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Chapter 1

Set Theory

1.1 Transfinite Recursion

Theorem 1.1.1 (Transfinite Induction) *Suppose C is a class of ordinals satisfying the follows:*

- $0 \in C$.
- $\alpha \in C \Rightarrow \alpha + 1 \in C$.
- *Suppose α is a limit ordinal, and* $(\forall \beta < \alpha \Rightarrow \beta \in C) \Rightarrow \alpha \in C$.

Then $C = \mathbf{On}$. This assertion remains valid when considering only ordinals less than a given ordinal θ

Proof. We only consider the case on a given ordinal θ. Suppose $C \neq \theta$, and let $\gamma = \min(\theta \setminus C)$. We have $\gamma \notin C$ and $\gamma \neq 0$, and the remainder of proof can be devided into several cases.

case 1. γ is a successor, so $\exists \beta \in \theta(\gamma = \beta + 1)$.

case 1a. $\beta \in C$, by definition we have $\gamma = \beta + 1 \in C$.

case 1b. $\beta \notin C$, then $\gamma < \beta < \gamma$.

case 2. γ is a limit ordinal.

case 2a. $\forall \beta < \gamma \ (\beta \in C)$, by definition we have $\gamma \in C$.

case 2b. $\exists \beta < \gamma \wedge \beta \notin C$, then $\gamma < \beta < \gamma$. □

To define a function whose domain is the ordinal θ, a formal approach can be outlined as follows. Initially, we assign $a(0)$ to be an element a_0 in $\mathbf{V}$. Subsequently, for any ordinal α satisfying $0 < \alpha < \theta$, we determine $a(\alpha)$ by previously established values $\{a(x)\}_{x<\alpha}$, which can be expressed as $a(\alpha) = G(\{a(x)\}_{x<\alpha})$, where G is a function mapping from $\mathbf{V}$ to $\mathbf{V}$.

For example, in the proof of Zermelo's Theorem, there exists a function from **On** to a nonempty set. Given a non-empty set S, it follows that $P(S)\setminus\{\varnothing\}$ is also non-empty. According to the Axiom of Choice, we have:

$$\prod_{A\in P(S)\setminus\{\varnothing\}} A\neq\varnothing.$$

which implies the existence of the function:

$$\begin{aligned} g: P(S)\setminus\{\varnothing\} &\to S \\ A &\mapsto x \text{ (an element belongs to } A). \end{aligned}$$

Next, we specify an arbitrary $a_0\in S$, and choose distinct elements $\Omega_0,\ \Omega_1\notin S$ with $\Omega_0\neq\Omega_1$, and define G as follows:

$$G(X)=\begin{cases} a_0 & X=\varnothing \\ g(S\setminus X) & X=\{a_x\}_{x<\alpha}\subsetneq S\ (\alpha\in\mathbf{On}) \\ \Omega_0 & X=S\vee X=S\sqcup\{\Omega_0\} \\ \Omega_1 & \text{else.} \end{cases}$$

Finally, we recursively define the function a by

$$a(\alpha)=G(\{a(x)\}_{x<\alpha})$$

Seems like we've defined a function from $\mathbf{On}\to S\sqcup\{\Omega_0\}$. However, in my opinion, our endeavors so far has not been adequate, because we have merely assigned an initial value to a and provided a procedure for updating its subsequent values.

Now back to the start. Does the function a exists (and even unique) when given information of its initial value $a(0)$ and the function G that prescribe its updates? This inquiry directs us toward the Transfinite Recursion.

Theorem 1.1.2 (Transfinite Recursion) *Given $G:\mathbf{V}\to\mathbf{V}$, for any ordinal θ, there exists a unique θ-sequence a such that for all ordinals $\alpha<\theta$, $a(\alpha)=G(a|_\alpha)$. In particular, there exists a unique function $a:\mathbf{On}\to\mathbf{V}$ so that for any ordinal α, $a(\alpha)=G(a|_\alpha)$.*

Proof. We consider the case involving a given ordinal θ, and initially demonstrate the uniqueness of the θ-*sequence*. Suppose both θ-*sequence* a and a' satisfy the recursive definitions

$$a(\alpha)=G(\{a(x)\}_{x<\alpha}),\ a'(\alpha)=G(\{a'(x)\}_{x<\alpha})$$

To prove uniqueness, we invoke the Transfinite Induction. Define a class C of ordinals as $C=\{\alpha<\theta: a(\alpha)=a'(\alpha)\}$. We then verify that C satisfies the three condition of Transfinite Induction. Firstly, notice that $a(0)=G(0)=a'(0)$ since any function constrained on an empty set is $\varnothing$. This implies that $0\in C$.

Secondly, suppose that $a(x)=a'(x)$ for all $x<\alpha$. Whether α is a successor or a limit ordinal, the equality $\{a(x)\}_{x<\alpha}=\{a'(x)\}_{x<\alpha}$ holds, so, consequently:

$$a(\alpha)=G(\{a(x)\}_{x<\alpha})=G(\{a'(x)\}_{x<\alpha})=a'(\alpha)$$

By the Transfinite Induction, we conclude that $C = \theta$.

Next, we establish the existence of a by adopting a method analogous to the proof of uniqueness. Define C as the class of ordinals satisfying:

$$C = \{\xi < \theta : \text{the } \xi\text{-}\textit{sequence } a[\xi] \text{ exists}\}.$$

Evidently, $a[0]$ exists and is naturally set to be 0, thus $0 \in C$. Assume that $0 < \beta < \theta$ and that for all $\xi < \beta$, the function $a[\xi]$ exists. We process to demonstrate the existence of $a[\beta]$.

Assert that if $\zeta < \eta$, and $a[\zeta]$, $a[\eta]$ exists, then $a[\eta]|_\zeta = a[\zeta]$. The reason for this assertion is:

$$a[\zeta]|_\eta(x) = G(\{a[\zeta]|_\eta(t)\}_{t<x})$$
$$a[\eta](x) = G(\{a[\eta](t)\}_{t<x})$$

Let $a[\beta](\xi) := G(a[\xi])$ $(\forall \xi < \beta)$, and it gives that for all $x < \beta$:

- $a[\beta]|_x = \{G(a[t])\}_{t<x} = \{G(a[x]|_t)\}_{t<x} = \{a[x](t)\}_{t<x}$;
- $a[\beta](x) = G(a[x]) := G(\{a[x](t)\}_{t<x})$;
- $a[\beta](x) = G(a[\beta]|_x)$.

Hence, $\beta \in C$. By the Transfinite Induction, it follows that $C = \theta$. $\square$

Chapter 2

Category Theory

2.1 Yoneda Lemma

定义函子:

$$\begin{aligned} \mathcal{C} &\xrightarrow{h_{\mathcal{C}}} \mathcal{C}^{\wedge} = \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}) \\ S &\longmapsto \mathrm{Hom}(\cdot, S) \end{aligned} \qquad \begin{aligned} \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\wedge} &\xrightarrow{\mathrm{ev}^{\wedge}} \mathsf{Set} \\ (S, A) &\longmapsto A(S) \end{aligned}$$

$$\begin{aligned} \mathcal{C} &\xrightarrow{k_{\mathcal{C}}} \mathcal{C}^{\vee} = \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}^{\mathrm{op}}) \\ S &\longmapsto \mathrm{Hom}(S, \cdot) \end{aligned} \qquad \begin{aligned} (\mathcal{C}^{\vee})^{\mathrm{op}} \times \mathcal{C} &\xrightarrow{\mathrm{ev}^{\vee}} \mathsf{Set} \\ (A, S) &\longmapsto A(S) \end{aligned}$$

Yoneda Lemma 说的是存在函子的同构 $\mathrm{Hom}(h_{\mathcal{C}}(\cdot), \cdot) \xrightarrow{\sim} \mathrm{ev}^{\wedge}(\cdot, \cdot)$, $\mathrm{Hom}(\cdot, k_{\mathcal{C}}(\cdot)) \xrightarrow{\sim} \mathrm{ev}^{\vee}(\cdot, \cdot)$. 以前者为例, 这一同构由下式给出:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}^{\wedge}}(h_{\mathcal{C}}(S), A) &\xrightarrow[\sigma_{S,A}]{1:1} A(S) \\ \phi &\longmapsto \phi_S(\mathrm{id}_S) \\ (\psi_T : m \mapsto A(m)(u_s))_{T \in \mathrm{Ob}(\mathcal{C})} &\longleftarrow u_S \end{aligned}$$

所以我们需要验证:

A. $\sigma = (\sigma_{S,A})_{(S,A)}$ 构成一个函子间的态射 $\mathrm{Hom}(h_{\mathcal{C}}(\cdot), \cdot) \to \mathrm{ev}^{\wedge}(\cdot, \cdot)$.

B. 每个 $\sigma_{S,A}$ 都是双射, 从而 σ 构成函子间的同构.

(A.) 对任意 $(f,\theta):(S_1,A_1)\to(S_2,A_2)$, 下图交换:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}^\wedge}(h_{\mathcal{C}}(S_1),A_1) & \xrightarrow{\sigma_{S_1,A_1}} & A_1(S_1) \\
{\scriptstyle \theta\circ-}\downarrow & & \downarrow{\scriptstyle \theta_{S_1}} \\
\mathrm{Hom}_{\mathcal{C}^\wedge}(h_{\mathcal{C}}(S_1),A_2) & \xrightarrow{\sigma_{S_1,A_2}} & A_2(S_1) \\
{\scriptstyle -\circ h_{\mathcal{C}}(f)}\downarrow & & \downarrow{\scriptstyle A_2(f)} \\
\mathrm{Hom}_{\mathcal{C}^\wedge}(h_{\mathcal{C}}(S_2),A_2) & \xrightarrow[\sigma_{S_2,A_2}]{} & A_2(S_2)
\end{array}$$

其中上方矩形的交换性很好验证. 现证下方矩形的交换性. 假设 $\psi\in\mathrm{Hom}_{\mathcal{C}^\wedge}(h_{\mathcal{C}}(S_1),A_2)$, 下方矩形要求 $A_2(f)(\psi_{S_1}(\mathrm{id}_{S_1}))=[\psi\circ h_{\mathcal{C}}(f)]_{S_2}(\mathrm{id}_{S_2})=\psi_{S_2}(f)$, 此式子成立因为下图交换:

$$\begin{array}{ccc}
\mathrm{Hom}(S_1,S_1) & \xrightarrow{\psi_{S_1}} & A_2(S_1) \\
{\scriptstyle f\circ-}\downarrow & & \downarrow{\scriptstyle A_2(f)} \\
\mathrm{Hom}(S_2,S_1) & \xrightarrow[\psi_{S_2}]{} & A_2(S_2)
\end{array}$$

(B.) 给定 $u_S\in A(S)$, 我们不禁要问 $(\psi_T:m\mapsto A(m)(u_s))_{T\in\mathrm{Ob}(\mathcal{C})}$ 是否是 $h_{\mathcal{C}}(S)$ 与 A 之间的态射. 对 $\mathcal{C}^{\mathrm{op}}$ 中的任何 $R\underset{\xi}{\to}T$, 可以证明下图交换:

$$\begin{array}{ccc}
\mathrm{Hom}(R,S) & \xrightarrow{\psi_R} & A(R) \\
{\scriptstyle f\circ-}\downarrow & & \downarrow{\scriptstyle A(f)} \\
\mathrm{Hom}(T,S) & \xrightarrow[\psi_T]{} & A(T)
\end{array}$$

因此这样定义的 ψ 确实是函子间态射; 另外注意到如此定义的 ψ 满足 $\psi_S(\mathrm{id}_S)=u_S$. 再者, 对任意 $\phi:h_{\mathcal{C}}(S)\to A$, 均有 $\phi_T(m)=A(m)(u_S)$; 从而 $\sigma_{S,A}$ 是双射.

2.2 Limits

Page 59 in the book states that when I is a small category and $\mathcal{C}$ is Set, both $\varinjlim$ and $\varprojlim$ exist.

Proposition 2.2.1 *Let s,t be the start and target map of morphisms $\mathrm{Mor}(I)\to\mathrm{Ob}(I)$. For any $\beta:I^{\mathrm{op}}\to\mathsf{Set}$, the limit $\varprojlim\beta$ exists.*

Proof. Define two maps as follows:

$$\prod_i\beta(i)\ \underset{g}{\overset{f}{\rightrightarrows}}\ \prod_\sigma\beta(s(\sigma))$$

$$(x_i)_i\ \xmapsto{\quad f\quad}\ \left[\sigma\mapsto x_{s(\sigma)}\right]$$

$$(x_i)_i\ \xmapsto{\quad g\quad}\ \left[\sigma\mapsto\beta(\sigma)(x_{t(\sigma)})\right]$$

Next, we let:

$$\begin{aligned}\varprojlim \beta &= \ker\left[\prod_i \beta(i) \rightrightarrows \prod_\sigma \beta(s(\sigma))\right] \\ &= \left\{(x_i)_i \in \prod_i \beta(i) : f((x_i)_i) = g((x_i)_i)\right\} \\ &= \left\{(x_i)_i \in \prod_i \beta(i) : \forall \sigma \in \mathrm{Hom}_I(i,j)(\beta(\sigma)(x_j) = x_i)\right\}\end{aligned}$$

Define a family of maps $p = (p_i)_i$, where $p_i((x_j)_j) = x_i$. We now need to verify (i): $p : \Delta(\varprojlim \beta) \to \beta$ is a morphism; (ii): $(\varprojlim \beta, p)$ is indeed the terminal object in (Δ/β). The verification of (i) is easy. As for (ii). For any (X, ξ), to make the diagram commute, the only way is to set $\phi(x) = (\xi_i(x))_i$. This establishes both the existence and uniqueness of $\phi : (X, \xi) \to (\varprojlim \beta, p)$:

$$\begin{tikzcd} X \arrow[d, "\phi"'] \arrow[dr, "\xi_i"] & \\ \varprojlim \beta \arrow[r, "p_i"'] & \beta(i) \end{tikzcd}$$

□

As for the definition of "ker" mentioned above, see the discussion of equalizer on page 63 of this book. Before we prove the existence of colimits, we first explain the equivalence relation generated by a relation.

Definition 2.2.2 *For any set X, let $R \subset X^2$ be a binary relation. Define*

$$S = \left\{(x,y) \in X^2 : \exists (x_i)_{i=0}^n \in X^{n+1} \wedge \forall 1 \le k \le n((x_{k-1}, x_k) \in R \vee (x_k, x_{k-1}) \in R)\right\}$$

*where $x_0 = x$, $x_n = y$. Then S is the **smallest** equivalence relation containing R.*

Proof. To show that S is an equivalence relation, the symmetry and transitivity are obvious. As for reflexivity, for $n = 0$, the definition of S implies $\forall x\, (x,x) \in S$. To show S is the **smallest** equivalence relation, let S' be any equivalence relation containing R. Suppose that $(x,y) \in S$. By definition, we have:

$$\exists (x_i)_{i=0}^n \in X^{n+1}\, (\forall 1 \le k \le n((x_{k-1}, x_k) \in R \vee (x_k, x_{k-1}) \in R)) \qquad \square$$

all such pairs (x_{k-1}, x_k) or (x_k, x_{k-1}) belong to S'. Thus, this implies $(x,y) \in S'$.

Proposition 2.2.3 *For any functor on small category $\alpha : I \to \mathsf{Set}$, the colimit $\varinjlim \alpha$ exists.*

Proof. Let $X = \bigsqcup_i \alpha(i)$. The notation $a \to b$ means there exists a σ such that $b = \alpha(\sigma)(a)$. All such pairs $\{(x,y) : x \to y\}$ forms a relation $R \subset X^2$. Define $\varinjlim \alpha = X/\sim$, where $\sim$ is the equivalence relation generated by R. Let $\iota = (\iota_i)_i$ where each ι_i is a map $\alpha(i) \to X/\sim$ sending $x \mapsto [x]_\sim$. It is straightforward to verify that

$\alpha \xrightarrow{\iota} \varinjlim \alpha$ is a morphism. For any object (L, ξ) in (α, Δ), to ensure the commutativity of the diagram, define the map $\phi : \varinjlim \alpha \to L$ that sending $[x_i]_\sim$ to $\xi_i(x_i)$.

$$\begin{tikzcd} \alpha(i) \arrow[r, "\iota_i"] \arrow[dr, "\xi_i"'] & \varinjlim \alpha \arrow[d, "\phi"] \\ & L \end{tikzcd}$$

We now need to verify that ϕ is well-defined. Suppose $x_i \sim x_j$, we must show $\xi_i(x_i) = \xi_j(x_j)$. By definition, there exists a sequence $(t_i)_{i=0}^n$ such that a diagram of the following form holds:

$$x_i = t_0 \to t_1 \leftarrow t_2 \leftarrow t_3 \to \cdots \leftarrow t_n = x_j$$

where the direction of arrows can be arbitrary. This reduces to iterating the same pattern, so it suffices to show that $t_0 \to t_1$ ensures $\xi_0(t_0) = \xi_1(t_1)$. This follows from the $\xi : \alpha \to \Delta(L)$ is a natural transformation:

$$\begin{tikzcd} \alpha(i_0) \arrow[r, "\alpha(\sigma)"] \arrow[dr, "\xi_0"'] & \alpha(i_1) \arrow[d, "\xi_1"] \\ & L \end{tikzcd}$$

□

需要注意一点, 在书图表 (2.13) 下方讲 Set 中的等化子时, 虽然是上面命题的退化情况, 却采用了略为不同的构造.

Proposition 2.2.4 (书命题 2.7.8.) *I 是小范畴时, $\alpha : I \to \mathcal{C}^\wedge$ or $\mathcal{C}^\vee$ 的余极限 $\varinjlim \alpha$ 存在. 同样 $\beta : I^{\mathrm{op}} \to \mathcal{C}^\wedge$ or $\mathcal{C}^\vee$ 的极限 $\varprojlim \beta$ 存在.*

Proof. (可参考 Riehl 2014 §3.4) 下面以 $\alpha : I \to \mathcal{C}^\wedge$ 为例. 此时若任意给出 $i \xrightarrow[I]{\sigma} j$ 和 $S \xrightarrow[\mathcal{C}^{\mathrm{op}}]{f} T$, 显然 $\alpha(i)(S), \alpha(\sigma)(f)$ 均是有定义的. 此外, 类似 $\alpha(i)(f), \alpha(\sigma)(\cdot)$ 等表示的定义也是自明的.

对任意 $S \in \mathrm{Ob}(\mathcal{C})$, 定义函子 $\alpha(\cdot)(S) : I \to \mathsf{Set}$, 其将 $i \in \mathrm{Ob}(I)$ 映射至 $\alpha(i)(S)$. 应用之前的命题可知对每个 S 都存在 $\varinjlim \alpha(\cdot)(S)$ 与相应的态射族 $(\iota_{i,S})_i$:

$$\alpha(\cdot)(S) \xrightarrow{(\iota_{i,S})_i} \Delta(\varinjlim \alpha(\cdot)(S))$$

定义 $\mathcal{C}^\wedge$ 中的元素 $\varinjlim \alpha$ 如下:

$$\begin{aligned} \mathcal{C}^{\mathrm{op}} &\longrightarrow \mathsf{Set} \\ S &\longmapsto \varinjlim \alpha(\cdot)(S) \\ \left[f : S \xrightarrow[\mathcal{C}^{\mathrm{op}}]{} T \right] &\longmapsto \varinjlim \alpha(\cdot)(f) \end{aligned}$$

我们断言下面资料构成了 (α/Δ) 中的初对象, 从而 $\varinjlim \alpha$ 就是 α 的余极限.

$$\alpha \xrightarrow{\iota=((\iota_{i,S})_S)_i} \Delta(\varinjlim \alpha)$$

接下来的验证分为三个部分:

A. ι 是函子间的态射.

B. 对任意 $(L,\xi) \in \mathrm{Ob}(\alpha/\Delta)$, 存在唯一的 $\phi : \varinjlim \alpha \to L$.

C. ϕ 是函子间的态射.

(A.) 归结于证明 $\alpha(i)(\cdot) \xrightarrow{(\iota_{i,S})_S} \varinjlim \alpha$ 是态射; 以及下图交换. 证明是显见的.

$$\begin{tikzcd}
\alpha(i) \arrow[r, "(\iota_{i,S})_S"] \arrow[d, "\alpha(\sigma)"'] & \varinjlim \alpha \\
\alpha(j) \arrow[ur, "(\iota_{j,S})_S"'] &
\end{tikzcd}$$

(B.) 给出对象 (L,ξ), 对任意 $S \in \mathrm{Ob}(\mathcal{C})$:

$$\begin{tikzcd}
\alpha(\cdot)(S) \arrow[r, "(\iota_{i,S})_i"] \arrow[dr, "(\xi_{i,S})_i"'] & \Delta(\varinjlim \alpha(\cdot)(S)) \arrow[d, "\exists!\Delta(\phi_S)"] \\
& \Delta(L(S))
\end{tikzcd} \quad \Rightarrow \quad \varinjlim \alpha \xrightarrow[\mathcal{C}^\wedge]{\exists!(\phi_S)_S} L$$

(C.) 我们的目标是验证对任意 $f : S \xrightarrow[\mathcal{C}^{\mathrm{op}}]{} T$, 下图交换:

$$\begin{tikzcd}
\varinjlim \alpha(\cdot)(S) \arrow[r, "\phi_S"] \arrow[d, "\varinjlim \alpha(\cdot)(f)"'] & L(S) \arrow[d, "L(f)"] \\
\varinjlim \alpha(\cdot)(T) \arrow[r, "\phi_T"'] & L(T)
\end{tikzcd}$$

证明又分为三步, 分别为:

(C1.) 下图交换:

$$\begin{tikzcd}
\alpha(\cdot)(S) \arrow[r, "(\xi_{i,S})_i"] \arrow[d, "\alpha(\cdot)(f)"'] & \Delta(L(S)) \arrow[d, "\Delta(L(f))"] \\
\alpha(\cdot)(T) \arrow[r, "(\xi_{i,T})_i"'] & \Delta(L(T))
\end{tikzcd}$$

这是 $\alpha \xrightarrow{\xi} \Delta(L)$ 的结果.

(C2.) $\varinjlim \Delta(L(S)) = L(S)$, $\varinjlim \Delta(L(f)) = L(f)$. 证明是容易的.

(C3.) $\varinjlim[(\xi_{i,S})_i] = \phi_S$. 这来源于下图:

$$\begin{tikzcd}
\alpha(\cdot)(S) \arrow[r, "(\iota_{i,S})_i"] \arrow[d, "(\xi_{i,S})_i"'] & \Delta(\varinjlim \alpha(\cdot)(S)) \arrow[d, "\Delta(\phi_S)"] \\
\Delta(L(S)) \arrow[r, "\mathrm{id}"'] & \Delta(\varinjlim \Delta(L(S)))
\end{tikzcd}$$

最后再在 C1. 的交换图中调用书 Lemma 2.7.4. $\square$

下面是一个比较困难的证明, 书中对此的说明仅寥寥数笔, 对于初学者是极其难懂的. 好在其他的范畴论入门教材基本都讲到了这一命题 (比如 Perrone 2024 §3.4.2).

Proposition 2.2.5 *对于 $\beta : I^{\mathrm{op}} \to \mathcal{C}$, 可定义 $\bar{\beta} : I^{\mathrm{op}} \to \mathcal{C}^\wedge$, 映 $i \mapsto h_{\mathcal{C}}(\beta(i)) = \mathrm{Hom}_{\mathcal{C}}(\cdot, \beta(i))$. 则 $\exists \varprojlim \beta$ 当且仅当 $\varprojlim \bar{\beta}$ 可表.*

对于 $\alpha : I \to \mathcal{C}$, 可定义 $\bar{\alpha} : I \to \mathcal{C}^\vee$, 映 $i \mapsto k_{\mathcal{C}}(\alpha(i)) = \mathrm{Hom}_{\mathcal{C}}(\alpha(i), \cdot)$. 则 $\exists \varinjlim \alpha$ 当且仅当 $\varinjlim \bar{\alpha}$ 可表.

Proof. 我们以 $\beta : I^{\mathrm{op}} \to \mathcal{C}$ 为例. 证明分为几个部分:

A. $\exists \varprojlim \beta \Rightarrow h_{\mathcal{C}}(\varprojlim \beta) \underset{\exists}{\xrightarrow{\sim}} \varprojlim \bar{\beta}$.

A1. $\forall S \in \mathrm{Ob}(\mathcal{C})\ \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta) \simeq \varprojlim \bar{\beta}(\cdot)(S)$.

A2. $\forall S \in \mathrm{Ob}(\mathcal{C})\ \mathrm{Hom}_{\mathcal{C}}(S, \varprojlim \beta) \simeq \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta)$.

B. $h_{\mathcal{C}}(D) \underset{\exists}{\xrightarrow{\sim}} \varprojlim \bar{\beta} \Rightarrow \exists \varprojlim \beta$.

(A1.)

$$\begin{aligned}
\varprojlim \bar{\beta}(\cdot)(S) &= \left\{ (\xi_i)_i \in \prod_I \bar{\beta}(i)(S) : \forall \sigma : i \xrightarrow{I} j (\xi_j \circ \beta(\sigma) = \xi_i) \right\} \\
&= \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta)
\end{aligned}$$

(A2.) 令 $\mathcal{C}^\wedge \ni A = \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(\cdot), \beta)$, 回顾 Yoneda 定理, 对于任意 S, 我们有:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{Hom}(\cdot, S), \mathrm{Hom}(\Delta(\cdot), \beta)) & \xleftrightarrow{1:1} & \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta) \\
\phi & \longmapsto & \phi_S(\mathrm{id}_S) \\
(\phi_T : f \mapsto u \circ \Delta(f))_{T \in \mathrm{Ob}(\mathcal{C})} & \longleftarrow & u
\end{array} \tag{2.1}$$

这是个极其实用的公式. 我们断言: $\Delta(S) \xrightarrow{u} \beta$ 是 (Δ/β) 中的终对象时 (即 $S = \varprojlim \beta$), u 的像 ϕ 是同构. 反之, 若 ϕ 是同构, 则 S 连同 $\phi_S(\mathrm{id}_S)$ 构成终对象. 后者的证明可见

(B.); 对于前者, 我们取 $S = \varprojlim \beta$, 然后验证对任意 S, ϕ_S 都是双射:

$$\mathrm{Hom}_{\mathcal{C}}(S, \varprojlim \beta) \overset{1:1}{\longleftrightarrow} \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta)$$

$$f \xmapsto[\phi_S]{\qquad\qquad} u \circ \Delta(f)$$

$$\varprojlim \lambda \xleftarrow[\psi_S]{\qquad\qquad} \lambda$$

$\psi_S \phi_S$ 是单位映射, 原因来自下图:

$$\begin{tikzcd}
\Delta(S) \arrow[r, "\Delta(f)"] \arrow[d, dashed, "\varprojlim"'] & \Delta(\varprojlim \beta) \arrow[r, "u"] \arrow[d, dashed, "\varprojlim"] & \beta \arrow[d, dashed, "\varprojlim"] \\
S \arrow[r, "f"'] & \varprojlim \beta \arrow[r, "\mathrm{id}"'] & \varprojlim \beta
\end{tikzcd}$$

$\phi_S \psi_S$ 也是单位映射, 即 $u \circ (\varprojlim \lambda) = \lambda$, 证明很容易.

(B.) 假设存在 D 使得 $h_{\mathcal{C}}(D) \xrightarrow[\phi]{\sim} \varprojlim \bar{\beta} = \mathrm{Hom}(\Delta(\cdot), \beta)$. 通过 (2.1) 式将 ϕ 映为 $u = \phi_D(\mathrm{id}_D)$. 对于任意 S, 观察到映射:

$$\mathrm{Hom}_{\mathcal{C}}(S, D) \overset{1:1}{\longleftrightarrow} \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta)$$

$$f \xmapsto[\phi_S]{\qquad\qquad} u \circ \Delta(f)$$

$$\psi_S(\lambda) \xleftarrow[\psi_S]{\qquad\qquad} \lambda$$

其中 ψ_S 为 ϕ_S 的逆. 断言 $\Delta(D) \xrightarrow{u}$ 给出 (Δ/β) 中的一个终对象. 对于任意 (S, λ), 上面映射给出了唯一一个 $\psi_S(\lambda) : S \to D$. 同时由于 $u \circ \Delta(\psi_S(\lambda)) = \phi_S \psi_S(\lambda) = \lambda$, 从而下图交换:

$$\begin{tikzcd}
\Delta(D) \arrow[r, "u"] & \beta \\
\Delta(S) \arrow[u, "\Delta(\psi_S(\lambda))"] \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

□

下面简单补充一下 $\alpha : I \to \mathcal{C}$ 的情况. 定义 $\bar{\alpha} : I \to \mathcal{C}^{\vee}$, 映 i 到 $\mathrm{Hom}_{\mathcal{C}}(\alpha(i), \cdot)$. 从而 $\bar{\alpha}(\cdot)(S) : I \to \mathsf{Set}^{\mathrm{op}}$, 这等同于 $I^{\mathrm{op}} \to \mathsf{Set}$. 然后就可以验证:

$$\varprojlim \bar{\alpha}(\cdot)(S) = \mathrm{Hom}_{\mathcal{C}^I}(\alpha, \Delta(S))$$
$$\mathrm{Hom}_{\mathcal{C}^{\vee}}(\mathrm{Hom}(\alpha, \Delta(\cdot)), \mathrm{Hom}(S, \cdot)) \simeq \mathrm{Hom}_{\mathcal{C}^I}(\alpha, \Delta(S))$$

几个比较短的小命题, 书中的证明要么没有, 要么过于简略. 故在此处补充上.

Proposition 2.2.6 *设 I, J 为小范畴, $\alpha : I \to \mathcal{C}^J$, 若对任意 $j \in J$, 存在 $\varinjlim_i \alpha(\cdot)(j)$, 则存在 $\varinjlim_i \alpha \in \mathcal{C}^J$.*

Proof. 令

$$\alpha(i)(j) \xrightarrow{\iota_{i,j}} \varinjlim_i \alpha(\cdot)(j)$$

再定义 $\mathcal{C}^J$ 中元素 $\varinjlim_i \alpha$ 如下:

$$j \longmapsto \varinjlim_i \alpha(\cdot)(j)$$

$$[\sigma : j \to j'] \longmapsto \varinjlim_i \alpha(\cdot)(\sigma)$$

可以验证 $\alpha \xrightarrow{((\iota_{i,j})_j)_i} \Delta_I\left(\varinjlim_i \alpha\right)$ 成为 (α/Δ) 中的初对象, 从而 $\varinjlim_i \alpha$ 正是 α 极限. 证明过程和 Proposition 2.2.4 非常类似. □

Proposition 2.2.7 *设 I, J 为小范畴, 假设 $\mathcal{C}$ 中具有以 I, J 为指标的 $\varinjlim$ (即对任意 $\alpha : I \to \mathcal{C}$, 都存在 $\varinjlim \alpha$, J 上同理). 考虑函子 $I \times J \to \mathcal{C}$, 存在典范同构*

$$\varinjlim_j \left(\varinjlim \alpha(\cdot, j)\right) \simeq \varinjlim \alpha \simeq \varinjlim_i \left(\varinjlim \alpha(i, \cdot)\right)$$

Proof. 对任意 j, $\alpha(\cdot, j)$ 定义为函子 $i \mapsto \alpha(i, j)$; 根据条件, 对任意 j 都存在 $\varinjlim \alpha(\cdot, j) \in \mathcal{C}$. 结合上一命题, I 到 $\mathcal{C}^J$ 的函子 $[i \to \alpha(i, \cdot)]$ 存在一个极限, 可用 $\left[j \mapsto \varinjlim \alpha(\cdot, j)\right]$ 表示, 具体来说其是函子:

$$j \longmapsto \varinjlim \alpha(\cdot, j)$$

$$[\sigma : j \to j'] \longmapsto \left[\varinjlim \alpha(\cdot, \sigma) : \varinjlim \alpha(\cdot, j) \to \varinjlim \alpha(\cdot, j')\right]$$

因此 $\varinjlim_j \left(\varinjlim \alpha(\cdot, j)\right)$ 是有定义的. 根据 Proposition 2.2.5, 只需证明 $\bar{\alpha}$ 是可表的, 这从下方可以看出:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}\left(S, \varinjlim_j \left(\varinjlim \alpha(\cdot, j)\right)\right) &\simeq \mathrm{Hom}_{\mathcal{C}^J}\left(\Delta_J(S), \left[j \mapsto \varinjlim \alpha(\cdot, j)\right]\right) \\
&\simeq \mathrm{Hom}_{(\mathcal{C}^J)^I}\left(\Delta_I(\Delta_J(S)), [i \mapsto \alpha(i, \cdot)]\right) \\
&\simeq \mathrm{Hom}_{\mathcal{C}^{I \times J}}\left(\Delta_{I \times J}(S), \alpha\right) \\
&= \varinjlim \bar{\alpha}(S)
\end{aligned}$$

□

Proposition 2.2.8 *设有小集合的无交并分解 $I = \bigsqcup_{j \in J} I_j$. 若范畴 $\mathcal{C}$ 具有以 J 和 I_j 的积, 则以 I 为指标的积也存在. 且存在唯一同构使下图交换:*

$$\begin{tikzcd}
\prod_{i \in I} X_i \arrow[rr, "\sim"] \arrow[dr, "p_k"'] & & \prod_{j \in J}\left(\prod_{i \in I_j} X_i\right) \arrow[dl, "p_k p_j"] \\
& X_k &
\end{tikzcd}$$

同样的断言对余积也成立.

Proof. 由于 J, I_j, I 都是小集合, 自然可以将其视为离散小范畴. 我们先给出几个函子的符号:

1. $\beta^{I_j} : I_j^{\mathrm{op}} \to \mathcal{C}$, 给出 $(X_\mu)_{\mu \in I_j}$;
2. $\beta^I : I^{\mathrm{op}} \to \mathcal{C}$, 给出 $(X_\mu)_{\mu \in I}$;
3. $\varprojlim \beta^{I_{(\cdot)}} : J \to C$, 其映 j 为 $\varprojlim \beta^{I_j} = \prod_{\mu \in I_j} X_\mu$.

此处我们再一次调用 (2.1), 证明的大体框架就是:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}\left(S, \varprojlim_j \left(\varprojlim \beta^{I_{(\cdot)}}\right)\right) &\simeq \mathrm{Hom}_{\mathcal{C}^J}\left(\Delta_J(S), \varprojlim \beta^{I_{(\cdot)}}\right) \\
&\simeq \prod_J \mathrm{Hom}_{\mathcal{C}}\left(S, \varprojlim \beta^{I_j}\right) \\
&\simeq \prod_J \left[\mathrm{Hom}_{\mathcal{C}^{I_j}}\left(\Delta_{I_j}(S), \beta^{I_j}\right)\right] \\
&\simeq \prod_J \left[\prod_{\mu \in I_j} \mathrm{Hom}_{\mathcal{C}}\left(S, \beta^{I_j}(\mu)\right)\right] \\
&\simeq \prod_{\mu \in I} \mathrm{Hom}_{\mathcal{C}}\left(S, \beta^I(\mu)\right) \\
&\simeq \mathrm{Hom}_{\mathcal{C}^I}\left(\Delta_I(S), \beta^I\right)
\end{aligned}$$

依据 Proposition 2.2.5, 可以说 $\varprojlim_j \left(\varprojlim \beta^{I_{(\cdot)}}\right)$, 或者表示为 $\prod_{j \in J}\left(\prod_{i \in I_j} X_i\right)$, 就是 β^I 的极限. 而两个极限之间存在唯一同构. □

书中的引理 2.7.13 证明思路是类似的.

Theorem 2.2.9 (书定理 2.8.3) *设 I 为小范畴, $\mathcal{C}$ 为范畴.*

1. *若对所有子集 $J \subset \mathrm{Mor}(I)$ 和 $\mathcal{C}$ 中对象族 $(X_j)_{j \in J}$ 都存在 $\prod_J X_j$; 而且对所有 $f, g : X \to Y$ 都存在 $\ker(f, g)$; 则 $\mathcal{C}$ 有所有以 I 为指标的 $\varprojlim$.*
2. *若对所有子集 $J \subset \mathrm{Mor}(I)$ 和 $\mathcal{C}$ 中对象族 $(X_j)_{j \in J}$ 都存在 $\coprod_J X_j$; 而且对所有 $f, g : X \to Y$ 都存在 $\mathrm{coker}(f, g)$; 则 $\mathcal{C}$ 有所有以 I 为指标的 $\varinjlim$.*

Proof. 在一切开始之前我们先声明要使用的符号. $\mathrm{Ob}(I)$ 和 $\mathrm{Mor}(I)$ 作为小集合, 自然可视作离散小范畴. 如果给定了 $\beta : I^{\mathrm{op}} \to \mathcal{C}$, 其中 I 由资料 $(\mathrm{Ob}(I), \mathrm{Mor}(I), s, t, \circ)$ 表示, 就可以定义如下两个离散范畴上的函子, 并且根据条件可得出它们的积:

$$\begin{aligned}
\beta_{\mathrm{Ob}} : \mathrm{Ob}(I) &\longrightarrow \mathcal{C} \\
i &\longmapsto \beta(i)
\end{aligned}
\qquad
\Delta_{\mathrm{Ob}(I)}\left(\prod \beta(i)\right) \xrightarrow{p} \beta_{\mathrm{Ob}}$$

$$\begin{aligned}
\beta_{\mathrm{Mor}} : \mathrm{Mor}(I) &\longrightarrow \mathcal{C} \\
\sigma &\longmapsto \beta(s(\sigma))
\end{aligned}
\qquad
\Delta_{\mathrm{Mor}(I)}\left(\prod \beta(s(\sigma))\right) \xrightarrow{q} \beta_{\mathrm{Mor}}$$

然后, 可以定义两个态射:

$$\prod_i \beta(i) \underset{N(\sigma)=\beta(\sigma)\circ p(t(\sigma))}{\overset{M(\sigma)=p(s(\sigma))}{\rightrightarrows}} \beta(s(\sigma))$$

将其放置到逗号范畴 $(\Delta/\beta_{\mathrm{Mor}})$ 中, 将会分别导出两个态射 $f = \varprojlim M, g = \varprojlim N$:

$$\begin{tikzcd} \Delta_{\mathrm{Mor}(I)}\left(\prod_\sigma \beta(s(\sigma))\right) \arrow[r, "q"] & \beta_{\mathrm{Mor}} \\ \Delta_{\mathrm{Mor}(I)}\left(\prod_i \beta(i)\right) \arrow[u, shift left, "\Delta(f)"] \arrow[u, shift right, "\Delta(g)"'] \arrow[ur, shift left, "M"] \arrow[ur, shift right, "N"'] & \end{tikzcd}$$

然后我们再定义一个范畴 I_0 为 $\bullet \rightrightarrows \bullet$. 函子 $\beta_0 : I_0^{\mathrm{op}} \to \mathcal{C}$ 给出 $\prod_i \beta(i) \overset{f}{\underset{g}{\rightrightarrows}} \prod_\sigma \beta(s(\sigma))$.

剩下的证明为:

A. $(\forall\sigma(q(\sigma)\circ\mu = q(\sigma)\circ\nu)) \Rightarrow \mu = \nu$;

B. $\forall L \in \mathrm{Ob}(\mathcal{C})\ \varprojlim \mathrm{Hom}(L, \beta(\cdot)) \simeq \mathrm{Hom}(L, \ker(f,g))$.

  B1. $\varprojlim \mathrm{Hom}(L, \beta(\cdot)) \simeq \mathrm{Hom}(\Delta_I(L), \beta)$;

  B2. $\mathrm{Hom}(\Delta_I(L), \beta) \simeq \{\psi \in \mathrm{Hom}(L, \prod_i \beta(i)) : f\psi = g\psi\}$;

  B3. $\{\psi \in \mathrm{Hom}(L, \prod_i \beta(i)) : f\psi = g\psi\} \simeq \mathrm{Hom}_{\mathcal{C}^{I_0}}(\Delta_{I_0}(L), \beta_0)$;

  B4. $\mathrm{Hom}_{\mathcal{C}^{I_0}}(\Delta_{I_0}(L), \beta_0) \simeq \mathrm{Hom}(L, \ker(f,g))$.

(A.) 对任意 $L \overset{\mu}{\underset{\nu}{\rightrightarrows}} \prod_\sigma \beta(s(\sigma))$, 若满足 $\forall\sigma(q(\sigma)\circ\mu = q(\sigma)\circ\nu)$, 从下图可知 $q\circ\Delta_{\mathrm{Mor}(I)}(\mu)$ 与 $q\circ\Delta_{\mathrm{Mor}(I)}(\nu)$ 合成同一个态射, 由终对象定义可知 $\mu = \nu$:

$$\begin{tikzcd} \Delta_{\mathrm{Mor}(I)}\left(\prod_\sigma \beta(s(\sigma))\right) \arrow[r, "q"] & \beta_{\mathrm{Mor}} \\ \Delta_{\mathrm{Mor}(I)}(L) \arrow[u, shift right, "\Delta(\nu)"] \arrow[u, shift left, "\Delta(\mu)"'] \arrow[ur] & \end{tikzcd}$$

(B1.) 老生常谈的证明了, 可参考 Proposition 2.2.5 的证明.
(B2.) 我们断言:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}^I}(\Delta_I(L), \beta) & \xrightarrow{1:1} & \{\psi \in \mathrm{Hom}_{\mathcal{C}}(L, \prod_i \beta(i)) : f\psi = g\psi\} \\ \phi & \longmapsto & \varprojlim_{\mathrm{Ob}(I)} \phi \\ p\circ\Delta_I(\psi) & \longleftarrow\!\shortmid & \psi \end{array}$$

设 $\phi \in \mathrm{Hom}(\Delta_{I^{\mathrm{op}}}(L), \beta)$. 首先注意到下面两图交换:

$$
\begin{tikzcd}
 & \beta(t(\sigma)) \arrow[dd, "\beta(\sigma)"] \\
L \arrow[ur, "\phi(t(\sigma))"] \arrow[dr, "\phi(s(\sigma))"'] & \\
 & \beta(s(\sigma))
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \beta(t(\sigma)) \\
L \arrow[ur, "\phi(t(\sigma))"] \arrow[dr, "\phi(s(\sigma))"'] \arrow[r, "\psi"] & \prod_I \beta(i) \arrow[u, "p(t(\sigma))"'] \arrow[d, "p(s(\sigma))"] \\
 & \beta(s(\sigma))
\end{tikzcd}
$$

由此推出 $\beta(\sigma) \circ p(t(\sigma)) \circ \psi = p(s(\sigma)) \circ \psi$. 再观察图:

$$
\begin{tikzcd}
L \arrow[r, "\psi"] & \prod_i \beta(i) \arrow[r, shift left, "M(\sigma)"] \arrow[r, shift right, "N(\sigma)"'] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & \beta(s(\sigma)) \\
 & \prod_\sigma \beta(s(\sigma)) \arrow[ur, "q(\sigma)"'] &
\end{tikzcd}
$$

又可推得 $q(\sigma) \circ f \circ \psi = q(\sigma) \circ g \circ \psi$. 调用 (A) 就有 $f \circ \psi = g \circ \psi$.

另一方面, 对任意 $\psi \in \mathrm{Hom}_{\mathcal{C}}\left(L, \prod_i \beta(i)\right)$ 且满足 $f\psi = g\psi$, 可推出

$$\forall \sigma\, (p(s(\sigma)) \circ \psi = \beta(\sigma) \circ p(t(\sigma)) \circ \psi)$$

从而对任意 $i \xrightarrow[I^{\mathrm{op}}]{\sigma} j$, 下图交换:

$$
\begin{tikzcd}
 & & \beta(i) \arrow[dd, "\beta(\sigma)"] \\
L \arrow[r, "\psi"] & \prod_i \beta(i) \arrow[ur, "p(i)"'] \arrow[dr, "p(j)"] & \\
 & & \beta(j)
\end{tikzcd}
$$

令 $\phi = p \circ \Delta_I(\psi)$, 即得 $\phi \in \mathrm{Hom}_{\mathcal{C}^I}\left(\Delta_I(L), \beta\right)$.

(B3.) 和 (B4.) 实际上可由一图道出:

$$
\begin{tikzcd}
L \arrow[r, "\psi"] \arrow[dr, dashed, "\exists!"'] & \prod_i \beta(i) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \prod_\sigma \beta(s(\sigma)) \\
 & \ker(f,g) \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

仔细来说, (B3.) 仅需如下双射:

$$\left\{\psi \in \mathrm{Hom}_{\mathcal{C}}\left(L, \prod_i \beta(i)\right) : f\psi = g\psi\right\} \xrightarrow{1:1} \mathrm{Hom}_{\mathcal{C}^{I_0^{\mathrm{op}}}}\left(\Delta_{I_0^{\mathrm{op}}}(L), \beta_0\right)$$

$$\psi \longmapsto (g\psi, \psi)$$

$$\psi_2 \longleftarrow\!\shortmid (\psi_1, \psi_2)$$

至于 (B4.), 注意到 $\varprojlim \beta_0 = \ker(f,g)$, 令 $\Delta_{I_0^{\mathrm{op}}}(\ker(f,g)) \xrightarrow{u} \beta_0$, 调用 (2.1), 有:

$$\begin{array}{ccc} \mathrm{Hom}(h_{\mathcal{C}}(\ker(f,g)), \mathrm{Hom}(\Delta(\cdot),\beta_0)) & \xleftarrow{\sim} & \mathrm{Hom}(\Delta(\ker(f,g)),\beta_0) \\ \phi = (\phi_L : f \mapsto u \circ \Delta(f))_L & \longleftarrow\!\shortmid & u \end{array}$$

同时 ϕ 也是自然同构, 对任意 L, 明细如下:

$$\begin{array}{ccc} \mathrm{Hom}(L,\ker(f,g)) & \xleftrightarrow{1:1} & \mathrm{Hom}(\Delta(L),\beta_0) \\ f & \longmapsto & u \circ \Delta_{I_0^{\mathrm{op}}}(f) \\ \varprojlim g & \longleftarrow\!\shortmid & g \end{array}$$

现在所有同构都已找到, 最后我们还要从中抽离出态射 $\Delta(\varprojlim \beta) \to \beta$. 这里依然参考 Proposition 2.2.5 的做法, 同构

$$h_{\mathcal{C}}(\ker(f,g)) \xrightarrow[\sigma]{\sim} \varprojlim \bar{\beta} = \mathrm{Hom}(\Delta_{I^{\mathrm{op}}}(\cdot),\beta)$$

由

$$\begin{array}{ccccccc} \mathrm{Hom}(L,\ker(f,g)) & \xrightarrow{\sim} & \mathrm{Hom}(\Delta(L),\beta_0) & \xrightarrow{\sim} & \{\psi \in \mathrm{Hom}(L,\prod_i \beta(i)) : \cdots\} & \xrightarrow{\sim} & \mathrm{Hom}(\Delta(L),\beta) \\ \Cup & & \Cup & & \Cup & & \Cup \\ a & \longmapsto & u \circ \Delta_{I_0^{\mathrm{op}}}(a) & \longmapsto & b = \left[L \xrightarrow{a} \ker(f,g) \to \prod_i \beta(i)\right] & \longmapsto & p \circ \Delta_{I^{\mathrm{op}}}(b) \end{array}$$

定义. 而 $\sigma_{\ker(f,g)}(\mathrm{id})$ 即为所求, 具体表示为态射:

$$\ker(f,g) \to \prod_i \beta(i) \xrightarrow{p(i)} \beta(i)$$

□

Chapter 3

Group Theory

3.1 Basics

Definition 3.1.1 设 G 为群, $E \subset G$ 为子集.

- $Z_G(E) := \{a \in G : \forall x \in E(xa = ax)\}$;
- $N_G(E) := \{a \in G : aEa^{-1} = E\}$;
- $Z_G = Z_G(G)$.

可以验证 $Z_G(E) \lhd N_G(E) < G$.

Corollary 3.1.2 若 $N, H < G$, 且 $H \subset N_G(N)$, 则:

- $NH = HN$;
- $NH < G$;
- $N \lhd NH$.

3.2 Free Group

Definition 3.2.1 设 $U : \mathsf{Mon} \to \mathsf{Set}$ 是遗忘函子. 对任意集合 X, 若 $(\mathbf{F}(X), \iota)$ 是逗号范畴 (j_X, U) 中的初对象, 则称 $\mathbf{F}(X)$ 称为自由幺半群. 自由群的定义无非是将 Mon 换成 Grp, 自由交换群则换成 Ab.

Proposition 3.2.2 自由是遗忘的左伴随.

Proof. 设 $\mathcal{C}$ 为一个代数系统的范畴, 如 Grp. 为了方便描述, 现定义集合:

$$\left\{(F, \iota) \in \mathrm{Ob}(\mathcal{C})^{\mathrm{Ob}(\mathsf{Set})} \times \left[\bigcup_X \mathrm{Hom}(X, UF(X))\right]^{\mathrm{Ob}(\mathsf{Set})} : P\right\}$$

其中 P 是条件: ι 映 X 为态射 $\iota(X) : X \to UF(X)$, 且使 $X \xrightarrow{\iota(X)} UF(X)$ 成为 (j_X, U) 中的初对象. 断言

$$\{(F,\iota) : P\} \xleftrightarrow{1:1} \{(\mathbf{F}, U, \phi) : \text{Adjoint pair}\}$$

现在来构造这一双射. 给定 (F,ι) 可以定义一个函子 $\mathbf{F}$, 映 X 为 FX, 而对于态射 $f : X \to Y$, $\mathbf{F}f$ 定义为如下态射:

$$\begin{tikzcd} X \arrow[r, "\iota(X)"] \arrow[d, "f"'] & U\mathbf{F}(X) \arrow[d, "\mathbf{F}f"] \\ Y \arrow[r, "\iota(Y)"'] & U\mathbf{F}(Y) \end{tikzcd} .$$

紧接着定义 $\phi : \mathrm{Hom}_{\mathcal{C}}(\mathbf{F}(\cdot), \cdot) \to \mathrm{Hom}_{\mathsf{Set}}(\cdot, U(\cdot))$. 对任意 $(L,\xi) \in \mathrm{Ob}(j_X, U)$, 由 $(UFX, \iota(X))$ 的初对象性质可导出唯一的 $\bar{\xi} : FX \to L$. 定义 $\phi_{X,L}$ 如下:

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{Set}}(X, U(L)) \arrow[r, "\sim", "\phi_{X,L}^{-1}"'] & \mathrm{Hom}_{\mathcal{C}}(\mathbf{F}(X), L) \\ \xi \arrow[r, mapsto] & \bar{\xi} \\ U(\bar{\xi}) \circ \iota(X) & \bar{\xi} \arrow[l, mapsto] \end{tikzcd}$$

还需验证 ϕ 确实是自然同构, 暂且略过.

反之, 给定伴随对 $(\mathbf{F}, U, \phi)$, F 就可直接定义为 $X \mapsto \mathbf{F}(X)$, 所以只要找到 ι 的定义即可完成任务. 任给 $\xi \in \mathrm{Hom}(X, U(L))$, 定义态射 η 为 $\phi_{X,L}^{-1}(\xi)$. 观察下面交换图:

$$\begin{tikzcd} \mathrm{id}_{\mathbf{F}(X)} \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \iota_X \arrow[ddd, mapsto] \\ & \mathrm{Hom}(\mathbf{F}(X), \mathbf{F}(X)) \arrow[r, "\sim", "\phi_{X,\mathbf{F}(X)}"'] \arrow[d, "\eta\circ-"'] & \mathrm{Hom}(X, U\mathbf{F}(X)) \arrow[d, "U(\eta)\circ-"] & \\ & \mathrm{Hom}(\mathbf{F}(X), L) \arrow[r, "\sim", "\phi_{X,L}"'] & \mathrm{Hom}(X, U(L)) & \\ \eta \arrow[rrr, mapsto] & & & \xi \end{tikzcd}$$

可以看出 $U(\eta) \circ \iota_X = \xi$. 至此得出结论, 对任意 (L,ξ), 存在唯一的 $\eta : FX \to L$ 使 $U(\eta) \circ \iota_X = \xi$ 成立. 从而 $(FX, (\iota_X)_X)$ 成为了 (j_X, U) 中的初对象, ι 的定义不言自明. □

构造**自由幺半群**. 对任意集合 X, 令

$$\mathbf{F}(X) = \{(x_i)_{i=1}^n \in X^n : n \in \mathbb{Z}_{\geq 0}\}$$

当 $n = 0$ 时, 元素空括号 $()$ 也可表示为 ϵ. 再在上述集合上赋予乘法:

$$((x_1, \cdots, x_m), (y_1, \cdots, y_n)) \mapsto (x_1, \cdots, x_m, y_1, \cdots, y_n)$$

容易验证 $\mathbf{F}(X)$ 是幺半群, 且满足自由幺半群的定义. 在没有歧义的情况下, $(x_1, \cdots, x_n)$ 可以记作 $x_1 \cdots x_n$. 这样一个元素称为一个**字**.

在定义融合积前先引入一个范畴. 资料 $E = (M_i, [f_i : A \to M_i])_{i\in I}$ 中 A 和 M_i 都是幺半群, f_i 是态射, I 是一个集合. 定义范畴 Pam_E 如下 (命名是 Pre-Amalgamated-product 的缩写):

- 元素: $(M, (\phi_i : M_i \to M)_i)$ 满足 $\phi_i f_i = \phi_j f_j$;
- 态射: $\psi : (M, (\phi_i : M_i \to M)_i) \to (M', (\phi'_i : M_i \to M')_i)$ 满足 $\forall i(\psi\phi_i f_i = \phi'_i f_i)$;
- 合成: $\psi' \circ \psi$ 就是一般的幺半群同态的合成.

Definition 3.2.3 *资料 E 的融合积 $\mathrm{Am}(E)$ 为 Pam_E 中的初对象.*

Proposition 3.2.4 *任给上述资料 E, 均存在 $\mathrm{Am}(E)$.*

Proof. 此证明中会多处出现重复括号, 这只是为了方便看出元素所属的集合, 在足够熟练时可根据自己的喜好简化. 设 $E = (M_i, f_i)_{i\in I}$, 取 $S = \bigsqcup_{i\in I} M_i$, 再取其自由幺半群 $\iota : S \to \mathbf{F}(S)$. 严格来讲, S 中的元素应表示为 (m_i, i), 其中 $m_i \in M_i$; 而 $\mathbf{F}(S)$ 中的元素应表示为 $((m_{i_1}, i_1), \cdots, (m_{i_s}, i_s))$. $\mathbf{F}(X)$ 上定义三个关系:

(i) $R_1 = \{(((x_i y_i, i)), ((x_i, i), (y_i, i))) : i \in I \wedge x_i, y_i \in M_i\}$;

(ii) $R_2 = \{(((1_i, i)), \epsilon) : i \in I\}$;

(iii) $R_3 = \{(((f_i(a), i)), ((f_j(a), j))) : i, j \in I \wedge a \in A\}$.

令 $R = R_1 \cup R_2 \cup R_3$; 以及 $\sim$, $\mathbf{F}(S)^2$ 上包含 R 且满足 (iv): $x \sim x' \wedge y \sim y' \Rightarrow xy \sim x'y'$ 的等价关系, 此关系的存在性显然, 比如直接取 $\sim = \mathbf{F}(S)^2$. 困难在于如何构造, 以及是否有最小的等价关系. 回顾生成子幺半群的构造, 设 $X \subset M$ 是幺半群 M 的一个非空子集, 则包含 X 的最小子幺半群可表示为:

$$\langle X \rangle = \{x_1 \cdots x_n : n \in \mathbb{Z}_{\geq 0} \wedge x_i \in X\}$$

当 $n = 0$ 时元素视作 1_M. 现在我们有幺半群 $\mathbf{F}(S)^2$ 和其子集 R, 我们断言 $\langle R \cup \Delta \cup R^{\mathrm{op}} \rangle$ **生成的等价关系** S 是满足 (i)-(iv) 的最小等价关系; 其中 $\Delta = \{(x, x) : x \in \mathbf{F}(S)\}$, $R^{\mathrm{op}} = \{(y, x) : (x, y) \in R\}$. 下面开始验证.

对任意 $(a, b), (c, d) \in S$, 可以获得类似下属式子的表达:

$$a = x_0 \longrightarrow x_1 \longleftarrow x_2 \longleftarrow x_3 = b$$

$$c = y_0 \longleftarrow y_1 = d$$

其中 $x_0 \to x_1$ 表示 $(x_0, x_1) \in \langle R \cup \Delta \cup R^{\mathrm{op}} \rangle$. 注意到 $x_0 \to x_1 \Leftrightarrow x_1 \to x_0$, 理由: 设 $(x_0, x_1) = (x_{01} \cdots x_{0n}, x_{11} \cdots x_{1n})$, 其中每个 $(x_{0i}, x_{1i}) \in R \cup \Delta \cup R^{\mathrm{op}}$. 因此每个 $(x_{1i}, x_{0i}) \in R \cup \Delta \cup R^{\mathrm{op}}$, 即得

$$(x_1, x_0) = (x_{11} \cdots x_{1n}, x_{01} \cdots x_{0n}) = (x_{11}, x_{01}) \cdots (x_{1n}, x_{0n}) \in \langle R \cup \Delta \cup R^{\mathrm{op}} \rangle$$

其次, 注意到对任意 $y \in \mathbf{F}(S)$ 均有 $y \to y$. 最后我们要来证明 S 对乘法封闭, 就拿上面两个特例 $(a,b),(c,d)$ 来说, 其等价于:

$$a = x_0 \longrightarrow x_1 \longrightarrow x_2 \longrightarrow x_3 = b$$

$$c = y_0 \longrightarrow y_1 \longrightarrow y_1 \longrightarrow y_1 = d$$

当 $(x_0,x_1),(y_0,y_1) \in \langle R \cup \Delta \cup R^{\mathrm{op}} \rangle$ 时, 显然 (x_0y_0, x_1y_1) 也属于该子幺半群. 所以我们有

$$ac = x_0y_0 \longrightarrow x_1y_1 \longrightarrow x_2y_1 \longrightarrow x_3y_1 = bd,$$

换言之 $(ac,bd) \in S$. 从而 S 是一个子幺半群. 假设 S' 是另一个满足要求的子幺半群兼等价关系, 由等价关系性质和 $R \subset S'$ 得 $R \cup \Delta \cup R^{\mathrm{op}} \subset S'$; 由子幺半群性质得 $\langle R \cup \Delta \cup R^{\mathrm{op}} \rangle \subset S'$; 再由生成等价关系的定义得 $S = \sim (\langle R \cup \Delta \cup R^{\mathrm{op}} \rangle) \subset S'$.

在这些定义下我们依次获得下述推论.

(1) 对于 $x \in \mathbf{F}(S)$, 我们用 $[x]$ 表示 $[x]_\sim$. 条件 (ii) 保证对任意 $i \in I$ 有

$$[\epsilon] = [((1_{M_i}, i))],$$

这是因为 $((1_{M_i}, i))R\epsilon \Rightarrow ((1_{M_i}, i)) \sim \epsilon$.

(2) 条件 (iv) 保证 $M = \mathbf{F}(S)/\sim$ 成为一个幺半群. $[x]\cdot[y] := [xy]$ 良定义当且仅当 (iv) 成立, 令 $1_M = [\epsilon]$, 显然其是 M 中的单位元.

(3) 条件 (i) 使得下述映射为幺半群同态:

$$\begin{array}{ccccccc} M_i & \hookrightarrow & S & \xhookrightarrow{\iota} & \mathbf{F}(S) & \xrightarrow{\pi} & \mathbf{F}(S)/\sim \\ m_i & \longmapsto & (m_i, i) & \longmapsto & ((m_i, i)) & \longmapsto & [((m_i, i))] \end{array}$$

我们令其为 φ_i.

(4) 对任意 $(M', (\varphi_i')_{i\in I}) \in \mathrm{Ob}(\mathsf{Pam}_E)$, 可定义出映射

$$\begin{array}{ccc} S & \xrightarrow{\sqcup_i \varphi_i'} & M' \\ m_i & \longmapsto & \varphi_i(m_i) \end{array}$$

而根据 $\mathbf{F}(S)$ 的泛性质, 有:

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[d, equal] & \mathbf{F}(S) \arrow[d, dashed, "\exists! g"] \\ S \arrow[r, "\sqcup_i \varphi_i'"'] & M' \end{tikzcd}$$

g 按如下程序工作:

$$((x_{i_1}, i_1), \cdots, (x_{i_s}, i_s)) \mapsto \varphi_{i_1}(x_{i_1}) \cdot \varphi_{i_2}(x_{i_2}) \cdots \varphi_{i_s}(x_{i_s}).$$

如果还可以证明 $\ker \pi \subset \ker g$, 就能使得下图交换:

$$\begin{tikzcd} \mathbf{F}(S) \arrow[r, two heads, "\pi"] \arrow[d, "g"'] & \mathbf{F}(S)/\sim \arrow[dl, dashed, "\exists!\ \psi"] \\ M' & \end{tikzcd}$$

这里我们依然举一个特例以略过繁琐的符号, 设 $a \sim 1_{\mathbf{F}(S)}$, 以及

$$a = x_0 \longrightarrow x_1 = 1_{\mathbf{F}(S)}$$

按照之前的定义就有该表达:

$$(x_0, x_1) = (u_1, v_1) \cdots (u_m, v_m) \in \langle R \cup \Delta \cup R^{\mathrm{op}} \rangle$$

其中每个 $(u_i, v_i) \in R \cup \Delta \cup R^{\mathrm{op}}$. 分类讨论有如下情况:

(a.1) $(u_1, v_1) \in \Delta$, 显然 $g(u_1) = g(v_1)$;

(a.2) $(u_1, v_1) \in R \Rightarrow g(u_1) = g(v_1)$;

- (a.2.1) $(u_1, v_1) \in R_1$, 设 $(u_1, v_1) = (((xy, i)), ((x, i), (y, i)))$, 且 $x, y \in M_i$; 可见 $g(u_1) = \varphi_i(xy) = \varphi_i(x)\varphi_i(y) = g(v_1)$;
- (a.2.2) $(u_1, v_1) \in R_2 \Rightarrow g(u_1) = g(v_1)$, 不难, 略过;
- (a.2.3) $(u_1, v_1) \in R_2 \Rightarrow g(u_1) = g(v_1)$, 不难, 略过;

(a.3) $(u_1, v_1) \in R^{\mathrm{op}} \Rightarrow g(u_1) = g(v_1)$, 无非是 (2) 的重复.

于是 $g(x_0) = g(u_1) \cdots g(u_m) = g(v_1) \cdots g(v_m) = g(x_1)$, 就有 $g(a) = 1_{M'}$.

依据资料 E 我们可以定义出 Pam_E 中的一个对象 $(M, (\varphi_i)_{i \in I})$, 其中 $M = \mathbf{F}(S)/\sim$, 而 φ_i 的定义见上方的第 (3) 推论. 根据第 (4) 推论, 对任意 $(M', (\varphi_i')_{i \in I}) \in \mathrm{Ob}(\mathsf{Pam}_E)$, 存在唯一的态射 g 和 ψ 在 Set 范畴下使下图交换:

$$\begin{tikzcd}
A \arrow[r, "f_i"] & M_i \arrow[r, hook] \arrow[dr, hook] \arrow[rrr, bend left, "\varphi_i"] \arrow[drr, bend right, "\varphi_i'"'] & S \arrow[r, "\iota"] \arrow[d, equal] & \mathbf{F}(S) \arrow[r, two heads, "\pi"] \arrow[d, "g"] & M \arrow[dl, "\psi"] \\
 & & S \arrow[r, "\sqcup_i \varphi_i'"'] & M' &
\end{tikzcd}$$

故此 $(M, (\varphi_i)_{i \in I})$ 就是我们要找的 $\mathrm{Am}(E)$. □

出于方便表示, 引入有限顺序乘法符号:

Definition 3.2.5 *对 $u \in \prod_{i \in I_u} X_i$, 其中 I_u 形如 $\{0, 1, \cdots, n-1\}$, 有函数*

$$f : \bigcup_{i \in I_u} X_i \to Y,$$

且 Y 上有乘法, 则:

$$\odot_{i \in I_u} f(u(i)) := f(u(0)) \cdot f(u(1)) \cdots f(u(n-1)).$$

所以, 对 $u \in I^n$, $\mathbf{F}(S)/\sim$ 中的元素形如 $\odot_{t\in I_u}[((x_u(t), u(t)))] = \odot_{t\in I_u}\varphi_{u(t)}(x_{u(t)})$. 若加上条件: 对任意 $i \in I$, 存在 $1_{M_i} \in H_i \subset M_i$, 使得:

$$\begin{aligned} A \times H_i &\xrightarrow{1:1} M_i \\ (a, h_i) &\longmapsto f_i(a)h_i \end{aligned} .$$

设 $\varphi_i f_i = g$, 那么此时

$$\odot_{t\in I_u}\varphi_{u(t)}(x_{u(t)}) = \odot_{t\in I_u}\varphi_{u(t)}[f_{u(t)}(a_{u(t)})\cdot h_{u(t)}] = \odot_{t\in I_u}[g(a_{u(t)})\cdot \varphi_{u(t)}(h_{u(t)})]$$

还可以进一步做微小的变换, 我们只观察累乘法的前两项

$$g(a_{u(0)})\cdot\varphi_{u(0)}(h_{u(0)})\cdot g(a_{u(1)})\cdot\varphi_{u(1)}(h_{u(1)})$$

注意到

$$\begin{aligned} \varphi_{u(0)}(h_{u(0)})\cdot g(a_{u(1)}) &= \varphi_{u(0)}(h_{u(0)})\cdot\varphi_{u(0)}[f_{u(0)}(a_{u(1)})] \\ &= \varphi_{u(0)}[\underbrace{h_{u(0)}\cdot f_{u(0)}(a_{u(1)})}_{\in M_{u(0)}}] \\ &= \varphi_{u(0)}[f_{u(0)}(a'_{u(0)})\cdot h'_{u(0)}] \\ &= g(a'_{u(0)})\cdot\varphi_{u(0)}(h'_{u(0)}) \end{aligned}$$

从而前两项可以化为

$$g(a_{u(0)})\cdot\varphi_{u(0)}(h_{u(0)})\cdot g(a_{u(1)})\cdot\varphi_{u(1)}(h_{u(1)}) = g(a_{u(0)})\cdot g(a'_{u(0)})\cdot\varphi_{u(0)}(h'_{u(0)})\cdot\varphi_{u(1)}(h_{u(1)})$$

重复这一过程, 得到融合积的即约表示:

$$\odot_{t\in I_u}\varphi_{u(t)}(x_{u(t)}) = g(a)\cdot\odot_{t\in I_u}\varphi_{u(t)}(h_{u(t)})$$

Theorem 3.2.6 给定资料 $(M_i : [f_i : A \to M_i])_{i\in I}$ 和它的融合积 $(M, (\varphi_i)_{i\in I})$, 态射 $g = \varphi_i f_i$, 以及一系列子集 $1_{M_i} \in H_i \subset M_i$, 还有双射 $A \times H_i \xrightarrow{1:1} M_i$. 则 M 中元素的既约表示唯一.

Proof. 定义集合

$$\Sigma = \left\{ (a, u, h) \in A \times \bigcup_{n\geq 0} I^n \times \bigcup_{\substack{I'\subset I \\ |I'|<\aleph_0}} \prod_{i\in I'} (H_i \setminus \{1_{M_i}\}) : u(i) \neq u(i+1) \wedge I' = \operatorname{im} u \right\}$$

细观该集合的元素, u 是一个 $I_u = \{0, 1, \cdots, n-1\} \to I$ 的映射, 且满足相邻的 $u(i) \neq u(i+1)$. $h \in \prod_{i\in \operatorname{im} u}(H_i \setminus \{1_{M_i}\})$. 定义映射

$$\begin{aligned} \Sigma &\xrightarrow{\quad\Phi\quad} M \\ (a, u, h) &\longmapsto g(a)\cdot\odot_{t\in I_u}\varphi_{u(t)}(h_{u(t)}) \\ (a, \varnothing, \varnothing) &\longmapsto g(a) \end{aligned}$$

根据先前的讨论已经可知此映射为满射, 下面证明它也是单射.

对任意 M_i 中的元素 $f_i(a'')s_i$, 考察 $\varphi_i(f_i(a'')s_i)\cdot\Phi(a,u,h)$, 展开有

$$g(a'')\cdot\varphi_i(s_i)\cdot g(a)\cdot\varphi_{u(0)}(h_{u(0)})\cdots\varphi_{u(n-1)}(h_{u(n-1)})$$

观察前面部分, 假设 $i\neq u(0)$:

$$\begin{aligned}&g(a'')\cdot\varphi_i(s_i)\cdot g(a)\cdot\varphi_{u(0)}(h_{u(0)})\\=&g(a'')\cdot\varphi_i[s_if_i(a)]\cdot\varphi_{u(0)}(h_{u(0)})\\=&g(a'')\cdot\varphi_i[f_i(a')s_i']\cdot\varphi_{u(0)}(h_{u(0)})\\=&g(a''a')\cdot\varphi_i(s_i')\cdot\varphi_{u(0)}(h_{u(0)})\end{aligned}$$

若 $i=u(0)$, 则考察:

$$\begin{aligned}&g(a'')\cdot\varphi_i(s_i)\cdot g(a)\cdot\varphi_{u(0)}(h_{u(0)})\\=&g(a'')\cdot\varphi_i[s_if_i(a)h_{u(0)}]\\=&g(a'')\cdot\varphi_i[f_i(a')s_i']\\=&g(a''a')\cdot\varphi_i(s_i')\end{aligned}$$

根据这一变形定义一系列幺半群作用 $\alpha_i: M_i\times\Sigma\to\Sigma$, 其将 $(f_i(a'')s_i,(a,u,h))$ 分下述情况映至 Σ:

$$\begin{cases}(a''a',(u(0),\cdots,u(n-1)),(s_i',h_{u(1)},\cdots,h_{u(n-1)})) & (u\neq\varnothing\wedge i=u(0)\wedge s_i'\neq1)\\(a''a',(u(1),\cdots,u(n-1)),(h_{u(1)},\cdots,h_{u(n-1)})) & (u\neq\varnothing\wedge i=u(0)\wedge s_i'=1)\\(a''a',(i,u(0),\cdots,u(n-1)),(s_i',h_{u(0)},\cdots,h_{u(n-1)})) & (u\neq\varnothing\wedge i\neq u(0)\wedge s_i'\neq1)\\(a''a',(u(0),\cdots,u(n-1)),(h_{u(0)},\cdots,h_{u(n-1)})) & (u\neq\varnothing\wedge i\neq u(0)\wedge s_i'=1)\\(a''a',(i),(s_i')) & (u=\varnothing\wedge s_i'\neq1)\\(a''a',\varnothing,\varnothing) & (u=\varnothing\wedge s_i'=1)\end{cases}$$

幺半群作用 $\alpha_A: A\times\Sigma\to\Sigma$ 则规定为映射:

$$(a',(a,u,h))\mapsto(a'a,u,h)$$

每个 α_i 导出一个映射 $p_i: M_i\to\mathrm{End}(\Sigma)$, 映 m_i 为 $\alpha(m_i,\cdot)$. 同样 α_A 导出 $p_A: A\to\mathrm{End}(\Sigma): a'\mapsto\alpha_A(a',\cdot)$. 由于 $(M,(\varphi_i)_i)$ 为 $\mathrm{Am}(E)$, 故存在唯一 ψ 使在 Pam_E 中有 $(M,(\varphi_i)_{i\in I})\xrightarrow{\psi}(\mathrm{End}(\Sigma),(p_i)_{i\in I})$, 从而下图交换

$$\begin{tikzcd}
 & & M \arrow[dd, "\psi"] \\
A \arrow[r, "f_i"] \arrow[urr, bend left, "h"] \arrow[drr, bend right, "p_A"'] & M_i \arrow[ur, "\varphi_i"] \arrow[dr, "p_i"] & \\
 & & \mathrm{End}(\Sigma)
\end{tikzcd}$$

定义

$$\begin{aligned}\alpha: M\times\Sigma &\longrightarrow \Sigma\\(m,\xi)&\longmapsto\psi(m)(\xi)\end{aligned}$$

任取 (a,u,h), 令 $m=\Phi(a,u,h)=g(a)\odot_{t\in I_u}\varphi_{u(t)}(h_{u(t)})$. 注意到

$$\begin{aligned}&\alpha(g(a)\odot_{t\in I_u}\varphi_{u(t)}(h_{u(t)}),(1_A,\varnothing,\varnothing))\\=&\psi\left[g(a)\odot_{t\in I_u}\varphi_{u(t)}(h_{u(t)})\right](1_A,\varnothing,\varnothing)\\=&\left[p_A(a)\circ\alpha_{u(0)}(h_{u(0)},\cdot)\circ\cdots\circ\alpha_{u(n-1)}(h_{u(n-1)},\cdot)\right](1_A,\varnothing,\varnothing)\\=&(a,u,h)\end{aligned}$$

所以 Φ 有左逆, 从而为单射. □

Definition 3.2.7 (Free product) 设 $A=\{1_A\}$, $E=(G_i,[f_i:A\to G_i])_{i\in I}$, 其中 G_i 是群, 则符号

$$*_{i\in I}G_i:=\mathrm{Am}(E)$$

称之为自由积.

构造自由群. 设 X 是一个集合, $I=X$, $G_x=\mathbb{Z}$. $\mathbb{Z}^{*X}:=*_{x\in X}G_x$ 为其自由积, 且有群同态 $\mathbb{Z}\xrightarrow{\varphi_x}\mathbb{Z}^{*X}$. 定义

$$\begin{aligned}\iota:X&\longrightarrow\mathbb{Z}^{*X}\\x&\longmapsto\varphi_x(1)\end{aligned}$$

注意此处 $\varphi_x(1)$ 中的 1 就是整数 1, 而非加法群的单位元. 我们断言 $(\mathbb{Z}^{*X},\iota)$ 是自由群. 任取 (G',f), 定义态射 $\varphi'_x:\mathbb{Z}\to G':a\mapsto f(x)^a$, 由融合积的泛性质得

$$\begin{tikzcd}A \arrow[r] & \mathbb{Z} \arrow[r, "\varphi_x"] \arrow[dr, "\varphi'_x"'] & \mathbb{Z}^{*X} \arrow[d, dashed, "\exists!"', "\psi"] \\ & & G'\end{tikzcd}$$

从而 $\psi(\iota(x))=\psi\varphi_x(1)=\varphi'_x(1)=f(x)$, 换言之下图交换:

$$\begin{tikzcd}X \arrow[r, "\iota"] \arrow[d, equal] & \mathbb{Z}^{*X} \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & G'\end{tikzcd}$$

Definition 3.2.8 设 $(M_i)_{i\in I}$ 是一族交换幺半群, 其直和定义为

$$\bigoplus_{i\in I}M_i:=\left\{(m_i)_{i\in I}\in\prod_{i\in I}M_i:\textit{Finite number of } m_i\neq 0\right\}$$

X 的**自由交换幺半群**可定义为 $\mathbb{Z}_{\geq 0}^{\oplus X}$, **自由交换群**可定义为 $\mathbb{Z}^{\oplus X}$.

Proposition 3.2.9 设 $X\subset G$, $\langle X\rangle_{\text{nor}}$ 为包含 X 的最小正规子群. 它可以这样来构造:

$$\langle X\rangle_{\text{nor}}=\langle\{gxg^{-1}:g\in G\wedge x\in X\}\rangle$$

Definition 3.2.10 (Group presentation) X 是一个集合, $\mathbf{F}(X)$ 是其自由群, $Y \subset \mathbf{F}(X)$, 则

$$\langle X|Y\rangle := \mathbf{F}(X)/\langle Y\rangle_{\text{nor}}$$

若 X 和 Y 均有限, 习惯上表示为 $\langle x_1, \cdots, x_n | y_1 = 1, \cdots, y_s = 1\rangle$

Proposition 3.2.11 任何群都可以表示成某个自由群的商.

Proof. 若子集 X 生成 G(实在不行取 $X = G$ 也可以), 注意到

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, equal] & \mathbf{F}(X) \arrow[d, dashed, "\exists!\psi"] \\ X \arrow[r, hook, "\subset"'] & G \end{tikzcd}$$

其中 ψ 定义为 $(x_1, \cdots, x_n) \mapsto x_1 \cdots x_n$, 显然是满同态, 进而有 $G \simeq \mathbf{F}(X)/\ker\psi$. $\square$

Chapter 4

Ring Theory

4.1 Polynomial Ring

Definition 4.1.1 设 M 是幺半群, R 是环, 幺半群环 $R[M]$ 中的元素定义为:

$$\{(r_m)_{m\in M} : \textit{Finite number of } r_m \neq 0_R\}$$

习惯上将上述元素表示为 $\sum_{m\in M} r_m m$.

$R[M]$ 的加法定义为

$$(r_m)_{m\in M} + (s_m)_{m\in M} = (r_m + s_m)_{m\in M}$$

乘法定义为

$$(r_m)_{m\in M} \cdot (s_m)_{m\in M} = \left(\sum_{\substack{x,y\in M\\ xy=m}} r_x s_y\right)_{m\in M}$$

我读到这里时产生了一些疑惑, 一是乘法是否封闭, 二是是否满足乘法结合律. 因此有了后面的没啥用的证明. 先给定两个元素 $(a_x)_x, (b_y)_y$, 由于

$$|\{(x,y)\in M^2 : xy = m \wedge a_x b_y \neq 0\}| \leq |\{(x,y)\in M^2 : a_x b_y \neq 0\}| < \aleph_0$$

所以每个 $\sum_{xy=m} a_x b_y$ 都有定义. 又因为

$$\bigsqcup_{m\in M} \{(x,y)\in M^2 : xy = m \wedge a_x b_y \neq 0\} = \{(x,y)\in M^2 : a_x b_y \neq 0\}$$

是有限集, 所以集合列

$$(\{(x,y)\in M^2 : xy = m \wedge a_x b_y \neq 0\})_{m\in M}$$

只能有有限个非空, 否则与 $\{(x,y)\in M^2 : a_x b_y \neq 0\}$ 有限矛盾. 故 $(\sum_{xy=m} a_x b_y)_{m\in M}$ 也在幺半群环集合中.

加上第三个元素 $(c_z)_z$, 我们来验证乘法结合律. 一切都是机械计算:

$$\begin{aligned}
[(a_x)_x(b_y)_y]\,(c_z)_z &= \left(\sum_{xy=s} a_x b_y\right)_s (c_z)_z \\
&= \left[\sum_{sz=t} \underbrace{\left(\sum_{xy=s} a_x b_y\right)}_{\text{Finite sum}} c_z\right]_t \\
&= \left(\sum_{(xy)z=t} (a_x b_y) c_z\right)_t \\
&= \left(\sum_{x(yz)=t} a_x (b_y c_z)\right)_t \\
&= (a_x)_x\,[(b_y)_y(c_z)_z]
\end{aligned}$$

$R[M]$ 上的乘法还有另一种表达, 即有限个元素的和, 证明略过:

$$\sum_x a_x x \cdot \sum_y b_y y = \sum_{(x,y)\in M^2} a_x b_y xy$$

注意到两个自然的同态:

$$\begin{aligned} M &\xrightarrow[\text{Mon}]{} R[M] & \qquad R &\xrightarrow[\text{Ring}]{} R[M] \\ m &\longmapsto 1_R \cdot m & r &\longmapsto r \cdot 1_M \end{aligned}$$

幺半群环同样可以作为某个范畴中的初对象来把握, 这个范畴是这样定义的: 给定 R 和 M; S 是一个环, 其对象形如 $(S, [f : R \xrightarrow[\text{Ring}]{} S], [g : M \xrightarrow[\text{Mon}]{} S])$, 且满足 $f(r)g(m) = g(m)f(r)$; 或者表示为图表

$$\begin{tikzcd}
R \arrow[dr, "f"] & \\
& S \\
M \arrow[ur, "g"'] &
\end{tikzcd}$$

其态射 $\phi : (S, f, g) \to (S', f', g')$ 使下图交换:

$$\begin{tikzcd}
R \arrow[dr, "f"] \arrow[drr, bend left, "f'"] & & \\
& S \arrow[r, "\phi"] & S' \\
M \arrow[ur, "g"'] \arrow[urr, bend right, "g'"'] & &
\end{tikzcd}.$$

Proposition 4.1.2 $(R[M], [R \to R[M]], [M \to R[M]])$ 是上述范畴的初对象. 就是说对任意的 (S, f, g) 满足 $f(r)g(m) = g(m)f(r)$, 存在唯一的 ϕ 使下图交换:

$$
\begin{tikzcd}
R \arrow[dr, hook] \arrow[rrd, bend left, "f"] & & \\
& R[M] \arrow[r, dashed, "\exists!\phi"] & S \\
M \arrow[ur, hook] \arrow[rru, bend right, "g"'] & &
\end{tikzcd}
$$

容易验证 ϕ 工作如 $\sum_m r_m m \mapsto \sum_m f(r_m)g(m)$.

Definition 4.1.3 (Polynomial ring) R 上以集合 $\mathcal{X}$ 为变元集的多项式环定义为 $R[\mathbf{CM}(\mathcal{X})]$, 其中 $\mathbf{CM}(\mathcal{X})$ 表示自由交换幺半群, 可用 $\mathbb{Z}_{\geq 0}^{\oplus \mathcal{X}}$ 构造.

同样的方法来造一个范畴, 元素为图表

$$
\begin{tikzcd}
R \arrow[dr, "f", "\text{Ring}"'] & \\
& S \\
\mathcal{X} \arrow[ur, "g", "\text{Set}"'] &
\end{tikzcd}
$$

且满足 (i)$\operatorname{im} f \subset S$ 交换, (ii)$f(r)g(x) = g(x)f(r)$. 态射 ϕ 使下图交换

$$
\begin{tikzcd}
R \arrow[dr, "f"] \arrow[rrd, bend left, "f'"] & & \\
& S \arrow[r, "\phi", "\text{Ring}"'] & S' \\
\mathcal{X} \arrow[ur, "g"] \arrow[rru, bend right, "g'"'] & &
\end{tikzcd}
$$

观察到自然的复合映射 $\mathcal{X} \to \mathbf{CM}(\mathcal{X}) \to R[\mathcal{X}]$, 我们有如下命题

Proposition 4.1.4 $(R[\mathcal{X}], [R \to R[\mathcal{X}]], [\mathcal{X} \to R[\mathcal{X}]])$ 为上述范畴的初对象.

Proof. 对任意元素 $(S, [f : R \to S], [g : \mathcal{X} \to S])$. 由于 $\operatorname{im} g$ 交换, 其生成的幺半群 $\langle \operatorname{im} g \rangle$ 也交换, 这同时也是 R 的一个乘性子集. 调用 $\mathbf{CM}(\mathcal{X})$ 的泛性质, 得唯一的幺半群同态 $\mathbf{CM}(\mathcal{X}) \to S$ 如下:

$$
\begin{tikzcd}
\mathcal{X} \arrow[r, "g"] \arrow[d, equal] & \langle \operatorname{im} g \rangle \arrow[r, hook, "\subset"] & S \\
\mathcal{X} \arrow[r] & \mathbf{CM}(\mathcal{X}) \arrow[u, dashed, two heads, "\exists!"] \arrow[ur, dashed, "\exists!\phi"'] &
\end{tikzcd}
$$

幺半群同态 ϕ 定义为 $(a_x)_x \mapsto g(x_1)^{a_1} \cdots g(x_n)^{a_n}$. 又因为 $g(x)f(r) = f(r)g(x)$, 故 ϕ 和 f 也以同样的方式交换, 即 $\phi(\boldsymbol{a})f(r) = f(r)\phi(\boldsymbol{a})$. 再调用 Proposition 4.1.2, 有了下面的图表

$$\begin{tikzcd}
R \arrow[dr, hook] \arrow[drr, bend left, "f"] & & \\
& R[\mathbf{CM}(\mathcal{X})] \arrow[r, dashed, "\exists!\psi"] & S \\
\mathbf{CM}(\mathcal{X}) \arrow[ur, hook] \arrow[urr, bend right, "\phi"'] & & \\
\mathcal{X} \arrow[u] \arrow[uurr, bend right, "g"'] & &
\end{tikzcd}$$

□

Proposition 4.1.5 若 R 是交换整环, 则 $R[\mathcal{X}]$ 也是.

Proof. 回想在高等代数的 $R[X]$ 中, 我们会取寻找最高次的 X^n 来解决问题. 虽然这里的问题每那么简单, 但思路是类似的. 对任意元素 $\sum_x a_x x, \sum_y b_y y \in R[\mathcal{X}]$, 可以表示为 $\sum_{i=1}^m a_{x_i} x_i$ 和 $\sum_{j=1}^n b_{y_j} y_j$, 并且所有 x_i 两两不同, y_j 亦然. 考虑集合 $\{x_i\}_i \times \{y_j\}_j$ 上的等价关系:

$$(x, y) \sim (x', y') \Leftrightarrow xy = x'y'$$

这就给出 $\{x_i\}_i \times \{y_j\}_j$ 的一个划分, 而我们希望找到某个 (x, y), 其等价类 $[x, y]$ 仅包含一个元素.

设 $x_i = (a_{i,t})_{t \in \mathcal{X}} = \sum_{t \in X_{a,i}} a_{i,t} t$; $X_{a,i} \subset \mathcal{X}$ 是一个有限集. 同样 $y_j = \sum_{t \in X_{b,j}} b_{j,t} t$, 集合

$$\bigcup_i X_{a,i} \cup \bigcup_j X_{b,j}$$

是一个有限集, 因此可以表示成 $\{t_1, \cdots, t_N\}$. 由此可以将 x_i 和 y_j 改写:

$$x_i = \sum_{\mu=1}^N a'_{i,t_\mu} t_\mu$$

$$y_j = \sum_{\mu=1}^N b'_{j,t_\mu} t_\mu$$

在 $\{x_i\}_i$ 上定义一个全序:

$$\sum_{\mu=1}^N a'_{k,t_\mu} t_\mu > \sum_{\mu=1}^N a'_{l,t_\mu} t_\mu \Leftrightarrow \exists 1 \leq \xi \leq N(a'_{k,t_1} = a'_{l,t_1}, \cdots, a'_{k,t_\xi} > a'_{l,t_\xi})$$

由于 x_i 两两不等, 所以存在一个最大元, 不妨就设为 x_1. 同理设 y_1 为 $\{y_j\}_j$ 中的最大元. 因此对任何 $(i, j) \neq (1, 1)$ 都有 $x_i y_j \neq x_1 y_1$. 又因为 $a_{x_1} b_{y_1} \neq 0$, 从而

$$\sum_{i=1}^m a_{x_i} x_i \cdot \sum_{j=1}^n b_{y_j} y_j = a_{x_1} b_{y_1} x_1 y_1 + \cdots \neq 0$$

□

若有两个集合 $\mathcal{X}, \mathcal{Y}$, 我们来证明 $R[\mathcal{X} \sqcup \mathcal{Y}] \simeq R[\mathcal{X}][\mathcal{Y}]$.

Proposition 4.1.6 现有资料

$$\begin{tikzcd}
R \arrow[dr, "f"] & \\
& S \\
\mathcal{X} \sqcup \mathcal{Y} \arrow[ur, "g"'] &
\end{tikzcd}$$

满足 $\operatorname{im} g$ 交换且 $g(t)f(r) = f(r)g(t)$. 以及

$$\begin{tikzcd}
R \arrow[dr] \arrow[drr, bend left, "f"] & & \\
& R[\mathcal{X}] \arrow[r, "\exists! \varphi"'] & S \\
\mathcal{X} \arrow[ur] \arrow[urr, bend right, "g|_{\mathcal{X}}"'] & &
\end{tikzcd}$$

则下面两个图表, 一侧的交换当且仅当另一侧的也交换:

$$\begin{tikzcd}
R[\mathcal{X}] \arrow[d] \arrow[dr, "\varphi"] & \\
R[\mathcal{X}][\mathcal{Y}] \arrow[r, "\psi"] & S \\
\mathcal{Y} \arrow[u] \arrow[ur, "g|_{\mathcal{Y}}"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
R \arrow[d] \arrow[dr, "f"] & \\
R[\mathcal{X}][\mathcal{Y}] \arrow[r, "\psi"] & S \\
\mathcal{X} \sqcup \mathcal{Y} \arrow[u] \arrow[ur, "g"'] &
\end{tikzcd}$$

Proof. (Left $\Rightarrow$ Right) 直接扩展左边的交换图即可.

$$\begin{tikzcd}
R \arrow[dr] \arrow[ddrr, bend left, "f"] & & \\
\mathcal{X} \arrow[r] \arrow[drr, "g|_{\mathcal{X}}"' near start] & R[\mathcal{X}] \arrow[d] \arrow[dr, "\varphi"] & \\
& R[\mathcal{X}][\mathcal{Y}] \arrow[r, "\psi"'] & S \\
& \mathcal{Y} \arrow[u] \arrow[ur, "g|_{\mathcal{Y}}"'] &
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
R \arrow[d] \arrow[dr, "f"] & \\
R[\mathcal{X}][\mathcal{Y}] \arrow[r, "\psi"] & S \\
\mathcal{X} \sqcup \mathcal{Y} \arrow[u] \arrow[ur, "g"'] &
\end{tikzcd}$$

(Right $\Rightarrow$ Left) 首先我们可以得出

$$\begin{tikzcd}
& R[\mathcal{X}] \arrow[d, "a"] \arrow[dr, "\psi a"] & \\
R \arrow[ur] \arrow[r] \arrow[rr, bend right, "f"'] & R[\mathcal{X}][\mathcal{Y}] \arrow[r, "\psi"] & S
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
R \arrow[d] \arrow[dr, "f"] & \\
R[\mathcal{X}] \arrow[r, "\psi a"'] & S
\end{tikzcd}$$

还有

$$\begin{tikzcd}
& R[\mathcal{X}] \arrow[d, "a"] \arrow[dr, "\psi a"] & \\
\mathcal{X} \arrow[ur] \arrow[r] \arrow[rr, bend right, "g|_{\mathcal{X}}"'] & R[\mathcal{X}][\mathcal{Y}] \arrow[r, "\psi"] & S
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
R[\mathcal{X}] \arrow[r, "\psi a"] & S \\
\mathcal{X} \arrow[u] \arrow[ur, "g|_{\mathcal{X}}"'] &
\end{tikzcd}$$

从图表

$$\begin{tikzcd}
R \arrow[d] \arrow[dr, "f"] & \\
R[\mathcal{X}] \arrow[r, "\psi a"] & S \\
\mathcal{X} \arrow[u] \arrow[ur, "g|_{\mathcal{X}}"'] &
\end{tikzcd}$$

以及该命题的条件得 $\psi a = \varphi$. 最后, 显然有

$$\begin{tikzcd}
& R[\mathcal{X}] \arrow[d] \arrow[dr, "\varphi"] & \\
R \arrow[ur] \arrow[r] & R[\mathcal{X}][\mathcal{Y}] \arrow[r, "\psi"] & S \\
& \mathcal{Y} \arrow[u] \arrow[ur, "g|_{\mathcal{Y}}"'] &
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
R[\mathcal{X}] \arrow[d, "a"] \arrow[dr, "\varphi"] & \\
R[\mathcal{X}][\mathcal{Y}] \arrow[r, "\psi"] & S \\
\mathcal{Y} \arrow[u] \arrow[ur, "g|_{\mathcal{Y}}"'] &
\end{tikzcd}$$

□

Corollary 4.1.7 $R[\mathcal{X} \sqcup \mathcal{Y}] \simeq R[\mathcal{X}][\mathcal{Y}]$.

Proof.

$$\begin{tikzcd}
R \arrow[dr, "f"] & \\
& S \\
\mathcal{X} \sqcup \mathcal{Y} \arrow[ur, "g"'] &
\end{tikzcd}$$

满足 $\operatorname{im} g$ 交换且 $g(t)f(r) = f(r)g(t)$. 而由 $R[\mathcal{X}]$ 的泛性质自然有条件:

$$\begin{tikzcd}
R \arrow[dr] \arrow[drr, bend left, "f"] & & \\
& R[\mathcal{X}] \arrow[r, "\exists!\varphi"'] & S \\
\mathcal{X} \arrow[ur] \arrow[urr, bend right, "g|_{\mathcal{X}}"'] & &
\end{tikzcd}$$

容易验证 $\varphi(u)g(x) = g(x)\varphi(u)$.

由 $R[\mathcal{X}][\mathcal{Y}]$ 的泛性质还可知

$$\begin{tikzcd}
R[\mathcal{X}] \arrow[dr] \arrow[drr, bend left, "\varphi"] & & \\
& R[\mathcal{X}][\mathcal{Y}] \arrow[r, "\exists!\psi"'] & S \\
\mathcal{Y} \arrow[ur] \arrow[urr, bend right, "g|_{\mathcal{Y}}"'] & &
\end{tikzcd}$$

调用上一命题的结论得到

$$\begin{tikzcd}
R \arrow[dr] \arrow[drr, bend left, "f"] & & \\
& R[\mathcal{X}][\mathcal{Y}] \arrow[r, "\exists!\psi"'] & S \\
\mathcal{X} \sqcup \mathcal{Y} \arrow[ur] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

□

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