Logical Equivalence and De Morgan's Laws

Recap: A compound proposition is a

- **tautology** if it is always true
- **contradiction** if it is always false
- **contingency** if it can be true or false depending on the truth value assignment of the propositions that make up the compound proposition.

Two compound propositions p and q are said to be **logically equivalent** (or simply **equivalent**) (notation: $p \cong q$, or $p \equiv q$ is also commonly used) if the column corresponding to p in the truth table for p is the same as the column corresponding to q in the truth table for q. (Remember that we always use the reverse-alphabetical truth value assignment to the ingredient propositions when we form the truth tables.)

Example: The compound propositions $p \to q$ and $\neg q \to \neg p$ are logically equivalent. (Observe that the second compound proposition is the contrapositive of the first one!)

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

So, $p \to q$ is logically equivalent to $\neg q \to \neg p$.

An alternative way for defining logical equivalence:

Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.
(Give a break here and try to convince yourself that this is true.)

details left to the reader

There are three main ways to verify logical equivalence of a given pair of compound propositions:

- using truth tables (seen before)

- using basic equivalences (up next)

(The table is slightly different than the one in the textbook. Use this one as your main reference.)

- using truth trees (maybe later)

we probably won't do this

$\cong$ $\equiv$ logically equivalent

	Conjunctive version	Disjunctive version	Algebraic analog
0) Implication Law	$A \to B \cong \neg A \vee B$	same	
1) Commutative laws	$A \wedge B \cong B \wedge A$	$A \vee B \cong B \vee A$	$2 + 3 = 3 + 2$
2) Associative laws	$A \wedge (B \wedge C) \cong (A \wedge B) \wedge C$	$A \vee (B \vee C) \cong (A \vee B) \vee C$	$2 + (3 + 4) = (2 + 3) + 4$
3) Distributive laws	$A \wedge (B \vee C) \cong (A \wedge B) \vee (A \wedge C)$	$A \vee (B \wedge C) \cong (A \vee B) \wedge (A \vee C)$	$2 \cdot (3 + 4) = (2 \cdot 3 + 2 \cdot 4)$
4) DeMorgan's laws	$\neg(A \wedge B) \cong \neg A \vee \neg B$	$\neg(A \vee B) \cong \neg A \wedge \neg B$	none
5) Double negation	$\neg(\neg A) \cong A$	same	$-(-2) = 2$
6) Complementarity	$A \wedge \neg A \cong (c)$ contradiction	$A \vee \neg A \cong (t)$ tautology	$2 + (-2) = 0$
7) Identity laws	$A \wedge t \cong A$	$A \vee c \cong A$	$7 + 0 = 7$
8) Domination	$A \wedge c \cong c$	$A \vee t \cong t$	$7 \cdot 0 = 0$
9) Idempotence	$A \wedge A \cong A$	$A \vee A \cong A$	$1 \cdot 1 = 1$

A	B	$\neg(A \wedge B)$	$\neg A$	$\neg B$	$\neg A \vee \neg B$
T	T	F	F	F	F
T	F	T	F	T	T
F	T	T	T	F	T
F	F	T	T	T	T

So, $\neg(A \wedge B) \cong \neg A \vee \neg B$

Examples: Use basic logical equivalences to show that

a) $\neg(p \to q) \cong \neg(\neg p \vee q)$

Start with $\neg(p \to q)$

$$\underset{\substack{(0)\\ \text{Implication}\\ \text{Law}}}{\cong} \neg(\neg p \vee q)$$

b) $\neg(p \vee (\neg p \wedge q)) \cong \neg p \wedge \neg q$

Start with $\neg(\overset{A}{p} \vee (\overset{B}{\neg p} \wedge \overset{C}{q}))$

$$\underset{\substack{(3)\\ \text{Distributive}\\ \text{(disjunctive)}}}{\cong} \neg((p \vee \neg p) \wedge (p \vee q))$$

$$\underset{\substack{(6)\\ \text{complementarity}}}{\cong} \neg(t \wedge (p \vee q))$$

$$\underset{\substack{(1)\\ \text{commutative}\\ \text{(conjunctive)}}}{\cong} \neg(\underbrace{(p \vee q)}_{A} \wedge t)$$

$$\underset{\substack{(7)\\ \text{identity law}\\ \text{(conjunctive)}}}{\cong} \neg(p \vee q)$$

$$\underset{\substack{(4)\\ \text{De Morgan's law}\\ \text{(disjunctive)}}}{\cong} \neg p \wedge \neg q$$

$\to$ This is what we wanted to show.

De Morgan's Laws in natural language:

Examples: Express the given sentence in the language of propositional logic. Then negate using De Morgan's Laws (and logical equivalences when necessary). Express the result back in natural language.

a) It's cold and snowy.

p: It's cold
q: It's snowy

$p \wedge q$

negate $p \wedge q$: $\neg(p \wedge q) \cong \neg p \vee \neg q$

=> It's not cold or it is not snowy.

b) We go to a Sigur Rós concert or we work on our mathematics assignment.

p: we go to a Sigur Rós concert
q: We work on our math assignment

$p \vee q$

negate $p \vee q$: $\neg(p \vee q) \cong \neg p \wedge \neg q$

=> we don't go to a Sigur Rós concert and we don't work on our math assignment.

c) I will pass this course only if I study hard.

$p \to q$
p only if q

p: I will pass this course
q: I study hard

$p \to q$

Use implication law : $p \to q \cong \neg p \vee q$

Negate $\neg p \vee q$: $\neg(\neg p \vee q) \cong \neg(\neg p) \wedge \neg q$

$\cong p \wedge \neg q$

I will pass this course and I won't study hard.

Arguments

An **argument** is a set of propositions in which one (called the **conclusion**) is claimed to follow from the others (called the **hypotheses** or **premises**). In other words, an argument is a proposition of the form

$$p_1 \wedge p_2 \wedge \ldots \wedge p_k \rightarrow c;$$

where $p_1, p_2, \ldots, p_k$ are the premises and c is the conclusion. Quite often a vertical notation is used, such as:

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ \underline{p_k} \\ \therefore c \end{array}$$

"therefore"

An argument $p_1 \wedge p_2 \wedge \ldots \wedge p_k \rightarrow c$ is **valid** if the conclusion c is true whenever all premises ($p_i's$) are true. Equivalently, an argument $p_1 \wedge p_2 \wedge \ldots \wedge p_k \rightarrow c$ is valid if the proposition $p_1 \wedge p_2 \wedge \ldots \wedge p_k \rightarrow c$ is a tautology. (WHY? How are these equivalent?)

→ left as an exercise

How to check the validity of an argument:

i) Using truth tables:

– We only check the rows in which all premises $p_1, p_2, \ldots, p_k$ are true.

ii) Using the **rules of inference**:

(This table is slightly different than the one in the textbook. Use this one as your main reference.)

Rules of Inference

Modus ponens	A $A \Longrightarrow B$ $\therefore B$
Modus tollens	$\neg B$ $A \Longrightarrow B$ $\therefore \neg A$
Hypothetical syllogism	$A \Longrightarrow B$ $B \Longrightarrow C$ $\therefore A \Longrightarrow C$
Disjunctive syllogism	$A \vee B$ $\neg B$ $\therefore A$
Constructive dilemma	$A \Longrightarrow B$ $C \Longrightarrow D$ $A \vee C$ $\therefore B \vee D$

see this on a truth table:

A	B	$\neg B$	$A \Rightarrow B$	$\neg A$
T	T	F	T	F
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

We look at the row(s) where all premises are true (last row in our case). We see that in those row(s) the conclusion is always true. So, Modus tollens is a valid argument.

Exercise:
Show that the hypothetical syllogism is a valid argument form.

$\boxed{\text{Assignment 1 problem}}$

Destructive dilemma	$\begin{array}{c} A \implies B \\ C \implies D \\ \neg B \vee \neg D \\ \hline \therefore \neg A \vee \neg C \end{array}$
Conjunctive simplification	$\begin{array}{c} A \wedge B \\ \hline \therefore A \end{array}$
Conjunctive addition	$\begin{array}{c} A \\ B \\ \hline \therefore A \wedge B \end{array}$
Disjunctive addition	$\begin{array}{c} A \\ \hline \therefore A \vee B \end{array}$
Absorption	$\begin{array}{c} A \implies B \\ \hline \therefore A \implies (A \wedge B) \end{array}$

try to see why this is valid

Example:

p_1: If $\overbrace{\text{Socrates is human}}^{A}$, then $\overbrace{\text{he is mortal}}^{B}$.

p_2: Socrates is human.

c: Socrates is mortal.

P_1: $A \to B$

P_2: A

c: B

This is a valid modus ponens argument.

$$\begin{array}{l} A \to B \\ A \\ \hline \therefore B \end{array}$$

Example:

A : I have wings

B : I can fly

p_1: If I have wings, I can fly.

p_2: I cannot fly.

c: I don't have wings.

$P_1 : A \to B$

$P_2 : \neg B$

$c : \neg A$

$$\begin{array}{l} A \to B \\ \neg B \\ \hline \therefore \neg A \end{array}$$

This is a valid modus tollens argument.

Example: Consider the following premises:

p_1: Logic is difficult or not many students like logic.

p_2: If mathematics is easy, then logic is not difficult.

Do the conclusions below make a valid arguments with these premises?

(a) c: Many students like logic only if mathematics is not easy.

(b) c: If mathematics is not easy, then logic is not difficult.

$\neg m \to \neg l$

Solutions:

$P_1 : l \vee \neg s$

$P_2 : m \to \neg l$

(a) $c : s \to \neg m$

(a) l: "logic is difficult"

s: "many students like logic"

m: "mathematics is easy"

$l \vee \neg s$

$m \rightarrow \neg l$

$\therefore s \rightarrow \neg m$

So, p_1: $l \vee \neg s$

p_2: $m \rightarrow \neg l$

c: $s \rightarrow \neg m$

Next we will show that c follows from the premises as a conclusion (that is, the argument is valid), using two different techniques.

We are considering the following argument:

$$\begin{array}{l} l \vee \neg s \\ \underline{m \rightarrow \neg l} \\ \therefore s \rightarrow \neg m \end{array}$$

(i) Use rules of inference and basic logical equivalences:

(1): $l \vee \neg s$ premise
(2): $\neg s \vee l$ from (1) by commutativity of "or"
(3): $s \rightarrow l$ from (2) by implication law
(4): $m \rightarrow \neg l$ premise
(5): $\neg m \vee \neg l$ from (4) by implication law
(6): $\neg l \vee \neg m$ from (5) by commutativity of "or"
(7): $l \rightarrow \neg m$ from (6) by implication law
(8) $s \rightarrow \neg m$ from (3) and (7) by hypothetical syllogism

$s \quad l$
$A \rightarrow B$
$l \quad B \rightarrow C \; \neg m$
$\therefore A \rightarrow C$
$s \quad \neg m$

Hence the argument is valid.

(ii) Use truth tables:

Make a column for l, s, m, p_1, p_2 and c each. Assign T/F to l, s, m (you'll have 8 rows). Consider only those rows where both p_1 and p_2 are true. Is c always true in these rows? If it is, then the argument is valid.

— Do this as an exercise.

s : T m : T

l	s	m	$l \vee \neg s$	$m \to \neg l$	$s \to \neg m$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	T
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	F	T	T
F	F	T	T	T	T
F	F	F	T	T	T

In all rows where all premises are true, we see that the conclusion $(s \to \neg m)$ is also true.

So, this is a valid argument

$$\begin{array}{l} l \vee \neg s \\ m \to \neg l \\ \hline \therefore\ s \to \neg m \end{array}$$

(b) l: "logic is difficult"

s: "many students like logic"

m: "mathematics is easy"

So, p_1: $l \vee \neg s$

p_2: $m \to \neg l$

c: $\neg m \to \neg l$

We are considering the following argument:

$$\boxed{\begin{array}{l} l \vee \neg s \\ \underline{m \to \neg l \qquad} \\ \therefore \neg m \to \neg l \end{array}}$$

(i) Use rules of inference and basic logical equivalences:

In what follows we will see that this argument is not valid. Therefore we save ourselves the futile attempt of using rules of inference to get from the premises to the conclusion.

(ii) Use truth tables:

Make a column for l, s, m, p_1, p_2 and c each. Assign T/F to l, s, m (you'll have 8 rows). Consider only those rows where both p_1 and p_2 are true.

Is c always true in these rows? If it is, then the argument is valid.

If not, then those rows where p_1 and p_2 are T, but c is F yield counterexamples, and we conclude that the argument is invalid.

m : F
l : T

l	s	m	$l \vee \neg s$	$m \to \neg l$	$\neg m \to \neg l$
T	T	T	T	F	T
T	T	F	T	T	F
T	F	T	T	F	T
T	F	F	T	T	F
F	T	T	F	T	T
F	T	F	F	T	T
F	F	T	T	T	T
F	F	F	T	T	T

We see that there are cases where all premises are true, but the conclusion is false. Therefore, the argument

$$\begin{array}{l} l \vee \neg s \\ m \to \neg l \\ \hline \therefore \neg m \to \neg l \end{array}$$

is invalid.