Complete Proofs in Methods in Algebra

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1 Yoneda Lemma

定义函子:

$$\begin{aligned} \mathcal{C} &\xrightarrow{h_{\mathcal{C}}} \mathcal{C}^{\wedge} = \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}) \\ S &\longmapsto \mathrm{Hom}(\cdot, S) \end{aligned} \qquad \begin{aligned} \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\wedge} &\xrightarrow{\mathrm{ev}^{\wedge}} \mathsf{Set} \\ (S, A) &\longmapsto A(S) \end{aligned}$$

$$\begin{aligned} \mathcal{C} &\xrightarrow{k_{\mathcal{C}}} \mathcal{C}^{\vee} = \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}^{\mathrm{op}}) \\ S &\longmapsto \mathrm{Hom}(S, \cdot) \end{aligned} \qquad \begin{aligned} (\mathcal{C}^{\vee})^{\mathrm{op}} \times \mathcal{C} &\xrightarrow{\mathrm{ev}^{\vee}} \mathsf{Set} \\ (A, S) &\longmapsto A(S) \end{aligned}$$

Yoneda Lemma 说的是存在函子的同构 $\mathrm{Hom}(h_{\mathcal{C}}(\cdot), \cdot) \xrightarrow{\sim} \mathrm{ev}^{\wedge}(\cdot, \cdot)$, $\mathrm{Hom}(\cdot, k_{\mathcal{C}}(\cdot)) \xrightarrow{\sim} \mathrm{ev}^{\vee}(\cdot, \cdot)$. 以前者为例, 这一同构由下式给出:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}^{\wedge}}(h_{\mathcal{C}}(S), A) &\xrightarrow[\sigma_{S,A}]{1:1} A(S) \\ \phi &\longmapsto \phi_S(\mathrm{id}_S) \\ (\psi_T : m \mapsto A(m)(u_s))_{T \in \mathrm{Ob}(\mathcal{C})} &\longleftarrow u_S \end{aligned}$$

所以我们需要验证:

A. $\sigma = (\sigma_{S,A})_{(S,A)}$ 构成一个函子间的态射 $\mathrm{Hom}(h_{\mathcal{C}}(\cdot), \cdot) \to \mathrm{ev}^{\wedge}(\cdot, \cdot)$.

B. 每个 $\sigma_{S,A}$ 都是双射, 从而 σ 构成函子间的同构.

(A.) 对任意 $(f, \theta) : (S_1, A_1) \to (S_2, A_2)$, 下图交换:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}^{\wedge}}(h_{\mathcal{C}}(S_1), A_1) \arrow[r, "\sigma_{S_1,A_1}"] \arrow[d, "\theta\circ-"'] & A_1(S_1) \arrow[d, "\theta_{S_1}"] \\
\mathrm{Hom}_{\mathcal{C}^{\wedge}}(h_{\mathcal{C}}(S_1), A_2) \arrow[r, "\sigma_{S_1,A_2}"] \arrow[d, "-\circ h_{\mathcal{C}}(f)"'] & A_2(S_1) \arrow[d, "A_2(f)"] \\
\mathrm{Hom}_{\mathcal{C}^{\wedge}}(h_{\mathcal{C}}(S_2), A_2) \arrow[r, "\sigma_{S_2,A_2}"'] & A_2(S_2)
\end{tikzcd}$$

其中上方矩形的交换性很好验证. 现证下方矩形的交换性. 假设 $\psi \in \mathrm{Hom}_{\mathcal{C}^{\wedge}}(h_{\mathcal{C}}(S_1), A_2)$, 下方矩形要求 $A_2(f)(\psi_{S_1}(\mathrm{id}_{S_1})) = [\psi \circ h_{\mathcal{C}}(f)]_{S_2}(\mathrm{id}_{S_2}) = \psi_{S_2}(f)$, 此式子成立因为下图交换:

$$\begin{tikzcd}
\mathrm{Hom}(S_1, S_1) \arrow[r, "\psi_{S_1}"] \arrow[d, "f\circ-"'] & A_2(S_1) \arrow[d, "A_2(f)"] \\
\mathrm{Hom}(S_2, S_1) \arrow[r, "\psi_{S_2}"'] & A_2(S_2)
\end{tikzcd}$$

(B.) 给定 $u_S \in A(S)$, 我们不禁要问 $(\psi_T : m \mapsto A(m)(u_s))_{T \in \mathrm{Ob}(\mathcal{C})}$ 是否是 $h_{\mathcal{C}}(S)$ 与 A 之间的态射. 对 $\mathcal{C}^{\mathrm{op}}$ 中的任何 $R \underset{\xi}{\to} T$, 可以证明下图交换:

$$\begin{tikzcd}
\mathrm{Hom}(R, S) \arrow[r, "\psi_R"] \arrow[d, "f\circ-"'] & A(R) \arrow[d, "A(f)"] \\
\mathrm{Hom}(T, S) \arrow[r, "\psi_T"'] & A(T)
\end{tikzcd}$$

因此这样定义的 ψ 确实是函子间态射; 另外注意到如此定义的 ψ 满足 $\psi_S(\mathrm{id}_S) = u_S$. 再者, 对任意 $\phi : h_{\mathcal{C}}(S) \to A$, 均有 $\phi_T(m) = A(m)(u_S)$; 从而 $\sigma_{S,A}$ 是双射.

2 Limits

Page 59 in the book states that when I is a small category and $\mathcal{C}$ is Set, both $\varinjlim$ and $\varprojlim$ exist.

Proposition 2.1 *Let s, t be the start and target map of morphisms $\mathrm{Mor}(I) \to \mathrm{Ob}(I)$. For any $\beta : I^{\mathrm{op}} \to \mathsf{Set}$, the limit $\varprojlim \beta$ exists.*

Proof. Define two maps as follows:

$$\prod_i \beta(i) \underset{g}{\overset{f}{\rightrightarrows}} \prod_\sigma \beta(s(\sigma))$$

$$(x_i)_i \xmapsto{\ f\ } \left[\sigma \mapsto x_{s(\sigma)}\right]$$

$$(x_i)_i \xmapsto{\ g\ } \left[\sigma \mapsto \beta(\sigma)(x_{t(\sigma)})\right]$$

Next, we let:

$$\begin{aligned}\varprojlim \beta &= \ker\left[\prod_i \beta(i) \rightrightarrows \prod_\sigma \beta(s(\sigma))\right] \\ &= \left\{(x_i)_i \in \prod_i \beta(i) : f((x_i)_i) = g((x_i)_i)\right\} \\ &= \left\{(x_i)_i \in \prod_i \beta(i) : \forall \sigma \in \mathrm{Hom}_I(i,j)(\beta(\sigma)(x_j) = x_i)\right\}\end{aligned}$$

Define a family of maps $p = (p_i)_i$, where $p_i((x_j)_j) = x_i$. We now need to verify (i): $p : \Delta(\varprojlim \beta) \to \beta$ is a morphism; (ii): $(\varprojlim \beta, p)$ is indeed the terminal object in (Δ/β). The verification of (i) is easy. As for (ii). For any (X, ξ), to make the diagram commute, the only way is to set $\phi(x) = (\xi_i(x))_i$. This establishes both the existence and uniqueness of $\phi : (X, \xi) \to (\varprojlim \beta, p)$:

$$\begin{tikzcd} X \arrow[d, "\phi"'] \arrow[dr, "\xi_i"] & \\ \varprojlim \beta \arrow[r, "p_i"'] & \beta(i) \end{tikzcd}$$

□

As for the definition of "ker" mentioned above, see the discussion of equalizer on page 63 of this book. Before we prove the existence of colimits, we first explain the equivalence relation generated by a relation.

Definition 2.2 *For any set X, let $R \subset X^2$ be a binary relation. Define*

$$S = \left\{(x, y) \in X^2 : \exists (x_i)_{i=0}^n \in X^{n+1} \wedge \forall 1 \le k \le n((x_{k-1}, x_k) \in R \vee (x_k, x_{k-1}) \in R)\right\}$$

*where $x_0 = x$, $x_n = y$. Then S is the **smallest** equivalence relation containing R.*

Proof. To show that S is an equivalence relation, the symmetry and transitivity are obvious. As for reflexivity, for $n = 0$, the definition of S implies $\forall x\, (x, x) \in S$. To show S is the **smallest** equivalence relation, let S' be any equivalence relation containing R. Suppose that $(x, y) \in S$. By definition, we have:

$$\exists (x_i)_{i=0}^n \in X^{n+1}\, (\forall 1 \le k \le n((x_{k-1}, x_k) \in R \vee (x_k, x_{k-1}) \in R))$$

□

all such pairs (x_{k-1}, x_k) or (x_k, x_{k-1}) belong to S'. Thus, this implies $(x, y) \in S'$.

Proposition 2.3 *For any functor on small category $\alpha : I \to \mathsf{Set}$, the colimit $\varinjlim \alpha$ exists.*

Proof. Let $X = \bigsqcup_i \alpha(i)$. The notation $a \to b$ means there exists a σ such that $b = \alpha(\sigma)(a)$. All such pairs $\{(x, y) : x \to y\}$ forms a relation $R \subset X^2$. Define $\varinjlim \alpha = X/\sim$, where $\sim$ is the equivalence relation generated by R. Let $\iota = (\iota_i)_i$ where each ι_i is a map $\alpha(i) \to X/\sim$ sending $x \mapsto [x]_\sim$. It is straightforward to verify that $\alpha \xrightarrow{\iota} \varinjlim \alpha$ is a morphism. For any object (L, ξ) in (α, Δ), to ensure the commutativity of the diagram, define the map $\phi : \varinjlim \alpha \to L$ that sending $[x_i]_\sim$ to $\xi_i(x_i)$.

$$\begin{tikzcd} \alpha(i) \arrow[r, "\iota_i"] \arrow[dr, "\xi_i"'] & \varinjlim \alpha \arrow[d, "\phi"] \\ & L \end{tikzcd}$$

We now need to verify that ϕ is well-defined. Suppose $x_i \sim x_j$, we must show $\xi_i(x_i) = \xi_j(x_j)$. By definition, there exists a sequence $(t_i)_{i=0}^n$ such that a diagram of the following form holds:

$$x_i = t_0 \to t_1 \leftarrow t_2 \leftarrow t_3 \to \cdots \leftarrow t_n = x_j$$

where the direction of arrows can be arbitrary. This reduces to iterating the same pattern, so it suffices to show that $t_0 \to t_1$ ensures $\xi_0(t_0) = \xi_1(t_1)$. This follows from the $\xi : \alpha \to \Delta(L)$ is a natural transformation:

$$\begin{tikzcd} \alpha(i_0) \arrow[r, "\alpha(\sigma)"] \arrow[dr, "\xi_0"'] & \alpha(i_1) \arrow[d, "\xi_1"] \\ & L \end{tikzcd}$$

□

需要注意一点, 在书图表 (2.13) 下方讲 Set 中的等化子时, 虽然是上面命题的退化情况, 却采用了略为不同的构造. 接下来我们来看书中命题 2.7.8.

Proposition 2.4 *I 是小范畴时, $\alpha : I \to \mathcal{C}^\wedge$ or $\mathcal{C}^\vee$ 的余极限 $\varinjlim \alpha$ 存在. 同样 $\beta : I^{\mathrm{op}} \to \mathcal{C}^\wedge$ or $\mathcal{C}^\vee$ 的极限 $\varprojlim \beta$ 存在.*

Proof. 下面以 $\alpha : I \to \mathcal{C}^\wedge$ 为例. 此时若任意给出 $i \xrightarrow[I]{\sigma} j$ 和 $S \xrightarrow[\mathcal{C}^{\mathrm{op}}]{f} T$, 显然 $\alpha(i)(S), \alpha(\sigma)(f)$ 均是有定义的. 此外, 类似 $\alpha(i)(f), \alpha(\sigma)(\cdot)$ 等表示的定义也是自明的.

对任意 $S \in \mathrm{Ob}(\mathcal{C})$, 定义函子 $\alpha(\cdot)(S) : I \to \mathsf{Set}$, 其将 $i \in \mathrm{Ob}(I)$ 映射至 $\alpha(i)(S)$. 应用之前的命题可知对每个 S 都存在 $\varinjlim \alpha(\cdot)(S)$ 与相应的态射族 $(\iota_{i,S})_i$:

$$\alpha(\cdot)(S) \xrightarrow{(\iota_{i,S})_i} \Delta(\varinjlim \alpha(\cdot)(S))$$

定义 $\mathcal{C}^\wedge$ 中的元素 $\varinjlim \alpha$ 如下:

$$\begin{aligned} \mathcal{C}^{\mathrm{op}} &\longrightarrow \mathsf{Set} \\ S &\longmapsto \varinjlim \alpha(\cdot)(S) \\ \left[f : S \xrightarrow[\mathcal{C}^{\mathrm{op}}]{} T \right] &\longmapsto \varinjlim \alpha(\cdot)(f) \end{aligned}$$

我们断言下面资料构成了 (α/Δ) 中的初对象, 从而 $\varinjlim \alpha$ 就是 α 的余极限.

$$\alpha \xrightarrow{\iota = ((\iota_{i,S})_S)_i} \Delta(\varinjlim \alpha)$$

接下来的验证分为三个部分:

A. ι 是函子间的态射.

B. 对任意 $(L, \xi) \in \mathrm{Ob}(\alpha/\Delta)$, 存在唯一的 $\phi : \varinjlim \alpha \to L$.

C. ϕ 是函子间的态射.

(A.) 归结于证明 $\alpha(i)(\cdot) \xrightarrow{(\iota_{i,S})_S} \varinjlim \alpha$ 是态射; 以及下图交换. 证明是显见的.

$$\begin{tikzcd}
\alpha(i) \arrow[r, "(\iota_{i,S})_S"] \arrow[d, "\alpha(\sigma)"'] & \varinjlim \alpha \\
\alpha(j) \arrow[ur, "(\iota_{j,S})_S"'] &
\end{tikzcd}$$

(B.) 给出对象 (L, ξ), 对任意 $S \in \mathrm{Ob}(\mathcal{C})$:

$$\begin{tikzcd}
\alpha(\cdot)(S) \arrow[r, "(\iota_{i,S})_i"] \arrow[dr, "(\xi_{i,S})_i"'] & \Delta(\varinjlim \alpha(\cdot)(S)) \arrow[d, "\exists!\Delta(\phi_S)"] \\
& \Delta(L(S))
\end{tikzcd} \quad \Rightarrow \quad \varinjlim \alpha \xrightarrow[\mathcal{C}^\wedge]{\exists!(\phi_S)_S} L$$

(C.) 我们的目标是验证对任意 $f : S \xrightarrow[\mathcal{C}^{\mathrm{op}}]{} T$, 下图交换:

$$\begin{tikzcd}
\varinjlim \alpha(\cdot)(S) \arrow[r, "\phi_S"] \arrow[d, "\varinjlim \alpha(\cdot)(f)"'] & L(S) \arrow[d, "L(f)"] \\
\varinjlim \alpha(\cdot)(T) \arrow[r, "\phi_T"'] & L(T)
\end{tikzcd}$$

证明又分为三步, 分别为:

(C1.) 下图交换:

$$\begin{tikzcd}
\alpha(\cdot)(S) \arrow[r, "(\xi_{i,S})_i"] \arrow[d, "\alpha(\cdot)(f)"'] & \Delta(L(S)) \arrow[d, "\Delta(L(f))"] \\
\alpha(\cdot)(T) \arrow[r, "(\xi_{i,T})_i"'] & \Delta(L(T))
\end{tikzcd}$$

这是 $\alpha \xrightarrow{\xi} \Delta(L)$ 的结果.

(C2.) $\varinjlim \Delta(L(S)) = L(S)$, $\varinjlim \Delta(L(f)) = L(f)$. 证明是容易的.

(C3.) $\varinjlim[(\xi_{i,S})_i] = \phi_S$. 这来源于下图:

$$\begin{tikzcd}
\alpha(\cdot)(S) \arrow[r, "(\iota_{i,S})_i"] \arrow[d, "(\xi_{i,S})_i"'] & \Delta(\varinjlim \alpha(\cdot)(S)) \arrow[d, "\Delta(\phi_S)"] \\
\Delta(L(S)) \arrow[r, "\mathrm{id}"'] & \Delta(\varinjlim \Delta(L(S)))
\end{tikzcd}$$

最后再在 C1. 的交换图中调用书 Lemma 2.7.4. □

接下来是一个比较困难的证明, 书中对此的说明仅寥寥数笔, 对于初学者是极其难懂的. 好在其他的范畴论入门教材基本都讲到了这一命题.

Proposition 2.5 对于 $\beta : I^{\mathrm{op}} \to \mathcal{C}$, 可定义 $\bar{\beta} : I^{\mathrm{op}} \to \mathcal{C}^\wedge$, 映 $i \mapsto h_{\mathcal{C}}(\beta(i)) = \mathrm{Hom}_{\mathcal{C}}(\cdot, \beta(i))$. 则 $\exists \varprojlim \beta$ 当且仅当 $\varprojlim \bar{\beta}$ 可表.

对于 $\alpha : I \to \mathcal{C}$, 可定义 $\bar{\alpha} : I \to \mathcal{C}^\vee$, 映 $i \mapsto k_{\mathcal{C}}(\alpha(i)) = \mathrm{Hom}_{\mathcal{C}}(\alpha(i), \cdot)$. 则 $\exists \varinjlim \alpha$ 当且仅当 $\varinjlim \bar{\alpha}$ 可表.

Proof. 我们以 $\beta : I^{\mathrm{op}} \to \mathcal{C}$ 为例. 证明分为几个部分:

A. $\exists \varprojlim \beta \Rightarrow h_{\mathcal{C}}(\varprojlim \beta) \xrightarrow[\exists]{\sim} \varprojlim \bar{\beta}$.

A1. $\forall S \in \mathrm{Ob}(\mathcal{C})$ $\mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta) \simeq \varprojlim \bar{\beta}(\cdot)(S)$.

A2. $\forall S \in \mathrm{Ob}(\mathcal{C})$ $\mathrm{Hom}_{\mathcal{C}}(S, \varprojlim \beta) \simeq \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta)$.

B. $h_{\mathcal{C}}(D) \xrightarrow[\exists]{\sim} \varprojlim \bar{\beta} \Rightarrow \exists \varprojlim \beta$.

(A1.)

$$\begin{aligned}\varprojlim \bar{\beta}(\cdot)(S) &= \left\{(\xi_i)_i \in \prod_I \bar{\beta}(i)(S) : \forall \sigma : i \xrightarrow{I} j(\xi_j \circ \beta(\sigma) = \xi_i)\right\} \\ &= \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta)\end{aligned}$$

(A2.) 令 $\mathcal{C}^\wedge \ni A = \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(\cdot), \beta)$, 回顾 Yoneda 定理, 对于任意 S, 我们有:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{Hom}(\cdot, S), \mathrm{Hom}(\Delta(\cdot), \beta)) & \xleftrightarrow{1:1} & \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta) \\ \phi & \longmapsto & \phi_S(\mathrm{id}_S) \\ (\phi_T : f \mapsto u \circ \Delta(f))_{T \in \mathrm{Ob}(\mathcal{C})} & \longleftarrow\!\shortmid & u \end{array} \tag{2.1}$$

这是个极其实用的公式. 我们断言: $\Delta(S) \xrightarrow{u} \beta$ 是 (Δ/β) 中的终对象时 (即 $S = \varprojlim \beta$), u 的像 ϕ 是同构. 反之, 若 ϕ 是同构, 则 S 连同 $\phi_S(\mathrm{id}_S)$ 构成终对象. 后者的证明可见 (B.); 对于前者, 我们取 $S = \varprojlim \beta$, 然后验证对任意 S, ϕ_S 都是双射:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(S, \varprojlim \beta) & \xleftrightarrow{1:1} & \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta) \\ f & \xmapsto[\phi_S]{} & u \circ \Delta(f) \\ \varprojlim \lambda & \xleftarrow[\psi_S]{} & \lambda \end{array}$$

$\psi_S \phi_S$ 是单位映射, 原因来自下图:

$$\begin{tikzcd} \Delta(S) \arrow[r, "\Delta(f)"] \arrow[d, dashed, "\varprojlim"'] & \Delta(\varprojlim \beta) \arrow[r, "u"] \arrow[d, dashed, "\varprojlim"] & \beta \arrow[d, dashed, "\varprojlim"] \\ S \arrow[r, "f"'] & \varprojlim \beta \arrow[r, "\mathrm{id}"'] & \varprojlim \beta \end{tikzcd}$$

$\phi_S \psi_S$ 也是单位映射, 即 $u \circ (\varprojlim \lambda) = \lambda$, 证明很容易.

(B.) 假设存在 D 使得 $h_{\mathcal{C}}(D) \xrightarrow[\phi]{\sim} \varprojlim \bar{\beta} = \mathrm{Hom}(\Delta(\cdot), \beta)$. 通过 (2.1) 式将 ϕ 映为 $u = \phi_D(\mathrm{id}_D)$. 对于任意 S, 观察到映射:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(S, D) & \xleftrightarrow{1:1} & \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta) \\ f & \xmapsto[\phi_S]{} & u \circ \Delta(f) \\ \psi_S(\lambda) & \xleftarrow[\psi_S]{} & \lambda \end{array}$$

其中 ψ_S 为 ϕ_S 的逆. 断言 $\Delta(D) \xrightarrow{u}$ 给出 (Δ/β) 中的一个终对象. 对于任意 (S, λ), 上面映射给出了唯一一个 $\psi_S(\lambda) : S \to D$. 同时由于 $u \circ \Delta(\psi_S(\lambda)) = \phi_S \psi_S(\lambda) = \lambda$, 从而下图交换:

$$\begin{tikzcd} \Delta(D) \arrow[r, "u"] & \beta \\ \Delta(S) \arrow[u, "\Delta(\psi_S(\lambda))"] \arrow[ur, "\lambda"'] & \end{tikzcd}$$

□

下面简单补充一下 $\alpha : I \to \mathcal{C}$ 的情况. 定义 $\bar{\alpha} : I \to \mathcal{C}^\vee$, 映 i 到 $\mathrm{Hom}_{\mathcal{C}}(\alpha(i), \cdot)$. 从而 $\bar{\alpha}(\cdot)(S) : I \to \mathsf{Set}^{\mathrm{op}}$, 这等同于 $I^{\mathrm{op}} \to \mathsf{Set}$. 然后就可以验证:

$$\varprojlim \bar{\alpha}(\cdot)(S) = \mathrm{Hom}_{\mathcal{C}^I}(\alpha, \Delta(S))$$

$$\mathrm{Hom}_{\mathcal{C}^\vee}(\mathrm{Hom}(\alpha, \Delta(\cdot)), \mathrm{Hom}(S, \cdot)) \simeq \mathrm{Hom}_{\mathcal{C}^I}(\alpha, \Delta(S))$$

接下来是几个比较短的小命题, 书中的证明要么没有, 要么过于简略. 故在此处补充上.

Proposition 2.6 设 I, J 为小范畴, $\alpha : I \to \mathcal{C}^J$, 若对任意 $j \in J$, 存在 $\varinjlim_i \alpha(\cdot)(j)$, 则存在 $\varinjlim_i \alpha \in \mathcal{C}^J$.

Proof. 令

$$\alpha(i)(j) \xrightarrow{\iota_{i,j}} \varinjlim_i \alpha(\cdot)(j)$$

再定义 $\mathcal{C}^J$ 中元素 $\varinjlim_i \alpha$ 如下:

$$j \longmapsto \varinjlim_i \alpha(\cdot)(j)$$

$$[\sigma : j \to j'] \longmapsto \varinjlim_i \alpha(\cdot)(\sigma)$$

可以验证 $\alpha \xrightarrow{((\iota_{i,j})_j)_i} \Delta_I \left(\varinjlim_i \alpha\right)$ 成为 (α/Δ) 中的初对象, 从而 $\varinjlim_i \alpha$ 正是 α 极限. 证明过程和 Proposition 2.4 非常类似. $\square$

Proposition 2.7 设 I, J 为小范畴, 假设 $\mathcal{C}$ 中具有以 I, J 为指标的 $\varinjlim$ (即对任意 $\alpha : I \to \mathcal{C}$, 都存在 $\varinjlim \alpha$, J 上同理). 考虑函子 $I \times J \to \mathcal{C}$, 存在典范同构

$$\varinjlim_j \left(\varinjlim \alpha(\cdot, j)\right) \simeq \varinjlim \alpha \simeq \varinjlim_i \left(\varinjlim \alpha(i, \cdot)\right)$$

Proof. 对任意 j, $\alpha(\cdot, j)$ 定义为函子 $i \mapsto \alpha(i, j)$; 根据条件, 对任意 j 都存在 $\varinjlim \alpha(\cdot, j) \in \mathcal{C}$. 结合上一命题, I 到 $\mathcal{C}^J$ 的函子 $[i \to \alpha(i, \cdot)]$ 存在一个极限, 可用 $\left[j \mapsto \varinjlim \alpha(\cdot, j)\right]$ 表示, 具体来说其是函子:

$$j \longmapsto \varinjlim \alpha(\cdot, j)$$

$$[\sigma : j \to j'] \longmapsto \left[\varinjlim \alpha(\cdot, \sigma) : \varinjlim \alpha(\cdot, j) \to \varinjlim \alpha(\cdot, j')\right]$$

因此 $\varinjlim_j \left(\varinjlim \alpha(\cdot, j)\right)$ 是有定义的. 根据 Proposition 2.5, 只需证明 $\bar{\alpha}$ 是可表的, 这从下方可以看出:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}\left(S, \varinjlim_j \left(\varinjlim \alpha(\cdot, j)\right)\right) &\simeq \mathrm{Hom}_{\mathcal{C}^J}\left(\Delta_J(S), \left[j \mapsto \varinjlim \alpha(\cdot, j)\right]\right) \\
&\simeq \mathrm{Hom}_{(\mathcal{C}^J)^I}\left(\Delta_I(\Delta_J(S)), [i \mapsto \alpha(i, \cdot)]\right) \\
&\simeq \mathrm{Hom}_{\mathcal{C}^{I \times J}}\left(\Delta_{I \times J}(S), \alpha\right) \\
&= \varinjlim \bar{\alpha}(S)
\end{aligned}$$

$\square$

Proposition 2.8 设有小集合的无交并分解 $I = \bigsqcup_{j \in J} I_j$. 若范畴 $\mathcal{C}$ 具有以 J 和 I_j 的积, 则以 I 为指标的积也存在. 且存在唯一同构使下图交换:

$$\begin{tikzcd}
\prod_{i \in I} X_i \arrow[rr, "\sim"] \arrow[dr, "p_k"'] & & \prod_{j \in J} \left(\prod_{i \in I_j} X_i\right) \arrow[dl, "p_k p_j"] \\
& X_k &
\end{tikzcd}$$

同样的断言对余积也成立.

Proof. 由于 J, I_j, I 都是小集合, 自然可以将其视为离散小范畴. 我们先给出几个函子的符号:

1. $\beta^{I_j} : I_j^{\mathrm{op}} \to \mathcal{C}$, 给出 $(X_\mu)_{\mu \in I_j}$;
2. $\beta^I : I^{\mathrm{op}} \to \mathcal{C}$, 给出 $(X_\mu)_{\mu \in I}$;
3. $\varprojlim \beta^{I_{(\cdot)}} : J \to C$, 其映 j 为 $\varprojlim \beta^{I_j} = \prod_{\mu \in I_j} X_\mu$.

此处我们再一次调用 (2.1), 证明的大体框架就是:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}\left(S, \varprojlim_j \left(\varprojlim \beta^{I_{(\cdot)}}\right)\right) &\simeq \mathrm{Hom}_{\mathcal{C}^J}\left(\Delta_J(S), \varprojlim \beta^{I_{(\cdot)}}\right)\\ &\simeq \prod_J \mathrm{Hom}_{\mathcal{C}}\left(S, \varprojlim \beta^{I_j}\right) \\ &\simeq \prod_J \left[\mathrm{Hom}_{\mathcal{C}^{I_j}}\left(\Delta_{I_j}(S), \beta^{I_j}\right)\right] \\ &\simeq \prod_J\left[\prod_{\mu\in I_j}\mathrm{Hom}_{\mathcal{C}}\left(S,\beta^{I_j}(\mu)\right)\right]\\ &\simeq \prod_{\mu\in I}\mathrm{Hom}_{\mathcal{C}}\left(S,\beta^I(\mu)\right) \\ &\simeq \mathrm{Hom}_{\mathcal{C}^I}\left(\Delta_I(S),\beta^I\right)\end{aligned}$$

依据 Proposition 2.5, 可以说 $\varprojlim_j \left(\varprojlim \beta^{I_{(\cdot)}}\right)$, 或者表示为 $\prod_{j\in J}\left(\prod_{i\in I_j} X_i\right)$, 就是 β^I 的极限. 而两个极限之间存在唯一同构. □

书中的引理 2.7.13 证明思路是类似的. 接下来我们将完成一个流程比较长的证明, 对应书的定理 2.8.3.

Theorem 2.9 *设 I 为小范畴, $\mathcal{C}$ 为范畴.*

1. *若对所有子集 $J \subset \mathrm{Mor}(I)$ 和 $\mathcal{C}$ 中对象族 $(X_j)_{j\in J}$ 都存在 $\prod_J X_j$; 而且对所有 $f, g : X \to Y$ 都存在 $\ker(f, g)$; 则 $\mathcal{C}$ 有所有以 I 为指标的 $\varprojlim$.*
2. *若对所有子集 $J \subset \mathrm{Mor}(I)$ 和 $\mathcal{C}$ 中对象族 $(X_j)_{j\in J}$ 都存在 $\coprod_J X_j$; 而且对所有 $f, g : X \to Y$ 都存在 $\mathrm{coker}(f, g)$; 则 $\mathcal{C}$ 有所有以 I 为指标的 $\varinjlim$.*

Proof. 在一切开始之前我们先声明要使用的符号. $\mathrm{Ob}(I)$ 和 $\mathrm{Mor}(I)$ 作为小集合, 自然可视作离散小范畴. 如果给定了 $\beta : I^{\mathrm{op}} \to \mathcal{C}$, 其中 I 由资料 $(\mathrm{Ob}(I), \mathrm{Mor}(I), s, t, \circ)$ 表示, 就可以定义如下两个离散范畴上的函子, 并且根据条件可得出它们的积:

$$\begin{aligned}\beta_{\mathrm{Ob}} : \mathrm{Ob}(I) &\longrightarrow \mathcal{C}\\ i &\longmapsto \beta(i)\end{aligned} \qquad \Delta_{\mathrm{Ob}(I)}\left(\prod \beta(i)\right) \xrightarrow{p} \beta_{\mathrm{Ob}}$$

$$\begin{aligned}\beta_{\mathrm{Mor}} : \mathrm{Mor}(I) &\longrightarrow \mathcal{C}\\ \sigma &\longmapsto \beta(s(\sigma))\end{aligned} \qquad \Delta_{\mathrm{Mor}(I)}\left(\prod \beta(s(\sigma))\right) \xrightarrow{q} \beta_{\mathrm{Mor}}$$

然后, 可以定义两个态射:

$$\begin{tikzcd} \prod_i \beta(i) \arrow[r, shift left, "M(\sigma)=p(s(\sigma))"] \arrow[r, shift right, "N(\sigma)=\beta(\sigma)\circ p(t(\sigma))"'] & \beta(s(\sigma)) \end{tikzcd}$$

将其放置到逗号范畴 $(\Delta/\beta_{\mathrm{Mor}})$ 中, 将会分别导出两个态射 $f = \varprojlim M, g = \varprojlim N$:

$$\begin{tikzcd} \Delta_{\mathrm{Mor}(I)}\left(\prod_\sigma \beta(s(\sigma))\right) \arrow[r, "q"] & \beta_{\mathrm{Mor}} \\ \Delta_{\mathrm{Mor}(I)}\left(\prod_i \beta(i)\right) \arrow[u, shift left, "\Delta(f)"] \arrow[u, shift right, "\Delta(g)"'] \arrow[ur, shift left, "M"] \arrow[ur, shift right, "N"'] & \end{tikzcd}$$

然后我们再定义一个范畴 I_0 为 $\bullet \rightrightarrows \bullet$. 函子 $\beta_0 : I_0^{\mathrm{op}} \to \mathcal{C}$ 给出 $\prod_i \beta(i) \overset{f}{\underset{g}{\rightrightarrows}} \prod_\sigma \beta(s(\sigma))$.

剩下的证明为:

A. $(\forall\sigma(q(\sigma)\circ\mu = q(\sigma)\circ\nu)) \Rightarrow \mu = \nu$;

B. $\forall L \in \mathrm{Ob}(\mathcal{C})\ \varprojlim \mathrm{Hom}(L, \beta(\cdot)) \simeq \mathrm{Hom}(L, \ker(f,g))$.

  B1. $\varprojlim \mathrm{Hom}(L, \beta(\cdot)) \simeq \mathrm{Hom}(\Delta_I(L), \beta)$;
  B2. $\mathrm{Hom}(\Delta_I(L), \beta) \simeq \{\psi \in \mathrm{Hom}(L, \prod_i \beta(i)) : f\psi = g\psi\}$;
  B3. $\{\psi \in \mathrm{Hom}(L, \prod_i \beta(i)) : f\psi = g\psi\} \simeq \mathrm{Hom}_{\mathcal{C}^{I_0}}(\Delta_{I_0}(L), \beta_0)$;
  B4. $\mathrm{Hom}_{\mathcal{C}^{I_0}}(\Delta_{I_0}(L), \beta_0) \simeq \mathrm{Hom}(L, \ker(f,g))$.

(A.) 对任意 $L \overset{\mu}{\underset{\nu}{\rightrightarrows}} \prod_\sigma \beta(s(\sigma))$，若满足 $\forall\sigma(q(\sigma)\circ\mu = q(\sigma)\circ\nu)$, 从下图可知 $q\circ\Delta_{\mathrm{Mor}(I)}(\mu)$ 与 $q\circ\Delta_{\mathrm{Mor}(I)}(\nu)$ 合成同一个态射, 由终对象定义可知 $\mu = \nu$:

$$\begin{tikzcd}
\Delta_{\mathrm{Mor}(I)}\left(\prod_\sigma \beta(s(\sigma))\right) \arrow[r, "q"] & \beta_{\mathrm{Mor}} \\
\Delta_{\mathrm{Mor}(I)}(L) \arrow[u, shift left, "\Delta(\mu)"'] \arrow[u, shift right, "\Delta(\nu)"] \arrow[ur] &
\end{tikzcd}$$

(B1.) 老生常谈的证明了, 可参考 Proposition 2.5 的证明.

(B2.) 首先注意到 $\mathrm{Hom}_{\mathcal{C}^{\mathrm{Ob}(I)}}(\Delta_{\mathrm{Ob}\,I}(L), \beta_{\mathrm{Ob}}) \simeq \mathrm{Hom}_{\mathcal{C}}(L, \prod_i \beta(i))$; 而 $\mathrm{Hom}_{\mathcal{C}^I}(\Delta_I(L), \beta)$ 中的元素必然也是 $\mathrm{Hom}_{\mathcal{C}^{\mathrm{Ob}(I)}}(\Delta_{\mathrm{Ob}(I)}(L), \beta_{\mathrm{Ob}})$ 中的元素, 反之则不一定. 我们断言:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}^I}(\Delta_I(L), \beta) & \xrightarrow{1:1} & \{\psi \in \mathrm{Hom}_{\mathcal{C}}(L, \prod_i \beta(i)) : f\psi = g\psi\} \\
\phi & \longmapsto & \varprojlim_{\mathrm{Ob}(I)} \phi \\
p\circ\Delta_I(\psi) & \longleftarrow\!\shortmid & \psi
\end{array}$$

对任意 $\phi \in \mathrm{Hom}_{\mathcal{C}^I}(\Delta_I(L), \beta)$, 以及任意 $i \xrightarrow[I^{\mathrm{op}}]{\sigma} j$, 下图交换:

$$\begin{tikzcd}
 & & \beta(i) \arrow[dd, "\beta(\sigma)"] \\
L \arrow[r, "\psi"] \arrow[urr, bend left, "\phi(i)"] \arrow[drr, bend right, "\phi(j)"'] & \prod_i \beta(i) \arrow[ur, "p(i)"'] \arrow[dr, "p(j)"] & \\
 & & \beta(j)
\end{tikzcd}$$

其中 $\psi = \varprojlim_{\mathrm{Ob}(I)} \phi$. 这表明 $\forall\sigma\,(p(s(\sigma))\circ\psi = \beta(\sigma)\circ p(t(\sigma))\circ\psi)$; 进而有 $\forall\sigma\,(q(\sigma)\circ f\circ\psi = q(\sigma)\circ g\circ\psi)$; 调用之前证明的 (A.) 可得 $f\circ\psi = g\circ\psi$.

另一方面, 对任意 $\psi \in \mathrm{Hom}_{\mathcal{C}}(L, \prod_i \beta(i))$ 且满足 $f\psi = g\psi$, 可推出 $\forall\sigma\,(p(s(\sigma))\circ\psi = \beta(\sigma)\circ p(t(\sigma))\circ\psi)$. 从而对任意 $i \xrightarrow[I^{\mathrm{op}}]{\sigma} j$, 下图交换:

$$\begin{tikzcd}
 & & \beta(i) \arrow[dd, "\beta(\sigma)"] \\
L \arrow[r, "\psi"] & \prod_i \beta(i) \arrow[ur, "p(i)"'] \arrow[dr, "p(j)"] & \\
 & & \beta(j)
\end{tikzcd}$$

令 $\phi = p\circ\Delta_I(\psi)$, 即得 $\phi \in \mathrm{Hom}_{\mathcal{C}^I}(\Delta_I(L), \beta)$.

(B3.) 仅需如下双射:

$$\begin{array}{ccc}
\{\psi \in \mathrm{Hom}_{\mathcal{C}}(L, \prod_i \beta(i)) : f\psi = g\psi\} & \xrightarrow{1:1} & \mathrm{Hom}_{\mathcal{C}^{I_0}}(\Delta_{I_0}(L), \beta_0) \\
\psi & \longmapsto & (\psi, g\psi) \\
\psi_1 & \longleftarrow\!\shortmid & (\psi_1, \psi_2)
\end{array}$$

(B4.) $\varprojlim \beta_0 = \ker(f, g)$, 调用 (2.1). $\square$