Chapter 1

Introduction

1.1 Basics

Definition 1.1.1 Suppose S is a semigroup, we have the following brief definitions:

(1) S is a null semigroup if $\forall x, y \in S(xy = 0)$;

(2) S is a left zero semigroup if $\forall x, y \in S(xy = x)$, dually one can define a right zero semigroup;

(3) $I \subset S$ is a proper ideal if $\{0\} \subset I \subsetneq S$ and $IS \subset S \wedge SI \subset S$;

(4) given a set X, the full transformation semigroup is defined as $(\mathrm{End}_{\mathsf{Set}}(X), \circ)$, where $\circ$ refers the composition of functions;

(5) a morphism $S \xrightarrow[\mathsf{Smg}]{\phi} \mathrm{End}(X)$ is a *representation* of S, and φ is faithful if it is injective;

(6) S is a rectangular band if $\forall a, b \in S(aba = a)$;

(7) $\langle a \rangle := \langle \{a\} \rangle_{\mathrm{smg}}$ is called a *monogenic semigroup.*

Proposition 1.1.2 Let S be a semigroup, the statements listed below are equivalent.

(1) S is a group;

(2) for all $a, b \in S$, there exists $x, y \in S$ such that $ax = b \wedge ya = b$;

(3) $\forall a \in S(aS = Sa = S)$.

Proof. It is easy to verify that (1) $\Rightarrow$ (3) and (2) $\Leftrightarrow$ (3). So we proceed to prove (3) $\Rightarrow$ (1), and it is suffices to show that S has the unique identity, and that for any element, its inverse exists and is unique. Let $ax = ya = a$, then

$$x = ax_1 = ay_1a = yay_1a = yx = ax_2ax = ax_2a = y_2a = y.$$

Thus, every element a in S has an identity ϵ_a such that $\epsilon_a a = a\epsilon_a = a$. Now, the issue lies in proving $\epsilon_a = \epsilon_b$ for any a, b in S, and the method is analogous:

$$\epsilon_a = by_1 = by_2b = by_2b\epsilon_b = \epsilon_a\epsilon_b = \epsilon_a ax_2a = ax_2a = x_1a = \epsilon_b.$$

As for the existence and uniqueness of inverse, it also follows the same manner, so we omit it here. $\square$

Theorem 1.1.3 Suppose that S is a semigroup and that $X = S^1$, then there exists a faithful representation

$$\varphi : S \to \mathrm{End}(X).$$

Proof. See [1, Theorem 1.1.2]. Simply stated,

$$\begin{aligned} S &\hookrightarrow \operatorname{End}(S^1) \\ a &\longmapsto [\varphi_a : x \mapsto xa]\,. \end{aligned}$$

□

Theorem 1.1.4 Let S be a semigroup, the following propositions are equivalent:

- ◇ S is a rectangular band;
- ◇ every $a \in S$ is an idempotent, and $abc = ac$ for all a, b, c in S;
- ◇ there exists a left zero semigroup L, and a right zero semigroup R, such that $S \simeq L \times R$;
- ◇ there exists two sets A, B such that $S \simeq A \times B$, in witch $A \times B$ is a semigroup with the multiplication defined as $(a_1, b_1)(a_2, b_2) = (a_1, b_2)$.

Proof. See [1, Theorem 1.1.3]. □

1.2 Monogenic Subsemigroup

To study the monogenic subsemigroup, we introduce the following concepts. Suppose a is an element in S, which has a finite order if not specified.

Definition 1.2.1

(1) $\langle a \rangle := \langle \{a\} \rangle_{\text{smg}}$;

(2) $\operatorname{ord}(a) := |\langle a \rangle|$;

(3) $\operatorname{idx}(a) := \min \{m \in \mathbb{Z}_{\geq 1} : \exists n \in \mathbb{Z}_{\geq 1}(a^m = a^n \wedge m \neq n)\}$;

(4) a semigroup is called *periodic* if all its elements are of finite order.

Let $m = \operatorname{idx}(a)$, $r = \operatorname{prd}$, clearly, $a, a^2, \cdots, a^{m+r-1}$ are mutually different, and $\langle a \rangle = \{a, \cdots, a^{m+r-1}\}$.

Let K_a be $\{a^m, \cdots, a^{m+r-1}\}$, we assert that it is a cyclic group. Consider the quotient ring $\mathbb{Z}/r\mathbb{Z}$, obviously, $\{[m], \cdots, [m+r-1]\} = \mathbb{Z}/r\mathbb{Z}$. Thus, there exists $0 \leq g \leq r-1$ such that $[m+g] = [1]$, which implies $\forall k([k] = [k(m+g)])$. Since $a^{(m+g)k} = a^{m+hr}a^{k-m} = a^m a^{k-m}$ for all $k > m$, the $a^{(m+g)k}$ exhaust K_a.

Proposition 1.2.2 Suppose a and b are elements of finite order in the same or different subsemigroups, then

$$\langle a \rangle \simeq \langle b \rangle \Leftrightarrow (\operatorname{idx}(a), \operatorname{prd}(a)) = (\operatorname{idx}(b), \operatorname{prd}(b)).$$

Proof. Suppose $\operatorname{idx}(a) = \operatorname{idx}(b) = m$ and $\operatorname{prd}(a) = \operatorname{ord}(b) = r$, the mapping defined below is an isomorphism.

$$\begin{aligned} \{a, \cdots, a^{m+r-1}\} &\xrightarrow{\sim} \{b, \cdots, b^{m+r-1}\} \\ a^k &\longmapsto b^k \end{aligned}$$

For the reverse, assume $\langle a \rangle \xrightarrow[\phi]{\sim} \langle b \rangle$, where ϕ maps a to b^ξ, it is straightforward to verify that $\langle b^\xi \rangle = \langle b \rangle$ and that $\operatorname{idx}(a) = \operatorname{idx}(b^\xi)$ and $\operatorname{prd}(a) = \operatorname{prd}(b^\xi)$. If $\xi = 1$, the proof is over. Otherwise, if $\xi > 1$, then there exists $\mu \geq 1$ such that $b^{\xi\mu} = b$, thus $\operatorname{idx}(b) = 1$, which implies $\langle b \rangle$ is a cyclic group. Hence, $\langle a \rangle$ is also a cyclic group with the generator $\phi^{-1}(b) = a^\zeta$. Since a is a generator, similarly, there exists an integer ν that makes $a^{\zeta\nu} = a$, and it follows that $\operatorname{idx}(a) = 1$. Thereby, $\operatorname{prd}(a) = |\langle a \rangle| = |\langle b \rangle| = \operatorname{prd}(b)$. □

Proposition 1.2.3 For any pair $(m, r) \in \mathbb{Z}^2_{\geq 1}$, there exists a semigroup S containing an element with idx of m and prd of r.

Proof. See [1, p.12]. Simply stated, the correspondence is given by $(m, r) \mapsto (12 \cdots m+1) \in S_{m+r}$. □

1.3 Relations

Given a set X, the power set $P(X^2)$ equipped with the multiplication defined as

$$R_1 \circ R_2 := \left\{(a,b) \in X^2 : \exists c \in X((a,c) \in R_1) \wedge (c,b) \in R_2\right\},$$

where R_i is the element in $P(X^2)$, forms a semigroup. To see this, it is suffices to verify $\circ$ is associative, which is obvious. Besides this, some brief definitions are listed as follows:

Definition 1.3.1

(1) $R(x) := \{y \in X : (x,y) \in R\}$, and so, R can be viewed as a mapping $x \mapsto R(x)$;

(2) $R(A) := \bigcup_{x \in A} R(x)$;

(3) $R^{\mathrm{op}} := \{(y,x) : (x,y) \in R\}$;

(4) $\Delta_X : \{(x,x) : x \in X\}$;

(5) if it is not specified, R^n represents $R \circ \cdots \circ R$ (n times);

(6) given a morphism $f : S \to S'$, then $\ker f := \{(x,y) \in S^2 : f(x) = f(y)\}$.

It can be easily verified that $(R_1 \circ R_2)^{\mathrm{op}} = R_2^{\mathrm{op}} \circ R_1^{\mathrm{op}}$, thus, $(R^n)^{\mathrm{op}} = (R^{\mathrm{op}})^n$. A commonly used conclusion is

$$(a,b) \in R^n \Leftrightarrow \exists (t_i)_{i=1}^n \in X^n \left(a = t_1 \to t_2 \to \cdots \to t_n = b\right),$$

where $t_i \to t_{i+1}$ means $t_i R t_{i+1}$.

We then introduce the definitions of partial orders and equivalent relations from this perspective.

Definition 1.3.2 A partial order is a relation satisfies the following conditions:

- ⋄ (reflective) $\Delta_X \subset R$;
- ⋄ (anti-symmetric) $R \cap R^{\mathrm{op}} = \Delta_X$;
- ⋄ (transitive) $R^2 \subset R$.

Besides, an equivalence relation satisfies:

- ⋄ (reflective) $\Delta_X \subset R$;
- ⋄ (symmetric) $R^{\mathrm{op}} \subset R$;
- ⋄ (transitive) $R^2 \subset R$.

Definition 1.3.3 Let $(S, \leq)$ be a partial-ordered set, U is a subset of S.

(1) $\min U$ is the minimal element of U if $\min U \in U$ and $\nexists a \in U(a < \min U)$;

(2) $\min^* U$ is the minimum element of U if $\min^* U \in U$ and $\forall a \in U(\min^* U \leq a)$;

(3) l is the lower bound of U if $\forall a \in U(l \leq a)$;

(4) $\inf U := \max^*\{l : \text{lower bounds of U}\}$ is the infimum of U;

(5) we say that S satisfies *minimal condition* if every nonempty subset of it has a minimal element;

(6) we say that S is a *complete lower semilattice* if $\forall U \subset X(\exists \inf U)$, and is a *lower semilattice* if $\forall\{x,y\} \subset X(\exists \inf\{x,y\})$;

(7) if S is a lower semilattice, the operation $(x,y) \mapsto \inf\{x,y\}$, as a binary function, denoted as $(\cdot) \wedge (\cdot)$, satisfies the condition of associativity; and for the upper-case, we denote $x \vee y$ by $\sup\{x,y\}$;

(8) we say that S is a *lattice* if it's both an upper semilattice and a lower semilattice.

Proposition 1.3.4 A semilattice $(S, \leq, \wedge)$ satisfies the following conditions:

- $\diamond$ $\forall x \in S(x \wedge x = x)$;
- $\diamond$ $\forall x, y \in S(x \wedge y = y \wedge x)$
- $\diamond$ $\forall x, y, z \in S((x \wedge y) \wedge z = x \wedge (y \wedge z))$;
- $\diamond$ $\forall x, y \in S(x = x \wedge y \Leftrightarrow x \leq y)$.

Thus, $(S, \wedge)$ forms a commutative semigroup, in which every element is idempotent. Conversely, suppose $(S, \cdot)$ is a semigroup satisfies

- $\diamond$ $\forall x \in S(xx = x)$;
- $\diamond$ $\forall x, y \in S(xy = yx)$;

then we can define a partial-order that $x \leq y \Leftrightarrow x = x \cdot y$. And so $(S, \leq, \cdot)$ forms a semilattice, where $x \cdot y = \inf\{x, y\}$.

Proposition 1.3.5 Given a set X, a partition $\mathcal{A}$ is a family of disjoint subsets of X satisfying $\bigsqcup \mathcal{A} = X$. There exists a bijection

$$\{R \in P(X^2) : \text{equivalent relation}\} \xleftrightarrow{1:1} \{\mathcal{A} \in P(X) : \text{partition}\}$$

$$R \longmapsto \{R(x)\}_{x \in X}$$

$$[R : (x, y) \in R \Leftrightarrow \exists A \in \mathcal{A}(x \in A \wedge y \in A)] \longleftarrow \mathcal{A}$$

1.4 Congruences

Definition 1.4.1 Let $(S, \cdot)$ be a semigroup, R is a relation on S. We have the following operations:

(1) $aR = a \cdot R := \{(ax, ay) : (x, y) \in R\}$, dually, $Ra := \{(xa, ya) : (x, y) \in R\}$, in addition, $aRb := \{(axb, ayb) : (x, y) \in R\}$;

(2) $S^1 R = S^1 \cdot R := \bigcup_{a \in S^1} aR$, $S^1 R S^1 = \bigcup_{(a,b) \in S^1 \times S^1} aRb$;

(3) $RR = R \cdot R := \{(x_1 x_2, y_1 y_2) : (x_i, y_i) \in R \wedge i \in \{1, 2\}\}$; furthermore, $R^{\cdot n} := R \cdot R \cdots R$ (n times).

Definition 1.4.2 (...)

(1) R is *left compatible* if $S^1 R \subset R$;

(2) dually, R is *right compatible* if $RS^1 \subset R$;

(3) R is a *congruence* (R is compatible) if $S^1 R \subset R \wedge RS^1 \subset R$, which is equivalent to $RR \subset R$.

The proof for the last assertion (3) is as follows. Since $\Delta_S \subset R$, $RR \subset R$ for any $a \in S^1$ and $(x, y) \in S$, $(ax, ay) \in R$. Conversely, assume $(x_1 x_2, y_1 y_2) \in RR$. Since $S^1 R \subset R \wedge RS^1 \subset R$, we obtain that $(x_1 x_2, x_1 y_2) \in R$ and $(x_1 y_2, x_2 y_2) \in R$. Thus, $(x_1 x_2, y_1 y_2) \in R$.

The conclusion below is often used in algebra, especially in situations where an equivalence relation and some operations are imposed on a set to give it an algebraic structure, for example, ideal of rings, the construction of amalgamated product and the construction of tensor product. Its core, precisely, is the concept of congruence in semigroup theory.

Proposition 1.4.3 Suppose R is an equivalence relation on a semigroup S, then

$$R(x)R(y) := R(xy) \text{ well defined} \Leftrightarrow R \text{ is a congruence.}$$

Proposition 1.4.4 The way to construct a certain relation is as follows.

(1) $\langle R\rangle_{\mathrm{eqv}} := \bigcup_{n\in\mathbb{Z}_{\geq 1}} [R \cup \Delta_S \cup R^{\mathrm{op}}]^n$ is the smallest equivalence relation containing R, where S can just be a set;

(2) $\langle R\rangle_{\mathrm{cpt}} := S^1RS^1$ is the smallest compatible relation containing R;

(3) $\langle R\rangle_{\mathrm{cge}} := \langle S^1RS^1\rangle_{\mathrm{eqv}}$ is the smallest congruence containing R.

Both set $\mathrm{Eqv}(S)$ of equivalences and $\mathrm{Cge}(S)$ of congruences on S are partially ordered by $\subset$. In fact, both are complete lattice. Take $\mathrm{Cge}(S)$ as an example, for any subset $\mathcal{U} \subset \mathrm{Cge}(S)$, it can be verified that $\inf\mathcal{U} = \bigcap\mathcal{U}$ and $\sup\mathcal{U} = \langle\bigcup\mathcal{U}\rangle_{\mathrm{cge}}$. Notice that for any $R_1, R_2 \in \mathrm{Cge}(S)$

$$\langle R_1 \cup R_2\rangle_{\mathrm{cge}} = \langle R_1 \cup R_2\rangle_{\mathrm{eqv}}\,, \tag{1.1}$$

so, both symbol $\wedge$ and $\vee$ on lattice $\mathrm{Eqv}(S)$ and $\mathrm{Eqv}(S)$ represent the same operations of sets.

Proposition 1.4.5 Suppose R_1, R_2 are equivalences, then

- $R_1 \vee R_2 = \langle R_1 \cup R_2\rangle_{\mathrm{eqv}} = \bigcup_{n\in\mathbb{Z}_{\geq 1}}(R_1 \cup R_2)^n = \bigcup_{n\in\mathbb{Z}_{\geq 1}}(R_1 \circ R_2)^n$;
- $R_1 \circ R_2 = R_2 \circ R_1 \Rightarrow R_1 \vee R_2 = R_1 \circ R_2$.

Proof. See [1, p.28]. □

1.5 Ideals

Definition 1.5.1 Let S be a semigroup and $I \in \mathrm{Idl}(S)$ be a proper ideal, then

(1) re is a mapping from the set of proper ideal of S to $\mathrm{Cge}(S)$, which is given by $I \mapsto I^2 \cup \Delta_S =: \mathrm{re}(I)$;

(2) elements in $\mathrm{im}\,\mathrm{re}$ are called *Rees ideals*;

(3) a morphism ϕ is called a *Ress morphism* if $\ker\phi$ is a Ress ideal.

Based on this, we obtain the following propositions.

Proposition 1.5.2 (...)

(1) Every $\mathrm{re}(I)$ is a congruence, thus,

(2) $S/\mathrm{re}(I) = \{I\} \sqcup \{\{x\} : x \in S \setminus I\}$ forms a semigroup;

(3) $I \in S/\mathrm{re}(I)$ is a zero element;

(4) suppose I is a proper ideal, there exists a bijection

$$\begin{aligned}\{I \subset J \subsetneq S :\ \text{ideal}\} &\xleftrightarrow{1:1} \{\bar{J} \subset S/\mathrm{re}(I) :\ \text{ideal}\}\\ J &\longmapsto \mathrm{re}\,I(J)\\ (\mathrm{re}\,I)^{-1}(\bar{J}) &\longleftarrow\!\shortmid\ \bar{J}.\end{aligned}$$

1.6 Free Semigroup

The definition of free semigroup is similar to other algebraic structures, that is, the initial object in the comma category (j_X, U). To be specific, $(\mathbf{F}(X), \iota)$ is the free semigroup of set X, if for any (S, f), where S is a semigroup and $f : X \to S$ is a function, there exists unique semigroup morphism ϕ that makes the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathbf{F}(X) \arrow[d, "\exists!"', "\phi"] \\ & S \end{tikzcd}$$

The construction is also straightforward, we omit it here.

Definition 1.6.1 Suppose Y is a relation on free semigroup $\mathbf{F}(X)$, let

$$\langle X|Y\rangle := \mathbf{F}(X)/\left\langle Y\right\rangle_{\mathrm{cge}}.$$

If there exists an epimorphism $\phi : \mathbf{F}(X) \to S$, a semigroup, such that $\ker\phi = \langle Y\rangle_{\mathrm{cge}}$, and hence $\langle X|Y\rangle \simeq S$, we say that S is presented.

Chapter 2

Green's Equivalences; Regular Semigroups

2.1 Green's Equivalences

Definition 2.1.1 Let S be a semigroup, and the follows are some basic concepts.

- S^1a is the principal left ideal of a, dually aS^1 is the principal right ideal, and S^1aS^1 is the principal ideal of a, denoted as (a), which is the *smallest* ideal containing a;
- $\mathcal{L}$ is an equivalence defined by $a\mathcal{L}b \Leftrightarrow S^1a = S^1b$,
- $\mathcal{R}$ is an equivalence defined by $a\mathcal{R}b \Leftrightarrow aS^1 = bS^1$,
- $\mathcal{J}$ is an equivalence defined by $a\mathcal{J}b \Leftrightarrow S^1aS^1 = S^1bS^1$,
- $\mathcal{H} := \mathcal{L} \cap \mathcal{R}$ is also an equivalence,
- $\mathcal{D} := \langle \mathcal{L} \cup \mathcal{R} \rangle_{\mathrm{eqv}} = \mathcal{L} \vee \mathcal{R}$, and it equals to $\mathcal{L} \circ \mathcal{R}$.

Proposition 2.1.2 These objects above possess some properties:

(1) $\mathcal{L}$ is a *right* congruence, and $\mathcal{R}$ is a *left* congruence;

(2) $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$, the proof can be found in [1, Proposition 2.1.3];

(3) $\mathcal{D} \subset \mathcal{J}$;

(4) suppose S, which has no identity, induces equivalences $\mathcal{L}, \mathcal{R}, \mathcal{D}, \cdots$, and S^1 induces $\mathcal{L}', \mathcal{R}', \mathcal{D}', \cdots$, and so, $\mathcal{L}' = \mathcal{L} \sqcup \{(1, 1)\}$, the same conclusion applies for the remaining equivalences.

We then can impose a partial order on $S/\mathcal{L}, S/\mathcal{R}$ and $S/\mathcal{J}$, to be specific,

- $\mathcal{L}(a) \leq \mathcal{L}(b) \Leftrightarrow S^1a \subset S^1b$,
- $\mathcal{R}(a) \leq \mathcal{R}(b) \Leftrightarrow aS^1 \subset bS^1$,
- $\mathcal{J}(a) \leq \mathcal{J}(b) \Leftrightarrow S^1aS^1 \subset S^1bS^1$.

Notice that for all $a \in S$ and $x, y \in S^1$,

- $\mathcal{L}(xa) \leq \mathcal{L}(a)$,
- $\mathcal{R}(ax) \leq \mathcal{R}(a)$,
- $\mathcal{J}(xay) \leq \mathcal{J}(a)$,
- $\mathcal{L}(a) \leq \mathcal{L}(b) \vee \mathcal{R}(a) \leq \mathcal{R}(b) \Rightarrow \mathcal{J}(a) \leq \mathcal{J}(b)$.

Noticing the property $\mathcal{D} \subset \mathcal{J}$, we are naturally led to ask when $\mathcal{D} = \mathcal{J}$, and the book [1] gives the following proposition:

Proposition 2.1.3

(1) when S is a group, $\mathcal{L} = \mathcal{R} = \mathcal{H} = \mathcal{J} = \mathcal{D} = S^2$;

(2) when S is a commutative semigroup, $\mathcal{L} = \mathcal{R} = \mathcal{H} = \mathcal{J} = \mathcal{D}$;

(3) when S is a periodic semigroup, then $\mathcal{D} = \mathcal{J}$ (see Proposition 2.1.4);

(4) when S is a semigroup, and both $S/\mathcal{L}$ and $S/\mathcal{R}$ as partial ordered sets satisfy the minimal condition, then $\mathcal{D} = \mathcal{J}$ (see Proposition 2.1.5).

Note that in the procedure of proving last proposition, we have to verify that if $S/\mathcal{L}$ possess minimal condition then so does $S^1/\mathcal{L}'$, where $\mathcal{L}'$ is originated from semigroup S^1, and that $\mathcal{D}' = \mathcal{J}' \Rightarrow \mathcal{D} = \mathcal{J}$. As for the former, let U' be any subset of $S^1/\mathcal{L}'$, then $U' = \{\mathcal{L}'(a) : a \in A \wedge A \subset S^1\}$. According to (4) of Proposition 2.1.2, we obtain that $a = 1 \Rightarrow \mathcal{L}'(a) = \{1\}$ and $a \in S \Rightarrow \mathcal{L}'(a) = \mathcal{L}(a)$, thus, if let U be $\{\mathcal{L}(a) : a \in A \setminus \{1\}\} \subset S/\mathcal{L}$, clearly it contains a minimal element $\mathcal{L}(m)$, which is also the minimal element of U'.

2.2 The $\mathcal{D}$-Classes

Proposition 2.2.1 The $\mathcal{D}$-classes of a semigroup S possess some properties, listed as follows.

(1) $\forall x \in \mathcal{D}(a) \Rightarrow \mathcal{L}(x) \subset \mathcal{D}(a) \wedge \mathcal{R}(x) \subset \mathcal{D}(a) \Rightarrow \mathcal{H}(x) \subset \mathcal{D}(a)$;

(2) $\mathcal{D}(a) = \bigcup_{t \in \mathcal{R}(a)} \mathcal{L}(t) = \bigcup_{t \in \mathcal{D}(a)} \mathcal{L}(t) = \bigcup_{t \in \mathcal{L}(a)} \mathcal{R}(t) = \bigcup_{t \in \mathcal{D}(a)} \mathcal{R}(t)$;

(3) $a\mathcal{D}b \Leftrightarrow \mathcal{R}(a) \cap \mathcal{L}(b) \neq \varnothing \Leftrightarrow \mathcal{R}(b) \cap \mathcal{L}(a) \neq \varnothing$;

(4) The intersection of an $\mathcal{L}$-class and an $\mathcal{R}$-class is either $\varnothing$ or a $\mathcal{H}$-class, conversely any $\mathcal{H}$-class is a intersection of an $\mathcal{L}$-class and an $\mathcal{R}$-class;

(5) Suppose $S = \bigsqcup_{i \in I} \mathcal{L}_i = \bigsqcup_{j \in J} \mathcal{R}_j$, then $S = \bigsqcup_{(i,j) \in I \times J} \mathcal{L}_i \cap \mathcal{R}_j$.

Notice that the data $\{\mathcal{L}_i \cup \mathcal{R}_j\}_{(i,j)} = \{\mathcal{H}(a) : a \in S\} \sqcup \{\varnothing\}$. Moreover, this partition of set S is always described as a table, each cell is either empty or an $\mathcal{H}$-class.

	$\mathcal{L}_1$	$\mathcal{L}_2$
$\mathcal{R}_1$		
$\mathcal{R}_2$		

Lemma 2.2.2 (Green's Lemma) Let S be a semigroup, and we denote by ρ_s the mapping $x \mapsto xs$, and by λ_s the mapping $x \mapsto sx$. Suppose $a\mathcal{R}b$, then there exists $s, s' \in S^1$ such that $as = b$ and $bs' = a$. One can conclude that:

- ◇ $\mathcal{L}(a) \underset{\rho_{s'}}{\overset{\rho_s}{\rightleftarrows}} \mathcal{L}(b) = \mathrm{id}$;
- ◇ $\forall x, x' \in \mathcal{L}(a)$, we have $x\mathcal{R}xs$, $x\mathcal{R}x' \Rightarrow xs\mathcal{R}x's$ and $x\mathcal{L}x' \Rightarrow xs\mathcal{L}x's$;
- ◇ $\forall x \in \mathcal{L}(a)$, $\mathcal{H}(x) \underset{\rho_{s'}}{\overset{\rho_s}{\rightleftarrows}} \mathcal{H}(xs) = \mathrm{id}$.

Dually, suppose $a\mathcal{L}b$, then there exists $s, s' \in S^1$ such that $sa = b$ and $s'b = a$. We obtain:

- ◇ $\mathcal{R}(a) \underset{\lambda_{s'}}{\overset{\lambda_s}{\rightleftarrows}} \mathcal{R}(b) = \mathrm{id}$;
- ◇ $\forall x, x' \in \mathcal{R}(a)$, we have $x\mathcal{L}xs$, $x\mathcal{L}x' \Rightarrow sx\mathcal{L}sx'$ and $x\mathcal{R}x' \Rightarrow sx\mathcal{R}sx'$;
- ◇ $\forall x \in \mathcal{R}(a)$, $\mathcal{H}(x) \underset{\lambda_{s'}}{\overset{\lambda_s}{\rightleftarrows}} \mathcal{H}(xs) = \mathrm{id}$.

Based on Green's Lemma, we have some corollaries.

Corollary 2.2.3

(1) $a\mathcal{D}b \Rightarrow |\mathcal{H}(a)| = |\mathcal{H}(b)|$;

(2) $ab \in \mathcal{H}(a) \Rightarrow \mathcal{H}(a) \xrightarrow[\sim]{\rho_b} \mathcal{H}(a)$;

(3) $ab \in \mathcal{H}(b) \Rightarrow \mathcal{H}(b) \xrightarrow[\sim]{\lambda_a} \mathcal{H}(b)$;

Proof. (1) Observe that $a\mathcal{D}b \Rightarrow a\mathcal{R}c \wedge c\mathcal{L}b$, by Green's Lemma, we have

$$\begin{array}{ccccccc} \mathcal{L}(a) & \underset{\rho_{s'}}{\overset{\rho_s}{\rightleftarrows}} & \mathcal{L}(c) & \quad & \mathcal{R}(c) & \underset{\lambda_{t'}}{\overset{\lambda_t}{\rightleftarrows}} & \mathcal{R}(b) \\ \cup & & \cup & & \cup & & \cup \\ \mathcal{H}(a) & \underset{\rho_{s'}}{\overset{\rho_s}{\rightleftarrows}} & \mathcal{H}(c), & \quad & \mathcal{H}(c) & \underset{\lambda_{t'}}{\overset{\lambda_t}{\rightleftarrows}} & \mathcal{H}(b). \end{array}$$

(2) $ab \in \mathcal{H}(a) \Rightarrow a\mathcal{R}ab$, and so

$$\begin{array}{ccc} \mathcal{L}(a) & \underset{\rho_{s'}}{\overset{\rho_b}{\rightleftarrows}} & \mathcal{L}(ab) \\ \cup & & \cup \\ \mathcal{H}(a) & \underset{\rho_{s'}}{\overset{\rho_b}{\rightleftarrows}} & \mathcal{H}(ab). \end{array}$$

(3) Similar to the previous one. □

Theorem 2.2.4 If H is an $\mathcal{H}$-class in a semigroup S, then either $HH \cap H = \varnothing$ or $HH = H$ and H is a subgroup of S.

Proof. Suppose $a, b \in H$ and $ab \in H$, then $a \in \mathcal{H}(ab)$ and $b \in \mathcal{H}(ab)$. By (2) and (3) of Corollary 2.2.3 above, there exists two isomorphisms $H \underset{\lambda_a}{\overset{\rho_b}{\rightleftarrows}} H$, which implies for any $h \in H$, $hb \in H$ and $ah \in H$. Apply these two proposition again, we obtain that $H \underset{\lambda_h}{\overset{\rho_h}{\rightleftarrows}} H$, furthermore, $HH = H$, and H is a group according to Proposition 1.1.2. □

Corollary 2.2.5 If e is an idempotent, then $\mathcal{H}(e)$ is a subgroup; no $\mathcal{H}$-class can contain more than one idempotent, since the idempotent in a group is identity.

2.3 Regular Semigroup

Definition 2.3.1 Let S be a semigroup, we then introduce some definitions.

- ⋄ $a \in S$ is *regular* if there exists $x \in S$ such that $axa = a$,
- ⋄ a' is the inverse of a if $a'aa' = a'$ and $aa'a = a$,
- ⋄ $\mathrm{inv}(a)$ is the set of all inverses of a.

It immediately follows the propositions below.

Proposition 2.3.2

(1) $\forall x, y \in S$, if $xyx = x$, then $xy\mathcal{R}x \wedge yx\mathcal{L}x$;

(2) if a is regular, both $\mathcal{L}(a)$ and $\mathcal{R}(a)$ are regular, thus $\mathcal{D}(a)$ is regular;

(3) any $\mathcal{D}(a)$ contains an idempotent is regular;

(4) let e be an idempotent, then e is a left identity in $\mathcal{R}(e)$, and is a right identity in $\mathcal{L}(e)$;

(5) a is regular $\Leftrightarrow$ a has inverse;

(6) if $y \in \mathrm{inv}(x)$, by (1) above, $yx \in \mathcal{R}(y) \cap \mathcal{L}(x) \wedge xy \in \mathcal{R}(x) \cap \mathcal{L}(y)$;

(7) if D is a regular class, then for any $a \in D$, both $\mathcal{L}(a)$ and $\mathcal{R}(a)$ contain idempotents.

Proof. (5) Suppose $axa = a$, namely, a is regular. Let $a' = xax$, and it's indeed an inverse.
(7) Assuming $axa = a$, and it follows that $xa\mathcal{L}a, ax\mathcal{R}a$, where both xa and ax are idempotent. □

Proposition 2.3.3 Let S be a semigroup, with the aid of "egg box", we have propositions as follows.

(1) Let a be an element of S, $a' \in \mathrm{inv}(a)$, then both $aa' \in \mathcal{L}(a) \cap \mathcal{R}(a')$ and $a'a \in \mathcal{L}(a') \cap \mathcal{R}(a)$ are idempotents. This can be illustrated by the table below.

a	$\exists aa'$
$\exists a'a$	a'

(2) Let a be an element of S, $e \in \mathcal{R}(a) \cap \mathcal{L}(b)$ and $f \in \mathcal{R}(b) \cap \mathcal{L}(a)$ are two idempotents, then there exists $a' \in \mathcal{H}(b)$ such that $a' \in \mathrm{inv}(a)$, that $aa' = e$ and $a'a = f$.

a	e
f	$\exists a'$

(3) In a semigroup S, no $\mathcal{H}$-calss contains more than one inverse of a.

(4) Let e, f be idempotents, then, $e\mathcal{D}f$ if and only if there exists a and $a' \in \mathrm{inv}(a)$ such that $aa' = e \wedge a'a = f$.

e	$\exists a$
$\exists a'$	f

Proof. (1) is the corollary of (6) of Proposition 2.3.2.
(2) From $a\mathcal{R}e$ it follows that $\exists x \in S^1(ax = e)$, let $a' = fxe$, thus, it can be verified that $aa'a = afxea = axa = ea = a$. The proof for $a'aa' = a', aa' = e$ and $a'a = f$ follows the same manner. Observe that $aa' = e \wedge fxe = a$, this implies $a'\mathcal{L}e$; similarly, $a'\mathcal{R}f$.
(3) Suppose a, a^* are two inverses of a in a single $\mathcal{H}(b)$. Then, by (1), aa' and aa^* are two idempotents in $\mathcal{R}(a) \cap \mathcal{L}(b)$, and it follows that $aa' = aa^*$. Similarly, $a'a = a^*a$. Hence, we obtain that

$$a^* = a^*aa^* = a^*aa' = a'aa' = a'.$$

(4) Suppose $e\mathcal{D}f$, then $\exists a \in \mathcal{R}(e) \cap \mathcal{L}(f)$. Besides this, by means of (2), there exists $a' \in \mathrm{inv}(a)$ such that $aa' = e \wedge a'a = f$. Conversely, if there exists a and $a' \in \mathrm{inv}(a)$ such that $aa' = e \wedge a'a = f$, then $e = aa' \in \mathcal{R}(a)$ and $f = a'a \in \mathcal{L}(a)$ by (1), thus, $e\mathcal{D}f$. □

The following propositions are the comprehensive application of the above propositions and the Green's Lemma.

Proposition 2.3.4 If H and K are two group $\mathcal{H}$-class in the same $\mathcal{D}$-class, then H and K are isomorphic.

Proof. Since the identity in a group is idempotent, H and K contain idempotents e and f respectively. Notice that $e\mathcal{D}f$, according to (4), we can find $a \in \mathcal{R}(e) \cap \mathcal{L}(f)$ and $\mathcal{R}(f) \cap \mathcal{L}(e) \ni a' \in \mathrm{inv}(a)$ that makes $aa' = e$ and $a'a = f$. In addition, we also have $ea = af = a, a'e = fa' = a'$. From $aa' = e \wedge ea = a$ and $a'a = f \wedge af = a$, one can construct the following isomorphisms by means of Green's Lemma.

$$\begin{array}{ccccccc}
\mathcal{L}(a) & \underset{\rho_a}{\overset{\rho_{a'}}{\rightleftarrows}} & \mathcal{L}(e) & & \mathcal{R}(a) & \underset{\lambda_a}{\overset{\lambda_{a'}}{\rightleftarrows}} & \mathcal{R}(f) \\
\cup & & \cup & & \cup & & \cup \\
\mathcal{H}(a) & \underset{\rho_a}{\overset{\rho_{a'}}{\rightleftarrows}} & \mathcal{H}(e) & & \mathcal{H}(a) & \underset{\lambda_a}{\overset{\lambda_{a'}}{\rightleftarrows}} & \mathcal{H}(f)
\end{array}$$

□

Proposition 2.3.5 Let a, b be elements in a $\mathcal{D}$-class. Then, $ab \in \mathcal{R}(a) \cap \mathcal{L}(b)$ if and only if $\mathcal{L}(a) \cap \mathcal{R}(b)$ contains an idempotent. (SIMPLIFY)

Proof. The content provided here can serve as a supplement of the original proof of [1, Proposition 2.3.7]. Suppose $ab \in \mathcal{R}(a) \cap \mathcal{L}(b)$, then there exists ξ, η such that

$$\begin{cases} ab = ab \\ ab\xi = a \end{cases} \wedge \begin{cases} ab = ab \\ \eta ab = b \end{cases},$$

thus, $b\xi = \eta ab\xi = \eta a$. Furthermore, we have $b = \eta ab \mathcal{R} \eta a$ and $a = ab\xi \mathcal{L} b\xi$ due to $ab\mathcal{R}a$ and $ab\mathcal{L}a$. Observe that

$$\begin{cases} \eta a = \eta a \\ a\eta a = ab\xi = a \end{cases} \wedge \begin{cases} b\xi = b\xi \\ b\xi b = \eta ab = b \end{cases},$$

we obtain $a\mathcal{L}\eta a$ and $b\mathcal{R}b\xi$. Hence

$$\mathcal{H}(b\xi) = \mathcal{R}(b\xi) \cap \mathcal{L}(b\xi) = \mathcal{R}(b) \cap \mathcal{L}(a).$$

By Green's Lemma, we have the following isomorphism

$$\begin{array}{ccc} \mathcal{L}(ab) & \underset{\rho_b}{\overset{\rho_\xi}{\rightleftarrows}} & \mathcal{L}(a) \\ \cup & & \cup \\ \mathcal{H}(b) & \underset{\rho_b}{\overset{\rho_\xi}{\rightleftarrows}} & \mathcal{H}(b\xi) = \mathcal{R}(b) \cap \mathcal{L}(a), \end{array}$$

and it's easy to verify $b\xi$ is an idempotent.

Conversely, suppose $\mathcal{L}(a) \cap \mathcal{R}(b)$ contains an idempotent e, then there exists $s, s', t, t' \in S^1$ such that

$$\begin{cases} te = a \\ t'a = e \end{cases} \wedge \begin{cases} es = b \\ bs' = e \end{cases}.$$

We found that

$$\begin{cases} ab = ab \\ abs' = a \end{cases} \Leftrightarrow ab\mathcal{R}a \wedge \begin{cases} ab = ab \\ t'ab = b \end{cases} \Leftrightarrow ab\mathcal{L}b,$$

thus $\mathcal{H}(ab) = \mathcal{R}(a) \cap \mathcal{L}(b)$. And the following isomorphism also stems from Green's Lemma.

$$\begin{array}{ccc} \mathcal{L}(a) & \underset{\rho_{s'}}{\overset{\rho_b}{\rightleftarrows}} & \mathcal{L}(ab) \\ \cup & & \cup \\ \mathcal{H}(a) & \underset{\rho_{s'}}{\overset{\rho_b}{\rightleftarrows}} & \mathcal{H}(ab) = \mathcal{R}(a) \cap \mathcal{L}(b), \end{array}$$

□

Definition 2.3.6 Let S be a semigroup, we have the following definitions:

- suppose $U < S$ is a subsemigroup, the green's equivalence $\mathcal{L}^U$ originated from U is defined as $\{(a, b) \in U^2 : U^1a = U^1b\}$, and the similar definitions apply to the remaining equivalences;
- $\mathrm{idm}(S)$ is the set of all idempotents of S;
- $R \in \mathrm{Eqv}(S)$ is *idempotent-separating* if $R \cap \mathrm{idm}(S)^2 = \Delta_{\mathrm{idm}(S)}$, that is, each R-class contains no more than one idempotent.

If S is regular, then $a\mathcal{L}b \Leftrightarrow S^1a = S^1b \Leftrightarrow Sa = Sb$, since $\exists b \in S(aba = a)$, which implies for all $a \in S$, $S^1a = Sa$. In fact, to define the Green's Equivalences on a regular semigroup, we can drop all reference to S^1.

Proposition 2.3.7 Let S be a regular semigroup and $a, b \in S$. Then

(1) $(a, b) \in \mathcal{L} \Leftrightarrow \exists a' \in \mathrm{inv}(a) \exists b' \in \mathrm{inv}(b)(a'a = b'b)$,

(2) $(a, b) \in \mathcal{R} \Leftrightarrow \exists a' \in \mathrm{inv}(a) \exists b' \in \mathrm{inv}(b)(aa' = bb')$,

(3) $(a, b) \in \mathcal{H} \Leftrightarrow \exists a' \in \mathrm{inv}(a) \exists b' \in \mathrm{inv}(b)(a'a = b'b \wedge aa' = bb')$.

Proof. Since S is regular, each element has an inverse. Suppose $a\mathcal{L}b$ and a', b' are inverses of a, b respectively. To prove (1), the following diagram says it all. □

As for the equivalences on subsemigroup U, it can be easily verified that $\mathcal{L}^U \subset \mathcal{L} \cap U^2$, the similar conclusions apply for remaining equivalences. However, this inclusion could be proper.

Proposition 2.3.8 If U is a regular subsemigroup of semigroup S, then all $\mathcal{L}, \mathcal{R}, \mathcal{H}$ satisfy $X^U = X \cap U^2$, where $X \in \{\mathcal{L}, \mathcal{R}, \mathcal{H}\}$.

Proposition 2.3.9 Suppose S is a regular semigroup and $\mathcal{C} \in \mathrm{Cge}(S)$, then $S/\mathcal{C}$ is regular.

Lemma 2.3.10 (Lallement) Suppose S is a regular semigroup, the following two propositions are equivalent.

(1) Given $\mathcal{C} \in \mathrm{Cge}(S)$, if $\mathcal{C}(a)$ is an idempotent in $S/\mathcal{C}$, then there exists an idempotent e such that $\mathcal{C}(a) = \mathcal{C}(e)$.

(2) Given morphism $\phi : S \to T$, if $\phi(a)$ is an idempotent, then there exists an idempotent $e \in S$ such that $\phi(e) = f$.

Proof. To proof (1), suppose $\mathcal{C}(a) = \mathcal{C}(a^2)$, let x be the inverse of a^2 and $e = axa$. We then proceed to prove the equivalence of these two propositions.

Suppose (1) holds, $f \in \operatorname{im}\phi$ is an idempotent. Clearly $\ker\phi = \{(a, b) : \phi(a) = \phi(b)\} \in \mathrm{Cge}(S)$. Let $a \in \phi^{-1}(f)$, it can be verified that $[\ker\phi](a) \in S/\ker\phi$ is an idempotent. And it follows that there exists idempotent $e \in S$ and $[\ker\phi](a) = [\ker\phi](e)$.

Conversely, suppose (2) holds, $\mathcal{C} \in \mathrm{Cge}(S)$, $\phi : S \to S/\mathcal{C}$ and $\phi(a)$ is an idempotent. Thus, there exists an idempotent e such that $\phi(a) = \phi(e)$. □

Proposition 2.3.11 If S is regular, $\mathcal{C} \in \mathrm{Cge}(S)$ is idempotent-separating iff $\mathcal{C} \subset \mathcal{H}$.

Proof. Assume $\mathcal{C} \subset \mathcal{H}$, it follows that

$$\Delta_{\mathrm{idm}(S)} \subset \mathcal{C} \cap \mathrm{idm}(S)^2 \subset \mathcal{H} \cap \mathrm{idm}(S)^2 \subset \Delta_{\mathrm{idm}(S)}.$$

For the converse, if $\mathcal{C}$ is idempotent-separating and $a\mathcal{C}b$. Let a' be the inverse of a, we can draw the following conclusions in sequence:

- $aa'\mathcal{C}ba'$
- $\mathcal{C}(ba') = \mathcal{C}(aa')$ is idempotent
- By (1) of Lemma 2.3.10, there exists idempotent $e \in S$ such that
$$\mathcal{C}(e) = \mathcal{C}(ba') \wedge \mathcal{R}(e) \le \mathcal{R}(ba') \wedge e = aa',$$
the assertion $e = aa'$ stems from $\mathcal{C}$ is idempotent separating
- By (1) of Proposition 2.3.3, $\mathcal{R}(a) = \mathcal{R}(aa') = \mathcal{R}(e) \le \mathcal{R}(ba') \le \mathcal{R}(b)$.

A dual argument shows that $\mathcal{L}(a) \le \mathcal{L}(b)$:

- $a'a\mathcal{C}a'b$
- $\mathcal{C}(a'b) = \mathcal{C}(a'a)$ is idempotent
- $\exists e \in S\,(e \text{ is idempotent} \wedge \mathcal{C}(e) = \mathcal{C}(a'b) \wedge \mathcal{L}(e) \le \mathcal{L}(a'b) \wedge e = a'a)$
- $\mathcal{L}(a) = \mathcal{L}(a'a) = \mathcal{L}(e) \le \mathcal{L}(a'b) \le \mathcal{L}(b)$.

Similarly, assume that $b' \in \mathrm{inv}(b)$ is chosen, then we have:

- $ab'\mathcal{C}bb'$
- $\mathcal{C}(ab') = \mathcal{C}(bb')$ is idempotent
- there exists idempotent $e \in S$ such that $\mathcal{C}(e) = \mathcal{C}(ab') \wedge \mathcal{R}(e) \le \mathcal{R}(ab') \wedge e = bb'$
- $\mathcal{R}(b) = \mathcal{R}(bb') = \mathcal{R}(e) \le \mathcal{R}(ab') \le \mathcal{R}(a)$. □

Chapter 3

Simple and 0-simple Groups

3.1 Basics

Definition 3.1.1 Let S be a semigroup, we then define the following concepts and symbols.

- ◇ $\mathrm{Idl}(S), \mathrm{Ldl}(S), \mathrm{Rdl}(S)$ refer the set of all ideals, left-ideals and right-ideals of S respectively.
- ◇ $\mathrm{Idl}^*(S), \mathrm{Ldl}^*(S), \mathrm{Rdl}^*(S)$ refer the set of all nonzero ideals, left-ideals and right-ideals of S respectively.
- ◇ S^* is the set of all nonzero elements of S.
- ◇ $\mathrm{idm}^* S$ is the set of all nonzero idempotents of S.
- ◇ S is simple if
  - ○ S does *not* have a 0,
  - ○ $\mathrm{Idl}(S) = \{S\}$.
- ◇ S is 0-simple if
  - ○ S has a 0,
  - ○ $S^2 \neq \{0\}$,
  - ○ $\mathrm{Idl}(S) = \{\{0\}, S\}$.
- ◇ $I \in \min \mathrm{Idl}^*(S)$ is a 0-minimal ideal.
- ◇ $I \in \min \mathrm{Idl}(S)$ is a minimal ideal.
- ◇ $K(S)$ is the unique minimal ideal (if it exists).

Proposition 3.1.2 Suppose S is a semigroup without a 0, the following statements are equivalent:

- ◇ S is simple;
- ◇ $\mathcal{J} = S^2$;
- ◇ $\forall a \in S (SaS = S)$.

Besides, if S is a semigroup with a 0, the following statements are equivalent:

- ◇ S is 0-simple;
- ◇ $\mathcal{J} = (S^*)^2 \sqcup \{(0, 0)\}$;
- ◇ $\forall a \in S^* (SaS = S)$.

Proposition 3.1.3 Let S be a semigroup, we have the following propositions.

(1) S has no 0, then $\min \operatorname{Idl} S$ is unique;

(2) S has no 0, then $I = \min \operatorname{Idl} S$ is simple;

(3) if I is the 0-minimal ideal, then either $I^2 = \{0\}$ or I is simple;

(4) $\{B \in \operatorname{Idl} S : I \subsetneq B \subsetneq J\} = \varnothing$, then $J/\operatorname{re} I$ is either 0-simple or null.

Definition 3.1.4 For any $a \in S$, we arrive at two complementary cases as follows.

(1) $\mathcal{J}(a) \in \min S/\mathcal{J}$, then

- $\diamond$ $\mathcal{J}(a) = (a) = K(S)$.

(2) $\mathcal{J}(a) \notin \min S/\mathcal{J}$, then

- $\diamond$ $U(a) := \{b \in (a) : \mathcal{J}(b) < \mathcal{J}(a)\} \neq \varnothing$;
- $\diamond$ $(a) = U(a) \sqcup \mathcal{J}(a)$, $U(a) = \bigsqcup\{\mathcal{J}(b) : (b) \subsetneq (a)\}$;
- $\diamond$ $(a)/\operatorname{re} U(a)$ is either 0-simple or null.

These results, $K(S)$ and $(a)/\operatorname{re} U(a)$, consist the *principal factors* of S.

3.2 Completely 0-simple Semigroups

Definition 3.2.1 Let S be a semigroup and $\operatorname{idm} S$ be the set of idempotents of S, we then introduce the follows:

- $\diamond$ a partial order on $\operatorname{idm} S$ is naturally defined as $f \leq e \Leftrightarrow f = fe = ef$;
- $\diamond$ $e \in \min \operatorname{idm}^* S$ is called a primitive idempotent;
- $\diamond$ S is completely 0-simple (CZS) if
  - $\circ$ S has a 0,
  - $\circ$ S is 0-simple,
  - $\circ$ $\exists \min \operatorname{idm}^* S$.

Proposition 3.2.2 Let S be a CZSS, in which e is a primitive idempotent, then

(1) $\mathcal{R}(e) = eS \setminus \{0\}$, dually, $\mathcal{L}(e) = Se \setminus \{0\}$;

(2) $\forall a \in S^*(\mathcal{L}(a) = aS \setminus \{0\})$, dually, $\forall a \in S^*(\mathcal{R}(a) = Sa \setminus \{0\})$;

(3) $\mathcal{D} = (S^*)^2 \sqcup \{(0,0)\}$;

(4) S is regular;

(5) $ab \neq 0 \Rightarrow (a \neq 0 \wedge b \neq 0 \wedge a\mathcal{D}b \wedge ab \in \mathcal{R}(a) \cap \mathcal{L}(b))$;

(6) for any $\mathcal{H}$-class $H \subset S^*$,

- $\diamond$ either $(\exists a, b \in H(ab \neq 0)) \Leftrightarrow H$ is a group $\Leftrightarrow (\forall a, b \in H(ab \neq 0))$;
- $\diamond$ or $(\exists a, b \in H(ab = 0)) \Leftrightarrow H^2 = \{0\} \Leftrightarrow (\forall a, b \in H(ab = 0))$.

Definition 3.2.3 Given sets I and Λ, a group G, and a mapping $P : \Lambda \times I \to G^0$, which can be viewed as a matrix, satisfying the condition of regular that $\forall \lambda \in \Lambda \exists i \in I(P(\lambda, i) \neq 0)$ and that $\forall i \in I \exists \lambda \in \Lambda(P(\lambda, i) \neq 0)$. The Rees matrix $M^0[G, I, \Lambda, P]$ contains the following matters:

- $\diamond$ $\{aE_{i,\lambda} : a \in G^0 \wedge (i, \lambda) \in I \times \Lambda\}$;
- $\diamond$ a binary operation imposed on the set above, that is,

$$\circ : (aE_{i,\lambda}, bE_{j,\mu}) \mapsto aE_{i,\lambda}PbE_{j,\mu} = (aP(\lambda, i)b)E_{i,\mu}.$$

It can be verified that $M^0[G,I,\Lambda,P]$ consists a CZSS (see [1, Lemma 3.2.2]).

Theorem 3.2.4 Any CZSS S is isomorphic to a Ress matrix $M^0[G,I,\Lambda,P]$.

Proof. Let $I=(S/\mathcal{R})\setminus\{0\}$ and $\Lambda=(S/\mathcal{L})\setminus\{0\}$. For any $a\in S\setminus\{0\}$, $\mathcal{R}(a)$ contains an idempotent e by (7) of Proposition 2.3.2, thus $\mathcal{R}(a)\cap\mathcal{L}(e)$ is a group. Similarly, $\mathcal{L}(a)$ contains an idempotent f that makes $\mathcal{L}(a)\cap\mathcal{R}(f)$ is a group. Hence, we may conclude that $\forall i\in I\exists\lambda\in\Lambda(i\cap\lambda$ is a group) and that $\forall\lambda\in\Lambda\exists i\in I(i\cap\lambda$ is a group). Suppose $G=\tilde{i}\cap\tilde{\lambda}$ is a group, and let $q\in\prod_\lambda(\lambda\cap\tilde{i})$, $r\in\prod_i(\tilde{\lambda}\cap i)$. Let the mapping $P:\Lambda\times I\to G^0$ send (λ,i) to $q(\lambda)r(i)$. Then, it can be verified that the following mapping is an isomorphism.

$$M^0[G,I,\Lambda,P] \xrightarrow{\sim} S$$

$$aE_{i,\lambda} \longmapsto r(i)aq(\lambda)$$

□

Proposition 3.2.5 If S is 0-simple, $L\in\min\mathrm{Ldl}^*S$, then

(1) $L^2\neq 0\Rightarrow\forall a\in L\setminus\{0\}(L=Sa)$;

(2) $S=LS=\bigcup_{s\in S}Ls$;

(3) when $Ls\neq\{0\}$, $Ls\in\min\mathrm{Ldl}^*S$.

Proposition 3.2.6 Let S be a CZSS containing at least one 0-minimal left-ideal and at least one 0-minimal right-ideal. Then, for every 0-minimal left-ideal L, there exists a 0-minimal right-ideal R such that

(1) $LR=S$;

(2) RL is a 0-group;

(3) the identity of RL is the primitive idempotent of RL.

Proposition 3.2.7 Suppose S has a 0, the following propositions are equivalent:

(1) S is 0-completely simple;

(2) S is group bounded, namely, $\forall a\in S\exists n$ such that a^n lies in a subgroup of S;

(3) $\exists\min S/\mathcal{L}\wedge\exists\min S/\mathcal{R}$;

(4) $\exists\min\mathrm{Ldl}^*S\wedge\exists\min\mathrm{Rdl}^*S$.

3.3 Completely Simple Semigroups

Most of the conclusions about the CSS are similar to that CZSS has.

Proposition 3.3.1 Let S be a CSS, in which e is a primitive idempotent, then

(1) $\mathcal{R}(e)=eS$, a dual conclusion applies to the case of $\mathcal{L}$;

(2) $\forall a\in S(\mathcal{R}(a)=aS^1)$, a dual conclusion applies to the case of $\mathcal{L}$;

(3) $\mathcal{D}=S^2$;

(4) S is regular;

(5) $\forall a\in S(\mathcal{R}(a)=aS)$, a dual conclusion applies to the case of $\mathcal{L}$;

(6) $\forall a,b\in S(ab\in\mathcal{R}(a)\cap\mathcal{L}(b))$;

(7) any $\mathcal{H}$-class $H\subset S$ is a group.

Proof. (1) Clearly $\mathcal{R}(e) \subset eS$. For the reverse, for any $a = es \in eS$, there exists $z, t \in S$ such that $zat = e$. Let $x = eze$, $y = te$ and $f = ayx$. Follow the manner analogous to [1, Lemma 3.2.4], it can be verified that $f^2 = f$ and $ef = fe = e$. Since $e \in \min\operatorname{idm} S$, it follows $e = f$ and $a \in \mathcal{R}(e)$.

(2) Clearly $\mathcal{R}(a) \subset aS^1$. Now suppose that $b \in aS^1$, and select $z, t \in S$ such that $zet = a$. Thus, $b = zeu$ for some $uinS$. By (1), we obtain that $eu \in eS = \mathcal{R}(e)$ and that $es \in \mathcal{R}(e)$, hence $zeu\mathcal{R}zet$.

(3) For any $a, b \in S$, it follows that $ab \in aS^1 \cap S^1b = \mathcal{R}(a) \cap \mathcal{L}(b)$, thus, $a\mathcal{D}b$.

(4) Observe that $D = S$, that D has an idempotent e, and apply (3) of Proposition 2.3.2.

(5) Since S is regular by (4), so for any $a \in S$, there exists $x \in S$ such that $a = axa \in aS$.

(6) The proof is excerpted from the process of proof of (2).

(7) It is sufficient to show that any $\mathcal{H}$-class satisfies $H^2 \cap H \neq \varnothing$ and apply the Green's Theorem. Certainly, $\forall a, b \in H$, we have $ab\mathcal{R}a, ab\mathcal{L}b, a\mathcal{R}b, a\mathcal{L}b$, thus $ab\mathcal{H}a$. □

Definition 3.3.2 Given sets I and Λ, group G, and mapping $P : \Lambda \times I \to G$, which can be viewed as a matrix. The Rees matrix $M[G, I, \Lambda, P]$ contains the following matters:

- ⋄ $\{aE_{i,\lambda} : a \in G \wedge (i, \lambda) \in I \times \Lambda\}$,
- ⋄ a binary operation imposed on the set above, that is,

$$\circ : (aE_{i,\lambda}, bE_{j,\mu}) \mapsto aE_{i,\lambda}PbE_{j,\mu} = (aP(\lambda, i)b)E_{i,\mu}.$$

It can be verified that $M[G, I, \Lambda, P]$ is a CSS. Note that we do not need to let P be regular, which is different from the case of CZSS, for P has no zero output.

Theorem 3.3.3 Every CSS is isomorphic to a Rees matrix $M[G, I, \Lambda, P]$, the process of construction is similar to the case of CZSS. In particular, P can be normal, in the sense that the first row and the first column of P only contain identity.

Proof. The proof of former can be found in 3.2.4. For the latter, see [1, Theorem 3.4.2]. □

Proposition 3.3.4 If S is simple and $L \in \min\operatorname{Ldl} S$, then

(1) $\forall a \in L (L = Sa)$;

(2) $S = LS = \bigcup_{s \in S} Ls$;

(3) every Ls belongs to $\min\operatorname{Ldl} S$.

Proof. (1) Sa is a left-ideal contained in L, thus it must equal to L.

(2) Observe that LS is an ideal of S.

(3) Suppose $B \subset Ls$ is a left-ideal. Let $A = \{x \in L : xs \in B\}$, a left-ideal contained in L, needs to coincide with L. So $B = As = L$. □

Proposition 3.3.5 Let S be a CSS containing at least one minimal left-ideal and at least one minimal right-ideal. Then, for every minimal left-ideal L, there exists a minimal right-ideal R such that

(1) $LR = S$;

(2) RL is a group;

(3) the identity of RL is the primitive idempotent.

Proof. (1) Clearly, LR is an ideal of S.

(2) It is sufficient to prove that for all $a \in RL$, $RLa = aRL = RL$. Observe that $RL \subset R \cap L$, so $a \in R$, and by (3) of Proposition 3.3.4, we obtain that $R = aS$. Thus, $S = LR = LaS$, where $La \subset L$ is a left-ideal on the ground of $a \in L$, so it follows that $La = L$. Hence, we conclude that $RLa = RL$. The proof of $aRL = RL$ proceeds in the similar manner.

(3) Suppose that $e \in RL$ is the identity of group, and that $f \leq e$, namely $ef = fe = f$. Observe that $eSe = eS^2e = RL$, thus, $f = efe \in RL$, which coincides with e. □

Proposition 3.3.6 If S is simple, the following propositions are equivalent:

(1) S is CS;

(2) S is CR, namely, every element of S lies in a subgroup of S;

(3) $\exists \min S/\mathcal{L} \wedge \exists \min S/\mathcal{R}$;

(4) $\exists \min \mathrm{Ldl}\, S \wedge \exists \min \mathrm{Rdl}\, S$.

Proof. (1) $\Rightarrow$ (2) According to (7) of Proposition 3.3.1, S is the disjoint union of group $\mathcal{H}$-calsses.

(2) $\Rightarrow$ (3) Suppose $\mathcal{J}(a) \leq \mathcal{J}(b)$, since S is simple, one can select u, x, y in S such that $a = ub$ and $b = xay = xuby$. We denote by $g = xu$, by g^{-1} the inverse of g in the group contains g, and by e the identity equal to $g^{-1}g$. Observe that $eb = egby = gby = b$, thus, $b = g^{-1}gb = g^{-1}xub = g^{-1}xa$. Furthermore, we have $a = ub$, and it implies that $a\mathcal{J}b$.

(3) $\Rightarrow$ (4) Assume the contrary situation that there is no $\min \mathrm{Ldl}\, S$. Then, for any $S^1a \in \mathrm{Ldl}\, S$, there exists a left-ideal $B \subsetneq S^1a$. For any $b \in B$, it follows that $S^1b \subset S^1B \subset B \subsetneq S^1a$, that is, $\mathcal{L}(b) < \mathcal{L}(a)$. And this means $S/\mathcal{L}$ has no minimal element.

(4) $\Rightarrow$ (1) By Proposition 3.3.5, S is a simple semigroup with a primitive idempotent. □

Proposition 3.3.7 Let S be a semigroup without 0, the following conditions are equivalent:

(1) S is CS;

(2) S is regular, and satisfies that for any $a, b, c \in S$,
$$(ca = cb \wedge ac = bc) \Rightarrow a = b;$$

(3) S is regular, and for all $a \in S$
$$aba = a \Rightarrow bab = b;$$

(4) S is regular and every idempotent is primitive.

Proof. See [1, Theorem 3.3.3]. □

Theorem 3.3.8 Two Rees matrix semigroups
$$M^0[G_1, I_1, \Lambda_1, P_1], M^0[G_2, I_2, \Lambda_2, P_2]$$
are isomorphic, if and only if there exists

- a group isomorphism $\theta : G_1 \xrightarrow{\sim} G_2$,
- a set isomorphism $\psi : I_1 \xrightarrow{\sim} I_2$,
- a set isomorphism $\chi : \Lambda_1 \xrightarrow{\sim} \Lambda_2$,
- two mappings $u \in G_2^{I_1}$ and $v \in G_2^{\Lambda_1}$,

such that
$$\theta(P_1(\lambda, i)) = v(\lambda)P_2(\chi(\lambda), \psi(i))u(i).$$

Proof. See [1, Theorem 3.4.1]. □

Theorem 3.3.9 Any Rees matrix $M^0[G, I, \Lambda, P]$ is isomorphic to another $M^0[G, I, \Lambda, R]$, where R is normal, in the sense that the first row and the first column of R only contain identity.

Proof. See [1, Theorem 3.4.3]. □

Chapter 4

Completely Regular Semigroups

4.1 Completely Regular Semigroups and Clifford Semigroups

Definition 4.1.1 A completely regular semigroup (CRS) contains data $(S, m(-,-), (-)')$, in which S is a set; $m : S^2 \to S$ is a mapping satisfying the condition of associative that for all a, b, c in S, $m(a, m(b, c)) = m(m(a.b), c)$, namely, (S, m) forms a semigroup; and $(-)'$ is a mapping satisfying

- $\diamond$ $\forall a \in S((a')' = a)$;
- $\diamond$ $\forall a \in S(aa' = a'a)$;
- $\diamond$ $\forall a \in S(aa'a = a)$.

Definition 4.1.2 A Clifford semigroup is a CRS $(S, \cdot, (-)')$ satisfies for all $, y \in S$

$$(xx')(yy') = (yy')(xx').$$

Definition 4.1.3 A center of a semigroup S is defined as

$$Z(S) = \{a \in S : \forall x \in S(xa = ax)\}.$$

From the definitions above, we immediately obtain some natures of CRS.

Proposition 4.1.4 Let S be a CRS, then

(1) for any $a \in S$, $a'aa' = a'$, thus $a' \in \mathrm{inv}(a)$;

(2) for any $a \in S$, all of $a', aa', a'a$ belong to $\mathcal{H}(a)$;

(3) if $e \in S$ is an idempotent, $e = e' = ee' = e'e$.

Proof. (1) Notice that $a'aa' = a'(a')'a' = a'$.

(2) Observe that a' is the inverse of a, that $aa' \in \mathcal{R}(a) \cap \mathcal{L}(a')$ and $a'a \in \mathcal{R}(a') \cap \mathcal{L}(a)$ and that $aa' = a'a$. Thus, $\mathcal{R}(a) \cap \mathcal{L}(a') = \mathcal{R}(a') \cap \mathcal{L}(a)$, which implies $\mathcal{R}(a) \cap \mathcal{R}(a') \neq \varnothing$ and $\mathcal{L}(a) \cap \mathcal{L}(a') \neq \varnothing$, so we may deduce that $a\mathcal{R}a'$ and $a\mathcal{L}a'$.

(3) Notice that $e = ee'e = eee' = ee', e' = e'ee' = e'e = ee' = e$. $\square$

Proposition 4.1.5 Let S be a semigroup, the following propositions are equivalent.

(1) S is CR;

(2) every element is S lies in a subgroup of S;

(3) every $\mathcal{H}$-class is a group.

Proof. (1) $\Rightarrow$ (2) Given a CRS $(S, \cdot, (-)')$, for any $a \in S$, let $e = a'a = aa'$. Then by (2), a belongs to a group $\mathcal{H}(e)$.

(2) $\Rightarrow$ (3) Given a semigroup $(S, \cdot)$ that each element a lies in a subgroup G_a of S, define the mapping $(-)'$ that sends a to its inverse in G_a. Then, from $aa' = a'a =: e_a$ and $e_a a = a e_a = a$, it follows that $a\mathcal{H}e_a$, where $\mathcal{H}(e_a)$ is a group, for e_a is an idempotent.

(3) $\Rightarrow$ (1) Suppose $(S, \cdot)$ is a semigroup that every $\mathcal{H}$-calss within it is a group. It's easy to verify that the mapping, $(-)'$, sending a to its inverse $a' \in \mathcal{H}(a)$, together with the presupposed structure on S, forms a CRS. □

Proposition 4.1.6 Suppose S is a semigroup, the following statements are equivalent:

(1) S is CS.

(2) S is CR, and for all $x, y \in S$, $xx' = (xyx)(xyx)'$;

(3) S is CR and simple.

Proposition 4.1.7 (1) $\Rightarrow$ (2) By (7) of Proposition 3.3.1, every $\mathcal{H}$-class of S is a group, so we let $(-)'$ be the mapping that sending a to its inverse within $\mathcal{H}(a)$. Let $x, y \in S$, by (6) of Proposition 3.3.1, we obtain that $xy \in \mathcal{R}(x) \cap \mathcal{L}(y)$ and that $xyx \in \mathcal{R}(xy) \cap \mathcal{L}(x)$, in which $\mathcal{R}(xy) = \mathcal{R}(x)$. Thus, $xyx \in \mathcal{H}(x)$. Hence, $xx' = (xyx)(xyx)'$.

(2) $\Rightarrow$ (3) Suppose $(S, \cdot, (-)')$ is a CRS and $a, b \in S$. Then

$$a = aa'a = a \cdot b \cdot a(aba)'a,$$

and so $\mathcal{J}(a) \leq \mathcal{J}(b)$. By interchanging the role of a and b, we may equally show that $\mathcal{J}(b) \leq \mathcal{J}(a)$. Thus, $\mathcal{J}(a) = \mathcal{J}(b)$ for any a, b in S, and so $\mathcal{J} = S^2$, which implies S is simple.

(3) $\Rightarrow$ (1) This is a direct conclusion of Proposition 3.3.6.

Regarding the last derivation above, we have an even stronger conclusion.

Proposition 4.1.8 Suppose $(S, \cdot, (-)')$ is simple and CR, then every idempotent of S is primitive.

Proof. Let e, f be two idempotents of S, and $e \leq f$, namely, $ef = fe = e$. Since S is simple, there exists $z, t \in S$ such that $e = zft$. Let $x = ezf$ and $y = fte$, then, it can be verified that

$$xfy = e, ex = xf = e, fy = ye = y.$$

On the other hand, S is CR, so x lies in a group $\mathcal{H}$-class, in which there exists an identity e_x satisfies $e_x x = x e_x = x$ and $x'x = xx' = e_x$. Observe that

$$f = e_x f = e_x e f = e_x x f y f = x f y = e,$$

so we may conclude that every idempotent of S is primitive. □

We Stipulate that a partially ordered set $(I, \leq)$ can be made into a category, where for any elements i, j of I,

$$i \to j \Leftrightarrow j \leq i.$$

In particular, a semilattice can also be made into a category in the same manner.

Definition 4.1.9 Let $(I, \leq, \wedge)$ be a semilattice, $\alpha : I \to \mathsf{Smg}$ be a functor and S be the set $\bigsqcup_{i \in I} \alpha(i)$. Define a multiplication on S as

$$(x, i) * (y, j) := (\alpha_{i \to i \wedge j}(x) \alpha_{j \to i \wedge j}(y), i \wedge j);$$

and a family of functions $\iota(i) : \alpha(i) \to S$ that maps x to (x, i) for every $i \in I$, which are in fact the natural inclusions. Then, it can be verified that:

(1) each $\iota(i) : \alpha(i) \to S$ is a morphism of semigroups, but the series of these morphisms, $\iota = [i \mapsto \iota(i)]$, may not consist a natural transformation between $\alpha \to \Delta_I(S)$;

(2) the set S, together with the multiplication defined above, forms a semigroup;

(3) for any i, j in I, $\alpha(i) * \alpha(j) \subset \alpha(i \wedge j)$;

(4) for any semigroup L and the morphism $f : \alpha \to \Delta_I(L)$, there exists unique morphism $\phi : S \to L$ which is given by $(x, i) \mapsto f(i)(x)$ such that for any $i \in I$ the following diagram commuts.

$$\begin{tikzcd} \alpha(i) \arrow[r, hook, "\iota(i)"] \arrow[dr, "f(i)"'] & S \arrow[d, dashed, "\exists! \, \phi"] \\ & L \end{tikzcd}$$

Hence, we call the semigroup $(S, *)$ the *strong semilattice of semigroups*.

Definition 4.1.10 Let $(I, \leq, \wedge)$ be a semilattice, $(S, *)$ be a semigroup, $\alpha : I \to \mathrm{Ob}(\mathsf{Smg})$ be a mapping and $(\iota(i) : \alpha(i) \hookrightarrow S)_{i \in I}$ be a series of semigroup monomorphisms. We say that S is the *lattice of semigroups* if

(1) $\alpha(i) * \alpha(j) \subset \alpha(i \wedge j)$;

(2) under the perspective of set, $S = \bigsqcup_{i \in I} \alpha(i)$.

The above two definitions, the strong semilattice of semigroups and the semilattice of semigroups, can generalize to the cases of CRSs, CSSs and groups.

Proposition 4.1.11 Suppose I is a semilattice and $\alpha : I \to \mathsf{Css}$ is a functor, then still is a CSS the semigroup $(S, *)$ constructed by Definition 4.1.9.

Proof. For any element (x, i) in S, let $(x, i)'$ be (x', i) where x' is the inverse of x within $\mathcal{H}(x)$ by the property of CSSs that every $\mathcal{H}$-calss is a group. It can be verified that $(-)'$ satisfies the conditions of (2) of Proposition 4.1.6, and so it is a CSS. □

Example 4.1.12 Let $\alpha : I \to \mathsf{Smg}$ be a functor, where I is a filtered category (a partial ordered set is of course filtered). Let S be the set $\bigsqcup_{i \in I} \alpha(i)$. The relation, $(x, i) \sim (y, j) \Leftrightarrow \exists k \in \mathrm{Ob}(I)$ such that $i \to k \leftarrow j$ and $\alpha_{i \to k}(x) = \alpha_{j \to k}(y)$, is an equivalence relation. We denote by $[x, i]$ the equivalence class of (x, i). Define a multiplication on $S/\sim$ as

$$[x, i][y, j] := [\alpha_{i \to k}(x)\alpha_{j \to k}(y), k],$$

and a family of functions $\iota(i) : \alpha(i) \to S/\sim$ that maps x to $[x, i]$. Then, it can be verified that:

(1) the multiplication defined above is well defined, thus $S/\sim$ forms a semigroup;

(2) each $\iota(i)$ is a morphism of semigroups, and the series of these morphisms $\iota = [i \mapsto \iota(i)]$ is a natural transformation between functors α and $\Delta_I(S)$;

(3) $S/\sim$, together with the morphism of functors ι, forms the colimit of α.

Proposition 4.1.13 Let $(S, \cdot, (-)')$ be a CRS, then

(1) $\forall a \in S(\mathcal{J}(a) = \mathcal{J}(a^2))$;

(2) $\forall a, b \in S(\mathcal{J}(ab) = \mathcal{J}(ba))$;

(3) $\mathcal{J}$ is a congruence, thus $(S/\mathcal{J}, *)$ is a semigroup, where $\mathcal{J}(a) * \mathcal{J}(b) := \mathcal{J}(ab)$;

(4) $\mathcal{J}(a) \leq \mathcal{J}(b) \Leftrightarrow \mathcal{J}(a) = \mathcal{J}(a) * \mathcal{J}(b)$, thus $(\mathcal{J}, \leq, \wedge)$ consists a semilattice, where $\mathcal{J}(a) \wedge \mathcal{J}(b) := \inf\{\mathcal{J}(a), \mathcal{J}(b)\} = \mathcal{J}(a) * \mathcal{J}(b)$;

(5) S is a semilattice of CSSs.

Proof. (1) Notice that $a'a^2 = a, aa = a^2$ and that $a^2a' = a, aa = a^2$, clearly $a\mathcal{H}a^2$, thus $a\mathcal{J}a^2$.

(2) For any $a, b \in S$,

$$\mathcal{J}(ab) = \mathcal{J}(abab) \leq \mathcal{J}(ba).$$

Similarly we have $\mathcal{J}(ba) \leq \mathcal{J}(ab)$, so $\mathcal{J}(ab) = \mathcal{J}(ba)$.

(3) If $a\mathcal{J}b$, then $b = xay, a = ubv$ for some x, y, u, v in S^1. If $c \in S$, then

$$\mathcal{J}(ca) = \mathcal{J}(cubv) \leq \mathcal{J}(cub) = \mathcal{J}(ubc) \leq \mathcal{J}(bc) = \mathcal{J}(cb);$$

similarly one can establish that $\mathcal{J}(cb) \leq \mathcal{J}(ca)$. By virtue of previous result we obtain $\mathcal{J}(ac) = \mathcal{J}(bc)$, and so $\mathcal{J}$ is a congruence.

(4) Hitherto we have established that $(S/\mathcal{J}, *)$ is a commutative semigroup, in which every element is idempotent. If $\mathcal{J}(a) = \mathcal{J}(ab)$, then $(a) = (ab) \subset (b)$, and it follows that $\mathcal{J}(a) \leq \mathcal{J}(b)$. Conversely, suppose $\mathcal{J}(a) \leq \mathcal{J}(b)$, there exists $x, y \in S^1$ such that $xby = a$, and so

$$\mathcal{J}(a) = \mathcal{J}(a^2) = \mathcal{J}(axby) \leq \mathcal{J}(axb) = \mathcal{J}(bax) \leq \mathcal{J}(ba) = \mathcal{J}(ab).$$

By $(ab) \subset (a)$ we also have $\mathcal{J}(ab) \leq \mathcal{J}(a)$. Thus, $\mathcal{J}(ab) = \mathcal{J}(a)$.

(5) We shall show that every $\mathcal{J}(p) \in S/\mathcal{J}$ is a CSS. Here we denote by A the set $\mathcal{J}(p)$ for convenience. The property $\mathcal{J}(p) \cdot \mathcal{J}(p) \subset \mathcal{J}(p^2) = \mathcal{J}(p)$ give that A is a semigroup. We assert that $\mathcal{J}^A = A^2$, thus it is simple, and due to it is also CR, it must be CS. For any $a, b \in A$, there exists $x, y, u, v \in S$ such that $xay = b, ubv = a$. There exists idempotents e, f such that $a \in \mathcal{H}(e) \subset A$ and $b \in \mathcal{H}(f) \subset A$ on the grounds of S is CR. Hence

$$fxayf = fbf = b, eubvf = eae = a;$$

it's clear that $\mathcal{J}(fx) \geq \mathcal{J}(fxayf) = \mathcal{J}(b) = A$, and similarly all of yf, eu, ve are in A. The equations above mean that $a\mathcal{J}^A b$, thus $\mathcal{J}^A = A^2$.

To show A is CR, observe that any $a \in A$ lies in a group $\mathcal{H}(a) \subset \mathcal{J}(p)$.

To complete this proof, define the mapping $\mathrm{id} : S/\mathcal{J} \to S/\mathcal{J} : \alpha \mapsto \alpha$, where $(S/\mathcal{J}, \leq, \wedge)$ can be viewed as a semilattice. On the other hand S is the disjoint union $\bigsqcup_{\alpha \in S/\mathcal{J}} \alpha$; and for any $\mathcal{J}$-calsses α and β, $\alpha \cdot \beta \subset \alpha \wedge \beta$; thus, S becomes a semilattice of CSSs. □

Theorem 4.1.14 Suppose S is a semigroup, then the following statements are equivalent:

(1) S is a Clifford semigroup;

(2) S is a semilattice of groups;

(3) S is a strong semilattice of groups;

(4) S is regular, and $\operatorname{idm} S \subset Z(S)$;

(5) S is regular, and $\mathcal{D}^S \cap (\operatorname{idm} S)^2 = \Delta_{\operatorname{idm} S}$.

Proof. (1) ⇒ (2) Let S be a Clifford semigroup, by (5) of Proposition 4.1.13, we first obtain that S is a semilattice of CSSs. Consider a particular CSS S_i. Since every idempotent $e = ee'$, the idempotents in S_i are commutative. By Proposition 4.1.8, every idempotent in S_i is primitive, then for any $e, f \in \operatorname{idm} S_i$,

$$e(ef) = ef(e) = ef, f(ef) = (ef)f = ef,$$

which means $ef \leq e$ and $ef \leq f$. Since both e, f are primitive, $e = ef = f$, that is, S_i contains only one idempotent. According to the property that any $\mathcal{H}$-class of the CSS S_i is a group, which naturally contains an idempotent, so we may deduce that S_i is a group on account of S_i has only one $\mathcal{H}$-class.

(2) ⇒ (3) This statement essentially says that the mapping $\alpha : I \to \mathrm{Ob}(\mathsf{Grp})$ within a semilattice of groups can be used to construct a functor. As will be evident in the proof below, the well-behaved properties of groups make this upgrade straightforward. We denote the element in $\alpha(i)$ by x_i. Observe that $\left(\bigsqcup_{i \in I} \alpha(i), \cdot\right)$ as a semilattice of groups, satisfies $\alpha(i) \cdot \alpha(j) \subset \alpha(i \wedge j)$ and every $(\alpha(i), \cdot)$ is the subsemigroup of $\left(\bigsqcup_{i \in I} \alpha(i), \cdot\right)$.

Suppose $i \to j$, define the mapping $\alpha_{i \to j} : \alpha(i) \to \alpha(j)$ as $x_i \mapsto e_j \cdot x_i$, where e_j is the identity of group $\alpha(j)$, and the remaining verifications proceed in these parts:

- ⋄ for any $i \to j$, $\alpha_{i\to j}$ is a morphism;
- ⋄ for any $i \in \mathrm{Ob}(I)$, $\alpha_{i\to i} = \mathrm{id}_{\alpha(i)}$;
- ⋄ for any $i \to j \to k$ within I, $\alpha_{j\to k} \circ \alpha_{i\to j} = \alpha_{i\to k}$;
- ⋄ for any $i, j \in \mathrm{Ob}(I)$, the multiplication that presupposed on S coincides with the multiplication defined as follows,

$$x_i * y_j = \alpha_{i\to i\wedge j}(x) \cdot \alpha_{j\to i\wedge j}(y).$$

See [1, Theorem 4.2.1] for details.

(3) ⇒ (4) Defining $x_i' := x_i^{-1}$ which lies in the group $\alpha(i)$, makes S a regular semigroup. Its idempotents are the identity elements e_i of the group $\alpha(i)$. For any $x_j \in \alpha(j) \subset S$,

$$e_i \cdot x_j = \alpha_{i\to i\wedge j}(e_i)\alpha_{j\to i\wedge j}(x_j) = e_{i\wedge j}\alpha_{j\to i\wedge j}(x_j) = \alpha_{j\to i\wedge j}(x_j) = x_j \cdot e_i,$$

thus, $\mathrm{idm}\, S \subset Z(S)$.

(4) ⇒ (5) Suppose $e\mathcal{D}f$, by (4) of Proposition 2.3.3 there exists a and its inverse a' such that $a'a = f, aa' = e$. So we have

$$e = e^2 = aa'aa' = afa' = faa' = a'aaa' = a'ae = a'ea = a'aa'a = f^2 = f.$$

(5) ⇒ (1) If a $\mathcal{D}$-calss contains a single idempotent e, we assert that $\mathcal{D}(e)$ is a group. By (3) of Proposition 2.3.2, $\mathcal{D}(e)$ is regular, so for any $a \in \mathcal{D}(e)$ there exists $x \in S^1$ such that $axa = a$. It follows that ax is an idempotent belongs to $\mathcal{R}(a) \subset \mathcal{D}(e)$, which must coincides with e due to the condition that e is the unique idempotent of $\mathcal{D}(e)$; similarly $xa = e$. From $ea = a$ and $ae = a$, we obtain $a\mathcal{H}e$, thus $\mathcal{D}(e) = \mathcal{H}(e)$.

On account of every element of S lies in a group, it is a CRS, and so is a semilattice of CSSs S_i. We assert that every S_i is a group. For every x, y in S_i, by (6) of Proposition 3.3.1, $xy \in \mathcal{R}(x) \cap \mathcal{L}(y)$, and so $x\mathcal{D}y$, thus $S_i \subset \mathcal{D}(y)$, which implies S_i is a group as we've established in (1) ⇒ (2) on the condition of S_i contains a single idempotent.

From (2) ⇒ (3) we now deduce that S is a strong semilattice of groups. Thus, one can define the operation $(-)'$ on S that sends x_i to its inverse x_i' within the group S_i, and it follows easily that for any $x_i, y_j \in S$

$$x_i x_i' y_j y_j' = e_i e_j = e_{i\wedge j} = e_j e_i = y_j y_j' x_i x_i'.$$

□

4.2 Varieties

Definition 4.2.1 An Ω-algebra contains data $(X, \Omega, \mathrm{ar}, \mathrm{ot})$, where X is a set and the set Ω is called operator domain. The arity ar is a function $\Omega \to \mathbb{Z}_{\geq 0}$ and ot is a function sends each $\omega \in \Omega$ to a certain operator $\mathrm{ot}(\omega) : X^{\mathrm{ar}(\omega)} \to X$. In the case that $\mathrm{ar}(\omega) = 0$, $\mathrm{ot}(\omega)$ is set to a constant function.

Particularly, when Ω contains two elements with the arity of 2 and 1 respectively, we call the system $(2, 1)$-algebra. A semigroup is a (2)-algebra.

In this section, if not specified, Ω is viewed as a finite set.

Definition 4.2.2 Given the data (Ω, ar), the category of Ω-algebras Ω-Aj contains the following data:

- ⋄ objects: all Ω-algebras in which every Ω-algebra as a set is a $\mathcal{U}$-small set, where $\mathcal{U}$ is a Grothendieck universe;
- ⋄ $\mathrm{Hom}((X, \Omega, \mathrm{ar}, \mathrm{ot}_X), (Y, \Omega, \mathrm{ar}, \mathrm{ot}_Y))$: function $f : X \to Y$ satisfies for all $\omega \in \Omega$,

$$f(\mathrm{ot}_X\, \omega(x_1, \cdots, x_{\mathrm{ar}\,\omega})) = \mathrm{ot}_Y\, \omega(f(x_1), \cdots, f(x_{\mathrm{ar}\,\omega}));$$

- ⋄ morphisms: the union of all Hom sets;
- ⋄ composition: the composition of functions.

Let I be a discrete $\mathcal{U}$-small category, Aj be a category of Ω-algebras and $\beta : I \to \mathsf{Aj}$ be a functor. The image of β is an Ω-algebra denoted as $(X_i, \Omega, \mathrm{ar}, \mathrm{ot}_i)$. For every $\omega \in \Omega$, define the function $\mathrm{ot}\,\omega : (\prod_i X_i)^{\mathrm{ar}\,\omega} \to \prod_i X_i$ as follows

$$(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{\mathrm{ar}\,\omega}) \mapsto (\mathrm{ot}\,\omega(x_{1,i}, \cdots, x_{\mathrm{ar}\,\omega,i}))_{i \in I}\,,$$

where $\boldsymbol{x}_k = (x_{k,i})_{i \in I}$. A family of projection $p_j : \prod_i X_i \to X_j$ is given by $(x_i)_i \mapsto x_j$. We may deduce that for any Ω-algebra L and the natural transformation $f : \Delta(L) \to \beta$, there exists unique morphism ϕ that makes the following diagram commutes, that is, Aj possesses products.

$$\begin{tikzcd}
\Delta(\prod_i X_i) \arrow[r, "p"] & \beta \\
\Delta(L) \arrow[u, dashed, "\exists!\,\phi"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Definition 4.2.3 For any set X, the *free algebra* of X, denoted as $\mathbf{F}(X)$, is the initial object within the comma category (j_X, U), in which $U : \mathsf{Aj} \to \mathsf{Set}$ is a forgetful functor. In other words, for any Ω-algebra L and the mapping $f : X \to L$, there exists unique *morphism* ϕ that makes the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & U\mathbf{F}(X) \arrow[d, dashed, "\phi"] \\
& UL
\end{tikzcd}$$

If every X has a free algebra, given a mapping $f : X \to Y$, by the definition above, there exist a unique morphism $\mathbf{F}(f)$ such that the diagram below commutes. This implies $\mathbf{F}$ can be made into a functor. In fact, $\mathsf{Set} \underset{U}{\overset{\mathbf{F}}{\rightleftarrows}} \mathsf{Aj}$ is an adjunction pair.

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "f"'] & U\mathbf{F}(X) \arrow[d, dashed, "\mathbf{F}f"] \\
Y \arrow[r, hook] & U\mathbf{F}(Y)
\end{tikzcd}$$

Free algebras exist, and a construction will be given here. Let F_0 be the set $X \sqcup \Omega \sqcup \{(,)\}$. Note that we are still examining this problem within the framework of the ZFC system, and both the elements of Ω and the parentheses "(", ")" can be replaced by some set. Besides this, we denote by $\mathbf{On}$ the class of all ordinals and by $\mathbf{V}$ the class of all sets.

Theorem 4.2.4 (Transfinite Recursion) Let $G : \mathbf{V} \to \mathbf{V}$ be a function, $\theta \in \mathbf{On}$, then there exists unique function $a : \theta \to \mathbf{V}$ such that for any $\alpha < \theta$

$$a(\alpha) = G(a|_\alpha).$$

Let S be the free semigroup $\mathbf{F}_{\mathsf{Smg}}(X \sqcup \Omega \sqcup \{(,)\})$. For any element $\omega \in \Omega$, define the function $H_\omega : S \to S$ as $(u_1, \cdots, u_{\mathrm{ar}\,\omega}) \mapsto \omega((u_1)(u_2)\cdots(u_{\mathrm{ar}\,\omega}))$. The set $\mathbf{F}$ is set to be the smallest set containing F_0 and closed under every H_ω, that is,

$$\mathbf{F} := \{T \subset S : F_0 \subset T \wedge (\forall \omega \in \Omega, \forall \boldsymbol{u} \in T^{\mathrm{ar}\,\omega}(H_\omega(\boldsymbol{u}) \in T))\}\,.$$

Furthermore, from F_0, one can recursively define a mapping $\mathbf{On} \to \mathbf{V}$:

$$\begin{aligned}
&F_0 = F_0 \\
&F_{\alpha+1} := F_\alpha \cup \{H_\omega(\boldsymbol{u}) : \omega \in \Omega \wedge \boldsymbol{u} \in (F_\alpha)^{\mathrm{ar}\,\omega}\} \\
&(\gamma \text{ is a limit ordinal})\ F_\gamma := \bigcup_{\beta<\gamma} F_\beta.
\end{aligned}$$

Lemma 4.2.5 $F_{\aleph_0+1} = F_{\aleph_0} = \mathbf{F}$.

Proof. It can be verified that $F_{\aleph_0}$ satisfies the properties that it contains F_0 and closed under every H_ω. By definition of $\mathbf{F}$, it follows that $\mathbf{F} \subset F_{\aleph_0}$. Conversely, since $F_0 \subset \mathbf{F}$, so $F_1 \subset \mathbf{F}$, and so every $F_\alpha \subset \mathbf{F}$, where $\alpha \leq \aleph_0$.

Clearly $F_{\aleph_0} \subset F_{\aleph_0+1}$, and the proof of reverse containment is straightforward. □

We assert that $\mathbf{F}$, together with the mapping $H : \omega \mapsto H_\omega$, forms an Ω-algebra, and even becomes the free ω-algebra of set X. The embedding mapping $\iota : X \hookrightarrow \mathbf{F}$ is obvious, and to let ϕ be a homomorphism and ensure the diagram

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & \mathbf{F} \arrow[d, dashed, "\phi"] \\ & L \end{tikzcd} \tag{4.1}$$

commutes, we define ϕ recursively as follows. Let $\theta = \aleph_0 + 1$ and $G : \mathbf{V} \to \mathbf{V}$ be the function:

$$\begin{aligned} \mathbf{V} &\xrightarrow{\quad G \quad} \mathbf{V} \\ \varnothing &\longmapsto [\phi_0 : F_0 \to L : x \mapsto f(x)] \\ \{f_\alpha : F_\alpha \to L\}_{\alpha \leq \lambda} &\longmapsto f_{\lambda+1} \\ \text{else} &\longmapsto \chi_0 \notin L, \end{aligned}$$

where $f_{\lambda+1}$ is defined as:

$$u \mapsto \begin{cases} f_\lambda(u) & u \in F_\lambda \\ \operatorname{ot}\omega(f_\lambda(a_1), \cdots, f_\lambda(a_{\operatorname{ar}\omega})) & u = H_\omega(a_1, \cdots, a_{\operatorname{ar}\omega}) \wedge a_i \in F_\lambda \wedge u \notin F_\lambda. \end{cases}$$

Thus, by the transfinite recursion, there exists unique $a : \theta \to \mathbf{V}$, such that sends

$$\begin{aligned} &0 \to [\phi_0 : F_0 \to L], \\ &1 \to [\phi_1 : F_1 \to L], \\ &\vdots \\ &\aleph_0 \to [\phi : F_{\aleph_0} \to L], \end{aligned}$$

in which $\phi|_{F_\alpha} = \phi_\alpha$ for any $\alpha < \aleph_0$. And so it can be verified that ϕ is a morphism.

To prove the uniqueness, suppose ϕ, ψ are morphisms that both make the diagram (4.1) commutes. Let

$$C = \{\alpha < \theta : \phi|_{F_\alpha} = \psi|_{F_\alpha}\},$$

then it follows that

- $0 \in C$:
- $\alpha \in C \Rightarrow \alpha + 1 \in C$;
- if γ is a limit ordinal and for every $\alpha < \gamma$, $\alpha \in C$, then $\gamma \in C$.

By transfinite induction, we may conclude that $C = \theta$.

Definition 4.2.6 The *variety* $\mathcal{V}$ is a subset of $\mathrm{Ob}(\mathsf{Aj})$, satisfies the properties:

- $S \in \mathcal{V}$ and T is the subalgebra of S, then $T \in \mathcal{V}$;
- $S \in \mathcal{V}$ and there exist an epimorphism $f : S \to T$, then $T \in \mathcal{V}$;
- given a small set I, $S_i \in \mathcal{V}$ for any $i \in I$, them $\prod_i S_i \in \mathcal{V}$.

Furthermore, a variety could be naturally made into a full subcategory of Aj.

Definition 4.2.7 Given a set X, the *$\mathcal{V}$-free algebra* $\mathbf{F}_{\mathcal{V}}(X)$ is an object of $\mathcal{V}$ satisfies for any $L \in \operatorname{Ob}\mathcal{V}$ and any mapping $f : X \to L$, there exist a unique morphism ψ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & \mathbf{F}_{\mathcal{V}}(X) \arrow[d, dashed, "\phi"] \\
& L
\end{tikzcd}$$

To construct the $\mathbf{F}_{\mathcal{V}}(X)$, and to facilitate the subsequent descriptions in this section, we define the following notations.

Definition 4.2.8 Let X be a set, then

- $\diamond$ for any $S \in \operatorname{Ob}\mathcal{V}$, $\operatorname{fker}(f, X, S) := \ker\left[\mathbf{F}(X) \xrightarrow{\bar{f}} S\right]$, where $\bar{f}$ is derived from the definition of free algebra;
- $\diamond$ $c_{\mathcal{V}}(X) := \bigcap_{(f,S)\in\mathcal{O}} \operatorname{fker}(f, X, S)$, where $\mathcal{O} = \{(f, S) \in \operatorname{Mor}(\mathsf{Set}) \times \operatorname{Ob}(\mathcal{V}) : f \in \operatorname{Hom}_{\mathsf{Set}}(X, S)\}$;
- $\diamond$ for u, v in $\mathbf{F}(X)$, $\operatorname{ceq}(X, u, v) := \left\{S \in \operatorname{Ob}(\mathcal{V}) : (u, v) \in \bigcap_{f\in\operatorname{Hom}_{\mathsf{Set}}(X,S)} \operatorname{fker}(f, X, S)\right\}$;
- $\diamond$ given a binary relation $R \subset \mathbf{F}(X)^2$, the equational class $\operatorname{ceq}(X, R) := \bigcap_{(u,v)\in R} \operatorname{ceq}(X, u, v)$.

Now we may observe that for any $S \in \operatorname{Ob}\mathcal{V}$ and mapping $f : X \to S$, there exist unique morphism $\hat{f} : \mathbf{F}(X)/c_{\mathcal{V}}(X)$, due to the commutative diagram below.

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, "f"'] & \mathbf{F}(X) \arrow[r, two heads] \arrow[d, dashed, "\exists!\,\bar{f}"] & \mathbf{F}(X)/c_{\mathcal{V}}(X) \arrow[dl, dashed, "\exists!"', "\hat{f}"] \\
& S &
\end{tikzcd}$$

And the issue lies in proving that $\mathbf{F}(X)/c_{\mathcal{V}}(X) \in \operatorname{Ob}\mathcal{V}$.

Notice that every $\mathbf{F}(X)/\operatorname{fker}(f, X, S) \simeq \operatorname{im}\bar{f} \subset S$ is a subalgebra, and so is an element of $\mathcal{V}$. Define a morphism Θ

$$\begin{aligned}
\mathbf{F}(X) &\longrightarrow \textstyle\prod_{(f,S)\in\mathcal{O}} \mathbf{F}(X)/\operatorname{fker}(f, X, S) \\
u &\longmapsto [(f, S) \mapsto \operatorname{fker}(f, X, S)(u)]
\end{aligned}$$

where $\mathcal{O} = \{(f, S) \in \operatorname{Mor}(\mathsf{Set}) \times \operatorname{Ob}(\mathcal{V}) : f \in \operatorname{Hom}_{\mathsf{Set}}(X, S)\}$. Then,

$$\Theta(u) = \Theta(v) \Leftrightarrow c_{\mathcal{V}}(X)(u) = c_{\mathcal{V}}(X)(v),$$

and it follows that

$$\mathbf{F}(X)/c_{\mathcal{V}}(X) \simeq \operatorname{im}\Theta \subset \prod_{(f,S)\in\mathcal{O}} \mathbf{F}(X)/\operatorname{fker}(f, X, S) \in \operatorname{Ob}\mathcal{V},$$

which implies $\mathbf{F}(X)/c_{\mathcal{V}}(X) \in \operatorname{Ob}\mathcal{V}$. Hence, $X \hookrightarrow \mathbf{F}(X) \twoheadrightarrow \mathbf{F}(X)/c_{\mathcal{V}}(X)$ consists the $\mathcal{V}$-free algebra of X.

Theorem 4.2.9 As a set, $\mathcal{V}$ is a variety if and on if it is an equational class $\operatorname{ceq}(X, R)$.

Proof. Given a set X and a relation $R \subset \mathbf{F}(X)^2$, we assert that $\operatorname{ceq}(X, R)$ is a variety. The proof proceeds in three parts:

(1) for any $S \in \operatorname{ceq}(X, R)$ and subalgebra $T \subset S$, $T \in \operatorname{ceq}(X, R)$, since for any $f \to T$ the following diagram commutes, and so $\forall (u, v) \in R(\bar{f}(u) = \bar{f}(v))$.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & \mathbf{F}(X) \arrow[d, dashed, "\bar{f}"] \arrow[dr, dashed, "\bar{f}"] & \\ & T \arrow[r, hook, "\subset"] & S \end{tikzcd}$$

(2) For any $S \in \operatorname{ceq}(X, R)$ and the epimorphism $\pi : S \to T$, $T \in \operatorname{ceq}(X, R)$. For any $f : S \to T$, let $g \in \prod_{x \in X} \pi^{-1} f(x)$, then $X \xrightarrow{g} S \xrightarrow{\pi} T = f$. By the commutativity of the diagram below, $\bar{f}(u) = \bar{f}(v)$ for any $(u, v) \in R$.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"] \arrow[ddr, "\exists g"'] & \mathbf{F}(X) \arrow[d, dashed, "\bar{f}"] \arrow[dd, dashed, bend left, "\bar{g}"] \\ & T \\ & S \arrow[u, two heads, "\pi"'] \end{tikzcd}$$

(3) Suppose I is small and $S_i \in \operatorname{ceq}(X, R)$, then so does $\prod_{i \in I} S_i$. This is from the following commutative diagram, in which p_i is the projection, which implies that

$$\forall j \in I(p_j \bar{f}(u) = p_j \bar{f}(v)) \Leftrightarrow \bar{f}(u) = \bar{f}(v)$$

for every $(u, v) \in R$.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"] \arrow[ddr] & \mathbf{F}(X) \arrow[d, dashed, "\bar{f}"] \arrow[dd, dashed, bend left, "p_j \bar{f}"] \\ & \prod_{i \in I} S_i \arrow[d, "p_j"] \\ & S_i \end{tikzcd}$$

Conversely, given a variety $\mathcal{V}$ and a countably finite set X, we assert that $\mathcal{V} = \operatorname{ceq}(X, c_{\mathcal{V}}(X))$. The containment $\mathcal{V} \subset \operatorname{ceq}(X, c_{\mathcal{V}}(X))$ is easy to verify. For the reverse, let $S \in \operatorname{ceq}(X, c_{\mathcal{V}}(X))$. Choose an infinite set Y that $|Y| \geq |X|$ and $|Y| \geq |S|$. We shall show that S is the image of morphism $[\mathbf{F}_{\mathcal{V}}(Y) \twoheadrightarrow S]$, and so $S \in \mathcal{V}$.

Suppose $(u, v) \in c_{\mathcal{V}}(Y)$, then $u, v \in \mathbf{F}(Y)$, each of them can be viewed as a series of finite length that every coordinate belongs to set $Y \sqcap \Omega \sqcup \{(,)\}$ respectively, and so could it be viewed as a function which has a finite domain contained in $\mathbb{Z}_{\geq 1}$ and a finite image contained in $Y \sqcap \Omega \sqcup \{(,)\}$. Let $Y_0 = (\operatorname{im} u \cup \operatorname{im} v) \cap Y$, clearly it is a finite set. Let $X_0 \subset X$, and we may find a function $\xi_0 : X \hookrightarrow Y$ that satisfies $\xi_0(X_0) = Y_0$.$\square$

Bibliography

[1] John M Howie. *Fundamentals of Semigroup Theory*. Oxford University Press, 1995.