Equivalence relations

An important family of relations is the equivalence relation. (We will soon see why it is important.)

Defn: A relation $\mathcal{R}$ on a set A is called an **equivalence relation** if it is reflexive, symmetric and transitive.

If $\mathcal{R}$ is an equivalence relation on a set A, then one defines the **equivalence class** of each element of A with respect to $\mathcal{R}$ as follows:

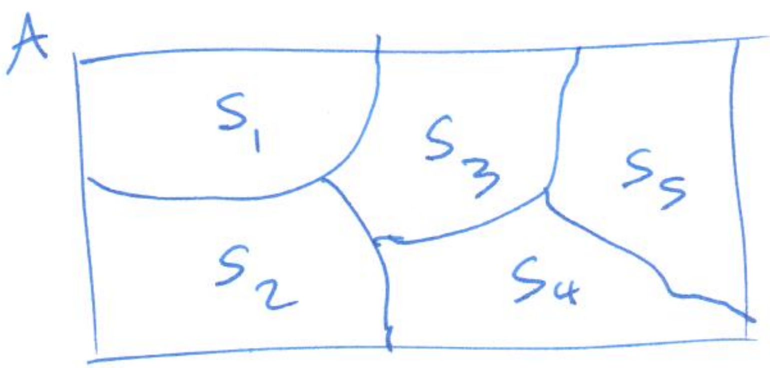
$$[a]_{\mathcal{R}} = \{b \in A : (a,b) \in \mathcal{R}\}$$

Note that $a \in [a]_{\mathcal{R}}$. (WHY?)

Defn: A **partition** of a set A is a collection of non-empty sets $\{S_1, S_2, \ldots\}$ of A with the property that

$A = S_1 \cup S_2 \cup \ldots$ where $S_i \cap S_j = \emptyset$ whenever $i \neq j$.

(Note that the collection $\{S_1, S_2, \ldots\}$ may contain an infinite number of sets.)



$A = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$

$S_i \cap S_j = \emptyset$ whenever $i \neq j$

ex:

$A = \{a, b, c, d, e, f\}$

$S_1 = \{a, b, e\}$

$S_2 = \{c\}$ $\quad S_3 = \{d, f\}$

$\{S_1, S_2, S_3\}$ is a partition of A because $A = S_1 \cup S_2 \cup S_3$ and $S_1 \cap S_2 = \emptyset$, $S_1 \cap S_3 = \emptyset$, $S_2 \cap S_3 = \emptyset$

Theorem: *If $\mathcal{R}$ is an equivalence relation on a set A, then the collection of equivalence classes with respect to A forms a partition of A.*

Moreover, if $\{S_1, S_2, \ldots\}$ is a partition of A, then there exists an equivalence relation on A with equivalence classes $S_1, S_2, \ldots$.

Example (of an equivalence relation): (**Congruence modulo m**)

Consider the set of integers $\mathbb{Z}$. Then $a \in \mathbb{Z}$ is said to be **congruent** to $b \in \mathbb{Z}$ **modulo** m if and only if

ex: $a = 7$, $b = 3$

≥ 1

$$a - b = km$$

$m = 4$

for some integer k.

$7 \equiv 3 \pmod 4$

because $7 - 3 = 4 = 1 \cdot 4$

Equivalently, a and b give the same remainder in $\{0, 1, \ldots, m-1\}$ when divided by m.

for ex: $8 \equiv 11 \pmod 3$ and observe that both 8 & 11 leave a remainder of 2 when divided by 3.

ex: $a = 5$ $b = 20$

$m = 5$

$5 \equiv 20 \pmod 5$

because $5 - 20 = -15 = -3 \cdot 5$

We write

$$a \equiv b \pmod{m}.$$

We ~~will~~ now show that the relation $\mathcal{R}$ on the set of integers $\mathbb{Z}$ defined as

$$\mathcal{R} = \{(a, b) : a \equiv b \pmod{m}\}$$

is an equivalence relation.

R is

(i) reflexive: Let $a \in \mathbb{Z}$. Then $a - a = 0 = 0 \cdot m$. So $(a,a) \in R$.

(ii) symmetric: Let $(a,b) \in R$, ($a, b \in \mathbb{Z}$). Then $a - b = k \cdot m$ for some $k \in \mathbb{Z}$. Then $b - a = -k \cdot m$ where $-k \in \mathbb{Z}$. So $(b,a) \in R$. This shows that R is symmetric.

(iii) transitive: Suppose $(a,b) \in R$ and $(b,c) \in R$ ($a, b, c \in \mathbb{Z}$). So, $a - b = k_1 \cdot m$ and $b - c = k_2 \cdot m$ ($k_1, k_2 \in \mathbb{Z}$)

$\rightarrow (a-b)+(b-c) = k_1 m + k_2 m$ So $a - c = (k_1 + k_2) m$ so $(a,c) \in R$

$a = 71$

$b = 13$

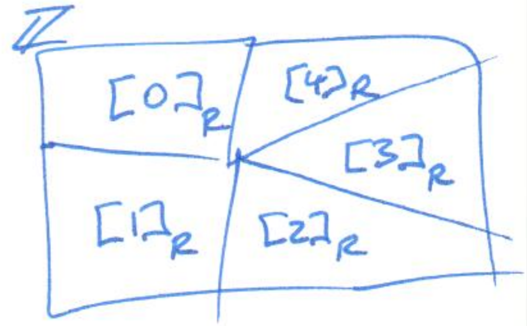
$m = 4$

$a \not\equiv b \pmod m$

because $71 - 13 = 58$ and 58 is not equal to $k \cdot 4$ for any integer k.

$[4]_R = \{\ldots, -6, -1, 4, 9, 14, 19, \ldots\}$

$[3]_R = \{\ldots, -7, -2, 3, 8, 13, 18, \ldots\}$



What are the equivalence classes of the equivalence relation

$$\mathcal{R} = \{(a, b) : a \equiv b \pmod 5\}?$$

$[5]_R = [0]_R = \{\ldots, -5, 0, 5, 10, 15, 20, \ldots\} = [10]_R = [-5]_R \ldots$

$[6]_R = [1]_R = \{\ldots, -9, -4, 1, 6, 11, \ldots\} = [11]_R = [-4]_R$

$[2]_R = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$

Example: Prove that $\mathcal{R}$ defined on $\mathbb{Z}$ as

$$\mathcal{R} = \{(a, b) : |a| = |b|\}$$

(This is R_6.)

is an equivalence relation and determine its equivalence classes.

We showed yesterday that R is reflexive, symmetric and transitive. So R is an equivalence relation.

$[1]_R = \{1, -1\}$

$[2]_R = \{2, -2\}$

$[3]_R = \{3, -3\}$

$\vdots$

$[0]_R = \{0\}$

Example: Prove that $\mathcal{R}$ defined on $\mathbb{Z} \setminus \{0\}$ as

$$\mathcal{R} = \{(a,b) : \frac{a}{b} = 2^k\} \text{ for some } k \in \mathbb{Z}$$

is an equivalence relation and determine its equivalence classes.

(a is chosen arbitrarily)

R is reflexive: Let $a \in \mathbb{Z} \setminus \{0\}$. Since $\frac{a}{a} = 1 = 2^0$ $(0 \in \mathbb{Z})$

we have that $(a,a) \in R$.

R is symmetric: Suppose $(a,b) \in R$ $(a,b \in \mathbb{Z} \setminus \{0\})$.

Then $\frac{a}{b} = 2^k$ for some $k \in \mathbb{Z}$.

Then $\frac{b}{a} = \frac{1}{2^k} = 2^{-k}$ where $-k \in \mathbb{Z}$.

So, $(b,a) \in R$.

R is transitive:

$\hookrightarrow$ left as an exercise

exercise

Example: Prove that $\mathcal{R}$ defined on $\mathbb{Z}^+ \times \mathbb{Z}^+$ as

$$\mathcal{R} = \{((a_1, a_2), (b_1, b_2)) : a_1 b_2 = a_2 b_1\}$$

is an equivalence relation and determine its equivalence classes.