ARAS $\frac{4!}{2!\,1!\cdot 1!} = \frac{24}{2} = 12$ arrangements of the letters in ARAS

2 A
1 R
1 S

AARS ARAS
AASR ARSA SRAA
ASAR SAAR RAAS
ASRA SARA RASA RSAA

Permutations of a Set with Repeated Elements:

Example: In how many ways can you arrange the letters in MISSISSAUGA?

11 letters

If all letters were distinct then we would have 11! arrangements.

S_1 S_2 S_3 S_4

2 I's
4 S's
2 A's
1 M
1 U
1 G

S_1MUGIIAS_2A$S_3$$S_4$ is the same word as S_4MUGIIAS_3A$S_1$$S_2$

$S_1 S_2 S_3 S_4$ can be permuted among themselves in 4! ways

There are $\frac{11!}{1!\cdot 1!\cdot 1!\cdot 4!\, 2!\, 2!}$ arrangements of the letters in MISSISSAUGA

Theorem: Suppose a collection consists of n objects of which

n_1 are of type 1 and are indistinguishable from each other,

n_2 are of type 2 and are indistinguishable from each other,

...

n_k are of type k and are indistinguishable from each other.

A_1 A_2 can be permuted in 2! ways

I_1 I_2 can be permuted in 2! ways.

Suppose that $n_1 + n_2 + \ldots + n_k = n$. Then the number of distinguishable permutations of the n objects is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

Combinations and the Binomial Theorem

Defn: Let n and r be non-negative integers with $r \le n$. An **r-combination** of a set of n elements is a subset of r of the n elements.

The number of subsets of size r (r-**combinations**) that can be chosen from a set of n elements is usually denoted by $\binom{n}{r}$ (read: "n choose r").

[Note that there are many alternative notations.] C^n_r

Numbers of the form $\binom{n}{r}$ are called **binomial coefficients**.

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

"n choose r"

Examples:

$$\binom{7}{2} = \frac{7!}{(7-2)!\,2!} = \frac{7!}{5!\,2!} = \frac{7\cdot 6\cdot \cancel{5\cdot 4\cdot 3\cdot 2\cdot 1}}{\cancel{5\cdot 4\cdot 3\cdot 2\cdot 1}\cdot 2\cdot 1}$$

$$= \frac{7\cdot 6}{2} = 21$$

$n = 7$
$r = 2$

$$\frac{7\cdot 6}{2!}$$

$$\binom{10}{4} = \frac{10\cdot 9\cdot 8\cdot 7}{4!} = \frac{10\cdot \overset{3}{\cancel{9}}\cdot \cancel{8}\cdot 7}{\cancel{24}} = 210$$

There are two distinct types of selecting r objects from a set of n elements.

In an ordered selection, it is not only what elements are chosen but also the order in which they are chosen that matters.

An ordered selection of r elements from a set of n elements is an r-**permutation** of the set.

In an unordered selection, on the other hand, the order in which the elements are chosen is irrelevant.

An unordered selection of r elements from a set of n elements is the same as a subset of size r or an r-**combination** of the set.

Relation between permutations and combinations:

Compare $P(4,2)$ to $\binom{4}{2}$.

$P(4,2) = 4 \cdot 3$

a b c d

ab
ac
ad
ba
bc
bd
ca
cb
cd
da
db
dc

$\frac{n!}{(n-r)!}$

In each permutation each selection of r things appear in $r!$ different copies.

$\frac{n!}{(n-r)!\,r!} = \binom{n}{r}$

Why is this the number of r element subsets of an n element set?

So if we forget about the ordering there are $\frac{P(n,r)}{r!}$ different selections (unordered)

$\frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\cdot r!}$

1 2 3 4 5 6

| • • • • |

$x_1 = 0$
$x_2 = 4$
$x_3 = 0$

separator locations 1 and 6

Number of Integer Solutions to Equations:

Example: How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 = \overset{r}{}\overset{n}{4}$$

where $x_1, x_2, x_3 \geq 0$?

15

sep. locations (2 and 3)

• | | • • •

x_1 x_2 x_3

$x_1 = 1$
$x_2 = 0$
$x_3 = 3$

sep. loc (2 and 5) "separators"

• | • • | •

x_1 x_2 x_3

$1 + 2 + 1$
x_1 x_2 x_3

| • • • | •

sep. location 1 and 5

$0 + 3 + 1$
x_1 x_2 x_3

x_1	x_2	x_3
4	0	0
3	1	0
3	0	1
2	2	0
2	1	1
2	0	2
1	3	0
1	2	1
1	1	2
1	0	3

x_1	x_2	x_3
0	4	0
0	3	1
0	2	2
0	1	3
0	0	4

There are 3 boxes and 4 indistinguishable objects to be placed into these boxes. In how many ways can this be done?

• • | • | •

box1 box2 box3

$$\binom{4+(3-1)}{3-1} = \binom{6}{2} = 15$$

to place the two separators we are "chosing" 2 of the 6 locations

$\binom{6}{2}$ ways

$\frac{6 \cdot 5}{2!} = 15$

Theorem: The number of ways to place n indistinguishable objects into r boxes is ...

$$\binom{\overset{\#\text{ of dots}}{n} + \overset{\#\text{ of separators}}{(r-1)}}{\underbrace{r-1}_{\#\text{ of separators}}}$$

$x_1 + x_2 \le 7 \quad x_1, x_2 \ge 0$

How many integer solutions?

Introduce x_3. $\underbrace{x_1 + x_2 + x_3}_{r} = \underbrace{7}_{n}$ $\quad x_1, x_2, x_3 \ge 0$

$\binom{n+(r-1)}{r-1} = \binom{9}{2} = \frac{9 \cdot 8}{2} = 36$

Example: How many integer solutions are there to the ~~equation~~ inequality

$$x_1 + x_2 + x_3 \le 4$$

where $x_1, x_2, x_3 \ge 0$?

Introduce x_4! Consider $x_1 + x_2 + x_3 + x_4 = 4$

Observe that $x_4 \ge 0$ (because $x_1 + x_2 + x_3 \le 4$)

Solve for $\underbrace{x_1 + x_2 + x_3 + x_4}_{r} = \underbrace{4}_{n}$ where $x_1, x_2, x_3, x_4 \ge 0$

$n = 4 \quad r = 4 \quad \binom{n+(r-1)}{r-1} = \binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3!} = 35$

Example: How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 9$$

where $x_1 \ge 2$, $x_2 \ge 0$, $x_3 \ge -3$, $x_4 \ge 4$?