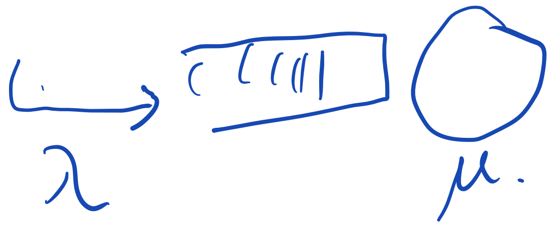
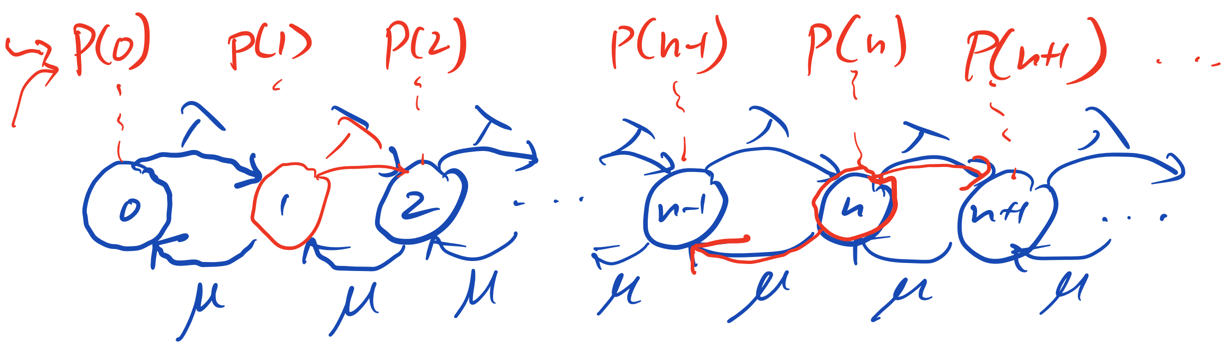
M/M/1 Queue. Birth-Death Model



State : Number of customers in System.



out going rate : $\lambda p(n) + \mu p(n) = (\lambda+\mu) p(n)$

$$\begin{cases} \lambda(n) = \lambda, & n = 0, 1, 2, 3, \ldots \\ \mu(n) = \mu, & n = 1, 2, 3, \ldots \end{cases}$$

Balance Equations

$$\lambda P(0) = \mu P(1) \qquad (0)$$

$$(\lambda+\mu) P(1) = \lambda P(0) + \mu P(2) \qquad (1)$$

$$\rightarrow (\lambda+\mu)p(2) = \lambda p(1) + \mu p(3) \quad (2)$$

$$\cdots$$

$$(\lambda+\mu)p(n) = \lambda p(n-1) + \mu p(n+1) \quad (n)$$

$$\cdots$$

$\rho = \frac{\lambda}{\mu}$

$\lambda < \mu$

$$(0) \rightarrow \quad p(1) = \frac{\lambda}{\mu} p(0) = \rho p(0)$$

$$(1) \rightarrow \quad \lambda p(1) + \cancel{\mu p(1)} = \cancel{\lambda p(0)} + \mu p(2)$$

$$(1') \cdots\cdots \quad \lambda p(1) = \mu p(2)$$

$$p(2) = \frac{\lambda}{\mu} p(1) = \rho p(1) = \rho^2 p(0)$$

$$(2) \rightarrow \quad \lambda p(2) + \cancel{\mu p(2)} = \cancel{\lambda p(1)} + \mu p(3)$$

$$(2') \quad \lambda p(2) = \mu p(3)$$

$$p(3) = \frac{\lambda}{\mu} p(2) = \rho \rho^2 p(0) = \rho^3 p(0)$$

$$\cdots$$

$$(n) \quad p(n) = \rho^n p(0)$$

$$\cdots$$

Since $p(0) + p(1) + p(2) + \cdots + p(n-1) + p(n) + p(n+1) + \cdots$

$$= \sum_{n=0}^{\infty} p(n) = 1$$

$$P(0) + \rho P(0) + \rho^2 P(0) + \cdots + \rho^n P(0) + \cdots = 1$$

$$P(0)\underbrace{\left[1 + \rho + \rho^2 + \cdots + \rho^n + \cdots\right]} = 1$$

$$P(0) = \frac{1}{\underbrace{1+\rho+\rho^2+\cdots+\rho^n+\cdots}} = \frac{1}{\frac{1}{1-\rho}}$$

$$= 1-\rho$$

$$\begin{cases} P(0) = 1-\rho. \\ P(1) = \rho P(0) = \rho(1-\rho) \\ P(2) = \rho^2 P(0) = \rho^2(1-\rho) \\ P(3) = \rho^3 P(0) = \rho^3(1-\rho) \\ \quad \cdots \\ P(n) = \rho^n(1-\rho) \\ \quad \cdots \end{cases}$$

$$\sum_{n=0}^{\infty} P(n) = 1$$

pdf : $\underline{p(n) = \rho^n (1-\rho)}$

where $\rho = \frac{\lambda}{\mu} < 1$.

ρ is Server utilization.

Mean number of customers in System

$$\bar{N} = \sum_{n=0}^{\infty} n p(n) = \sum_{n=1}^{\infty} n p(n) = \sum_{n=1}^{\infty} n \rho^n (1-\rho)$$

$$= (1-\rho) \sum_{n=1}^{\infty} n \rho^n$$

$$\sum_{i=0}^{\infty} i x^i = \frac{x}{(1-x)^2} \qquad (|x| < 1)$$

$$\bar{N} = (1-\rho) \frac{\rho}{(1-\rho)^2} = \frac{\rho}{1-\rho}$$

$$p(n) = \rho^n (1-\rho)$$

$$\bar{N} = \frac{\rho}{1-\rho}, \quad \bar{T} = \frac{\bar{N}}{\lambda} = \frac{\frac{\rho}{1-\rho}}{\lambda}$$

$$\bar{T} = \frac{\rho}{\lambda(1-\rho)} = \frac{\frac{\lambda}{\mu}}{\lambda(1-\rho)}$$

$$= \frac{1}{\mu(1-\rho)}$$

M/M/1

$$\boxed{\begin{array}{l} \bar{N} = \frac{\rho}{1-\rho}, \quad \bar{T} = \frac{1}{\mu(1-\rho)} \\ P(n) = \rho^n(1-\rho) \end{array}}$$

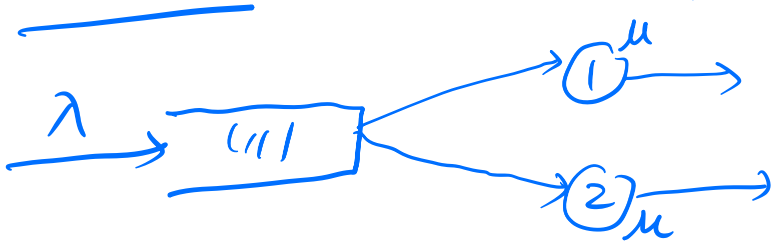
Assume $\underline{\lambda = 5, \quad \mu = 10}$

$$\rho = \frac{1}{2}$$

$$P(0) = \frac{1}{2}, \quad P(1) = \frac{1}{4}, \quad P(2) = \frac{1}{8} \cdots$$

$$\bar{N} = 1, \quad \bar{T} = \frac{1}{5}.$$

<u>M/M/2</u> Birth-Death process



$\lambda(n) = \lambda, \quad n = 0, 1, 2, \dots$

$\mu(n) = \min(2, n)\mu, \quad n = 1, 2, 3, \dots$

Balance Equations:

$$\mu p(1) = \lambda p(0)$$

$$(\lambda + \mu) p(1) = \lambda p(0) + 2\mu p(2)$$

$$(\lambda + 2\mu) p(2) = \lambda p(1) + 2\mu p(3)$$

$$\cdots$$

$$(\lambda + 2\mu) p(n) = \lambda p(n-1) + 2\mu p(n+1)$$

$$\cdots$$

$$\boxed{p(0) = \frac{2-\rho}{2+\rho}, \qquad \bar{N} = \frac{4\rho}{4-\rho^2}}$$

$$\bar{T} = \frac{\bar{N}}{\lambda} = \frac{4}{\mu(4-\rho^2)}$$

$\bar{N}$, $P(n)$, $\bar{T}$