Network of SSQ

Network of SSQ

- Assume that $k \geq 1$ single-server service nodes are indexed by $s=1, 2, \ldots, k$. The index $s=0$ is reserved for the "super node" that represents the *exterior* of the network – the *source* of the jobs that flowing into the network and the *sink* of the jobs flowing out of the network.
- Each service node has its own queue and its own type of queueing discipline (FIFO, LIFO, ect.), its own service-time distribution and infinite capacity. The service rate for node i is μ_i , $i \in \{1, 2, \ldots, k\}$.

Node-transition probability

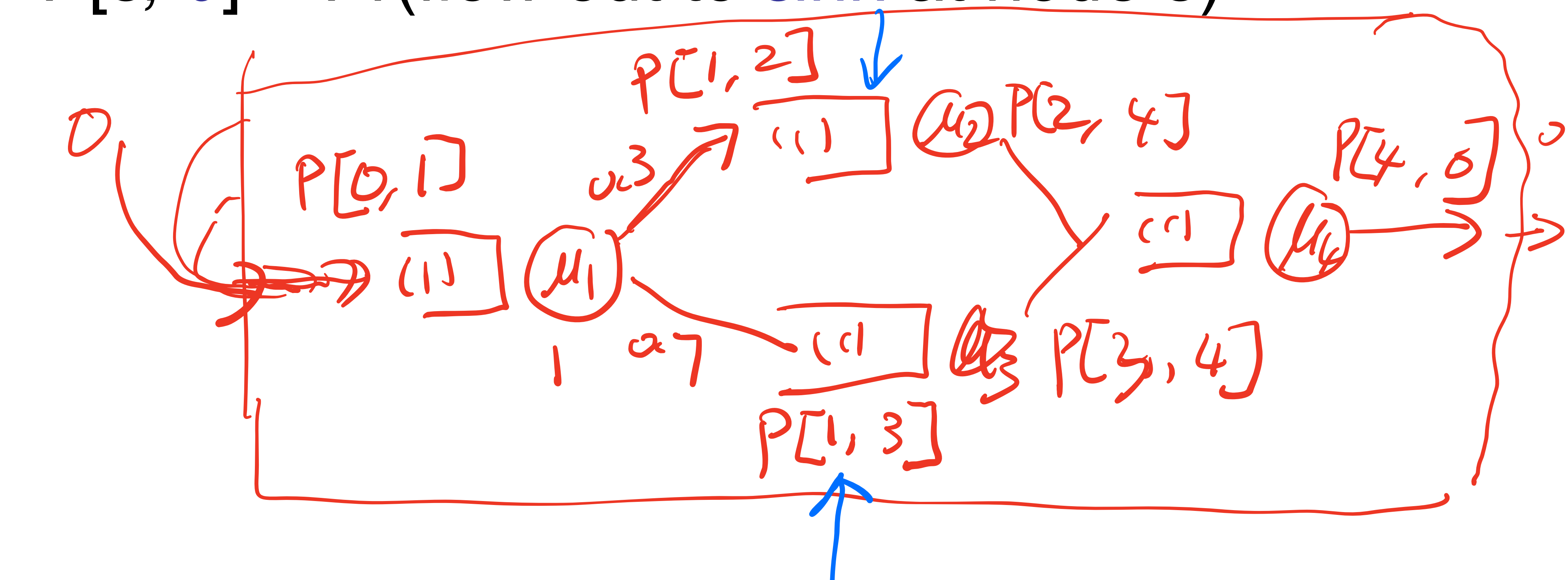
s=0

$P[s, s'] = \Pr(s \to s')$

$\quad = \Pr(\text{transition from node } s \text{ to node } s')$

$P[0, s'] = \Pr(\text{external arrival at node } s')$

$P[s, 0] = \Pr(\text{flow out to } \textit{sink} \text{ at node } s)$

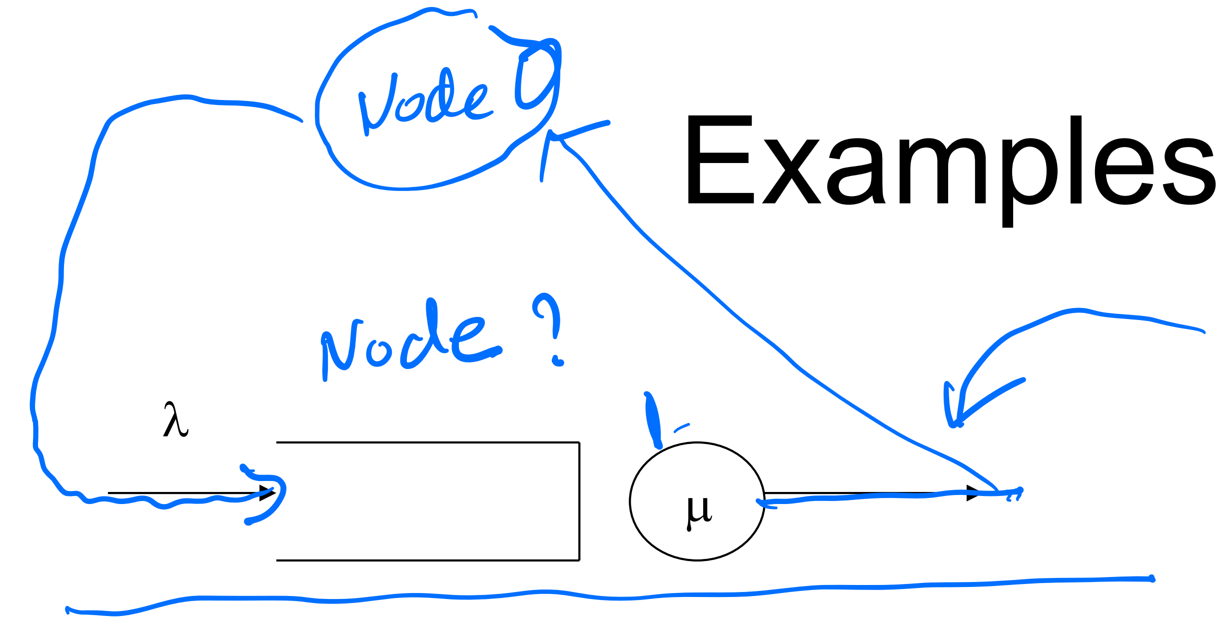


Node-transition probability matrix

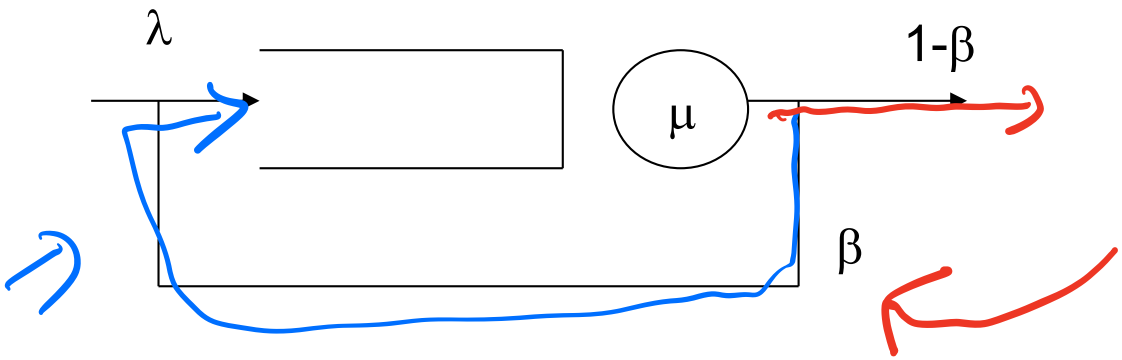
$$\begin{bmatrix} p[0,0], p[0,1], p[0,2], \ldots, p[0,k] \\ p[1,0], p[1,\ 1], p[1,2],\ \ldots, p[1,k] \\ p[2,0], p[2,1], p[2,2], \ldots, p[2,k] \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ p[k,0], p[k,1], p[k,2], \ldots, p[k,k] \end{bmatrix}$$

- $p[i, j]$, $i, j \in \{0, 1, 2, \ldots, k\}$.

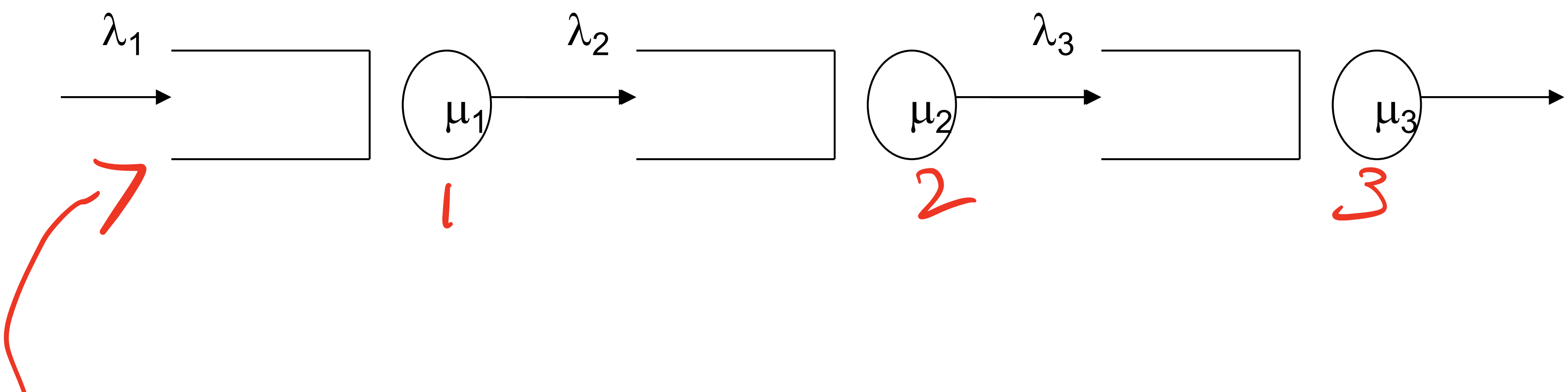
Examples



$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 1 \\ 1-\beta & \beta \end{bmatrix}$$



$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Open and closed network

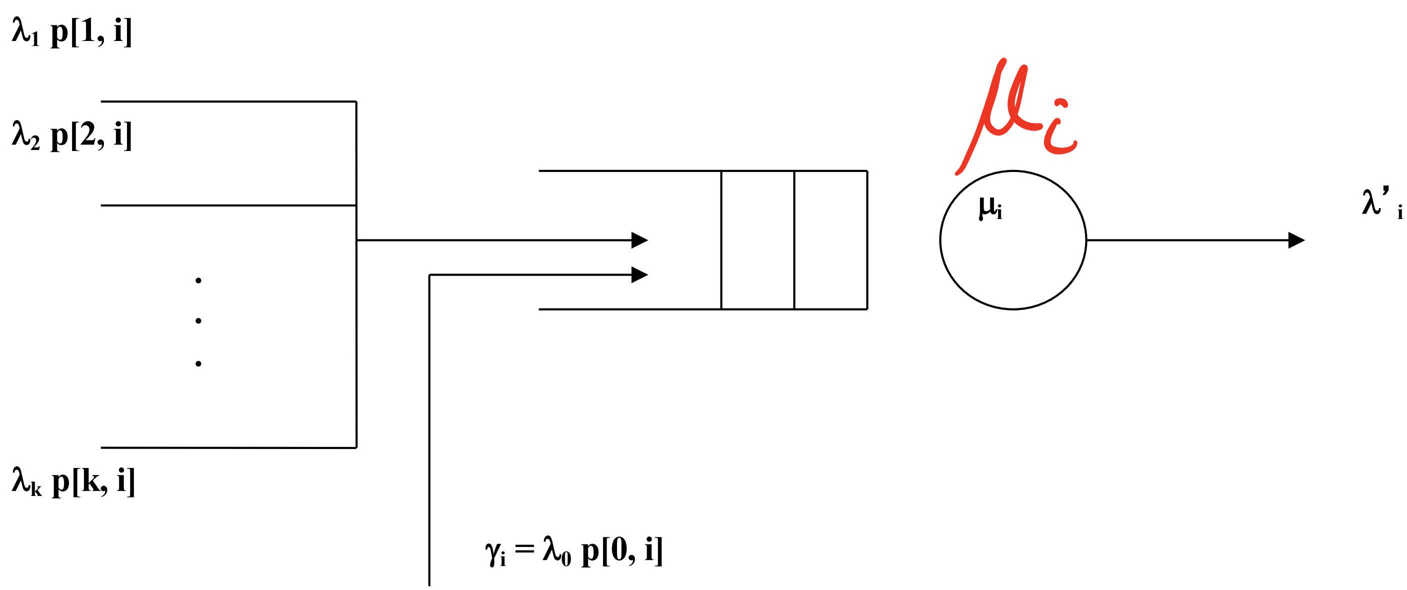
- **Definition 1**

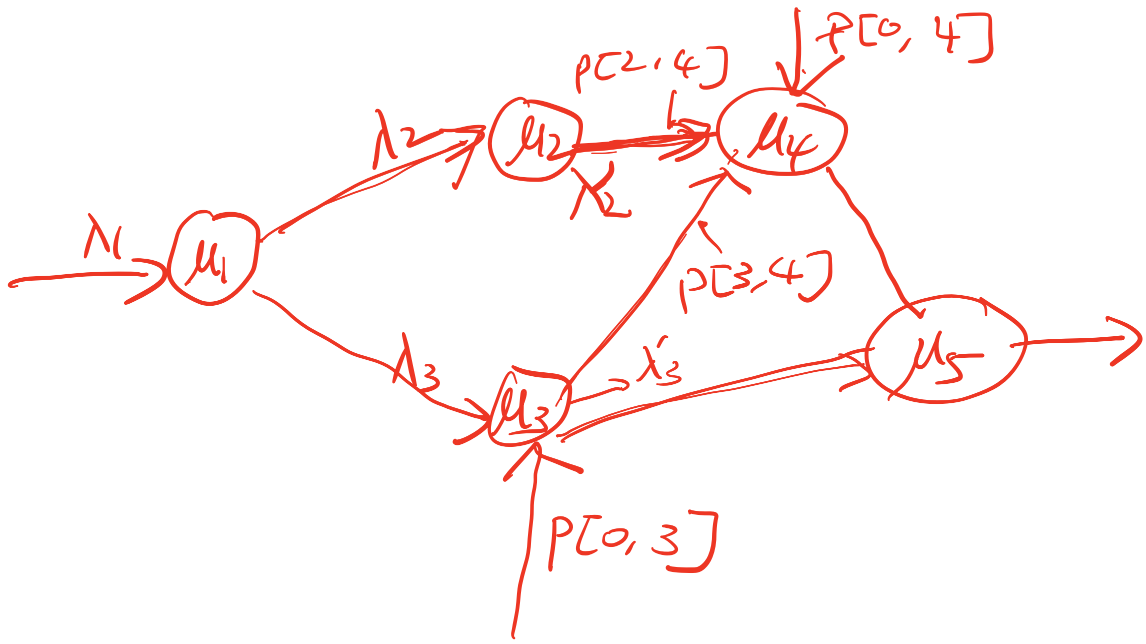
  If $P[0, s'] = P[s, 0] = 0$, for all $s, s' \in \{0, 1, 2, \ldots, k\}$, then the network is said to be closed. Otherwise, it is an open network.

Total arrival rate

- **Definition 2** For each service node *i*, let λ_i be the ***total* arrival rate**. Then

$$\lambda_i = \lambda_0 p[0,i] + \lambda_1 p[1,i] + \lambda_2 p[2,i] + \ldots + \lambda_k p[k,i], \quad i \in \{1,2,\ldots k\}$$





$$\lambda_4 = P[0,4]\,\lambda_0 + P[2,4]\,\lambda_2' + P[3,4]\cdot\lambda_3'$$

Steady State Analysis

If the system can reach steady state, then:

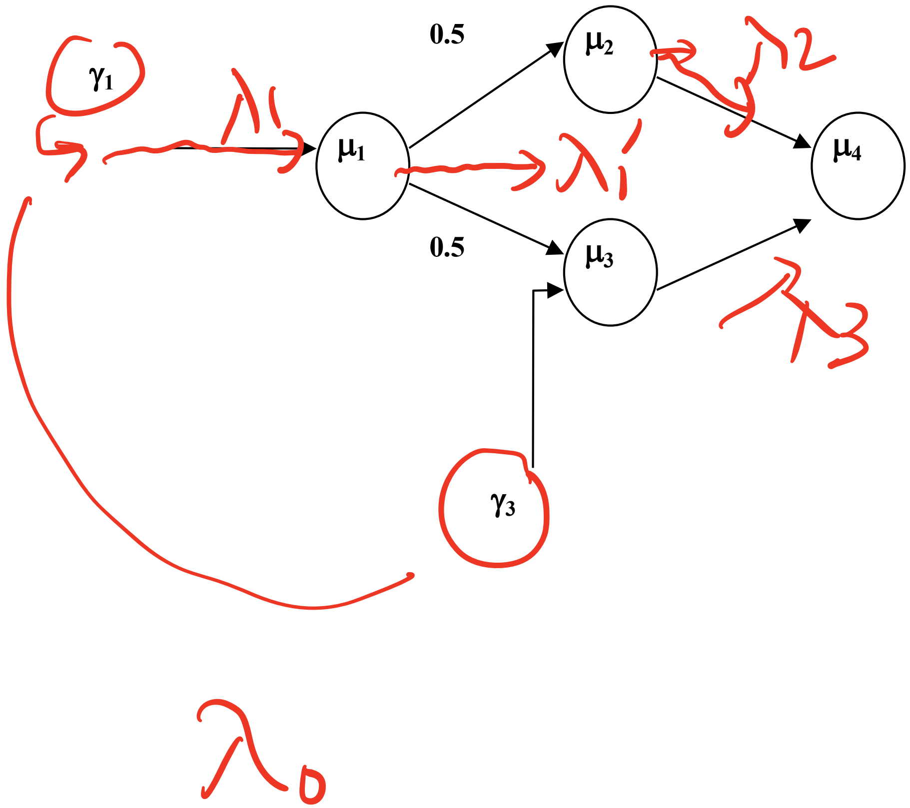
1) For every node, the flow-in rate = the flow-out rate (balance equation).

$$\lambda_i = \lambda_0 p[0,i] + \lambda_1 p[1,i] + \lambda_2 p[2,i] + \ldots + \lambda_k p[k,i] = \lambda'_i$$

2) Each server utilization:

$$\rho_i = \frac{\lambda_i}{\mu_i} < 1, \quad i \in \{1,2,\ldots,k\}$$

Example – Open Network



p[1,2]=0.5, p[1,3]=0.5,
p[2,4]=1, p[3,4]=1,

$\gamma_1 = \lambda_0$ p[0, 1],
$\gamma_3 = \lambda_0$ p[0, 3]
(γ_1 and γ_3 are known external arrival rates),

all other p[*i*, *j*]=0.

Balance Equations $(\lambda_1' = \lambda_1)$

Node 1 $\quad \lambda_1 = \nu_1$

(2) $\quad \lambda_2 = 0.5\lambda_1' = 0.5\lambda_1 = 0.5\nu_1$

(3) $\quad \lambda_3 = 0.5\lambda_1 + \nu_3 = 0.5\nu_1 + \nu_3$

(4) $\quad \lambda_4 = \lambda_3' + \lambda_2' = \lambda_2 + \lambda_3$

$$= 0.5\lambda_1 + 0.5\lambda_1 + \nu_3$$

$$= \lambda_1 + \nu_3$$

$$= \nu_1 + \nu_3$$

Solution:

$$\lambda_1 = \nu_1$$

$$\lambda_2 = 0.5\nu_1$$

$$\lambda_3 = 0.5\nu_1 + \nu_3$$

$$\lambda_4 = \nu_1 + \nu_3$$

Balance equations - open network

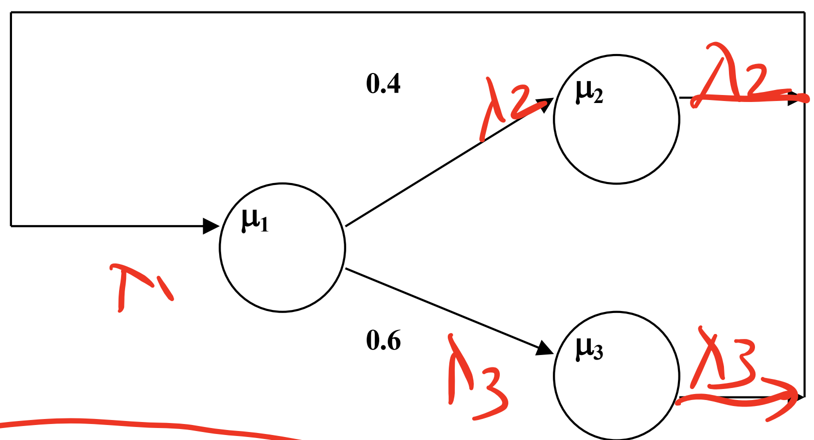
$\lambda_1 = \gamma_1$
$\lambda_2 = 0.5\lambda_1 = 0.5\gamma_1$
$\lambda_3 = 0.5\lambda_1 + \gamma_3 = 0.5\gamma_1 + \gamma_3$
$\lambda_4 = \lambda_2 + \lambda_3 = \gamma_1 + \gamma_3$

- In an open network, λ_i can be found from γ_i (the external arrival rates).
- Given μ_i, sever utilization can be found:

$$\rho_i = \frac{\lambda_i}{\mu_i}, i \in \{1,2,\ldots,k\}$$

Balance equations - closed network

- Central server model



Mean service time at each server:

x_1=0.05, x_2=0.08, x_3=0.04,

The mean service rates:

μ_1=1/ 0.05, μ_2= 1/0.08, μ_3=1/ 0.04

Balance equation for each node:

$\lambda_1= \lambda_2+\lambda_3$
$\lambda_2=0.4\lambda_1$
$\lambda_3=0.6\lambda_1$

$$\frac{\lambda_2}{\lambda_1} = 0.4, \qquad \frac{\lambda_3}{\lambda_1} = 0.6$$

$$\frac{\rho_2}{\rho_1} = \frac{\lambda_2 \mu_1}{\lambda_1 \mu_2} = 0.4 \times \frac{0.08}{0.05} = 0.64, \quad \frac{\rho_3}{\rho_1} = \frac{\lambda_3 \mu_1}{\lambda_1 \mu_3} = 0.6 \times \frac{0.04}{0.05} = 0.48$$

$$\frac{\lambda_2}{\lambda_1} = 0.4$$

$$\frac{\lambda_3}{\lambda_1} = 0.6$$

The upper bound server utilization: $\rho_1<1 \Rightarrow \rho_2<0.64, \rho_3<0.48$.

$\lambda_1 = \lambda_2 + \lambda_3$

$\lambda_2 = 0.4\lambda_1$

$\lambda_3 = 0.6\lambda_1$ $\qquad$ μ_1 : Service rate

$$\frac{\rho_2}{\rho_1} = \frac{\frac{\lambda_2}{\mu_2}}{\frac{\lambda_1}{\mu_1}} = \frac{\lambda_2 \mu_1}{\lambda_1 \mu_2}$$

$$= 0.4 \times \frac{\frac{1}{0.05}}{\frac{1}{0.08}}$$

Service Time $= 0.05$

Service Rate $\mu_2 = \frac{1}{0.05}$

$$= 0.4 \times \frac{0.08}{0.05}$$

$$\frac{\rho_2}{\rho_1} = 0.64 \Rightarrow \rho_2 = 0.64\rho_1$$

$$\frac{\rho_3}{\rho_1} = 0.48 \Rightarrow \rho_3 = 0.48\rho_1$$

$\rho_1 < 1, \quad \rho_2 < 0.64$

$\rho_3 < 0.48$

Review

- Introduction to discrete event simulation
- Examples: SSQ, inventory systems, optimization
- Monte Carlo Simulation
- Lehmer random number generators
- Empirical tests of randomness
- Discrete event simulation - exponential and geometric Variates
- Simulation languages – GPSS
- Discrete Random Variables - Statistical Models and Generations
- Continuous random variables
- Output analysis – Single Server Queue
- Output analysis – Birth Death models
- Network of single-server-queue