Chapter 1

Set Theory

1.1 Ordinals

Proposition 1.1.1 Suppose C is a class of ordinals, then $\inf C := \bigcap C$ is an ordinal and $\inf C \in C$.

Proof. It is easy to see $\bigcap C$ is a total-ordered set. For any $x \in \bigcap C$, observe that

$$t \in x \in \bigcap C \Rightarrow \forall \alpha \in C(x \in \alpha) \wedge t \in x \Rightarrow \forall \alpha \in C(t \in \alpha) \Rightarrow t \in \bigcap C,$$

so we conclude that $x \subset \bigcap C$. $\square$

When α is an ordinal, then so is $\alpha \sqcup \{\alpha\}$, its total-order and transitivity are both obvious. And there is a common conclusion about this:

$$\alpha \sqcup \{\alpha\} = \inf\{\beta : \beta > \alpha\}.$$

To see this, [4, Lemma 1.2.9] has already demonstrated that for any two arbitrary ordinals α and β, $\alpha \subsetneq \beta \Leftrightarrow \alpha \in \beta \Leftrightarrow \alpha < \beta$, and we also have

$$\alpha < \inf\{\beta : \beta > \alpha\} \leq \alpha \sqcup \{\alpha\} \Rightarrow \alpha \subsetneq \inf\{\beta : \beta > \alpha\} \subset \alpha \sqcup \{\alpha\}.$$

Proposition 1.1.2 Suppose S is a set of ordinals, then $\sup S := \bigcup S$ is also an ordinal.

Proof. $\bigcup S$ has total-order with respect to $\in$, which is evident. The issue lies in proving that any subset $T \in \bigcup S$ has a minimal element. Since $T \neq \varnothing$, we can choose an $\alpha \in S$ such that $\alpha \cap T \neq \varnothing$. Let m be the minimal element in $\alpha \cap T$, it is the minimal element in T. Otherwise, let $m' < m \in T$, then m' would be the minimal element in $\alpha \cap T$, witch leads to contradiction. $\square$

Any ordinal α satisfies $\alpha = \{x : x < \alpha\}$. However, limit ordinal also possess another property:

$$\alpha = \{x : x < \alpha\} = \bigcup \{x : x < \alpha\},$$

the reason is as follows:

$$\begin{aligned} \alpha \text{ is a limit ordinal} &\Leftrightarrow \nexists x(\alpha = x + 1) \\ &\Leftrightarrow \forall x \in \alpha(x + 1 \in \alpha) \\ &\Leftrightarrow \alpha = \bigcup \{x : x < \alpha\}. \end{aligned}$$

1.2 Transfinite Recursion

Theorem 1.2.1 (Transfinite Induction) Suppose C is a class of ordinals satisfies the following properties:

(i) $0 \in C$,

(ii) $\alpha \in C \Rightarrow \alpha + 1 \in C$,

(iii) if α is a limit ordinal and $\forall \beta < \alpha \Rightarrow \beta \in C$, $\alpha \in C$;

then $C = \mathbf{On}$. If we only select elements on a given ordinal θ, the conclusion still holds, provided that $\mathbf{On}$ is replaced by θ.

Proof. Consider the case restricted on θ, assume that $C \neq \theta$ and $\gamma = \min(\theta \setminus C)$. It follows that $\gamma \notin C$ and $\gamma \neq 0$. The remain steps can be categorized into the follows cases, all cases lead to a contradiction:

case 1. γ is a successor, then $\exists \beta \in \theta (\gamma = \beta + 1)$,

case 1a. $\beta \in C$, by definition, we have $\gamma = \beta + 1 \in C$,

case 1b. $\beta \notin C$, then $\min(\theta \setminus C) = \gamma < \beta < \gamma = \beta + 1$,

case 2. γ is a limit ordinal,

case 2a. $\forall \beta < \gamma \ (\beta \in C)$, by definition, we have $\gamma \in C$,

case 2b. $\exists \beta < \gamma \wedge \beta \notin C$, similarly, we conclude that $\gamma < \beta < \gamma$. □

Theorem 1.2.2 (Transfinite Recursion) Given a function $G : \mathbf{V} \to \mathbf{V}$, where $\mathbf{V}$ is the proper class of all sets, and G is an operation in first-order logic. For any ordinal θ, there exists a unique function $a : \theta \to \mathbf{V}$ such that for every $\alpha < \theta$,

$$a(\alpha) = G(\Gamma(a|_\alpha)) = G(\{(a(x), x)\}_{x \in \alpha}). \tag{1.1}$$

In particular, θ can be replaced by $\mathbf{On}$.

Proof. (Uniqueness) We primarily use transfinite induction. Suppose a and a' both satisfy (1.1), and let $C = \{\alpha \in \mathbf{On} : a'(\alpha) = a(\alpha)\}$. It can be observed that C satisfies all conditions for transfinite induction, hence $C = \theta$.

(Lemma) Before proving existence, we first insert a lemma. Let

$$C = \{\xi \in \theta : \exists \text{mapping } a_\xi : \xi \to \mathbf{V} \left[\forall \alpha \in \xi \left(a_\xi(\alpha) = G(\Gamma(a_\xi|_\alpha))\right)\right]\},$$

for $\mu, \nu \in C$ where $\mu < \nu$, we assert that $a_\nu = a_\mu|_\nu$. The procedure can follow the proof of uniqueness above, provided that a_ν and $a_\mu|_\nu$ both satisfy (1.1) on the set ν.

(Existence) To prove existence, again, we verify that C satisfies the three conditions of transfinite induction one by one. (i) Let $a_0 = \varnothing$, it is apparent without further explanation that $0 \in C$. (ii) For $\xi \in C$, we let

$$a_{\xi+1}(\alpha) = \begin{cases} a_\xi(\alpha) & (\alpha \in \xi) \\ G(\Gamma(a_\xi)) & (\alpha = \xi) \end{cases};$$

it can be verified that $a_{\xi+1}(\alpha) = G(\Gamma(a_{\xi+1}|_\alpha))$. Thus, $\xi + 1 \in C$. (iii) If ξ is a limit ordinal and for all $\eta < \xi (\eta \in C)$, we set $a_\xi(\alpha)$ be $G(\Gamma(a_\alpha))$ for every $\alpha \in \xi$, and it follows that

$$\begin{aligned}\Gamma(a_\xi|_\alpha) &= \{(a_\xi(x), x)\}_{x \in \alpha < \xi} = \{(G(\Gamma(a_x)), x)\}_{x \in \alpha < \xi} \\ &= \{(G(\Gamma(a_\alpha|_x)), x)\}_{x \in \alpha < \xi} = \{(a_\alpha(x), x)\}_{x \in \alpha < \xi} \\ &= \Gamma(a_\alpha).\end{aligned}$$

Thus,

$$a_\xi(\alpha) = G(\Gamma(a_\alpha)) = G(\Gamma(a_\xi|_\alpha)),$$

which implies that $\xi \in C$. □

Chapter 2

Category Theory

2.1 Yoneda Lemma

Define functors as follows.

$$\begin{aligned} \mathcal{C} &\xrightarrow{h_{\mathcal{C}}} \mathcal{C}^{\wedge} = \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}) \\ S &\longmapsto \mathrm{Hom}(\cdot, S) \end{aligned} \qquad \begin{aligned} \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\wedge} &\xrightarrow{\mathrm{ev}^{\wedge}} \mathsf{Set} \\ (S, A) &\longmapsto A(S) \end{aligned}$$

$$\begin{aligned} \mathcal{C} &\xrightarrow{k_{\mathcal{C}}} \mathcal{C}^{\vee} = \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}^{\mathrm{op}}) \\ S &\longmapsto \mathrm{Hom}(S, \cdot) \end{aligned} \qquad \begin{aligned} (\mathcal{C}^{\vee})^{\mathrm{op}} \times \mathcal{C} &\xrightarrow{\mathrm{ev}^{\vee}} \mathsf{Set} \\ (A, S) &\longmapsto A(S) \end{aligned}$$

The Yoneda Lemma states that there exists isomorphism of functors

$$\mathrm{Hom}(h_{\mathcal{C}}(\cdot), \cdot) \xrightarrow{\sim} \mathrm{ev}^{\wedge}(\cdot, \cdot),\ \mathrm{Hom}(\cdot, k_{\mathcal{C}}(\cdot)) \xrightarrow{\sim} \mathrm{ev}^{\vee}(\cdot, \cdot).$$

Taking the former as an example, the isomorphism is given by

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}^{\wedge}}(h_{\mathcal{C}}(S), A) &\xrightarrow[\sigma_{S,A}]{1:1} A(S) \\ \phi &\longmapsto \phi_S(\mathrm{id}_S) \\ (\psi_T : m \mapsto A(m)(u_s))_{T \in \mathrm{Ob}(\mathcal{C})} &\longleftarrow u_S. \end{aligned}$$

Proof. What we need to verify is enumerated as follows:

(a) $\sigma = (\sigma_{S,A})_{(S,A)}$ forms a morphism between functors $\mathrm{Hom}(h_{\mathcal{C}}(\cdot), \cdot) \to \mathrm{ev}^{\wedge}(\cdot, \cdot)$,

(b) each $\sigma_{S,A}$ is a bijection, thus σ is an isomorphism.

(a) For each $(f, \theta) : (S_1, A_1) \to (S_2, A_2)$, the following diagram commutes.

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}^{\wedge}}(h_{\mathcal{C}}(S_1), A_1) \arrow[r, "\sigma_{S_1,A_1}"] \arrow[d, "\theta\circ -"'] & A_1(S_1) \arrow[d, "\theta_{S_1}"] \\ \mathrm{Hom}_{\mathcal{C}^{\wedge}}(h_{\mathcal{C}}(S_1), A_2) \arrow[r, "\sigma_{S_1,A_2}"] \arrow[d, "-\circ h_{\mathcal{C}}(f)"'] & A_2(S_1) \arrow[d, "A_2(f)"] \\ \mathrm{Hom}_{\mathcal{C}^{\wedge}}(h_{\mathcal{C}}(S_2), A_2) \arrow[r, "\sigma_{S_2,A_2}"'] & A_2(S_2) \end{tikzcd}$$

The commutativity of the upper rectangle is easily to verify by tracing the image of any element in $\mathrm{Hom}_{\mathcal{C}^\wedge}(h_{\mathcal{C}}(S_1), A_1)$ under such mappings.

$$\begin{tikzcd}
\phi \arrow[r, mapsto] \arrow[d, mapsto] & \phi_{S_1}(\mathrm{id}_{S_1}) \arrow[d, mapsto] \\
\theta \circ \phi \arrow[r, mapsto] & \theta_{S_1}\phi_{S_1}(\mathrm{id}_{S_1})
\end{tikzcd}$$

We now prove the commutativity of the lower rectangle. Assuming that $\psi \in \mathrm{Hom}_{\mathcal{C}^\wedge}(h_{\mathcal{C}}(S_1), A_2)$, the lower rectangle requires that

$$A_2(f)(\psi_{S_1}(\mathrm{id}_{S_1})) = [\psi \circ h_{\mathcal{C}}(f)]_{S_2}(\mathrm{id}_{S_2}) = \psi_{S_2}(f),$$

which is ensured by the commutativity of the diagram below.

$$\begin{tikzcd}
\mathrm{Hom}(S_1, S_1) \arrow[r, "\psi_{S_1}"] \arrow[d, "f\circ -"'] & A_2(S_1) \arrow[d, "A_2(f)"] \\
\mathrm{Hom}(S_2, S_1) \arrow[r, "\psi_{S_2}"'] & A_2(S_2)
\end{tikzcd}$$

(b) Given $u_S \in A(S)$, we cannot help but ask whether $(\psi_T : m \mapsto A(m)(u_s))_{T\in\mathrm{Ob}(\mathcal{C})}$ is a morphism between $h_{\mathcal{C}}(S)$ and A, the answer is affirmative. This is because for any $R \xrightarrow[\xi]{} T$ in $\mathcal{C}^{\mathrm{op}}$, the commutativity of the diagram below can be verified. Additionally, the bijection is also straightforward to verify, and we omit it here.

$$\begin{tikzcd}
\mathrm{Hom}(R, S) \arrow[r, "\psi_R"] \arrow[d, "f\circ -"'] & A(R) \arrow[d, "A(f)"] \\
\mathrm{Hom}(T, S) \arrow[r, "\psi_T"'] & A(T)
\end{tikzcd}$$

□

2.2 Limits

[4, p.59] states that when I is a small category and $\mathcal{C}$ is Set, both $\varinjlim$ and $\varprojlim$ exist. Note that we always regard I as a small category if not specified.

Proposition 2.2.1 Let s, t be the start and target map of morphisms $\mathrm{Mor}(I) \to \mathrm{Ob}(I)$. For any $\beta : I^{\mathrm{op}} \to \mathsf{Set}$, the limit $\varprojlim \beta$ exists.

Proof. Define two mappings as

$$\prod_i \beta(i) \overset{f}{\underset{g}{\rightrightarrows}} \prod_\sigma \beta(s(\sigma))$$

$$(x_i)_i \overset{f}{\longmapsto} \left[\sigma \mapsto x_{s(\sigma)}\right]$$

$$(x_i)_i \overset{g}{\longmapsto} \left[\sigma \mapsto \beta(\sigma)(x_{t(\sigma)})\right],$$

then, let

$$\begin{aligned}
\varprojlim \beta &= \ker\left[\prod_i \beta(i) \rightrightarrows \prod_\sigma \beta(s(\sigma))\right] \\
&= \left\{(x_i)_i \in \prod_i \beta(i) : f((x_i)_i) = g((x_i)_i)\right\} \\
&= \left\{(x_i)_i \in \prod_i \beta(i) : \forall \sigma \in \mathrm{Hom}_I(i,j)(\beta(\sigma)(x_j) = x_i)\right\}.
\end{aligned}$$

Define a family of maps $p = (p_i)_i$, where $p_i((x_j)_j) = x_i$. We now need to verify (i) $p : \Delta(\varprojlim \beta) \to \beta$ is a morphism; (ii) $(\varprojlim \beta, p)$ is indeed the terminal object in (Δ/β). The verification of (i) is easy. As for (ii), for any (X, ξ), to make the diagram below commute, the only way is to set $\phi(x) = (\xi_i(x))_i$. This establishes both the existence and uniqueness of $\phi : (X, \xi) \to (\varprojlim \beta, p)$.

$$\begin{tikzcd} X \arrow[d, "\phi"'] \arrow[dr, "\xi_i"] & \\ \varprojlim \beta \arrow[r, "p_i"'] & \beta(i) \end{tikzcd}$$

□

As for the definition of "ker" mentioned above, see the discussion of equalizer in [4, p.63]. Before we prove the existence of colimits, we first explain the equivalence relation generated by a relation.

Definition 2.2.2 Given a set X, let $R \subset X^2$ be a binary relation, and

$$\langle R \rangle_{\text{eqv}} := \left\{ (x, y) \in X^2 : \exists (x_i)_{i=0}^n \in X^{n+1} \wedge \forall 1 \le k \le n ((x_{k-1}, x_k) \in R \vee (x_k, x_{k-1}) \in R) \right\},$$

where $x_0 = x$ and $x_n = y$. Then $\langle R \rangle_{\text{eqv}}$ is the **smallest** equivalence relation containing R.

Proof. (This proof is taken from [1, §1.4]) We denote by $P(X^2)$ the power set of X^2, and by R the element of it, i.e., a binary relation. $P(X^2)$ forms a semigroup with the multiplication $\circ$ defined as

$$R_2 \circ R_1 := \left\{ (x, y) \in X^2 : \exists a \in X ((x, a) \in R_1 \wedge (a, y) \in R_2) \right\}.$$

Therefore, $(x, y) \in R^n$, in which R^n represents the multiplication of n copies of R, has an alternative description, namely, there exists a sequence $(t_i)_{i=1}^n$ such that

$$x = t_1 \to t_2 \to \cdots t_n = y,$$

where $t_i \to t_j$ refers $(t_i, t_j) \in R$.

Let R^{op} be $\{(y, x) : (x, y) \in R\}$, and it's easy to verify that $(R_2 \circ R_1)^{\text{op}} = R_1^{\text{op}} \circ R_2^{\text{op}}$. Furthermore, let $\Delta_X = \{(x, x) : x \in X\}$. R is called an equivalence relation if it satisfies (i) reflexivity: $\Delta \subset R$, (ii) symmetry: $R^{\text{op}} \subset R$ and (iii) transitivity: $R \circ R \subset R$. We assert that for any relation R,

$$\langle R \rangle_{\text{eqv}} = \bigcup_{n \ge 1} [R \cup \Delta_X \cup R^{\text{op}}]^n$$

is the smallest equivalence relation containing R.

The proof for reflexivity and symmetry is obvious, we proceed to prove the transitivity. Suppose $(x, y), (y, z) \in \langle R \rangle_{\text{eqv}}$, then, there exists two integers m and n such that (x, y) in $[R \cup \Delta_X \cup R^{\text{op}}]^m$ and (y, z) in $[R \cup \Delta_X \cup R^{\text{op}}]^n$. Thus, there exists two sequences $(a_i)_{i=1}^m$, $(b_i)_{i=1}^n$ satisfying

$$x = a_1 \to \cdots a_m = y \wedge y = b_1 \to \cdots b_n = z.$$

Concatenating $(a_i)_{i=1}^m$ and $(b_i)_{i=1}^n$, we obtain $(x, z) \in \langle R \rangle_{\text{eqv}}$. □

Proposition 2.2.3 For any functor $\alpha : I \to \mathsf{Set}$, where I is a small category, the colimit $\varinjlim \alpha$ exists.

Proof. Let $X = \bigsqcup_i \alpha(i)$, where any element in X can be represented as (x_i, i), in which $x_i \in \alpha(i)$. We use the symbol $(x_i, i) \to (x_j, j)$ to denote there exists a morphism $\sigma : i \to j$ such that $x_j = \alpha(\sigma)(x_i)$. All such pairs $\{(a, b) \in X^2 : a \to b\}$ forms a relation R on X^2. Define $\varinjlim \alpha$ as $X/\sim$, where $\sim$ is the equivalence relation generated by R, and let $\iota = (\iota_i)_i$, where each ι_i is a map from $\alpha(i)$ to $X/\sim$ that sending $x_i \mapsto [(x_i, i)]$. It is straightforward to verify that $\alpha \xrightarrow{\iota} \varinjlim \alpha$ is a morphism. We assert that the above $(\varinjlim \alpha, \iota)$ is the initial object in the category (j_α/Δ). Indeed, for any given object (L, ξ), the only way to define $\phi : \varinjlim \alpha \to L$ making the diagram below commutative is to let it sending $[(x_i, i)]$ to $\xi_i(x_i)$.

$$\begin{tikzcd} \alpha(i) \arrow[r, "\iota_i"] \arrow[dr, "\xi_i"'] & \varinjlim \alpha \arrow[d, "\phi"] \\ & L \end{tikzcd}$$

At this point, the matter is not yet concluded, as it is necessary to verify ϕ is well-defined. Suppose $(x_a, a) \sim (x_b, b)$, then there exists a sequence $(t_i, i)_{i=0}^n$ such that adjacent elements can always be connected by an arrow, for example, it resembles as

$$(x_a, a) = (t_0, i_0) \to (t_1, i_1) \leftarrow (t_2, i_2) \leftarrow (t_3, i_3) \to \cdots \leftarrow (t_n, i_n) = (x_b, b).$$

And we only need to prove $(t_0, i_0) \to (t_1, i_1)$ ensures $\xi_0(t_0) = \xi_1(t_1)$, which is guaranteed by the following diagram.

$$\begin{tikzcd} \alpha(i_0) \arrow[r, "\alpha(\sigma)"] \arrow[dr, "\xi_0"'] & \alpha(i_1) \arrow[d, "\xi_1"] \\ & L \end{tikzcd}$$

□

Proposition 2.2.4 Suppose α is a functor from I to $\mathcal{C}^\wedge$ (or $\mathcal{C}^\vee$), the colimit $\varinjlim \alpha$ exists. Similarly, the limit $\varprojlim \beta$ exist for any functor β from I^{op} to $\mathcal{C}^\wedge$ (or $\mathcal{C}^\vee$).

Proof. (An analogous proof can be found in [3, §3.4]) Taking the case $\alpha : I \to \mathcal{C}^\wedge$ as an example, if $i \xrightarrow[I]{\sigma} j$ and $S \xrightarrow[\mathcal{C}^{\text{op}}]{f} T$ are given, such representations like

$$\alpha(i)(S), \alpha(\sigma)(f), \alpha(i)(f), \alpha(\sigma)(\cdot)$$

etc. are self-evident.

Applying Proposition 2.2.1, it can be seen that for every S, there exists the colimit $\varinjlim \alpha(\cdot)(S)$ of functor $\alpha(\cdot)(S)$ and a family of morphisms $(\iota_{i,S})_i$ corresponding to it.

$$\alpha(\cdot)(S) \xrightarrow{(\iota_{i,S})_i} \Delta(\varinjlim \alpha(\cdot)(S))$$

We define $\varinjlim \alpha$, which is an element in $\mathcal{C}^\wedge$, as follows

$$\mathcal{C}^{\text{op}} \longrightarrow \mathsf{Set}$$

$$S \longmapsto \varinjlim \alpha(\cdot)(S)$$

$$\left[f : S \xrightarrow[\mathcal{C}^{\text{op}}]{} T\right] \longmapsto \varinjlim \alpha(\cdot)(f).$$

We assert that the data below constitutes the initial object in (j_α, Δ), and hence $\varinjlim \alpha$ is the colimit of α

$$\alpha \xrightarrow{\iota = ((\iota_{i,S})_S)_i} \Delta(\varinjlim \alpha).$$

The verification proceeds in three parts:

(a) ι is a morphism between functors,

(b) for any $(L, \xi) \in \text{Ob}(\alpha/\Delta)$, there exists unique $\phi : \varinjlim \alpha \to L$,

(c) ϕ is a morphism between functors.

The proof for (a) reduces to showing $\alpha(i)(\cdot) \xrightarrow{(\iota_{i,S})_S} \varinjlim \alpha$ is a morphism and the commutativity of the following diagram, we omit it here.

$$\begin{tikzcd} \alpha(i) \arrow[r, "(\iota_{i,S})_S"] \arrow[d, "\alpha(\sigma)"'] & \varinjlim \alpha \\ \alpha(j) \arrow[ur, "(\iota_{j,S})_S"'] & \end{tikzcd}$$

As for (b), given any object (L, ξ), the proof is described by

$$\begin{tikzcd} \alpha(\cdot)(S) \arrow[r, "(\iota_{i,S})_i"] \arrow[dr, "(\xi_{i,S})_i"'] & \Delta(\varinjlim \alpha(\cdot)(S)) \arrow[d, "\exists!\Delta(\phi_S)"] \\ & \Delta(L(S)) \end{tikzcd} \quad \Rightarrow \quad \varinjlim \alpha \xrightarrow[\mathcal{C}^\wedge]{(\phi_S)_S} L.$$

Our goal is to verify for any $f : S \xrightarrow[\mathcal{C}^{\mathrm{op}}]{} T$, the following diagram commutes.

$$\begin{tikzcd} \varinjlim \alpha(\cdot)(S) \arrow[r, "\phi_S"] \arrow[d, "\varinjlim \alpha(\cdot)(f)"'] & L(S) \arrow[d, "L(f)"] \\ \varinjlim \alpha(\cdot)(T) \arrow[r, "\phi_T"'] & L(T) \end{tikzcd}$$

Since $\alpha \xrightarrow{\xi} \Delta(L)$, the diagram

$$\begin{tikzcd} \alpha(\cdot)(S) \arrow[r, "(\xi_{i,S})_i"] \arrow[d, "\alpha(\cdot)(f)"'] & \Delta(L(S)) \arrow[d, "\Delta(L(f))"] \\ \alpha(\cdot)(T) \arrow[r, "(\xi_{i,T})_i"'] & \Delta(L(T)) \end{tikzcd}$$

commutes. Moreover, it is straightforward to show that $\varinjlim \Delta(L(S)) = L(S)$ and $\varinjlim \Delta(L(f)) = L(f)$. Then, form

$$\begin{tikzcd} \alpha(\cdot)(S) \arrow[r, "(\iota_{i,S})_i"] \arrow[d, "(\xi_{i,S})_i"'] & \Delta(\varinjlim \alpha(\cdot)(S)) \arrow[d, "\Delta(\phi_S)"] \\ \Delta(L(S)) \arrow[r, "\mathrm{id}"'] & \Delta(\varinjlim \Delta(L(S))), \end{tikzcd}$$

we obtain $\varinjlim[(\xi_{i,S})_i] = \phi_S$. Finally, we invoke the property that limit preserves morphisms. □

Proposition 2.2.5 Given $\beta : I^{\mathrm{op}} \to \mathcal{C}$, define a functor $\bar{\beta} : I^{\mathrm{op}} \to \mathcal{C}^\wedge$ that sending i to $h_{\mathcal{C}}(\beta(i)) = \mathrm{Hom}_{\mathcal{C}}(\cdot, \beta(i))$ with its natural definition of morphisms, then $\varprojlim \beta$ exists if and only if $\varprojlim \bar{\beta}$ is representable.

Similarly, for $\alpha : I \to \mathcal{C}$, let $\bar{\alpha} : I \to \mathcal{C}^\vee$ that sending i to $k_{\mathcal{C}}(\alpha(i)) = \mathrm{Hom}_{\mathcal{C}}(\alpha(i), \cdot)$, then $\varinjlim \alpha$ exists if and only if $\varinjlim \bar{\alpha}$ is representable.

Proof. (An analogous proof can be found in [2, §3.4.2]) Take $\beta : I^{\mathrm{op}} \to \mathcal{C}$ as an example, our proof can be divided into two parts:

(a) $\exists \varprojlim \beta \Rightarrow h_{\mathcal{C}}(\varprojlim \beta) \xrightarrow[\exists]{\sim} \varprojlim \bar{\beta}$,

(b) $h_{\mathcal{C}}(D) \xrightarrow[\exists]{\sim} \varprojlim \bar{\beta} \Rightarrow \exists \varprojlim \beta$.

To prove (a), we first observe that

$$\begin{aligned} \varprojlim \bar{\beta}(\cdot)(S) &= \left\{ (\xi_i)_i \in \prod_I \bar{\beta}(i)(S) : \forall \sigma : i \xrightarrow{I} j(\xi_j \circ \beta(\sigma) = \xi_i) \right\} \\ &= \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta). \end{aligned}$$

Recall the Yoneda Lemma, let $\mathcal{C}^\wedge \ni A = \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(\cdot), \beta)$, for any $S \in \mathrm{Ob}(\mathcal{C})$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{Hom}(\cdot, S), \mathrm{Hom}(\Delta(\cdot), \beta)) &\xleftrightarrow{1:1} \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta) \\ \phi &\longmapsto \phi_S(\mathrm{id}_S) \\ (\phi_T : f \mapsto u \circ \Delta(f))_{T \in \mathrm{Ob}(\mathcal{C})} &\longleftarrow\!\shortmid\; u. \end{aligned} \tag{2.1}$$

This formula will be used multiple times in subsequent sections. We claim that if $\Delta(S) \xrightarrow{u} \beta$ is the terminal object in (Δ, β), that is, $S = \varprojlim \beta$, then the image ϕ of u is an isomorphism. Conversely, if ϕ is an isomorphism, then S together with $u = \phi_S(\mathrm{id}_S)$ forms a terminal object in (Δ, β). The proof of the latter will be discussed later. For the former, we take $S = \varprojlim \beta$, and then verify that for any $S \in \mathrm{Ob}(\mathcal{C})$, ϕ_S is a bijection. To be specific, (2.1) provides the following bijection.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(S, \varprojlim \beta) & \xleftrightarrow{1:1} & \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta) \\ f & \xmapsto[\phi_S]{\qquad\qquad} & u \circ \Delta(f) \\ \varprojlim \lambda & \xleftarrow[\psi_S]{\qquad\qquad} & \lambda \end{array}$$

It can be verified both $\psi_S \phi_S$ and $\phi_S \psi_S$ are identities. As for the $\psi_S \phi_S$, the reason is illustrated by the following diagram, which is **not** a commutative diagram. The part above the dashed lines represents the composition of morphisms before taking the limit, while the part below it represents the composition after taking the limits. In fact, we can also see that limit-taking operation induces a functor. And for the converse, the proof is straightforward, and we omit it here.

$$\begin{tikzcd} \Delta(S) \arrow[r, "\Delta(f)"] \arrow[d, dashed, "\varprojlim"'] & \Delta(\varprojlim \beta) \arrow[r, "u"] \arrow[d, dashed, "\varprojlim"] & \beta \arrow[d, dashed, "\varprojlim"] \\ S \arrow[r, "f"'] & \varprojlim \beta \arrow[r, "\mathrm{id}"'] & \varprojlim \beta \end{tikzcd}$$

To prove (b), suppose there exists an object D and an isomorphism ϕ such that $h_{\mathcal{C}}(D) \xrightarrow[\phi]{\sim} \varprojlim \bar{\beta} = \mathrm{Hom}(\Delta(\cdot), \beta)$. Applying (2.1), let $u = \phi_D(\mathrm{id}_D)$, and for any object S in $\mathcal{C}$, we have

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(S, D) & \xleftrightarrow{1:1} & \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta) \\ f & \xmapsto[\phi_S]{\qquad\qquad} & u \circ \Delta(f) \\ \psi_S(\lambda) & \xleftarrow[\psi_S]{\qquad\qquad} & \lambda, \end{array}$$

in which ψ_S is the inverse of ϕ_S. We assert that (D, u) forms a terminal object in (Δ, β). For any object (S, λ) in (Δ, β), the mapping defined above gives a unique $\psi_S(\lambda) : S \to D$, while since $u \circ \Delta(\psi_S(\lambda)) = \phi_S \psi_S(\lambda) = \lambda$, the following diagram commutes.

$$\begin{tikzcd} \Delta(D) \arrow[r, "u"] & \beta \\ \Delta(S) \arrow[u, "\Delta(\psi_S(\lambda))"] \arrow[ur, "\lambda"'] & \end{tikzcd}$$

The case of $\alpha : I \to \mathcal{C}$ is briefly supplemented below. Define $\bar{\alpha} : I \to \mathcal{C}^{\vee}$ that mapping i to $\mathrm{Hom}_{\mathcal{C}}(\alpha(i), \cdot)$, thus, $\bar{\alpha}(\cdot)(S)$ is a functor from I to $\mathsf{Set}^{\mathrm{op}}$, which is equal to the functor between I^{op} and Set. Then, the remaining proof can be verified through similar steps, and the essence of the proof lies in the formula

$$\varprojlim \bar{\alpha}(\cdot)(S) = \mathrm{Hom}_{\mathcal{C}^I}(\alpha, \Delta(S)) \simeq \mathrm{Hom}_{\mathcal{C}^{\vee}}(\mathrm{Hom}(\alpha, \Delta(\cdot)), \mathrm{Hom}(S, \cdot)). \qquad \square$$

Proposition 2.2.6 Suppose I, J are small categories and $\alpha : I \to \mathcal{C}^J$. If there exists $\varinjlim_i \alpha(\cdot)(j)$ for any $j \in J$, then $\varinjlim_i \alpha \in \mathcal{C}^J$ exists.

Proof. Let

$$\alpha(i)(j) \xrightarrow{\iota_{i,j}} \varinjlim_i \alpha(\cdot)(j),$$

And define $\varinjlim_i \alpha$, which is the element in $\mathcal{C}^J$, as follows.

$$\begin{aligned} j &\longmapsto \varinjlim_i \alpha(\cdot)(j) \\ [\sigma : j \to j'] &\longmapsto \varinjlim_i \alpha(\cdot)(\sigma) \end{aligned}$$

It can be verified that $\alpha \xrightarrow{((\iota_{i,j})_j)_i} \Delta_I \left(\varinjlim_i \alpha\right)$ becomes the initial object in (α, Δ), thus, $\varinjlim_i \alpha$ is indeed the limit of α. The proof is similar to Proposition 2.2.4. $\square$

Proposition 2.2.7 Suppose I, J are small categories and $\mathcal{C}$ has all colimits with index I and J, that is, any functor α from I or J to $\mathcal{C}$, its colimit $\varinjlim \alpha$ exists. Then, for any functor $I \times J \to \mathcal{C}$, there exists a canonical isomorphism

$$\varinjlim_j \left(\varinjlim \alpha(\cdot, j)\right) \simeq \varinjlim \alpha \simeq \varinjlim_i \left(\varinjlim \alpha(i, \cdot)\right).$$

Proof. Given $j \in J$, the functor $\alpha(\cdot, j)$ that maps i to $\alpha(i, j)$ has a colimit $\varinjlim \alpha(\cdot, j)$. Then, we can define a functor as

$$\begin{aligned} j &\longmapsto \varinjlim \alpha(\cdot, j) \\ [\sigma : j \to j'] &\longmapsto \left[\varinjlim \alpha(\cdot, \sigma) : \varinjlim \alpha(\cdot, j) \to \varinjlim \alpha(\cdot, j')\right]. \end{aligned}$$

Thus, according to Proposition 2.2.5, we obtain

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}\left(S, \varinjlim_j \left(\varinjlim \alpha(\cdot, j)\right)\right) &\simeq \mathrm{Hom}_{\mathcal{C}^J}\left(\Delta_J(S), \left[j \mapsto \varinjlim \alpha(\cdot, j)\right]\right) \\ &\simeq \mathrm{Hom}_{(\mathcal{C}^J)^I}\left(\Delta_I(\Delta_J(S)), [i \mapsto \alpha(i, \cdot)]\right) \\ &\simeq \mathrm{Hom}_{\mathcal{C}^{I\times J}}\left(\Delta_{I\times J}(S), \alpha\right) \\ &= \varinjlim \bar{\alpha}(S), \end{aligned}$$

and it follows that $\varinjlim_j \left(\varinjlim \alpha(\cdot, j)\right)$ is the colimit of α. $\square$

Theorem 2.2.8 Suppose I is a small category, if the product $\prod_{j\in J} X_j$ exists for all subsets $J \subset \mathrm{Mor}(I)$ and families of objects $(X_j)_{j\in J}$ in $\mathrm{Ob}(\mathcal{C})$, and the equalizer $\ker(f, g)$ exists for all $f, g : X \to Y$, then $\mathcal{C}$ has all limits indexed by I.

Similarly, if all $\coprod_J X_j$ and all $\mathrm{coker}(f, g)$ exists, $\mathcal{C}$ has all colimits indexed by I.

Proof. Since $\mathrm{Ob}(I)$ and $\mathrm{Ob}(J)$ are $\mathcal{U}$-small sets, each of them can be seen as a discrete category. Given a functor $\beta : I^{\mathrm{op}} \to \mathcal{C}$, where I is represented by the data $(\mathrm{Ob}(I), \mathrm{Mor}(I), s, t, \circ)$, we can define the following two functors on discrete categories discussed above, and according to the given conditions, their products all exist.

$$\begin{aligned} \beta_{\mathrm{Ob}} : \mathrm{Ob}(I) &\longrightarrow \mathcal{C} \\ i &\longmapsto \beta(i) \end{aligned} \qquad \Delta_{\mathrm{Ob}(I)}\left(\prod \beta(i)\right) \xrightarrow{p} \beta_{\mathrm{Ob}}$$

$$\begin{aligned} \beta_{\mathrm{Mor}} : \mathrm{Mor}(I) &\longrightarrow \mathcal{C} \\ \sigma &\longmapsto \beta(s(\sigma)) \end{aligned} \qquad \Delta_{\mathrm{Mor}(I)}\left(\prod \beta(s(\sigma))\right) \xrightarrow{q} \beta_{\mathrm{Mor}}$$

Then, define two functors as

$$\prod_i \beta(i) \overset{M(\sigma)=p(s(\sigma))}{\underset{N(\sigma)=\beta(\sigma)\circ p(t(\sigma))}{\rightrightarrows}} \beta(s(\sigma)).$$

Respectively place them into the comma category $(\Delta_{\mathrm{Mor}(I)}, \beta_{\mathrm{Mor}})$, and they will all induces corresponding morphisms f and g: $\prod_i \beta(i) \to \prod_\sigma \beta(s(\sigma))$. Here, we take $N(\sigma)$ as an example, note that the limit of $\Delta_{\mathrm{Mor}(I)}\left(\prod_i \beta(i)\right)$ is $\prod_i \beta(i)$, thus, its induced isomorphism is nothing but the result of taking limit.

$$\begin{tikzcd}
\Delta_{\mathrm{Mor}(I)}\left(\prod_\sigma \beta(s(\sigma))\right) \arrow[r, "q"] & \beta_{\mathrm{Mor}} \\
\Delta_{\mathrm{Mor}(I)}\left(\prod_i \beta(i)\right) \arrow[u, "\Delta(g)"] \arrow[r, "\mathrm{id}"'] & \Delta_{\mathrm{Mor}(I)}\left(\prod_i \beta(i)\right) \arrow[u, "N"']
\end{tikzcd}$$

To describe the equalizer, it is necessary to define a category I_0 as $\bullet \rightrightarrows \bullet$. And the functor $\beta_0 : I_0^{\mathrm{op}} \to \mathcal{C}$ gives that $\prod_i \beta(i) \overset{f}{\underset{g}{\rightrightarrows}} \prod_\sigma \beta(s(\sigma))$.

The remaining proof proceeds in following parts.

(a) $(\forall \sigma \in \mathrm{Mor}(I)(q(\sigma) \circ \mu = q(\sigma) \circ \nu)) \Rightarrow \mu = \nu$,

(b) $\forall L \in \mathrm{Ob}(\mathcal{C}) \left(\varprojlim \mathrm{Hom}(L, \beta(\cdot)) \simeq \mathrm{Hom}(L, \ker(f,g)) \right)$, and this can be divided into more detailed steps:

  (b1) $\varprojlim \mathrm{Hom}(L, \beta(\cdot)) \simeq \mathrm{Hom}(\Delta_I(L), \beta)$,

  (b2) $\mathrm{Hom}(\Delta_I(L), \beta) \simeq \{\psi \in \mathrm{Hom}(L, \prod_i \beta(i)) : f\psi = g\psi\}$,

  (b3) $\{\psi \in \mathrm{Hom}(L, \prod_i \beta(i)) : f\psi = g\psi\} \simeq \mathrm{Hom}_{\mathcal{C}^{I_0}}(\Delta_{I_0}(L), \beta_0)$,

  (b4) $\mathrm{Hom}_{\mathcal{C}^{I_0}}(\Delta_{I_0}(L), \beta_0) \simeq \mathrm{Hom}(L, \ker(f,g))$. $\square$

2.3 Enriched Category

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