CS450 HW2.

part 1: $Ax = b = \hat{A}\hat{x}$

So $(A+\delta A)(x+\delta x) = b$

$$\cancel{Ax}^{\,b} + \delta A x + A\delta x + \cancel{\delta A \delta x}^{\,0} = b$$

So $\Leftrightarrow \delta A x + A \delta x = 0$

$$\delta A x = -A\delta x$$

$\Rightarrow \quad \|\delta A x\|_1 = \|-A\delta x\|_1$

Since $\max_{ij} \left|\frac{\delta a_{ij}}{a_{ij}}\right| \le \varepsilon$

$\Rightarrow \max_{ij} |\delta a_{ij}| \le \varepsilon |a_{ij}|$.

For 1-norm of A: $\max_{col=j} \sum_{row\,i} |a_{ij}| = \|A\|_1$

So $\max_{col=j} \sum_{row\,i} |\delta a_{ij}| \le \max_{col=j} \sum_{row\,i} |a_{ij}| \times \varepsilon$

$\Leftrightarrow \quad \|\delta A\|_1 \le \varepsilon \|A\|$

$\Rightarrow$ $\|\delta A\|_1 = \frac{\|-A\delta x\|_1}{\|x\|_1} \le \varepsilon \|A\|$

$\Leftrightarrow \frac{\|\delta x\|_1}{\|x\|_1} \le \varepsilon \|A\| \|A^{-1}\|$

$\Leftrightarrow \frac{\|\delta x\|_1}{\|x\|_1} \le \varepsilon \| |A| |A^{-1}| \|_1$

Part 2:

First we prove: $|DA| = |D||A|$

$\Rightarrow$ $D = \begin{vmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_n \end{vmatrix}$ $A = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$

$DA = \begin{vmatrix} \lambda_1 a_{11} & \lambda_1 a_{12} & \cdots & \lambda_1 a_{1n} \\ \lambda_2 a_{21} & \cdots & \cdots & \lambda_2 a_{2n} \\ \vdots & & \ddots & \vdots \\ \lambda_n a_{n1} & \cdots & \cdots & \lambda_n a_{nn} \end{vmatrix}$

So it is just $\lambda_i \times a_{ij}$. So we can see:

$|\lambda_i| \times |a_{ij}| = |\lambda_i a_{ij}|$

$\Leftrightarrow$ $|DA| = |D|\,|A|$.

Also D^{-1} is a diagonal matrix as well.

So: $K_{CR}(DA) = \|\,|(DA)^{-1}|\,|(DA)|\,\|$

$= \|\,|D^{-1}|\,|A^{-1}|\,|D|\,|A|\,\|$

$= \|\,|D^{-1}D|\,|A^{-1}|\,|A|\,\|$

$= \|\,I\,|A^{-1}|\,|A|\,\|$

$= \|\,|A^{-1}|\,|A|\,\| = K_{CR}(A)$