# Application of Stokes and Divergence theorem

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## 1 Introduction

The application of Stokes' theorem and Divergence theorem is ubiquitous in the theory of Electromagnetism. It primarily helps us to convert Maxwell's equations from integral formulations into differential forms.

Electromagnetism is the study of the electric and the magnetic interaction between charged particles. There are two types of charged particles namely, positive and negative. The electric interaction between charged particles is governed by the Coulomb's Law (1).

$$\vec{F} = \frac{1}{4\pi\varepsilon}\frac{Qq}{R^2}\hat{r} \tag{1}$$

Where:
- $\vec{F}$: is the force on charge $q$ due to the charge $Q$.
- $\varepsilon$: is called permittivity of free space.
- $R$: is the distance between the charged particles $q$ and $Q$
- $\hat{r}$: is the unit vector going from $Q$ to $q$

For a more mathematical description of electric interaction, we rather consider electric fields $\vec{E} = \frac{\vec{F}}{q}$, instead of the Coulomb Force. The electric field yields a vector field in $\mathbf{R}^3$. Similarly, Magnetic force is described by the Magnetic Field $\vec{B}$. However, unlike the electric field, the magnetic field generated by a charged particle is velocity dependent which makes it a non-conservative force.

The phenomenon of Electromagnetism is governed by four fundamental laws. They are Gauss's laws for the electric field and the magnetic field, Faraday's law of induction, and Ampere's law. They are known as Maxwell's equations. Stokes' theorem and Divergence theorem helps us transform Maxwell's equations from the integral form into differential form. There are multiple reasons for turning Maxwell's equation from the integral form into differential form. One of the main reasons for doing this is because integral formulation is suitable for only basic situations like vectors fields on spheres or cylinders. As soon as the situation becomes asymmetrical, the integral formulation is rendered useless. Moreover, the differential formulation helps us to frame Electromagnetism in larger physics of Gauge theory.

## 2 Gauss's Law:

Gauss's law (2) states that the total flux of $\vec{E}$ in a closed surface is the net charge enclosed by the surface.

$$\oiint_S \vec{E} \cdot \hat{n} dS = \frac{Q}{\varepsilon} \tag{2}$$

Now, we'll use the Divergence theorem to change Gauss's law from the integral form to the differential form.

$$\oiint_S \vec{E} \cdot \hat{n} dS = \iiint_V \nabla \cdot \vec{E} dV = \frac{Q}{\varepsilon} \tag{3}$$

Let $\rho$ be the charge density such that $\rho = \frac{dq}{dV}$, then the total charge enclosed in the volume is

$$Q = \iiint_V \rho dV \tag{4}$$

Combining equation (4) and equation (3) we get

$$\iiint_V \nabla \cdot \vec{E} dV = \iiint_V \frac{\rho}{\varepsilon} dV \tag{5}$$

Therefore, the differential form for Gauss's law is

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon} \tag{6}$$

## 3 Gauss's Law of Magnetism:

Gauss's law of magnetism is the observation that magnets come in two polarities, such that the magnetic field from one pole ends into another. Consequently, there are as many magnetic fields coming in as leaving out of the closed surface, therefore the net magnetic flux denoted by $\Phi_B$ is zero. We can describe this in integral equation as:

$$\Phi_B = \oiint_S \vec{B} \cdot dS = 0 \tag{7}$$

Similarly, we can use Divergence theorem to transform it into differential form.

$$\oiint_S \vec{B} \cdot dS = \iiint \nabla \cdot \vec{B} dV = 0 \tag{8}$$

Therefore, the differential form for Gauss's law is

$$\nabla \cdot \vec{B} = 0 \tag{9}$$

## 4 Faraday's law of induction:

If we get a copper coil and move a magnet back and forth inside the coil, then the magnet will generate a voltage or emf (electromotive force). In other words, the change in flux over time ($\frac{d\Phi_B}{dt}$) induces a voltage or electromotive force and it can be represented using line integral along the closed loop. Then Faraday's law of induction can be stated as:

$$\oint_C \vec{E} \cdot dl = -\frac{d\Phi_B}{dt} \tag{10}$$

Using Stoke's theorem we can restate the equation into differential form

$$\oint_C \vec{E} \cdot dl = \iint_S \nabla \times \vec{E} \cdot dS = -\frac{d\Phi_B}{dt} \tag{11}$$

Since,

$$\Phi_B = \iint_S \vec{B} \cdot dS \tag{12}$$

then,

$$\frac{d\Phi_B}{dt} = \iint_S \frac{d\vec{B}}{dt} \cdot dS \tag{13}$$

Thus, combining equation (11) and (13), we get

$$\iint_S \nabla \times \vec{E} \cdot dS = -\iint_S \frac{d\vec{B}}{dt} \cdot dS \tag{14}$$

Henceforth, we get the differential form for Faraday's law of induction:

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt} \tag{15}$$

## 5 Ampere's Law:

Consider a wire with flowing currents I, the the current will generate a magnetic field around the wire. This observation is known as Ampere's law:

$$\oint_C \vec{B} \cdot dl = \mu I \tag{16}$$

where $\mu$ vacuum permeability constant. Let $\vec{J}$ be the current density such that

$$I = \iint_S \vec{J} \cdot dS \tag{17}$$

Similarly, we can use Stokes' theorem and we get:

$$\oint_C \vec{B} \cdot dl = \iint_S \nabla \times \vec{B} \cdot ds = \mu I \tag{18}$$

now, combining equation (17) and (18), we get

$$\iint_S \nabla \times \vec{B} \cdot ds = \iint_S \mu \vec{J} \cdot dS \tag{19}$$

Therefore, we get the Ampere's law in differential form.

$$\nabla \times \vec{B} = \mu \vec{J} \tag{20}$$

However, this equation is inconsistent with the rest of the Maxwell's equations because

$$\nabla \cdot (\nabla \times \vec{B}) = \mu(\nabla \cdot \vec{J}) \tag{21}$$

which is not necessarily equal to zero. To make this equation consistent, we must add an extra term, Then, we get the our fourth Maxwell's equation

$$\nabla \times \vec{B} = \mu \vec{J} + \mu\varepsilon \frac{d\vec{E}}{dt} \tag{22}$$

and the term $\varepsilon \frac{d\vec{E}}{dt}$ is known as displacement current.

# 6 Conclusion

We now have the four Maxwell's Equations in differential form using Divergence and Stokes theorem

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{\rho}{\varepsilon} \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} &= -\frac{d\vec{B}}{dt} \\ \nabla \times \vec{B} &= \mu \vec{J} + \mu\varepsilon \frac{d\vec{E}}{dt} \end{aligned} \tag{23}$$