Assignment 4

Question 1

Q1. X pdf $\longrightarrow f(x) = ax$

(a) $\sum_{1}^{n} f(x) = 1$

$\Rightarrow \sum_{x=1}^{n} ax = 1$

$\Rightarrow a = \frac{1}{\sum_1^n x} \Rightarrow \boxed{a = \frac{2}{n(n+1)}}$

(b) cdf $\cdot\ F(x) = F(x-1) + f(x)$

$F(x) = \sum_{1}^{x} f(x)$

$= \sum_{1}^{x} ax$

$= a \sum x$

$= \frac{2}{n(n+1)} \cdot \frac{x(x+1)}{2}$

$\boxed{\text{cdf} \rightarrow F(x) = \frac{x(x+1)}{n(n+1)}}$

mean $= \sum_{1}^{n} x f(x) = \sum_{1}^{n} x \cdot ax = a \sum_{1}^{n} x^2$

$= \frac{2}{n(n+1)} * \frac{n(n+1)(2n+1)}{6}$

$\boxed{\text{mean} = \frac{2n+1}{3}}$

~~varia~~ standard deviation $= \sqrt{\sum_{n} x^2 f(x) - \mu^2}$

$\sigma = \sqrt{\sum_{n} x^2 \cdot ax - \left(\frac{2n+1}{3}\right)^2}$

$= \sqrt{a \sum x^3 - \frac{(2n+1)^2}{(3)^2}}$

$$\sigma^2 = \frac{2}{n(n+1)} * \frac{n^2(n+1)^2}{4} - \frac{(2n+1)^2}{9}$$

$$= \frac{n(n+1)}{2} - \frac{(4n^2+4n+1)}{9}$$

$$= \frac{1}{18}\left(9n^2+9n - 8n^2 - 8n - 2\right)$$

$$= \frac{1}{18}(n^2+n-2) \quad = \frac{1}{18}(n^2+2n-n-2)$$

$$= \frac{1}{18}(n(n+2) - 1(n+2))$$

$$\sigma^2 = \frac{(n-1)(n+2)}{18}$$

$$\boxed{\text{standard deviation} = \sigma = \sqrt{\frac{(n-1)(n+2)}{18}}}$$

Question 2

Q2. exponential distribution for random variates

$$X = -\mu \ln(1-u)$$

cdf $F(x) = P(X \leq x) = P(-\mu \ln(1-u) \leq x)$

$$= P(+\ln(1-u) \leq -x/\mu)$$

$$= P(1-u \leq e^{-x/\mu})$$

$$= P(u \geq 1-e^{-x/\mu})$$

$$\boxed{\text{cdf} \longrightarrow F(x) = 1-e^{-x/\mu}}$$

$$pdf = \frac{\partial}{\partial x} cdf = \frac{\partial}{dx}\left(1 - e^{-x/\mu}\right)$$

$$= \frac{\partial}{\partial x} 1 - \frac{\partial}{dx} e^{-x/\mu}$$

$$= 0 - \left(-\frac{1}{\mu}\right) e^{-x/\mu}$$

$$\boxed{pdf \rightarrow f(x) = \frac{1}{\mu} e^{-x/\mu} = \lambda e^{-\lambda x} \quad , \quad \lambda = \frac{+1}{\mu}}$$

let's say $k = \frac{1}{\mu}$

$$\text{mean} = \int x f(x)\, dx$$

$$= \int x \cdot k e^{-kx} dx = k \cdot \int_0^\infty x e^{-kx} dx$$

$u = x \quad \frac{dv}{dx} = e^{-kx}$

$\frac{du}{dx} = 1 \quad v = \frac{-e^{-kx}}{k}$

$du = dx$

$$= k\left(u \cdot v - \int v\, du\right)$$

$$= k\left(\frac{-xe^{-kx}}{k} + \int \frac{e^{-kx}}{k} dx\right)$$

$$\text{mean} = -xe^{-kx} - \frac{e^{-kx}}{k} + C$$

$$\text{mean} = -e^{-kx}\left(x + \frac{1}{k}\right) + C \Big|_0^\infty$$

$$\text{mean} = \frac{1}{k} = \mu$$ → well that makes sense

$E(X)$

$$(\text{standard deviation})^2 = \text{variance} = \int_0^\infty x^2 f(x)\, dx - \mu^2$$

$u = x^2 \quad \frac{dv}{dx} = e^{-kx}$

$du = 2x\, dx \quad v = \frac{-e^{-kx}}{k}$

$$= \int_0^\infty x^2 \cdot k e^{-kx} dx - \mu^2$$

$$= k \int_0^\infty x^2 \cdot e^{-kx} dx - \mu^2$$

$$\frac{-x^2 e^{-kx}}{k} + \int \frac{+e^{-kx}}{k} \cdot 2x\, dx$$

$$\frac{-x^2 e^{-kx}}{k}\Big|_0^\infty + \frac{2}{k} \int_0^\infty \overbrace{x e^{-kx} dx}^{\mu} = \left(\frac{2}{k} \cdot \frac{1}{k}\right)$$

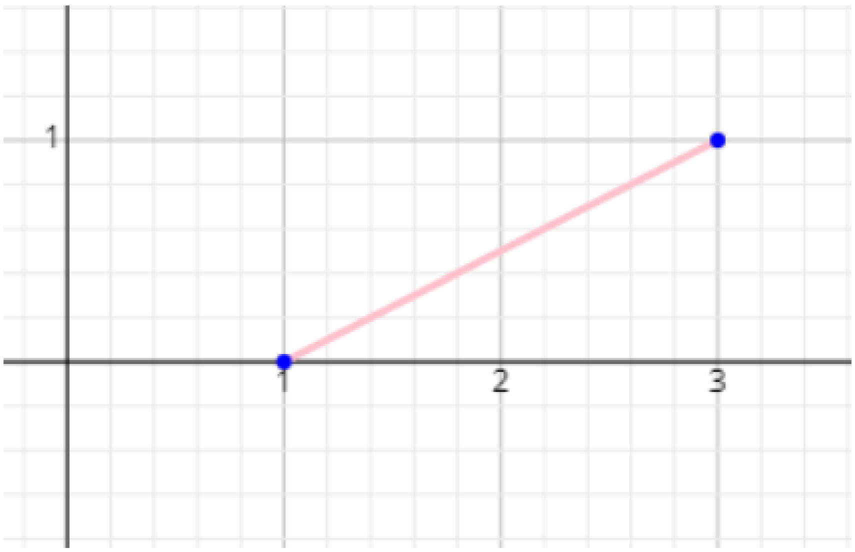
$$= \frac{2}{k} \times \frac{1}{k} - \mu^2 = \frac{2}{k^2} - \mu^2$$

$$= 2\mu^2 - \mu^2$$

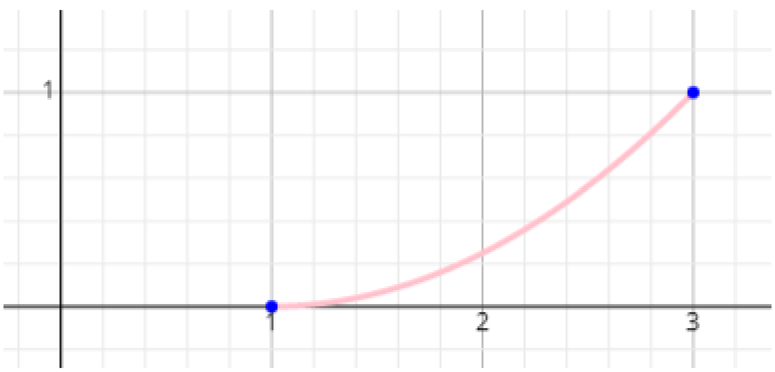
$\boxed{\text{variance} = \mu^2}$ = $E(X)^2$

Question 3

Graph of PDF



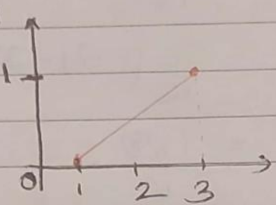
Graph of CDF



Q3 continuous random variable

$$f(x) = \begin{cases} \frac{x}{2} - \frac{1}{2} & , 1 \le x \le 3 \\ 0 & , \text{elsewhere} \end{cases}$$

(a) pdf graph



(b) cdf $F(x) = F(x-1) + f(x)$

$\text{cdf} = \int_1^x \text{pdf}$

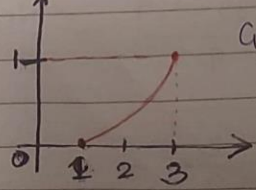
$$F(x) = \int_1^x f(x)\,dx = \int_1^x \frac{x}{2} - \frac{1}{2}\,dx$$

$$= \left(\frac{x^2}{4} - \frac{x}{2}\right)_1^x$$

$$= \frac{x^2 - 1}{4} - \frac{x-1}{2}$$

$$= \frac{x^2}{4} - \frac{x}{2} - \frac{1}{4} + \frac{1}{2}$$

$$\boxed{\text{cdf } F(x) = \frac{x^2}{4} - \frac{x}{2} + \frac{1}{4}}$$



Graph of cdf

(c) mean $= \int_1^3 x f(x)\,dx$

$$= \int_1^3 x\left(\frac{x}{2} - \frac{1}{2}\right) dx$$

$$= \frac{1}{2} \int_1^3 x^2 - x \; dx$$

$$= \frac{1}{2}\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Bigg|_1^3 = \frac{1}{2} \times \frac{1}{6}\left(2x^3 - 3x^2\right)_1^3$$

$$= \frac{1}{12}\Big(\left(2(3)^3 - 3(3)^2\right) - \left(2(1) - 3(1)\right)\Big)$$

$$= \frac{1}{12}(54 - 27 + 1)$$

$$= \frac{28}{12}$$

$$\boxed{\text{mean} = \frac{7}{3}} = \mu$$

variance $= \int_1^3 x^2 f(x) - \mu^2$

$$= \int_1^3 x^2\left(\frac{x}{2} - \frac{1}{2}\right) dx - \mu^2 = \frac{1}{2}\int_1^3 (x^3 - x^2)\,dx - \mu^2$$

$$= \frac{1}{2}\left(\frac{x^4}{4} - \frac{x^3}{3}\right)_1^3 - \mu^2$$

$$= \frac{1}{24}\Big(\left(3(3)^4 - 4(3)^3\right) - \left(3(1) - \cancel{4}(1)\right)\Big) - \mu^2$$

$$= \frac{17}{3} - \left(\frac{7}{3}\right)^2$$

$$= \cancel{\frac{17 \times 49}{}} \; \frac{17(3) - 49}{9} = \frac{51 - 49}{9}$$

$$\boxed{\text{variance} = \sigma^2 = \frac{2}{9}}$$

(d) Inverse Transformation Method

$$u = F(x)$$
$$u = \frac{x^2}{4} - \frac{x}{2} + \frac{1}{4}$$
$$4u = x^2 - 2x + 1$$
$$4u = (x-1)^2$$

$$\pm\sqrt{4u} = x - 1$$
$$x = 1 \pm \sqrt{4u}$$

for $(-\sqrt{4u})$, x values do not stay within our domain, which is $1 \leq x \leq 3$

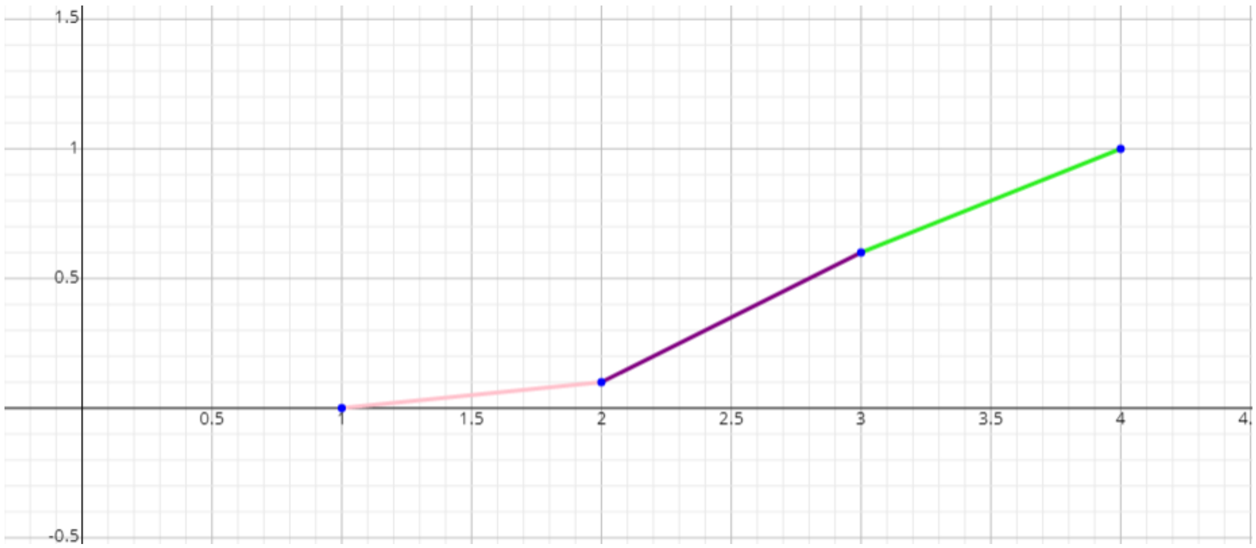
$\therefore$ Random variate Generator is

$$\boxed{x = 1 + 2\sqrt{u}}$$

$$\boxed{F^{-1}(u) = 1 + 2\sqrt{u} \quad 0 \leq u \leq 1}$$

Question 4

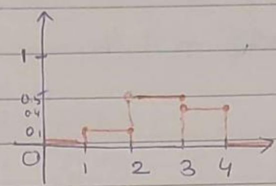
Graph of CDF



Q4. continuous random variable X

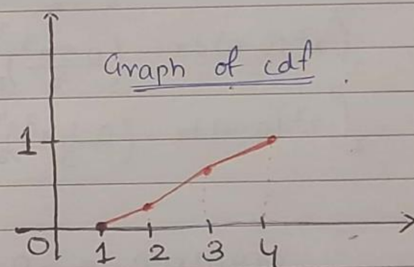
$$f(x) = \begin{cases} 0.1 & 1 \le x \le 2 \\ 0.5 & 2 < x \le 3 \\ 0.4 & 3 < x \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

(a) pdf graph



(b) cdf $\int_0^x \text{pdf}\, dx$ cdf $= \int_0^n \text{pdf} = 1$

$F(x_1) = F(x_0) + f(x_1)$



$1 \le x \le 2$ $F(x) = 0 + \int_1^x 0.1\, dx$

$= 0.1x\big|_1^x$

$$\boxed{F(x) = 0.1x - 0.1}$$

$2 \le x \le 3$ $F(x) = 0.1 + \int_2^x 0.5\, dx$

$= 0.1 + 0.5x - 0.5(2)$

$= 0.5x - 1.0 + 0.1$

$$\boxed{F(x) = 0.5x - 0.9}$$

$3 < x \le 4$ $F(x) = 0.1 + 0.5 + \int_3^x 0.4 \cdot dx$

$= 0.6 + 0.4x - 0.4(3)$

$= 0.4x - 1.2 + 0.6$

$$\boxed{F(x) = 0.4x - 0.6}$$

(c) mean $= \int x f(x)\,dx$

$$\mu = \int_1^2 x(0.1)\,dx + \int_2^3 x(0.5)\,dx + \int_3^4 (0.4)x\,dx$$

$$= \left.\frac{0.1x^2}{2}\right|_1^2 + \left.\frac{0.5x^2}{2}\right|_2^3 + \left.\frac{0.4x^2}{2}\right|_3^4$$

$$= \frac{0.1}{2}(4-1) + \frac{0.5}{2}(9-4) + \frac{0.4}{2}(16-9)$$

$$= \frac{0.3}{2} + \frac{2.5}{2} + \frac{2.8}{2}$$

$$= 0.15 + 1.25 + 1.4$$

$$\boxed{\text{mean } \mu = 2.8}$$

variance $= \int x^2 f(x)\,dx - \mu^2$

$$\int_1^2 x^2(0.1)\,dx = \left(\frac{0.1x^3}{3}\right)_1^2 = \frac{0.1}{3}(8-1) = \frac{0.7}{3} = 0.233$$

$$\int_2^3 x^2(0.5)\,dx = \left(\frac{0.5x^3}{3}\right)_2^3 = \frac{0.5}{3}(27-8) = \frac{9.5}{3} = 3.167$$

$$\int_3^4 x^2(0.4)\,dx = \left(\frac{0.4x^3}{3}\right)_3^4 = \frac{0.4}{3}(64-27) = \frac{14.8}{3} = 4.933$$

$$\text{variance} = 0.233 + 3.167 + 4.933 - (2.8)^2$$

$$= 8.333 - 7.84$$

$$\boxed{\text{variance} = \sigma^2 = 0.493}$$

(d) Inverse Transformation Method

$1 \leq x \leq 2$

$$u = F(x)$$
$$u = 0.1x - 0.1$$
$$u + 0.1 = x(0.1)$$
$$\boxed{x = \frac{u+0.1}{0.1}}$$

$2 < x \leq 3$

$$u = F(x)$$
$$u = 0.5x - 0.9$$
$$0.5x = u + 0.9$$
$$\boxed{x = \frac{u+0.9}{0.5}}$$

$3 < x \leq 4$

$$u = F(x)$$
$$u = 0.4x - 0.6$$
$$0.4x = u + 0.6$$
$$\boxed{x = \frac{u+0.6}{0.4}}$$

∴ Random variate generator, $0 \leq u_{*} < 1$

$$\boxed{F^{-1}(u) = \begin{cases} \dfrac{u+0.1}{0.1} & u \leq 0.1 \\ \dfrac{u+0.9}{0.5} & 0.1 < u \leq 0.6 \\ \dfrac{u+0.6}{0.4} & u > 0.6 \end{cases}}$$

Question 5

Q5. 1 cake, 1 baker

mean service time 1 cake = 15 minutes = $\bar{s}$

mean arrival rate (λ) = ?

mean interarrival rate ($\bar{r}$) = $\frac{1}{\lambda}$ = ?

mean service rate $\mu = \frac{1}{\bar{s}} = \frac{1}{15}$

max mean length ≤ 5

mean no. of customers ≤ 5

$$N_c \leq 5$$

$$\frac{\rho}{1-\rho} \leq 5 \quad , \quad \rho \rightarrow \text{server utilization}$$

Traffic intensity / server utilization

$$\rho = \frac{\lambda}{\mu}$$

$$\frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}} \leq 5$$

$$\frac{\lambda}{\mu-\lambda} \leq 5$$

$$\lambda \leq 5(\mu-\lambda)$$

$$\lambda \leq 5\mu - 5\lambda$$

$$6\lambda \leq 5\left(\frac{1}{15_3}\right)$$

$$\boxed{\lambda \leq \frac{1}{18}} \Rightarrow \boxed{\bar{r} = \frac{1}{\lambda} = 18}$$

$\therefore$ mean time between arrivals (exponentially distributed) accepted is if it 18 minutes between each customer

$$\underset{\downarrow \atop \bar{r}}{\text{18 minutes}}$$

Question 6

Q6. Discrete time finite-state birth-death process

A , B

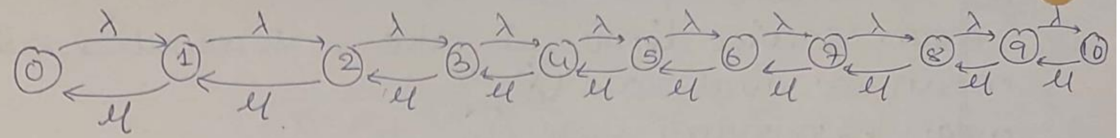
money $= K$, $A = a$, $B = K - a$

$P(A) = \lambda$, $P(B) = 1-\lambda$

$K = 10$

$\lambda = 0.51$

(a)



(b)

$$\lambda (P0) = \mu P(1)$$

$$(\lambda+\mu)P(1) = \lambda P(0) + \mu P(2)$$

$$(\lambda+\mu)P(2) = \lambda (P1) + \mu P(3)$$

$$(\lambda+\mu)P(n) = \lambda P(n-1) + \mu P(n+1) \longrightarrow P(n) = \text{state of probability of being in state } n.$$

(c) $P(n) = ?$, $n = 0, 1, \ldots, 10$

$\lambda = 0.51$, $\mu = 1$.

$$\rho = \frac{\lambda}{\mu} = 0.51$$

$$P(n) = \rho^n (1-\rho) = (0.51)^n (0.49)$$

$P(0) = \underline{0.49}$

$P(1) = 0.51 * 0.49 = \underline{0.2499}$

$P(2) = (0.51)^2 * 0.49 = \underline{0.127}$

$P(3) = (0.51)^3 * 0.49 = \underline{0.064}$

$P(4) = (0.51)^4 * 0.49 = \underline{0.033}$

$P(5) = (0.51)^5 * 0.49 = \underline{0.01690}$

$P(6) = (0.51)^6 * 0.49 = \underline{0.00862}$

$P(7) = (0.51)^7 * 0.49 = \underline{0.00439}$

$P(8) = (0.51)^8 * 0.49 = \underline{0.002242}$

$P(9) = (0.51)^9 * 0.49 = \underline{0.0011437}$

$P(10) = (0.51)^{10} * 0.49 = \underline{0.0005833}$

(d) The probability of being in a higher state decreases by a lot. ~~Thou~~ lower probability.