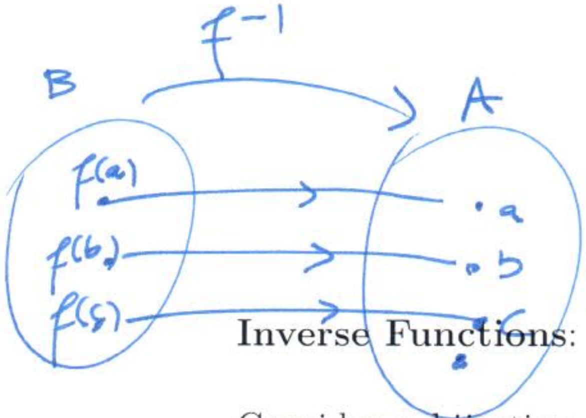
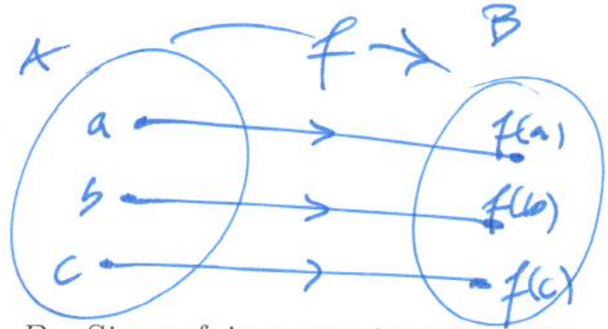
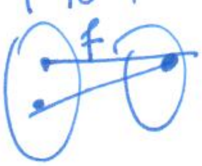



**Inverse Functions**:

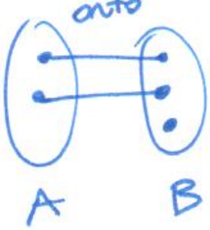
Consider a bijection $f : A \to B$. Since $f$ is a one-to-one correspondence, each element in $B$ corresponds to exactly one element in $A$.

not 1-to-1




This observation allows us to define a function $f^{-1} : B \to A$ that assigns each element in $B$ to its preimage in $A$; $f^{-1}$ is called the **inverse function** of $f$. (Note that since $f$ is a bijection, each element in $B$ has a preimage containing one element only. In general, when $f$ is not necessarily a bijection, the preimage of an element on $B$ can be a set of any cardinality.)

not onto




**Defn**: Let $f : A \to B$ be a bijection. The **inverse** of $f$ is a function $f^{-1} : B \to A$ defined as:

$$f^{-1}(b) = a \leftrightarrow f(a) = b.$$

$f^{-1}(3) = 7$

$3 = f(7)$

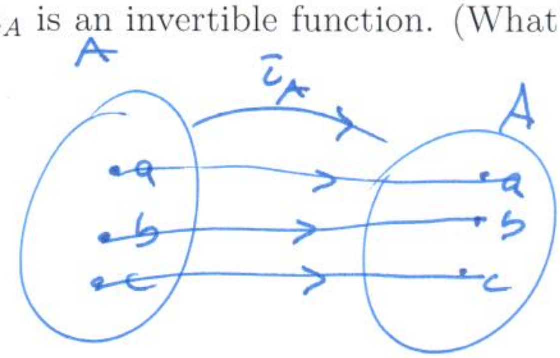
A function that has an inverse is called **invertible**.

in either case inverse from B to A cannot be defined

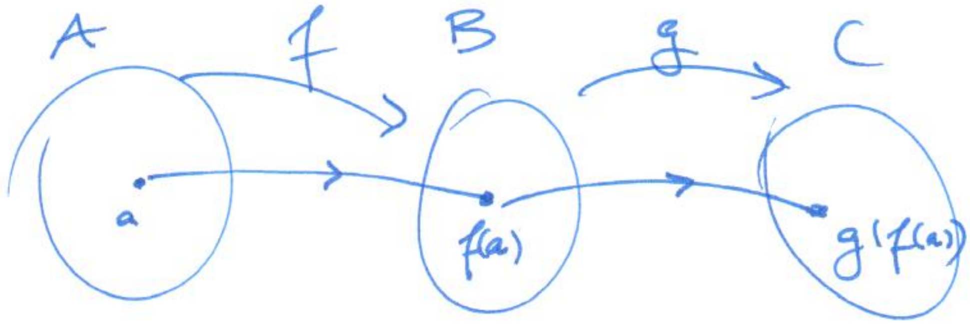
We conclude that a function is invertible if and only if it's a bijection.

**Defn**: The **identity function** on a set $A$, denoted by $\iota_A : A \to A$, is the function defined as $\iota_A(x) = x$ for all $x \in A$.

Observe that $\iota_A$ is an invertible function. (What's its inverse?)



$g \circ f$




In general $g \circ f \neq f \circ g$

**Defn**: Let $f : A \to B$ and $g : B \to C$ be two functions. The **composition** $g \circ f$ of $f$ and $g$ is the function $g \circ f : A \to C$ defined as

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in A$$

Observe that $g : B \to A$ is the inverse of $f : A \to B$ if and only if $g \circ f = \iota_A$ and $f \circ g = \iota_B$

Think about this a little bit.
(over the reading week)

elements in the domain are ordered pairs of the $(a,b)$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}-\{0\}$ ($b$ is a non-zero integer)

domain codomain

**Exercises on Functions**:
Let $f : \mathbb{Z} \times (\mathbb{Z} - \{0\}) \to \mathbb{Q}$ be a function defined as $f(m,n) = \dfrac{m}{n}$.
Is $f$ one-to-one? Is $f$ onto?

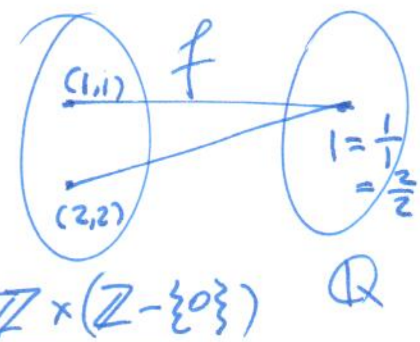
(Note that the function $f$ takes an ordered pair as input and gives a rational number as output.)

$f$ is not one-to-one

because $(1,1) \in \mathbb{Z} \times (\mathbb{Z}-\{0\})$ and

$(2,2) \in \mathbb{Z} \times (\mathbb{Z}-\{0\})$, but

$$f(1,1) = f(2,2)$$
$$\frac{1}{1} = 1 = \frac{2}{2}$$




Is $f$ onto?

Consider an arbitrary element, say $y$, from the codomain $\mathbb{Q}$. Since $y \in \mathbb{Q}$, we can write $y = \dfrac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. Then we see that $f(\underbrace{(a,b)}_{\in \mathbb{Z} \times (\mathbb{Z}-\{0\})}) = \dfrac{a}{b} = y$. So we conclude that any element from the codomain is in the image of the domain. Therefore, $f$ is onto.

codomain


can't get these numbers as output from $f(x)$

Let $f : (0, +\infty) \to (0, +\infty)$ be a function defined by $f(x) = 5x^2 + 7$ $= 5x^2 + 7$
Is $f$ one-to-one? Is $f$ onto?

$f$ is not onto because for example

$2 \in (0, +\infty)$ (the codomain), but

$f(x) \neq 2$ for any $x \in (0, +\infty)$ (the domain)

( Observe that $f(x) = 5x^2 + 7 = 2$

$\Rightarrow 5x^2 = -5$

$\Rightarrow x^2 = -1$

no solution for $x \in (0, +\infty)$ )

$f$ is 1-to-1 :

Suppose that $f(x) = f(y)$
for arbitrary $x, y \in (0, +\infty)$ (the domain)

$f(x) = 5x^2 + 7 \qquad 5y^2 + 7 = f(y).$

So $5x^2 + 7 = 5y^2 + 7 \quad / -7$

$\Rightarrow 5x^2 = 5y^2 \quad / \div 5$

$\Rightarrow x^2 = y^2 \quad / \sqrt{\ }$

$\Rightarrow x = \pm y$ Then since $x, y \in (0, +\infty)$ we can't have $x = -y$.



$\Rightarrow x = y$.

So, $f(x) = f(y) \Rightarrow x = y$, which shows that $f$ is 1-to-1.


Let $A$ be a set with 3 elements and $B$ be a set with 5 elements. Determine whether the statements below are true or false.

Correct

1. There is no onto function from $A$ to $B$.

power set

$|\mathcal{P}(A)| = 2^{|A|} = 2^3$ elements

2. There is no one-to-one function from $\mathcal{P}(A)$ to $\mathcal{P}(B)$. $|\mathcal{P}(B)| = 2^5$ elements
3. There is no one-to-one function from $\mathcal{P}(A)$ to $B$.
4. There is no onto function from $A \times B$ to $\mathcal{P}(B)$.
5. There is no one-to-one function from $\mathcal{P}(\mathcal{P}(A))$ to $\mathcal{P}(A \times B)$.

2. False, there are such one-to-one functions

4

$|A \times B| = 3 \cdot 5 = 15$

$|\mathcal{P}(B)| = 32 = 2^{|B|} = 2^5$

no such onto function exists.

Finish 3 & 5