Pigeonhole Principle

If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Example: What is the smallest number of students we need in a class to guarantee that

(a) at least 2 have birthday on the same day of the year?

In the "worst"-case scenario (365 days) 365 students might have all different birthdays. So there needs to be 365+1 students to guarantee that at least two have the same birthday.

(b) at least 2 have birthday in the same month?

J F M A M J J A S O N D

12 + 1 = 13 students

(c) at least 3 have birthday in the same month?

In the worst-case scenario, there can be 12 + 12 = 24 students with collectively two birthdays in each month; so we need 12 + 12 + 1 = 25 students to guarantee that there are at least 3 sharing the same birthday month.

Example: How many socks do you need to pull out of the drawer in the dark to guarantee that you get a pair of the same colour if the drawer contains

(a) 10 white and 10 black socks?

2+1=3 socks (worst-case scenario would be to draw two socks of different colour! Then the third sock guarantees a uniform pair)

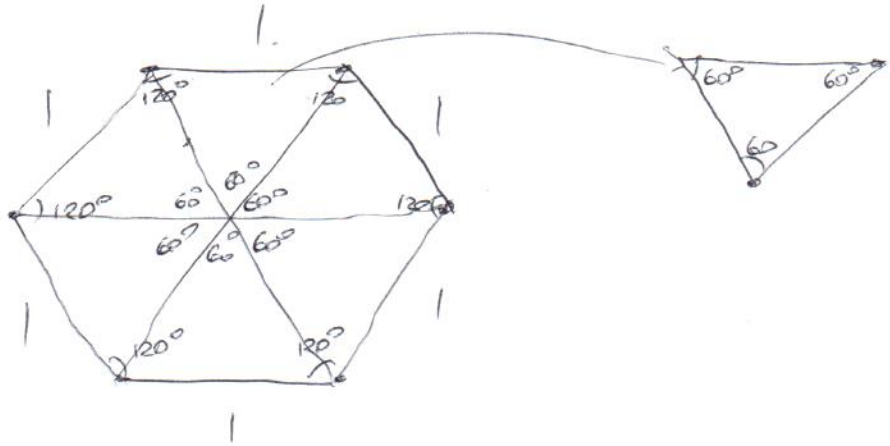
(b) 10 pairs of socks, each pair of a different colour?

10 + 1 = 11 socks.

It's possible to draw 10 socks with no matching pairs. Then the 11th sock will guarantee a uniform pair.

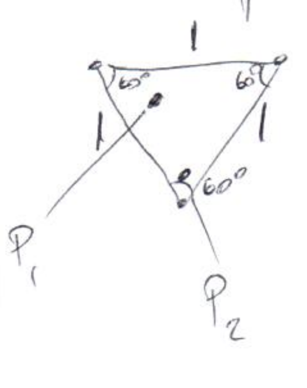
6-gon

Example: Show that no matter how 7 points are chosen in the interior or on the perimeter of a regular hexagon with side length 1, at least two of these points will be at distance at most 1.



7 points go in 6 triangles.

By Pigeonhole Principle there is at least one triangle which has two or more of the 7 points.



In such a triangle the longest distance between any two points is $\underline{1}$ (basic geometry).

So, we showed that there are at least two points that are at distance at most 1.

ex: 7, 12, 5, 6

6 & 12 $6-12=-6$ divisible by 3
$(-6=3\cdot(-2))$

ex: 1, 4, 5, 9
4 & 1 $4-1=3$ divisible by 3 $(3=1\cdot 3)$

Example: Use the pigeonhole principle to show that any set of four integers contains a pair of elements whose difference is divisible by 3.

Consider the set of integers. Any integer will have remainder 0, 1 or 2 from division by 3
(ex: $22=7\cdot 3+1$ → remainder $17=5\cdot 3+2$ → remainder)
$9=3\cdot 3+0$ → remainder

Consider the 3 groups of integers according to their remainder from division by 3.

[0]
[1] ⇒ 3 separate groups of integers
[2]
equivalence class

Example: Show that at any party (with at least two people!) there are at least two people who have the same number of friends.

4 integers
3 groups (boxes)

At least one group will contain at least two of the 4 integers.
So there is a pair of integers within the same equivalence class with respect to division by 3, call them x & y.
Then $x-y$ is divisible by 3.

Suppose that there are n people. $(n \geq 2)$

A, B, C, D, E

A - B
A - C friendship
C - D
C - E

A: 2 friends
B: 1 friend
C: 3 friends
E: 1 friend

B & E have the same number of friends.

You can have at most $n-1$ friends among n people including yourself.

It's also possible that you have 0 friends among the group of n people including yourself.

0, . . . , $n-1$

If there's someone with 0 friends, then noone can have $n-1$ friends.
Similarly if there's someone who's friends with everyone, then there cannot be anyone with 0 friends.

So, the number of friends anyone can have either ranges from 0 to $n-2$ or from 1 to $n-1$

Case 1: $[0, n-2]$

There are $n-1$ options (from 0 to $n-2$).

~~By Pigeonho~~

Case 2: $[1, n-1]$

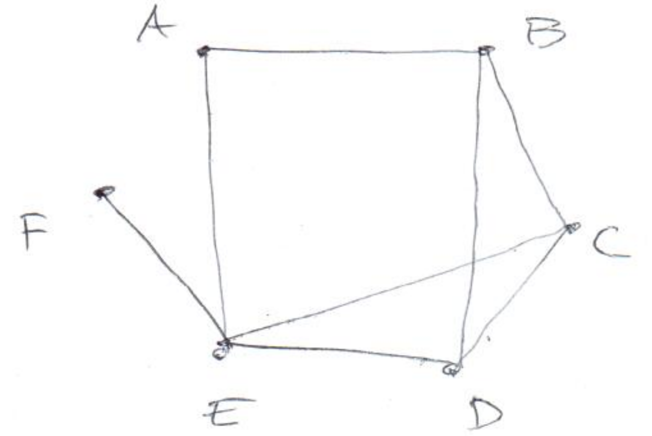
There are $n-1$ options (from 1 to $n-1$).

In either case, by Pigeonhole Principle, at least two of the n people will have the same number of friends.

(Pigeonhole Principle: objects: n people
boxes: $n-1$ possible number of friends)

n people can be represented by <u>vertices</u> (dots)

$n = 6$



Draw an <u>edge</u> (line) between friends

ex: A – B

A – E

B – C

B – D

C – D

C – E

E – F

The <u>degree</u> of a vertex is the number of edges at that vertex.

$\deg(A) = 2$, $\deg(B) = 3$, $\deg(C) = 3$

$\deg(D) = 3$, $\deg(E) = 4$, $\deg(F) = 1$

In the previous example we proved that in any graph (with at least two vertices) there are at least two vertices with the same degree.

Graph Theory

A **graph** G is an ordered pair (V, E), where

$V = V(G)$ is a non-empty set of vertices — the **vertex set** of G;

$E = E(G)$ is a set of edges — the **edge set** of G; and

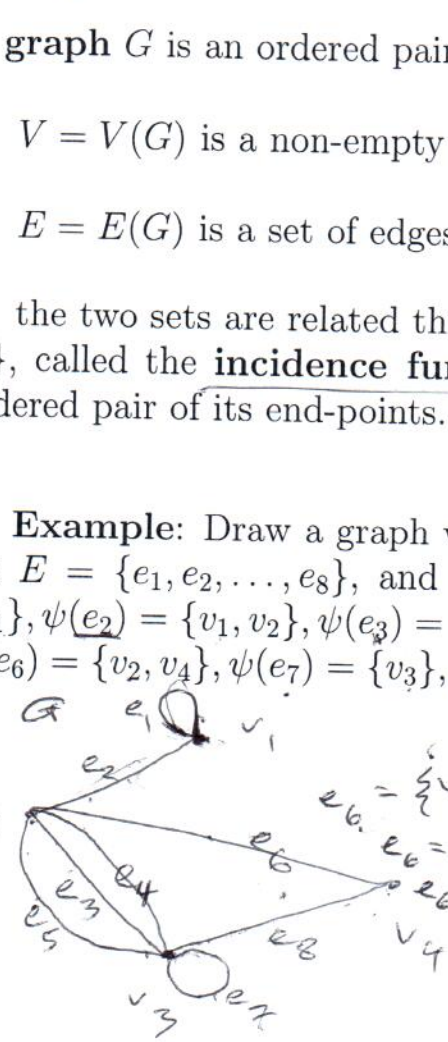
the two sets are related through a function $\psi_G : E \to \{u, v : u, v \in V\}$, called the **incidence function**, assigning to each edge the unordered pair of its end-points.

Example: Draw a graph with vertex set $V = \{v_1, v_2, v_3, v_4\}$, edge set $E = \{e_1, e_2, \ldots, e_8\}$, and incidence function defined by $\psi(e_1) = \{v_1\}, \psi(e_2) = \{v_1, v_2\}, \psi(e_3) = \{v_2, v_3\}, \psi(e_4) = \{v_2, v_3\}, \psi(e_5) = \{v_2, v_3\}, \psi(e_6) = \{v_2, v_4\}, \psi(e_7) = \{v_3\}, \psi(e_8) = \{v_3, v_4\}$.

$\deg_G v_1 = 3$

degree sequence $(3, 5, 6, 2)$

$|E| = 8$



$e_6 = \{v_2, v_4\}$

$e_6 = v_2 v_4$

$e_6 = v_4 v_2$

e_1 is a loop

e_7 is a loop

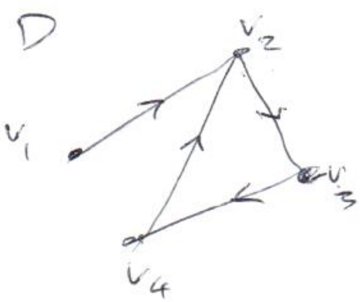
e_3, e_4 and e_5 are parallel edges

An edge e in a graph G is called a **loop** if $\psi_G(e) = \{u\}$ for some vertex $u \in V(G)$ (so, a loop is an edge whose endpoints is the same vertex). Distinct edges e_1 and e_2 in a graph G are called **parallel** or **multiple** if $\psi_G(e_1) = \psi_G(e_2)$ (so, two edges are parallel if they have the same endpoints). A **simple graph** is a graph without loops and without multiple edges.

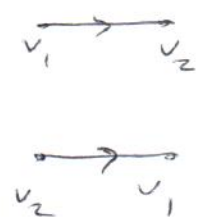
To simplify notation, in a graph the incidence function may be omitted. We may write $e = \{u, v\}$ instead of $\psi(e) = \{u, v\}$. In a simple graph, we may also write shortly $e = uv$ instead of $e = \{u, v\}$, omitting the braces. Note that uv is an unordered pair of vertices u and v; thus, $uv = vu$.

In a **directed graph (digraph)** D, each edge has an associated direction. Directed edges are called **arcs**, and the **arc set** is usually denoted by A. Therefore, in a directed graph the arcs are ordered pairs.

Example:



$(v_1, v_2) \neq (v_2, v_1)$

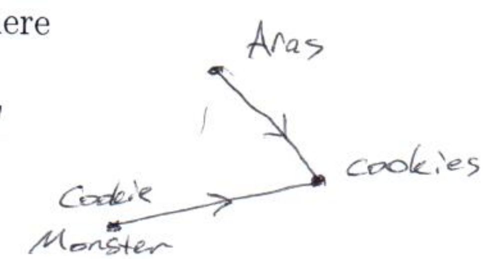


$A = \{\underbrace{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_2)}_{\text{arcs}}\}$

Graphs and digraphs have many applications in sciences, often representing network systems.

Niche overlap graphs in ecology: $G = (V, E)$ where
$V = \{\text{species in an ecosystem}\}$
$uv \in E \leftrightarrow$ species u and v compete for resources

Predator-prey graphs: $D = (V, A)$ where
$V = \{\text{species in an ecosystem}\}$
$(u, v) \in A \leftrightarrow$ species u preys on species v

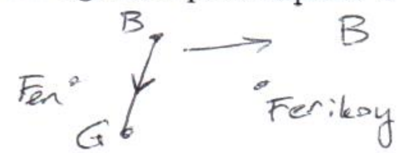


Friendship graphs: $G = (V, E)$ where
$V = \{\text{people in a group}\}$
$uv \in E \leftrightarrow$ persons u and v are friends

teams

Beşiktaş
Fenerbahçe
Galatasaray
Feriköyspor

Tournament graphs: $D = (V, A)$ where
$V = \{\text{participants in a round-robin tournament}\}$
$(u, v) \in A \leftrightarrow$ participant u won a match against participant v



$\longrightarrow$ B won a match against G

Exercise: The **intersection graph** of a collection of sets $\{A_1, \ldots, A_n\}$ is the graph with vertices $A_1, \ldots, A_n$, where vertices A_i and A_j are adjacent if and only if their intersection is non-empty.

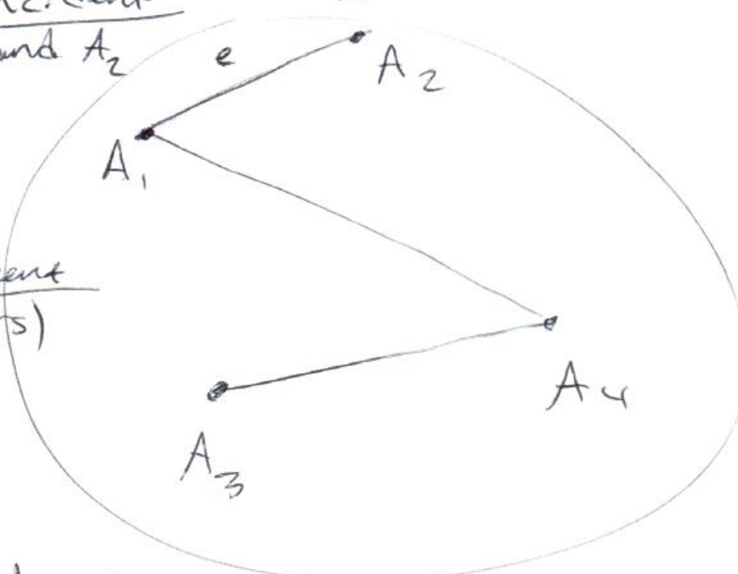
Construct the intersection graph for the following collection of sets:

$A_1 = \{1, 3, 4, 5\}$, $A_2 = \{1, 3, 5\}$, $A_3 = \{2, 6\}$, $A_4 = \{2, 4\}$.

G

e is incident with A_1 and A_2

A_1 & A_2 are adjacent (neighbours)



$A_1 \cap A_2 = \{1,3,5\} \neq \emptyset$

$A_1 \cap A_3 = \emptyset$

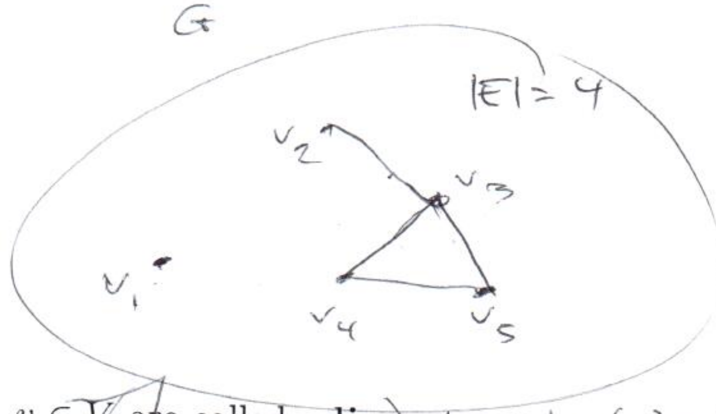
$A_1 \cap A_4 = \{4\}$

$A_2 \cap A_3 = \emptyset$

$A_2 \cap A_4 = \emptyset$

$A_3 \cap A_4 = \{2\}$

$\deg_G A_1 = 2$



vertex set

edge set

Graph Terminology

Let $G = (V, E)$ be a graph. Vertices $u, v \in V$ are called **adjacent** or **neighbours** in G if uv is an edge of G.

An edge uv is said to be **incident** with each of its endpoints u and v.

$deg_G(v_1) = 0$
v_1 is an isolated vertex

The **degree** of a vertex $u \in V$, denoted by $deg_G(u)$, is the number of edges of G incident with vertex u, each loop counting twice.

A vertex of degree 0 is called **isolated**.

If $V = \{v_1, v_2, \dots, v_n\}$, then the sequence $(deg_G(v_1), deg_G(v_2), \dots, deg_G(v_n))$ is called a **degree sequence** of G.

Example:

degree sequence of G:
(0, 1, 3, 2, 2)
$\| = deg_G(v_1)$
$deg_G(v_2)$

Theorem (The *Handshake Theorem*): In any graph $G = (V, E)$,

$$\sum_{u \in V} deg_G(u) = 2|E|.$$

Number of hands shaken is twice the number of handshakes.

vertex set

edge set

size of the set E (the # of edges)

Proof: Any edge (or loop) contributes two to the sum of degrees. Therefore the sum of the degrees is twice the number of edges.

Example: How many edges are there in a graph with 5 vertices and degree sequence (2,3,3,4,4)?

degree sum $= 2+3+3+4+4 = 16$

By Handshake Theorem, $16 = 2|E|$

$\Rightarrow |E| = \frac{16}{2} = 8$

There are 8 edges

Example: A graph with 8 edges has twice as many vertices of degree 3 as there are vertices of degree 2 (and no other vertices). How many vertices are of degree 2?

8 edges

$|V_3|$ # of vertices of degree 3: $2x$

$|V_2|$ # of " " degree 2: x

Find # of vertices of degree 2.

By Handshake theorem, the degree sum is $8 \cdot 2 = 16$.

$16 = 3(2x) + 2(x)$

$16 = 6x + 2x = 8x \Rightarrow x = 2 \Rightarrow$ So there are 2 vertices of degree 2 and 4 vertices of degree 3

Corollary: Every graph has an even number of vertices of odd degree.

Proof: