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TOTAL POINTS: /15

Trent University MATH 2600 - Discrete Structures - Winter 2020

Test 2

READ ME: Any attempts for cheating on graded work will be dealt with according to the university policies. Show all your work. Explain your solutions when appropriate.

Problem 0) (0 points): Use this space to draw a picture. No restriction on the theme.

I'm not talented enough to draw but enjoy this meme. Stay safe!

Problem 1) (2 points): Let $A = \{w, x\}$ and $B = \{y\}$. Write down all relations from A to B. How many functions are there from A to B?

Problem 1:

$A = \{w, x\}$

$B = \{y\}$

$A \times B = \{(w,y), (x,y)\}$

A relation from A to B is a subset of $A \times B$ or the Cartesian product of A and B.

The relations are:

1. $\emptyset$, null set is a subset of $A \times B$
2. $\{(w,y)\}$
3. $\{(x,y)\}$
4. $\{(w,y), (x,y)\}$

A relation from A to B can be a function if $\forall a \in A$, $\exists b \in B$ such that (a,b) belongs to the relation and is unique.

Therefore $\{(w,y), (x,y)\}$ is a function.

Hence, there is one function from A to B.

Problem 2) (2 points): Let $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ be a function given as $f(x, y) = |x| + 2y$ (where $\mathbb{R}^+$ is the set of positive real numbers).

Answer the following questions, and explain your answers.

(a) Is f one-to-one?

(b) Is f onto?

Problem 2

$f : R \times R^+ \to R$

$f(x,y) = |x| + 2y$

(a) To prove that $f(x,y)$ is one-to-one, we must prove that $f(x_1, y_1) = f(x_2, y_2)$ and $(x_1, y_1) = (x_2, y_2)$

Consider $(x_1, y_1) = (1, 3)$
and $(x_2, y_2) = (-1, 3)$

$f(1,3) = |1| + 2(3) = 7$

$f(-1,3) = |-1| + 2(3) = 7$

Here, $(1,3) \neq (-1,3)$

But, $f(1,3) = f(-1,3)$

Therefore, $f(x,y)$ is not one-to-one

(b) $f(x,y) = |x| + 2y$

$\therefore$ $x \in R \Rightarrow |x| \geqslant 0$
$\therefore$ $y \in R^+ \Rightarrow 2y > 0$

$\therefore$ $|x| + 2y > 0$
$f(x,y) > 0 \quad \forall (x,y) \in R \times R^+$

From this, we can observe that there is no $(x,y) \in R \times R^+$ such that $f(x,y) = -1 \in R$

$\therefore$ $f(x,y)$ is not onto

Problem 3) (2 points): Let A and B be any two subsets of a universal set $\mathcal{U}$. Does $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$ always hold? If your answer is "yes", then prove it; otherwise show a counterexample.

Problem 3

Prove that $P(A) \cup P(B) = P(A \cup B)$

Counterexample :

Consider $A = \{4\}$ and $B = \{8\}$

$P(A) = \{\emptyset, \{4\}\}$ $\emptyset$, null set is a subset of every set

$P(B) = \{\emptyset, \{8\}\}$

$P(A) \cup P(B) = \{\emptyset, \{4\}, \{8\}\}$

$P(A \cup B) = P(\{4, 8\}) = \{\emptyset, \{4\}, \{8\}, \{4, 8\}\}$

Therefore, $P(A) \cup P(B) \neq P(A \cup B)$

Problem 4) (3 points): Let A be the set of all functions from $\mathbb{Z}$ to $\mathbb{R}$. We define a relation $\mathcal{R}$ on A as $\mathcal{R} = \{(f, g)$ such that for all $x \in \mathbb{Z}, f(x) - g(x) = c$ for some constant $c \in \mathbb{Z}\}$. Show that $\mathcal{R}$ is an equivalence relation on A.

Problem 4

We know that R can only be an equivalence relation, only if R is reflexive, symmetric and transitive.

R is reflexive

$R = \{(f,g)$ such that $\forall x \in Z, f(x) - g(x) = c\}$

$\forall x \in Z, f(x) - f(x) = 0$ and $0 \in Z$

$\therefore$ R is reflexive

R is symmetric

Suppose $(f,g) \in R$

$f(x) - g(x) = c, \forall x \in Z$ for some constant $c \in Z$

$g(x) - f(x) = -c, \forall x \in Z$ where $-c \in Z$

$\therefore (g,f) \in R$

if $(f,g) \in R$ and $(g,f) \in R$

Therefore, R is symmetric

R is transitive

Suppose (f,g) and $(g,h) \in R$

$f(x) - g(x) = c_1$, $g(x) - h(x) = c_2$ $\forall x \in Z$ for some constants $c_1, c_2 \in Z$

$f(x) - g(x) + g(x) - h(x) = c_1 + c_2$ $\forall x \in Z$ and $c_1 + c_2 \in Z$

$f(x) - h(x) = c_1 + c_2$ $\forall x \in Z$ and $c_1 + c_2 \in Z$

$(f,h) \in R$

So if $(f,g), (g,h) \in R$ then $(f,h) \in R$

Therefore, the relation R is transitive

Since R is reflexive, symmetric and transitive,

R is an equivalence relation on A.

Problem 5) (1 point): You have a set of building-blocks which contains blocks of heights 1, 3 and 4 centimeters. (Other dimensions irrelevant.) You are constructing towers by piling blocks directly on top of one another. (A tower of height 7 cm could be obtained using seven blocks of height 1; one block of height 3 and one block of height 4; 2 blocks of height 3 and one block of height 1; etc.)

Let b_n be the number of ways to construct a tower of height n cm using blocks from the set. Assume that there is an unlimited supply of blocks of each size. Find a recurrence relation for b_n. (You are not required to solve the recurrence relation.)

Problem 5

Let b_n be the no. of ways to construct a tower of height n.

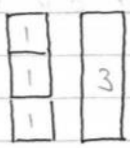
The last block on the tower can be either 1cm, 3cm or 4cm

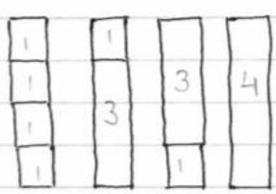
The tower height under the first block is $n-1, n-3, n-4$ respectively.

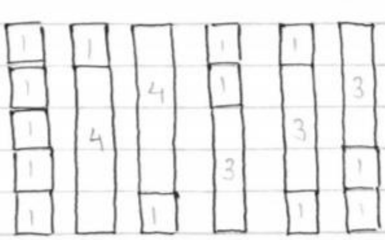
$$b_n = b_{n-1} + b_{n-3} + b_{n-4}$$

$n=1$  $b_1 = b_{1-1} = b_0 = 1$ way from $\{1\}$

$n=2$  $b_2 = b_{2-1} = b_1 = 1$ way from $\{1,1\}$

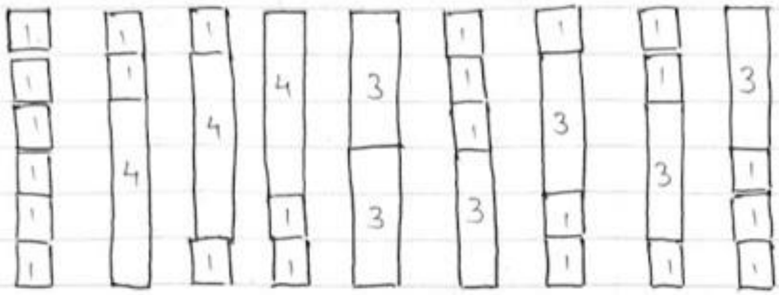
$n=3$  $b_3 = b_{3-1} + b_{3-3} = b_2 + b_0 = 1+1$
$= 2$ ways from $\{1,1,1\}$ and $\{3\}$

$n=4$  $b_4 = b_{4-1} + b_{4-3} + b_{4-4} = b_3 + b_1 + b_0 = 2+1+1$
$= 4$ ways from $\{1,1,1,1\}, \{1,3\}$
$\{3,1\}$ $\{4\}$

$n=5$  $b_5 = b_{5-1} + b_{5-3} + b_{5-4}$
$= b_4 + b_2 + b_1 = 4+1+1$
$= 6$ ways
$\{1,1,1,1,1\}$, $(2 \times \{4,1\})$
$(3 \times \{1,1,3\})$

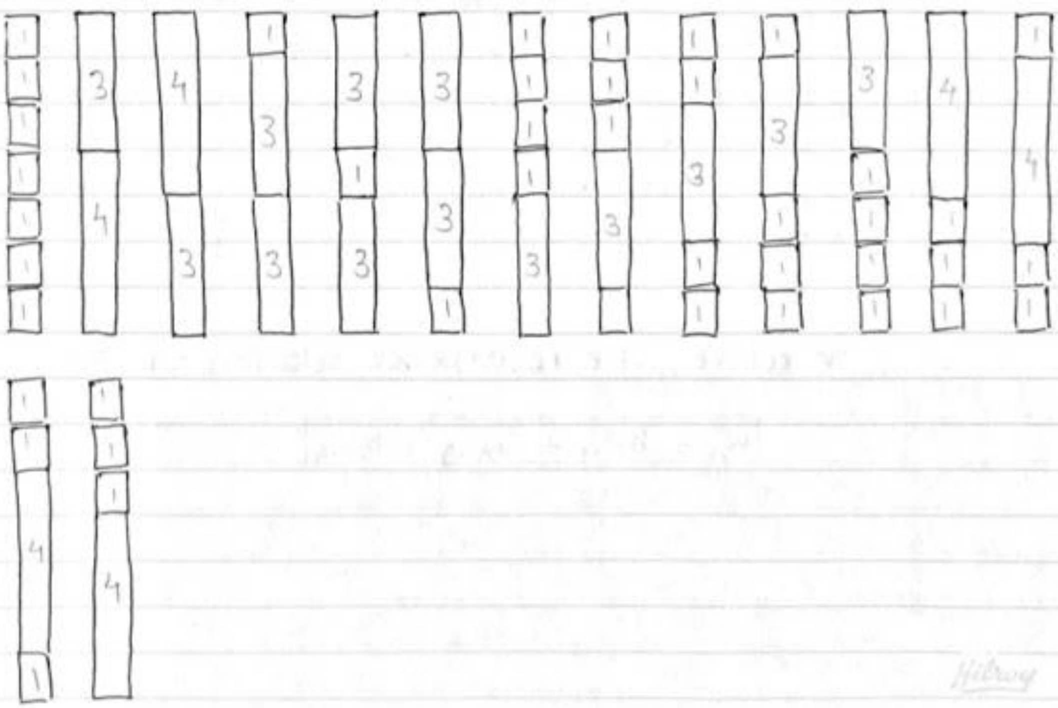
$n = 6 \quad b_6 = b_{6-1} + b_{6-3} + b_{6-4} = b_5 + b_3 + b_2 = 6 + 2 + 1$

$= 9$ ways from $\{1,1,1,1,1,1\}$ $(3 \times \{4,1,1\})$
$\{3,3\}$ $(4 \times \{1,1,1,3\})$



$n = 7 \quad b_7 = b_{7-1} + b_{7-3} + b_{7-4} = b_6 + b_4 + b_3 = 9 + 4 + 2$

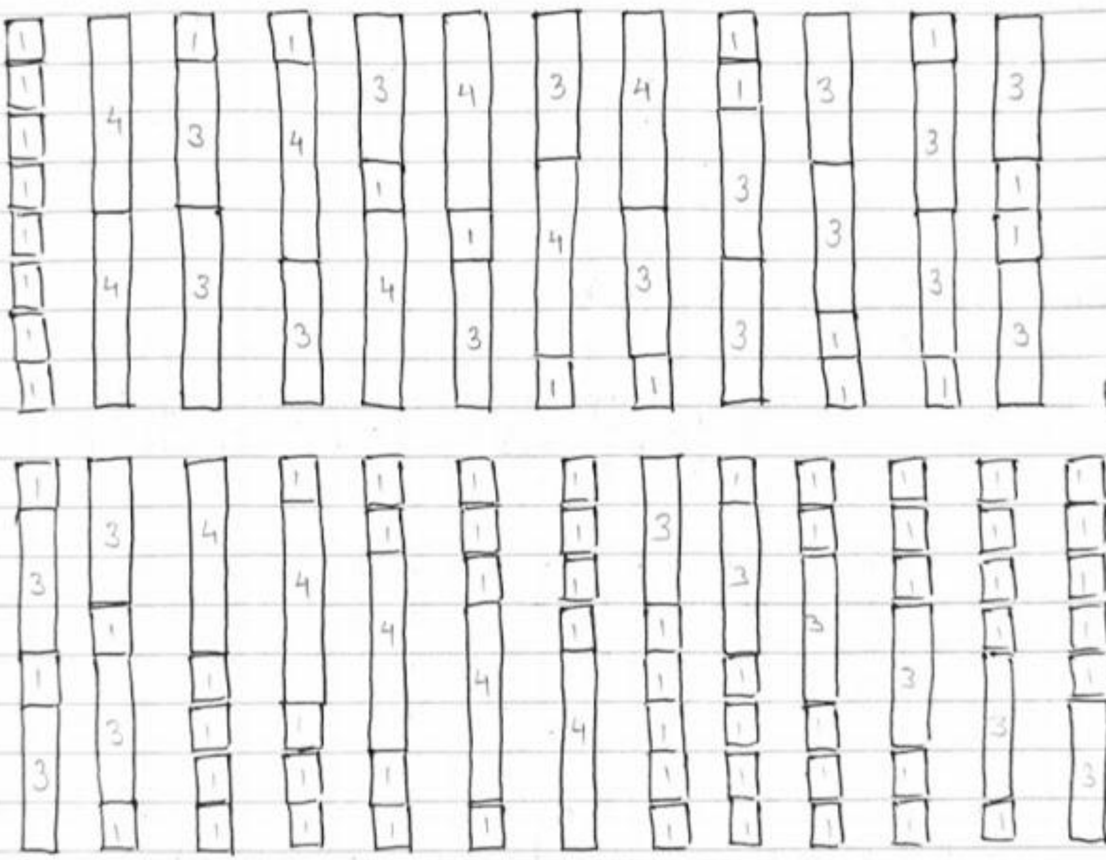
$\therefore b_7 = 15$ ways from $\{1,1,1,1,1,1,1\}$ $(2 \times \{3,4\})$
$(3 \times \{1,3,3\})$ $(5 \times \{1,1,1,1,3\})$
$(4 \times \{4,1,1,1\})$



$n = 8 \qquad b_8 = b_{8-1} + b_{8-3} + b_{8-4} = b_7 + b_5 + b_4$

$= 15 + 6 + 4$

$\therefore\ b_8 = 25$ ways from $\{1,1,1,1,1,1,1,1\}$ $\{4,4\}$ $(6 \times \{3,4,1\})$ $(6 \times \{3,3,1,1\})$ $(5 \times \{4,1,1,1,1\})$ $(6 \times \{3,1,1,1,1,1\})$



Therefore, the rec rrence relation for b_n is

$$\boxed{b_n = b_{n-1} + b_{n-3} + b_{n-4}}$$

Problem 6) (2 points): Use mathematical induction to show that for all integers $n \geq 1$, $4^{n+1} + 5^{2n-1}$ is divisible by 21.

Problem 6

We have to prove $4^{n+1} + 5^{2n-1}$ is divisible by 21. $\forall\, n \geq 1$

$P(n) = 4^{n+1} + 5^{2n-1} = 21m \quad (m \in \mathbb{Z})$

Base case : Show that $P(n)$ is true when $n=1$, ie $P(1)$

$$\text{LHS}: \quad 4^{1+1} + 5^{2-1} = 4^2 + 5^1$$
$$= 16 + 5$$
$$= 21$$
$$= 21 \times 1 \quad (m = 1)$$
$$= \text{RHS}$$

Therefore, $P(n)$ is true for $n=1$

Induction step : Suppose $P(k)$ is true $(k \geq 1)$, show that $P(k+1)$ follows.

We suppose that $P(k) = 4^{k+1} + 5^{2k-1} = 21m \quad (m \in \mathbb{Z})$ is true

We must prove that $P(k+1) = 4^{k+2} + 5^{2k+1} = 21m$

$$\text{LHS}: \quad 4^{k+2} + 5^{2k+1} = 4^{k+1+1} + 5^{2k-1+2}$$
$$= 4^{k+1} \cdot 4^1 + 5^{2k-1} \cdot 5^2$$
$$= 4^{k+1} \cdot 4^1 + 5^{2k-1} \cdot 25$$
$$= 4^{k+1} \cdot 4^1 + (21+4) \cdot 5^{2k-1}$$
$$= 4(4^{k+1}) + 21(5^{2k-1}) + 4(5^{2k-1})$$
$$= 4(4^{k+1} + 5^{2k-1}) + 21(5^{2k-1})$$

Hilroy

$= 4(21m) + 21(5^{2k-1})$

$= 21(4m + 5^{2k-1})$

Therefore $4^{k+2} + 5^{2k+1} = 21(4m + 5^{2k-1})$

Since $21(4m + 5^{2k-1})$ is divisible by 21,
$P(n)$ is true for $P(k+1)$

$\therefore$

By Base Case and Induction step, we proved that the given proposition is true for all $n \geq 1$.

Problem 7) (1 point): Let d_n be the number of derangements of n elements. Express the answer to the following problem in terms of d_n (for the appropriate value of n).

A machine inserts 8 letters into 8 envelopes randomly (one in each). In how many ways can the machine insert the letters into envelopes so that exactly one of the 8 letters go into the correct envelope?

Problem 7

Formula for derangement is

$$d_n = n! \sum_{m=0}^{n} \frac{(-1)^m}{m!}$$

$$d_8 = 8! \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} \right)$$

$$\underline{\underline{d_8 = 14833}}$$

Therefore, we can insert the letters into envelopes in 14833 ways.

Problem 8) (2 points): A relation $\mathcal{R}$ on a set A is said to be *cyclic* if the following implication holds:

$$\big((a,b) \in \mathcal{R}\big) \wedge \big((b,c) \in \mathcal{R}\big) \to (c,a) \in \mathcal{R}.$$

Show that if $\mathcal{R}$ is reflexive and cyclic, then it is symmetric and transitive.

Problem 8

$((a,b) \in R) \wedge ((b,c) \in R) \to (c,a) \in R$

I: The Relation R is reflexive if $(a,a) \in R \ \forall a \in D$ where D is the domain on which R is defined

$\therefore$ R is reflexive

II: The Relation R is cyclic only if for $a,b,c \in R$ $(a,b) \in R$ and $(b,c) \in R \Rightarrow (c,a) \in R$

$\therefore$ R is cyclic

III: From II, we can replace b with c to get

$(a,b) \in R$ and $(b,b) \in R \Rightarrow (b,a) \in R$

Therefore, $(a,b) \in R \Rightarrow (b,a) \in R \ \forall a,b$ in D

$\therefore$ R is symmetric

IV: Consider $(a,b) \in R$ and $(b,c) \in R$

$(b,a) \in R$ and $(c,b) \in R$ (since R is symmetric)

$(c,b) \in R$ and $(b,a) \in R$

$\Rightarrow (a,c) \in R$ (since R is cyclic)

$\therefore (a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$

$\therefore$ R is transitive