Solving Recurrence Relations

We want to find an explicit formula to a given recurrence relation rather than having to calculate each term of the sequence one by one.

Defn: A **linear homogeneous recurrence relation of degree k with constant coefficients** is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}, \quad (\ddagger)$$

where c_1, c_2, ..., $c_k \in \mathbb{R}$ and $c_k \neq 0$.

"Linear" means that $a_{n-1}, a_{n-2}, \ldots, a_{n-k}$ appear in separate terms and to the first power.

"Homogeneous" means that there is no constant term.

"Degree k" denotes that the expression for a_n depends on the previous k terms $a_{n-1}, a_{n-2}, \ldots, a_{n-k}$ (some of these terms may not appear, but a_{n-k} has to).

Such a recurrence relation is completely (and uniquely) determined by the values of the k initial conditions $a_1 = t_1, a_2 = t_2, \ldots, a_k = t_k$.

Examples:

$$a_n = 7a_{n-1} + 8a_{n-2} - 3a_{n-4} + 2a_{n-5}$$

recurrence relation of degree 5

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \qquad (\ddagger)$$

$c_1, c_2, \ldots, c_k \in \mathbb{R}$

$c_k \neq 0$

$$\frac{r^n}{r^{n-k}} = c_1 \frac{r^{n-1}}{r^{n-k}} + c_2 \frac{r^{n-2}}{r^{n-k}} + \ldots + c_k \frac{r^{n-k}}{r^{n-k}}$$

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \ldots + c_k$$

Fact: $a_n = r^n$ is a solution to the recurrence relation ($\ddagger$) if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}.$$

We divide both sides by r^{n-k} and then subtract the right hand side from the left, and we get

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_{k-1} r^1 - c_k = 0.$$

This equation is called the **characteristic equation** of the recurrence relation ($\ddagger$).

So $a_n = r^n$ is a solution to ($\ddagger$) if and only if r is a solution of the characteristic equation.

The solutions of the characteristic equation are called the **characteristic roots** of the recurrence relation.

We will use characteristic roots to explicitly solve (some) given recurrence relations.

Solving linear homogeneous recurrence relations of degree 2 with constant coefficients:

We state a couple of very useful theorems without proof. (For the proofs you may consult most discrete mathematics textbooks – Rosen is a classic. These notes also cover the topic properly.)

degree 2

Theorem: (two distinct roots) (could start at a_0 too)

Suppose a sequence $\{a_n\} = a_1, a_2, a_3, \ldots$ satisfies a recurrence relation $a_k = Aa_{k-1} + Ba_{k-2}$ for all integers $k \geq 3$, where A and $B \neq 0$ are some constants.

If the characteristic equation $r^2 - Ar - B = 0$ has two distinct real roots α and β, then $\{a_n\}$ is given by the explicit formula

$$a_n = K_1\alpha^n + K_2\beta^n,$$

where K_1 and K_2 are chosen so that

$$a_1 = K_1\alpha + K_2\beta$$

$$a_2 = K_1\alpha^2 + K_2\beta^2.$$

if a_0, a_1 are given, then ...

$a_0 = K_1\alpha^0 + K_2\beta^0 = K_1 + K_2$

$a_1 = K_1\alpha^1 + K_2\beta^1 = K_1\alpha + K_2\beta$

Example: Solve the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ $(n \geq 3)$ with the initial conditions $a_1 = 0$ and $a_2 = 6$.

$a_n = a_{n-1} + 2a_{n-2}$

$r^n = r^{n-1} + 2r^{n-2}$

$\frac{r^n}{r^{n-2}} = \frac{r^{n-1} + 2r^{n-2}}{r^{n-2}}$

$r^2 = r + 2$

characteristic equation $r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$\alpha = 2 \quad \beta = -1$

$\alpha \neq \beta$

$a_n = K_1\alpha^n + K_2\beta^n = K_1 \cdot 2^n + K_2 \cdot (-1)^n$

I: $0 = a_1 = K_1\alpha^1 + K_2\beta^1 = K_1\alpha + K_2\beta$

II: $6 = a_2 = K_1\alpha^2 + K_2\beta^2$

I: $0 = 2K_1 - K_2$

II: $6 = 4K_1 + K_2$

$\Rightarrow$ (I+II) $6 = 6K_1 \Rightarrow K_1 = 1 \Rightarrow K_2 = 2$

$\boxed{a_n = 1 \cdot 2^n + 2 \cdot (-1)^n}$

Example: Solve the Fibonacci sequence $F_n = F_{n-1} + F_{n-2}$ $(n \geq 3)$, $F_1 = 1$ and $F_2 = 2$.

exercise

Theorem: (single root) — could start at a_0 too.

Suppose a sequence $\{a_n\} = a_1, a_2, a_3, \ldots$ satisfies a recurrence relation $a_k = Aa_{k-1} + Ba_{k-2}$ for all integers $k \geq 3$, where A and $B \neq 0$ are some constants.

If the characteristic equation $r^2 - Ar - B = 0$ has a single (real) root α, then $\{a_n\}$ is given by the explicit formula

$$a_n = K_1\alpha^n + K_2 n\alpha^n,$$

where K_1 and K_2 are chosen so that

$$a_1 = K_1\alpha + K_2\alpha$$

$$a_2 = K_1\alpha^2 + 2K_2\alpha^2.$$

Example: Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ $(n \geq 3)$ with the initial conditions $a_1 = 9$ and $a_2 = 36$.

$$a_n = 6a_{n-1} - 9a_{n-2}$$

$$\frac{r^n}{r^{n-2}} = \frac{6r^{n-1} - 9r^{n-2}}{r^{n-2}}$$

$$r^2 = 6r - 9$$

$$r^2 - 6r + 9 = 0$$

$$(r-3)^2 = 0$$

$$\Rightarrow \alpha = 3$$

$$a_n = K_1\alpha^n + K_2 n\alpha^n$$

$$9 = a_1 = K_1 \cdot 3^1 + K_2 \cdot 3^1$$

$$36 = a_2 = K_1 \cdot 3^2 + 2 \cdot K_2 \cdot 3^2$$

$$\text{I}: \; 9 = 3K_1 + 3K_2$$

$$\text{II}: \; 36 = 9K_1 + 18K_2$$

$$\text{I}: \; 9 = 3K_1 + 3K_2$$

$$-3\text{I} + \text{II}: \; 9 = 9K_2 \Rightarrow K_2 = 1$$

From I: $K_1 = 2$

$$\Rightarrow a_n = 2 \cdot 3^n + 1 \cdot n \cdot 3^n$$

$$\boxed{a_n = (2+n) \cdot 3^n}$$

After finding the formula, check your formula for the initial conditions.

Generalization to degree $> k$: The last two theorems can actually be generalized to higher degree linear homogeneous recurrence relations, [redacted]

Theorem: (degree k with k distinct roots)

Suppose a sequence $\{a_n\}$ satisfies a recurrence relation of degree k with the characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$ ($c_1, \ldots, c_k$ are constants) with k distinct real roots $\alpha_1, \ldots, \alpha_k$, then $\{a_n\}$ is given by the explicit formula

$$a_n = K_1 \alpha_1^n + K_2 \alpha_2^n + \ldots + K_k \alpha_k^n,$$

where the constants $K_1, K_2, \ldots, K_k$ are determined by setting up a system of k linear equations using the values of the initial conditions $a_1, a_2, \ldots, a_k$ (just like we did for degree 2 recurrences).

Theorem: (degree k, not all roots distinct)

Suppose a sequence $\{a_n\}$ satisfies a recurrence relation of degree k with the characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$ ($c_1, \ldots, c_k$ are constants) with j distinct real roots $\alpha_1, \ldots, \alpha_j$ of positive multiplicities $m_1, \ldots, m_j$ ($m_1 + \ldots + m_j = k$), then $\{a_n\}$ is given by the explicit formula

$$a_n = (K_{1,1} + K_{1,2} n \ldots + K_{1,m_1} n^{m_1 - 1}) \alpha_1^n$$

$$+ (K_{2,1} + K_{2,2} n + \ldots, K_{2,m_2} n^{m_2 - 1}) \alpha_2^n$$

$$+ \ldots + (K_{j,1} + K_{j,2} n + \ldots + K_{j,m_j} n^{m_j - 1}) \alpha_j^n,$$

where the constants $K_{i,j}$ are determined by setting up a system of k linear equations using the values of the initial conditions $a_1, a_2, \ldots, a_k$ (just like we did for degree 2 recurrences).

Derangements

Problem: There are n guests at a party. The guest list has the guests numbered $1, 2, \ldots, n$. Each guest has a hat, which they leave at the coatroom when they enter. After the party, each guest takes a hat at random [well, don't ask me why, but you may find an interesting discussion here on the question of whether word problems in mathematics should actually make sense]. How likely is it that no guest gets their own hat?

Let us number the hats accordingly with $1, 2, \ldots, n$ such that the hat belonging to guest i is labelled i as well.

A **derangement** of $\{1, 2, \ldots, n\}$ is a permutation π of $\{1, 2, \ldots, n\}$ such that $\pi(i) \neq i$ for each i ($1 \leq i \leq n$).

Let us denote by d_n the number of derangements of $\{1, 2, \ldots, n\}$.

So we need to find $d_n/n!$.

Express d_n as a recurrence relation.

$d_1 = 0$

$d_2 = 1 \quad (2,1)$

$d_3 = 2 \quad (2,3,1), (3,1,2)$

$d_1 = 0$, $d_2 = 1$

$d_3 = 2$

$d_4 = ?$

2143
2341
2413
3142
3412
3421
4123
4312
4321

$d_4 = 9$

Ex: $n=5$

guests	1	2	3	4	5
forbidden hats $f(i)$	2	4	3	5	1
hat taken Case 1	4	2			
hat taken Case 2	4	$\neq 2$			

guest	3	4	5
forbidden hats	3	5	1

hat 4 is forbidden for guest 2

Forbid exactly one hat for each guest
Also make sure that each hat is forbidden for exactly one guest.

We could have stated the same problem requiring that each guest has exactly 1 forbidden choice for hats, say $f(i)$, and that each hat is forbidden for exactly one guest. (In our case for guest i the forbidden hat is the hat labelled i, so $f(i) = i$.)

Now assume that guest 1 takes hat $i \neq f(1)$. There are $n-1$ choices for i.

Consider guest j whose forbidden hat is i.

Two scenarios are possible, depending on whether or not guest j takes hat $f(i)$ in return:

(i) Guest j takes hat $f(i)$. Now the problem reduces to $n-2$ guests and $n-2$ hats where each guest has a forbidden hat (among the remaining hats) and each remaining hat is forbidden for exactly one remaining guest. This can be done in d_{n-2} ways.

(ii) Guest j does not take hat $f(i)$. Now the problem reduces to $n-1$ guests and $n-1$ hats where each guest has a forbidden hat (forbidden hat for guest j is now $f(i)$) and each remaining hat is forbidden for exactly one remaining guest. This can be done in d_{n-1} ways.

We have found

$$\boxed{d_n = (n-1)(d_{n-1} + d_{n-2}).}$$

$d_5 = (5-1)(9+2) = 44$

$d_6 = (6-1)(44+9) = 265$

We can now confirm that $d_4 = 9$, $d_5 = 44$, etc.

Note that the recurrence relation we have found for d_n is not linear, so we cannot apply the technique we learned earlier on for finding an explicit formula for d_n.

$d_n = n d_{n-1} - d_{n-1} + n d_{n-2} - d_{n-2}$

We'll take another approach. $d_n - n d_{n-1} = -d_{n-1} + n d_{n-2} - d_{n-2}$

$= -(d_{n-1} - (n-1) d_{n-2})$

$$\boxed{d_n = (n-1)(d_{n-1} + d_{n-2})}$$

$d_{n-1} - (n-1) d_{n-2}$

$= -(d_{n-2} - (n-2) d_{n-3})$

$$\Rightarrow d_n - nd_{n-1} = -(d_{n-1} - (n-1)d_{n-2})$$

Observe that the expression on the right is the negative of the expression on the left with n replaced by $n-1$; this helps to iterate easily.

$d_n - n d_{n-1} = -(d_{n-1} - (n-1) d_{n-2})$

$= (-1)^2 (d_{n-2} - (n-2) d_{n-3})$

$n = 5$

$n - 4 = 1$

$n - 3 = 2$

$$\Rightarrow d_n - nd_{n-1} = (-1)^2(d_{n-2} - (n-2)d_{n-3})$$

$= (-1)^3 (d_{n-3} - (n-3) d_{n-4})$

$\cdots$

after many iterations

$$\boxed{\Rightarrow d_n - nd_{n-1} = (-1)^{n-2}(d_2 - 2d_1)}$$

With $d_1 = 0$ and $d_2 = 1$, we have

$d_2 = 1$, $d_1 = 0$

$$\boxed{\Rightarrow d_n - nd_{n-1} = (-1)^{n-2} = (-1)^n.}$$

$(-1)^n = (-1)^{n-2} \cdot (-1)^2$, $(-1)^2 = 1$

$\Rightarrow (-1)^n = (-1)^{n-2}$

This is better, but we still don't have an explicit formula. We make the following claim.

Claim:

$$\boxed{d_n = n! \sum_{m=0}^{n} \frac{(-1)^m}{m!}, \quad n \geq 1}$$