Complete Proofs in Chapter 2: Category Theory

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1 Limits

Page 59 in the book states that when I is a small category and $\mathcal{C}$ is Set, both $\varinjlim$ and $\varprojlim$ exist.

Proposition 1.1 *Let s, t be the start and target map of morphisms $\mathrm{Mor}(I) \to \mathrm{Ob}(I)$. For any $\beta : I^{\mathrm{op}} \to \mathsf{Set}$, the limit $\varprojlim \beta$ exists.*

Proof. Define two maps as follows:

$$\prod_i \beta(i) \underset{g}{\overset{f}{\rightrightarrows}} \prod_\sigma \beta(s(\sigma))$$

$$(x_i)_i \xmapsto{\ f\ } \left[\sigma \mapsto x_{s(\sigma)}\right]$$

$$(x_i)_i \xmapsto{\ g\ } \left[\sigma \mapsto \beta(\sigma)(x_{t(\sigma)})\right]$$

Next, we let:

$$\begin{aligned}
\varprojlim \beta &= \mathrm{Ker}\left[\prod_i \beta(i) \rightrightarrows \prod_\sigma \beta(s(\sigma))\right] \\
&= \left\{(x_i)_i \in \prod_i \beta(i) : f((x_i)_i) = g((x_i)_i)\right\} \\
&= \left\{(x_i)_i \in \prod_i \beta(i) : \forall \sigma \in \mathrm{Hom}_I(i,j)(\beta(\sigma)(x_j) = x_i)\right\}
\end{aligned}$$

Define a family of maps $p = (p_i)_i$, where $p_i((x_j)_j) = x_i$. We now need to verify (i): $p : \Delta(\varprojlim \beta) \to \beta$ is a morphism; (ii): $(\varprojlim \beta, p)$ is indeed the terminal object in (Δ/β). The verification of (i) is easy. As for (ii). For any (X, ξ), to make the diagram commute, the only way is to set $\phi(x) = (\xi_i(x))_i$. This establishes both the existence and uniqueness of $\phi : (X, \xi) \to (\varprojlim \beta, p)$:

$$\begin{tikzcd}
X \arrow[d, "\phi"'] \arrow[dr, "\xi_i"] & \\
\varprojlim \beta \arrow[r, "p_i"'] & \beta(i)
\end{tikzcd}$$

$\square$

As for the definition of "Ker" mentioned above, see the discussion of equalizer on page 63 of this book. Before we prove the existence of colimits, we first explain the equivalence relation generated by a relation.

Definition 1.2 *For any set X, let $R \subset X^2$ be a binary relation. Define*

$$S = \left\{(x,y) \in X^2 : \exists (x_i)_{i=0}^n \in X^{n+1} \wedge \forall 1 \leq k \leq n((x_{k-1}, x_k) \in R \vee (x_k, x_{k-1}) \in R)\right\}$$

where $x_0 = x$, $x_n = y$. Then S is the ***smallest*** *equivalence relation containing R.*

Proof. To show that S is an equivalence relation, the symmetry and transitivity are obvious. As for reflexivity, for $n = 0$, the definition of S implies $\forall x\, (x,x) \in S$. To show S is the **smallest** equivalence relation, let S' be any equivalence relation containing R. Suppose that $(x,y) \in S$. By definition, we have:

$$\exists (x_i)_{i=0}^n \in X^{n+1}\ (\forall 1 \leq k \leq n((x_{k-1}, x_k) \in R \vee (x_k, x_{k-1}) \in R))$$

$\square$

all such pairs (x_{k-1}, x_k) or (x_k, x_{k-1}) belong to S'. Thus, this implies $(x,y) \in S'$.

Proposition 1.3 *For any functor on small category $\alpha : I \to \mathsf{Set}$, the colimit $\varinjlim \alpha$ exists.*

Proof. Let $X = \bigsqcup_i \alpha(i)$. The notation $a \to b$ means there exists a σ such that $b = \alpha(\sigma)(a)$. All such pairs $\{(x, y) : x \to y\}$ forms a relation $R \subset X^2$. Define $\varinjlim \alpha = X/\sim$, where $\sim$ is the equivalence relation generated by R. Let $\iota = (\iota_i)_i$ where each ι_i is a map $\alpha(i) \to X/\sim$ sending $x \mapsto [x]_\sim$. It is straightforward to verify that $\alpha \xrightarrow{\iota} \varinjlim \alpha$ is a morphism. For any object (L, ξ) in (α, Δ), to ensure the commutativity of the diagram, define the map $\phi : \varinjlim \alpha \to L$ that sending $[x_i]_\sim$ to $\xi_i(x_i)$.

$$\begin{tikzcd} \alpha(i) \arrow[r, "\iota_i"] \arrow[dr, "\xi_i"'] & \varinjlim \alpha \arrow[d, "\phi"] \\ & L \end{tikzcd}$$

We now need to verify that ϕ is well-defined. Suppose $x_i \sim x_j$, we must show $\xi_i(x_i) = \xi_j(x_j)$. By definition, there exists a sequence $(t_i)_{i=0}^n$ such that a diagram of the following form holds:

$$x_i = t_0 \to t_1 \leftarrow t_2 \leftarrow t_3 \to \cdots \leftarrow t_n = x_j$$

where the direction of arrows can be arbitrary. This reduces to iterating the same pattern, so it suffices to show that $t_0 \to t_1$ ensures $\xi_0(t_0) = \xi_1(t_1)$. This follows from the $\xi : \alpha \to \Delta(L)$ is a natural transformation:

$$\begin{tikzcd} \alpha(i_0) \arrow[r, "\alpha(\sigma)"] \arrow[dr, "\xi_0"'] & \alpha(i_1) \arrow[d, "\xi_1"] \\ & L \end{tikzcd}$$

□

需要注意一点, 在书图表 (2.13) 下方讲 Set 中的等化子时, 虽然是上面命题的退化情况, 却采用了略为不同的构造. 接下来我们来看书中命题 2.7.8.

Proposition 1.4 *I 是小范畴时, $\alpha : I \to \mathcal{C}^\wedge$ or $\mathcal{C}^\vee$ 的余极限 $\varinjlim \alpha$ 存在. 同样 $\beta : I^{\mathrm{op}} \to \mathcal{C}^\wedge$ or $\mathcal{C}^\vee$ 的极限 $\varprojlim \beta$ 存在.*

Proof. 下面以 $\alpha : I \to \mathcal{C}^\wedge$ 为例. 此时若任意给出 $i \xrightarrow[I]{\sigma} j$ 和 $S \xrightarrow[\mathcal{C}^{\mathrm{op}}]{f} T$, 显然 $\alpha(i)(S), \alpha(\sigma)(f)$ 均是有定义的. 此外, 类似 $\alpha(i)(f), \alpha(\sigma)(\cdot)$ 等表示的定义也是自明的.

对任意 $S \in \mathrm{Ob}(\mathcal{C})$, 定义函子 $\alpha(\cdot)(S) : I \to \mathsf{Set}$, 其将 $i \in \mathrm{Ob}(I)$ 映射至 $\alpha(i)(S)$. 应用之前的命题可知对每个 S 都存在 $\varinjlim \alpha(\cdot)(S)$ 与相应的态射族 $(\iota_{i,S})_i$:

$$\alpha(\cdot)(S) \xrightarrow{(\iota_{i,S})_i} \Delta(\varinjlim \alpha(\cdot)(S))$$

定义 $\mathcal{C}^\wedge$ 中的元素 $\varinjlim \alpha$ 如下:

$$\begin{aligned} \mathcal{C}^{\mathrm{op}} &\longrightarrow \mathsf{Set} \\ S &\longmapsto \varinjlim \alpha(\cdot)(S) \\ \left[f : S \xrightarrow[\mathcal{C}^{\mathrm{op}}]{} T \right] &\longmapsto \varinjlim \alpha(\cdot)(f) \end{aligned}$$

我们断言下面资料构成了 (α/Δ) 中的初对象, 从而 $\varinjlim \alpha$ 就是 α 的余极限.

$$\alpha \xrightarrow{\iota = ((\iota_{i,S})_S)_i} \Delta(\varinjlim \alpha)$$

接下来的验证分为三个部分:

A. ι 是函子间的态射.

B. 对任意 $(L, \xi) \in \mathrm{Ob}(\alpha/\Delta)$, 存在唯一的 $\phi : \varinjlim \alpha \to L$.

C. ϕ 是函子间的态射.

(A.) 归结于证明 $\alpha(i)(\cdot) \xrightarrow{(\iota_{i,S})_S} \varinjlim \alpha$ 是态射; 以及下图交换. 证明是显见的.

$$\begin{tikzcd} \alpha(i) \arrow[r, "(\iota_{i,S})_S"] \arrow[d, "\alpha(\sigma)"'] & \varinjlim \alpha \\ \alpha(j) \arrow[ur, "(\iota_{j,S})_S"'] & \end{tikzcd}$$

(B.) 给出对象 (L, ξ), 对任意 $S \in \mathrm{Ob}(\mathcal{C})$:

$$\begin{tikzcd} \alpha(\cdot)(S) \arrow[r, "(\iota_{i,S})_i"] \arrow[dr, "(\xi_{i,S})_i"'] & \Delta(\varinjlim \alpha(\cdot)(S)) \arrow[d, "\exists! \Delta(\phi_S)"] \\ & \Delta(L(S)) \end{tikzcd} \quad \Rightarrow \quad \varinjlim \alpha \xrightarrow[\mathcal{C}^\wedge]{\exists!(\phi_S)_S} L$$

(C.) 我们的目标是验证对任意 $f : S \xrightarrow[\mathcal{C}^{\mathrm{op}}]{} T$, 下图交换:

$$\begin{tikzcd} \varinjlim \alpha(\cdot)(S) \arrow[r, "\phi_S"] \arrow[d, "\varinjlim \alpha(\cdot)(f)"'] & L(S) \arrow[d, "L(f)"] \\ \varinjlim \alpha(\cdot)(T) \arrow[r, "\phi_T"'] & L(T) \end{tikzcd}$$

证明又分为三步, 分别为:

(C1.) 下图交换:

$$\begin{tikzcd} \alpha(\cdot)(S) \arrow[r, "(\xi_{i,S})_i"] \arrow[d, "\alpha(\cdot)(f)"'] & \Delta(L(S)) \arrow[d, "\Delta(L(f))"] \\ \alpha(\cdot)(T) \arrow[r, "(\xi_{i,T})_i"'] & \Delta(L(T)) \end{tikzcd}$$

这是 $\alpha \xrightarrow{\xi} \Delta(L)$ 的结果.

(C2.) $\varinjlim \Delta(L(S)) = L(S)$, $\varinjlim \Delta(L(f)) = L(f)$. 证明是容易的.

(C3.) $\varinjlim[(\xi_{i,S})_i] = \phi_S$. 这来源于下图:

$$\begin{tikzcd} \alpha(\cdot)(S) \arrow[r, "(\iota_{i,S})_i"] \arrow[d, "(\xi_{i,S})_i"'] & \Delta(\varinjlim \alpha(\cdot)(S)) \arrow[d, "\Delta(\phi_S)"] \\ \Delta(L(S)) \arrow[r, "\mathrm{id}"'] & \Delta(\varinjlim \Delta(L(S))) \end{tikzcd}$$

最后再在 C1. 的交换图中调用书 Lemma 2.7.4. □

接下来是一个比较困难的证明, 书中对此的说明仅寥寥数笔, 对于初学者是极其难懂的. 好在其他的范畴论入门教材基本都讲到了这一命题.

Proposition 1.5 *对于 $\beta : I^{\mathrm{op}} \to \mathcal{C}$, 可定义 $\bar{\beta} : I^{\mathrm{op}} \to \mathcal{C}^\wedge$, 映 $i \mapsto h_{\mathcal{C}}(\beta(i)) = \mathrm{Hom}_{\mathcal{C}}(\cdot, \beta(i))$. 则 $\exists \varprojlim \beta$ 当且仅当 $\varprojlim \bar{\beta}$ 可表.*

对于 $\alpha : I \to \mathcal{C}$, 可定义 $\bar{\alpha} : I \to \mathcal{C}^\vee$, 映 $i \mapsto k_{\mathcal{C}}(\alpha(i)) = \mathrm{Hom}_{\mathcal{C}}(\alpha(i), \cdot)$. 则 $\exists \varinjlim \alpha$ 当且仅当 $\varinjlim \bar{\alpha}$ 可表.

Proof. 我们以 $\beta : I^{\mathrm{op}} \to \mathcal{C}$ 为例. 证明分为几个部分:

A. $\exists \varprojlim \beta \Rightarrow h_{\mathcal{C}}(\varprojlim \beta) \xrightarrow[\exists]{\sim} \varprojlim \bar{\beta}$.

A1. $\forall S \in \mathrm{Ob}(\mathcal{C})$ $\mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta) \simeq \varprojlim \bar{\beta}(\cdot)(S)$.

A2. $\forall S \in \mathrm{Ob}(\mathcal{C})$ $\mathrm{Hom}_{\mathcal{C}}(S, \varprojlim \beta) \simeq \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta)$.

B. $h_{\mathcal{C}}(D) \xrightarrow[\exists]{\sim} \varprojlim \bar{\beta} \Rightarrow \exists \varprojlim \beta$.

(A1.)

$$\begin{aligned}\varprojlim \bar{\beta}(\cdot)(S) &= \left\{(\xi_i)_i \in \prod_I \bar{\beta}(i)(S) : \forall \sigma : i \xrightarrow{I} j (\xi_j \circ \beta(\sigma) = \xi_i)\right\} \\ &= \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta)\end{aligned}$$

(A2.) 令 $\mathcal{C}^\wedge \ni A = \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(\cdot), \beta)$, 回顾 Yoneda 定理, 对于任意 S, 我们有:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{Hom}(\cdot, S), \mathrm{Hom}(\Delta(\cdot), \beta)) & \xleftrightarrow{1:1} & \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta) \\ \phi & \longmapsto & \phi_S(\mathrm{id}_S) \\ (\phi_T : f \mapsto u \circ \Delta(f))_{T \in \mathrm{Ob}(\mathcal{C})} & \longleftarrow & u \end{array} \tag{1.1}$$

这是个极其实用的公式. 我们断言: $\Delta(S) \xrightarrow{u} \beta$ 是 (Δ/β) 中的终对象时 (即 $S = \varprojlim \beta$), u 的像 ϕ 是同构. 反之, 若 ϕ 是同构, 则 S 连同 $\phi_S(\mathrm{id}_S)$ 构成终对象. 后者的证明可见 (B.); 对于前者, 我们取 $S = \varprojlim \beta$, 然后验证对任意 S, ϕ_S 都是双射:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(S, \varprojlim \beta) & \xleftrightarrow{1:1} & \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta) \\ f & \xmapsto[\phi_S]{} & u \circ \Delta(f) \\ \varprojlim \lambda & \xleftarrow[\psi_S]{} & \lambda \end{array}$$

$\psi_S \phi_S$ 是单位映射, 原因来自下图:

$$\begin{tikzcd} \Delta(S) \arrow[r, "\Delta(f)"] \arrow[d] & \Delta(\varprojlim \beta) \arrow[r, "u"] \arrow[d] & \beta \arrow[d] \\ S \arrow[r, "f"'] & \varprojlim \beta \arrow[r, "\mathrm{id}"'] & \varprojlim \beta \end{tikzcd}$$

$\phi_S \psi_S$ 也是单位映射, 即 $u \circ (\varprojlim \lambda) = \lambda$, 证明很容易.

(B.) 假设存在 D 使得 $h_{\mathcal{C}}(D) \xrightarrow[\phi]{\sim} \varprojlim \bar{\beta} = \mathrm{Hom}(\Delta(\cdot), \beta)$. 通过 (1.1) 式将 ϕ 映为 $u = \phi_D(\mathrm{id}_D)$. 对于任意 S, 观察到映射:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(S, D) & \xleftrightarrow{1:1} & \mathrm{Hom}_{\mathcal{C}^{I^{\mathrm{op}}}}(\Delta(S), \beta) \\ f & \xmapsto[\phi_S]{} & u \circ \Delta(f) \\ \psi_S(\lambda) & \xleftarrow[\psi_S]{} & \lambda \end{array}$$

其中 ψ_S 为 ϕ_S 的逆. 断言 $\Delta(D) \xrightarrow{u}$ 给出 (Δ/β) 中的一个终对象. 对于任意 (S, λ), 上面映射给出了唯一一个 $\psi_S(\lambda) : S \to D$. 同时由于 $u \circ \Delta(\psi_S(\lambda)) = \phi_S \psi_S(\lambda) = \lambda$, 从而下图交换:

$$\begin{tikzcd} \Delta(D) \arrow[r, "u"] & \beta \\ \Delta(S) \arrow[u, "\Delta(\psi_S(\lambda))"] \arrow[ur, "\lambda"'] & \end{tikzcd}$$

□

下面简单补充一下 $\alpha : I \to \mathcal{C}$ 的情况. 定义 $\bar{\alpha} : I \to \mathcal{C}^\vee$, 映 i 到 $\mathrm{Hom}_{\mathcal{C}}(\alpha(i), \cdot)$. 从而 $\bar{\alpha}(\cdot)(S) : I \to \mathsf{Set}^{\mathrm{op}}$, 这等同于 $I^{\mathrm{op}} \to \mathsf{Set}$. 然后就可以验证:

$$\varprojlim \bar{\alpha}(\cdot)(S) = \mathrm{Hom}_{\mathcal{C}^I}(\alpha, \Delta(S))$$
$$\mathrm{Hom}_{\mathcal{C}^\vee}(\mathrm{Hom}(\alpha, \Delta(\cdot)), \mathrm{Hom}(S, \cdot)) \simeq \mathrm{Hom}_{\mathcal{C}^I}(\alpha, \Delta(S))$$