-1.7^2

X

$\Pr[X=1] = 0.8$ $\Pr[X=0] = 0.2$

Mean : $\mu = 0.8 \times 1 + 0.2 \times 0$

$= 0.8$

0.64
0.2
1.28

0.128
0.04
0.168

$\sigma^2 = ?$ $\sigma^2 = \Sigma (x-\mu)^2 \cdot f(x)$

$f(0)$
0.2

$= (1-0.8)^2 \, 0.8 + (0-0.8)^2 \, 0.2$

$= (0.2)^2 \times 0.8 + 0.64 \times 0.2$

$= 0.04 \times 0.8 + 0.64 \times 0.2$

$\sigma^2 = \Sigma x^2 f(x) - \mu^2$ $= 0.16$

$= 1 \cdot 0.8 + 0 \cdot 0.2$

$- 0.64$

$\sigma = 0.4$

$= 0.8 - 0.64 = 0.16$

$\sigma = \sqrt{0.16} = 0.04$

$$\sigma^2 = \sum (x-\mu)^2 f(x) = \boxed{\sum x^2 f(x) - \mu^2}$$

$$\sigma^2 = \sum_x (x-\mu)^2 f(x)$$

$$\sigma^2 = \sum_x x^2 f(x) - \mu^2$$

Proof: $\sigma^2 = \sum_x (x-\mu)^2 f(x)$

$$= \sum_x (x^2 - 2\mu x + \mu^2) f(x)$$

$$= \sum_x \left[x^2 f(x) - 2\mu x f(x) + \mu^2 f(x) \right]$$

$$= \sum_x x^2 f(x) - 2\mu \sum_x x f(x) + \sum_x \mu^2 f(x)$$

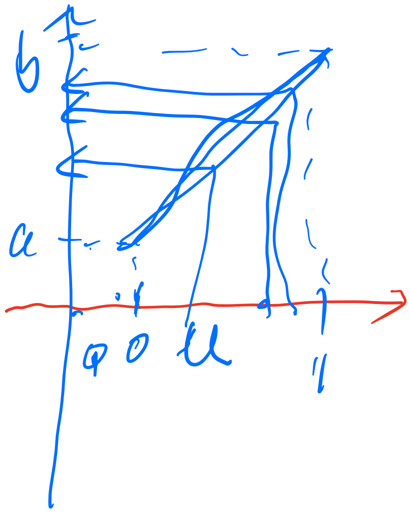
$$= \sum_x x^2 f(x) - 2 \cdot \mu \cdot \mu + \mu^2 \underbrace{\sum f(x)}_{1}$$

$$= \sum x^2 f(x) - 2\mu^2 + \mu^2$$

$$\sigma^2 = \sum x^2 f(x) - \mu^2$$

$u \in (0, u, 1)$

$x \in (a, u, b)$



$x = a + (b-a)\cdot u$

$u=0 \Longrightarrow x = a$

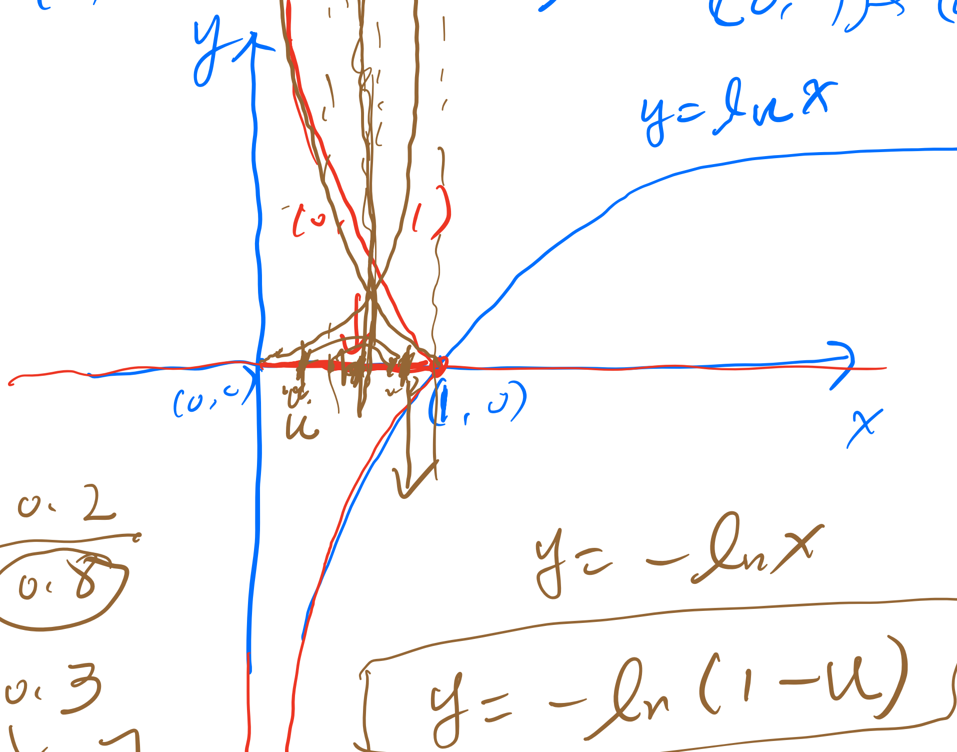
$u=1 \Longrightarrow x = b$

$y = -\ln x$

$u \in (0, 1]$?

$x \in (0, +\infty)$

$(0, 1) \to (0, +\infty)$



$u = 0.2$

0.8

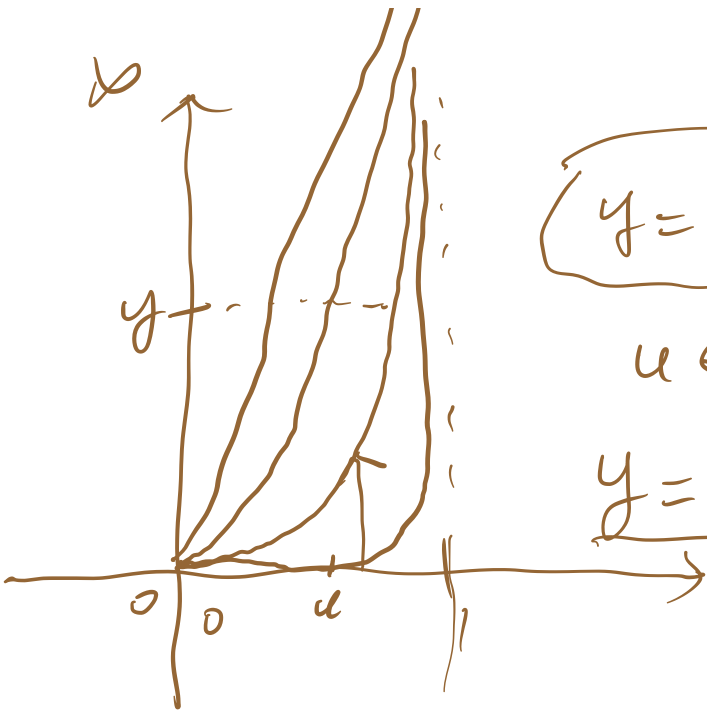
$u = 0.3$

$u = 0.7$

$y = -\ln x$

$\boxed{y = -\ln(1-u)}$

$\|$



$$\boxed{y = -\ln(1-u)}$$

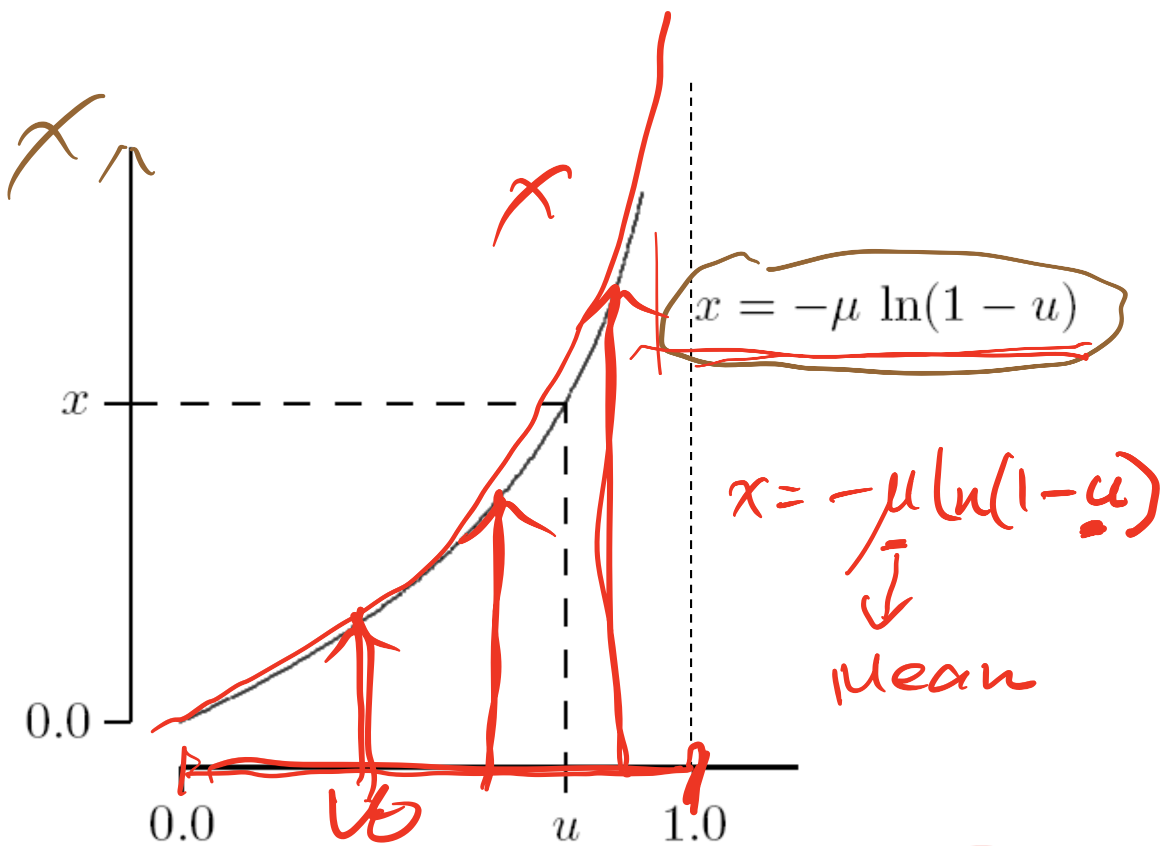
$u \in (0, 1)$

$y = -\mu \ln(1-u)$

Exponential Random Variates

The transformation is monotone increasing, one-to-one, and onto

$$
\begin{aligned}
& 0 < u < 1 \\
\Leftrightarrow\ & 0 < (1 - u) < 1 \\
\Leftrightarrow\ & -\infty < \ln(1 - u) < 0 \\
\Leftrightarrow\ & 0 < -\mu \ln(1 - u) < \infty \\
\Leftrightarrow\ & 0 < x < \infty
\end{aligned}
$$



$$x = -\mu \ln(1 - u)$$

$[0, +\infty)$

$u \in [0, 1)$

Exponential Random Variate

- Parameter μ: $$x = -\mu \ln(1-u)$$

$$\bar{x} = \mu \qquad \text{(sample mean)}$$

$$\sigma^2 = \mu^2 \qquad \text{(variance)}$$

$$\sigma = \mu \qquad \text{(standard deviation)}$$

- Continuous uniform - Parameters *a* and *b*:

$x = a + (b-a) \cdot u$

$$x = a + (b-a) * \text{Random()};$$

$[0, 1)$

$$\bar{x} = \frac{a+b}{2}$$

$$\sigma^2 = \frac{(b-a)^2}{12}$$

Generating an Exponential Random Variate

```
double Exponential(double μ)      /* use μ > 0.0 */
{
        return (-μ * log(1.0 - Random()));
}
```

$-\mu$ u

- The parameter μ specifies the sample mean.
- In the single-server service node simulation, use Exponential(μ) interarrival times:

$$a_i = a_{i-1} + \text{Exponential}(\mu);\ i = 1, 2, 3, \ldots, n.$$

Exponential Interarrival Time

```
double Exponential(double m)          // generate an Exponential random variate, m >0.0
{
     return (-m * log(1.0 - Random()));
}
double Uniform(double a, double b)  // generate a Uniform random variate, a < b
 {
      return (a + (b - a) * Random());
 }
double GetArrival(void)                    // generate the next arrival time
 {
     static double arrival = START;
     arrival +=   Exponential(2.0);
     return (arrival);
}
double GetService(void)                   // generate the next service time
{
     return (Uniform(1.0, 2.0));
}
```

arrival = arrival + (exponential(2.0))

Geometric Random Variates

- The Geometric(p) random variate is the discrete analog to a continuous Exponential(μ) random variate

  Let x = Exponential(μ) = $-\mu \ln(1 - u)$

  $y = \lfloor x \rfloor$ and let $p = \Pr(y \neq 0)$,

It can be proved that

$$p = Pr(y \neq 0) = exp(-1/\mu)$$

Use p as a parameter for the Geometric Random Variate,

$$y = \lfloor \ln(1 - u) / \ln(p) \rfloor$$

where $p = \Pr(y \neq 0)$.

μ: mean

return long(-μ ln(1-u))

$$x = -\mu \ln(1-u)$$

$$y = \lfloor x \rfloor = \lfloor -\mu \ln(1-u) \rfloor$$

$$x \in (0, +\infty)$$

$$y \in \{0, 1, 2, 3, 4, \ldots\}$$

$$\lfloor 0.5 \rfloor = 0$$

$$\lfloor 5.756 \rfloor = 5$$

$$\lfloor 101.00001 \rfloor = 101$$

$$\lfloor 99.0000 \rfloor = 99$$

$$x = -\mu \ln(1-u)$$

$$y = \lfloor x \rfloor \qquad x \in (0, +\infty)$$

$$Pr(y \neq 0) \qquad \boxed{x \geq 1}$$

For what value of

x $y \neq 0$? $\qquad x = 0.6$

$$y = \lfloor x \rfloor \qquad \lfloor x \rfloor = \lfloor 0.6 \rfloor$$

$$y \neq 0 \qquad = 0$$

$$Pr[y \neq 0] = Pr[x \geq 1]$$

$$= Pr[-\mu \ln(1-u) \geq 1]$$

$$= Pr[u \geq 1 - e^{-\frac{1}{\mu}}]$$

$$-\mu \ln(1-u) \geq 1$$

$$e^{\ln(1-u)} \leq e^{-\frac{1}{\mu}}$$

$$1-u \leq e^{-\frac{1}{\mu}}$$

$$\boxed{u \geq 1 - e^{-\frac{1}{\mu}}}$$

$$\Pr\left[u \geq \underbrace{1 - e^{-\frac{1}{\mu}}}_{r}\right]$$

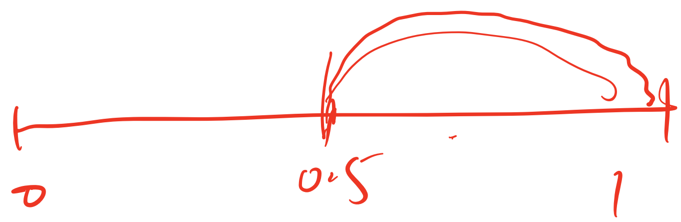
$$e^{-\frac{1}{\mu}} = \frac{1}{e^{\frac{1}{\mu}}} > 1$$

$$< 1$$

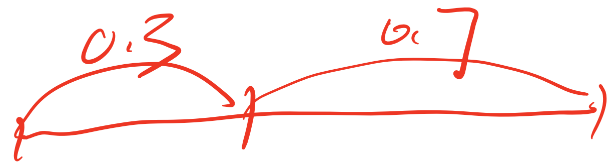
$$0 < r < 1$$

$$\Pr[u \geq r] \quad \underline{0 < r \leq 1}$$

Assume that $r = 0.5$



$r = 0.3$



$$\Pr[u \geq r] = 1 - r$$

$$1 - \left[1 - e^{-\frac{1}{\mu}}\right] = e^{-\frac{1}{\mu}}$$

$$P = \Pr[y \neq 0] = e^{-\frac{1}{\mu}}$$

$$P = e^{-\frac{1}{\mu}}$$

$$\ln P = \ln e^{-\frac{1}{\mu}} = -\frac{1}{\mu} \frac{\ln e}{1}$$

$$\ln p = -\frac{1}{\mu}$$

$$\boxed{\mu = -\frac{1}{\ln p}}$$

$$y = \lfloor x \rfloor = \lfloor -\mu \ln(1-u) \rfloor$$

$$= \left\lfloor \frac{\ln(1-u)}{\ln p} \right\rfloor$$

$$p = \Pr[y \neq 0]$$

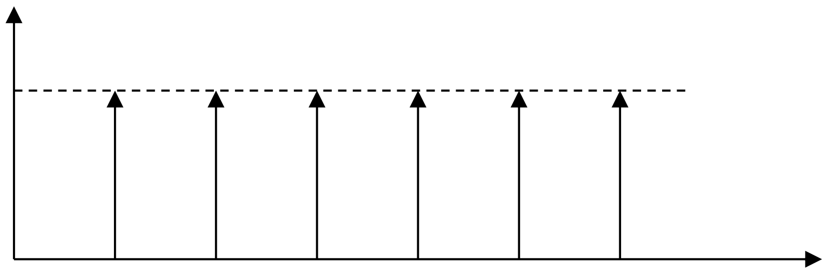
Discrete Uniform

- Parameters *a* and *b*

$$P(n) = \frac{1}{b-a+1} \qquad \text{for } n = a, a+1, \ldots, b-1, b$$

$$\bar{x} = \frac{a+b}{2} \qquad \text{(mean)}$$

$$\sigma^2 = \frac{(b-a)(b-a-2)}{12} \qquad \text{(variance)}$$



Geometric Random Variates

- With parameter p: $y = \lfloor \ln(1 - u) / \ln(p) \rfloor$

$$P(n) = P^n(1 - P), \quad n = 0,\ 1,\ 2, \ldots$$

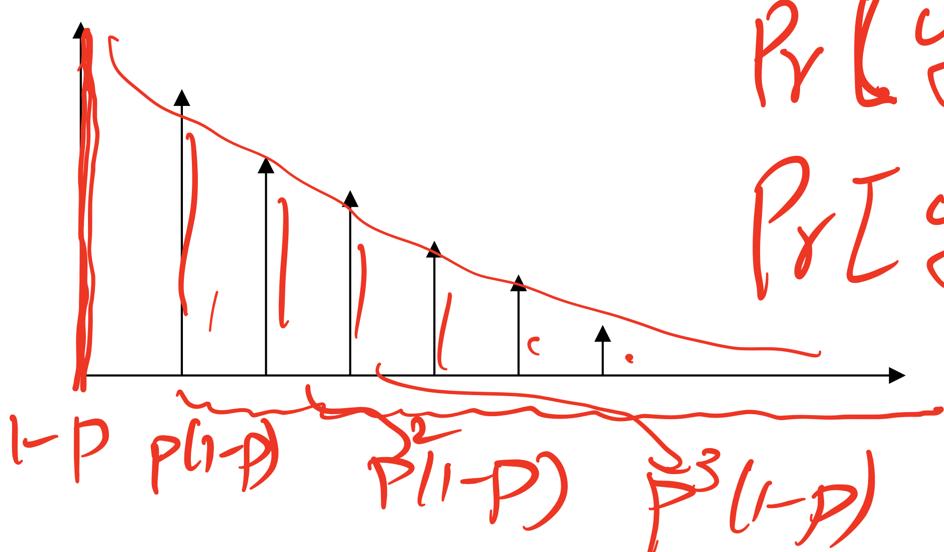
$Pr[y = n]$

$\bar{x}$

$$\bar{x} = \frac{p}{1-p} \qquad \text{(mean)}$$

$$\sigma^2 = \frac{p}{(1-p)^2} \qquad \text{(standard deviation)}$$

pdf



$Pr[y = 0] = 1 - P$

$Pr[y \neq 0] = P$

Generating a Geometric Random Variate

$Pr(y \neq 0)$

```
long Geometric(double p) /* use 0.0 < p < 1.0 */
{
   return ((long) (log(1.0 - Random()) / log(p)));
}
```

$U \in [0, 1)$

- The mean of a Geometric(p) random variate is $p/(1-p)$.
- If p is close to zero then the mean will be close to zero.
- If p is close to one, then the mean will be large.

$$\frac{p}{1-p}$$

Example of Service Times

- Assume that jobs arrive at random with a steady-state arrival rate of 0.5 jobs per minute.
- Assume that Job service times are composite with two components:
  - The number of service tasks is

    1 + Geometric(0.9) 0.9

  - The time (in minutes) per task is $Pr(y \neq 0) = 0.9$

    Uniform(0.1, 0.2)

Get Service Method

```
double GetService(void)
{
   long k;
   double sum = 0.0;
   long tasks = 1 + Geometric(0.9);
   for (k = 0; k < tasks; k++)
   sum += Uniform(0.1, 0.2);
   return (sum);
}
```

10

0, 1, 2, 3 . . .