Full Name: Punyaja Mishra
Student Number: 0660001

TOTAL POINTS: /30

Trent University MATH 2600 - Discrete Structures
Instructor: Aras Erzurumluoˇglu

TAKE-HOME FINAL EXAM (due 11:00 am on April 15th)

READ ME: You may use any course material and the internet for reference, but you are required to solve the problems independently and follow the principles of academic integrity as stated within the university policies.

Show all your work. Explain your solutions when appropriate.

0) (0 points): Prove something.

Punyaja Mishra
0660001

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MATH-2600H
Final Exam

Problem 0) I can prove that 1+1 = window
So, we all know 1+1 = 2
but actually 1+1 = window
How? well here is the proof :-

1 + 1
one + one

now ⇒ the equal to sight

$\therefore 1+1 =$ there, a window

1) **(3 points): On the Island of Knights and Knaves we have two people A and B.**

A says: B is a knight. B says: I am a knight or we are both knaves.

Is A a knight? Is B a knight? Show all your work. (The problem may have a single solution, multiple solutions, or no solution at all.)

Problem 1) A says: B is a knight
B says: I am a knight or we are both knaves.

p: A is a knight
q: B is a knight

A : q
B : $q \oplus (\sim p \wedge \sim q)$

p	q (A says)	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$q \oplus (\sim p \wedge \sim q)$ (B says)
T	T	F	F	F	T
F	T	T	F	F	T
T	F	F	T	F	F
F	$\boxed{F}$	T	T	T	T

Let's assume A is a knave, that mean B is not a knight.

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If B is knave, then he is lying.
But he says, either B is a knight OR A & B both are knaves

Since, A and B are indeed knaves, this means that B is telling the truth, like the last row, which makes B ~~is~~ a knight.

∴ Both $\boxed{A}$ and $\boxed{B}$ are $\boxed{\text{knights}}$.

2) **(2 points): Consider the following relation on the set A = {1,2,3,4}: R = {(1,1),(2,2),(2,3),(3,2),(3,3),(3,4),(4,1)} What properties does the relation R possess? Circle the correct responses.**

R is reflexive. T F

R is symmetric. T F

R is antisymmetric. T F

R is transitive. T F

Problem 2) $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (2,2), (2,3), (3,2), (3,3), (3,4), (4,1)\}$

R is reflexive — False

R is symmetric — False

R is antisymmetric — False

R is transitive — False

3) **(3 points): Let A = {−4,−3,0,1,6,9,10}, and define an equivalence relation R on A as follows: xRy ⇔ x ≡ y (mod 5). Find the partition of A into equivalence classes with respect to R. (You need to find the equivalence classes of A, and then determine which elements in A belong to which equivalence classes.)**

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blem 3) $A = \{-4, -3, 0, 1, 6, 9, 10\}$

$xRy \iff x \equiv y \pmod 5$

$\Rightarrow x \equiv y \,\%\, 5 \quad \Rightarrow \quad x \% 5 = y \% 5$

R is a congruence modulo

the equivalence relation is reflexive, symmetric & transitive.

to prove R is equivalence

① Reflexive

$xRx \Rightarrow x \equiv x \pmod 5$

$\Rightarrow x \equiv x \,\%\, 5$

This is always true

∴ R is reflexive

② symmetric

$xRy \qquad\qquad yRx$

$x \equiv y \,\%\, 5 \;\; -① \qquad y \equiv x \,\%\, 5 \;\; -②$

① = ② always as x & $y \,\%\, 5$ will always be same for same x & y.

③ transitive

Let xRy & yRz be true

$x \equiv y \,\%\, 5 \qquad y \equiv z \,\%\, 5$

$\Rightarrow x \% 5 = y \% 5 \;\; -① \qquad y \% 5 = z \% 5 \;\; -②$

adding ① & ②

$x \% 5 = z \% 5$

$x \equiv z \,\%\, 5$

$\Rightarrow xRz$

R is Transitive.

$\therefore$ R is an equivalence relation

Possible equivalence classes for a congruence modulo 5 :

$[0]_R = [5]_R$
$[1]_R = [6]_R$
$[2]_R = [7]_R$
$[3]_R = [8]_R$
$[4]_R = [9]_R$ ... and so on

They are equal because any number modulo 5 can have a remainder from $\{0, 1, 2, 3, 4\}$ or just repitition.

$\therefore$ equivalence classes are
$[0]_R, [1]_R, [2]_R, [3]_R, [4]_R$.

now for elements of A

$-4 = -5+1 \equiv 1 \pmod 5$
$-3 = -5+2 \equiv 2 \pmod 5$
$0 = 0 * 5 \equiv 0 \pmod 5$
$1 = 0+1 \equiv 1 \pmod 5$
$6 = 5+1 \equiv 1 \pmod 5$
$9 = 5+4 \equiv 4 \pmod 5$
$10 = 5*2 \equiv 0 \pmod 5$

$\therefore$ $[0]_R = \{0, 10\}$
$[1]_R = \{-4, 1, 6\}$
$[2]_R = \{-3\}$
$[3]_R = \{\text{null set}\}$
$[4]_R = \{9\}$

so we get following equivalence classes

$\{-4,1,6\}$; $\{-3\}$; $\{9\}$; $\{0,10\}$

4) (4 points): Let f : R→R×Z be a function defined by f(x) = (x−3,4).

Which of the following statements about f are true? If you claim the statement is true, prove it; otherwise, give a concrete counterexample.

f is injective. T F Justification (proof or counterexample):
f is surjective. T F Justification (proof or counterexample):

Problem 4) $R \longrightarrow R \times Z$

$f(x) = (x-3, 4)$

* f is injective $=$ $\boxed{\text{true}}$

let's suppose x_1 and $x_2 \in R$ and $x_1 \neq x_2$
assuming f is not injective.

$\Rightarrow f(x_1) = f(x_2)$

$\Rightarrow (x_1 - 3, 4) = (x_2 - 3, 4)$

$\because 4 \in Z$ and is a constant

$\Rightarrow x_1 - 3 = x_2 - 3$

$\Rightarrow x_1 = x_2$

this contradicts our assumption.

$\therefore f$ is injective

Hence Proved.

* f is surjective $\boxed{\text{false}}$

$R \longrightarrow R \times Z$

$f(x) = (x-3, 4)$

there is no $x \in R$, for which

$f(x) = (x-3, 1)$

$\because 4$ is constant

Thus not all $y \in Z$ are a result of $f(x)$

$\therefore f$ is not surjective.

5) (4 points): Use Mathematical Induction to prove that 2n+2+32n+1 is divisible by 7 for all integers n ≥ 0. Clearly state the proposition to be proved. Express the basis of induction, the induction hypothesis and the induction step. Indicate where the induction hypothesis is used in your proof.

Problem 5) $2^{n+2} + 3^{2n+1}$ is divisible by 7, $n \geq 0$

$P(n) = 2^{n+2} + 3^{2n+1}$ is divisible by 7, $n \geq 0$

BI: Show $P(n_0) = P(0)$ is true

$P(n_0) = P(0) = 2^{0+2} + 3^{2(0)+1}$

$= 2^2 + 3^1$

$= 4 + 3$

$= 7 \quad \Rightarrow 7 \% 7 = 0$

we know 7 is divisible by 7.

$\therefore$ BI is completed as $P(n_0) = P(0)$ is true

IS: Let's assume/suppose $P(K)$ is true for $K \geq 0$ and show that $P(K+1)$ follows.

So we suppose

$P(K) = 2^{K+2} + 3^{2K+1}$ is divisible by 7.

(a)

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$P(K+1) = 2^{(K+1)+2} + 3^{2(K+1)+1}$ is divisible by 7

$= 2^{K+1+2} + 3^{2K+2+1}$

$= 2^{(K+2)+1} + 3^{(2K+1)+2}$

$= 2^{K+2} \cdot 2 + 3^{2K+1} \cdot 3^{2}$

$= 2 \cdot 2^{K+2} + 3^{2K+1} \cdot 9$

$= 2 \cdot 2^{K+2} + (7+2) \cdot 3^{2K+1}$

$= 2 \cdot 2^{K+2} + 7 \cdot 3^{2K+1} + 2 \cdot 3^{2K+1}$

$= 2 \cdot \underbrace{\left(2^{K+2} + 3^{2K+1}\right)} + \underline{7} \cdot 3^{2K+1}$

by (*) $2^{K+2} + 3^{2K+1}$ is divisible by 7
and
$7 \cdot 3^{2K+1}$ is divisible by 7 as it is being multiplied by 7.

addition of 2 terms divisible by 7 will yield a sum divisible by 7

$\therefore$ P(K+1) is true
$\therefore$ IS is completed.

So, BI & IS is completed and hence prove proposition for all integers $n \geq 0$ for P(n).

Hence proved.

6) (3 points): Solve the recurrence relation an = 8an−1 −16an−2 (n ≥ 2) with the initial conditions a0 = 3 and a1 = 14. Show all your work.

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Problem 6) $a_n = 8a_{n-1} - 16a_{n-2}$ $(n \geq 2)$

$a_0 = 3$, $a_1 = 14$

$a_n = 8a_{n-1} - 16a_{n-2}$

$\Rightarrow r^n = 8r^{n-1} - 16r^{n-2}$

dividing by r^{n-2}

$$\frac{r^n}{r^{n-2}} = \frac{8r^{n-1}}{r^{n-2}} - \frac{16r^{n-2}}{r^{n-2}}$$

$r^2 = 8r - 16$

$r^2 - 8r + 16 = 0$

$\Rightarrow r^2 - 4r - 4r + 16 = 0$

$\Rightarrow r(r-4) - 4(r-4) = 0$

$\Rightarrow (r-4)^2 = 0$

$\therefore$ root $= \boxed{\alpha = 4}$

by theorem of single root when solving recurrence relation

$$a_n = k_1\alpha^n + k_2 n\alpha^n$$

for $n=0$

$a_0 = a_n = 3$

$a_0 = k_1\alpha^0 + k_2(0)\alpha^0$

$a_0 = k_1$

$\boxed{k_1 = 3}$ — ①

for $n=1$

$a_1 = a_n = 14$

$a_1 = k_1\alpha^1 + k_2(1)\cdot\alpha^1$

$14 = k_1(4) + k_2(4)$

substituting ①

$14 = (3)(4) + 4k_2$

$14 = 12 + 4k_2$

$14 - 12 = 4k_2$

$k_2 = \frac{2}{4}$

$\boxed{k_2 = \frac{1}{2}}$

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$\therefore$ On solving recurrence relation we get

$$\boxed{a_n = 3 \cdot 4^n + \frac{1}{2} \cdot n \cdot 4^n}$$

7) (3 points): Does a simple graph with 8 vertices exist with degree sequence (2,2,4, 4,5,5,7,7)? If yes, then draw an example of such a graph; otherwise explain why it cannot exist.

Problem 7) degree sequence (2, 2, 4, 4, 5, 5, 7, 7)

let degrees be $V_1 = 2$ $V_3 = 4$ $V_5 = 5$ $V_7 = 7$
$V_2 = 2$ $V_4 = 4$ $V_6 = 5$ $V_8 = 7$

where $V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8$ are degrees of the respective vertices.

$\because$ degrees of V_7 & V_8 $= 7$
and total no. of vertices $= 8$
$\therefore$ V_7 and V_8 will be joined to ~~each~~ all other vertices.

Since degree of V_1 & $V_2 = 2$
and V_7 and V_8 are adjacent to ALL vertices
$\Rightarrow V_2 \& V_1$ are adjacent to V_7 and V_8
and thus completing their degree of adjacent.

Similarly V_3, V_4, V_5, V_6 are adjacent to V_7 and V_8.
Thus V_3 and V_4 can be adjacent to only 2 more vertices $\because$ their degree is 4 and they are adjacent to V_7 and V_8
So, $4 - 2 = 2$

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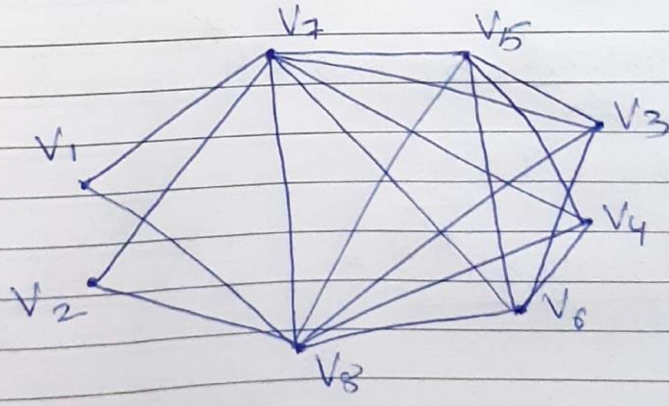
Similarly degree of V_5 & $V_6 = 5$
$\forall$ $5 - 2 = 3$
thus V_5 & V_6 can be adjacent to 3 more vertices.

$\Rightarrow$ V_5 can be adjacent to V_3, V_4, V_6
$\Rightarrow$ V_6 can be adjacent to V_3, V_4, V_5
This fulfils every vertices' degree correctly.

$\therefore$ A graph with degree sequence
$(2, 2, 4, 4, \overset{V_1}{5}, \overset{V}{5}, 7, 7)$ is possible



8) (3 points): What is the coefficient of x^3y^7 in the expansion of $(3x-4y)^{10}$? Show all your work.

Problem 8) x^3y^7 $(3x-4y)^{10}$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$(3x-4y)^{10} = \sum_{k=0}^{10} \binom{10}{k} (3x)^{10-k} (-4y)^k$$

so to find coeff of x^3y^7

$\Rightarrow 10-k=3$

$\boxed{k=7}$

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$\therefore$ coeff of x^3y^7 term in expansion is

$$\cancel{\binom{n}{k}} \cdot \binom{10}{7}\cdot(3x)^3\cdot(-4y)^7$$

$$=\binom{10}{7}\cdot 3^3\cdot(-4)^7\cdot x^3y^7$$

$\therefore$ coeff of x^3y^7 is

$${}^{10}C_7 \times 3^3 \times (-4)^7$$

$$= \frac{-10!}{7!\,3!}\times 3^3\times 4^7$$

$$= \frac{-10\times 9\times \cancel{8}^{\,4}}{\cancel{3}\times\cancel{2}\times 1}\times\cancel{3}\times 3\times 3\times 4^7$$

$$=-360\times 9\times 4^7$$

$$=\boxed{-3240\times 4^7}$$

9) (3 points): Draw an example of a connected bipartite simple graph with 9 vertices and 10 edges that has an Euler tour. Verify that your graph has these properties.

Problem 9) connected bipartite simple graph

9 vertices {
10 edges } Euler tour

we can split 9 vertices as
(4 + 5) as in a bipartite
simple graph

Since the graph has euler tour, thus
every vertex should have an even degree

and they can only be joined with a vertex on opposite set. Also, to keep in mind <u>10 edges</u> in total.

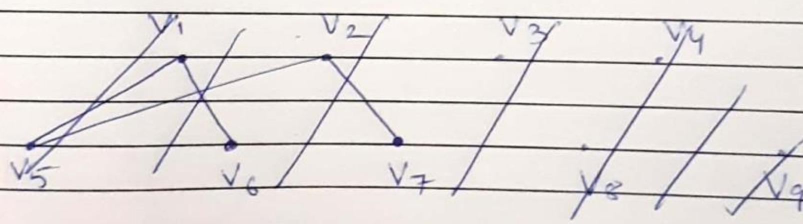
so let's say V_1, V_2, V_3, V_4 are in set of 4 vertices and

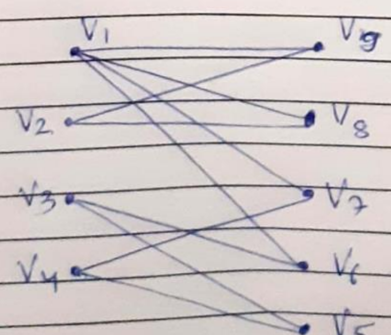
V_5, V_6, V_7, V_8, V_9 are in set of 5 vertices.

So one partite has 4 vertices and another 5.

to get 10 edges while having Euler tour, all vertices of partite with 5 vertices can have degree 2.

And another partite with 4 vertices can have 3 vertices of degree 2 and 1 vertex with degree 4.





Thus Euler tour will be

$V_1 V_9 V_2 V_8 V_1 V_7 V_4 V_5 V_3 V_6 V_1$

10) (2 points): In how many ways can you arrange all letters in NOGOJIWANONG?

Problem 10) NOGOJIWANONG

repeated letters are N, O, G

repeatitition $N = 3$

$O = 3$

$G = 2$

total no. of letters $= 12$

$\therefore$ no. of ways to arrange all letters

$$= \boxed{\frac{12!}{3!\,3!\,2!}}$$