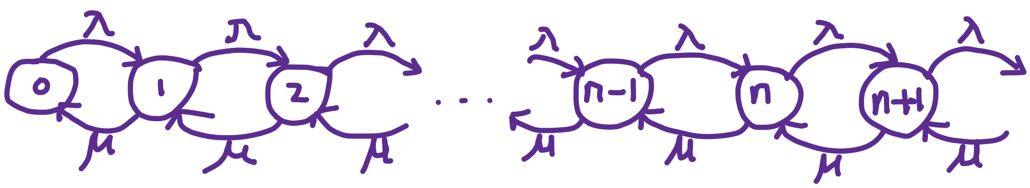
Birth-Death Models

Example 1. M/M/1 Queue

$$\begin{cases} \lambda(n) = \lambda, & n = 0,1,2,\dots \\ \mu(n) = \mu, & n = 1,2,3,\dots \end{cases}$$

State Transition Diagram



Balance Equations

$$\lambda P(0) = \mu P(1)$$
$$(\lambda + \mu)P(1) = \lambda P(0) + \mu P(2)$$
$$(\lambda + \mu)P(2) = \lambda P(1) + \mu P(3)$$
$$\dots$$
$$(\lambda + \mu)P(n) = \lambda P(n-1) + \mu P(n+1)$$
$$\dots$$

where P(n) is the probability for the system to be in State n.

Let $\rho = \frac{\lambda}{\mu}$, assume $\lambda < \mu,\ so\ \rho < 1$ is the server utilization.

Solve the balance equations:

$$P(1) = \frac{\lambda}{\mu}P(0) = \rho P(0)$$
$$P(2) = \frac{\lambda}{\mu}P(1) = \rho^2 P(0)$$
$$P(3) = \frac{\lambda}{\mu}P(2) = \rho^3 P(0)$$
$$\dots$$
$$P(n) = \frac{\lambda}{\mu}P(n-1) = \rho^n P(0)$$
$$\dots$$

Since $\sum_{n=0}^{\infty} P(n) = 1 \implies P(0) + \rho P(0) + \rho^2 P(0) + \cdots + \rho^n P(0) + \cdots = 1$
So

$$P(0) \sum_{n=0}^{\infty} \rho^n = 1 \implies P(0) = \frac{1}{\sum_{n=0}^{\infty} \rho^n} = 1 - \rho$$

$$P(n) = \rho^n P(0) = \rho^n (1 - \rho)$$

Next, the mean number of customers in the system can be calculated as:

$$\bar{N} = \sum_{n=0}^{\infty} nP(n) = \sum_{n=0}^{\infty} n\rho^n (1 - \rho) = (1 - \rho) \sum_{n=0}^{\infty} n\rho^n$$

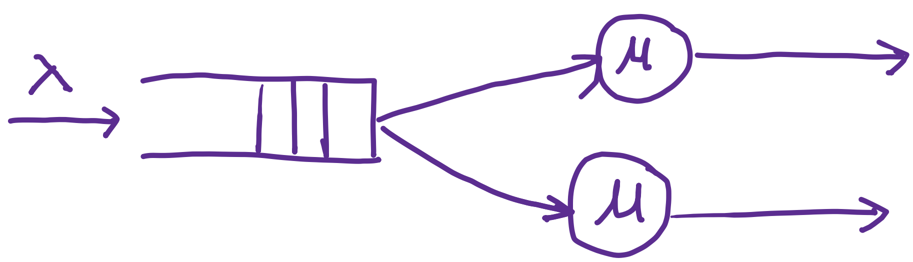
By the result $\sum_{n=0}^{\infty} ix^i = \frac{x}{(1-x)^2}$, we have

$$\bar{N} = (1 - \rho) \frac{\rho}{(1 - \rho)^2} = \frac{\rho}{1 - \rho}$$

By Little's Results, the mean response time:

$$\bar{T} = \frac{\bar{N}}{\lambda} = \frac{1}{\mu(1 - \rho)}$$

Example 2. M/M/2 Queue



System equations:

$$\begin{cases} \lambda(n) = \lambda, & n = 0,1,2,\ldots \\ \mu(n) = \min(2, n)\mu, & n = 1,2,3,\ldots \end{cases}$$

State Transition Diagram:

Balance equations:

$$\mu P(1) = \lambda P(0)$$
$$(\lambda + \mu)P(1) = \lambda P(0) + 2\mu P(2)$$
$$(\lambda + 2\mu)P(2) = \lambda P(1) + 2\mu P(3)$$
$$\dots$$
$$(\lambda + 2\mu)P(n) = \lambda P(n-1) + 2\mu P(n+1)$$
$$\dots$$

Solve the balance equations:

$$P(1) = \frac{\lambda}{\mu}P(0) = \rho P(0) = 2\frac{\rho}{2}P(0)$$
$$P(2) = \frac{1}{2}\left(\frac{\lambda}{\mu}\right)P(1) = \frac{1}{2}\rho^2 P(0) = 2\left(\frac{\rho}{2}\right)^2 P(0)$$
$$P(3) = \frac{1}{2}\frac{\lambda}{\mu}P(2) = 2\left(\frac{\rho}{2}\right)^3 P(0)$$
$$\dots$$
$$P(n) = \left(\frac{1}{2}\right)^{n-1}\left(\frac{\lambda}{\mu}\right)^n P(0) = 2\left(\frac{1}{2}\right)^n \rho^n P(0) = 2\left(\frac{\rho}{2}\right)^n P(0)$$

$$\dots$$

Since $\sum_{n=0}^{\infty} P(n) = 1 \implies P(0) + \sum_{n=1}^{\infty} 2\left(\frac{\rho}{2}\right)^n P(0) = 1$

So

$$P(0)\left(1 + \sum_{n=1}^{\infty} 2\left(\frac{\rho}{2}\right)^n\right) = 1$$

$$P(0) = \frac{1}{1 + 2\sum_{n=1}^{\infty}\left(\frac{\rho}{2}\right)^n} = \frac{1}{1 + 2\frac{\rho}{2-\rho}} = \frac{2-\rho}{2+\rho}$$

$$P(n) = 2\left(\frac{\rho}{2}\right)^n P(0) = \frac{2(2-\rho)}{2+\rho}\left(\frac{\rho}{2}\right)^n$$

Next, the mean number of customers in the system can be calculated as:

$$\bar{N} = \sum_{n=0}^{\infty} nP(n) = \sum_{n=0}^{\infty} n\frac{2(2-\rho)}{2+\rho}\left(\frac{\rho}{2}\right)^n = \frac{4\rho}{4-\rho^2}$$ (2+p)(2-p)

By Little's Results, the mean response time:

$$\bar{T} = \frac{\bar{N}}{\lambda} = \frac{\frac{4\rho}{4-\rho^2}}{\lambda} = \frac{4}{\mu(4-\rho^2)}$$

So we obtain the results for M/M/2 queue:

$$\bar{N} = \frac{4\rho}{4-\rho^2}, \quad \bar{T} = \frac{4}{\mu(4-\rho^2)}$$

Basic steps for Birth-Death models:

1. Draw State Transition Diagram
2. Obtain the balance equations
3. Solve the balance equations
4. The mean number of customers in system

$$\bar{N} = \sum_{n=0}^{\infty} nP(n)$$

5. The mean response time (Little's results)

$$\bar{T} = \frac{\bar{N}}{\lambda}$$

P(n) (n=0, 1, 2, 3, …) is the equilibrium distribution of the number of customers in system.