'MATH-2600H'

Assignment-04

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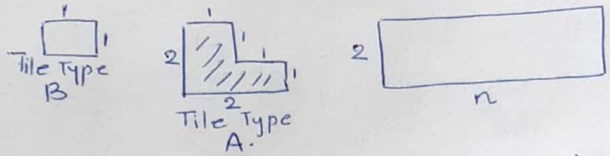
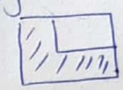
Q1.



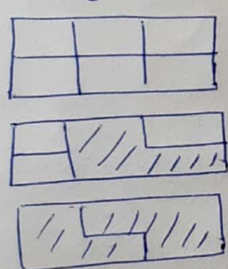
(a) a_1 is 2×1
a_2 is 2×2
a_3 is 2×3

Since only Tile Type B can be used for 2×1, thus, $\boxed{a_1 = 1}$...

For 2×2, one way is to use only Type B  then, one type A, and 3 type B.  and then it can be rotated type A 4 times.

$\boxed{a_2 = 5}$...

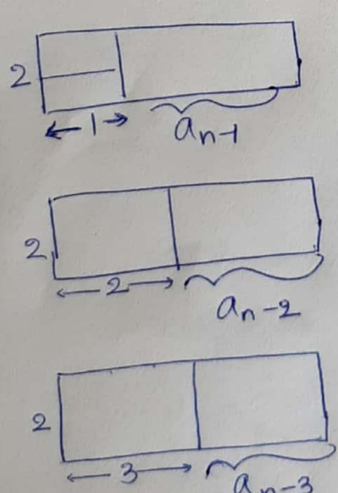
For 2×3



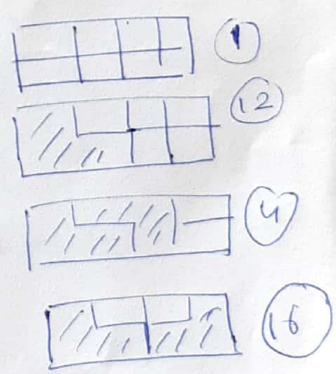
considering all possible configurations and permutations. when using type A, one tile and type A 2 tiles.

$\boxed{a_3 = 11}$...

(b)



On finding a_4: after all possible rotations & configurations



$a_4 = 33$.

$= 22 + 10 + 1$

$= 2(11) + 2(5) + 1$

$a_4 = 2(a_3) + 2(a_2) + a_1$

$\therefore$ recurrence relation

$$\boxed{a_n = 2a_{n-1} + 2a_{n-2} + a_{n-3}}$$

Q2) $\forall n \geq 2 \in \mathbb{Z}$

$$\prod_{j=2}^{n}\left(1-\frac{1}{j^2}\right)=\frac{n+1}{2n}$$

$$P(n)\ \prod_{j=2}^{n}\left(1-\frac{1}{j^2}\right)=\frac{n+1}{2n}, \quad n\geq 2$$

BI: Show that $P(n_0)=P(2)$ is true

LHS of $P(2)$	RHS of $P(2)$
$\prod_{j=2}^{n}\left(1-\frac{1}{j^2}\right)$	$\frac{2+1}{2(2)}$
$1-\frac{1}{(2)^2}=\frac{4-1}{4}=\frac{3}{4}$	$=\frac{3}{4}$

LHS of $P(2)$ = RHS of $P(2)$

$\therefore$ BI is completed, as $P(2)$ is true.

IS: $P(K)=\prod_{j=2}^{K}\left(1-\frac{1}{j^2}\right)=\frac{K+1}{2K}, \quad K\geq 2$ —— (1)

We suppose $P(K)$ is true where $K\geq 2$ and show that $P(K+1)$ follows. Let's suppose (1) is the $P(K)$ statement.

$$P(K+1)=\prod_{j=2}^{K+1}\left(1-\frac{1}{j^2}\right)=\frac{K+1+1}{2(K+1)}$$

LHS of $P(K+1)$

$$\prod_{j=2}^{K+1}\left(1-\frac{1}{j^2}\right)=\underbrace{\prod_{j=2}^{K}\left(1-\frac{1}{j^2}\right)}\times\left(1-\frac{1}{(K+1)^2}\right)$$

$$=\frac{\cancel{(K+1)}}{2K}\times\frac{(K+1)^2-1}{(K+1)^{\cancel{2}}} \quad \{\text{from } (1)\}$$

$$=\frac{(K+1)^2-1}{2K(K+1)}$$

$$=\frac{K^2+2K+1-1}{2K(K+1)}$$

$$=\frac{K^2+2K}{2K(K+1)}$$

$$=\frac{\cancel{K}(K+2)}{2\cancel{K}(K+1)}=\boxed{\frac{K+1+1}{2(K+1)}} \Rightarrow \text{RHS of } P(K+1)$$

LHS=RHS

$\therefore P(K+1)$ is true

$\therefore$ IS is completed.

By BI and IS, we prove the proposition is true for all integers $n\geq 2$.

Q3. $\sum_{i=1}^{n} (-1)^i \cdot i^2 = \frac{(-1)^n n(n+1)}{2}$ for all positive integers

$\Rightarrow \cancel{n \geq 0}$ $n>0$

$P(n) = \sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n \cdot n(n+1)}{2.}$, $n>0$

BI: show that $P(n_0) = P(1)$ is true.

LHS of P(1)

$\sum_{i=1}^{n} (-1)^i \cdot i^2 \cancel{= (-1)^n (n)(n}$

$= (-1)^1 (1)^2$

$= \boxed{-1}$

RHS of P(1)

$\frac{(-1)^1 \cdot 1 \cdot (1+1)}{2}$

$= \frac{-1 \cdot 2}{2} = \boxed{-1}$

LHS of P(1) = RHS of P(1)

$\therefore$ P(1) is true

$\therefore$ BI is completed.

IS: Suppose P(K) is true where $n>0$, then show that P(K+1) is true / P(K+1) follows.

$P(K) = \sum_{i=1}^{K} (-1)^i \cdot i^2 = \frac{(-1)^K (K)(K+1)}{2}$ — (1)

$P(K+1) = \sum_{i=1}^{K+1} (-1)^i i^2 = \frac{(-1)^{K+1} (K+1)(K+1+1)}{2.}$

LHS of P(K+1)

$\sum_{i=1}^{K+1} (-1)^i i^2 = \underbrace{\sum_{i=1}^{K} (-1)^i \cdot i^2}_{} + \cancel{K}(-1)^{K+1} (K+1)^2 = \frac{(-1)^K K(K+1)}{2} + (-1)^{K+1}(K+1)^2$

{from (1) statement} $= \frac{(-1)^{K+1}(K+1)}{2}\left[\frac{K}{-1} + 2(K+1)\right]$

$= \frac{(-1)^{K+1} \cdot (K+1)}{2} [-K+2K+2]$

$= \frac{(-1)^{K+1} \cdot (K+1)}{2} [K+2]$

$= \underbrace{\frac{(-1)^{K+1} \cdot (K+1)(K+1+1)}{2.}}_{\text{RHS of } P(K+1)}$

$\therefore$ LHS = RHS

$\therefore$ P(K+1) is true

$\therefore$ IS is completed

$\therefore$ BI and IS are completed, we prove the proposition is true for all integers $n>0$.

Q4) for $n \geq 12$ cents, they can be form postage of 4-cent and 5-cent stamps.

$a_n = 4a + 5b$ (a : no. of 4-cent stamp
b : no. of 5-cent stamp)

BI : Show that 12, 13, 14, 15 cent can form postage of cents (4, 5).

$12 = 4+4+4$ $P(12)$ is true
$13 = 4+4+5$ $P(13)$ is true
$14 = 4+5+5$ $P(14)$ is true
$15 = 5+5+5$ $P(15)$ is true

IS: $P(j)$ is true for $12 \leq j \leq k$, and we know $P(12)$, $P(13)$, $P(14)$, $P(15)$ is true. Show that $P(j)$ is true for ~~k>15, we~~ to $P(k+1)$ is true too, that we can form postage of k+1 cents., for $k > 15$, since we proved till then. Show that $12 \leq j \leq k+1$

$k \geq 15$

subtracting 3 from both sides

$k-3 \geq 12$; $k-3 < k$

we can form postage of $k-3$ cents using just 4-cent & 5-cent stamps, since, $k-3 \geq 12$; $k-3 < k$

$\Rightarrow k-3 = 4a+5b$.

adding '4' both sides

$k-3+4 = 4a+4b+5$

$k+1 = 4(a+1)+5b$

Adding '4' is like one more 4-cent stamp. Since, we could form postage of $k-3$ cents then by adding one 4-cent, we can form postage of $k+1$ cents.

$\Rightarrow$ $P(j)$ is true for $12 \leq j \leq k+1$.

$\therefore$ every amount of postage $n \geq 12$ cents can be made from 4-cent and 5-cent stamps.

Q5.> $a_n = 11a_{n-1} - 28a_{n-2}$ → recurrence relation

$r^n = 11r^{n-1} - 28r^{n-2}$

$a_0 = 5$

$a_1 = 6$.

$$\frac{r^n}{r^{n-2}} = \frac{11r^{n-1}}{r^{n-2}} - \frac{28r^{n-2}}{r^{n-2}}$$

$r^2 = 11r - 28$

$\boxed{r^2 - 11r + 28 = 0}$ → characteristic equation.

$r^2 - 4r - 7r + 28 = 0$

$r(r-4) - 7(r-4) = 0$

$(r-7)(r-4) = 0$.

$\alpha = 7$, $\beta = 4$ → 2 distinct roots

$a_n = k_1\alpha^n + k_2\beta^n$

$a_n = k_1(7)^n + k_2(4)^n$

$a_0 = 5 = k_1\alpha^1 + k_2\beta^1$

$5 = k_1(7)^1 + k_2(4)^1 \Longrightarrow 5 = 7k_1 + 4k_2$ — (1)

$a_1 = 6 = k_1\alpha^2 + k_2\beta^2$

$6 = k_1(7)^2 + k_2(4)^2 \Longrightarrow 6 = 49k_1 + 16k_2$ — (2)

multiplying (1) by 7, then subtract (2) from it.

(1)×7 − (2)

$35 = \cancel{49k_1} + 28k_2$

$-6 = \cancel{49k_1} + 16k_2$

$29 = 12k_2$

$k_2 = \frac{29}{12}$

subst. k_2 in (1)

$5 = 7k_1 + 4 \cdot \left(\frac{29}{12}\right)$

$\frac{15-29}{3} = 7k_1$

$k_1 = \frac{-14}{21} = -\frac{2}{3}$

$k_1 = -\frac{2}{3}$.

$\therefore \boxed{a_n = \frac{-2}{3}(7)^n + \frac{29}{12}(4)^n}$