Propositional Logic

We need to understand the rules of logic in order to define, understand and evaluate mathematical statements. These rules form the basis of automated reasoning (think of computer science). To understand propositional logic we need some terminology.

Proposition (statement): a declarative sentence that is either true (T) or false (ϕ or **F**), but not both. T and ϕ are called the **truth values**. (The truth value of a proposition cannot be ambiguous.)

(Note for the curious: There are different types of logic where there are more than two truth value options for a given proposition. See "fuzzy logic" for example.)

Examples:

There is an elephant in my water bottle.

↳ this is a false proposition

There is no elephant in my water bottle.

↳ true proposition

Propositional variable: a symbol (conventionally one of the letters $p, q, r, s, \ldots$) that represents an unknown proposition.

p:

Logical connectives are operators used to connect multiple propositions, thereby forming a **compound proposition**.

Logical connectives are:

$\sim$ $\neg$ **"negation"** (~~the~~ some textbooks use $\sim$)
$\wedge$ **"conjunction"**
$\vee$ **"disjunction"**
$\oplus$ **"exclusive or"**
$\rightarrow$ **"implication"**
$\leftrightarrow$ **"biconditional"**

We want to be able to assign a truth value to a given compound proposition. In fact, the truth value of a compound proposition completely depends on the truth values of the ingredient propositions. To cover all possible scenarios for the truth values of the ingredient propositions we construct a **truth table**.

Consider a compound proposition q that involves ingredient propositions denoted by the propositional variables $p_1, p_2, \ldots, p_k$.

The corresponding truth table has columns for $p_1, p_2, \ldots, p_k$ and q (conventionally in this order). It has a row for each distinct truth assignment to the propositional variables $p_1, p_2, \ldots, p_k$. These rows are conventionally listed in reverse-alphabetical order (that is if you think of T as 1 and ϕ as 0):

$p_1 \quad p_2 \quad p_3 \quad \ldots \quad p_k$ $\qquad \underbrace{(p_1 \vee p_2) \leftrightarrow p_3}_{\text{compound proposition}}$

TTT ... TTT
TTT ... TTϕ
TTT ... TϕT
TTT ... Tϕ ϕ
...
$\phi\phi\phi \ldots \phi\phi$T
$\phi\phi\phi \ldots \phi\phi\phi$

Question: How many rows are there?

Two compound propositions are called **equivalent** if they have the same truth value in all the columns in their respective truth tables.

	p	q	r	(some compound proposition)
111	T	T	T	
110	T	T	F	
101	T	F	T	
100	T	F	F	
011	F	T	T	
010	F	T	F	
001	F	F	T	
000	F	F	F	

3 propositions → truth table has 8 rows

→ k propositions → truth table has 2^k rows

Logical connectives (operators)

Name Compound proposition	Truth value	Corresponding English expressions
Negation $\neg p$	true *if and only if* p is false	"not p" "it is not the case that p"
Conjunction $p \wedge q$	true *if and only if* both p and q are true	"p and q" "p but q"
Disjunction $p \vee q$	false *if and only if* both p and q are false	"p or q (or both)" "p unless q" "p or else q
Exclusive or $p \oplus q$	true *if and only if* exactly one of p and q is true	"either p or q" "p or q but not both"
Implication $p \rightarrow q$	false *if and only if* p is true and q is false	"p implies q" "if p, then q" "p only if q" "p is sufficient for q" "q follows from p" "q if p" "q when(ever) p" "q is necessary for p"
Biconditional $p \leftrightarrow q$	true *if and only if* p and q have the same truth value	"p if and only if q" "p is necessary and sufficient for q" "if p, then q, and conversely"

ex: $(p \wedge (\neg q)) \leftrightarrow (p \vee q)$

Precedence of logical operators:
1) $\neg$; 2) $\wedge$; 3) $\vee$; 4) $\oplus$; 5) $\rightarrow$; 6) $\leftrightarrow$

Implication Terminology:

$$p \rightarrow q$$

p **hypothesis** (or **premise**) q **conclusion** (or **consequence**)

Propositions related to the implication $p \rightarrow q$:

- The **converse** of $p \rightarrow q$: $q \rightarrow p$
- The **contrapositive** of $p \rightarrow q$: $\neg q \rightarrow \neg p$
- The **inverse** of $p \rightarrow q$: $\neg p \rightarrow \neg q$

A relevant equivalence: Form the truth tables for $p \rightarrow q$ and for its contrapos. to see that they're equivalent.

$p \rightarrow q \cong \underline{\neg q \rightarrow \neg p}$

So, the contrapositive of an implication is equivalent to itself! This is useful when proving mathematical statements.

A compound proposition that is always true is called a **tautology**.

A compound proposition that is always false is called a **contradiction**.

A compound proposition that is not a tautology or a contradiction is called a **contingency**.

p	q	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

We observe that the contrapositive of an implication is equivalent to itself.

$(\neg p \oplus q) \vee (\neg q \oplus p)$

this is a contingency

p	q	$\neg p \oplus q$	$\neg q \oplus p$	$(\neg p \oplus q) \vee (\neg q \oplus p)$
T	T	T	T	T
T	F	F	F	F
F	T	F	F	F
F	F	T	T	T

Proposition or not?

- "Ottawa is the capital of Canada." — prop. T
- "1+2=3." — prop. T
- "2+2=3." — prop. F
- "$1 + x = 3$." — If we know the value of x, then this is a prop.
- "There exists an integer x such that $1 + x = 3$." — prop. T
- "For all integers x, the equation $1 + x = 3$ holds." — prop. F
- "Eat your vegetables." — not a prop.
- "What time is it?" — not a prop.
- "You are smart." — prop. if we have a way of evaluating smartness.
- "It is winter now." — If the location is known, then prop.
- "Will you marry me?" — not a prop.
- "Alice asked Bob to marry her." — If we know these people and their lives, then this is a proposition

p	$p \Leftrightarrow p$
T	T
F	T

$p \Leftrightarrow p$ is a tautology.

$p \vee \neg p \rightsquigarrow$ tautology

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Note that the negation of any tautology is a contradiction.

About "$\to$" (**conditional sentences**):

Suppose you go to interview for a job at a store and the owner of the store makes you the following promise.

"If $\underbrace{\text{you show up for work Monday morning}}_{p}$, then $\underbrace{\text{you will get the job}}_{q}$."

Under what circumstances are you justified in saying the owner spoke falsely? That is, under what circumstances is the above sentence false?

p: "you show up for work Monday morning"

q: "you will get the job"

$p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

If you show up for work Monday morning but you don't get the job, then you can claim that the owner spoke falsely.

Tables for logical connectives:

negation

p	$\neg p$
T	F
F	T

conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

exclusive or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

biconditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Note that the biconditional is the negation of ~~the~~ the exclusive or

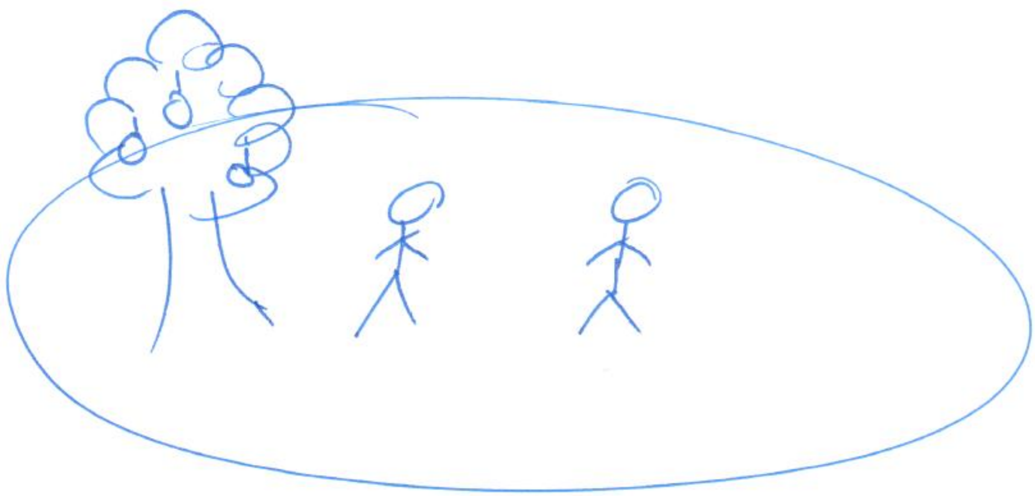
$$p \leftrightarrow q \underset{\text{equivalent}}{\equiv} \neg(p \oplus q)$$

implication

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$$p \underset{\text{equivalent}}{\cong} q$$

Examples: (Knights and Knaves)



On the Island of Knights and Knaves,
there are two types of people:
knights : they always speak the truth
knaves : they always lie.

You hear the following conversation:

Person A says: I'm a knight if B is a knave.

Person B says: I'm a knave and A is a knight.

Determine (if possible) if A and B is a knight or knave.