A **complete graph** K_n:

K_6

each vertex in K_n has degree $n-1$

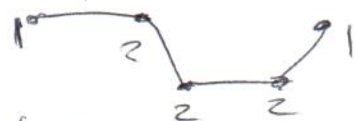
Then, by Handshake Th. there are $\frac{n(n-1)}{2}$ edges.

A **path** P_n of length n:

↳ # of edges

P_3

P_4



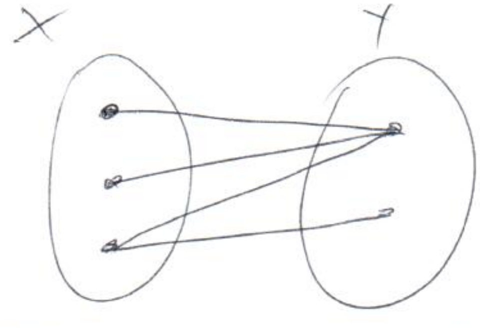
any vertex except for ~~the~~ two has degree 2.

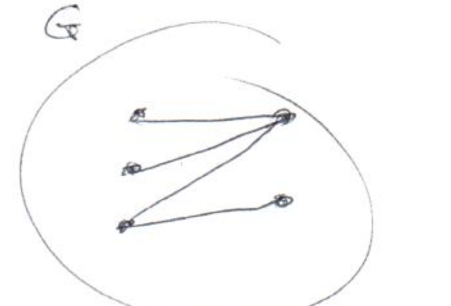
A **cycle** C_n of length n:
$(n \geq 3)$

C_6

any vertex in C_n has degree 2 (as long as $n \geq 3$)

C_3



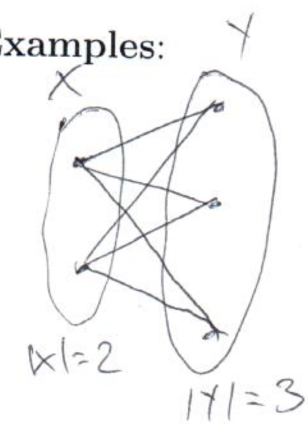


Bipartite graphs: A graph $G = (V, E)$ is called **bipartite** if V can be partitioned into two subsets X and Y such that every edge of G has one endpoint in X and the other endpoint in Y. If this is the case, then $\{X, Y\}$ is called a bipartition of G, and subsets X and Y are the two parts.

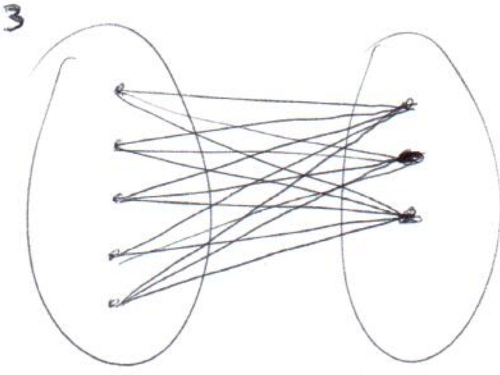
A **complete bipartite graph** $K_{m,n}$ (for $m, n \geq 1$) is a simple graph with $m + n$ vertices. The vertex set partitions into sets X and Y of cardinalities m and n, and each pair of vertices from distinct parts are adjacent.

Examples:

$K_{2,3}$



$K_{5,3}$

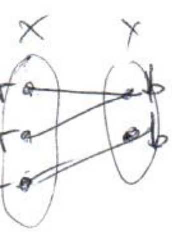


Theorem: The following statements about a graph G are equivalent:

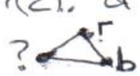
(1) G is bipartite,

(2) The vertices of G can be coloured with 2 colours (say, red and blue) so that the endpoints of each edge receive distinct colours,

(3) G has no subgraph that is a cycle of odd length.



(2) ⇒ (3) Contrapositive ¬(3) ⇒ ¬(2). G has a subgraph that is a cycle of odd length.



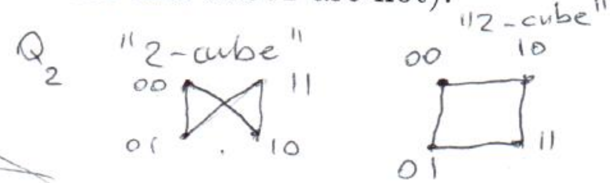
? needs a third colour to guarantee the endpoints of each edge have distinct colours. So ¬(2) follows.

The implications (1) ⇒ (2), (2) ⇒ (1), and (2) ⇒ (3) are easy to see. (3) ⇒ (2) is a bit more challenging.

(1) ⇒ (2) If G is bipartite, then we have a bipartition of the vertices into sets X & Y s.t all edges have one endpoint and one endpoint in Y. Then color all vertices in X with red and all vertices in Y with blue. So (2) follows

(2) ⇒ (1) Let X be the set of red vertices and Y be the set of blue vertices. Then (1) follows

Example: The k-**cube** Q_k (for $k = 1, 2, \ldots$) is the graph whose vertices are the binary strings of length k, and two vertices are adjacent if and only if they differ in exactly one bit (e.g. for $k = 5$, vertices 01001 and 11001 are adjacent, while 01001 and 00011 are not).



(i) Draw Q_k for $k = 2, 3$.

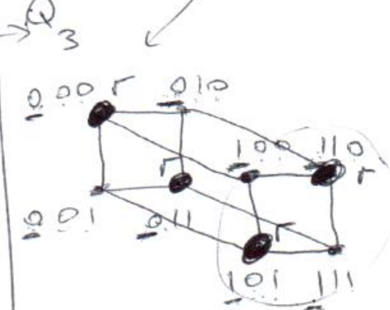
k coordinates (k bits) each is 0 or 1. So there 2^k vertices

(ii) Show that Q_k has 2^k vertices.

(iii) Show that Q_k is k-regular.

(iv) Show that Q_k has $k2^{k-1}$ edges.

(v) Show that Q_k is bipartite.



(iii) Consider any vertex v. Changing each bit yields a vertex adjacent to v. There are k bits in vertices in Q_k. So there are k adjacent vertices for each vertex. Therefore the degree of each vertex is k, and hence Q_k is k-regular.

(v) Let X be the set of vertices with an even # of 1's and Y be the set of vertices with an odd number of 1's. Then there cannot be any edge within X or within Y. So Q_k is bipartite.

(iv) Each vertex has degree k (by iii). There are 2^k vertices (by ii). So, the degree sum is $k \cdot 2^k$. Then by Handshake Theorem, we have $\frac{k \cdot 2^k}{2} = k \cdot 2^{k-1}$ edges.

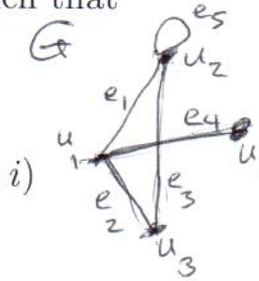
Matrices Associated to Graphs

Let G be a graph with $V(G) = \{u_1, u_2, \dots, u_n\}$, $E(G) = \{e_1, e_2, \dots, e_m\}$, and incidence function ψ_G. Then, the **incidence matrix** of G is an $n \times m$ matrix M with entries m_{ij} such that

$m_{ij} = 2$ if $\psi_G(e_j) = \{u_i\}$

$m_{ij} = 1$ if $\psi_G(e_j) = \{u_i, u_k\}$ $(k \neq i)$

$m_{ij} = 0$ otherwise.

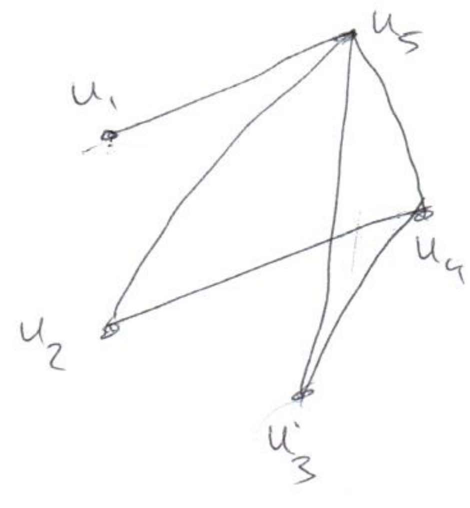


$$M = \begin{array}{c|ccccc} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline u_1 & 1 & 1 & 0 & 1 & 0 \\ u_2 & 1 & 0 & 1 & 0 & 2 \\ u_3 & 0 & 1 & 1 & 0 & 0 \\ u_4 & 0 & 0 & 0 & 1 & 0 \end{array}$$

The **adjacency matrix** of G is an $n \times n$ matrix A with entries a_{ij} where a_{ij} is the number of edges with endpoints u_i and u_j.

→ # of vertices

Example:



$$A = \begin{array}{c|ccccc} & u_1 & u_2 & u_3 & u_4 & u_5 \\ \hline u_1 & 0 & 0 & 0 & 0 & 1 \\ u_2 & 0 & 0 & 0 & 1 & 1 \\ u_3 & 0 & 0 & 0 & 1 & 1 \\ u_4 & 0 & 1 & 1 & 0 & 1 \\ u_5 & 1 & 1 & 1 & 1 & 0 \end{array}$$

Any adjacency matrix is symmetric.

What are the row and column sums in M?

column sums are all equal to 2

each row sum is the degree of the corresponding vertex

Now consider a simple graph G. Find the adjacency matrix A. What are the row and column sums in A?

example on p. 144

The row/column sums in a simple graph give us the degrees of the corresponding vertices.

Walks, Trails, Paths, Cycles

Let $G = (V, E)$ be a graph with the incidence function ψ_G. Let $x, y \in V$ and $k \in \mathbb{N}$.

An **(x, y)-walk** of length k in G is an alternating sequence W of vertices and edges $W = v_0 e_1 v_1 e_2 v_2 \ldots v_{k-1} e_k v_k$ such that

$v_0 \overset{\shortparallel}{} x$ $\quad v_k \overset{\shortparallel}{} y$

length = # of edges

$$v_0, v_1, \ldots, v_k \in V,$$

$$e_1, e_2, \ldots, e_k \in E,$$

$$v_0 = x \text{ and } v_k = y, \text{ and}$$

$$\psi_G(e_i) = v_{i-1} v_i \text{ for all } i = 1, 2, \ldots, k.$$

A walk $W = v_0 e_1 v_1 e_2 v_2 \ldots v_{k-1} e_k v_k$ is called **closed** if $v_0 = v_k$, and **open** otherwise.

A walk $W = v_0 e_1 v_1 e_2 v_2 \ldots v_{k-1} e_k v_k$ is called a **trail** if its edges are pairwise distinct.

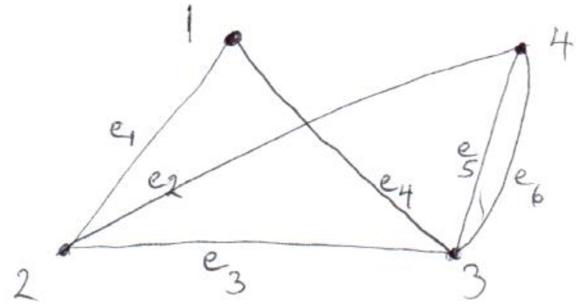
A walk $W = v_0 e_1 v_1 e_2 v_2 \ldots v_{k-1} e_k v_k$ is called a **path** if its vertices are pairwise distinct.

A walk $W = v_0 e_1 v_1 e_2 v_2 \ldots v_{k-1} e_k v_k$ is called a **cycle** if $v_0 = v_k$ while its **internal vertices** $v_1, \ldots, v_{k-1}$ are pairwise distinct.

In a simple graph, a walk $W = v_0 e_1 v_1 e_2 v_2 \ldots v_{k-1} e_k v_k$ can be simply denoted by its sequence of vertices as $W = v_0 v_1 v_2 \ldots v_{k-1} v_k$.

W is a path => W is a trail => W is a walk

Example: The following graph is given.



trail: distinct edges

path: distinct vertices

(i) Find a (1,3)-walk of length 5 that is not a trail.

$1e_1 2e_2 4e_5 3e_4 1e_4 3$

length 5
starts at 1
finishes at 3
not a trail
because e_4
is used
more than
once

(ii) Find a (1,3)-trail of length 3 that is not a path.

$1e_4 3e_6 4e_5 3$

length 3
(3 edges)

starts 1
ends at 3
trail because
no edges
repeated
not a path
because
a vertex (3)
is repeated

(iii) Find a (1,3)-path of length 3.

$1e_1 2e_2 4e_5 3$ path
because
no vertices repeated

has length 3
because
3 edges are
used.

(iv) Find a closed walk of length 5 that contains vertex 2 and is not a trail.

$1e_4 3e_5 4e_2 2e_3 3e_4 1$

not a trail because e_4 is
repeated

closed
because
starts and ends
at the same
vertex

length 5 because
5 edges used
contains vertex 2

(v) Find a cycle of length 4 containing vertex 2.

$1e_1 2e_2 4e_5 3e_4 1$

length 4 because 4 edges
used

it's a cycle because
all internal vertices distinct
contains vertex 2

and starts
and ends
at the same
vertex

Connected Graphs

A graph $G = (V, E)$ is called **connected** if for any $x, y \in V$ there exists an (x, y)-path (or equivalently, (x, y)-walk) in G.

A graph that is not connected is called **disconnected**.

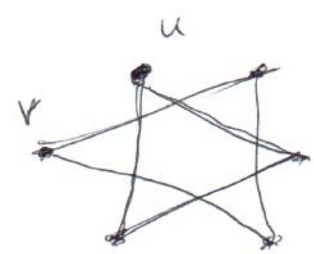
Fact: In any graph G, for any vertices x and y, there exists an (x, y)-path in G if and only if there exists an (x, y)-walk.

Example: Which one(s) of the graphs below are connected?

~~not~~

disconnected

connected



~~not~~

disconnected

(no (u,v)-walk exists)