Statements with Multiple Quantifiers

The existential quantifier and the universal quantifier may appear together in one statement.

The order in which they appear affects the meaning of the statement.

If you want to establish the truth of a statement of the form

$$\forall x \in D, \exists y \in E \text{ such that } P(x, y),$$

your challenge is to allow someone else to pick whatever element x in D they wish and then you must find an element y in E that "works" for that particular x.

If you want to establish the truth of a statement of the form

$$\exists x \in D \text{ such that } \forall y \in E, P(x, y),$$

your job is to find one particular x in D that will "work" for any y in E anyone might choose to challenge you with.

Example: Find the truth value of the following statements.

(i) $\forall x \in \mathbb{R}$, $\exists\ y \in \mathbb{R}$ such that $x + y = 0$.

True for any choice of x we have $y = -x$ so that $x+y=0$

(ii) $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$ $x + y = 0$.

False

This statement claims that there's a fixed value of x that makes $x+y=0$ for all choices of y.

When negating statement with multiple quantifiers, we simply use the De Morgan's laws repeatedly (from left to right).

Example: Let $D = E = \{-2, -1, 0, 1, 2\}$. Write negations for each of the following statements and determine which is true, the given statement or its negation.

F because $x=-2$ is a counter-example

(i) $\forall x \in D$, $\exists y \in E$ such that $x + y = 1$.

negation: $\neg(\forall x \in D, \exists y \in E$ such that $x+y=1)$

$\exists x \in D\ \neg(\exists y \in E$ such that $x+y=1)$

$\exists x \in D.\ \forall y \in E.\ \neg(x+y=1)$

$\exists x \in D\ \forall y \in E\ x+y \neq 1$

negation is T

F

(ii) $\exists x \in D$ such that $\forall y \in E$, $x + y = -y$.

negation: $\neg(\exists x \in D$ s.t $\forall y \in E, x+y=-y)$

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$\forall x \in D\ \exists y \in E\ x+y \neq -y$

original statement is false: consider $y=-2$

(iii) $\forall x \in D$, $\exists y \in E$ such that $xy \geq y$.

original statement is true.

negation: $\neg(\forall x \in D, \exists y \in E\ xy \geq y)$

$\exists x \in D\ \forall y \in E\ xy \ngeq y$

$\exists x \in D\ \forall y \in E\ xy < y$

Note that for $y=0$ we see that no choice of x works.

(iv) $\exists x \in D$ such that $\forall y \in E$, $x \leq y$.

True: for $x=-2$ we have $x \leq y\ \forall y \in E$

negation: $\neg(\exists x \in D\ \forall y \in E\ x \leq y)$

$\forall x \in D\ \exists y \in E\ x \nleq y$

$\forall x \in D\ \exists y \in E\ x > y$

Proofs

First some terminology:

A **theorem** is a mathematical statement that is known to be true.

An **axiom** (**postulate**) is an assumption accepted without proof.

A **proof** is a sequence of statements forming an argument that shows that a theorem is true.

A **lemma** is a short theorem used in the proof of another theorem.

A **corollary** is a theorem that is an immediate consequence of another theorem.

A **conjecture** is a mathematical statement whose truth value is still unknown (usually conjectures are statements that are thought to be true but hard to prove). Once proved (if it is indeed true), it becomes a theorem.

Goldbach conjecture: Every positive even integer greater than or equal to 4 can be written as the sum of two prime numbers.

A **fallacy** is an incorrect reasoning.

Some common fallacies:

Fallacy of affirming the conclusion:

$$\begin{array}{l} p \to q \\ \underline{q \qquad} \\ \therefore p \end{array}$$

This is an invalid argument. (Make a truth table to see this.)

$4 = 2+2$
$6 = 3+3$
$8 = 3+5$
$10 = 3+7$
$12 = 5+7$
$14 = 7+7$
$16 = 5+11$
$18 = 7+11$