1 Basics

Definition 1.1 *S is a semigroup, $0 \in S$ is a zero if $\forall x \in S(0x = x0 = 0)$.*

Let S be a semigroup, we can adjoin an identity or zero on it. For example, if S has no identity, then set S^1 be $S \sqcup \{1\}$, in witch $1 \notin S$. Note that if S is a monoid with identity 1_0, its no longer the identity in S^1, since $11_0 \neq 1_0$.

Example 1.2

1. *Null semigroup: $\forall x, y(xy = 0)$;*
2. *left (right) zero semigroup: $\forall x, y(xy = x)$;*
3. *S is a total-ordered set, $xy := \min\{x, y\}$.*

Proposition 1.3 *Suppose S is a semigroup, the follows are equivalent:*

1. *S is a group;*
2. *for all $a, b \in S$, there exists $x, y \in S$ such that $ax = b \wedge ya = b$;*
3. *$\forall a \in S(aS = Sa = S)$.*

Proof. (3. $\Rightarrow$ 1.) The assumption implies that $\forall a, b \in S \exists c(a = bc)$. Suppose that $ax = ya = a$, observe that $x = ax_1 = ay_1a = yay_1a = yx$ and $y = y_2a = ax_2a = ax_2ax = yx$, thus $x = y$. If $ax = xa = a \wedge by = yb = b$, we use the same methodology, and it follows that $x = by_1 = by_2b = by_2by = xy$, $y = ax_1 = ax_2a = xax_2a = xy$. Hence, we conclude that $\forall a \in S \exists! e \in S(ae = ea = a)$. As for the inverse, for any $a \in S$, suppose $ax = ya = e$. Since $x = ex = yax = ye = y$, the inverse exists and is even unique.

(1. $\Rightarrow$ 3.) $\forall a, b \in S(b = ba^{-1}a = aa^{-1}b)$.

(2. $\Leftrightarrow$ 3.) Obvious. □

Proposition 1.4 *A semigroup with zero is a 0-group if and only if*

$$\forall a \in S \setminus \{0\}(aS = Sa = S).$$

Proof. See [M.Howie 1995, Proposition 1.1.1]. □

Definition 1.5 *$I \subset S$ is a proper ideal if $\{0\} \subset I \subsetneq S$ and is an ideal.*

Definition 1.6 *Given a set X, the full transformation semigroup is defined as $(\mathrm{End}_{\mathsf{Set}}(X), \circ)$.*

Definition 1.7 *A morphism $\varphi : S \xrightarrow[\mathsf{Smg}]{} \mathrm{End}(X)$ is a representation of S. We say that φ is faithful if it is injective.*

Theorem 1.8 *Suppose S is a semigroup, $X = S^1$, then there exists a faithful representation $\varphi : S \to \mathrm{End}(X)$.*

Proof. See [M.Howie 1995, Theorem 1.1.2]. Simply stated,

$$\begin{aligned} S &\hookrightarrow \mathrm{End}(S^1) \\ a &\longmapsto [\varphi_a : x \mapsto xa] \end{aligned}$$

□

Definition 1.9 *A semigroup S is a retangular band if*

$$\forall a, b(aba = a)$$

Theorem 1.10 *Suppose S is a semigroup, the follows are equivalent:*

1. *S is a retangular band;*

2. *any $a \in S$ is an idempotent, and $abc = ac$;*

3. *there exists a left zero semigroup L, and a right zero semigroup R, such that $S \simeq L \times R$;*

4. *there exists sets A, B such that $S \simeq A \times B$, in witch $(a_1, b_1)(a_2, b_2) = (a_1, b_2)$.*

Proof. See [M.Howie 1995, Theorem 1.1.3]. $\square$

$A \subset S$ is a subset, the generated sub-semigroup is defined as:

$$\langle A \rangle := \bigcap_{\substack{S' < S \\ A \subset S'}} S',$$

it can be verified that

$$\langle A \rangle = \{a_1 \cdots a_n : n \in \mathbb{Z}_{\geq 1} \wedge a_i \in A\}.$$

If $A = \{a\}$, call $\langle a \rangle = \langle A \rangle$ a monogenic semigroup.

Definition 1.11 *The order of a in S is defined as:*

$$\mathrm{ord}(a) = |\langle a \rangle|.$$

When $\mathrm{ord}(a)$ is finite, let

$$m = \mathrm{idx}(a) := \min \{m \in \mathbb{Z}_{\geq 1} : \exists n \in \mathbb{Z}_{\geq 1}(a^m = a^n \wedge m \neq n)\},$$

and

$$r = \mathrm{prd}(a) := \min \left\{r \in \mathbb{Z}_{\geq 1} : a^{m+r} = a^m\right\}.$$

We may deduce that $a, a^2, \cdots, a^{m+r-1}$ are mutually different. Furthermore, $\langle a \rangle = \{a, \cdots, a^{m+r-1}\}$.

Let K_a be $\{a^m, \cdots, a^{m+r-1}\}$. We assert that it is a cyclic group. Consider the quotient ring $\mathbb{Z}/r\mathbb{Z}$, clearly $\{[m], \cdots, [m+r-1]\} = \mathbb{Z}/r\mathbb{Z}$. Thus, there exists $0 \leq g \leq r-1$ such that $[m+g] = [1]$, which implies that $\forall k([k] = [k(m+g)])$. Since $a^{(m+g)k} = a^{m+hr}a^{k-m} = a^m a^{k-m}$ for all $k > m$, the $a^{(m+g)k}$ exhaust K_a.

References

[M.H95] John M.Howie. *Fundamentals of Semigroup Theory*. 1995.