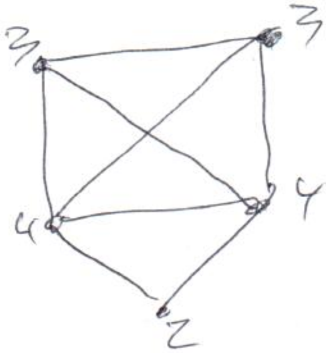
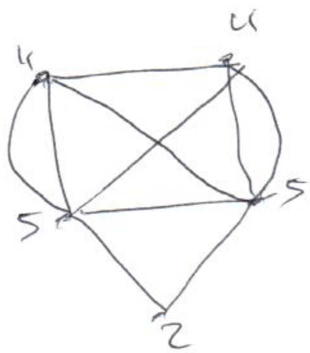
J_1



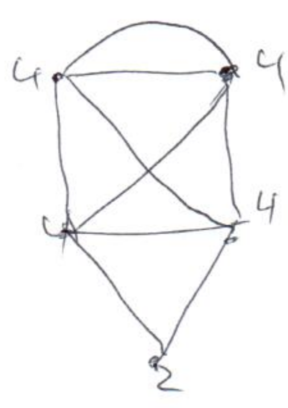
No, we cannot start at a vertex, go through each edge exactly once and come back to the starting vertex.

J_2



No.

J_3



Yes.

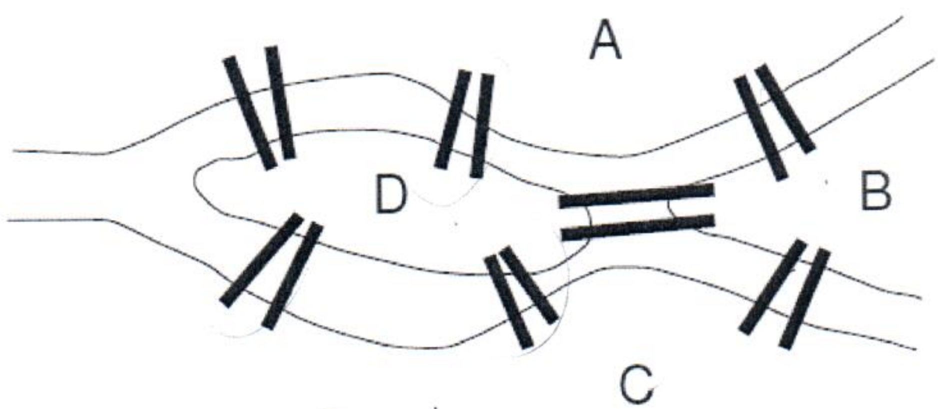
J_4

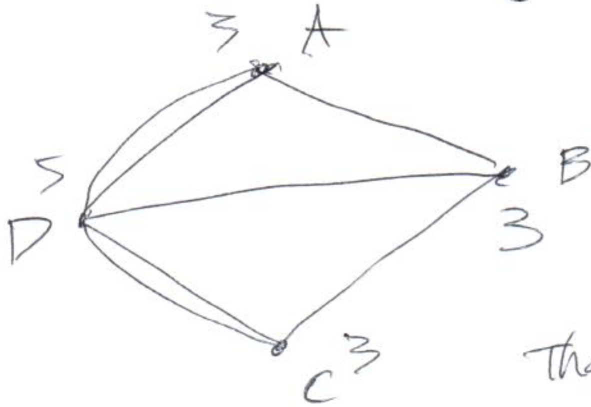
Yes

Euler Tours and Euler Trails

In 1736, while stationed in St. Petersburg, the Swiss mathematician Leonhard Euler took interest in the following puzzle.

The townsfolk of Königsberg, Prussia (now Kaliningrad in Russia) take long Sunday walks. They wonder if it is possible to walk around the town, traverse each of the seven bridges of Königsberg exactly once and return to the starting point.





The degrees are not all even; so, no such tour possible.

This puzzle is equivalent to the question whether the graph ~~below~~ above admits a closed trail that traverses each edge of the graph (exactly once). In Euler's honour, such a closed trail is called an Euler tour.

An **Euler tour** in a graph G is a closed trail of G traversing each edge of G (being a trail it does not repeat any edge).

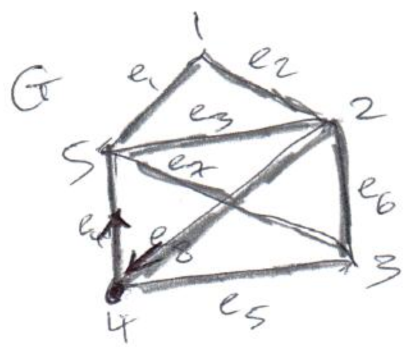
An **Euler trail** in a graph G is an open trail of G traversing each edge of G (again, being a trail it does not repeat any edge).

Example: Does the graph that represents Königsberg have an Euler tour?

No.

Example: For each of the graphs below, determine if it has an Euler tour, an Euler trail or neither.

We have many examples already.



G admits no Euler tour, but it admits an Euler trail.

$4 e_4 5 e_1 1 e_2 2 e_3 5 e_7 3 e_6 2 e_8 4 e_5 3$

Theorem: Let G be a connected graph. Then G has an Euler tour if and only if it has no vertices of odd degree.

Proof: ($\Rightarrow$) Suppose G has an Euler tour T. Each time an internal vertex x is visited by the tour, 2 edges of the graph are traversed, contributing 2 to the degree of x. Hence $deg_G(x)$ is even for all internal vertices of the tour. For the initial (= terminal) vertex of the tour (say u), the initial edge of the tour adds 1 to $deg_G(u)$, each visit of u as an internal vertex of the tour adds 2, and the last edge of the tour adds 1 to $deg_G(u)$. Hence $deg_G(u)$ is also even. Therefore, if G admits an Euler tour, then every vertex of G has even degree.

($\Leftarrow$) Suppose G has no vertices of odd degree. We construct an Euler tour in G as follows. Starting from any vertex, let T be a longest trail (that is, any trail that can not be extended). Since every vertex in G has even degree and T can not be extended, T must be a closed trail. If T traverses all edges of G, then T is the required Euler tour of G. Otherwise, remove the edges of T from G (as well as any isolated vertices) to obtain a graph G'. Note that the degrees of all vertices in G' are even. Since G is connected, T and G' must have a common vertex, say w. Now let T' be a longest trail in G' starting from vertex w. As above, T' must in fact be a closed trail. Join T and T' into a longer closed trail T'', using their common vertex w. (That is, we obtain T'' by traversing T up to vertex w, then traverse T' entirely from w, then continue along T back to its initial vertex.) Now replace T by T'', and continue the process until all edges of G have been used up. Since G has a finite number of edges, this process indeed terminates, producing an Euler tour of G.

Theorem: Let G be a connected graph. Then G has an Euler trail if and only if it has exactly two vertices of odd degree.