Sets

Defn: A **set** is a collection of distinct objects; these objects are then called **elements** of the set. A set can have a finite (even zero) or an infinite number of elements. Conventionally we use capital letters to denote sets. To express that an element x is in a set A, we write $x \in A$.

The elements of a set can be numbers, functions, and pretty much anything else.

Examples: $A = \{\text{elephant, monkey, chipmunk, Aras}\}$

set of natural numbers $\{1, 2, 3, 4, \ldots\}$ — set of rational numbers

Important sets of numbers: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}^-, \mathbb{Z}^+$ → positive integers

set of integers

set of real numbers

There are two common ways of denoting sets:

(i) **list notation**

Example: $S = \{8, 12, 16, 20\}$

(ii) **set-builder notation**

Example: $S = \{4i | i \in \mathbb{Z}, 2 \le i \le 5\}$ or $S = \{4i : i \in \mathbb{Z}, 2 \le i \le 5\}$

"such that" "such that"

$\emptyset$ $\{\}$

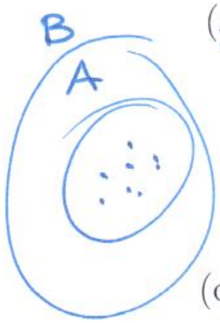
Defns: The set that has no elements is called the **empty set**, denoted by $\emptyset$ or $\{\}$.

$x \in A \Rightarrow x \in B$

A set A is called a **subset** of a set B, denoted by $A \subseteq B$, if $(x \in A \Rightarrow x \in B)$ for all x.

↳ subset

Note that a subset A of a set B can possibly be equal to B.



In fact, two sets A and B are **equal** if and only if $A \subseteq B$ and $B \subseteq A$.

A subset A of a set B is called a **proper subset**, denoted by $A \subset B$ (or $A \subsetneq B$), if $A \subseteq B$ and $A \neq B$.

$A \subseteq B$

Examples:

$A = \{1, 2, 3\}$ $\quad B \subseteq A$

$B = \{1, 2\}$ $\quad B \subset A$

$C = \{1, 2\}$ $\quad B \subseteq C$

$B \not\subset C$ (B is not a proper subset of C)

$|A| = 3$

$|B| = 2$

$B = C$

If a set is finite, then we can count its elements. The number of elements in a finite set A is referred to as the **cardinality** of A, denoted by $|A|$.

The set of subsets of a set A is called the **power set** of A, denoted by $\mathcal{P}(A)$.

$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

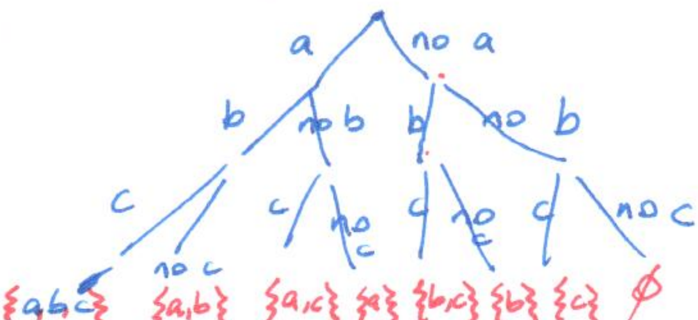
If $|A| = n$, then $|\mathcal{P}(A)| = 2^n$. (**WHY?**)

Example: $A = \{a, b, c\}$

$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

$|\mathcal{P}(A)| = 8$

$|\mathcal{P}(A)| = 2^n$ if $|A| = n$



The **Cartesian product** of a set A with a set B, denoted by $A \times B$, is defined as:

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

The cardinality of a Cartesian product can be found as follows:

$$|A \times B| = |A| \cdot |B|$$

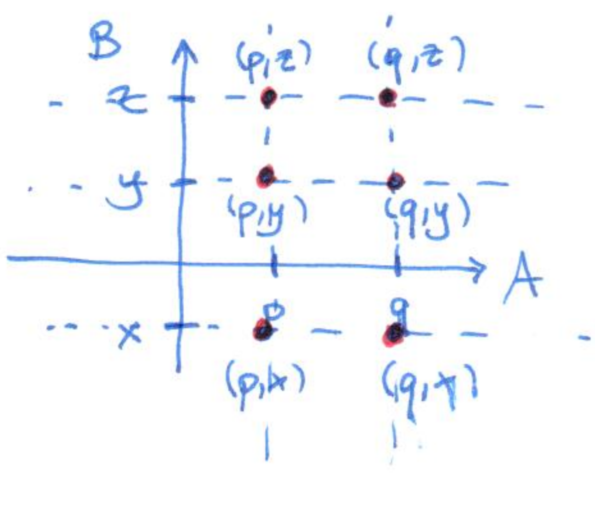
Example:

$A = \{p, q\}$

$B = \{x, y, z\}$

$A \times B = \{(p,x), (p,y), (p,z), (q,x), (q,y), (q,z)\}$

$|A \times B| = 6$, $|A| = 2$, $|B| = 3$



$\underbrace{|A \times B|}_{=6} = \underbrace{|A|}_{=2} \cdot \underbrace{|B|}_{=3}$

Table of Set Identities

Prove : (use set identities)

$$\overline{(A-\overline{B})} \cap \overline{(A-C)} = \overline{A} \cup (C-B)$$

	Identity	Name
(1)	$A - B = A \cap \overline{B}$	Set Difference
(2)	$A \cup \overline{A} = \mathcal{U}$	Complement Laws
(3)	$A \cap \overline{A} = \emptyset$	
(4)	$A \cup \emptyset = A$	Identity Laws
(5)	$A \cap \mathcal{U} = A$	
(6)	$A \cup \mathcal{U} = \mathcal{U}$	Domination Laws
(7)	$A \cap \emptyset = \emptyset$	
(8)	$A \cup A = A$	Idempotent Laws
(9)	$A \cap A = A$	
(10)	$\overline{\overline{A}} = A$	Complementation Law
(11)	$A \cup B = B \cup A$	Commutative Laws
(12)	$A \cap B = B \cap A$	
(13)	$A \cup (B \cup C) = (A \cup B) \cup C$	Associative Laws
(14)	$A \cap (B \cap C) = (A \cap B) \cap C$	
(15)	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive Laws
→ (16)	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	
(17)	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's Laws
(18)	$\overline{A \cap B} = \overline{A} \cup \overline{B}$	

Proofs with Sets, Functions

Let us prove *rigorously* that identity (17) is indeed correct.

Example: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

To prove equality we need to prove:
(i) $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$, and
(ii) $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

(i): Let $x \in \overline{A \cup B}$. Then $x \notin A \cup B$.
So $x \notin A$ and $x \notin B$. Then $x \in \overline{A}$ and $x \in \overline{B}$.
Therefore, $x \in \overline{A} \cap \overline{B}$.

(ii) Let $x \in \overline{A} \cap \overline{B}$. Then $x \in \overline{A}$ and $x \in \overline{B}$.
So, $x \notin A$ and $x \notin B$. Then $x \notin A \cup B$.
Therefore, $x \in \overline{A \cup B}$.

~~(ii):~~

By (i) & (ii), we showed that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Example: Prove rigorously: $A \subseteq B \leftrightarrow A \cap B = A$

We want to show an equivalence. Therefore we need to prove:
(i) $A \subseteq B \rightarrow A \cap B = A$ and
(ii) $A \cap B = A \rightarrow A \subseteq B$

(i): To prove (i) we need to assume $A \subseteq B$, and show that $A \cap B = A$ follows.

Assume $\boxed{A \subseteq B.}$ To show $A \cap B = A$ we need to prove:
(ia) $A \cap B \subseteq A$, and
(ib) $A \subseteq A \cap B$

Proof of (ia): Let $x \in A \cap B$. So $x \in A$ and $x \in B$. So we showed that if $x \in A \cap B$, then $x \in A$. Hence $A \cap B \subseteq A$.

Proof of (ib): Let $x \in A$. From the assumption $A \subseteq B$, we have that $x \in B$. Then $x \in A \cap B$. So we showed that $A \subseteq A \cap B$.

Prove this

(ii): To prove (ii) we need to assume $A \cap B = A$, and show that $A \subseteq B$ follows.

Let $x \in A$. Since $A \cap B = A$, $x \in A \cap B$. Then $x \in A$ and $x \in B$. So we ~~sh~~ showed that $x \in A$ implies $x \in B$; therefore $A \subseteq B$.

By parts (i) and (ii), we proved that
$$A \subseteq B \leftrightarrow A \cap B = A.$$

$A - B = A \cap \overline{B}$

Example: Let A, B, C be some subsets of a universal set. Use set identities to prove that

$$\overline{(A - \overline{B})} \cap \overline{(A - C)} = \overline{A} \cup (C - B).$$

Our goal is to start with $\overline{(A-\overline{B})} \cap \overline{(A-C)}$ and apply set identities to it to get $\overline{A} \cup (C-B)$.

$$\begin{aligned}
\overline{(A-\overline{B})} \cap \overline{(A-C)} &\overset{(1)}{=} \overline{(A \cap \overline{\overline{B}})} \cap \overline{(A-C)} \\
&\overset{(10)}{=} \overline{(A \cap B)} \cap \overline{(A-C)} \\
&\overset{(18)}{=} (\overline{A} \cup \overline{B}) \cap \overline{(A-C)} \\
&\overset{(1)}{=} (\overline{A} \cup \overline{B}) \cap \overline{(A \cap \overline{C})} \\
&\overset{(18)}{=} (\overline{A} \cup \overline{B}) \cap (\overline{A} \cup \overline{\overline{C}}) \\
&\overset{(10)}{=} (\overline{A} \cup \overline{B}) \cap (\overline{A} \cup C) \\
&\overset{(16)}{=} \overline{A} \cup (\overline{B} \cap C) \\
&\overset{(12)}{=} \overline{A} \cup (C \cap \overline{B}) \\
&\overset{(1)}{=} \overline{A} \cup (C - B)
\end{aligned}$$