

ARAS

$$\frac{4!}{2!1!1!1!} = \frac{24}{2} = 12 \text{ arrangements of the letters in ARAS}$$

2A  
1R  
1S

AARS  
AASR  
ASAR  
ASRA

ARAS  
ARSA  
SAAR  
SARA

SRAA  
RAAS  
RASA

RSAA

### Permutations of a Set with Repeated Elements:

**Example:** In how many ways can you arrange the letters in MISSISSAUGA?

11 letters

if all letters were distinct then we would have 11! arrangements.

2 I's  
4 S's  
2 A's  
1 M  
1 U  
1 G

$S_1 S_2 S_3 S_4$

$S_1 M U G I I A S_2 A S_3 S_4$  —  $S_1 S_2 S_3 S_4$

is the same word as can be permuted

$S_4 M U G I I A S_3 A S_1 S_2$  among themselves in 4! ways

There are  $\frac{11!}{1!1!1!4!2!2!}$  arrangements of the letters in MISSISSAUGA

**Theorem:** Suppose a collection consists of  $n$  objects of which

$n_1$  are of type 1 and are indistinguishable from each other,

$n_2$  are of type 2 and are indistinguishable from each other,

...

$n_k$  are of type  $k$  and are indistinguishable from each other.

Suppose that  $n_1 + n_2 + \dots + n_k = n$ . Then the number of distinguishable permutations of the  $n$  objects is

$$\frac{n!}{n_1!n_2!\dots n_k!}$$

$A_1, A_2$   
can be  
permuted  
in 2!  
ways

$I_1, I_2$   
can be  
permuted  
in 2!  
ways.

## Combinations and the Binomial Theorem

**Defn:** Let  $n$  and  $r$  be non-negative integers with  $r \leq n$ . An  $r$ -**combination** of a set of  $n$  elements is a subset of  $r$  of the  $n$  elements.

The number of subsets of size  $r$  ( $r$ -**combinations**) that can be chosen from a set of  $n$  elements is usually denoted by  $\binom{n}{r}$  (read: " $n$  choose  $r$ ").

[Note that there are many alternative notations.]

$C_r^n$

Numbers of the form  $\binom{n}{r}$  are called **binomial coefficients**.

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

**Examples:**

" $n$  choose  $r$ "

$$\binom{7}{2} = \frac{7!}{(7-2)!2!} = \frac{7!}{5!2!} = \frac{7 \cdot 6 \cdot \cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \cdot 2 \cdot 1} = \frac{7 \cdot 6}{2} = 21$$

$$n = 7$$

$$r = 2$$

$$\frac{7 \cdot 6}{2!}$$

$$\binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} = \frac{10 \cdot 9 \cdot \cancel{8 \cdot 7}}{\cancel{2 \cdot 2}} = 210$$

There are two distinct types of selecting  $r$  objects from a set of  $n$  elements.

In an ordered selection, it is not only what elements are chosen but also the order in which they are chosen that matters.

An ordered selection of  $r$  elements from a set of  $n$  elements is an  **$r$ -permutation** of the set.

In an unordered selection, on the other hand, the order in which the elements are chosen is irrelevant.

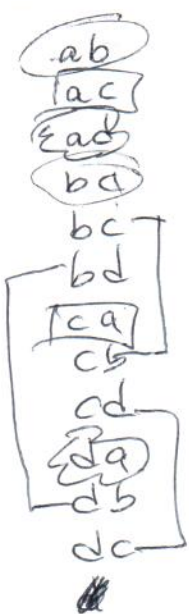
An unordered selection of  $r$  elements from a set of  $n$  elements is the same as a subset of size  $r$  or an  **$r$ -combination** of the set.

### Relation between permutations and combinations:

Compare  $P(4, 2)$  to  $\binom{4}{2}$ .

$$P(4, 2) = 4 \cdot 3$$

a b c d



$$\frac{n!}{(n-r)!}$$

In each permutation each selection of  $r$  things appear in  $r!$  different copies.

So if we forget about the ordering there are  $\frac{P(n, r)}{r!}$  different selections (unordered)

$$\frac{n!}{(n-r)! \cdot r!} = \binom{n}{r}$$

Why is this the number of  $r$  element subsets of an  $n$  element set?

$$\frac{P(n, r)}{r!} = \frac{n!}{(n-r)! \cdot r!}$$



1 2 3 4 5 6



$$\begin{aligned} x_1 &= 0 \\ x_2 &= 4 \\ x_3 &= 0 \end{aligned}$$

separator locations 1 and 6

## Number of Integer Solutions to Equations:

**Example:** How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 = 4$$

15

sep. locations where  $x_1, x_2, x_3 \geq 0$   
(2 and 3)

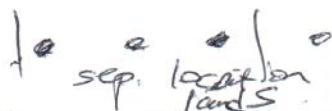


$$\begin{aligned} x_1 &= 1 \\ x_2 &= 0 \\ x_3 &= 3 \end{aligned}$$



$$1 + 2 + 1$$

$$x_1 \quad x_2 \quad x_3$$



$$0 + 3 + 1$$

$$x_1 \quad x_2 \quad x_3$$

$x_1$	$x_2$	$x_3$
4	0	0
3	1	0
3	0	1
2	2	0
2	1	1
2	0	2
1	3	0
1	2	1
1	1	2
1	0	3

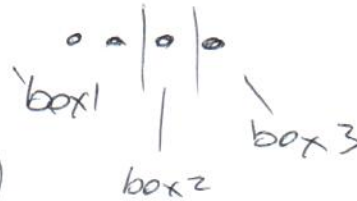
$x_1$	$x_2$	$x_3$
0	4	0
0	3	1
0	2	2
0	1	3
0	0	4

There are 3 boxes and 4 indistinguishable objects to be placed into these boxes. In how many ways can this be done?

to place the two separators we are "choosing" 2 of the 6 locations



$$\binom{4 + (3 - 1)}{3 - 1} = \binom{6}{2} = 15$$



**Theorem:** The number of ways to place  $n$  indistinguishable objects into  $r$  boxes is ...

$$\binom{n + (r - 1)}{r - 1}$$

# of dots      # of separators

# of separators

$$x_1 + x_2 \leq 7 \quad x_1, x_2 \geq 0$$

How many integer solutions?

Introduce  $x_3$  .  $x_1 + x_2 + x_3 = 7$   $x_1, x_2, x_3 \geq 0$

$$\binom{n+r-1}{r-1} = \binom{9}{2} = \frac{9 \cdot 8}{2} = 36$$

**Example:** How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 \leq 4$$

inequality

where  $x_1, x_2, x_3 \geq 0$ ?

Introduce  $x_4$  ~~!~~ consider

$$x_1 + x_2 + x_3 + x_4 = 4$$

observe that  $x_4 \geq 0$  (because  $x_1 + x_2 + x_3 \leq 4$ )

Solve for  $x_1 + x_2 + x_3 + x_4 = 4$  where  $x_1, x_2, x_3, x_4 \geq 0$

$$n = 4 \quad r = 4 \quad \binom{n+r-1}{r-1} = \binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3!} = 35$$

**Example:** How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 9$$

where  $x_1 \geq 2, x_2 \geq 0, x_3 \geq -3, x_4 \geq 4$ ?