$x_1 + x_2 \le 7 \qquad x_1, x_2 \ge 0$

How many integer solutions?

Introduce x_3. $\underbrace{x_1 + x_2 + x_3}_{r} = \underbrace{7}_{n} \qquad x_1, x_2, x_3 \ge 0$

$\binom{n+(r-1)}{r-1} = \binom{9}{2} = \frac{9 \cdot 8}{2} = 36$

Example: How many integer solutions are there to the ~~equation~~ inequality

$$x_1 + x_2 + x_3 \le 4$$

where $x_1, x_2, x_3 \ge 0$?

Introduce x_4! Consider

$$x_1 + x_2 + x_3 + x_4 = 4 \qquad \boxed{x_4 \ge 0}$$

Observe that $x_4 \ge 0$ (because $x_1 + x_2 + x_3 \le 4$)

Solve for $\underbrace{x_1 + x_2 + x_3 + x_4}_{r} = \underbrace{4}_{n}$ where $x_1, x_2, x_3, x_4 \ge 0$

$n = 4 \quad r = 4$

$\binom{n+(r-1)}{r-1} = \binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3!} = 35$

Example: How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 9$$

where $x_1 \ge 2$, $x_2 \ge 0$, $x_3 \ge -3$, $x_4 \ge 4$?

$x_1 = x_1' + 2 \quad (x_1' \ge 0)$

$x_2 = x_2' + 0 \quad (x_2' \ge 0)$

$x_3 = x_3' - 3 \quad (x_3' \ge 0)$

$x_4 = x_4' + 4 \quad (x_4' \ge 0)$

$x_1' + \cancel{2} + x_2' + \cancel{0} + x_3' - \cancel{3} + x_4' + \cancel{4} = 9$

$x_1', x_2', x_3', x_4' \ge 0$

$x_1' + x_2' + x_3' + x_4' = 6$

$n = 6$

$r = 4$

$\binom{n+r-1}{r-1} = \binom{9}{3}$

$= \frac{9 \cdot 8 \cdot 7}{3!} = \frac{\overset{12}{\cancel{9 \cdot 8}} \cdot 7}{6} = 84$

Pascal's Triangle:

Pascal's triangle is a neat way for finding binomial coefficients. Here is how we construct it.

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \cdots$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Observation 1:

$$\binom{n}{r} = \binom{n}{n-r}$$

$$\binom{4}{1} = \binom{4}{4-1}$$

$$\binom{4}{1} = \binom{4}{3}$$

[Why does this hold?]

$$\binom{7}{2} = \binom{7}{7-2}$$

$$\binom{7}{2} = \binom{7}{5}$$

$$\frac{7\cdot 6}{2} = 21$$

Observation 2:

famous property of Pascal's triangle.

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

$n = 5$
$r = 3$

$$\binom{5}{3-1} + \binom{5}{3} = \binom{6}{3}$$

[Prove this combinatorially!]

$$\frac{n!}{(n-r)!\,r!}$$

$1 \quad 2 \quad 3 \quad x \quad n \quad n+1$

# of ways choosing r things from n+1 things

$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$

Fix one of the $n+1$ elements, call it x.

To choose r things from $n+1$ things, there are two cases: Case 1: x is one of r things chosen

Case 2: x is not one of the r things chosen

Case 1: We need to choose $r-1$ more things from the remaining n things: $\binom{n}{r-1}$

Case 2: We need to choose r things from the remaining n things $\binom{n}{r}$.

Together $\binom{n}{r-1} + \binom{n}{r}$ ways.

So, $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$

Exercise: Expand $(a+b)^3$.

$= 1 \cdot a^3 + 3a^2b + 3ab^2 + 1 \cdot b^3$

Exercise: Expand $(a+b)^6$.

$= 1 \cdot a^6 + 6a^5 \cdot b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6a^1b^5 + 1 \cdot b^6$

Binomial Theorem:

Given any real numbers a and b and any non-negative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Example: Expand $(a+b)^6$.

$n = 6$

$$(a+b)^6 = \sum_{k=0}^{6} \binom{6}{k} a^{6-k} \cdot b^k$$

$= \binom{6}{0} a^{6-0} \cdot b^0 + \binom{6}{1} a^{6-1} \cdot b^1 + \binom{6}{2} a^{6-2} b^2 + \binom{6}{3} a^{6-3} b^3 + \binom{6}{4} a^{6-4} b^4 + \binom{6}{5} a^{6-5} b^5 + \binom{6}{6} a^{6-6} b^6$

$= 1a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + 1b^6$

Given any real numbers a and b and any non-negative integer n

BINOMIAL THEOREM

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Now we'll see why the theorem holds. First consider $n = 0$ separately.

$n=0$

$$(a+b)^0 = \sum_{k=0}^{0} \binom{0}{k} a^{0-k} b^k$$

$$1 \quad \checkmark \quad \binom{0}{0} a^{0-0} b^0 = 1 \cdot 1 \cdot 1 = 1$$

To understand why the theorem holds consider the example below.

Example: Consider $(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b)$.
What is the coefficient of a^3b^2 in $(a+b)^5$?

$k=2$ gives $\binom{5}{3} a^3 b^2$

To get a^3b^2 from this expansion after carrying out the multiplications we need the a's of 3 of the 5 terms. The remaining two terms will contribute b's. There are $\binom{5}{3}$ ways of selecting 3 of these 5 factors for the a's. So the coefficient of a^3b^2 is $\binom{5}{3}$.

Find the coefficient of a^6b^3 in $(a+b)^9$.

$\binom{9}{6}$ (or equally $\binom{9}{3}$)

$$= \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84$$

The terms a and b can be replaced with other terms:

Example: Expand $(\underset{a}{2x} - \underset{b}{3})^{\overset{=n}{4}}$ using the binomial theorem.

$a = 2x$
$b = -3$
$n = 4$

Binomial Theorem

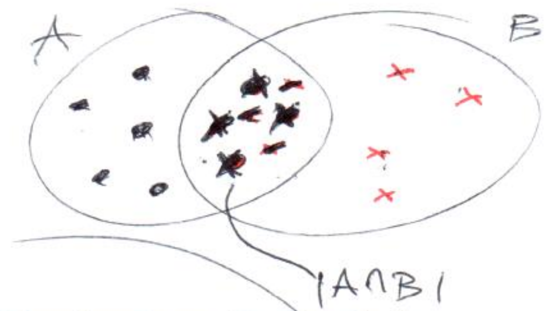
$$(2x + (-3))^4 = \sum_{k=0}^{4} \binom{4}{k} (2x)^{4-k} \cdot (-3)^k$$

$$= \binom{4}{0}(2x)^4 \cdot (-3)^0 + \binom{4}{1} \cdot (2x)^3 \cdot (-3)^1 + \binom{4}{2} \cdot (2x)^2 \cdot (-3)^2 + \binom{4}{3} \cdot (2x)^1 \cdot (-3)^3 + \binom{4}{4} \cdot (2x)^0 \cdot (-3)^4$$

A neat induction proof of the binomial theorem is also available (on pages 598-600 in the textbook).

$$= (2x)^4 - 12 \cdot (2x)^3 + 54 \cdot (2x)^2 - 108 \cdot (2x)^1 + 81 \cdot (2x)^0$$

$$= 16x^4 - 96x^3 + 216x^2 - 216x + 81$$



$|A \cup B| = |A| + |B| - |A \cap B|$

Inclusion-Exclusion Principle

The Inclusion-Exclusion Principle for two tasks:

If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

In the terminology of sets, the Inclusion-Exclusion principle simply translates to the following:

If A and B are finite sets, then $\boxed{|A \cup B| = |A| + |B| - |A \cap B|.}$

Example: How many bit strings of length eight either start with 1 or end with 00? (binary)

A 1 _ _ _ _ _ _ _ $\rightarrow 2^7$ $|A| = 2^7$

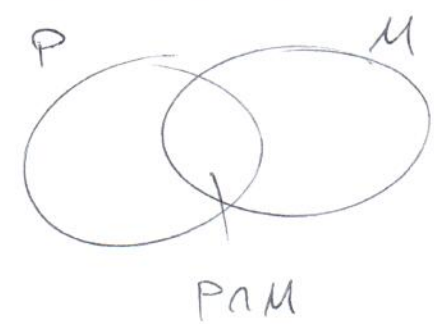
B _ _ _ _ _ _ 0 0 $\rightarrow 2^6$ $|B| = 2^6$

$|A \cap B| = 2^5$

$A \cap B$ 1 _ _ _ _ _ 0 0 $\rightarrow 2^5$ $|A \cup B| = 2^7 + 2^6 - 2^5$

$= 128 + 64 - 32 = 160$

Example: A company receives 350 applications for a job. Suppose that 220 of these applicants majored in psychology, 147 majored in mathematics, and 51 majored both in psychology and in mathematics. How many of these applicants majored neither in psychology nor in mathematics?



$|P| = 220$

$|M| = 147$

$|P \cap M| = 51$

$350 - \underline{|P \cup M|}$
316

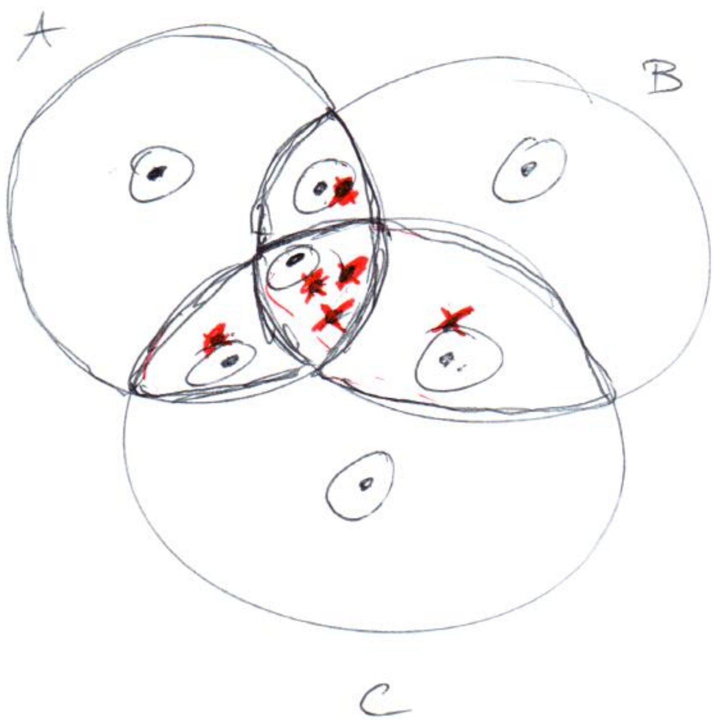
$= 34$ applicant major neither in psyc nor in math

$|P \cup M| = |P| + |M| - |P \cap M|$

$= 220 + 147 - 51 = 316$

Inclusion-Exclusion Principle for 3 Sets:
Let A, B, C be finite sets.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$\left\lfloor \frac{7}{3} \right\rfloor = 2$

$7 \div 3 = 2, \ldots$

$\frac{9}{2} = 4.5 \quad \left\lfloor \frac{9}{2} \right\rfloor = 4$

We can extend the Inclusion-Exclusion Principle to situations with more than two tasks.

Example: How many integers are there between ~~1~~ 100 and 999 that are divisible by 3, 5 or 7?

Let T be the set of integers between ~~1~~ 100 and 999 that are divisible by 3,5 or 7. We want to find $|T|$.

Let T_3 be the set of integers between ~~1~~ 100 and 999 that are divisible by 3,

T_5 be the set of integers between ~~1~~ 100 and 999 that are divisible by 5, and

T_7 be the set of integers between ~~1~~ 100 and 999 that are divisible by 7.

First find $|T_3|, |T_5|$ and $|T_7|$.

$$|T| = |T_3| + |T_5| + |T_7| - |T_3 \cap T_5| - |T_3 \cap T_7| - |T_5 \cap T_7| + |T_3 \cap T_5 \cap T_7|$$

$$|T_3| = \left\lfloor \frac{999}{3} \right\rfloor - \left\lfloor \frac{\cancel{100}\ 99}{3} \right\rfloor = 333 - 33 = 300$$

$\left\lfloor \frac{999}{3} \right\rfloor$ = # integers between 1 and 999 divisible by 3

$\left\lfloor \frac{99}{3} \right\rfloor$ = # integers between 1 and ~~100~~ 99 divisible by 3

$$|T_5| = \left\lfloor \frac{999}{5} \right\rfloor - \left\lfloor \frac{99}{5} \right\rfloor = 199 - 19 = 180$$

$$|T_7| = \left\lfloor \frac{999}{7} \right\rfloor - \left\lfloor \frac{99}{7} \right\rfloor = 142 - 14 = 128$$

Now let $T_{3,5}$ be the set of integers between 1 and 999 that are divisible by both 3 and 5,

$T_{3,7}$ be the set of integers between 1 and 999 that are divisible by both 3 and 7, and

$T_{5,7}$ be the set of integers between 1 and 999 that are divisible by both 5 and 7.

Find $|T_{3,5}|$, $|T_{3,7}|$ and $|T_{5,7}|$.

$|T_{3,5}|$ = least common multiple $lcm(3,5) = 15$ $\left\lfloor \frac{999}{15} \right\rfloor - \left\lfloor \frac{99}{15} \right\rfloor = 66 - 6 = 60$

$|T_{3,7}|$ $lcm(3,7) = 21$ $\left\lfloor \frac{999}{21} \right\rfloor - \left\lfloor \frac{99}{21} \right\rfloor = 47 - 4 = 43$

$|T_{5,7}|$ $lcm(5,7) = 35$ $\left\lfloor \frac{999}{35} \right\rfloor - \left\lfloor \frac{99}{35} \right\rfloor = 28 - 2 = 26$

Finally let $T_{3,5,7}$ be the set of integers between 1 and 999 that are divisible by 3, 5 and 7.

Find $|T_{3,5,7}|$. $lcm(3,5,7) = 105$ $\left\lfloor \frac{999}{105} \right\rfloor - \left\lfloor \frac{99}{105} \right\rfloor = 9 - 0 = 9$

Then $|T| = |T_3| + |T_5| + |T_7| - |T_{3,5}| - |T_{3,7}| - |T_{5,7}| + |T_{3,5,7}|$.

300 180 128 60 43 26 9

$= 488$