Predicates and Quantifiers

Consider the following statement:

Aras doesn't have a driver's license.

Using propositional logic (that is the type of logic we have been dealing with) we cannot reach the following conclusion:

There exists a person in this room who doesn't have a driver's license.

This example suggests that we need a more powerful type of logic (namely, **predicate logic**) to be able to work with such statements that include quantifiers (in our example the quantifier is "there exists").

To understand predicate logic, we need to introduce the notion of a predicate first.

Consider the statement "x is greater than 6". It has two parts:

The first part is the **variable** x, which is the subject of the statement.

The second part is the **predicate**, "is greater than 6", which refers to a property that the subject of the statement can have.

The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

We can denote the statement "x is greater than 6" by $P(x)$, where P denotes the predicate "is greater than 6" and x is the variable. The statement $P(x)$ is also said to be the value of the propositional function P at x. Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value.

If a domain is not specified, then we take all elements of $\mathbb{R}$ as the domain (except for values that don't make sense)

Usually a predicate comes with a specified domain. In our case the domain may be $\mathbb{R}$, the set of all real numbers.

Example: Let $P(x)$ denote the statement "$x > 6$" with domain $\mathbb{R}$. What are the truth values of $P(-4)$ and $P(9)$?

$P(x)$: $x > 6$ domain: $\mathbb{R}$

$P(-4)$: $-4 > 6$ False

$P(9)$: $9 > 6$ True

If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x.

The truth set of $P(x)$ is the set $\{x \in D | P(x)\}$.

Example Let $Q(n)$ be the predicate "n is a factor of 8". Find the truth set of $Q(n)$ if

(i) the domain of n is the set of all positive integers

Truth set $\{n \in \mathbb{Z}^+ \mid Q(n)\}$

$= \{1, 2, 4, 8\}$

(ii) the domain of n is the set of all integers.

Truth set $\{n \in \mathbb{Z} \mid Q(n)\}$

$= \{\pm 1, \pm 2, \pm 4, \pm 8\}$

We can also have statements that involve more than one variable. Consider the statement "$x = y + 3$." We can denote this statement by $Q(x, y)$, where x and y are variables and Q is the predicate. When values are assigned to the variables x and y, the statement $Q(x, y)$ has a truth value.

Example: Let $Q(x, y)$ denote the statement "$x = y + 7$". What are the truth values of the propositions $Q(1, 5)$ and $Q(9, 2)$?

x y x y

$Q(1,5)$: $1 = 5+7$ False

$Q(9,2)$: $9 = 2+7$ True

$\boxed{Q(9,2) \neq Q(2,9)}$

$(Q(2,9)$: $2 = 9+7$ False$)$

In general, a statement involving n variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$.

A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function P at the n-tuple $(x_1, x_2, \ldots, x_n)$, and P is called a **n-ary predicate**.

Now that we have some understanding of predicates, we may move on to quantifiers.

We will focus on two types of quantification:

i) **universal quantification**, which tells us that a predicate is true for every element under consideration, and

ii) **existential quantification**, which tells us that there is one or more elements under consideration for which the predicate is true.

The Universal Quantifier: Many mathematical statements assert that a property is true for all values of a variable in a particular domain. Such a statement is expressed using universal quantification.

The universal quantification of $P(x)$ for a particular domain is the proposition that asserts that $P(x)$ is true for all values of x in this domain.

A value for x for which $P(x)$ is false is called a **counterexample** to the universal statement.

Note that the meaning of the universal quantification of $P(x)$ changes when we change the domain.

The universal quantification of $P(x)$ is the statement "$P(x)$ for all values of x in the domain."

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the **universal quantifier**. We read $\forall x P(x)$ as "for all x $P(x)$" or "for every x $P(x)$." An element for which $P(x)$ is false is a counterexample of $\forall x P(x)$.

Example: Let $P(x)$ be the statement "$x + 3 > x$". What is the truth value of $\forall x P(x)$, where the domain consists of all real numbers?

$\forall x P(x)$ True

$P(x) : x + 3 > x$

$D : \mathbb{R}$

$\forall x \quad x+3 > x$

$\leftrightarrow \quad \forall x \quad 3 > 0$

Other ways of expressing universal quantification besides "**for all**" and "**for every**": "**all of**", "**given any**", "**for arbitrary**", "**for each**", "**for any**", etc.

So how do you show that a universal quantification $\forall x P(x)$ is false?

– A single counterexample is all we need!

Example: Let $P(x)$ be the statement "$x^2 \geq x$". What is the truth value of $\forall x P(x)$, where the domain consists of all real numbers?

$D: \mathbb{R}$

$\forall x P(x)$

Consider $x = \frac{1}{2}$

$P(\frac{1}{2}): (\frac{1}{2})^2 \geq \frac{1}{2}$ False

$\underbrace{\forall x \; x^2 \geq x}$

False because $x = \frac{1}{2} \in D$ is a counterexample

A useful thing to note is that whenever we can list all elements in the domain, say $x_1, x_2, \ldots, x_n$, then $\forall x P(x)$ is equivalent to the conjunction $P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$.

Example: Let $P(x)$ be the statement "$x^2 < 42$". What is the truth value of $\forall x P(x)$, where the domain consists of all positive integers not exceeding 6?

$D = \{1, 2, 3, 4, 5, 6\}$

$\forall x P(x) \cong P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5) \wedge P(6)$

$1^2 < 42 \quad 2^2 < 42 \quad 3^2 < 42$

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So, $\forall x P(x)$ is true.

The existential quantifier: Many mathematical statements assert that there is an element with a certain property. Such statements are expressed using existential quantification. With existential quantification, we form a proposition that is true if and only if $P(x)$ is true for at least one value of x in the domain.

The existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

there exists

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the existential quantifier.

Other ways of expressing existential quantification besides "**there exists**": "for some", "for at least one", etc.

[Question: Can you express $\exists x P(x)$ using the universal quantifier $\forall$?] and maybe some logical connectives

Example: Let $P(x)$ denote the statement "$x > 8467839486389$". What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

$\exists x P(x)$ is true
for ex consider
$x = 8467839486390$

$$\exists x P(x) \cong \neg(\forall x \neg P(x))$$

Example: Let $Q(x)$ denote the statement "$x = x + 1$". What is the truth value of the quantification $\exists x Q(x)$, where the domain consists of all real numbers?

$\exists x\, Q(x)$ is false.

A useful thing to note is that whenever we can list all elements in the domain, say $x_1, x_2, \ldots, x_n$, then $\exists x P(x)$ is equivalent to the disjunction $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$.

Example: Let $P(x)$ be the statement "$x^2 < 2$". What is the truth value of $\exists x P(x)$, where the domain consists of all positive integers not exceeding 8?

$D = \{8, 7, 6, 5, 4, 3, 2, 1\}$

$P(x) : x^2 < 2$

$$\exists x P(x) \cong \underbrace{\underset{F}{\underset{8^2<2}{P(8)}} \vee \underset{F}{\underset{7^2<2}{P(7)}} \vee \cdots \vee \underset{T}{\underset{1^2<2}{P(1)}}}_{T}$$

So $\exists x P(x)$ is T

Note that there exist other quantifiers as well. One important example is the uniqueness quantifier. The **uniquesness quantifier** is denoted by $\exists!$, and $\exists! x P(x)$ states that there is exactly one element x which has the property $P(x)$.

uniqueness quantifier

Precedence of quantifiers: The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional logic.

Example: $\forall x P(x) \vee Q(x)$ means $(\forall x P(x)) \vee Q(x)$. $\underbrace{\exists x P(x)} \wedge Q(x)$

$(\exists x P(x)) \wedge Q(x)$

Logical Equivalences Involving Quantifiers:

We take a slightly different approach when showing logical equivalences involving quantifiers (compare this to logical equivalences we have seen in propositional logic).

Example: Show that $\forall x(P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$ are logically equivalent (where the same domain is used throughout).

Strategy: To show the logical equivalence of these statements we show that they always have the same truth value, no matter what the predicates P and Q are, and no matter which common domain is used.

Consider some predicates P and Q, with a common domain. To show the given logical equivalence we'll do two things:

First, we show that if $\forall x(P(x) \wedge Q(x))$ is true, then $\forall x P(x) \wedge \forall x Q(x)$ is true. Second, we show that if $\forall x P(x) \wedge \forall x Q(x)$ is true, then $\forall x(P(x) \wedge Q(x))$ is true.

The idea is to take an arbitrary element satisfying the left hand side, and then to show that it also satisfies the right hand side; and then vice versa.

Need to show:

$$\forall x(P(x) \wedge Q(x)) \cong \forall x P(x) \wedge \forall x Q(x)$$

(1) : $\underbrace{\forall x(P(x) \wedge Q(x))}_{\text{suppose}} \Rightarrow \underbrace{\forall x P(x) \wedge \forall x Q(x)}_{\text{show}}$

(1) So, suppose that $\forall x(P(x) \wedge Q(x))$ is true. This means that if a is in the domain, then $P(a) \wedge Q(a)$ is true. Hence, $P(a)$ is true and $Q(a)$ is true. Because $P(a)$ is true and $Q(a)$ is true for every element in the domain, we conclude that $\forall x P(x)$ and $\forall x Q(x)$ are both true, which means that $\forall x P(x) \wedge \forall x Q(x)$ is true.

(2) : $\underbrace{\forall x P(x) \wedge \forall x Q(x)}_{\text{suppose}} \Rightarrow \underbrace{\forall x (P(x) \wedge Q(x))}_{\text{show}}$

(2) Now, suppose that $\forall x P(x) \wedge \forall x Q(x)$ is true. Then $\forall x P(x)$ is true and $\forall x Q(x)$ is true. So, if a is in the domain, then $P(a)$ is true and $Q(a)$ is true. It follows that for all a, $P(a) \wedge Q(a)$ is true, therefore $\forall x(P(x) \wedge Q(x))$ is true.

We conclude that $\forall x(P(x) \wedge Q(x)) \cong \forall x P(x) \wedge \forall x Q(x)$.

Observe that this logical equivalence shows that we can distribute a universal quantifier over a conjunction. We can also distribute an existential quantifier over a disjunction. [Who wants to say how?]

$$\exists x(P(x) \vee Q(x)) \cong \exists x P(x) \vee \exists Q(x)$$

However, we cannot distribute a universal quantifier over a disjunction. [Examples anyone?]

$$\forall x(P(x) \vee Q(x)) \not\cong \forall x P(x) \vee \forall x Q(x)$$

Neither can we distribute an existential quantifier over a conjunction.

$$\exists x(P(x) \wedge Q(x)) \not\cong \exists x P(x) \wedge \exists x Q(x)$$

(maybe an assignment problem)

Consider $D = \mathbb{R}$

$P(x)$: $x \geq 0$

$Q(x)$: $x < 0$

$\forall x(P(x) \vee Q(x))$ T

$\underbrace{\forall x P(x)}_{F} \vee \underbrace{\forall x Q(x)}_{F}$ F

$\exists x P(x) \cong \neg(\forall x \neg P(x))$

Negating Quantified Expressions:

Consider the following statement:

$P(x)$: x likes predicate logic

"Every student in this class likes predicate logic."

$\forall x P(x)$

D: {students in this class}

We write this statement as $\forall x P(x)$, where $P(x)$ is the statement "x likes predicate logic" and the domain is the students in this class.

The negation of this statement is "It is not the case that every student in this class likes predicate logic."

This is equivalent to "There is a student in this class who doesn't like predicate logic." We can write this as $\exists x \neg P(x)$.

We've been considering the logical equivalence $\neg\forall x P(x) \cong \exists x \neg P(x)$. Next we show it more precisely.

$\forall x P(x) \cong \neg(\exists x \neg P(x))$

$\neg\forall x P(x) \cong \exists x \neg P(x)$:

Note that $\neg\forall x P(x)$ is true if and only if $\forall x P(x)$ is false.

Also, note that $\forall x P(x)$ is false if and only if there is an element x in the domain for which $P(x)$ is false.

This holds if and only if there is an element x in the domain for which $\neg P(x)$ is true.

Finally, "there is an element x in the domain for which $\neg P(x)$ is true" can be written as "$\exists x \neg P(x)$ is true".

We have shown that $\neg\forall x P(x)$ is true if and only if $\exists x \neg P(x)$ is true; so they are logically equivalent.

How do we negate an existential quantification?

Consider the proposition "There is a student in this class who likes predicate logic." This is the existential quantification $\exists x Q(x)$, where $Q(x)$ is the statement "x likes predicate logic".

The negation of this statement is the proposition "It is not the case that there is a student in this class who likes predicate logic."

This is equivalent to "Every student in this class does not like predicate ~~calculus"~~ logic". We can write this as $\forall x \neg Q(x)$.

Let us now show the equivalence $\underline{\neg \exists x Q(x) \cong \forall x \neg Q(x)}$ more precisely.

non-assignment exercise

The two equivalences we have shown about the negation of quantifiers are called De Morgan's laws for quantifiers.

De Morgan's laws for quantifiers:

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x\, P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x\, P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Examples: What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 5)$?

$\forall x\,(x^2 > x)$

negation: $\neg \forall x\,(x^2 > x)$

$\cong \exists x\, \neg(x^2 > x)$

$\cong \exists x\,(x^2 \leq x)$

$\exists x\,(x^2 = 5)$

negation: $\neg \exists x(x^2 = 5)$

$\cong \forall x\, \neg(x^2 = 5)$

$\cong \forall x\,(x^2 \neq 5)$

Examples: Find the negation of the following statement: "If a computer program has more than 100000 lines, then it contains a bug."

Domain: all computer programs that have more than 100000 lines

$Q(x)$: x contains a bug

given statement: $\forall x\, Q(x)$

negation: $\neg \forall x\, Q(x) \cong \exists x\, \neg Q(x)$

negation translated back to English:
"There is a computer program that has more than 100000 lines, which doesn't contain a bug."