Chapter 1

Introduction

1.1 Basics

Some brief definitions are listed as follows.

(11BD1) S is a null semigroup if $\forall x, y \in S(xy = 0)$,

(11BD2) S is a left zero semigroup if $\forall x, y \in S(xy = x)$, the definition for right zero semigroup is obvious,

(11BD3) $I \subset S$ is a proper ideal if $\{0\} \subset I \subsetneq S$ and $IS \subset S \wedge SI \subset S$,

(11BD4) given a set X, the full transformation semigroup is defined as $(\mathrm{End}_{\mathsf{Set}}(X), \circ)$, where $\circ$ refers the composition of functions,

(11BD5) a morphism $S \xrightarrow[\mathsf{Smg}]{\phi} \mathrm{End}(X)$ is a *representation* of S, and φ is faithful if it is injective,

(11BD6) a semigroup S is a rectangular band if $\forall a, b \in S(aba = a)$,

(11BD7) $\langle\{a\}\rangle_{\mathrm{smg}}$ is called a *monogenic semigroup*.

Proposition 1.1.1 Suppose S is a semigroup, the propositions listed below are equivalent.

(a) S is a group,

(b) for all $a, b \in S$, there exists $x, y \in S$ such that $ax = b \wedge ya = b$,

(c) $\forall a \in S(aS = Sa = S)$.

Proof. It is easy to demonstrate that (a) $\Rightarrow$ (c) and (b) $\Leftrightarrow$ (c). So we proceed to prove (c) $\Rightarrow$ (a), and it is suffices to show that S has the unique identity, and that for any element, its inverse exists and is unique. Let $ax = ya = a$, and it follows that

$$x = ax_1 = ay_1a = yay_1a = yx = ax_2ax = ax_2a = y_2a = y.$$

Thus, we may conclude that every element a in S has an identity ϵ_a such that $\epsilon_a a = a\epsilon_a = a$. Now, the issue lies in proving $\epsilon_a = \epsilon_b$ for any a, b in S, and the method is analogous:

$$\epsilon_a = by_1 = by_2b = by_2b\epsilon_b = \epsilon_a\epsilon_b = \epsilon_a ax_2a = ax_2a = x_1a = \epsilon_b.$$

As for the existence and uniqueness of inverse, it also follows the same manner, so we omit it here. $\square$

Theorem 1.1.2 Suppose S is a semigroup, $X = S^1$, then there exists a faithful representation

$$\varphi : S \to \mathrm{End}(X).$$

Proof. See [1, Theorem 1.1.2]. Simply stated,

$$\begin{aligned} S &\hookrightarrow \mathrm{End}(S^1) \\ a &\longmapsto [\varphi_a : x \mapsto xa] . \end{aligned}$$

□

Theorem 1.1.3 Suppose S is a semigroup, the following propositions are equivalent.

- ◇ S is a rectangular band (11BD6),
- ◇ every $a \in S$ is an idempotent, and $abc = ac$ for all a, b, c in S,
- ◇ there exists a left zero semigroup L, and a right zero semigroup R, such that $S \simeq L \times R$,
- ◇ there exists two sets A, B such that $S \simeq A \times B$, in witch $A \times B$ is a semigroup with the multiplication defined as $(a_1, b_1)(a_2, b_2) = (a_1, b_2)$.

Proof. See [1, Theorem 1.1.3]. □

1.2 Monogenic Subsemigroup

To study the monogenic subsemigroup, we introduce the following concepts. Suppose a is an element in S, and its order is finite if not specified.

(12MSD1) $\langle a \rangle := \langle \{a\} \rangle_{\mathrm{smg}}$,

(12MSD2) $\mathrm{ord}(a) := |\langle a \rangle|$,

(12MSD3) $\mathrm{idx}(a) := \min \{m \in \mathbb{Z}_{\geq 1} : \exists n \in \mathbb{Z}_{\geq 1}(a^m = a^n \wedge m \neq n)\}$,

(12MSD4) $\mathrm{prd}(a) := \min \{r \in \mathbb{Z}_{\geq 1} : a^{m+r} = a^m\}$,

(12MSD5) a semigroup is called *periodic* if all its elements are of finite order.

Let $m = \mathrm{idx}(a)$, $r = \mathrm{prd}$, clearly, $a, a^2, \cdots, a^{m+r-1}$ are mutually different, and $\langle a \rangle = \{a, \cdots, a^{m+r-1}\}$.

Let K_a be $\{a^m, \cdots, a^{m+r-1}\}$, we assert that it is a cyclic group. Consider the quotient ring $\mathbb{Z}/r\mathbb{Z}$, obviously, $\{[m], \cdots, [m + r - 1]\} = \mathbb{Z}/r\mathbb{Z}$. Thus, there exists $0 \leq g \leq r - 1$ such that $[m + g] = [1]$, which implies $\forall k([k] = [k(m + g)])$. Since $a^{(m+g)k} = a^{m+hr}a^{k-m} = a^m a^{k-m}$ for all $k > m$, the $a^{(m+g)k}$ exhaust K_a.

Proposition 1.2.1 Suppose a and b are elements of finite order in the same or different subsemigroups, then

$$\langle a \rangle \simeq \langle b \rangle \Leftrightarrow (\mathrm{idx}(a), \mathrm{prd}(a)) = (\mathrm{idx}(b), \mathrm{prd}(b)).$$

Proof. Suppose $\mathrm{idx}(a) = \mathrm{idx}(b) = m$ and $\mathrm{prd}(a) = \mathrm{ord}(b) = r$, the mapping defined below is an isomorphism.

$$\begin{aligned} \{a, \cdots, a^{m+r-1}\} &\xrightarrow{\sim} \{b, \cdots, b^{m+r-1}\} \\ a^k &\longmapsto b^k \end{aligned}$$

For the reverse, assume $\langle a \rangle \xrightarrow[\phi]{\sim} \langle b \rangle$, where ϕ maps a to b^ξ, it is straightforward to verify that $\langle b^\xi \rangle = \langle b \rangle$ and that $\mathrm{idx}(a) = \mathrm{idx}(b^\xi)$, $\mathrm{prd}(a) = \mathrm{prd}(b^\xi)$. If $\xi = 1$, the proof is over. On the other hand, if $\xi > 1$, then there exists $\mu \geq 1$ such that $b^{\xi\mu} = b$, thus, $\mathrm{idx}(b) = 1$, which implies $\langle b \rangle$ is a cyclic group. Hence, $\langle a \rangle$ is also a cyclic group with the generator $\phi^{-1}(b) = a^\zeta$. Since a is a generator, similarly, there exists an integer ν that makes $a^{\zeta\nu} = a$, and it follows that $\mathrm{idx}(a) = 1$. Furthermore, observe that $\mathrm{prd}(a) = |\langle a \rangle| = |\langle b \rangle| = \mathrm{prd}(b)$. □

Proposition 1.2.2 For any pair $(m, r) \in \mathbb{Z}^2_{\geq 1}$, there exists a semigroup S containing an element with idx of m and prd of r.

Proof. See [1, p.12]. Simply stated, the correspondence is given by $(m, r) \mapsto (12 \cdots m + 1) \in S_{m+r}$. □

1.3 Relations

Given a set X, the power set $P(X^2)$ equipped with the multiplication defined as

$$R_1 \circ R_2 := \left\{(a,b) \in X^2 : \exists c \in X((a,c) \in R_1) \wedge (c,b) \in R_2\right\},$$

where R_i is the element in $P(X^2)$, forms a semigroup. To see this, it is suffices to verify $\circ$ is associative, which is obvious. Some brief definitions are listed as follows.

(13RD1) $R(x) := \{y \in X : (x,y) \in R\}$,

(13RD2) $R(A) := \bigcup_{x\in A} R(x)$,

(13RD3) $R^{\mathrm{op}} := \{(y,x) : (x,y) \in R\}$,

(13RD4) $\Delta_X : \{(x,x) : x \in X\}$,

(13RD5) if it is not specified, R^n represents $R \circ \cdots \circ R$ (n times),

(13RD6) given a morphism $f : S \to S'$, then $\ker f := \{(x,y) \in S^2 : f(x) = f(y)\}$.

It can be easily verified that $(R_1 \circ R_2)^{\mathrm{op}} = R_2^{\mathrm{op}} \circ R_1^{\mathrm{op}}$, thus, $(R^n)^{\mathrm{op}} = (R^{\mathrm{op}})^n$. A commonly used conclusion is

$$(a,b) \in R^n \Leftrightarrow \exists (t_i)_{i=1}^n \in X^n \left(a = t_1 \to t_2 \to \cdots \to t_n = b\right),$$

where $t_i \to t_{i+1}$ means $t_i R t_{i+1}$.

We then introduce the definitions of partial orders and equivalent relations from this perspective. A partial order is a relation satisfies the following conditions,

- (reflective) $\Delta_X \subset R$,
- (anti-symmetric) $R \cap R^{\mathrm{op}} = \Delta_X$,
- (transitive) $R^2 \subset R$.

On another hand, R is an equivalence relation is the conditions below are all hold,

- (reflective) $\Delta_X \subset R$,
- (symmetric) $R^{\mathrm{op}} \subset R$,
- (transitive) $R^2 \subset R$.

Given a partial ordered set X, we have the following concepts, note that the definition of max/supre/upper-case is analogous.

(13LTC1) Suppose $U \subset X$, $m \in U$ is the *minimal* element if $\nexists a \in U(a < m)$,

(13LTC2) suppose $U \subset X$, $m \in U$ is the *minimum* element if $\forall a \in U(m \leq a)$,

(13LTC3) suppose $U \subset X$, $l \in X$ is the *lower bound* of U if $\forall a \in U(l \leq a)$.

(13LTC4) We say that X satisfies *minimal condition* if every nonempty subset of it has a minimal element.

(13LTC5) Suppose $U \subset X$, $i \in X$ is the *infimum*, denoted as $\inf U$, if i is the maximum element of all lower bounds of U.

(13LTC6) We say that X is a *complete lower semilattice* if $\forall U \subset X(\exists \inf U)$, and is a *lower semilattice* if $\forall\{x,y\} \subset X(\exists \inf\{x,y\})$. If X is a lower semilattice, the operation $(x,y) \mapsto \inf\{x,y\}$ forms a binary function, denoted as $(\cdot) \wedge (\cdot)$, as for the upper-case, we denote $x \vee y$ by $\sup\{x,y\}$.

(13LTC7) We say that X is a *lattice* if it's both an upper semilattice and a lower semilattice.

In addition, in a lower semilattice, it can be verified that

- $\diamond$ $x \leq y \Leftrightarrow x = x \wedge y$,
- $\diamond$ $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, that is, $(X, \wedge)$ forms a semigroup.

Proposition 1.3.1 Given a set X, a partition $\mathcal{A}$ is a family of disjoint subsets of X satisfying $\bigsqcup \mathcal{A} = X$. Then, there exists a bijection

$$\{R \in P(X^2) : \text{equivalence relation}\} \xleftrightarrow{1:1} \{\mathcal{A} \in P(X) : \text{partition}\}$$

$$R \longmapsto \{R(x)\}_{x\in X}$$

$$[R : (x,y) \in R \Leftrightarrow \exists A \in \mathcal{A}(x \in A \wedge y \in A)] \longleftarrow \mathcal{A}$$

1.4 Congruences

Let S be a semigroup, R is a relation on S, here are some definitions:

(14SRD1) $aR = a \cdot R := \{(ax, ay) : (x,y) \in R\}$, for the reverse, $Ra := \{(xa, ya) : (x,y) \in R\}$, in addition, $aRb := \{(axb, ayb) : (x,y) \in R\}$.

(14SRD2) $S^1R = S^1 \cdot R := \bigcup_{a\in S^1} aR$, the definition of RS^1 is analogous, furthermore, S^1RS^1 represents $\bigcup_{(a,b)\in S^1\times S^1} aRb$.

(14SRD3) $RR = R \cdot R := \{(x_1x_2, y_1y_2) : (x_i, y_i) \in R \wedge i \in \{1,2\}\}$,

(14SRD4) $R^{\cdot n} := R \cdot R \cdots R$ (n times).

(14SRD5) We say that R is *left compatible* if $S^1R \subset R$, similar to *right compatible*.

(14SRD6) We say that R is *compatible* if $S^1R \subset R \wedge RS^1 \subset R$, which is equivalent to $RR \subset R$.

We possess to prove the assertion in the last definition above. Since $\Delta_S \subset R$, $RR \subset R$ ensures for all $a \in S^1$ and $(x,y) \in S$, $(ax, ay) \in R$. Conversely, assume $(x_1x_2, y_1y_2) \in RR$. Since $S^1R \subset R \wedge RS^1 \subset R$, we obtain that $(x_1x_2, x_1y_2) \in R$ and $(x_1y_2, x_2y_2) \in R$. Thus, $(x_1x_2, y_1y_2) \in R$.

The conclusion below is often used in algebra, especially in situations where an equivalence relation and some operations are imposed on a set to give it an algebraic structure, for example, ideal of rings, the construction of amalgamated product and the construction of tensor product. Its core, precisely, is the concept of congruence in semigroup theory.

Proposition 1.4.1 Suppose R is an equivalence relation on a semigroup S, then

$$R(x)R(y) := R(xy) \text{ well defined} \Leftrightarrow R \text{ is a congruence.}$$

The following constructions are also important.

(a) $\langle R\rangle_{\text{eqv}} := \bigcup_{n\in\mathbb{Z}_{\geq 1}} [R \cup \Delta_S \cup R^{\text{op}}]^n$ is the smallest equivalence relation containing R, where S can just be a set.

(b) $\langle R\rangle_{\text{cpt}} := S^1RS^1$ is the smallest compatible relation containing R.

(c) $\langle R\rangle_{\text{cge}} := \langle S^1RS^1\rangle_{\text{eqv}}$ is the smallest congruence containing R.

Proof. The proof for (a) is on [1, Proposition 1.4.9], the proof for (b) and (c) can be found in p.25-p.26 in the same book. ☐

Both set $\text{Eqv}(S)$ of equivalences and $\text{Cge}(S)$ of congruences on S are partially ordered by $\subset$. In fact, both are complete lattice. Take $\text{Cge}(S)$ as an example, for any subset $\mathcal{U} \subset \text{Cge}(S)$, it can be verified that $\inf\mathcal{U} = \bigcap \mathcal{U}$ and $\sup\mathcal{U} = \langle\bigcup\mathcal{U}\rangle_{\text{cge}}$. Notice that for any $R_1, R_2 \in \text{Cge}(S)$

$$\langle R_1 \cup R_2\rangle_{\text{cge}} = \langle R_1 \cup R_2\rangle_{\text{eqv}}. \tag{1.1}$$

Thus, both symbol $\wedge$ and $\vee$ on lattice $\text{Eqv}(S)$ and $\text{Eqv}(S)$ represent the same operations of sets. Here are some conclusions for supplementation, which can be found in [1, p.28]. Suppose R_1, R_2 are equivalences, then

(14CP1) $R_1 \vee R_2 = \langle R_1 \cup R_2 \rangle_{\text{eqv}} = \bigcup_{n \in \mathbb{Z}_{\geq 1}} (R_1 \cup R_2)^n = \bigcup_{n \in \mathbb{Z}_{\geq 1}} (R_1 \circ R_2)^n$,

(14CP2) $R_1 \circ R_2 = R_2 \circ R_1 \Rightarrow R_1 \vee R_2 = R_1 \circ R_2$.

Example 1.4.2 Let G be a group, $E \subset G^2$ be an equivalence, and $N = E(1_G)$ (14RD1), which is typically denoted as $[1_G]$ in other books. Then,

$$[a][b] = [ab] \text{ well-defined} \Leftrightarrow N \lhd G \wedge (aEb \Leftrightarrow ab^{-1} \in E).$$

If E is an equivalence on a ring R, which is given by the data $(R, +, -, 0_R, \cdot, 1_R)$. Let $I = [0_R]$, then,

$$[a] + [b] = [a + b] \text{ well-defined} \Leftrightarrow I < R \wedge (aEb \Leftrightarrow a - b \in I).$$

It is nothing but a corollary of the former assertion in the case of an Abelian group $(R, +, -, 0_R)$. Based on this, to let

$$[a][b] = [ab]$$

well-defined, again, we consider it on the semigroup $(R, \cdot)$, and this requires $RE \subset E \wedge ER \subset E$ (14SRD2). It is easy to verify that $RE \subset E \wedge ER \subset E \Rightarrow RI \subset I \wedge IR \subset I$. Conversely, suppose $RI \subset I \wedge IR \subset I$, it follows that

$$aEb \Leftrightarrow a - bE0 \Leftrightarrow a - b \in I \Rightarrow \forall r \in R(ra - rb \in I) \Leftrightarrow \forall r \in R(raErb) \Rightarrow RE \subset E,$$

the procedure of proving $RE \subset E$ follows the same manner.

1.5 Ideals

Some definitions are listed below.

(15IDD1) ρ is a mapping from the set of proper ideal of S to $\mathrm{Cge}(S)$, which is given by $I \mapsto I^2 \cup \Delta_S$,

(15IDD2) elements in $\operatorname{im} \rho$ are called *Rees ideals*,

(15IDD3) a morphism ϕ is called a *Ress morphism* if $\ker \phi$ (15RD6) is a Ress ideal.

Based on this, we obtain some properties:

- ◇ each $\rho(I)$ is a congruence, thus, $S/\rho(I) = \{I\} \sqcup \{\{x\} : x \in S \setminus I\}$ forms a semigroup,
- ◇ $I \in S/\rho(I)$ is a zero element.
- ◇ Above all, suppose I is a proper ideal, there exists a bijection

$$\begin{array}{ccc} \{I \subset J \subsetneq S : \text{ ideal}\} & \xleftrightarrow{1:1} & \{\bar{J} \subset S/\rho(I) : \text{ ideal}\} \\ J & \longmapsto & \rho(I)(J) \\ \rho(I)^{-1}(\bar{J}) & \longleftarrow & \bar{J}. \end{array}$$

1.6 Free Semigroup

The definition of free semigroup is similar to other algebraic structures, that is, the initial object in the comma category (j_X, U). To be specific, $(\mathbf{F}(X), \iota)$ is the free semigroup of set X, if for any (S, f), where S is a semigroup and $f : X \to S$ is a function, there exists unique semigroup morphism ϕ that makes the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathbf{F}(X) \arrow[d, "\exists!"', "\phi"] \\ & S \end{tikzcd}$$

The construction is also straightforward, we omit it here.

Definition 1.6.1 Suppose Y is a relation on free semigroup $\mathbf{F}(X)$, let

$$\langle X|Y\rangle := \mathbf{F}(X)/\langle Y\rangle_{\mathrm{cge}}.$$

If there exists an epimorphism $\phi : \mathbf{F}(X) \to S$, a semigroup, such that $\ker\phi = \langle Y\rangle_{\mathrm{cge}}$, and hence $\langle X|Y\rangle \simeq S$, we say that S is presented.

Chapter 2

Green's Equivalences; Regular Semigroups

2.1 Green's Equivalences

Suppose S is a semigroup, here are some basic concepts.

- $S^1 a$ is the *principal left ideal* of a, same as the right-case,
- $\mathcal{L}$ is an equivalence defined by $a\mathcal{L}b \Leftrightarrow S^1 a = S^1 b$,
- $\mathcal{R}$ is an equivalence defined by $a\mathcal{R}b \Leftrightarrow aS^1 = bS^1$,
- $\mathcal{J}$ is an equivalence defined by $a\mathcal{J}b \Leftrightarrow S^1 a S^1 = S^1 b S^1$,
- $\mathcal{H} := \mathcal{L} \cap \mathcal{R}$ is also an equivalence,
- $\mathcal{D} := \langle \mathcal{L} \cup \mathcal{R} \rangle_{\text{eqv}} = \mathcal{L} \vee \mathcal{R}$, which is equal to $\mathcal{L} \circ \mathcal{R}$, the reason is illustrated by (21GBP13) and (21CP2).

These objects possess these short properties:

(21GBP11) "Mutual divisibility" means if $a\mathcal{L}b$, then a, b can divide each other, that is, $\exists x, y \in S^1$ such that $ax = b \wedge by = a$. Same as the right-cese.

(21GBP12) $\mathcal{L}$ is a *right* congruence, and $\mathcal{R}$ is a *left* congruence.

(21GBP13) $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$, the proof can be found in [1, Proposition 2.1.3].

(21GBP14) Obviously, $\mathcal{D} \subset \mathcal{J}$,

(21GBP15) Suppose S, which has no identity, induces equivalences $\mathcal{L}, \mathcal{R}, \mathcal{D}, \cdots$, and S^1 induces $\mathcal{L}', \mathcal{R}', \mathcal{D}', \cdots$. Then, $\mathcal{L}' = \mathcal{L} \sqcup \{(1,1)\}$, the same conclusions apply for the remaining equivalences.

We then can impose a partial order on $S/\mathcal{L}, S/\mathcal{R}$ and $S/\mathcal{J}$; to be specific,

- $\mathcal{L}(a) \leq \mathcal{L}(b) \Leftrightarrow S^1 a \subset S^1 b$,
- $\mathcal{R}(a) \leq \mathcal{R}(b) \Leftrightarrow aS^1 \subset bS^1$,
- $\mathcal{J}(a) \leq \mathcal{J}(b) \Leftrightarrow S^1 a S^1 \subset S^1 b S^1$.

Notice that for all $a \in S$ and $x, y \in S^1$,

- $\mathcal{L}(xa) \leq \mathcal{L}(a)$,
- $\mathcal{R}(ax) \leq \mathcal{R}(a)$,
- $\mathcal{J}(xay) \leq \mathcal{J}(a)$,

- $\mathcal{L}(a) \leq \mathcal{L}(b) \vee \mathcal{R}(a) \leq \mathcal{R}(b) \Rightarrow \mathcal{J}(a) \leq \mathcal{J}(b)$.

Noticing the property $\mathcal{D} \subset \mathcal{J}$, we are naturally led to ask when $\mathcal{D} = \mathcal{J}$, and the book [1] gives the following proposition:

(21GBP21) If S is a group, then $\mathcal{L} = \mathcal{R} = \mathcal{H} = \mathcal{J} = \mathcal{D} = S^2$.

(21GBP22) If S is a commutative semigroup, then $\mathcal{L} = \mathcal{R} = \mathcal{H} = \mathcal{J} = \mathcal{D}$.

(21GBP23) If S is a periodic semigroup (21MSD5), then $\mathcal{D} = \mathcal{J}$ (see Proposition 2.1.4).

(21GBP24) If S is a semigroup, and both $S/\mathcal{L}$ and $S/\mathcal{R}$ as partial ordered sets satisfy the minimal condition (21LTC4), then $\mathcal{D} = \mathcal{J}$ (see Proposition 2.1.5).

Note that in the procedure of proving (21GBP24), we have to verify that if $S/\mathcal{L}$ possess minimal condition then so does $S^1/\mathcal{L}'$, where $\mathcal{L}'$ is originated from semigroup S^1, and that $\mathcal{D}' = \mathcal{J}' \Rightarrow \mathcal{D} = \mathcal{J}$. As for the former, let U' be any subset of $S^1/\mathcal{L}'$, then $U' = \{\mathcal{L}'(a) : a \in A \wedge A \subset S^1\}$. According to (21GBP15), we obtain that $a = 1 \Rightarrow \mathcal{L}'(a) = \{1\}$ and $a \in S \Rightarrow \mathcal{L}'(a) = \mathcal{L}(a)$, thus, let $U = \{\mathcal{L}(a) : a \in A \setminus \{1\}\} \subset S/\mathcal{L}$, clearly it contains a minimal element $\mathcal{L}(m)$, which is also the minimal element of U'.

2.2 The $\mathcal{D}$-Classes

Here are some properties of $\mathcal{D}$-Classes:

(22CDP1) $\forall x \in \mathcal{D}(a) \Rightarrow \mathcal{L}(x) \subset \mathcal{D}(a) \wedge \mathcal{R}(x) \subset \mathcal{D}(a) \Rightarrow \mathcal{H}(x) \subset \mathcal{D}(a)$.

(22CDP2) $\mathcal{D}(a) = \bigcup_{t \in \mathcal{R}(a)} \mathcal{L}(t) = \bigcup_{t \in \mathcal{D}(a)} \mathcal{L}(t) = \bigcup_{t \in \mathcal{L}(a)} \mathcal{R}(t) = \bigcup_{t \in \mathcal{D}(a)} \mathcal{R}(t)$.

(22CDP3) $a\mathcal{D}b \Leftrightarrow \mathcal{R}(a) \cap \mathcal{L}(b) \neq \varnothing \Leftrightarrow \mathcal{R}(b) \cap \mathcal{L}(a) \neq \varnothing$.

(22CDP4) The intersection of an $\mathcal{L}$-class and an $\mathcal{R}$-class is either $\varnothing$ or a $\mathcal{H}$-class, conversely any $\mathcal{H}$-class is a intersection of an $\mathcal{L}$-class and an $\mathcal{R}$-class.

(22CDP5) Suppose $S = \bigsqcup_{i \in I} \mathcal{L}_i = \bigsqcup_{j \in J} \mathcal{R}_j$, then $S = \bigsqcup_{(i,j) \in I \times J} \mathcal{L}_i \cap \mathcal{R}_j$.

Notice that the data $\{\mathcal{L}_i \cup \mathcal{R}_j\}_{(i,j)} = \{\mathcal{H}(a) : a \in S\} \sqcup \{\varnothing\}$. Moreover, this partition of set S is always described as a table, each cell is either empty or an $\mathcal{H}$-class.

	$\mathcal{L}_1$	$\mathcal{L}_2$
$\mathcal{R}_1$		
$\mathcal{R}_2$		

The following result is usually known as Green's Lemma. We denote by ρ_s the mapping $x \mapsto xs$, and by λ_s the mapping $x \mapsto sx$.

(22GLM1) Suppose $a\mathcal{R}b$, there exists $s, s' \in S^1$ such that $as = b$ and $bs' = a$, then,

- $\mathcal{L}(a) \underset{\rho_{s'}}{\overset{\rho_s}{\rightleftarrows}} \mathcal{L}(b) = \mathrm{id}$,
- $\forall x, x' \in \mathcal{L}(a)$, we have $x\mathcal{R}sx$, $x\mathcal{R}x' \Rightarrow xs\mathcal{R}x's$ and $x\mathcal{L}x' \Rightarrow xs\mathcal{L}x's$,
- $\forall x \in \mathcal{L}(a),\quad \mathcal{H}(x) \underset{\rho_{s'}}{\overset{\rho_s}{\rightleftarrows}} \mathcal{H}(xs) = \mathrm{id}\,.$

(22GLM2) Suppose $a\mathcal{L}b$, then there exists $s, s' \in S^1$ such that $sa = b$ and $s'b = a$, we obtain

- $\mathcal{R}(a) \underset{\lambda_{s'}}{\overset{\lambda_s}{\rightleftarrows}} \mathcal{R}(b) = \mathrm{id}$,
- $\forall x, x' \in \mathcal{R}(a)$, we have $x\mathcal{L}xs$, $x\mathcal{L}x' \Rightarrow sx\mathcal{L}sx'$ and $x\mathcal{R}x' \Rightarrow sx\mathcal{R}sx'$,
- $\forall x \in \mathcal{R}(a),\quad \mathcal{H}(x) \underset{\lambda_{s'}}{\overset{\lambda_s}{\rightleftarrows}} \mathcal{H}(xs) = \mathrm{id}\,.$

Based on this, we have some corollaries:

(22GLMC1) $a\mathcal{D}b \Rightarrow |\mathcal{H}(a)| = |\mathcal{H}(b)|$.

Proof. Observe that $a\mathcal{D}b \Rightarrow a\mathcal{R}c \wedge c\mathcal{L}b$, by Green's Lemma, we have

$$\begin{array}{ccccccc} \mathcal{L}(a) & \underset{\rho_{s'}}{\overset{\rho_s}{\rightleftarrows}} & \mathcal{L}(c) & & \mathcal{R}(c) & \underset{\lambda_{t'}}{\overset{\lambda_t}{\rightleftarrows}} & \mathcal{R}(b) \\ \cup & & \cup & & \cup & & \cup \\ \mathcal{H}(a) & \underset{\rho_{s'}}{\overset{\rho_s}{\rightleftarrows}} & \mathcal{H}(c), & & \mathcal{H}(c) & \underset{\lambda_{t'}}{\overset{\lambda_t}{\rightleftarrows}} & \mathcal{H}(b). \end{array}$$

□

(22GLMC2) $ab \in \mathcal{H}(a) \Rightarrow \mathcal{H}(a) \xrightarrow[\sim]{\rho_b} \mathcal{H}(a)$.

Proof. $ab \in \mathcal{H}(a) \Rightarrow a\mathcal{R}ab$, and it follows that

$$\begin{array}{ccc} \mathcal{L}(a) & \underset{\rho_{s'}}{\overset{\rho_b}{\rightleftarrows}} & \mathcal{L}(ab) \\ \cup & & \cup \\ \mathcal{H}(a) & \underset{\rho_{s'}}{\overset{\rho_b}{\rightleftarrows}} & \mathcal{H}(ab). \end{array}$$

□

(22GLMC3) $ab \in \mathcal{H}(b) \Rightarrow \mathcal{H}(b) \xrightarrow[\sim]{\lambda_a} \mathcal{H}(b)$.

(22GLMC4) (Green's Theorem) If H is an $\mathcal{H}$-class in a semigroup S, then either $HH \cap H = \varnothing$ or $HH = H$ and H is a subgroup of S.

Proof. Suppose $a, b \in H$ and $ab \in H$, then $a \in \mathcal{H}(ab)$ and $b \in \mathcal{H}(ab)$. By (22GLMC3) and (22GLMC2) above, there exists two isomorphisms $H \underset{\lambda_a}{\overset{\rho_b}{\rightleftarrows}} H$. This implies for any $h \in H$, $hb \in H$ and $ah \in H$. Apply these two proposition again, we obtain that $H \underset{\lambda_h}{\overset{\rho_h}{\rightleftarrows}} H$, furthermore, $HH = H$, and it can be concluded that H is a group by Proposition 1.1.1. □

(22GLMC5) If e is an idempotent, then $\mathcal{H}(e)$ is a subgroup. No $\mathcal{H}$-class can contain more than one idempotent, since the idempotent in a group is identity.

2.3 Regular Semigroup

Here are some simple definitions.

- ◇ $a \in S$ is *regular* if there exists $x \in S$ such that $axa = a$,
- ◇ a' is the inverse of a if $a'aa' = a'$ and $aa'a = a$,
- ◇ $\mathrm{inv}(a)$ is the set of all inverses of a.

These propositions are straightforward to verify:

(23RGP1) $\forall x, y \in S$, if $xyx = x$, then $xy\mathcal{R}x \wedge yx\mathcal{L}x$.

(23RGP2) If a is regular, both $\mathcal{L}(a)$ and $\mathcal{R}(a)$ are regular, thus $\mathcal{D}(a)$ is regular.

(23RGP3) Any $\mathcal{D}(a)$ contains an idempotent is regular.

(23RGP4) Let e be an idempotent, then e is a left identity in $\mathcal{R}(e)$, and is a right identity in $\mathcal{L}(e)$.

(23RGP5) a is regular $\Leftrightarrow$ a has inverse.

Proof. Suppose $axa = a$, namely, a is regular. Let $a' = xax$, and it is indeed an inverse. □

(23RGP6) If $y \in \mathrm{inv}(x)$, by (23RGP1), $yx \in \mathcal{R}(y) \cap \mathcal{L}(x) \wedge xy \in \mathcal{R}(x) \cap \mathcal{L}(y)$.

(23RGP7) Suppose D is a regular class, then for any $a \in D$, both $\mathcal{L}(a)$ and $\mathcal{R}(a)$ contain idempotents.

Proof. Assuming $axa = a$, and it follows that $xa\mathcal{L}a, ax\mathcal{R}a$. □

The following propositions can be memorized with the aid of "eggbox".

(23REB1) Let a be an element of a semigroup S, $a' \in \mathrm{inv}(a)$, then both $aa' \in \mathcal{L}(a) \cap \mathcal{R}(a')$ and $a'a \in \mathcal{L}(a') \cap \mathcal{R}(a)$ are idempotents. This can be illustrated by the table below.

a	$\exists aa'$
$\exists a'a$	a'

(23REB2) Let a be an element of a semigroup S, $e \in \mathcal{R}(a) \cap \mathcal{L}(b)$ and $f \in \mathcal{R}(b) \cap \mathcal{L}(a)$ are two idempotents, then there exists $a' \in \mathcal{H}(b)$ such that $a' \in \mathrm{inv}(a)$, $aa' = e$ and $a'a = f$.

a	e
f	$\exists a'$

Proof. From $a\mathcal{R}e$ it follows that $\exists x \in S^1(ax = e)$, let $a' = fxe$, thus, it can be verified that $aa'a = afxea = axa = ea = a$ (23RGP4). The proof for $a'aa' = a', aa' = e$ and $a'a = f$ follows the same manner. Observe that $aa' = e \wedge fxe = a$, this implies $a'\mathcal{L}e$; similarly, $a'\mathcal{R}f$. □

(23REB3) In a semigroup S, no $\mathcal{H}$-calss contains more than one inverse of a.

Proof. Suppose a, a^* are two inverses of a in a single $\mathcal{H}(b)$. Then, by (23REB1), aa' and aa^* are two idempotents in $\mathcal{R}(a) \cap \mathcal{L}(b)$, and it follows that $aa' = aa^*$ (23GLMC5). Similarly, $a'a = a^*a$. Hence, we obtain that

$$a^* = a^*aa^* = a^*aa' = a'aa' = a'.$$ □

(23REB4) Let e, f be idempotents, then, $e\mathcal{D}f$ if and only if there exists a and $a' \in \mathrm{inv}(a)$ such that $aa' = e \wedge a'a = f$.

e	$\exists a$
$\exists a'$	f

Proof. Suppose $e\mathcal{D}f$, then $\exists a \in \mathcal{R}(e) \cap \mathcal{L}(f)$ (23CDP3). Besides this, according to (23REB2), we have $\exists a' \in \mathrm{inv}(a)$ such that $aa' = e \wedge a'a = f$.

Conversely, if there exists a and $a' \in \mathrm{inv}(a)$ such that $aa' = e \wedge a'a = f$, then $e = aa' \in \mathcal{R}(a)$ and $f = a'a \in \mathcal{L}(a)$ by (23REB1), thus, $e\mathcal{D}f$. □

The following propositions are the comprehensive application of the above propositions and the Green's Lemma.

Proposition 2.3.1 If H and K are two group $\mathcal{H}$-class in the same $\mathcal{D}$-class, then H and K are isomorphic.

Proof. Since the identity in a group is idempotent, H and K contain idempotents e and f respectively. Notice that $e\mathcal{D}f$, by (23REB4), we can find $a \in \mathcal{R}(e) \cap \mathcal{L}(f)$ and $\mathcal{R}(f) \cap \mathcal{L}(e) \ni a' \in \mathrm{inv}(a)$ that makes $aa' = e$ and $a'a = f$. In addition, we also have $ea = af = a, a'e = fa' = a'$. From $aa' = e \wedge ea = a$ and $a'a = f \wedge af = a$, one can construct the following isomorphisms by utilizing Green's Lemma (23GLM1).

$$\begin{array}{ccccccc}
\mathcal{L}(a) & \underset{\rho_a}{\overset{\rho_{a'}}{\rightleftarrows}} & \mathcal{L}(e) & & \mathcal{R}(a) & \underset{\lambda_a}{\overset{\lambda_{a'}}{\rightleftarrows}} & \mathcal{R}(f) \\
\cup & & \cup & & \cup & & \cup \\
\mathcal{H}(a) & \underset{\rho_a}{\overset{\rho_{a'}}{\rightleftarrows}} & \mathcal{H}(e) & & \mathcal{H}(a) & \underset{\lambda_a}{\overset{\lambda_{a'}}{\rightleftarrows}} & \mathcal{H}(f)
\end{array}$$ □

Proposition 2.3.2 Let a, b be elements in a $\mathcal{D}$-class. Then, $ab \in \mathcal{R}(a) \cap \mathcal{L}(b)$ if and only if $\mathcal{L}(a) \cap \mathcal{R}(b)$ contains an idempotent.

Proof. The content provided here can serve as a supplement of the original proof of [1, Proposition 2.3.7]. Suppose $ab \in \mathcal{R}(a) \cap \mathcal{L}(b)$, then there exists ξ, η such that

$$\begin{cases} ab = ab \\ ab\xi = a \end{cases} \wedge \begin{cases} ab = ab \\ \eta ab = b \end{cases},$$

thus, $b\xi = \eta ab\xi = \eta a$. Furthermore, we have $b = \eta ab \mathcal{R} \eta a$ and $a = ab\xi \mathcal{L} b\xi$ due to $ab\mathcal{R}a$ and $ab\mathcal{L}a$. Observe that

$$\begin{cases} \eta a = \eta a \\ a\eta a = ab\xi = a \end{cases} \wedge \begin{cases} b\xi = b\xi \\ b\xi b = \eta ab = b \end{cases},$$

we obtain $a\mathcal{L}\eta a$ and $b\mathcal{R}b\xi$. Hence

$$\mathcal{H}(b\xi) = \mathcal{R}(b\xi) \cap \mathcal{L}(b\xi) = \mathcal{R}(b) \cap \mathcal{L}(a).$$

By Green's Lemma, we have the following isomorphism

$$\begin{array}{ccc} \mathcal{L}(ab) & \underset{\rho_b}{\overset{\rho_\xi}{\rightleftarrows}} & \mathcal{L}(a) \\ \cup & & \cup \\ \mathcal{H}(b) & \underset{\rho_b}{\overset{\rho_\xi}{\rightleftarrows}} & \mathcal{H}(b\xi) = \mathcal{R}(b) \cap \mathcal{L}(a), \end{array}$$

and it's easy to verify $b\xi$ is an idempotent.

Conversely, suppose $\mathcal{L}(a) \cap \mathcal{R}(b)$ contains an idempotent e, then there exists $s, s', t, t' \in S^1$ such that

$$\begin{cases} te = a \\ t'a = e \end{cases} \wedge \begin{cases} es = b \\ bs' = e \end{cases}.$$

We found that

$$\begin{cases} ab = ab \\ abs' = a \end{cases} \Leftrightarrow ab\mathcal{R}a \wedge \begin{cases} ab = ab \\ t'ab = b \end{cases} \Leftrightarrow ab\mathcal{L}b,$$

thus $\mathcal{H}(ab) = \mathcal{R}(a) \cap \mathcal{L}(b)$. And the following isomorphism also stems from Green's Lemma.

$$\begin{array}{ccc} \mathcal{L}(a) & \underset{\rho_{s'}}{\overset{\rho_b}{\rightleftarrows}} & \mathcal{L}(ab) \\ \cup & & \cup \\ \mathcal{H}(a) & \underset{\rho_{s'}}{\overset{\rho_b}{\rightleftarrows}} & \mathcal{H}(ab) = \mathcal{R}(a) \cap \mathcal{L}(b), \end{array}$$

□

2.4 Regular Semigroup

Some definitions are listed below.

- ◇ Suppose $U < S$ is a subsemigroup, the green's equivalence $\mathcal{L}^U$ originated from U is defined as $\{(a, b) \in U^2 : U^1 a = U^1 b\}$. The similar definitions apply to the remaining equivalences.
- ◇ $\mathrm{idm}(S)$ is the set of all idempotents of S.
- ◇ $R \in \mathrm{Eqv}(S)$ is *idempotent-separating* if $R \cap \mathrm{idm}(S)^2 = \Delta_{\mathrm{idm}(S)}$, that is, each R-class contains no more than one idempotent.

If S is regular, then $a\mathcal{L}b \Leftrightarrow S^1 a = S^1 b \Leftrightarrow Sa = Sb$, since $\exists b \in S(aba = a)$, which implies for all $a \in S$, $S^1 a = Sa$. In fact, to define the Green's Equivalences on a regular semigroup, we can drop all reference to S^1.

Proposition 2.4.1 Let S be a regular semigroup and $a, b \in S$. Then

(a) $(a, b) \in \mathcal{L} \Leftrightarrow \exists a' \in \operatorname{inv}(a) \exists b' \in \operatorname{inv}(b)(a'a = b'b)$,

(b) $(a, b) \in \mathcal{R} \Leftrightarrow \exists a' \in \operatorname{inv}(a) \exists b' \in \operatorname{inv}(b)(aa' = bb')$,

(c) $(a, b) \in \mathcal{H} \Leftrightarrow \exists a' \in \operatorname{inv}(a) \exists b' \in \operatorname{inv}(b)(a'a = b'b \wedge aa' = bb')$.

Proof. Since S is regular, each element has an inverse. Suppose $a\mathcal{L}b$ and a', b' are inverses of a, b respectively. To prove (a), the following diagram says it all. □

As for the equivalences on subsemigroup U, it can be easily verified that $\mathcal{L}^U \subset \mathcal{L} \cap U^2$, the similar conclusions apply for remaining equivalences. However, this inclusion could be proper.

Example 2.4.2 Suppose S is the free group generated by set $\{a\}$, namely,

$$S = \mathbf{F}_{\mathsf{Grp}}(\{a\}) = \{\cdots, a^{-1}, e, a, \cdots\}.$$

And $U = \{a, a^2, \cdots\}$ is a subsemigroup of it. Then,

$$\mathcal{L}^U = \cdots = \mathcal{J}^U = \Delta_U,$$

while

$$\mathcal{L} \cap U^2 = \cdots = \mathcal{J}^U \cap U^2 = U^2.$$

Proposition 2.4.3 If U is a regular subsemigroup of semigroup S, then all $\mathcal{L}, \mathcal{R}, \mathcal{H}$ satisfy $X^U = X \cap U^2$, where $X \in \{\mathcal{L}, \mathcal{R}, \mathcal{H}\}$.

Proposition 2.4.4 Suppose S is a regular semigroup and $\mathcal{C} \in \operatorname{Cge}(S)$, then $S/\mathcal{C}$ is regular.

The following result usually known as Lallement's Lemma. Suppose S is a regular semigroup, there are two equivalent propositions.

(24LAL1) Given $\mathcal{C} \in \operatorname{Cge}(S)$; if $\mathcal{C}(a)$ is an idempotent in $S/\mathcal{C}$, then there exists an idempotent e such that $\mathcal{C}(a) = \mathcal{C}(e)$.

(24LAL2) Given morphism $\phi : S \to T$; if $\phi(a)$ is an idempotent, then there exists an idempotent $e \in S$ such that $\phi(e) = f$.

Proof. To proof (24LAL1), suppose $\mathcal{C}(a) = \mathcal{C}(a^2)$, let x be the inverse of a^2 and $e = axa$. We then proceed to prove the equivalence of these two propositions.

Suppose (24LAL1) holds, $f \in \operatorname{im}\phi$ is an idempotent. Clearly $\ker\phi = \{(a, b) : \phi(a) = \phi(b)\} \in \operatorname{Cge}(S)$. Let $a \in \phi^{-1}(f)$, it can be verified that $[\ker\phi](a) \in S/\ker\phi$ is an idempotent. And it follows that there exists idempotent $e \in S$ and $[\ker\phi](a) = [\ker\phi](e)$.

Conversely, suppose (24LAL2) holds, $\mathcal{C} \in \operatorname{Cge}(S)$, $\phi : S \to S/\mathcal{C}$ and $\phi(a)$ is an idempotent. Thus, there exists an idempotent e such that $\phi(a) = \phi(e)$. □

Proposition 2.4.5 If S is regular, $\mathcal{C} \in \operatorname{Cge}(S)$ is idempotent-separating iff $\mathcal{C} \subset \mathcal{H}$.

Proof. Assume that $\mathcal{C} \subset \mathcal{H}$, we may notice that

$$\Delta_{\operatorname{idm}(S)} \subset \mathcal{C} \cap \operatorname{idm}(S)^2 \subset \mathcal{H} \cap \operatorname{idm}(S)^2 \subset \Delta_{\operatorname{idm}(S)}.$$

For the converse, if $\mathcal{C}$ is idempotent-separating and $a\mathcal{C}b$. Let a' be the inverse of a, we can draw the following conclusions in sequence:

- $aa'\mathcal{C}ba'$
- $\mathcal{C}(ba') = \mathcal{C}(aa')$ is idempotent
- (24LAL1) there exists idempotent $e \in S$ such that $\mathcal{C}(e) = \mathcal{C}(ba') \wedge \mathcal{R}(e) \leq \mathcal{R}(ba') \wedge e = aa'$, the last assertion stems from $\mathcal{C}$ is idempotent separating

- (24REB1) $\mathcal{R}(a) = \mathcal{R}(aa') = \mathcal{R}(e) \leq \mathcal{R}(ba') \leq \mathcal{R}(b)$.

A dual argument shows that $\mathcal{L}(a) \leq \mathcal{L}(b)$:

- $a'a\mathcal{C}a'b$
- $\mathcal{C}(a'b) = \mathcal{C}(a'a)$ is idempotent
- $\exists e \in S\,(e$ is idempotent $\wedge\, \mathcal{C}(e) = \mathcal{C}(a'b) \wedge \mathcal{L}(e) \leq \mathcal{L}(a'b) \wedge e = a'a)$
- $\mathcal{L}(a) = \mathcal{L}(a'a) = \mathcal{L}(e) \leq \mathcal{L}(a'b) \leq \mathcal{L}(b)$.

Similarly, assume that $b' \in \mathrm{inv}(b)$ is chosen, then we have:

- $ab'\mathcal{C}bb'$
- $\mathcal{C}(ab') = \mathcal{C}(bb')$ is idempotent
- there exists idempotent $e \in S$ such that $\mathcal{C}(e) = \mathcal{C}(ab') \wedge \mathcal{R}(e) \leq \mathcal{R}(ab') \wedge e = bb'$
- $\mathcal{R}(b) = \mathcal{R}(bb') = \mathcal{R}(e) \leq \mathcal{R}(ab') \leq \mathcal{R}(a)$. $\square$

Bibliography

[1] John M Howie. *Fundamentals of Semigroup Theory.* Oxford University Press, 1995.