STOCHASTIC DIFFERENTIAL EQUATION

St. Xavier's College, Kolkata

NAME – SAMYA MUKHERJEE

ROLL – 35

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SUPERVISED BY – SUCHARITA ROY

DECLARATION:

I affirm that I have identified all my sources and that no part of my dissertation paper uses unacknowledged material.

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ABSTRACT:

Stochastic Differential Equations (SDEs) serve as indispensable tools in modelling systems subject to random fluctuations, finding wide-ranging applications across various disciplines including physics, biology, finance, and engineering. This dissertation delves into the theoretical underpinnings, diverse applications, and computational techniques associated with stochastic differential equations. Beginning with an overview of the fundamental concepts and properties of SDEs, this work progresses to explore advanced topics such as methods for solving SDEs. How to solve a linear SDE and graphical understanding after introducing a bit of stochastic calculus and Ito integration properties and uses.

CONTENTS

CHAPTER 1

MOTIVATION

- Financial markets trade investments into stocks of a company, commodities (e.g. oil, gold), bonds, or derivatives. Bonds evolve in a predictive way, but stocks and commodities do not. They are risky assets, because their value is affected by randomness. Our goal is to model the price of such a risky asset. Many processes in science, technology, and engineering can be described very accurately with ordinary differential equations (ODEs)

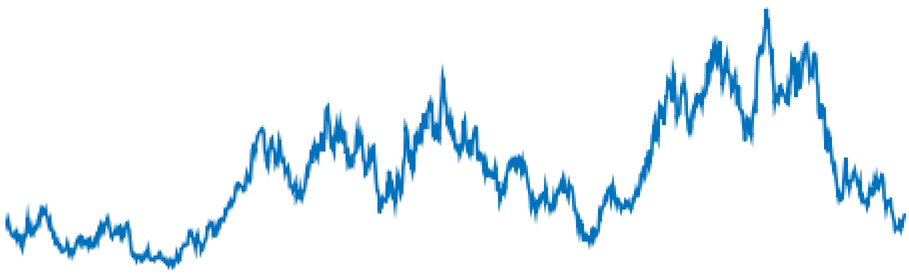
$\frac{dx}{dt}$ = f(t,x) --(1.1)

Here, X = X(t) is the time-dependent solution, and f = f (t, X) is a given function which depends on what is supposed to be modelled. But the solution of an ODE has no randomness – it is a purely deterministic object. If we know its value X(t0) at a given time t0, then by solving the ODE we can compute its value X(t) at any later time t ≥ 0. This is certainly not true for stocks. In order to model risky assets, we have to put some randomness into the dynamics. A first and rather naive approach to do so is to add a term which generates "random noise":

$\frac{dx}{dt}$ = f(t,x) + g(t,x)z(t)---(1.2)

[Here $\frac{dx}{dt}$ =f(t,x) denotes differential equation and g(t,x)z(t) denotes random noise]

- The next questions are obviously how to choose g(t, X), and how to define z(t) in a mathematically sound way. But even if these problems can be solved, Equation (1) is dubious. The solution of an ODE is, by definition, a differentiable function, whereas the chart of a stock typically looks like this:



It can be expected that such a function is continuous, but not differentiable. This raises the question whether (1) makes sense at all. Then what is the correct way to define a stochastic differential equation?

- The goal my dissertation is to give an informal introduction to stochastic differential equation (SDEs) of Itˆo type. They are based on the Itˆo integral, which can be thought of as an extension of the classical Riemann-Stieltjes integral to cases where both the integrand and the integrator can be stochastic processes. Another important tool is the Itˆo-Doeblin formula, which is a stochastic counterpart of the classical chain rule.
- Basically the stochastic differential equation is summation of ordinary differential equation and gaussian white noise.

Here the equation (1.1) is slope of x(t) this is the expected slope which is never same as the actual slope due to gaussian white noise.

We basically form the equation as:

$\frac{dx}{dt} = f(x) + €_{t+dt}$ $[€_{t+dt} \sim N(0,1)]$ --(1.3)

$€_{t+dt}$ denotes the different intensities

$\frac{dx}{dt} = f(x) +$ g(x) $€_{t+dt}$, $[€_{t+dt} \sim N(0,1)]$ --- (1.4)

Here g(x) denotes the intensity

Here dx_t denotes the return of an asset for the trading period t to t+dt

G (x_t) is the stochastic conditional volatility function

$€_{t+dt}$ is the realized return shock in t+dt

CHAPTER 2

STOCHASTIC PROCESS:

2.1

DEFINITION:

A stochastic process is a random variable that moves across times. It is defined on a probability space (Ω, A, P).

- Ω is the space of all possible outcomes
- A is the subset of admissible events
- P is the probability of each admissible events and hence called a probability measure

P: A →[0,1] {this is basically a mapping}

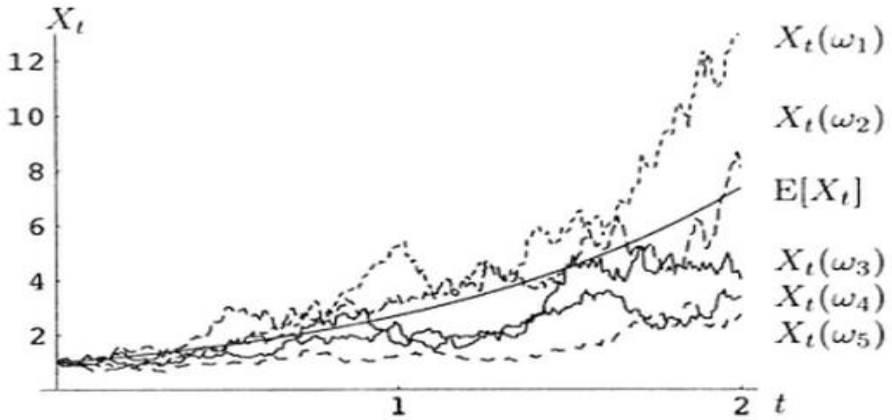
X be the random variable where X: Ω→R

Stochastic process: { X_t } t: t* Ω → R [t is the set of time it can take values from 1 to infinity]

(i) A collection {X(t)|t ≥ 0} of random variables is called a stochastic process.

(ii) For each point ω ∈ Ω, the mapping t → X(t, ω) is the corresponding sample path.

- The idea is that if we run an experiment and observe the random values of X(·) as time evolves, we are in fact looking at a sample path {X(t, ω)| t ≥ 0} for some fixed ω ∈ Ω. If we rerun the experiment, we will in general observe a different sample path.



- By the formal definition, stochastic process is basically a family of random variables {X(θ)}, indexed by a parameter θ, where θ belongs to some index set Θ. Θ will represent time. If Θ is a set of integers, representing specific time points, we have a stochastic process in discrete time and we shall replace the general subscript θ by n. So I will discuss about the discrete time process {Xn}.

- If Θ is the real line (or some interval of the real line) we have a stochastic process in continuous time and we shall replace the general subscript θ by t and change the notation slightly, writing X(t) rather than Xt. The reason that we introduce the rather abstract notion of an index set Θ, rather than just working with time, is that we sometimes want to study spatial processes as well as temporal processes. In a spatial process, Θ would be a vector, representing location in space rather than time. For example, we might have a process {X(u,v)}, representing a random variable that varies across two-dimensional space}. Here, X(u,v) represents the value of the process at position (u, v). We can even have processes that evolve in both time and space, so called spatio-temporal processes.

- A stochastic process is simply a process that develops in time according to probabilistic rules. We shall be particularly concerned with stationary processes, in which the probabilistic rules do not change with time

- Therefore, we are often interested in conditional distributions of the form Pr(Xt_k|Xt_{k-1}, Xt_{k-2} , . . . , Xt_1) for some set of times t_k > t_{k-1} > . . . > t_1. In general, this conditional distribution will depend upon values of Xt_{k-1}, Xt_{k-2} , . . . , Xt_1 . However, we shall focus particularly in this module on processes that satisfy the Markov property, which says that
Pr(Xt_k |Xt_{k-1}, Xt_{k-2} , . . . , Xt_1) = Pr(Xt_k | Xt_{k-1},).

• **A counting process** is a process X(t) in discrete or continuous time for which the possible values of X(t) are the natural numbers (0, 1, 2, . . .) with the property that X(t) is a nondecreasing function of t. Often, X(t) can be thought of as counting the number of ‘events’ of some type that have occurred by time t. The basic example of a counting process is the Poisson process, which we shall study in some detail.

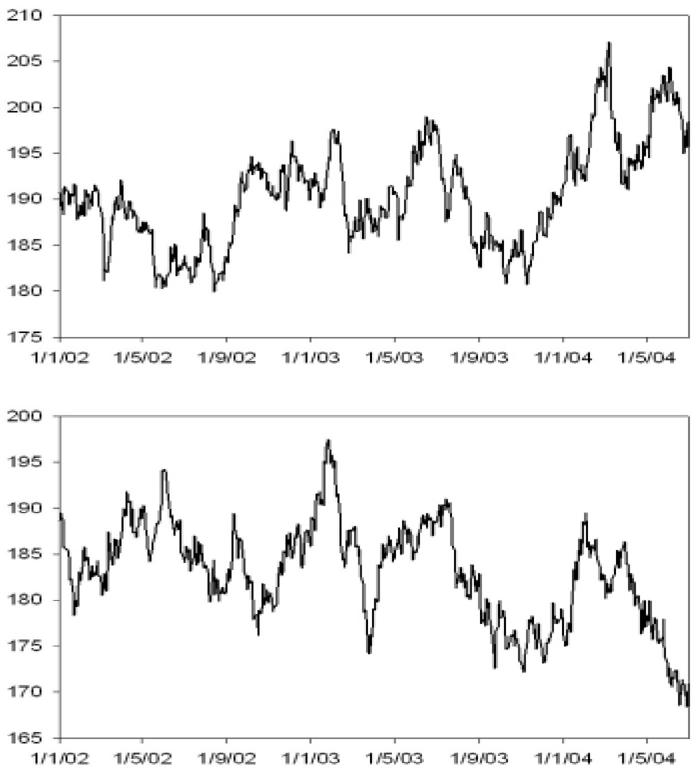
• **A sample path** of a stochastic process is a particular realisation of the process, i.e. a particular set of values X(t) for all t (which may be discrete of continuous), generated according to the (stochastic) ‘rules’ of the process.

• **The increments of a process** are the changes X(t) − X(s) between time points s and t (s < t). Processes in which the increments for non-overlapping time intervals are independent and stationary (i.e. dependent only on the lengths of the time intervals, not the actual times) are of particular importance. Random walks are a good example. General processes of this type are called L´evy processes, and include Brownian motion.

2.2

EXAMPLE OF A STOCHASTIC PROCESS:

In this section, I offer an example of stochastic process, to give some idea of the wide range of application areas



: Exchange rate between British pound and Japanese yen over the period 1/1/02 to 31/7/04.

Upper graph shows true exchange rate, lower graph shows a simulation of a random walk model. The upper panel of Figure 1 shows the exchange rate between the British pound and the Japanese yen from 1st January 2002 to 31st July 2004. Over this period, the average exchange rate was 191.1 yen to one pound. The lower panel shows a simulation of a type of stochastic process called a random walk. We will be studying random walks in Section 2 of this module. In the random walk model, the daily changes in exchange rate are independent normal random variables with zero mean and standard deviation of 1.206 (matching the standard deviation in the observed data). Whilst the detailed patterns are of course different, the two series have a similar structure. Note that in the random walk model, upward and downward movements in the exchange rate are equally likely, and there is no scope for making money through currency speculation except by luck.

CHAPTER 3

STOCHASTIC CALCULUS

3.1

STOCHASTIC PROCESS:

Stochastic calculus is a branch of mathematics that deals with random processes and their integration techniques. It is primarily used in the field of probability theory and mathematical finance to model and analyse systems involving random variations over time. Stochastic calculus is the area of mathematics that deals with processes containing a stochastic component and thus allows the modelling of random systems. Many stochastic processes are based on functions which are continuous, but nowhere differentiable. This rules out differential equations that requires mostly the use of derivative terms, since they are unable to be defined on non-smooth functions. Instead, a theory of integration is required where integral equations do not need the direct definition of derivative terms. In quantitative finance, the theory is known as Ito's calculus.

The most use of stochastic calculus is in finance, which is through modelling the random motion of an asset price in the Black-Scholes-model .The physical process of Brownian motion in particular, a geometric Brownian motion, is used as a model of asset prices via the Wiener process .This process is in fact represented by a stochastic differential equation, which, in despite its name, is an integral equation .

3.2

FUNCTION OF BIONOMIAL MODEL

The binomial model provides one means of deriving the Black-Scholes equation. A fundamental tool of stochastic calculus, which is currently known as Ito's lemma, allows us to derive it in an alternative manner.

3.3

ITOS LEMMA:

Itô's Lemma, named after the Japanese mathematician Kiyoshi Itô, is a fundamental result in stochastic calculus. It provides a formula for finding the differential of a function of a stochastic process.

Formally, suppose we have a stochastic process X(t) described by a stochastic differential equation (SDE) of the form:

$dX(t)=\mu(t,X(t))dt+\sigma(t,X(t))dW(t)$

where W(t) is a Wiener process (Brownian motion),$\mu(t,X(t))$ represents the drift term, and $\sigma(t,X(t))$ represents the diffusion term

Now, let f(t,X(t)) be a differentiable function of t and X(t). Ito's Lemma states that if Y(t)=f(t,X(t)), then the differential of Y(t) can be expressed as:

$dY(t)=(\partial t/\partial f+\mu(t,X(t))\partial X/\partial f+21\sigma 2(t,X(t))\partial X2/\partial 2f)dt+\sigma(t,X(t))\partial X/\partial f dW(t)$

This formula allows us to compute the differential of a function of a stochastic process, taking into account both the deterministic part (the drift term) and the random part (the diffusion term) of the process.

Itô's Lemma is widely used in mathematical finance for pricing derivatives and risk management. It provides a tool for expressing the change in the value of a derivative security as a function of the underlying asset's movement and volatility.

3.4

Fundamental difference between ordinary calculus and stochastic calculus

The fundamental difference between stochastic calculus and an ordinary calculus is that stochastic calculus allows the derivative to have a random component involve in it which is determined by a particular motion, which is actually known as Brownian motion. The derivative of a random variable has both the deterministic component and random component involved in it, which is rather normally distributed. That means it follows a normal distribution.

3.5

Aim of using stochastic calculus:

In this dissertation, I will utilise the theory of stochastic calculus to derive the black Sholes formula for a contingent claim. For this, I need to assume that my asset price will never be negative. A vanilla equity such as stock always has this property. A standard Brownian motion cannot be used as a model here, since there is a nonzero probability of the price becoming negative. A geometric Brownian motion is used instead, where the logarithm of the stock price has a stochastic behaviour, we will form a stochastic differential equation for this asset price movement and solve it to provide the path of the stock price. In order to price my contingent claim, I will note that the price of the claim mainly depends upon the price of asset, and that by clever construction of a portfolio of claims and assets. We will estimate the stochastic components by cancellation. I can then finally use a no- arbitrage argument to price a European call option via the derived Black-Scholes equation.

CHAPTER 4

MARKOV AND MARTINGLE PROPERTY

INTRODUCTORY PART:

In order to formally define the concept of Brownian motion and utilise it as a basis for an asset price model, it is necessary to define the Markov and Martingale properties. These provide an intuition as to how an asset price will behave over time.

The Markov property states that a stochastic process essentially has "no memory". This means that the conditional probability distribution of the future states of the process are independent of any previous state, with the exception of the current state. The Martingale property states that the future expectation of a stochastic process is equal to the current value, given all known information about the prior events.

Both of these properties are extremely important in modelling asset price movements.

MARKOV PROPERTY

A sensible way to introduce the Markov property is through a sequence of random variables Zi, which can take one of two values from the set $\{1,-1\}$. This is known as a coin toss. We can calculate the expectations of

Z_i: E(Z_i) = 0 ; E(Zi^2) = 1 ; E ($Z_i Z_k$) = 0 ---(4.1)

The key point is that the expectation of Z_i has no dependence on any previous values within the sequence. Let us take the partial sums of our random variables within our coin toss, which we will denote by S_i.

$$S_i = \sum_{k=1}^{i} Z_i \text{---------------------------------------(4.2)}$$

We can now calculate the expectations of our partial sums, using the linearity of the expectation operator:

$$\mathrm{E}(S_i) = 0 \;;\; \mathrm{E}(S_i{}^2) = E\left(Z_1{}^2 + 2Z_1Z_2 + \cdots\right) = 1 \text{--------------------------------------(4.3)}$$

We see that, again, there is no dependence on the expectation of S_i of any previous value within the sequence of partial sums. We can extend this to discuss conditional expectation. Conditional expectation is the expectation of a random variable with respect to some conditional probability distribution. Hence, we can ask that if i=4 (i.e. we carry out four-coin tosses), what does this mean for the expectation of S_5 ?

$$\mathrm{E}(S_5 \mid Z_1, Z_2, Z_3, Z_4) = S_4 \text{--}(4.4)$$

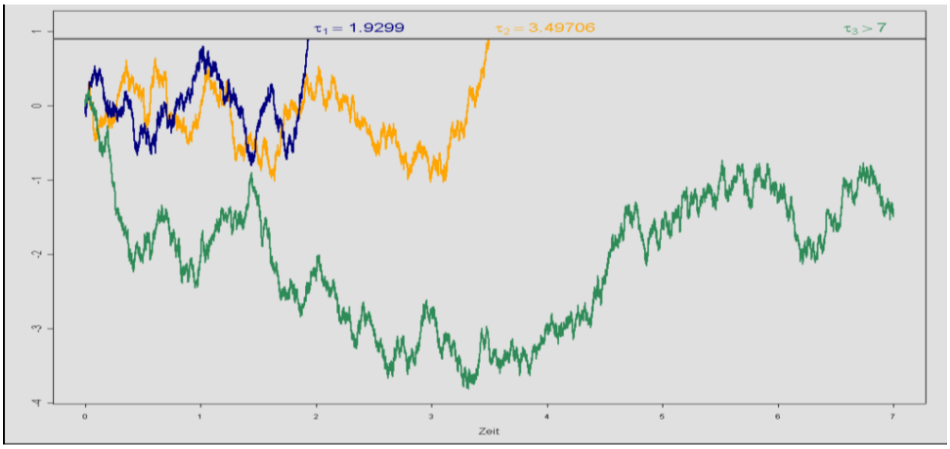
That is, the expected value of S_{i} is only dependent upon the previous value $S_{\mathrm{i}-1}$, not on any values prior to that. This is known as the Markov Property. Essentially, there is no memory of past events beyond the point our variable is currently at within the sequence. Nearly all financial models discussed in these articles will possess the Markov property.

THE MARTINGALE PROPERTY

An additional property that holds for our sequence of partial sums is the Martingale property. It states that the conditional expectation of the sequence of partial sums S_{i}, is simply the current value:

$\mathrm{E}(S_i | S_K, K<i) = S_K$

Essentially, the martingale property ensures that in a "fair game", knowledge of the past will be of no use in predicting future winnings.



Above is the Stopped Brownian motion which is an example of a martingale. It can model an even coin-toss betting game with the possibility of bankruptcy.

Uses of Martingale process

The simplest of these strategies was designed for a game in which the gambler wins their stake if a coin comes up heads and loses it if the coin comes up tails. The strategy had the gambler double their bet after every loss so that the first win would recover all previous losses plus win a profit equal to the original stake. As the gambler's wealth and available time jointly approach infinity, their probability of eventually flipping heads approaches 1, which makes the martingale betting strategy seem like a sure thing. However, the exponential growth of the bets eventually bankrupts its users due to finite bankrolls. Stopped Brownian motion, which is a martingale process, can be used to model the trajectory of such games.

CHAPTER 5

GEOMETRIC BROWNIAN MOTION SOLVING BY ITO'S LEMMA:

The usual model for the time-evolution of an asset price S(t) is given by the geometric Brownian motion, represented by the following stochastic differential equation:

$dS(t)=\mu S(t)dt+\delta S(t)dB(t)$---(5.1)

Note that the coefficients μ and δ, representing the drift and volatility of the asset, respectively, are both constant in this model. In more sophisticated models they can be made to be functions of t, S(t) and other stochastic processes.

The solution S(t) can be found by the application of Ito's Lemma to the stochastic differential equation.

Dividing through by S(t) in the above equation leads to:

$\frac{dS(t)}{S(t)}= \mu dt+ \delta\, dB(t)$---(5.2)

Notice that the left hand side of this equation looks similar to the derivative of logS(t). Applying Ito's Lemma to logS(t) gives:

$$d(\log S(t))=(\log S(t))'(\mu)S(t)dt+(\log S(t))'\,\delta S(t)dB(t)+\frac{1}{2}(\log S(t))''\delta^2 S(t)^2 dt \text{-------------------(5.3)}$$

This becomes:

$$d(\log S(t))= \mu dt + \delta\, dB(t)-\frac{1}{2}\,\delta^2 dt=(\mu-\frac{1}{2}\,\delta^2)\, dt+ \delta\, dB(t) \text{---(5.4)}$$

This is an Ito drift-diffusion process. It is a standard Brownian motion with a drift term. Since The above formula is simply shorthand for an integral formula, we can write this as:

$$\log S(t)-\log S(0)= (\mu-\frac{1}{2}\,\delta^2)\, dt + \delta\, dB(t) \text{---(5.5)}$$

Finally, taking the exponential of this equation gives:

$$S(t)=S(0)e^{(\mu-\frac{1}{2}\delta^2)\, dt\, +\, \delta\, dB(t)} \text{--(5.6)}$$

This is the solution the stochastic differential equation. In fact it is one of the only analytical solutions that can be obtained from stochastic differential equations.

CHAPTER 6

STOCHASTIC INTEGRAL AND ITS PROPERTY

Stochastic integration is a mathematical operation that extends the concept of integration from deterministic functions to stochastic processes. It plays a crucial role in the theory of stochastic calculus and is essential for understanding and solving stochastic differential equations (SDEs) and related problems in probability theory, finance, physics, and other fields.The main type of stochastic integration used in the context of stochastic calculus is the Ito integral, named after the Japanese mathematician Kiyosi Ito. The Ito integral is defined with respect to a stochastic process W(t), typically a Brownian motion or Wiener process, and is denoted by $\int_0^t H(s)dW(s)$, where H(s) is a deterministic function (often called the integrand).The integral $\int_0^t H(s)dW(s)$, represents the accumulated effect of the integrand H(s) with respect to the stochastic process W(t) over the interval from 0 to t. Intuitively, it can be understood as the limit of Riemann sums in which the integrand is multiplied by the increments of the stochastic process and summed over smaller and smaller intervals as the partition becomes finer.A stochastic integral of the function f=f(t) is a function W=W(t), t∈[0,T] given by: W(t)=$\int_0^t f(s)dB(s)$=lim n→∞$\sum_{k=1}^{n} f(t_{k-1})(B(t_k) - B(t_{k-1}))$------------------------------(6.1)

where t_k=$\frac{kt}{N}$.

Note that the function f is non-anticipatory, in the sense that it is evaluated within the summation sign at time t_{k-1}. This means that it has no information as to what the random variable at $X(t_k)$ is. Supposing that f represented some portfolio allocation based on B, then if it were not evaluated at t_{k-1}, but rather at t_k, we would be able to anticipate the future and modify the portfolio accordingly.

The previous expression provided for W(t) is an integral expression and thus is well-defined for a non-differentiable variable, B(t), due the property of finiteness as well as the chosen mean and variance. However, we wish to be able to write it in differential form: dW=f(t)d(B)

KEY PROPERTY OF ITO INTEGRAL:

- **LINEARITY**: The Ito integral is a linear operator, meaning that it satisfies the linearity property: $\int_0^t (aH1(s) + bH2(s))dW(s)$ =$a \int_0^t H1(s)dW(s) + b \int_0^t H2(s)dW(s)$ for constants a and b and integrable functions H1(s) and H2(s).
- **ITO ISOMETRY:** The Ito integral preserves the L2 norm, or Euclidean norm, of the integrand: E [$\int_0^t H1(s)dW(s)$]^2=E [$\int_0^t H1(s)$^2 ds], where E[·] denotes the expected value.
- **QUADRATIC VARIATION:** Itô integrals have quadratic variation properties. Specifically, the quadratic variation of an Itô integral is given by:
  $<\int_0^t H(s)dW(s)> = \int_0^t H(s)^2 ds$

  where H(s) is the integrand and ⟨·⟩ denotes the quadratic variation. This property is essential for calculating the variance and covariance of stochastic integrals

- **ITO'S CHAIN RULE**: Let's consider two stochastic processes X(t) and Y(t), and a function f(x,y) that is twice continuously differentiable with respect to its arguments. Then, Ito's chain rule states:
- df(X(t),Y(t))=$\frac{\partial f}{\partial x}$dX(t) + $\frac{\partial f}{\partial y}$ dY(t)+$\frac{1}{2}\frac{\partial^2 f}{\partial x^2}$ d(X(t))^2+$\frac{1}{2}\frac{\partial^2 f}{\partial y^2}$ d(Y(t))^2 + $\frac{\partial^2 f}{\partial x\, \partial y}$d(X(t))d(Y(t))
- df(X(t),Y(t)) is the differential of the function f(X(t),Y(t)
- $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are partial derivative of f with respect to x and y evaluated at x(t),y(t)
- $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x\, \partial y}$ are the second-order partial derivatives of f with respect to x and y, evaluated at (X(t),Y(t)).
- dX(t) and dY(t) are the differentials of the stochastic processes X(t) and Y(t) respectively.
- Ito's chain rule accounts for the additional randomness present in the stochastic processes X(t) and Y(t), providing terms involving the quadratic variations and cross-variations of the processes.
- Ito's chain rule is a fundamental tool in stochastic calculus and is widely used in the analysis of stochastic differential equations (SDEs) and their applications in finance, physics, engineering, and other fields. It allows for the computation of the differential of composite functions involving stochastic processes, enabling the modelling and analysis of complex systems subject to random fluctuations.

CHAPTER 7

CONCEPT OF LINEAR STOCHASTIC DIFFERENTIAL EQUATION

A linear stochastic differential equation (LSDE) is an equation of the form

$$\mathrm{d}X_t = \big(a_1(t)X_t + a_2(t)\big)\,\mathrm{d}t + \big(b_1(t)X_t + b_2(t)\big)\,\mathrm{d}B_t, \qquad X_0 = x_0,$$

where ai and bi (i = 1, 2) are non-random functions bounded on every finite interval [0, T](e.g. continuous)*;* it is easy to check that the coefficients of an LSDE satisfy the Lipschitz conditions, and thus the equation has a unique solution. If ai and bi are constants, then the LSDE is called autonomous; if a2 = b2 = 0, then it is called homogeneous.

To solve a linear SDE, you typically follow these steps

- **Identification of Coefficients**: Determine the coefficients a(t) and b(t) from the given SDE. These coefficients represent the deterministic drift and stochastic diffusion terms, respectively
- **Integration**: Integrate both sides of the SDE. The integration process involves solving deterministic integrals for a(t)Xtdt and stochastic integrals for b(t)XtdWt. Often, this integration can be challenging, especially for stochastic terms, and may require techniques like Ito's calculus.
- **Applying Initial/Boundary Conditions (if any):** If initial or boundary conditions are provided, apply them to the solution obtained after integration.
- **Simulation (optional):** If a closed-form solution is not obtainable, or if you want to validate your solution, you can simulate the stochastic process numerically using methods like Euler-Maruyama, Milstein, or Monte Carlo simulations.

CHAPTER 8

BROWNIAN MOTION(WIENNER PROCESS)

Consider a continuous real-valued time interval [0, T], with T>0. In this interval N coin tosses will be carried out, which each take a time T/N. Hence the coin tosses will be spaced equally in time. Concurrently the payoff returned from each coin toss will be modified. The sequence of discrete random variables representing the coin toss is $Z_i \in \{-1,1\}$. A further sequence of discrete random variables, $Z_i \in \{\sqrt{T/N}, -\sqrt{T/N}\}$, can also be defined. This definition of such a sequence of discrete random variables is used to provide a very specific quadratic variation of the coin toss.

The quadratic variation of a sequence of DRVs is defined as the sum of the squared differences of the current and previous terms:

$$\sum_{k=1}^{i}(S_K - S_{k-1})^2$$

For Z_i, the first coin toss random variable sequence, the quadratic variation is given by:

$$\sum_{k=1}^{i}(S_K - S_{k-1})^2 = i$$

For $\widehat{Z_\imath}$ the quadratic variation of partial sum $\widehat{S_\imath}$ is:

$$\sum_{k=1}^{n}(\widehat{S_k} - \widehat{S_{k-1}})^2 = N * (\sqrt{T/N})^2 = T$$

Thus, by construction, the quadratic variation of the amended coin toss $\widehat{Z_\imath}$, is simply the total duration of all tosses T. Importantly, note that both the Markov and Martingale properties are retained by $\widehat{Z_\imath}$ As N→∞ the random walk coin toss does not diverge. If the value of the asset at time t, with t∈ [0, T], is given by S(t), then its conditional expectation at the end of the interval, given that S(0)=0, is E(S(T))=0 with a variance of $E(S(T)^2) = T$

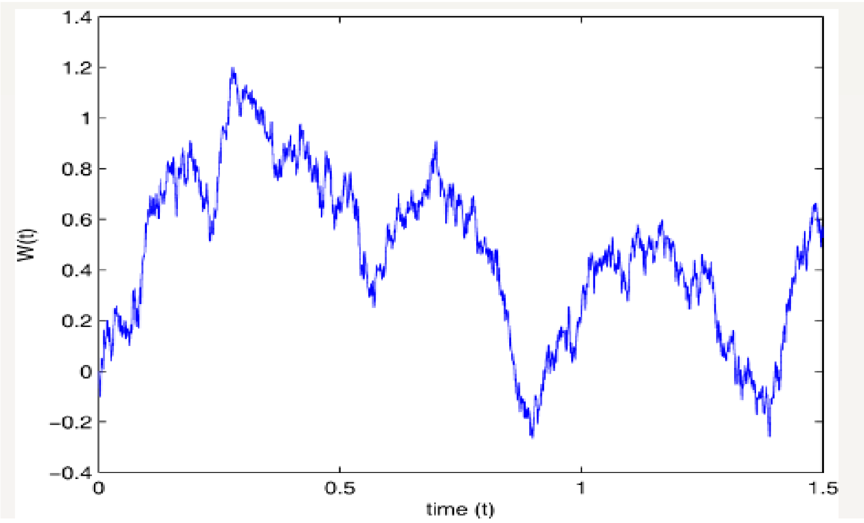
A sequence of random variables B(t) is a Brownian motion if B(0)=0, and for all t,s such that s<t, B(t)−B(s) is normally distributed with variance t−s and the distribution of B(t)−B(s) is independent of B(r) for r≤s.

PROPERTIES OF BROWNIAN MOTION:

- Brownian motions are finite. The construction of $\widehat{Z_\imath}$ was chosen carefully in order that in the limit of large N, B was both finite and non-zero.
- Brownian motions have unbounded variation. This means that if the sign of all negative gradients were switched to positive, then B would hit infinity in an arbitrarily short time period.
- Brownian motions are continuous. Although Brownian motions are continuous everywhere they are differentiable nowhere. Essentially this means that a

Brownian motion has fractal geometry. This has important implications regarding the choice of calculus methods used when Brownian motions are to be manipulated.

- Brownian motions satisfy both the Markov and Martingale properties. The conditional distribution of B(t) given information until s<t is dependent only on B(s) and, given information until s<t, the conditional expectation of B(t) is B(s).
- Brownian motions are strongly normally distributed. This means that, for s<t, s,t∈[0,T], that B(t)−B(s) is normally distributed with mean zero and variance t−s.



CHAPTER 9

SOLVING A LINEAR SDE

Let's consider a simple linear stochastic differential equation (SDE) in one dimension:

$dX_t = aX_t dt + bX_t dW_t$--(9.1)

- X_t is the stochastic process.
- a and b are constants.
- W_t is a Wiener process (Brownian motion).

This equation represents a linear SDE with both drift (deterministic) and diffusion (stochastic) terms.

To solve this SDE, we can use the method of integrating factors. We start by dividing both sides by X_t to get the SDE in a more standard form:

$\frac{dX_t}{X_t}$=adt+bdW_t---(9.1)

Now, we integrate both sides. The integral of $\frac{1}{X_t}$ with respect to X_t is the natural logarithm ln$|X_t|$. The integral of the right-hand side involves the cumulative sum of the deterministic term and the stochastic integral. Integrating the stochastic term involves Ito's integral.

Applying the integration, we get:

$\int\frac{1}{X_t}$ d X_t=∫a dt+∫b dW_t--(9.2)

ln$|X_t|$=at + bW_t + c [c is the constant term of integration] {from 9.2}

Now, we exponentiate both sides to eliminate the logarithm:

$|Xt|=e^{at+bWt+C}$---(9.3)

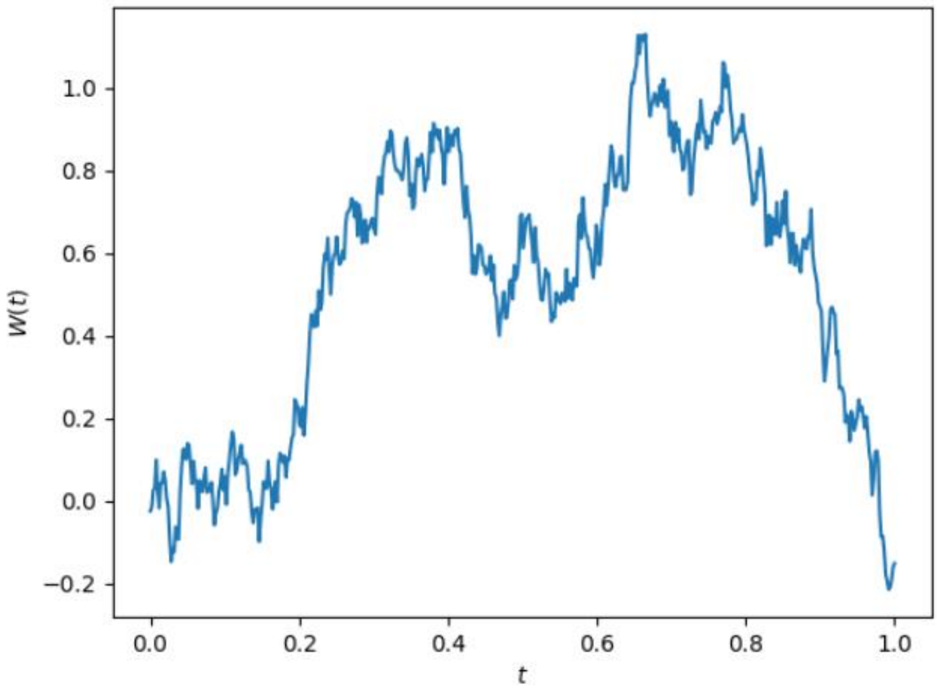
Where $A=\pm e^C$ is another constant of integration.

So, the solution to the linear SDE $dX_t=aX_tdt+bX_tdW_t$ is given by $X_t=Ae^{at+bW_t}$, where A is a constant. This solution represents a stochastic process where the deterministic term aX_t represents exponential growth or decay over time, while the stochastic term $bX_t\ dW_t$ introduces randomness into the process.

CHAPTER 10

PYTHON CODING OF THAT LINEAR SDE

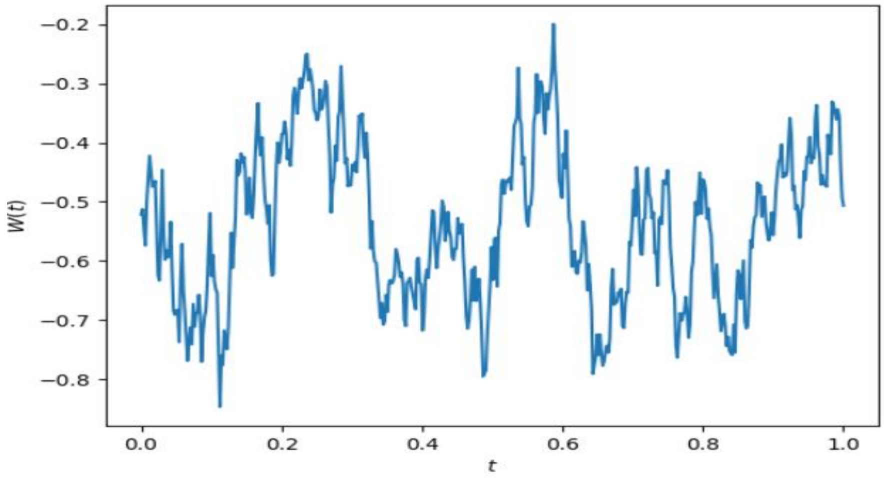
```
import numpy as np
import matplotlib.pyplot as plt
T = 1.0 # End time
N = 500 # Number of steps
dt = T / N
t = np.linspace(0, T, N)

W = np.zeros(N)
dW = np.zeros(N)

dW[0] = np.random.randn()
W[0] = dW[0]

# This defines the Wiener process, or Brownian motion
for j in range(1, N):
    dW[j] = np.sqrt(dt) * np.random.randn()
    W[j] = W[j-1] + dW[j]

plt.plot(t, W)
plt.xlabel(r'$t$')
plt.ylabel(r'$W(t)$')
plt.show()
```



I WILL VECTORIZE IT NOW

```
T = 1.0 # End time
N = 500 # Number of steps
dt = T / N
t = np.linspace(0, T, N)
dW = np.sqrt(dt) * np.random.randn(N) # All increments are independent, so
initialize the lot in one go
W = np.cumsum(dW) # The Brownian motion is the cumulative sum of all
increments
plt.plot(t, W)
plt.xlabel(r'$t$')
plt.ylabel(r'$W(t)$')
plt.show()
```

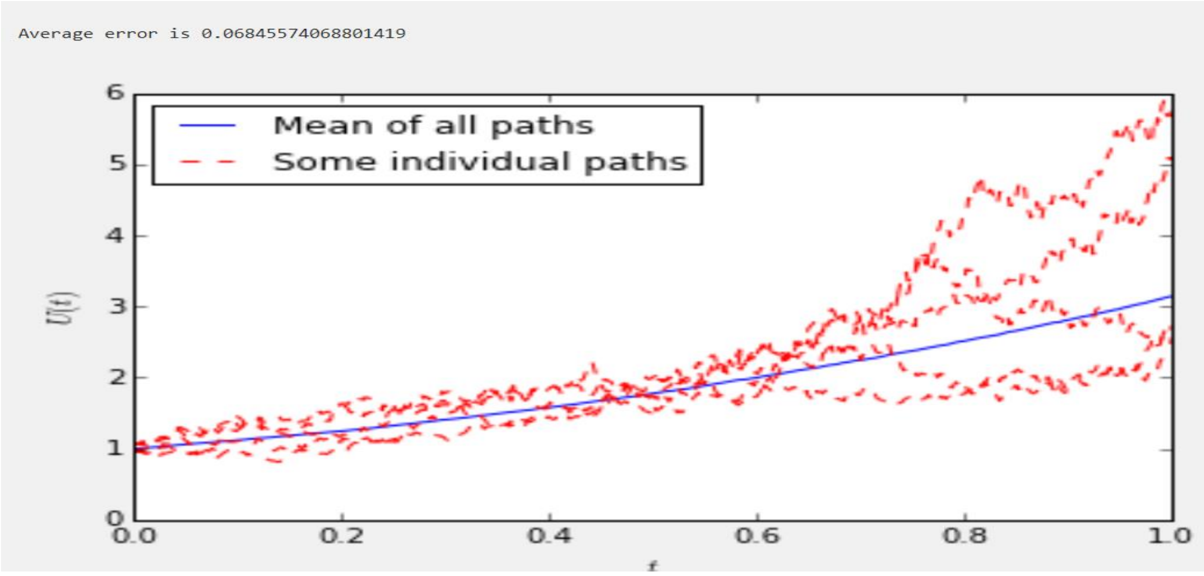
GRAPH



A single path by itself is not very useful. Instead we should look at an ensemble of paths, and of functions of paths.

```
T = 1.0 # End time
N = 500 # Number of steps
dt = T / N
t = np.linspace(0, T, N)

M = 1000 # Will construct M paths simultaneously

dW = np.sqrt(dt) * np.random.randn(M, N) # Now construct the increments for
all points and paths
W = np.cumsum(dW, 1) # Take the cumulative sum for each path separately

U = np.exp(t + 0.5*W) # Construct a function of the Brownian path
Umean = np.mean(U, 0) # Take the mean across all paths

# Compare against the expected average solution
print("Average error is {}".format(np.linalg.norm((Umean - np.exp(9*t/8)),
np.inf)))

plt.plot(t, Umean, 'b-')
for i in range(5):
    plt.plot(t, U[i, :], 'r--')
plt.xlabel(r'$t$')
plt.ylabel(r'$U(t)$')
plt.legend(('Mean of all paths', 'Some individual paths'), loc = 'upper
left')

plt.show()
```

GRAPH



CHAPTER 11

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