Let $f : \mathbb{Q} \to \mathbb{Q} \times \mathbb{Q}$ be a function defined as $f(x) = (x^3, x^4)$.

Is $f$ injective?

Suppose $f(x) = f(y)$ for some $x, y \in \mathbb{Q}$ (domain).
Then $(x^3, x^4) = (y^3, y^4)$. So $x^3 = y^3$ and $x^4 = y^4$.
Then from $x^3 = y^3$ we get $x = y$. So, we have shown that $f(x) = f(y) \Rightarrow x = y$.
Therefore, $f$ is injective (one-to-one).

Is $f$ onto? (surjective)

Consider $(1,0) \in \underbrace{\mathbb{Q} \times \mathbb{Q}}^{\text{codomain}}$.
We claim that $(1,0)$ is not in the image of $f$. To show this, suppose $(1,0)$ is in the image.
Then there must be $x \in \mathbb{Q}$ such that $f(x) = (1,0)$. This means that $(x^3, x^4) = (1,0)$.
So, $x^3 = 1$ and $x^4 = 0$. $x^4 = 0 \Rightarrow x = 0$, which contradicts $x^3 = 1$, which proves our claim.
Therefore, $f$ is not onto.

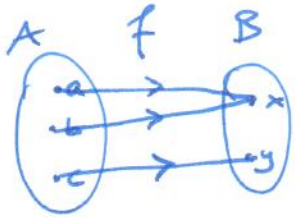
Is $f$ a bijection? Is $f$ invertible?

not a bijection because it's not onto.
not invertible since it's not a bijection.

## Relations

When defining a function we require that the function takes each element from the domain to exactly one element in the codomain.

Given a function $f : A \to B$, one can associate it to a subset $G_f$ of $A \times B$, where elements are some ordered pairs $(a, b)$ $(a \in A, b \in B)$.




$G_f = \{(a,x), (b,x), (c,y)\}$

Note that since $f$ is a function from $A$ to $B$, for each $x \in A$, there is exactly one $(x, y) \in G_f$.

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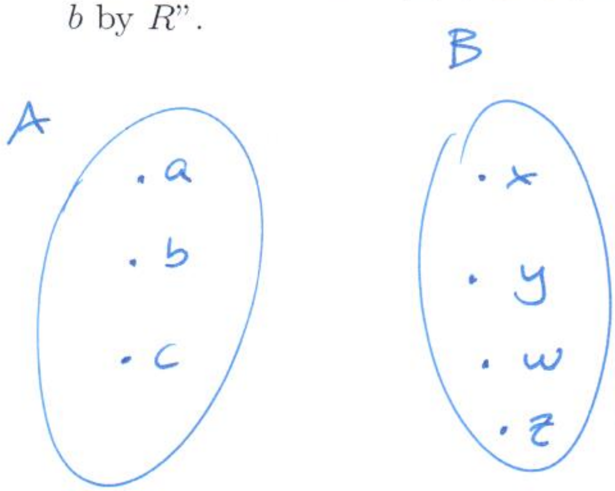
Now we introduce a new notion, called a **relation**, which will serve as a generalization of the notion of function:

**Defn**: A **binary relation** $\mathcal{R}$ from a set $A$ to a set $B$ is a subset of $A \times B$. We write $(a, b) \in \mathcal{R}$ (or $a\mathcal{R}b$) to denote that "$a$ is related to $b$ by $R$".




$A \times B = \{(a,x), (a,y), (a,w), (a,z), (b,x), (b,y), (b,w), (b,z), (c,x), (c,y), (c,w), (c,z)\}$

$R_1 = \{(a,x), (b,y), (b,w)\}$ is a relation

$(a,x) \in R_1$

$a R_1 x$

$R_2 = \{(c,x), (c,y), (c,w), (c,z)\}$ is a relation

$(c,y) \in R_2$

$c R_2 y$

$A = \{a, b, c\}$  $\mathcal{P}(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

$|A| = 3$

$|\mathcal{P}(A)| = 8 = 2^3$  $|\mathcal{P}(A)| = 2^{|A|}$

From this definition we observe that a relation is similar to a function in that it relates elements of a set $A$ to elements of a set $B$, but there is a significant difference: unlike functions, a relation is not required to assign each element of $A$ to exactly one element in $B$.

Fact: For finite sets $A$ and $B$, the number of relations from $A$ to $B$ is $|\mathcal{P}(A \times B)| = 2^{|A \times B|} = 2^{|A||B|}$

→ power set

One can also define a relation from a set to itself.

**Defn**: A binary relation from $A$ to $A$ (i.e. a subset of $A \times A$) is referred to as a **binary relation on** $A$.

**Properties of Relations**

$A = \{a, b, c\}$

A relation on $A$ is a subset of $A \times A$

$= \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c)\}$

**Defn**: A relation $\mathcal{R}$ on a set $A$ is

- **reflexive** if for all $a \in A$, $(a, a) \in \mathcal{R}$
- **symmetric** if for all $a, b \in A$, $(a, b) \in \mathcal{R} \Rightarrow (b, a) \in \mathcal{R}$
- **antisymmetric** if for all $a, b \in A$, $((a, b) \in \mathcal{R}$ and $(b, a) \in \mathcal{R}) \Rightarrow a = b$
- **transitive** if for all $a, b, c \in A$, $((a, b) \in \mathcal{R}$ and $(b, c) \in \mathcal{R}) \Rightarrow (a, c) \in \mathcal{R}$

**Examples**: For each relation defined on $\mathbb{Z}$, determine if it is reflexive, symmetric, antisymmetric or transitive.

$1 R_1 1$ $\quad -3 R_1 -3$ $\quad -4 \not R_1 3$

$(-4,3) \notin R_1$

- $\mathcal{R}_1 = \{(a,b) : a = b\}$

$R_1$ is reflexive because for any $a \in \mathbb{Z}$, $a=a$. (So $\forall a \in \mathbb{Z}, a R_1 a$)

$R_1$ is symmetric because for any $a,b \in \mathbb{Z}$, $a=b \Rightarrow b=a$. So, $(a,b) \in R_1 \Rightarrow (b,a) \in R_1$.

$R_1$ is antisymmetric because for any $a,b \in \mathbb{Z}$ s.t $a=b$ and $b=a$, it follows that $a=b$.
So, $(a,b) \in R_1$ and $(b,a) \in R_1 \Rightarrow a=b$.

$R_1$ is transitive because for all $a,b,c \in \mathbb{Z}$, s.t $a=b$ and $b=c$, we have $b=c$. So if $(a,b) \in R_1$ and $(b,c) \in R_1$, then $(a,c) \in R_1$.

- $\mathcal{R}_2 = \{(a,b) : a \geq b\}$

reflexive ?: Let $a \in \mathbb{Z}$ be an arbitrary element. Since $a \geq a$, we have that $(a,a) \in R_2$.
Therefore, $R_2$ is reflexive.

symmetric ?: $R_2$ is not symmetric because for example $(1,0) \in R_2$ (since $1 \geq 0$), but $(0,1) \notin R_2$ (because $0 \not\geq 1$).

antisymmetric ?: Let $a,b \in \mathbb{Z}$ s.t $(a,b) \in R_2$ and $(b,a) \in R_2$.
Then $a \geq b$ and $b \geq a$. Therefore, $a=b$, which shows that $R_2$ is antisymmetric.

transitive ?: Let $a,b,c \in \mathbb{Z}$. Suppose $(a,b) \in R_2$ and $(b,c) \in R_2$. Then $a \geq b$ and $b \geq c$. So $a \geq b \geq c$, hence $a \geq c$. Therefore $(a,c) \in R_2$. This means that $R_2$ is transitive.

Let $a,b,c \in \mathbb{Z}$. Suppose $(a,b) \in R_3$ and $(b,c) \in R_3$. So, $a<b$ and $b<c$. Then $a<b<c$, hence $a<c$. So $(a,c) \in R_3$. This shows that $R_3$ is transitive

- $\mathcal{R}_3 = \{(a,b) : a < b\}$ reflexive ?: Consider $2 \in \mathbb{Z}$. Since $2 \nless 2$, we have that $(2,2) \notin R_3$. Therefore, $R_3$ is not reflexive.

$R_3$ is not symmetric because for example $(2,3) \in R_3$ $(2<3)$, but $(3,2) \notin R_3$ $(3 \nless 2)$.

Let $a,b \in \mathbb{Z}$ s.t $(a,b) \in R_3$ and $(b,a) \in R_3$. Then, $a<b$ and $b<a$. Since $a<b \wedge b<a$ is never possible, it is a false proposition making $a<b \wedge b<a \Rightarrow a=b$ true. So, we showed that $(a,b) \in R_3$ and $(b,a) \in R_3$ implies $a=b$, so $R_3$ is antisymmetric.

- $\mathcal{R}_4 = \{(a,b) : a|b\}$ "a divides b"

exercise

- $\mathcal{R}_5 = \{(a,b) : a = 2b+1\}$

exercise

Let $a, b, c \in \mathbb{Z}$. Suppose that $(a,b) \in R_6$ and $(b,c) \in R_6$. Then $|a| = |b|$ and $|b| = |c|$. So $|a| = |b| = |c|$, hence $|a| = |c|$, which means that $(a,c) \in R_6$. Therefore $R_6$ is transitive.

- $\mathcal{R}_6 = \{(a,b) : |a| = |b|\}$ Let $a \in \mathbb{Z}$ be an arbitrary elt.

Since $|a| = |a|$, we have $(a,a) \in R_6$. Therefore, $R_6$ is reflexive.

$R_6$ is symmetric: Suppose $a, b \in \mathbb{Z}$ s.t $(a,b) \in R_6$. So, $|a| = |b|$. Then $|b| = |a|$, so $(b,a) \in R_6$.

$R_6$ is not antisymmetric because $(2,-2) \in R_6$ and $(-2,2) \in R_6$, but $2 \neq -2$.

- $\mathcal{R}_7 = \{(a,b) : a \leq b + 5\}$ Let $a \in \mathbb{Z}$ be an arbitrary element

Since $a \leq a + 5$, $(a,a) \in R_7$. Therefore, $R_7$ is reflexive.

$R_7$ is not symmetric. To see this, consider $a = 2 \in \mathbb{Z}$ and $b = 9 \in \mathbb{Z}$. $(a,b) \in R_7$ because $2 \leq 9 + 5$, but $(b,a) \notin R_7$ because $9 \nleq 2 + 5$.

$R_7$ is not antisymmetric because $(1,2) \in R_7$ $(1 \leq 2+5)$ and $(2,1) \in R_7$ $(2 \leq 1+5)$ but $1 \neq 2$.

Consider $a = 6$, $b = 3$ and $c = 0$ $(a, b, c \in \mathbb{Z})$.

Observe that $(a,b) \in R_7$ $(6 \leq 3+5)$ and $(b,c) \in R_7$ $(3 \leq 0+5)$, but $(a,c) \notin R_7$ $(6 \nleq 0+5)$.

So, $R_7$ is not transitive.