Summary on
Discrete Random Variables

1. Bernoulli (p)

   X={0, 1}, 1-> success, 0-> failure

   Pr(X=1) = p, Pr(X=0)=1-p

2. Discrete uniform (*a*, *b*)
3. Geometric (*p*) $\rightarrow$ P. = Pr[Y ≠ 0]
4. Binomial (*n*, *p*)

   f(x): probability of *x* successes in *n* independent Bernoulli trials.

n p H T H T T

Summary on Discrete Random Variables- cont.

5. Poisson (μ) – limiting case of binomial when

$$n \to \infty,\ p \to 0 \quad \text{and} \quad np \to \lambda$$

Pdf

$$f(n) = \frac{\lambda^n e^{-\lambda}}{n!}, n = 0, 1, 2, \ldots$$

Model arrival process with a large population. If interarrival time has exponential distribution, arrival time follow Poisson distribution.

Summary on Discrete Random Variables- cont.

6. Empirical Distribution

– based on observed data

- use the Inverse-Transform method to generate the random variate

$u \in (0,1)$

$u < 0.2$

$u < 0.7$

u

x	$P(x)$
1	0.2
2	0.5
3	0.3

Random Variae Generation

Special methods for some
common distributions

Bernoulli Trial

Parameter: $Pr(X=1) = p$

(a) Generate u

(b) If $u<p$ then variate =1

else variate =0

Binomial

(a) Perform algorithm for Bernoulli trial n times

(b) Variate = number of successes

Geometric

(a) Perform a series of Bernoulli trials until the first failure is encountered

(b) Variate = number of successes before the first failure

Discrete Uniform

(a) Generate u

(b) Variate = a + Integer (u*(b-a+1))

Empirical Distribution

- Based on observed data
- **Inverse Transformation Method**

- Example:

  $P(1) = 0.1$, $P(2)=0.5$

  $P(3) = 0.2$, $P(4)=0.15$, $P(5)=0.05$

Inverse Transformation Method

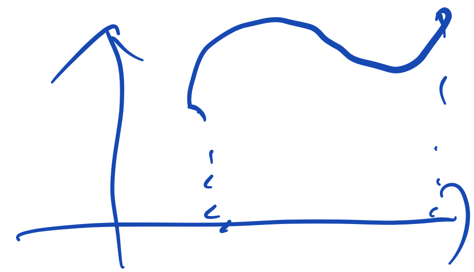
- Suppose u is a random number uniformly distributed between 0 and 1, the following table can be used to select the variate:

u	**SelectedVariate**
0–0.1	1
0.1–0.6	2
0.6–0.8	3
0.8–0.95	4
0.95–1.0	5

Σ Σ

Algorithm

$\sum_{i=1}^{n} P(x_i)$



(a) Generate u

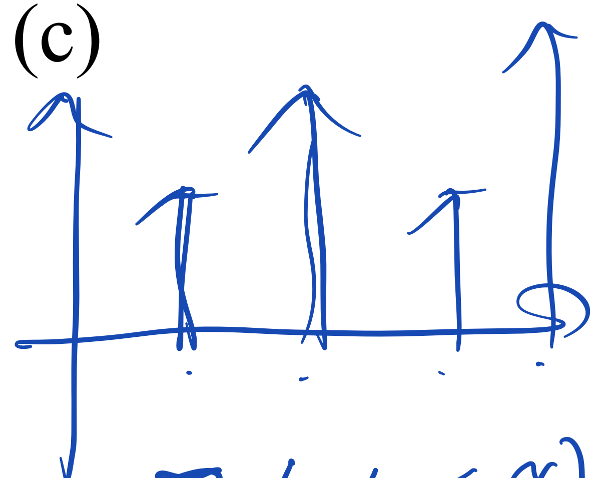
(b) i=1

(c) u=u-p(i)

(d) If u<0 then random variate is given by i

Else increment i and go to (c)

pdf



cdf

$F(x) = Pr(X \le x)$

$= \sum P(x)$

Continuous Random Variables

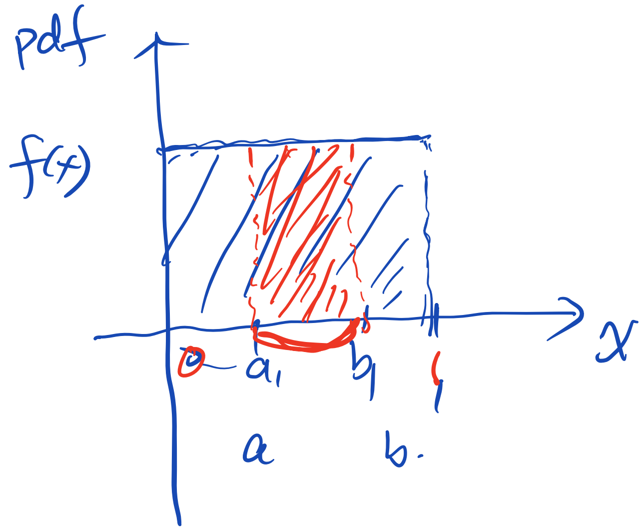
- A random variable X is continuous if its values are over a continuous interval.
- Properties of pdf:

$$(a) f(x) >= 0$$

$$(b) \Pr(a \le x \le b) = \int_a^b f(x)\,dx$$

$$(c) \text{ By definition: } \int_a^\infty f(x)\,dx = 1$$

$$\int_0^1 f(x)\,dx = 1$$



pdf :

$$f(x) \geqslant 0$$

$$\int_a^b f(x)\,dx = 1$$

$$\Pr(a_1 \leq X \leq b_1) = \int_{a_1}^{b_1} f(x)\,dx$$

pdf : $f(x)$

cdf : $F(x) = \Pr(X \leq x) = \int_a^x f(t)\,dt$

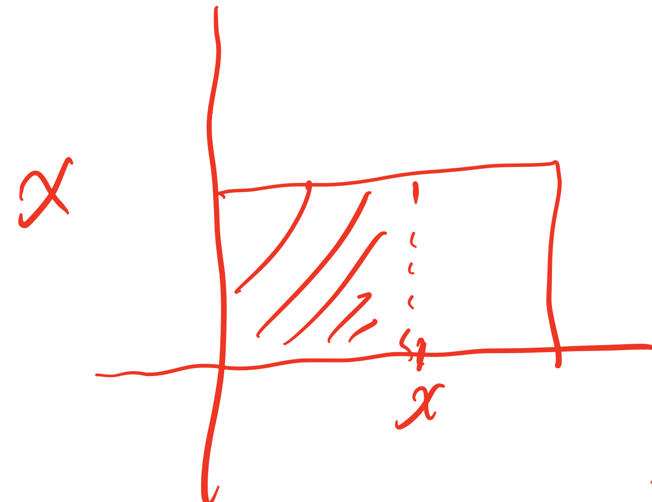
pdf

$$f(x) = F'(x) = \frac{dF}{dx}$$

$$\left(\int_0^x f(t)\,dt\right)'_x = f(x)$$

Continuous Random Variables – cont.

- Properties of cdf:



$F(x)=\Pr(X\le x)$

$$F(x)=\Pr(X\le x)=\int_0^x f(t)\,dt$$

pdf and cdf: $f(x)=F'(x)=\dfrac{dF}{dx}$

Mean: $\mu=\displaystyle\int_0^\infty xf(x)\,dx$

Standard deviation: $\sigma=\sqrt{\displaystyle\int_0^\infty x^2 f(x)\,dx-\mu^2}=\sqrt{\displaystyle\int_0^\infty (x-\mu)^2 f(x)\,dx}$

Variance: σ^2

$$\sigma^2=\int_0^\infty (x-\mu)^2 f(x)\,dx$$

$$\mu = \sum f(x_i)\, x_i$$

$$\mu = \int_0^\infty f(x)\, dx$$

$$\sigma^2 = \sum (x-\mu)^2 f(x)$$ discrete

$$= \sum x^2 f(x) - \mu^2$$

$$\sigma^2 = \int_0^\infty (x-\mu)^2 f(x)\, dx$$ continuous

$$= \int_0^\infty x^2 f(x)\, dx - \mu^2$$

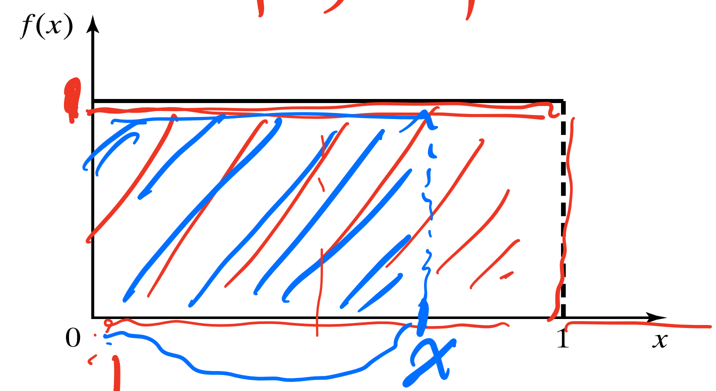
Continuous Uniform (0, 1)

$\Pr(X \le \frac{1}{2}) = \frac{1}{2}$

$\Pr(X \le \frac{3}{4}) = 75\%$

pdf

$f(x) = 1$



$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$\int_0 f(x)\,dx = 1$

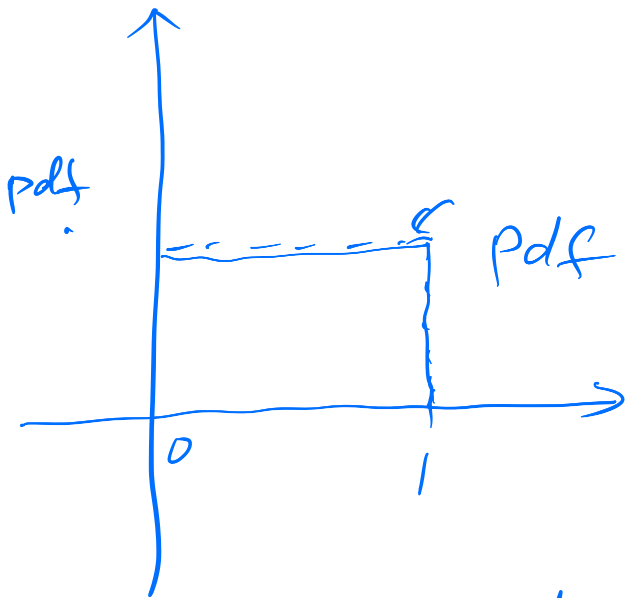
$$E(R) = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

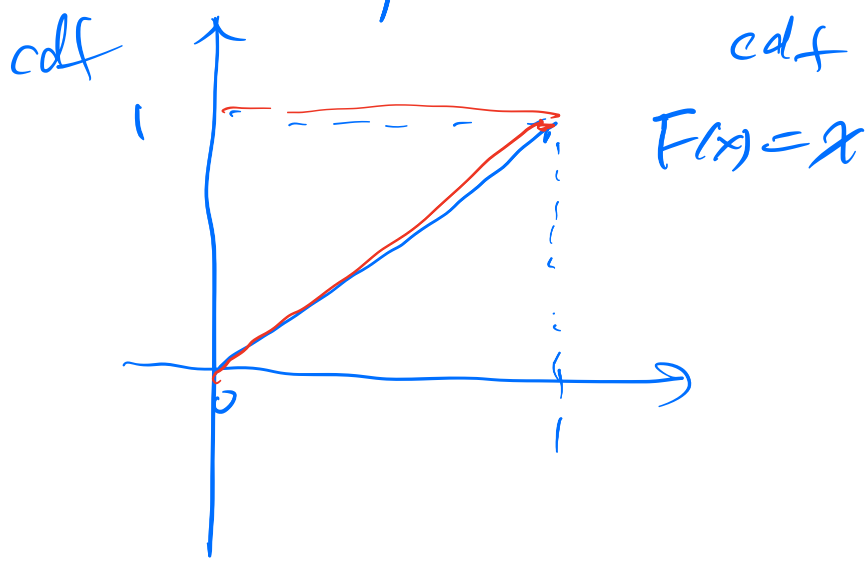
$\int_0^x f(t)\,dt$

cdf: $F(x) = \Pr(X \le x)$

$= \int_0^x f(t)\,dt = \int_0^x 1\,dt$ $\quad = \int_0^x 1\,dt$

$F(x) = x$ $\quad = x$





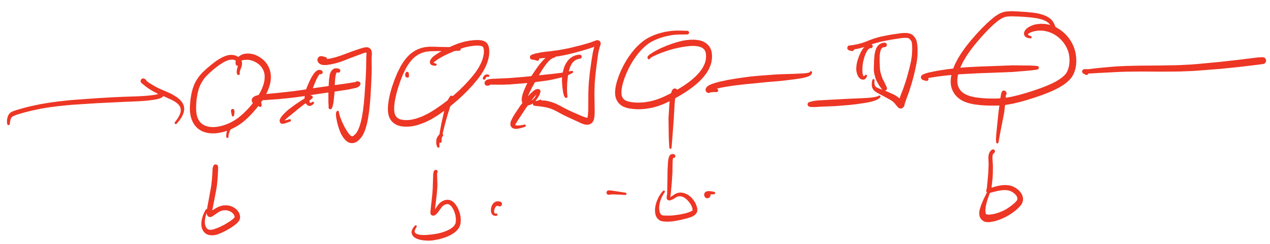
$$\mu = \int_0^1 x\,dx = \frac{1}{2}.$$

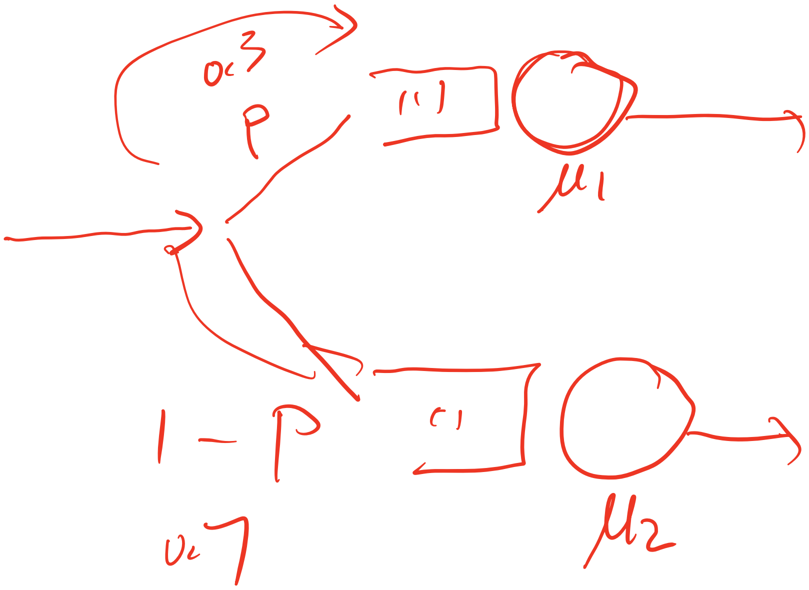
Examples of Continuous Random Variables

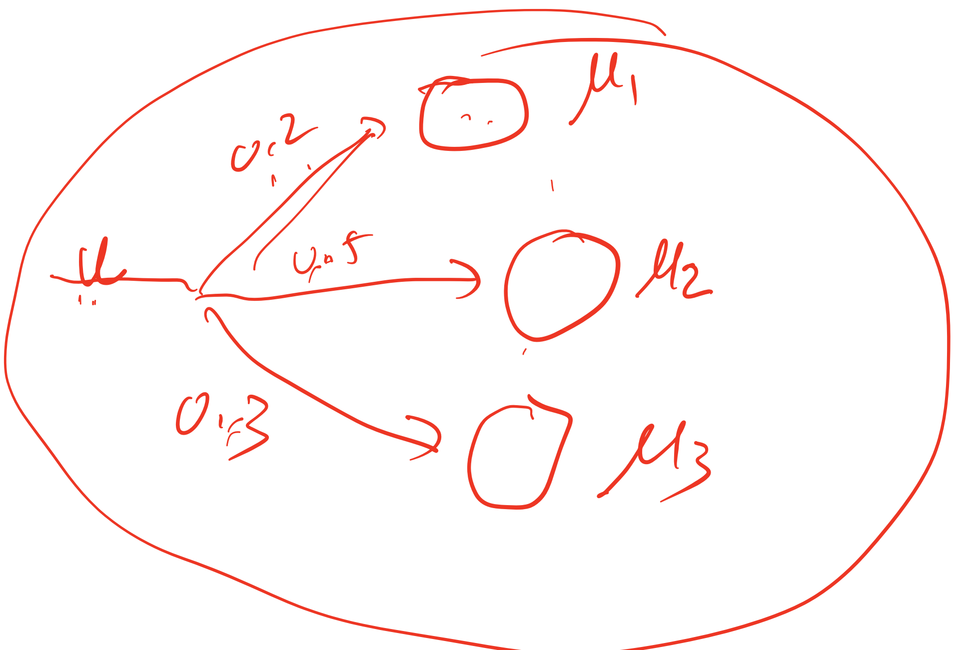
- Continuous Uniform (a, b)
- Exponential (μ) $x = -\mu \ln(1-u)$
- Erlang (b, n) – Combination of exponential random variable, sum of n exponentially distributed random variables each with parameter b:

  X=X1+X2+…Xn

  where each X1, X2, …, Xn are exponential (b)
- Hyper exponential – weighted sum of exponentials







$u \in (0, 1)$

If $u < 0.2$ — RVEXPO$(2, \mu_1)$

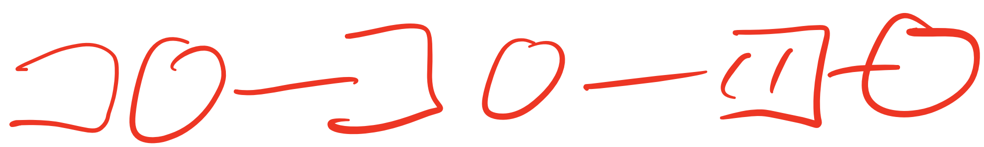
Else If $u < 0.7 \longrightarrow$ RVEXPO$(3, \mu_2)$

Else $\rightarrow$ RVEXPO(1, μ_3)

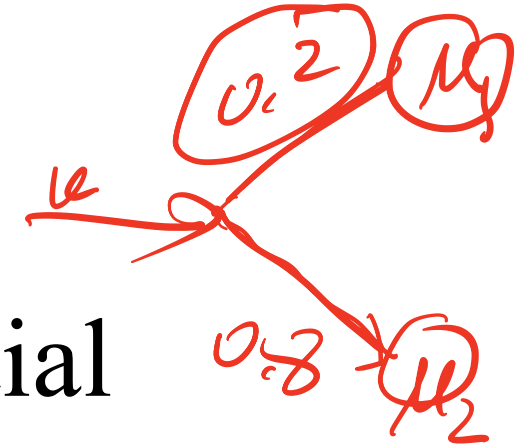
Variate generation for Erlang Distribution

(a) Generate k exponential variates, each with parameter μ

(b) Variate = sum of these k variates



Variate generation for Hyper-exponential



(a)Generate a random number for Bernoulli trial

(b)If less than p

generate an exponential variate with parameter μ_1

Else

generate an exponential variate with parameter μ_2

The Inverse-Transform Technique for Continuous Distributions

cdf

$F(x) = u$

$x = F^{-1}(u)$

- Algorithm

  Step 1: Compute the *cdf* for *F(x)*

  Step 2: Set $F(x)=u$ on the range of x

  u is uniform distribution over (0,1)

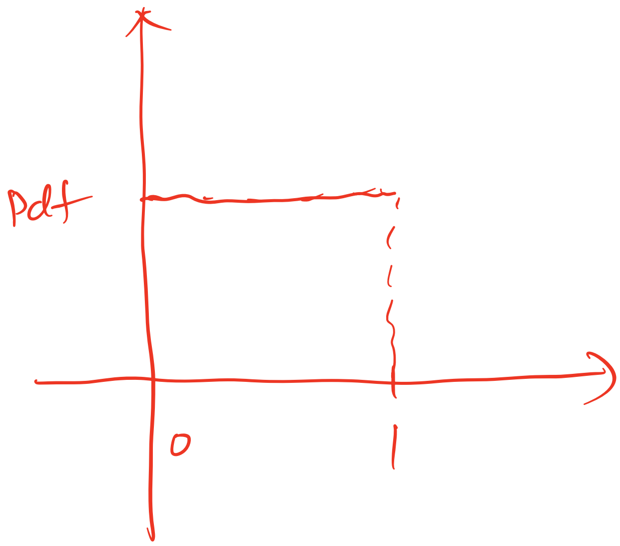
  Step 3: Solve the equation $F(x)=u$ in term of *u*,

$$x=F^{-1}(u) \tag{1}$$

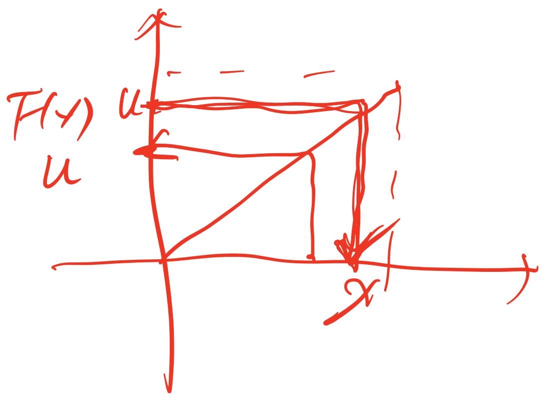
// (1) is called the random-variate generator

Step 4: Generate $u_1, u_2, \ldots,$

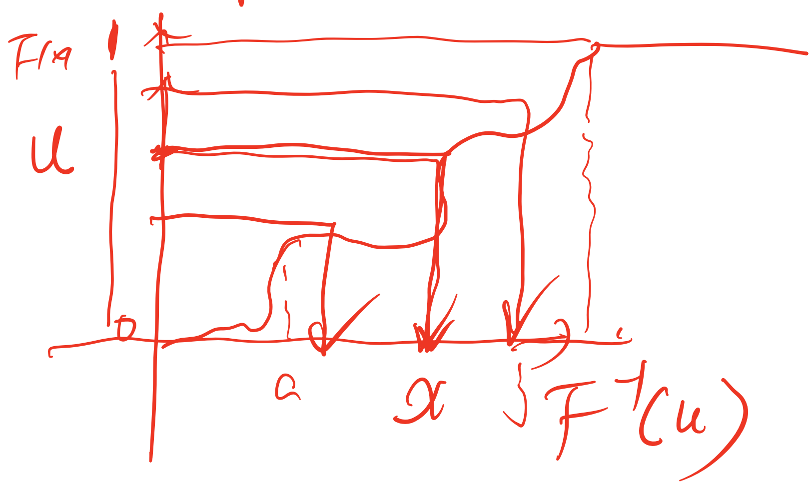
$$x_1=F^{-1}(u_1),\ x_2=F^{-1}(u_2),\ x_3=F^{-1}(u_3),\ \ldots$$



Pdf
0
1



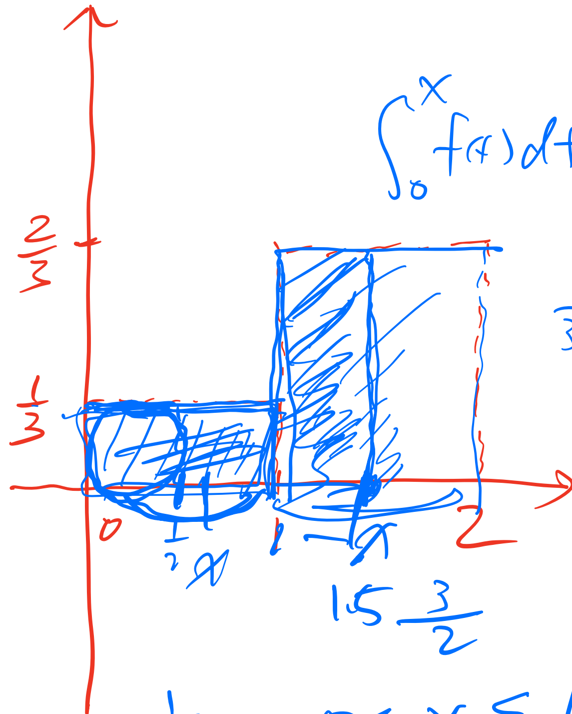
F(x)
u
u
x



F(x)
u
0
a
x
F^-1(u)

pdf

$$\int_0^x f(x)\,dt$$



$$\frac{1}{3}+\frac{2}{3}=1$$

$$Pr\left(X \le \frac{1}{2}\right)$$

$$= \int_0^{\frac{1}{2}} \frac{1}{3}\,dt$$

$$= \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$

$$f(x) = \begin{cases} \frac{1}{3} & 0 < x \le 1 \\ \frac{2}{3} & 1 < x \le 2 \end{cases}$$

cdf:

$$F(x) = Pr(X \le x) \qquad x \le 1$$

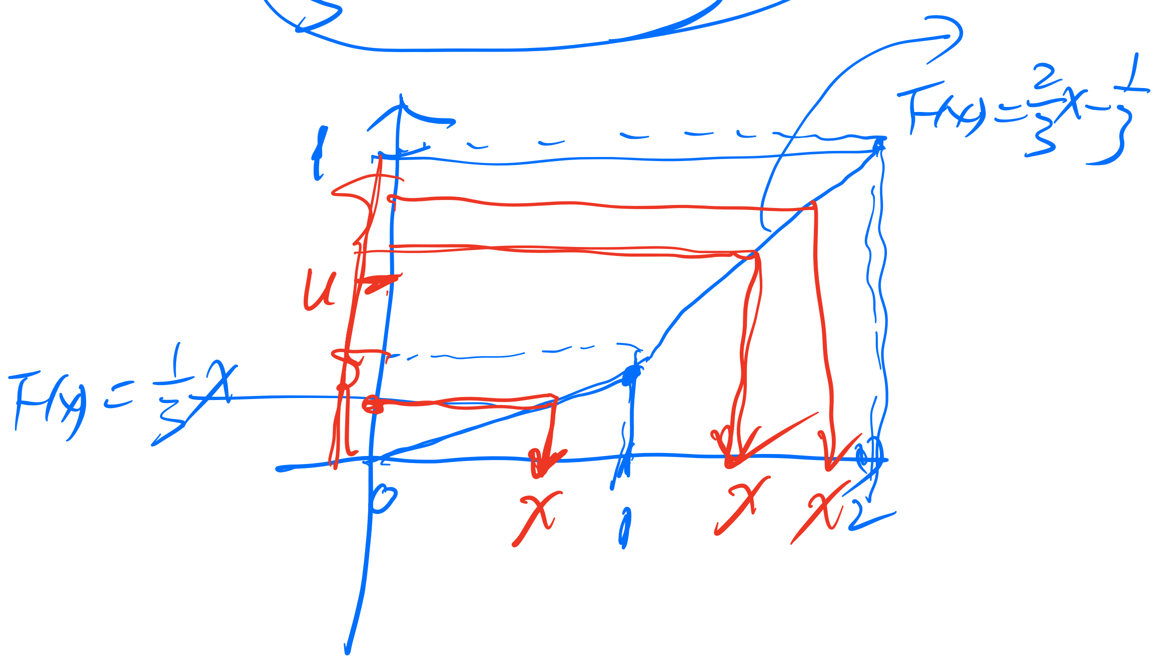
$$= \int_0^x \frac{1}{3}\,dt = \frac{1}{3}x$$

$$F(x) = Pr(X \le x) \qquad \underline{1 < x \le 2}$$

$$= \frac{1}{3} + (x-1)\frac{2}{3}$$

$$= \frac{1}{3} + \frac{2}{3}x - \frac{2}{3}$$

$$= \frac{2}{3}x - \frac{1}{3}$$



$$F(x) = \begin{cases} \frac{1}{3}x & 0 < x \leq 1 \\ \frac{2}{3}x - \frac{1}{3} & 1 < x \leq 2 \end{cases}$$

$0 < u < 1$

$\frac{1}{3}x = u \Rightarrow x = 3u$

u

$\frac{1}{3}x \quad 0 \leq x \leq 1$

$0 \leq u \leq \frac{1}{3}$