Quantities with Errors

15 points

Let $x, y \in \mathbb{R}^n$ be vectors of real numbers. Consider the problem of computing the dot product

$$\mathbf{x}^T\mathbf{y} = \sum_{i=1}^{n} x_i y_i.$$

On an actual machine, the above computation will be subject to rounding errors that accumulate over the course of computation.

Part 1

Let us model the effect of rounding a number by saying that the rounded number $\tilde{x}$ should satisfy

$$\tilde{x} = x(1+\delta)$$

where δ is a small quantity. Assuming that δ does not depend on the number x, what (in exact arithmetic) does the expression $\sum_{i=1}^{n} \tilde{x}_i \tilde{y}_i$ evaluate to?

Part 2

Arithmetic operations are also subject to rounding on a computer. Let $\oplus$, $\otimes$ denote the operation of rounded addition and rounded multiplication as done by a computer. Let us model these operations by saying that

$$x \oplus y = (x+y)(1+\delta)$$

$$x \otimes y = xy(1+\delta)$$

where δ is another small quantity.
Suppose the dot product is evaluated by means of the expression:

$$(x_1 \otimes y_1) \oplus (x_2 \otimes y_2) \oplus \cdots \oplus (x_n \otimes y_n).$$

The expression is evaluated from left to right.
How many times do $\otimes$ and $\oplus$ appear in the expression? Assuming that δ does not depend on the inputs, what does this expression evaluate to? (You may ignore the effect of input rounding as described in Part 1.)

Part 3

Suppose you had a function $\mathrm{fma}(a, x, b)$ which satisfies

$$\mathrm{fma}(a, x, b) = (ax+b)(1+\delta)$$

where δ is a small quantity. (See Fused multiply-add (https://en.wikipedia.org/wiki/Multiply%E2%80%93accumulate_operation) for some context.)
Then the dot product can be evaluated as

$$\mathrm{fma}(x_n, y_n, \mathrm{fma}(x_{n-1}, y_{n-1}, \cdots\ \mathrm{fma}(x_2, y_2, \mathrm{fma}(x_1, y_1, 0)))).$$

How many times is fma applied in the expression? Assuming that δ does not depend on the inputs, what does this expression evaluate to? (You may once again ignore the effect of input rounding as described in Part 1.)

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