Discrete-Event Simulation

Exponential and Geometric Variates

Geometric Distribution

- A coin has p as its probability of a head. Toss it until the first tail occurs. If X is the number of heads,

$$\mathscr{X} = \{x \mid x = 0, 1, 2, ...\}$$

$p^n(1-p)$

- The pdf is

$Pr(\cancel{X}=x)$ — $f(x) = p^x(1 - p)$, $x = 0, 1, 2, ...$

- X is Geometric(p) and the set of possible values is infinite

p

$$\Sigma f(x) = 1 \text{ ?}$$

$$\sum_{x=0}^{\infty} p^x (1-p) = (1-p)(1 + p + p^2 + p^3 + ...) = 1$$

$Pr(X=1) = P$, $Pr(X=0) = 1 - P$

$H \to p, \quad T \to$

$$
\begin{array}{lcll}
 & & X & Pr \\
T & \longrightarrow & 0 & 1-p \\
HT & \longrightarrow & 1 & p(1-p) \\
HHT & \longrightarrow & 2 & p \cdot p(1-p) \\
HHHT & \longrightarrow & 3 & p^3(1-p) \\
\vdots & & & \vdots \\
\underbrace{HHHHH\cdots H}_{100}T & \longrightarrow & 100 & p^{100}(1-p)
\end{array}
$$

$$X \left\{ 0, 1, 2, 3, 4, \ldots \right\}$$

$$1-p \qquad p(1-p) \qquad p^2(1-p)$$

$$(1-p) + p(1-p) + p^2(1-p) + p^3(1-p) + \cdots$$

$$= \sum_{x=0}^{\infty} p^x (1-p) = (1-p) \sum_{x=0}^{\infty} p^x$$

$$= (1-p) \cdot \left[1 + p + p^2 + p^3 + p^4 + \cdots p^n + p^{n+1} + \cdots \right]$$

$$= (1-p) \frac{1 - p^n}{1-p} \qquad p^n \to 0 \qquad p<1$$

$$\longrightarrow \cancel{(1-p)} \frac{1}{\cancel{1-p}} = 1$$

Cumulative Distribution Function

- The cumulative distribution function (cdf) of the discrete random variable X is the real-valued function F(·) for each $x \in \mathcal{X}$:

$$\boxed{F(x) = \Pr(X \le x)} = \sum_{t \le x} f(t)$$

- If X is Equilikely(1, 6),

x	1	2	3	4	5	6
$f(x) = \Pr(X=x)$	1/6	1/6	1/6	1/6	1/6	1/6
$F(x)=\Pr(X\le x)$	1/6	2/6	3/6	4/6	5/6	1

$F(2) = \Pr(X \le 2) = \Pr(X \le 1) + \Pr(X = 2)$

Examples - cdf

$= F(1) + f(2)$

a x b

$(1, \ldots, 15)$

- If X is Equilikely(a, b) then the cdf is

$\frac{1}{15} + \frac{1}{15} + \frac{1}{\ldots} \ldots$

$$F(x) = \sum_{t=a}^{x} \frac{1}{b-a+1} = \frac{x-a+1}{b-a+1}, \qquad x = a, a+1, \ldots, b$$

$b-a+1$

- If X is Geometric(p) then the cdf is

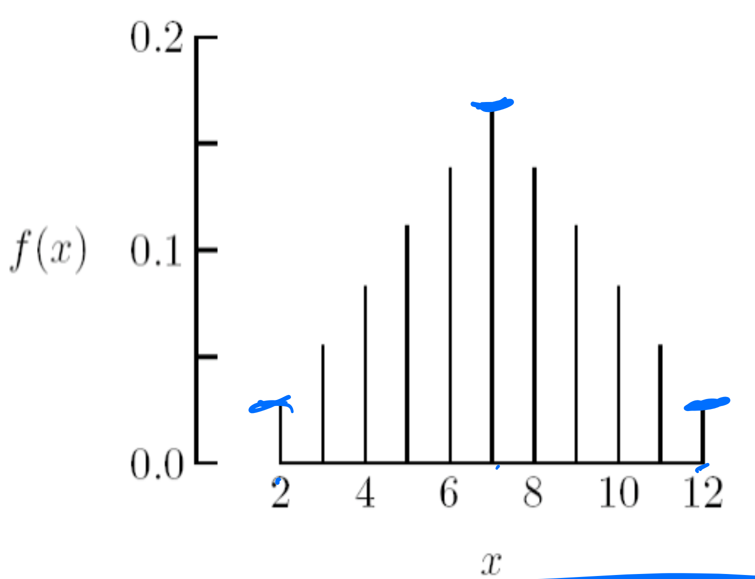
$$F(x) = \sum_{t=0}^{x} p^t (1-p) = (1-p)(1+p+\ldots+p^x) = 1-p^{x+1},$$

$$x = 0, 1, 2, \ldots$$

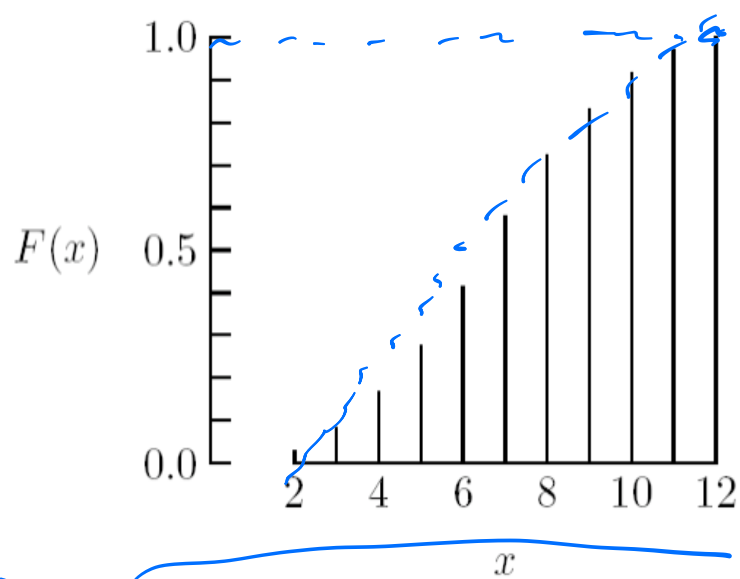
$$= \cancel{1-p} \cdot \frac{1-p^{x+1}}{\cancel{1-p}} = 1-p^{x+1}$$

pdf and cdf – sum of two dice

- No simple equation for F(·) for sum of two dice.



pdf



cdf

Relationship Between cdfs and pdfs

- A cdf can be generated from its corresponding pdf by recursion.

For example, $\mathscr{X} = \{x \mid x = \text{a, a} + 1, \ldots, \text{b}\}$:

pdf $f(x)$

$$F(a) = f(a),$$
$$F(x) = F(x - 1) + f(x), \quad x = a + 1, a + 2, \ldots, b.$$

- A pdf can be generated from its corresponding cdf by subtraction.

$$f(a) = F(a)$$
$$f(x) = F(x) - F(x - 1), \quad x = a + 1, a + 2, \ldots, b.$$

- A discrete random variable can be defined by specifying either its pdf or its cdf.

Other cdf Properties

- A cdf is strictly monotone increasing:

  $$\text{if } x_1 < x_2 \Rightarrow F(x_1) < F(x_2)$$

- The cdf values are bounded between 0.0 and 1.0.
- Monotonicity of $F(\cdot)$ is the basis to generate discrete random variates.

$0 < u < 1$ $\qquad F(x)$

cdf

Mean and Variance

- The mean is the measure of the central tendency of a random variable where the variance is the spread of possible values around the mean.
- The mean of a random variable is also known as the expected value:

$$E[X] = \sum_{x} x f(x) = \mu$$

$\mu = \Sigma x f(x)$

Mean and Standard Deviation

- The mean μ of the discrete random variable X is

$$\mu = \sum_x x f(x)$$

- The corresponding standard deviation σ is

$$\sigma = \sqrt{\sum_x (x-\mu)^2 f(x)} \quad \text{or} \quad \sigma = \sqrt{\left(\sum_x x^2 f(x)\right) - \mu^2}$$

- The variance is σ^2.

$$\sigma^2 = \sum_x (x - \underset{\text{Mean.}}{\mu})^2 f(x)$$

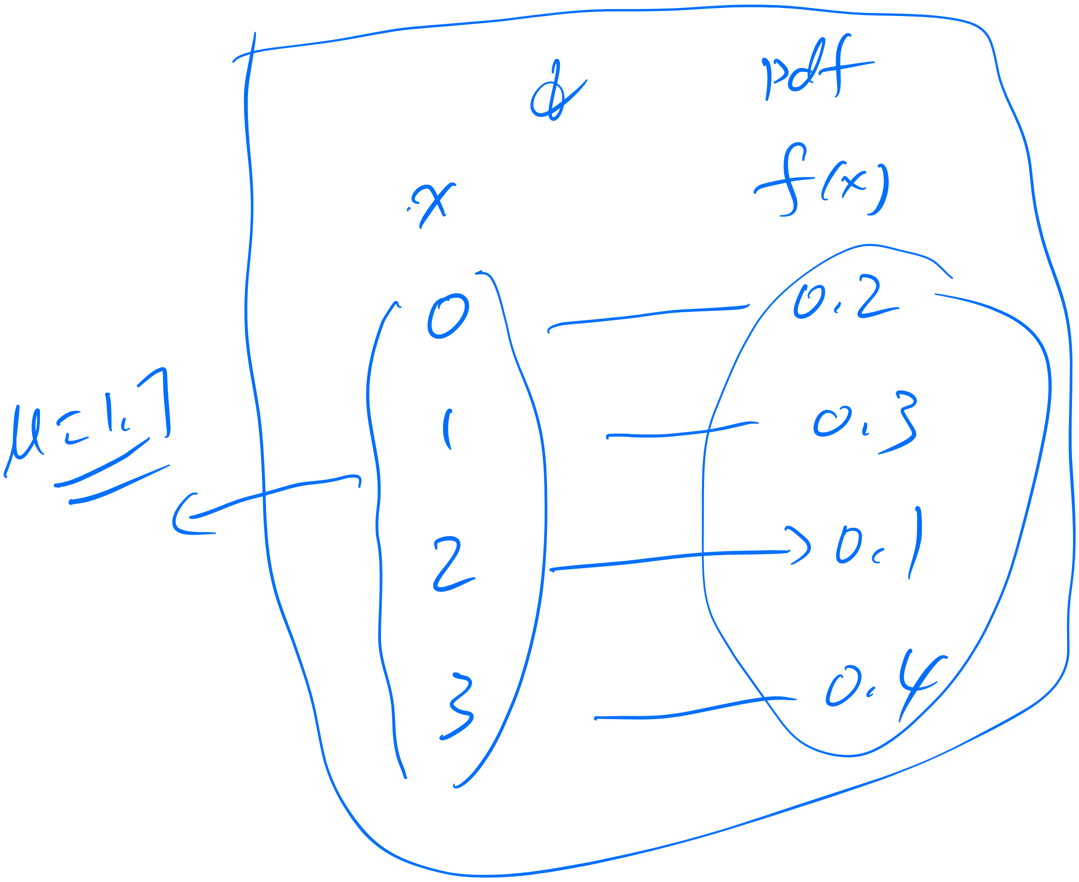
Example

- If X is Geometric (p) random variable, what are the mean, variance and standard deviation?

p

$p(1-p)$

$p^2(1-p)$



Mean: $\mu = \cancel{0 \times 0.2} + 1 \times 0.3 + 2 \times 0.1 + 3 \times 0.4$

$= 0.3 + 0.2 + 1.2$

$= 0.5 + 1.2 = 1.7$

$\sigma^2 = (0-1.7)^2 \cdot 0.2 + (1-1.7)^2 \cdot 0.3 + (2-1.7)^2 \cdot 0.1 + (3-1.7)^2 \cdot 0.4$

$\sigma =$