$d_5 = (5-1)(9+2) = 44$

$d_6 = (6-1)(44+9) = 265$

We can now confirm that $d_4 = 9$, $d_5 = 44$, etc.

Note that the recurrence relation we have found for d_n is not linear, so we cannot apply the technique we learned earlier on for finding an explicit formula for d_n.

$d_n = n d_{n-1} - d_{n-1} + n d_{n-2} - d_{n-2}$

We'll take another approach. $d_n - n d_{n-1} = -d_{n-1} + n d_{n-2} - d_{n-2}$

$$\boxed{d_n = (n-1)(d_{n-1} + d_{n-2})}$$

$= -(d_{n-1} - (n-1) d_{n-2})$

$d_{n-1} - (n-1) d_{n-2} = -(d_{n-2} - (n-2) d_{n-3})$

$$\Rightarrow d_n - nd_{n-1} = -(d_{n-1} - (n-1)d_{n-2}),$$

Observe that the expression on the right is the negative of the expression on the left with n replaced by $n-1$; this helps to iterate easily.

$d_n - n d_{n-1} = -(d_{n-1} - (n-1) d_{n-2})$

$= (-1)^2 (d_{n-2} - (n-2) d_{n-3})$

$n = 5$

$n - 4 = 1$

$n - 3 = 2$

$$\Rightarrow d_n - nd_{n-1} = (-1)^2(d_{n-2} - (n-2)d_{n-3})$$

$= (-1)^3 (d_{n-3} - (n-3) d_{n-4})$

$\cdots$

$\Rightarrow$ after many iterations

$$\boxed{\Rightarrow d_n - nd_{n-1} = (-1)^{n-2}(d_2 - 2d_1)}$$

With $d_1 = 0$ and $d_2 = 1$, we have

$d_2 = 1$, $d_1 = 0$

$$\boxed{\Rightarrow d_n - nd_{n-1} = (-1)^{n-2} = (-1)^n.}$$

$(-1)^n = (-1)^{n-2} \cdot \underbrace{(-1)^2}_{=1}$

$\Rightarrow (-1)^n = (-1)^{n-2}$

This is better, but we still don't have an explicit formula. We make the following claim.

Claim:

$$\boxed{d_n = n! \sum_{m=0}^{n} \frac{(-1)^m}{m!}, \quad n \geq 1}$$

[This claim may seem as if it is made out of the blue (it may even remind you of the famous cartoon by Sidney Harris), but the point is that I want to show you an application of mathematical induction here, considering that mathematical induction is a very strong tool in combinatorics.]

We prove the claim by mathematical induction.

Base case: Check the claim for $n = 1$.

Induction step: Suppose that the claim is true for $n = k$, and show that it hold for $n = k + 1$. (Take advantage of the identity [redacted] that we have just proved: $d_n - nd_{n-1} = (-1)^n$.)

We prove this claim by induction.

$P(n)$: $d_n = n! \sum_{m=0}^{n} \frac{(-1)^m}{m!}$ $(n \geq 1)$

BI: $n_0 = 1$ We check $d_1 = n_0! \sum_{m=0}^{n_0} \frac{(-1)^m}{m!}$

Check $d_1 = 1! \sum_{m=0}^{1} \frac{(-1)^m}{m!}$

$d_1 = 0$

$1! \sum_{m=0}^{1} \frac{(-1)^m}{m!} = \frac{(-1)^0}{0!} + \frac{(-1)^1}{1!} = 1 + (-1) = 0$

So we see that $P(n_0)$ holds, and therefore the basis of induction is complete.

$$P(n): \quad d_n = n! \sum_{m=0}^{n} \frac{(-1)^m}{m!} \quad (n \geq 1)$$

IS : Suppose that $P(k)$ is true $(k \geq 1)$.

So, we suppose that $d_k = k! \sum_{m=0}^{k} \frac{(-1)^m}{m!}$. (*)

We want to show that $P(k+1)$ is true.

we want to show that $\boxed{d_{k+1} = (k+1)! \sum_{m=0}^{k+1} \frac{(-1)^m}{m!}}$.

Remember that we found $d_n - n d_{n-1} = (-1)^n$. $\quad (\overset{\text{sub}}{n = k+1})$

$$\Rightarrow d_{k+1} - (k+1) d_k = (-1)^{k+1}$$

Using (*), we have:

$$d_{k+1} - (k+1)\left(k! \cdot \sum_{m=0}^{k} \frac{(-1)^m}{m!}\right) = (-1)^{k+1}$$

$$\Rightarrow d_{k+1} = (k+1)\left(k! \sum_{m=0}^{k} \frac{(-1)^m}{m!}\right) + (-1)^{k+1}$$

$$\Rightarrow d_{k+1} = (k+1) \cdot \left(k! \left(\sum_{m=0}^{k+1} \frac{(-1)^m}{m!} - \underbrace{\frac{(-1)^{k+1}}{(k+1)!}}_{k+1^{st} \text{ term}}\right)\right) + (-1)^{k+1}$$

$$\Rightarrow d_{k+1} = \underbrace{(k+1) \cdot k!}_{=(k+1)!} \cdot \sum_{m=0}^{k+1} \frac{(-1)^m}{m!} - \cancel{\underbrace{(k+1)k!}_{(k+1)!}} \cdot \frac{(-1)^{k+1}}{\cancel{(k+1)!}} + (-1)^{k+1}$$

$$\Rightarrow d_{k+1} = (k+1)! \sum_{m=0}^{k+1} \frac{(-1)^m}{m!} - \cancel{(-1)^{k+1}} + \cancel{(-1)^{k+1}}$$

$$\Rightarrow d_{k+1} = (k+1)! \sum_{m=0}^{k+1} \frac{(-1)^m}{m!},$$ which finishes the IS and also the proof.

The answer to the hat problem is then $\dfrac{d_n}{n!} = \dfrac{n! \sum_{m=0}^{n} \dfrac{(-1)^m}{m!}}{n!} = \sum_{m=0}^{n} \dfrac{(-1)^m}{m!}$.

Example: At a party there are n men and n women. In how many ways can the n women choose male partners for the first dance?

$W_1 \quad M_1$
$W_2 \quad M_2$
$\vdots \quad \vdots$
$W_n \quad M_n$

$\underset{W_1}{\underline{n}}$ options $\quad \underset{W_2}{\underline{n-1}}$ options $\quad \underset{W_3}{\underline{n-2}}$ options $\quad \cdots \quad \underset{W_n}{\underline{1}}$ option

So, $n(n-1)(n-2)\cdot \ldots \cdot 2 \cdot 1 = n!$ different ways.

How many ways are there for the second dance if the women are required to dance with a different male partner than before?

d_n (See that in the case each woman has exactly one "forbidden" partner. So this problem is equivalent to the hat problem.)

Counting

In this section we will learn some very elementary combinatorial techniques to prove certain mathematical statements, but first we learn how to count. Seriously. $1, 2, 3, \ldots$.

Well, of course we will count things in a smarter way!

Sum Rule: Suppose that a procedure is carried out by performing exactly one of the k tasks $T_1, T_2, \ldots, T_k$, where there are n_i ways to carry out task T_i (for each $i \in \{1, \ldots, k\}$). Then there are $n_1 + n_2 + \ldots + n_k$ ways to carry out the procedure.

Example: A student can choose a project from one of four lists. The four lists contain 3, 15, 7 and 17 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

$L_1 \quad L_2 \quad L_3 \quad L_4$

$3 \quad 15 \quad 7 \quad 17$

$3 + 15 + 7 + 17 = 42$

Example: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

$|P_6| = (26+10)^6 - 26^6$ → # of password with 6 characters not containing any digit

$|P_7| = (26+10)^7 - 26^7$

$|P_8| = (26+10)^8 - 26^8$

$|P| = |P_6| + |P_7| + |P_8|$