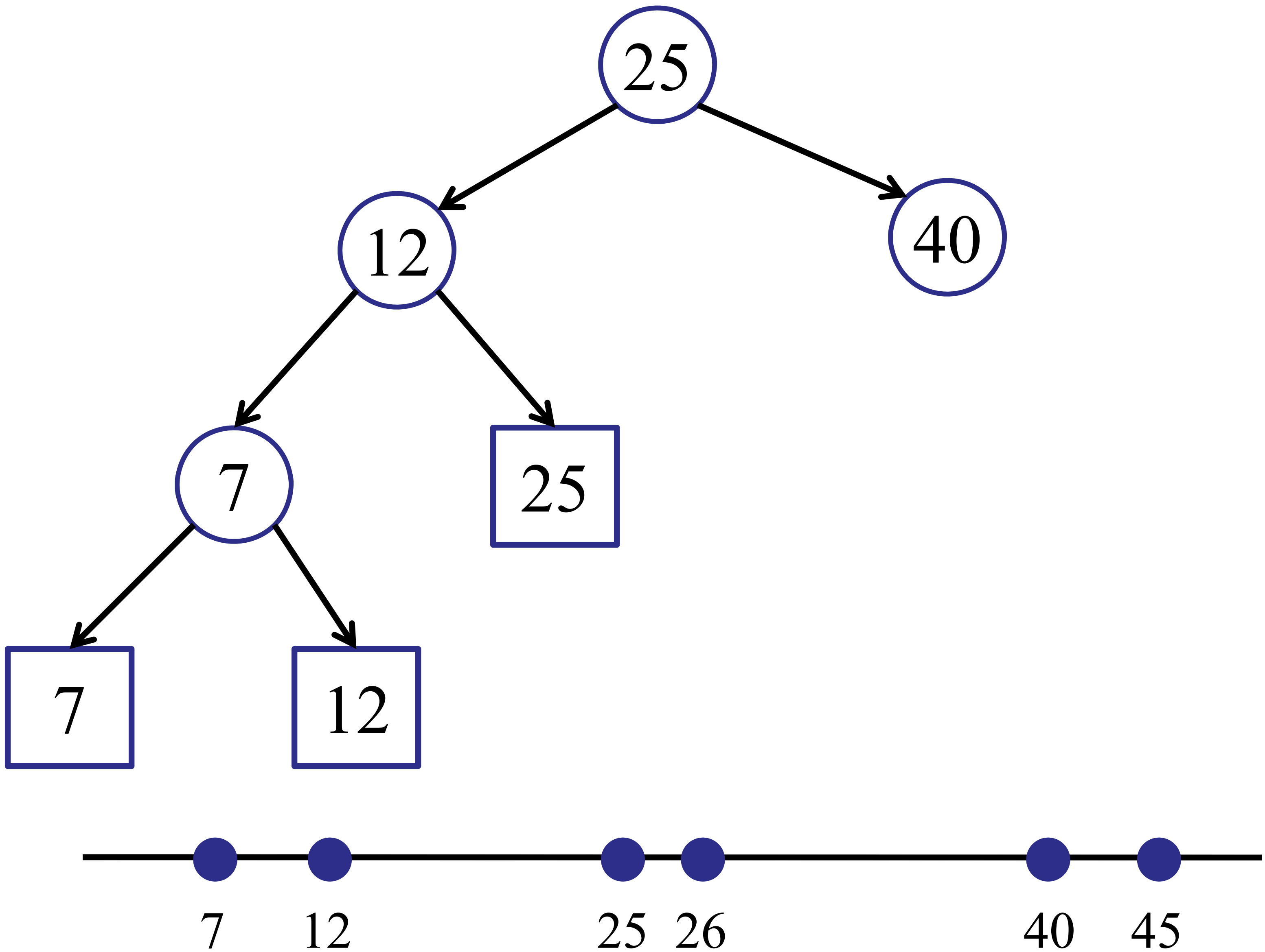
Example



Example

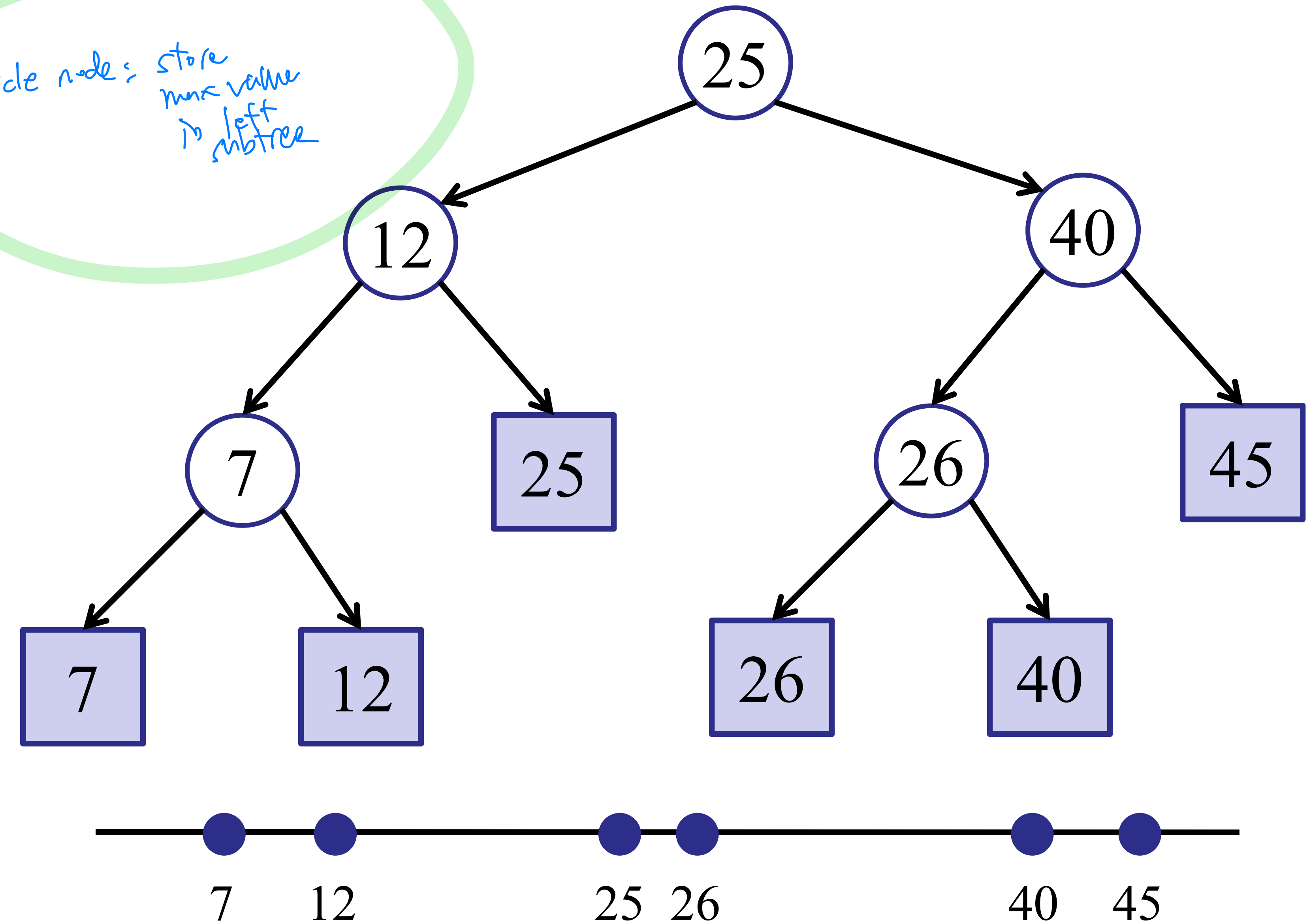


Example

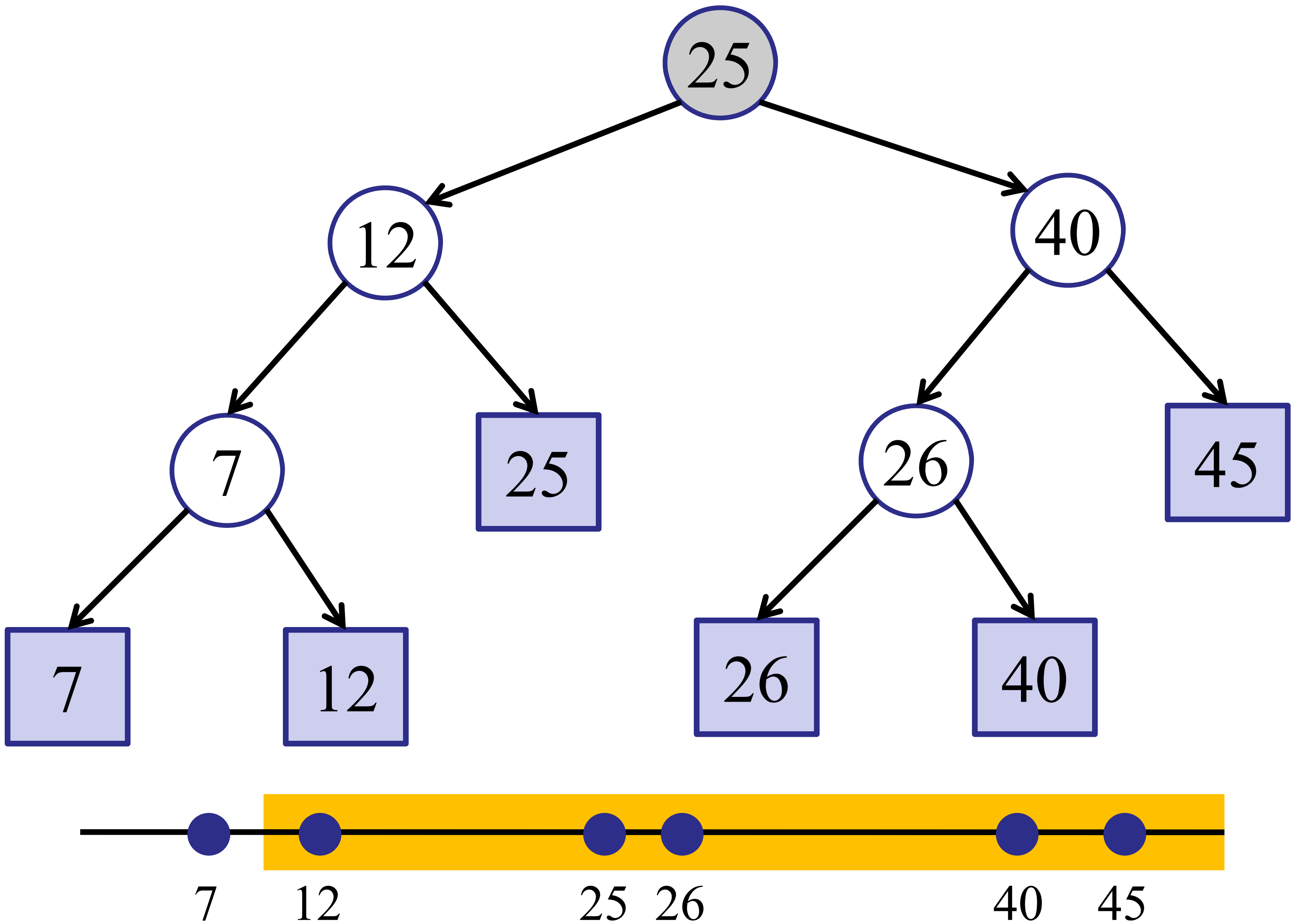


Note: BST Property

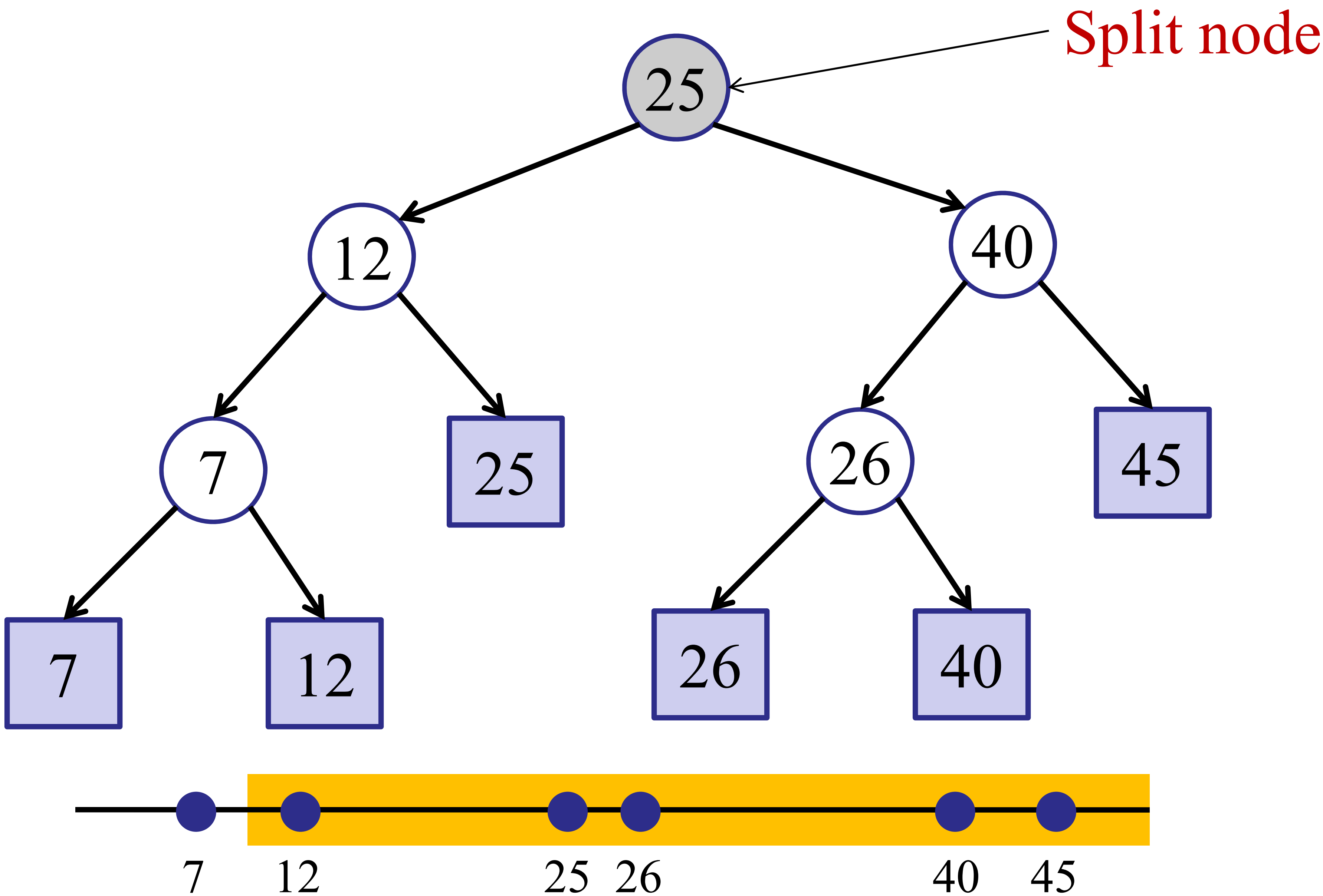
circle node: store max value in left subtree



Example: query(10, 50)



Example: query(10, 50)



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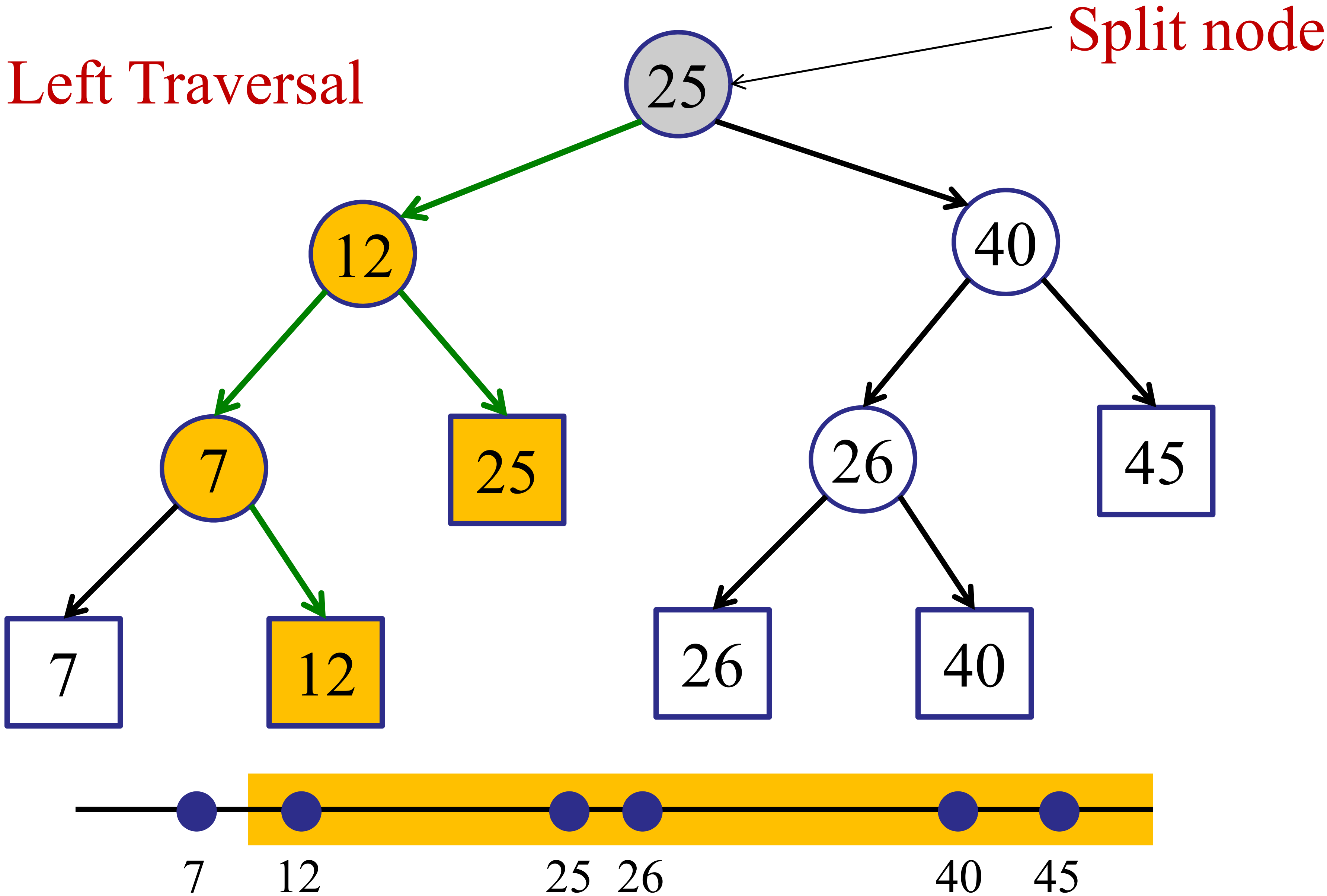

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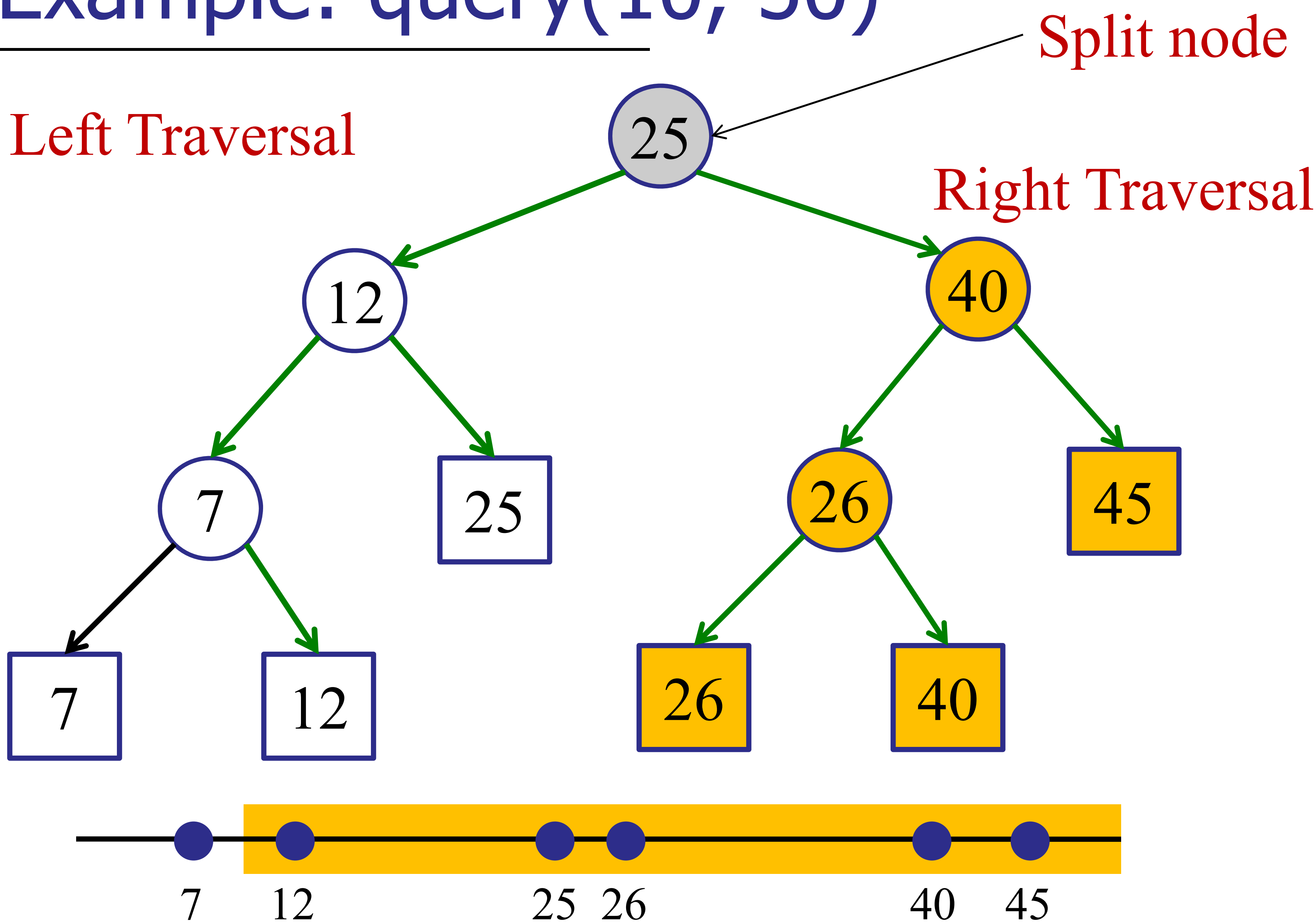
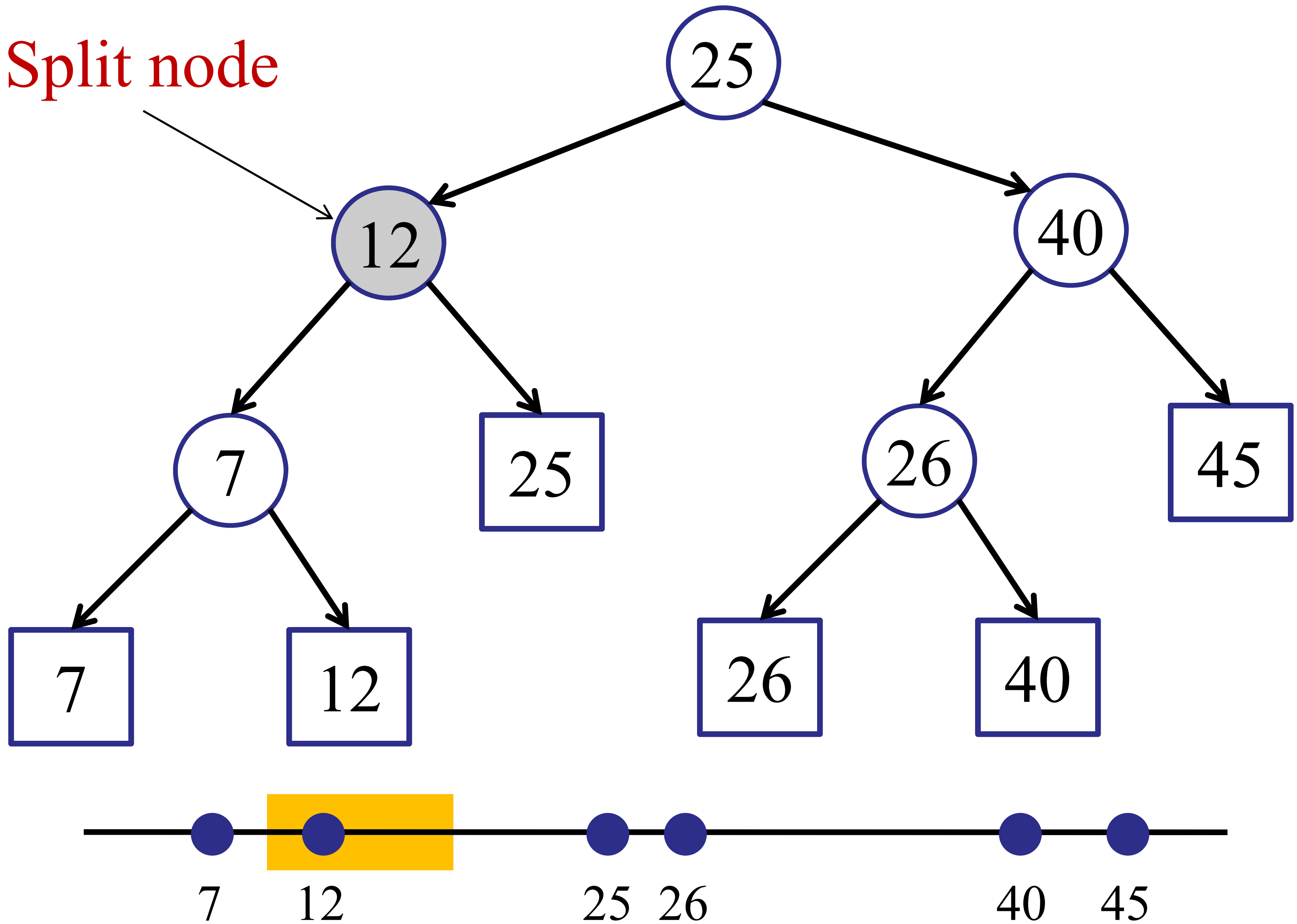
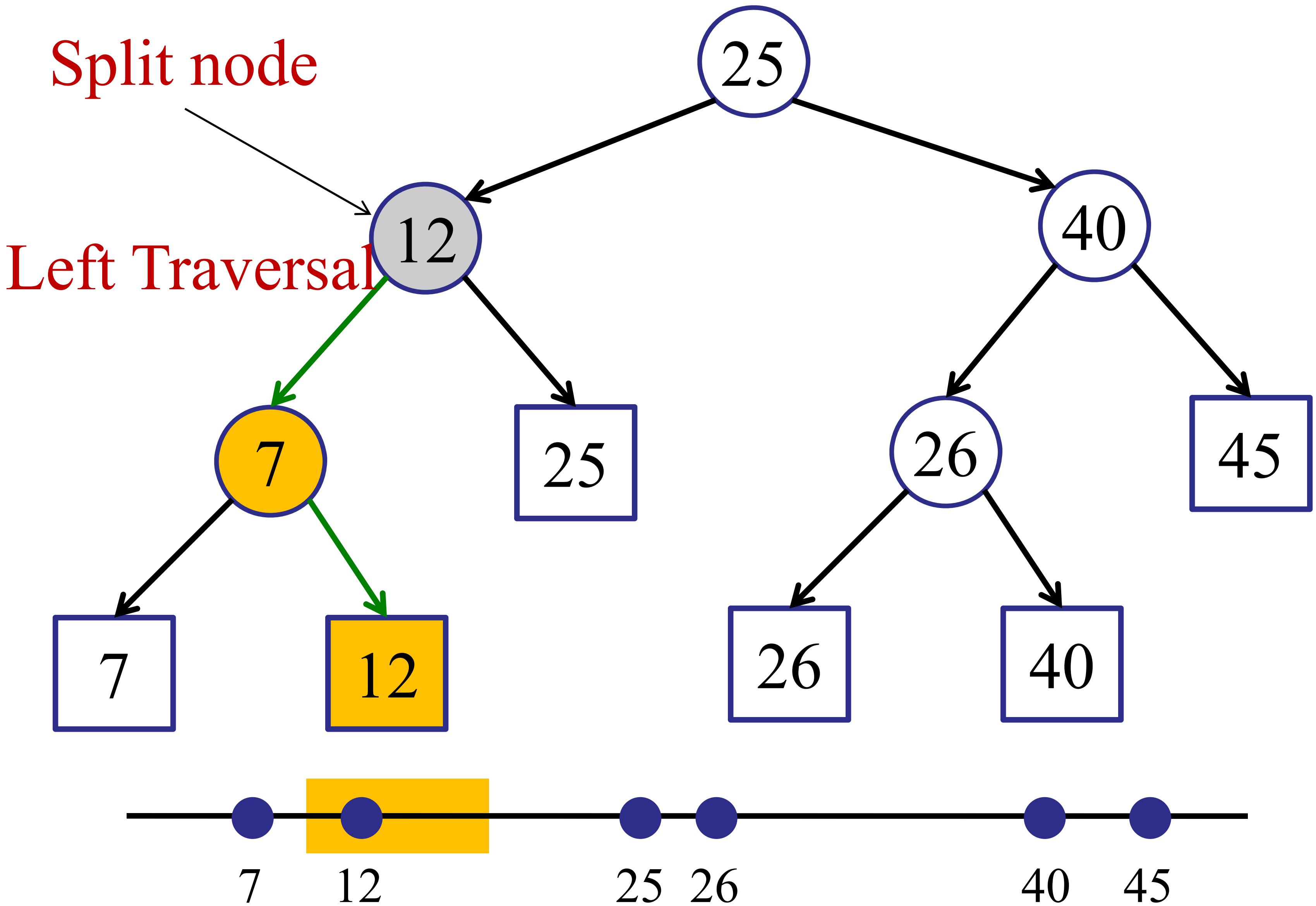


Example: query(8, 20)



Example: query(8, 20)



One Dimensional Range Queries

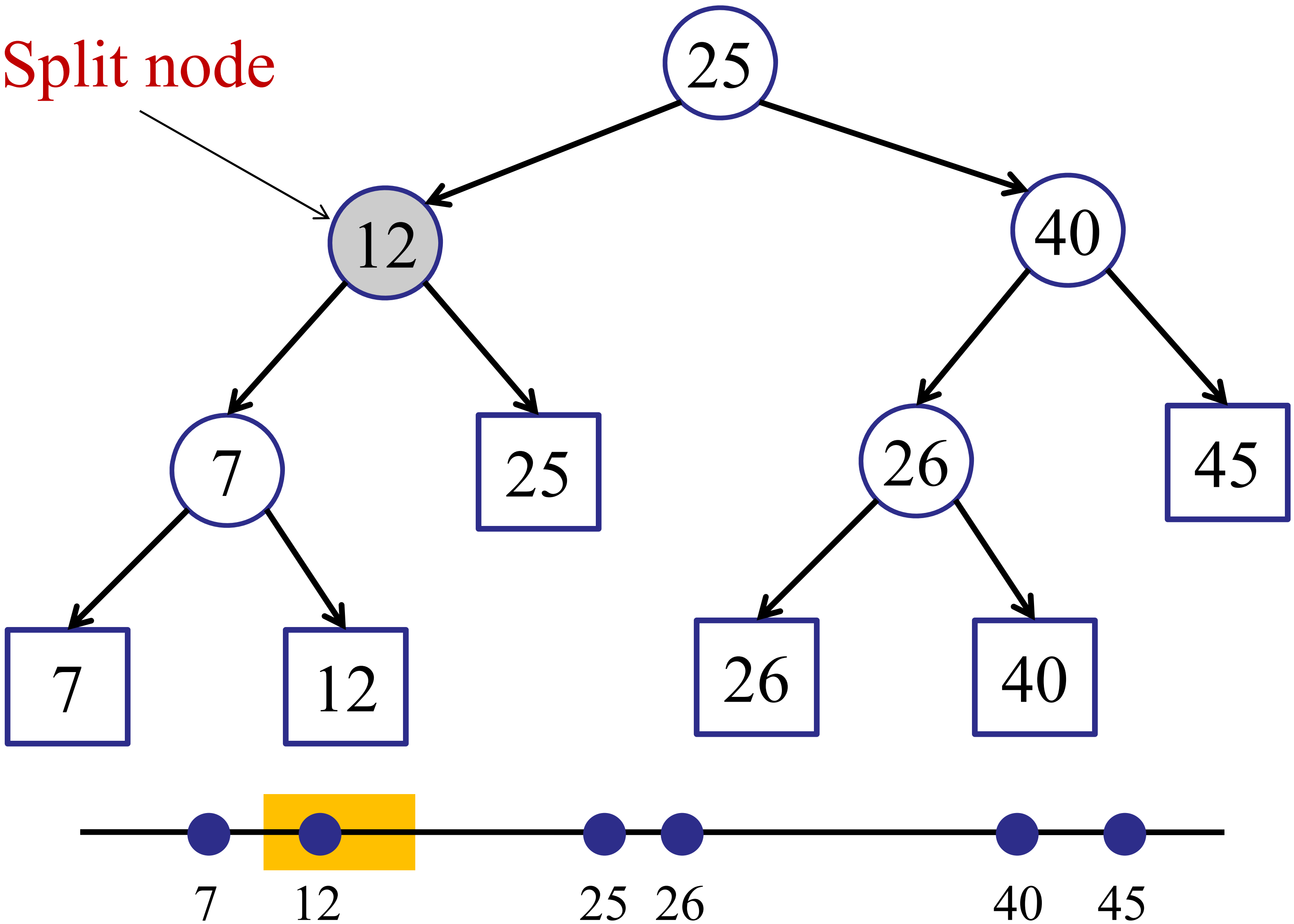
Algorithm:

- Find “split” node.
- Do left traversal.
- Do right traversal.

One Dimensional Range Queries

```
FindSplit(low, high)
    v = root;
    done = false;
    while !done {
        if (high <= v.key) then v=v.left;
        else if (low > v.key) then v=v.right;
        else (done = true);
    }
    return v;
```

Example: query(8, 20)



One Dimensional Range Queries

Algorithm:

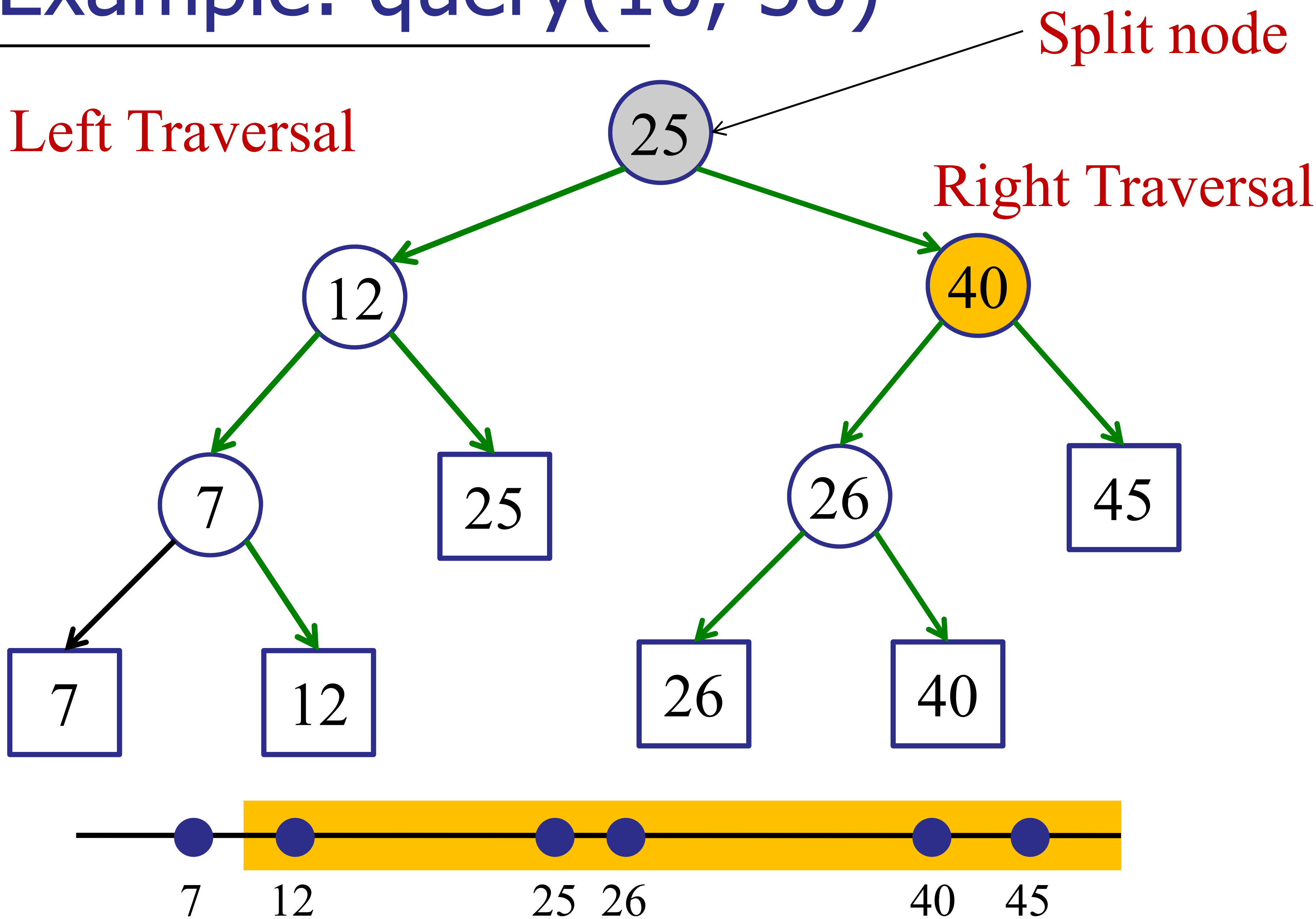
- v = FindSplit(low, high);
- LeftTraversal(v, low, high);
- RightTraversal(v, low, high);

One Dimensional Range Queries

```
RightTraversal(v, low, high)
    if (v.key <= high) {
        all-leaf-traversal(v.left);
        RightTraversal(v.right, low, high);
    }
    else {
        RightTraversal(v.left, low, high);
    }
}
```

low high

Example: query(10, 50)



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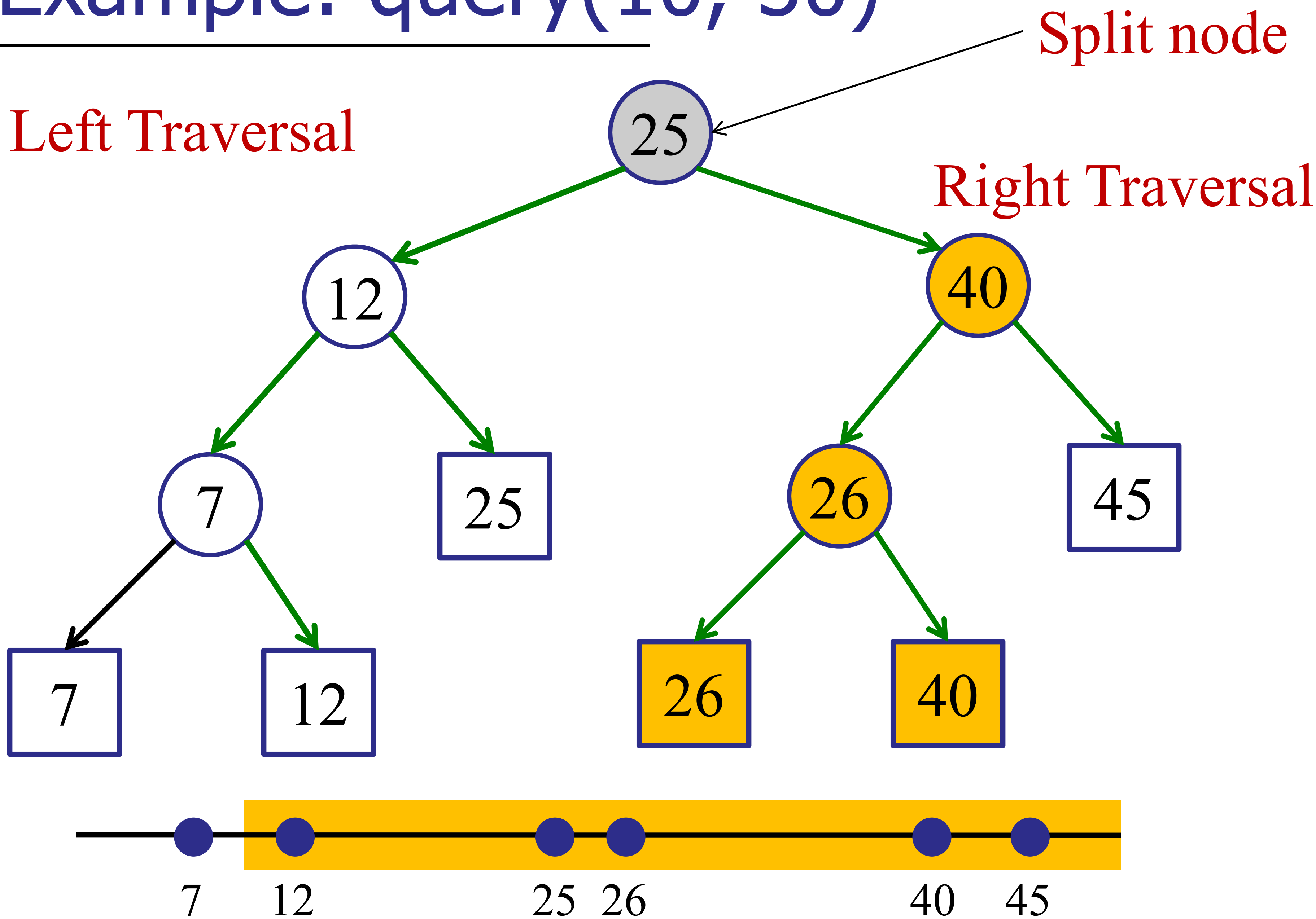
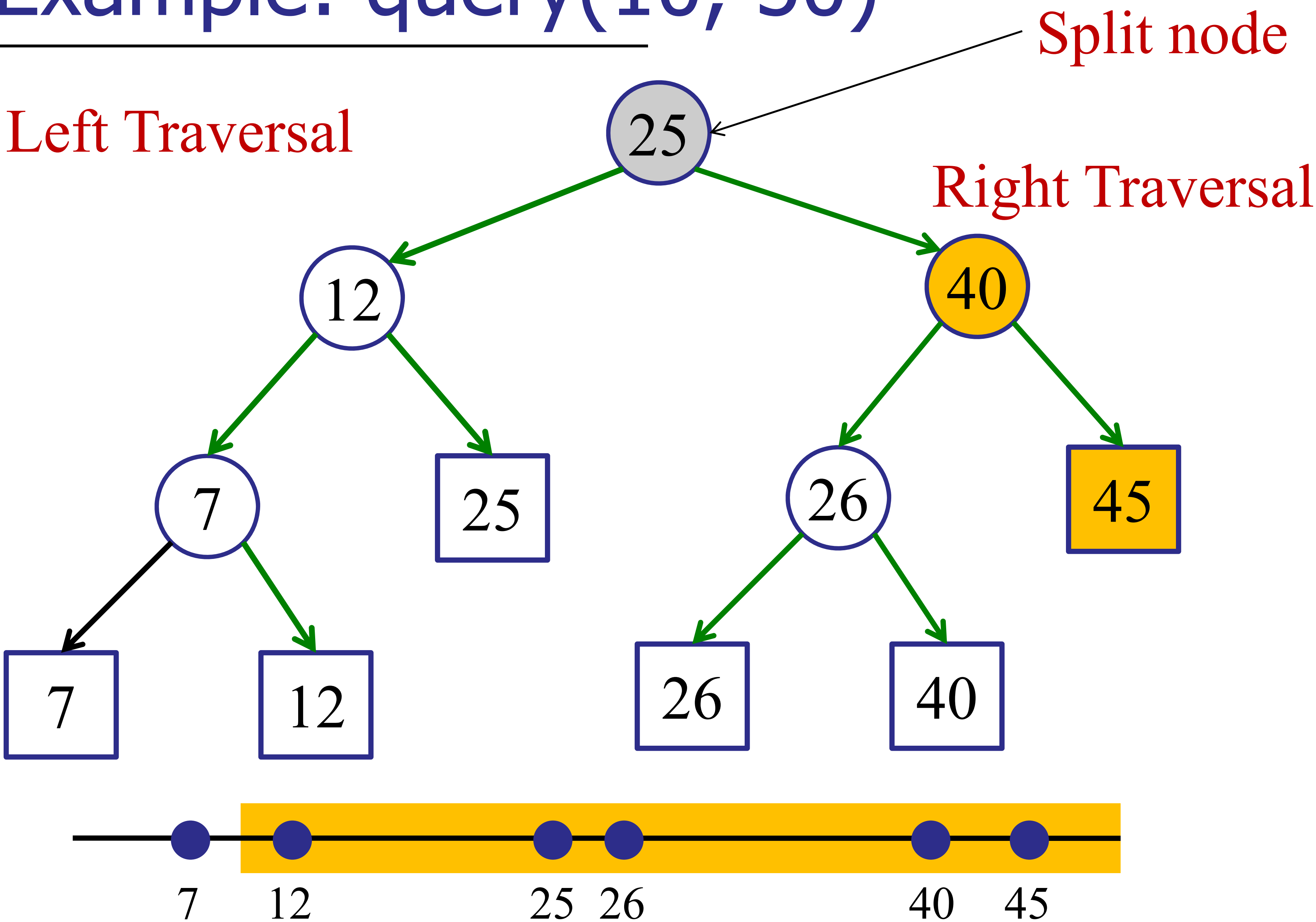


One Dimensional Range Queries

```
RightTraversal(v, low, high)
    if (v.key <= high) {
        all-leaf-traversal(v.left);
        RightTraversal(v.right, low, high);
    }
    else {
        RightTraversal(v.left, low, high);
    }
}
```

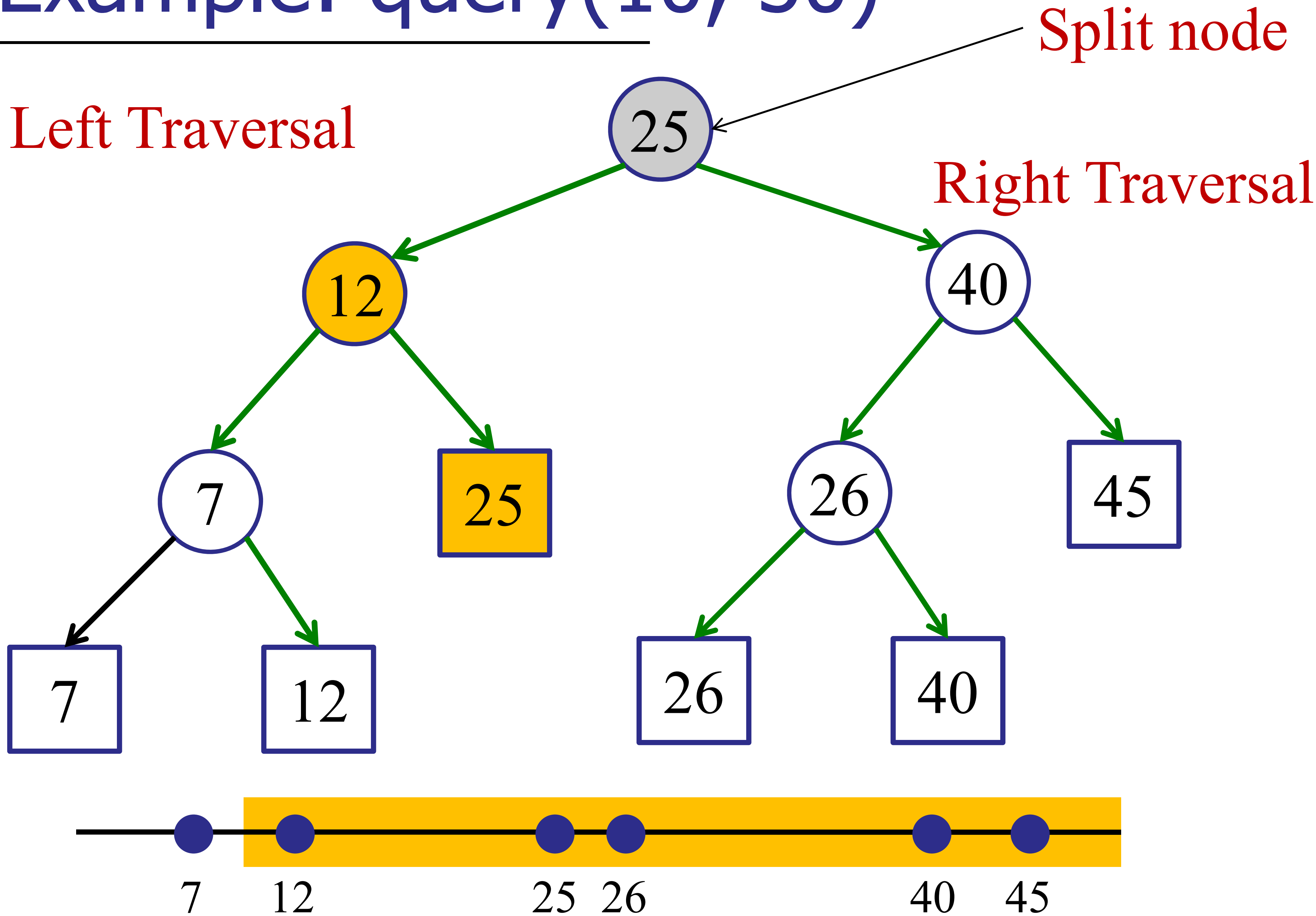
Example: query(10, 50)



One Dimensional Range Queries

```
LeftTraversal(v, low, high)
    if (low <= v.key) {
        all-leaf-traversal(v.right);
        LeftTraversal(v.left, low, high);
    }
    else {
        LeftTraversal(v.right, low, high);
    }
}
```

Example: query(10, 50)



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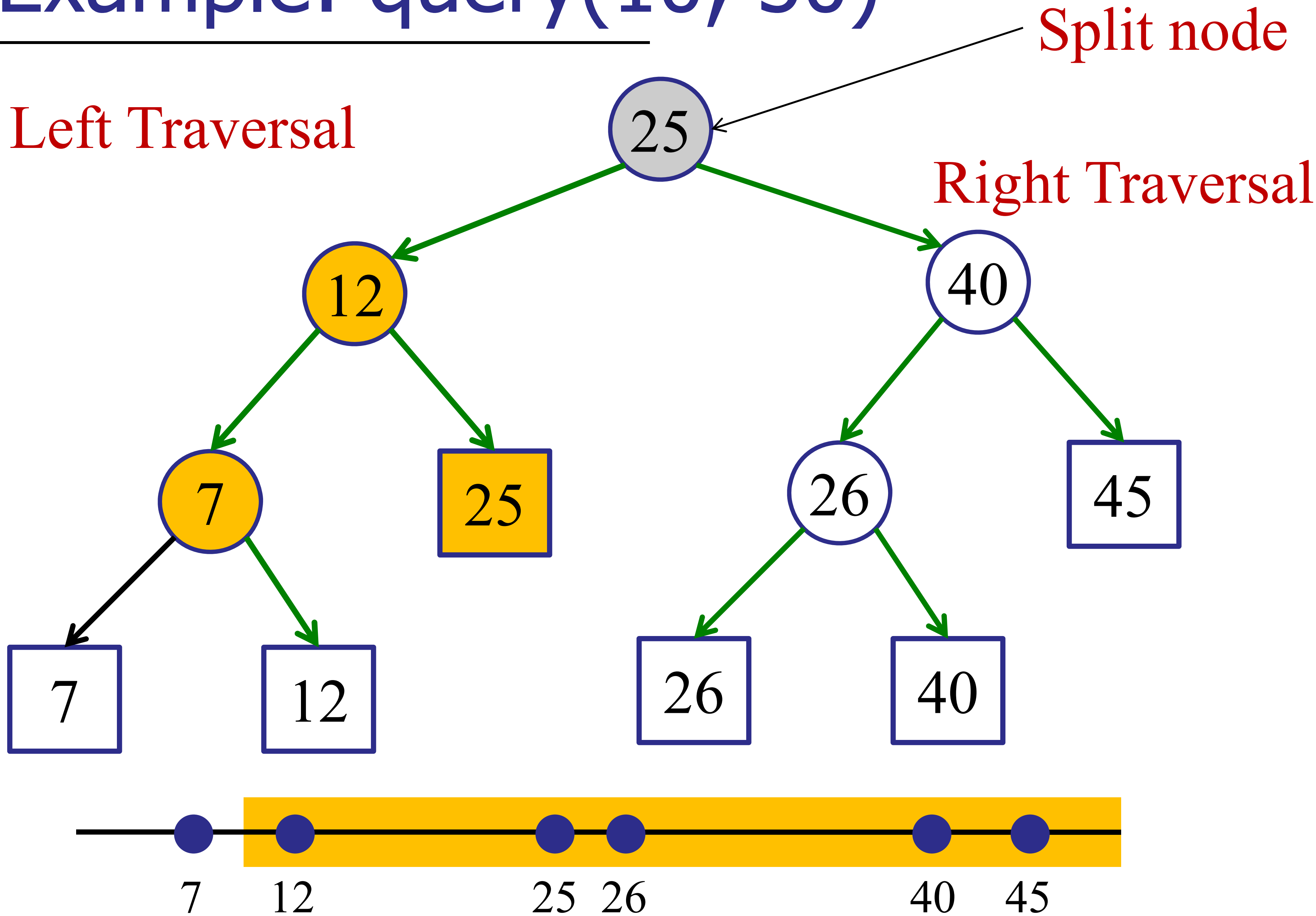
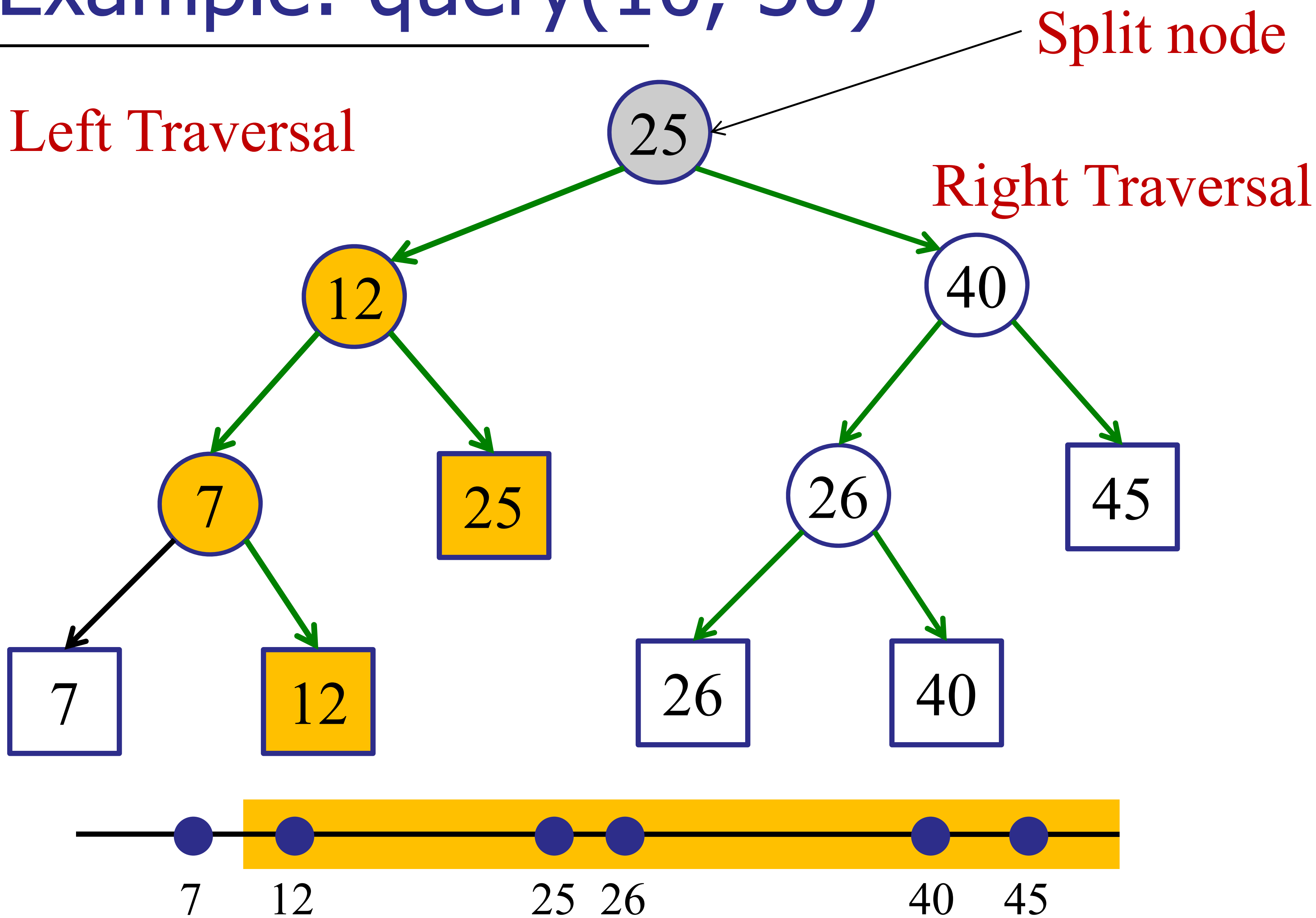


Example: query(10, 50)



One Dimensional Range Queries

```
LeftTraversal(v, low, high)
    if (low <= v.key) {
        all-leaf-traversal(v.right);
        LeftTraversal(v.left, low, high);
    }
    else {
        LeftTraversal(v.right, low, high);
    }
}
```

One Dimensional Range Queries

Algorithm:

- v = FindSplit(low, high);
- LeftTraversal(v, low, high);
- RightTraversal(v, low, high);

Analysis

Query time:

- Finding split node: O(log n)
- Left Traversal:

  At every step, we either:

  1. Output all right sub-tree and recurse left.
  2. Recurse right.
- Right Traversal:

  At every step, we either:

  1. Output all left sub-tree and recurse right.
  2. Recurse left.

Analysis

Left Traversal:

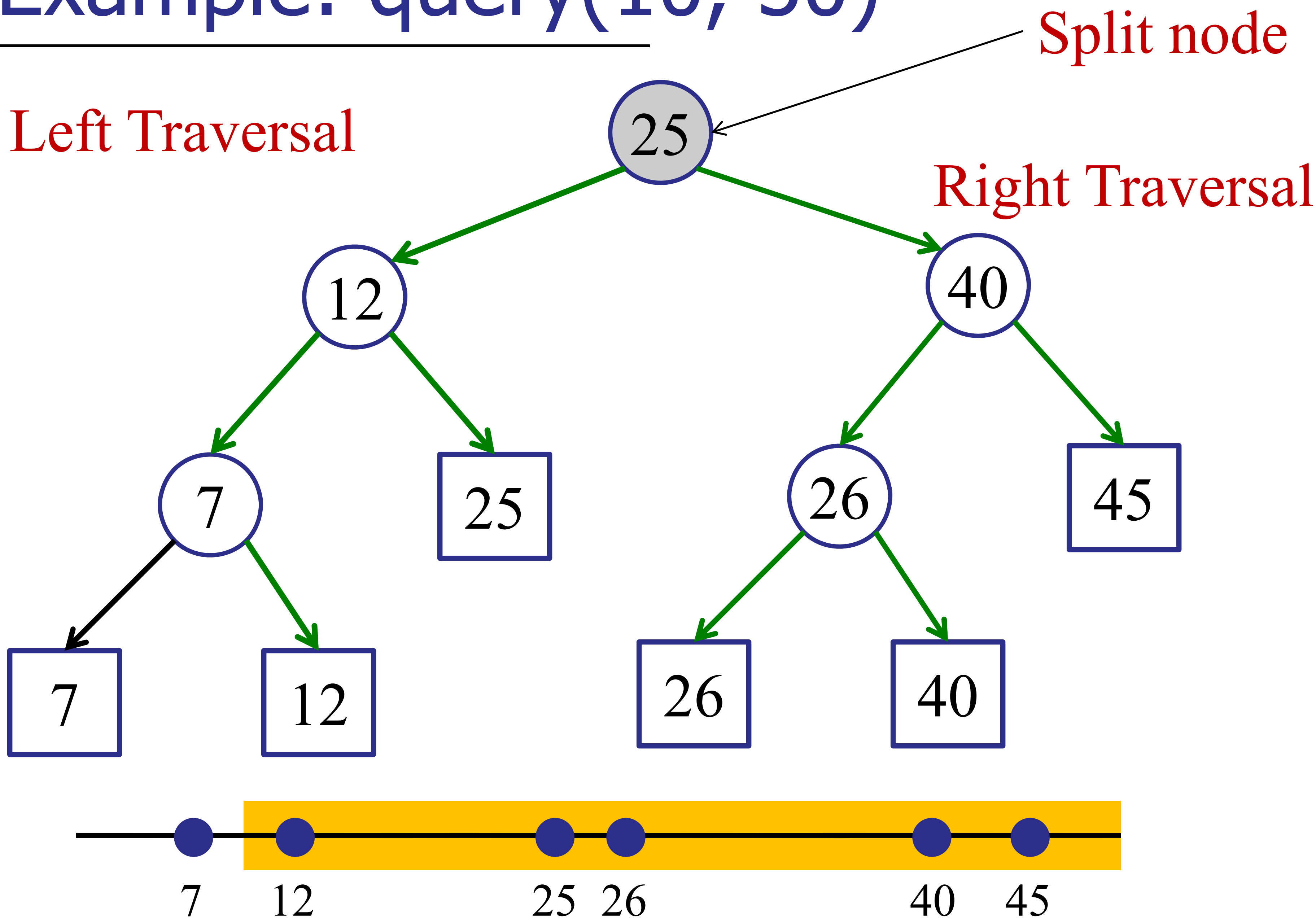
At every step, we either:

1. Output all right sub-tree and recurse left.
2. Recurse right.

Counting:

1. Recurse at most O(log n) times (i.e., option 2).
2. How expensive is "output all sub-tree" (i.e., option 1)?

Example: query(10, 50)



Analysis

Left Traversal:

At every step, we either:

1. Output all right sub-tree and recurse left.
2. Recurse right.

Counting:

1. Recurse at most O(log n) times (i.e., option 2).
2. How expensive is "output all sub-tree" (i.e., option 1)?

    ➔ O(k), where k is number of items found.

Analysis

Query time complexity:

$O(k + \log n)$

where k is the number of points found.

Preprocessing (buildtree) time complexity:

$O(n \log n)$

Total space complexity:

$O(n)$

One Dimensional Range Queries

What if you just want to know *how many* points are in the range?

One Dimensional Range Queries

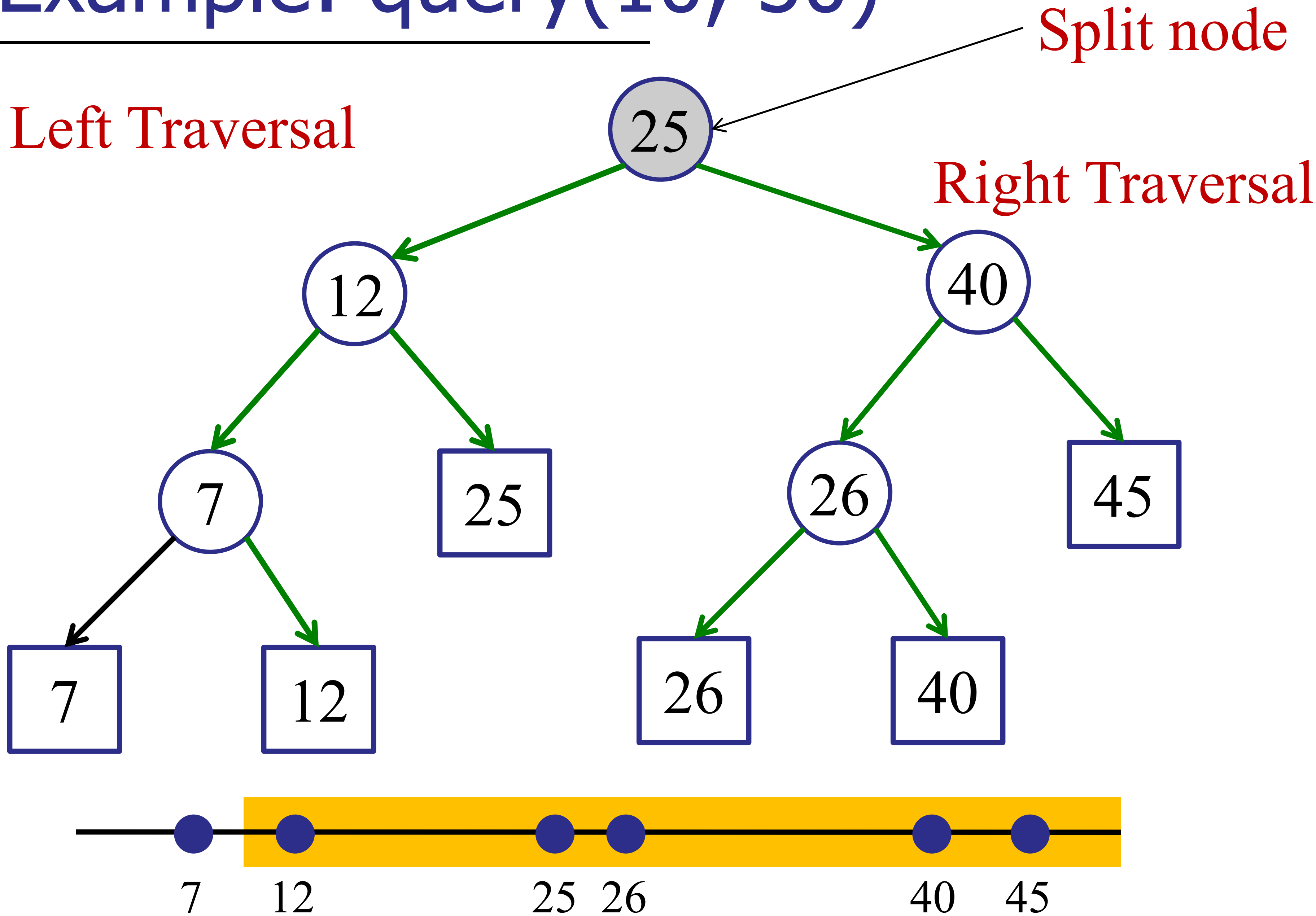
What if you just want to know *how many* points are in the range?

- Augment the tree!
- Keep a count of the number of nodes in each sub-tree.
- Instead of walking entire sub-tree, just remember the count.

One Dimensional Range Queries

```
LeftTraversal(v, low, high)
        if (low <= v.key) {
                all-leaf-traversal(v.right);
                total += v.right.count;
                LeftTraversal(v.left, low, high);
        }
        else {
                LeftTraversal(v.right, low, high);
        }
}
```

Example: query(10, 50)



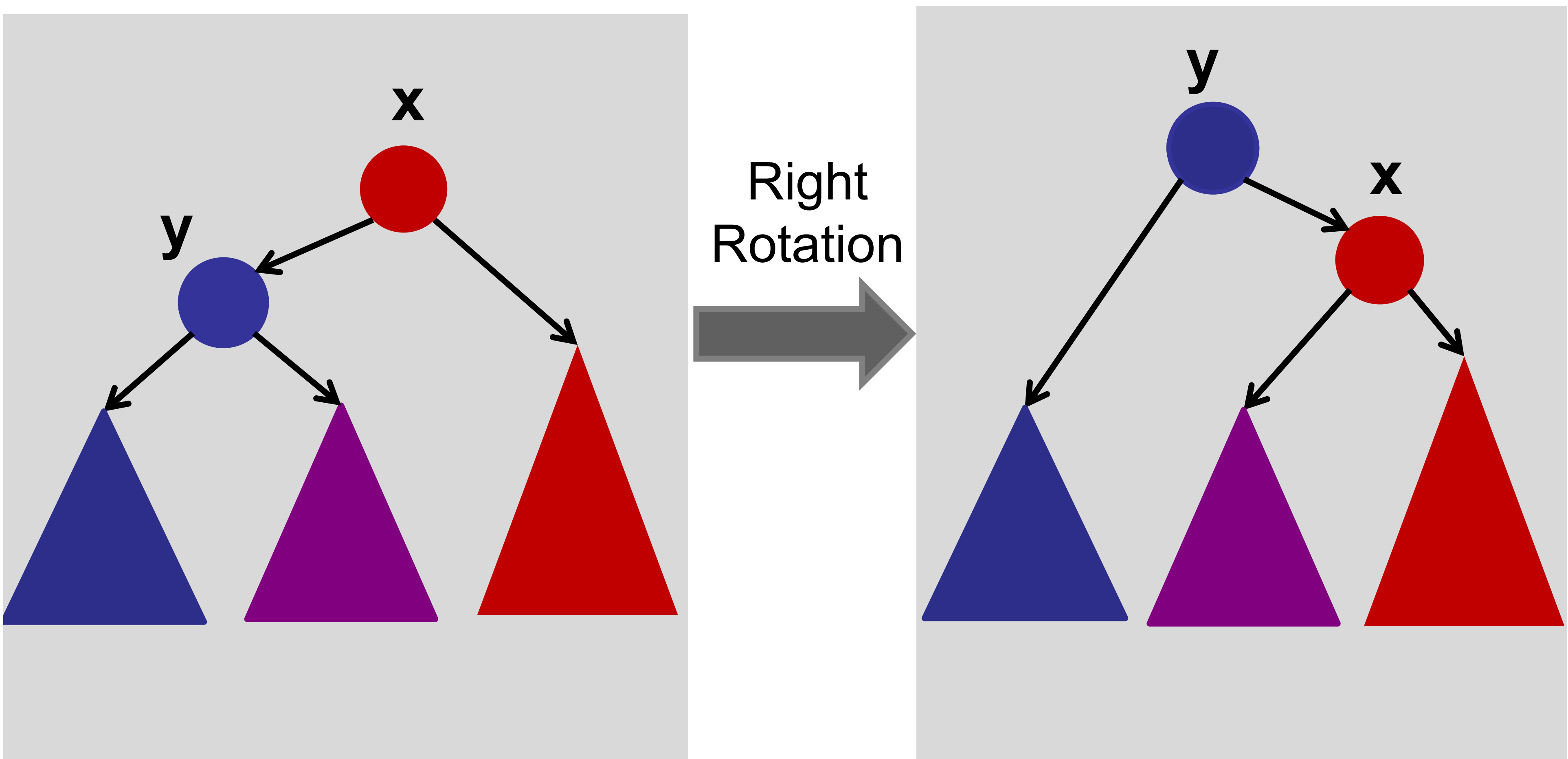
1D Range Tree

Done??

One Dimensional Range Queries

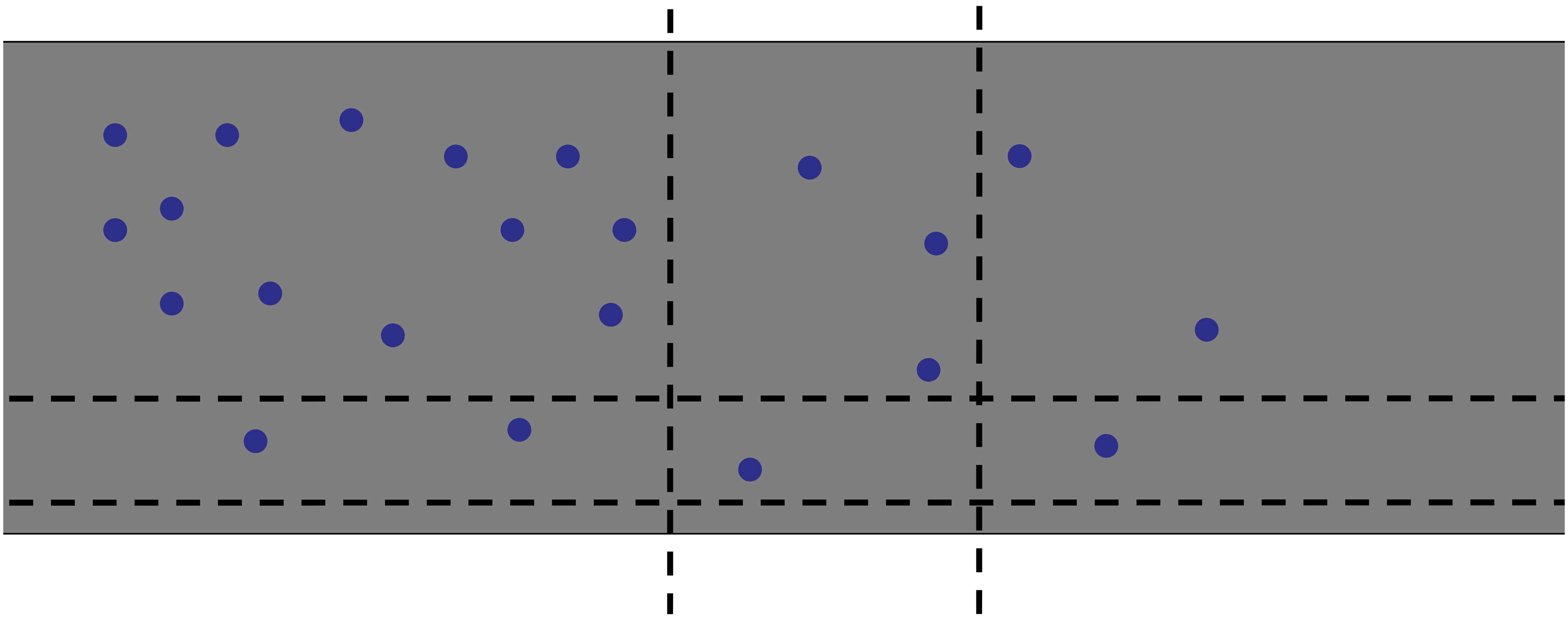
What about dynamic updates?

- Need to verify rotations!



Two Dimensional Range Tree

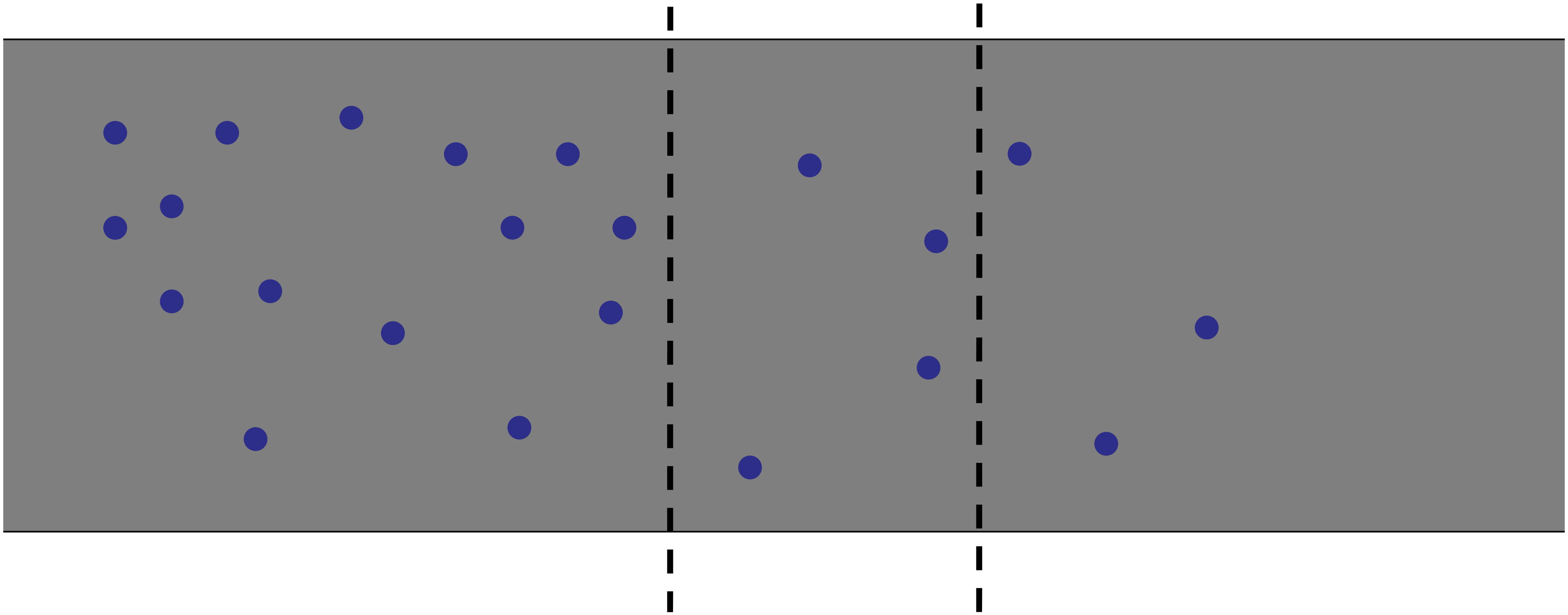
Ex: search for all points between dashed lines.



Two Dimensional Range Tree

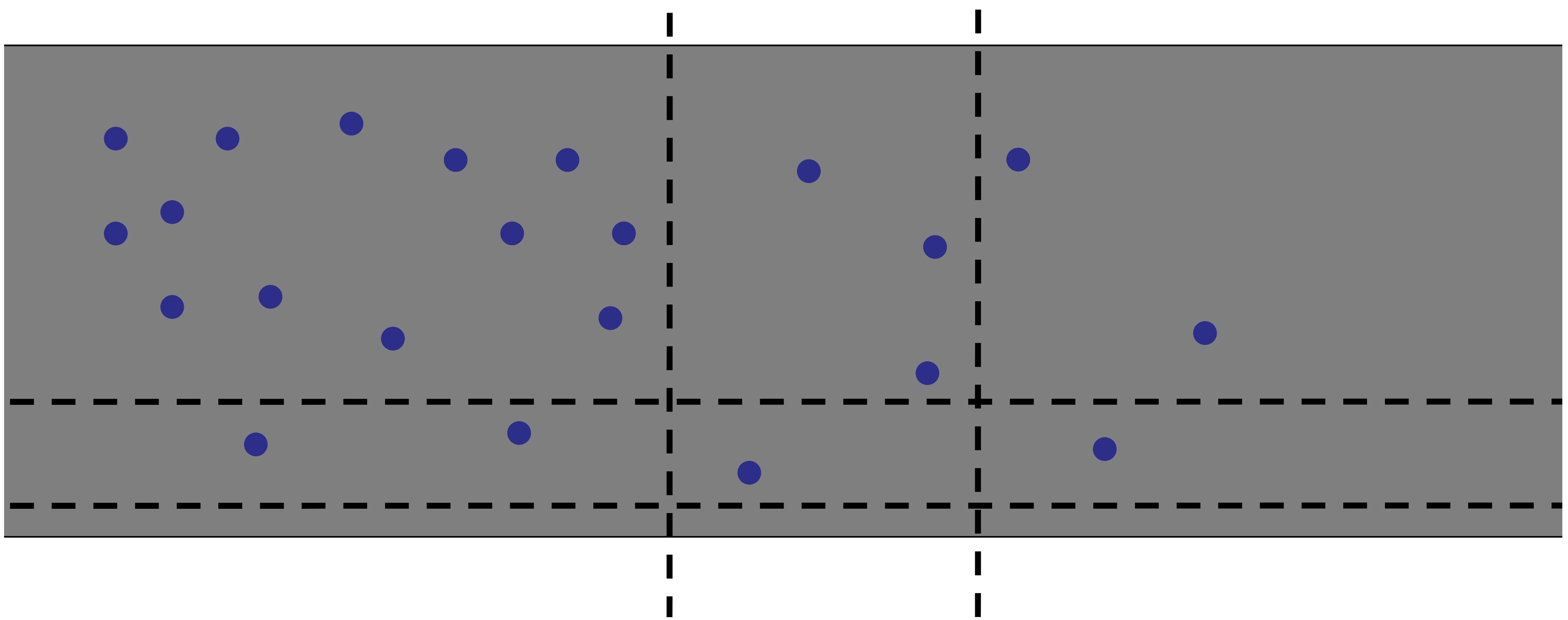
Step 1:

- Create a 1d-range-tree on the x-coords.



Two Dimensional Range Tree

Problem: can't enumerate entire sub-trees, since there may be too many nodes that don't satisfy the y-restriction.



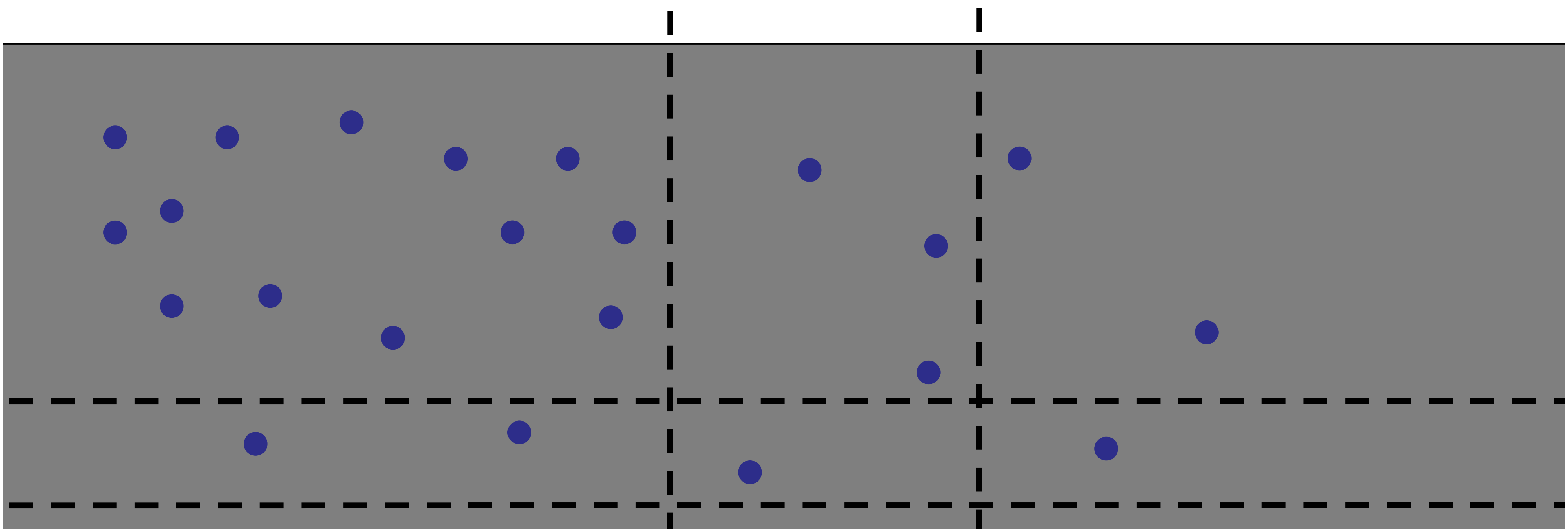
One Dimensional Range Queries

```
LeftTraversal(v, low, high)
    if (v.key >= low) {
            all-leaf-traversal(v.right);
            LeftTraversal(v.left, low, high);
    }
    else {
            LeftTraversal(v.right, low, high);
    }
}
```

Two Dimensional Range Tree

Solution: Augment!

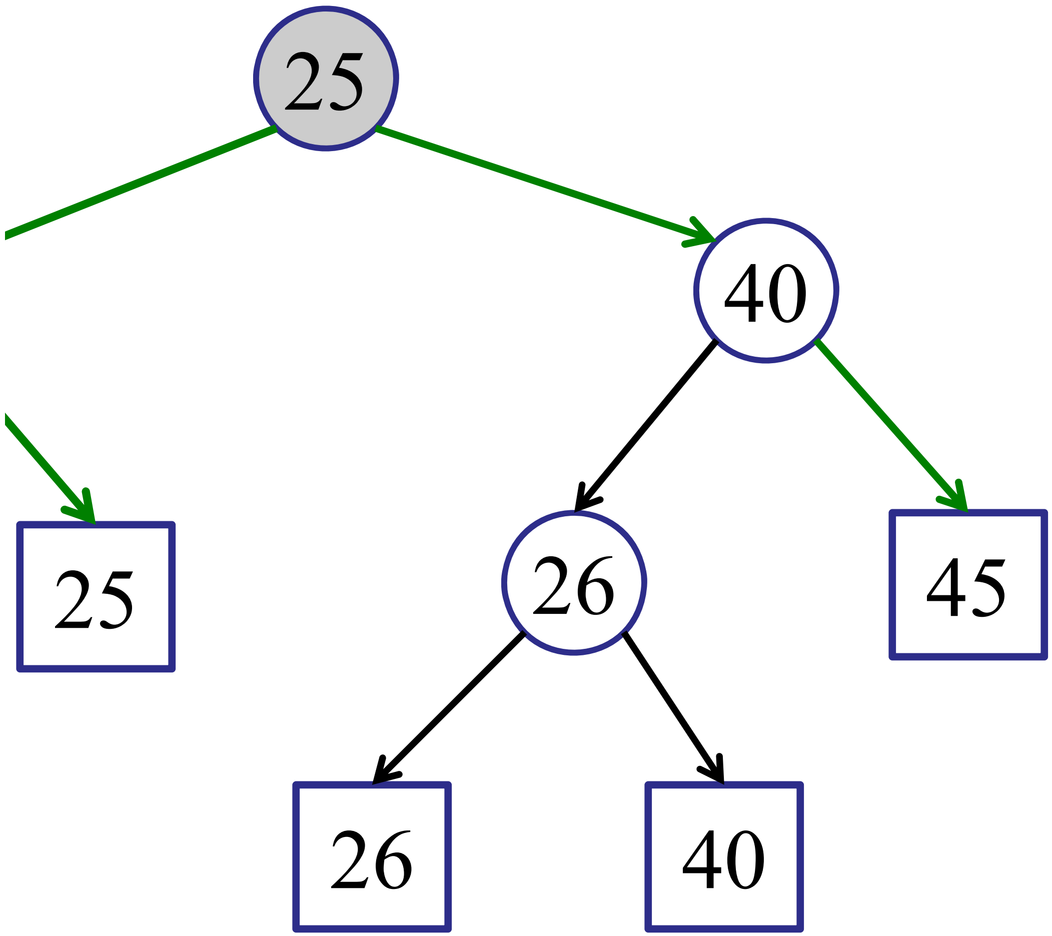
- Each node in the x-tree has a set of points in its sub-tree.
- Store a y-tree at each x-node containing all the points in the sub-tree.



One Dimensional Range Queries

```
LeftTraversal(v, low, high)
  if (v.key.x >= low.x) {
        ytree.search(low.y, high.y);
        LeftTraversal(v.left, low, high);
  }
  else {
        LeftTraversal(v.right, low, high);
  }
}
```

Example:



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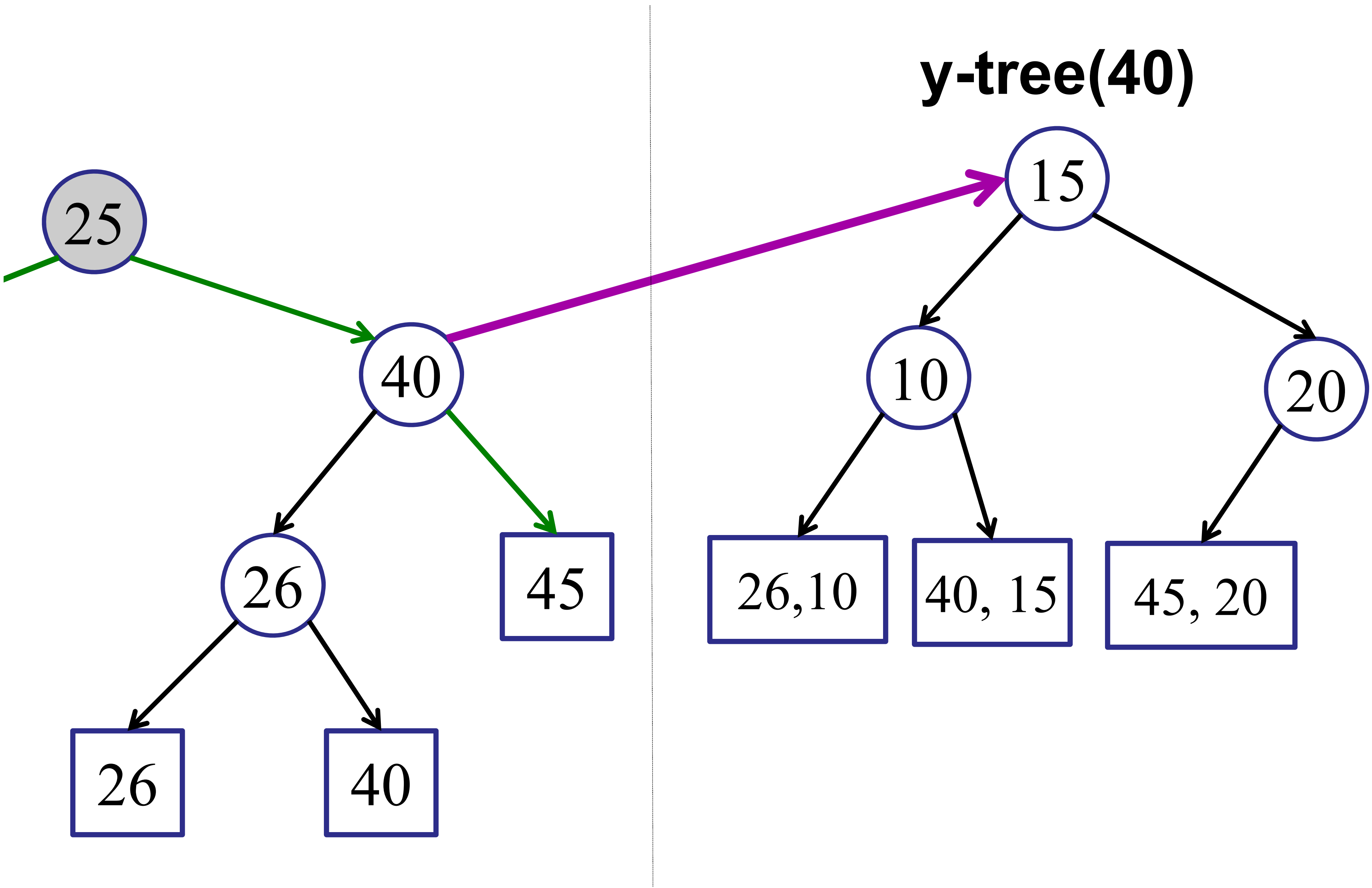
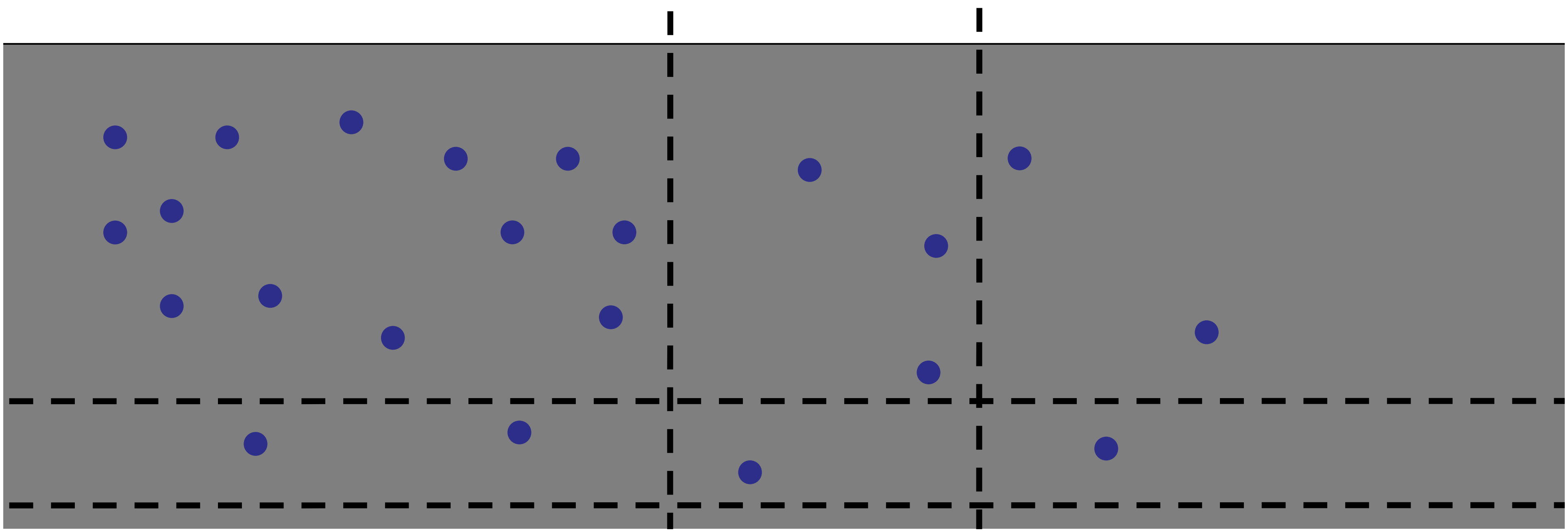


Two Dimensional Range Tree

Idea:

- Build an x-tree using only x-coordinates.
- For every node in the x-tree, build a y-tree out of nodes in subtree using only y-coordinates.



Analysis

Query time: $O(\log^2 n + k)$

- $O(\log n)$ to find split node.
- $O(\log n)$ recursing steps
- $O(\log n)$ y-tree-searches of cost $O(\log n)$
- $O(k)$ enumerating output

Analysis

Space complexity: O(n log n)

- Each point appears in at most one y-tree per level.
- There are O(log n) levels.

➔ Each node appears in at most O(log n) y-trees.

- The rest of the x-tree takes O(n) space.

Analysis

Building the tree: O(n log n)

- Tricky…
- Left as a puzzle… ☺

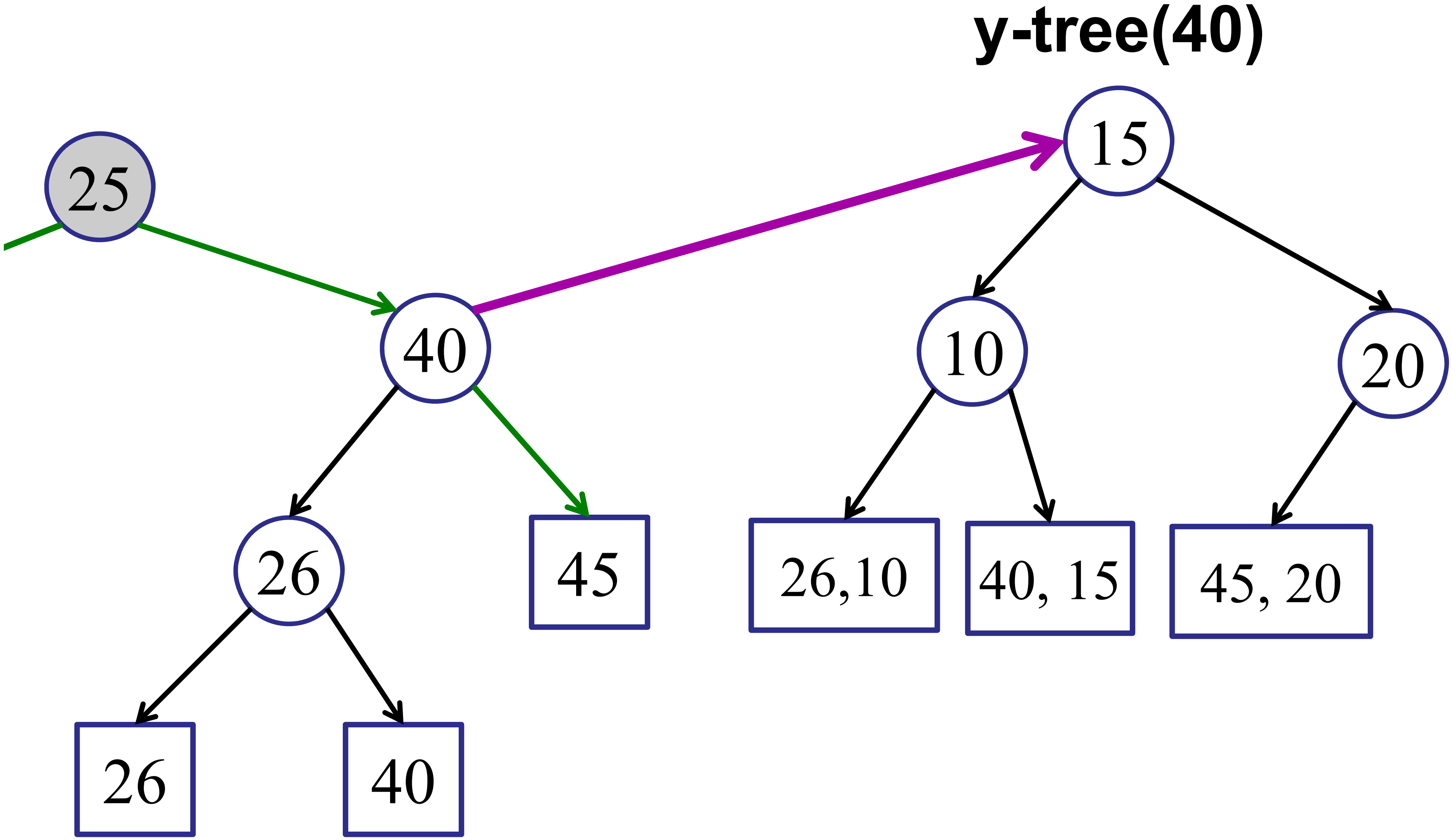
Challenge of the Day…

Dynamic Trees

What about inserting/deleting nodes?

- Hard!
- How do you do rotations?
- Every rotation you may have to entirely rebuild the y-trees for the rotated nodes.
- Cost of rotate: $O(n)$!!!!

Example:



d-dimensional

What if you want high-dimensional range queries?

- Query cost: $O(\log^d n + k)$
- buildTree cost: $O(n \log^{d-1} n)$
- Space: $O(n \log^{d-1} n)$

Idea:

- Store d–1 dimensional range-tree in each node of a 1D range-tree.
- Construct the d–1-dimensionsal range-tree recursively.

Curse of Dimensionality

What if you want high-dimensional range queries?

- Query cost: $O(\log^d n + k)$
- buildTree cost: $O(n \log^{d-1} n)$
- Space: $O(n \log^{d-1} n)$

Idea:

- Store d–1 dimensional range-tree in each node of a 1D range-tree.
- Construct the d–1-dimensionsal range-tree recursively.

Real World (aside)

kd-Trees

- Alternate levels in the tree:
  - vertical
  - horizontal
  - vertical
  - horizontal
- Each level divides the points in the plane in half.

Real World (aside)

kd-Trees

- Alternate levels in the tree
- Each level divides the points in the plane in half.
- Query cost: $O(\sqrt{n})$ worst-case
  - Sometimes works better in practice for many queries.
  - Easier to update dynamically.
  - Good for other types of queries: e.g., nearest-neighbor

Today

Three examples of augmenting BSTs

1. Order Statistics

2. Intervals

3. Orthogonal Range Searching