AGEC 652 - Lecture 5.2

Constrained optimization

Part B: Using Julia

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Course roadmap

1. Intro to Scientific Computing
2. Numerical operations and representations
3. Systems of equations
4. Function approximation (Skipped)
5. **Optimization**
    - 5.1 Unconstrained optimization
    - **5.2 Constrained optimization**
    - A) **Theory and solution algorithms** ← *You are here*
    - B) Constrained optimization in Julia
6. Structural estimation

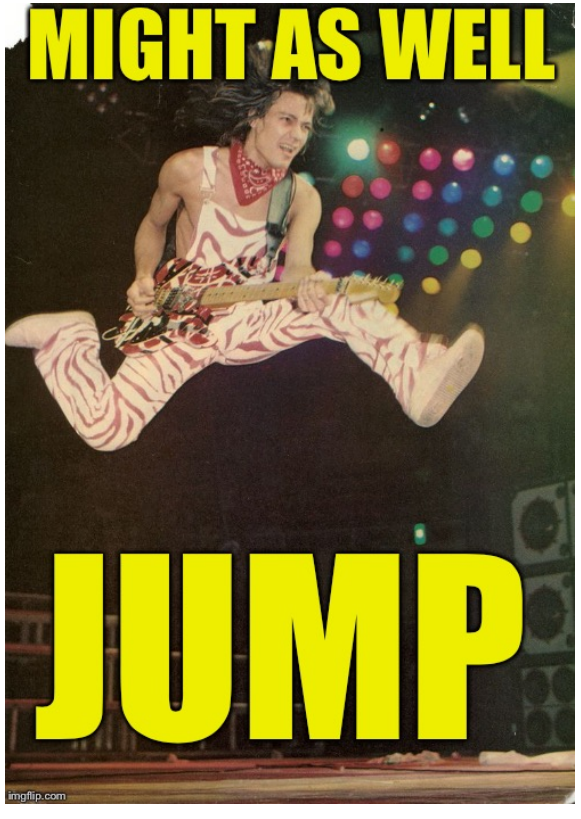
*These slides are based on Miranda & Fackler (2002), Nocedal & Wright (2006), Judd (1998), and course materials by Ivan Rudik and Florian Oswald.

Constrained optimization in Julia

Constrained optimization in Julia

We are going to cover a cool package called `JuMP.jl`

- It offers a whole modeling language inside Julia
- You define your model and plug it into one of the many solvers available
- It's like GAMS and AMPL... *but FREE and with a full-fledged programming language around it*



Constrained optimization in Julia

Most solvers can be accessed directly in their own packages

- Like we did to use `Optim.jl`
- These packages are usually just a Julia interface for a solver programmed in another language

But `JuMP` gives us a unified way of specifying our models and switching between solvers

`JuMP` specifically designed for constrained optimization but works with unconstrained too

- With more overhead relative to using `Optim` or `NLopt` directly

Getting stated with JuMP

There are 5 key steps:

1) Initialize your model and solver:

- `mymodel = Model(SomeOptimizer)`

2) Declare variables (adding any box constraints)

- `@variable(mymodel, x >= 0)`

3) Declare the objective function

- If linear: `@objective(mymodel, Min, 12x + 20y)`
- If nonlinear: `@NLobjective(mymodel, Min, 12x^0.7 + 20y^2)`

Getting stated with JuMP

4) Declare constraints

- If linear: `@constraint(mymodel, c1, 6x + 8y >= 100)`
- If nonlinear: `@NLconstraint(mymodel, c1, 6x^2 - 2y >= 100)`

5) Solve it

- `optimize!(mymodel)`
- Note the `!`, so we are modifying `mymodel` and saving results in this object

Follow along!

Let's use `JuMP` to solve the illustrative problem from the first slides

We will use solver `Ipopt`, which stands for *Interior Point Optimizer*. It's a free solver we can access through package `Ipopt.jl`

```
using JuMP, Ipopt;
```

Follow along: function definition

Define the function:

$$\min_x -exp\left(-(x_1x_2 - 1.5)^2 - (x_2 - 1.5)^2\right)$$

```
f(x1,x2) = -exp.(-(x1.*x2 - 3/2).^2 - (x2-3/2).^2);
```

Follow along: initialize model

Initialize the model for `Ipopt`

```
model = Model(Ipopt.Optimizer)
```

```
## A JuMP Model
## Feasibility problem with:
## Variables: 0
## Model mode: AUTOMATIC
## CachingOptimizer state: EMPTY_OPTIMIZER
## Solver name: Ipopt
```

You can set optimzer parameters like this

- There are TONS of parameters you can adjust (see the manual)

```
# This is relative tol. Default is 1e-8
set_optimizer_attribute(model, "tol", 1e-6)
```

Follow along: declare variables

We will focus on non-negative values

```
@variable(model, x1 >=0)
```

```
## x1
```

```
@variable(model, x2 >=0)
```

```
## x2
```

- You could type `@variable(model, x1)` to declare a x_1 as a free variable

Follow along: declare objective

We will focus on non-negative values

```
@NLobjective(model, Min, f(x1, x2))
```

`JuMP` will use autodiff (with `ForwardDiff` package) by default. If you want to use your define gradient and Hessian, you need to "register" the function like this

```
register(model, :my_f, n, f, grad, hessian)
```

- `:my_f` is the name you want to use inside `model`, `n` is the number of variables `f` takes, and `grad hessian` are user-defined functions

Follow along: solving the model

First, let's solve the (mostly) unconstrained problem

- Not really unconstrained because we defined non-negative `x1` and `x2`

Checking our model

```
print(model)
```

```
## Min f(x1, x2)
## Subject to
##  x1 >= 0.0
##  x2 >= 0.0
```

Follow along: solving the model

```
optimize!(model)
```

```
##
## ******************************************************************************
## This program contains Ipopt, a library for large-scale nonlinear optimization.
##  Ipopt is released as open source code under the Eclipse Public License (EPL).
##          For more information visit https://github.com/coin-or/Ipopt
## ******************************************************************************
##
## This is Ipopt version 3.14.4, running with linear solver MUMPS 5.5.1.
##
## Number of nonzeros in equality constraint Jacobian...:        0
## Number of nonzeros in inequality constraint Jacobian.:        0
## Number of nonzeros in Lagrangian Hessian.............:        0
##
## Total number of variables............................:        2
##                     variables with only lower bounds:        2
##                variables with lower and upper bounds:        0
##                     variables with only upper bounds:        0
## Total number of equality constraints.................:        0
## Total number of inequality constraints...............:        0
##         inequality constraints with only lower bounds:        0
##    inequality constraints with lower and upper bounds:        0
```

Follow along: solving the model

The return message is rather long and contains many details about the execution. You can turn this message off with

```
set_silent(model);
```

We can check minimizers with

```
unc_x1 = value(x1)
```

```
## 1.0000000326687912
```

```
unc_x2 = value(x2)
```

```
## 1.4999999423446968
```

```
unc_obj = objective_value(model)
```

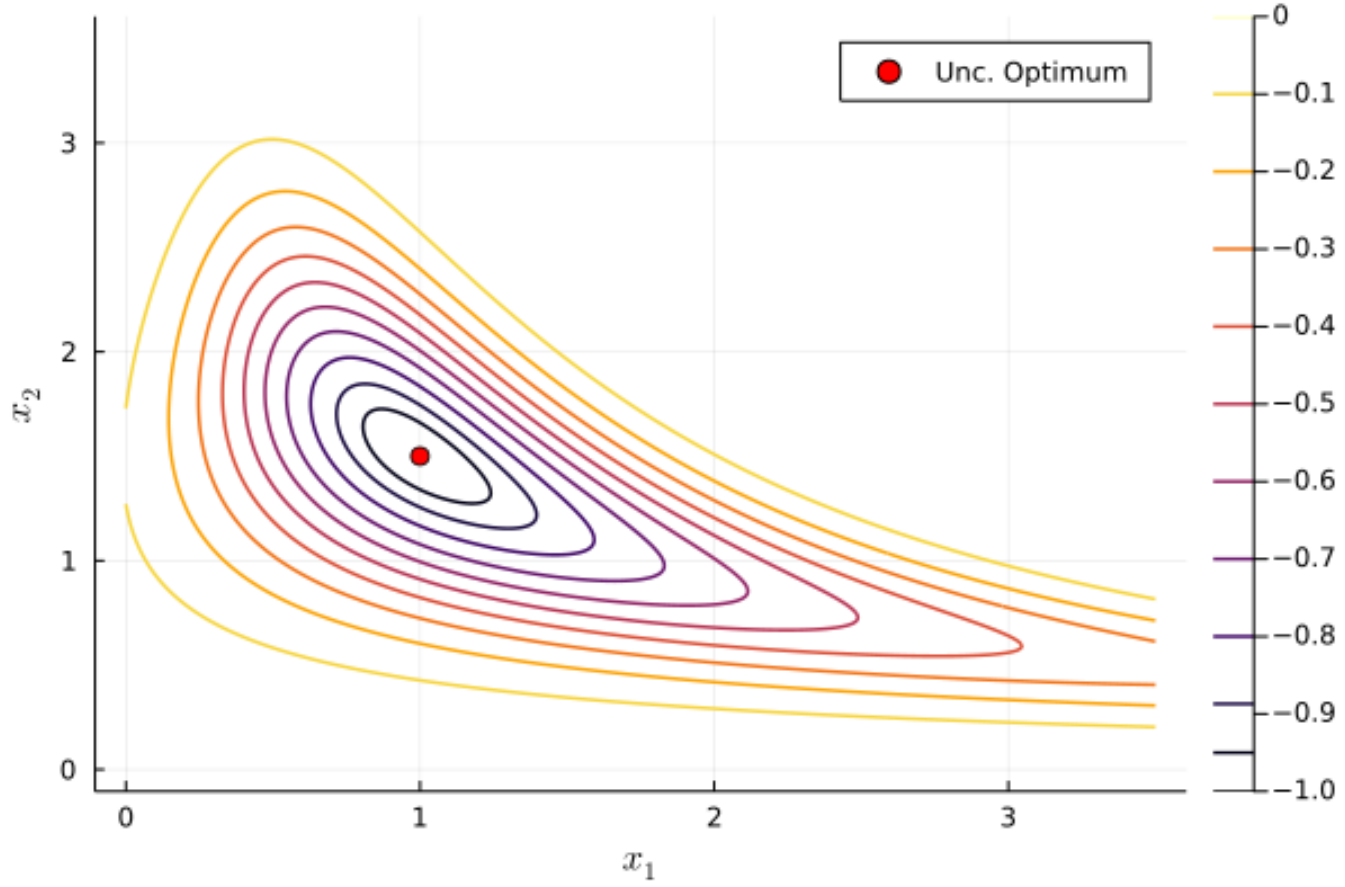
Follow along: solving the model

And the minimum with

```
unc_obj = objective_value(model)
```

```
## -0.9999999999999966
```

Follow along: solving the model



Follow along: declaring constraints

Let's add a nonlinear equality constraint $-x_1 + x^2 = 0$ and re-solve the model

```
@NLconstraint(model, -x1 +x2^2 == 0)
```

```
## (-x1 + x2 ^ 2.0) - 0.0 == 0
```

```
print(model)
```

```
## Min f(x1, x2)
## Subject to
##  x1 >= 0.0
##  x2 >= 0.0
##  (-x1 + x2 ^ 2.0) - 0.0 == 0
```

```
optimize!(model)
```

Follow along: solving the equality constrained model

```
eqcon_x1 = value(x1)
```

```
## 1.3578043097074932
```

```
eqcon_x2 = value(x2)
```

```
## 1.1652486042512478
```

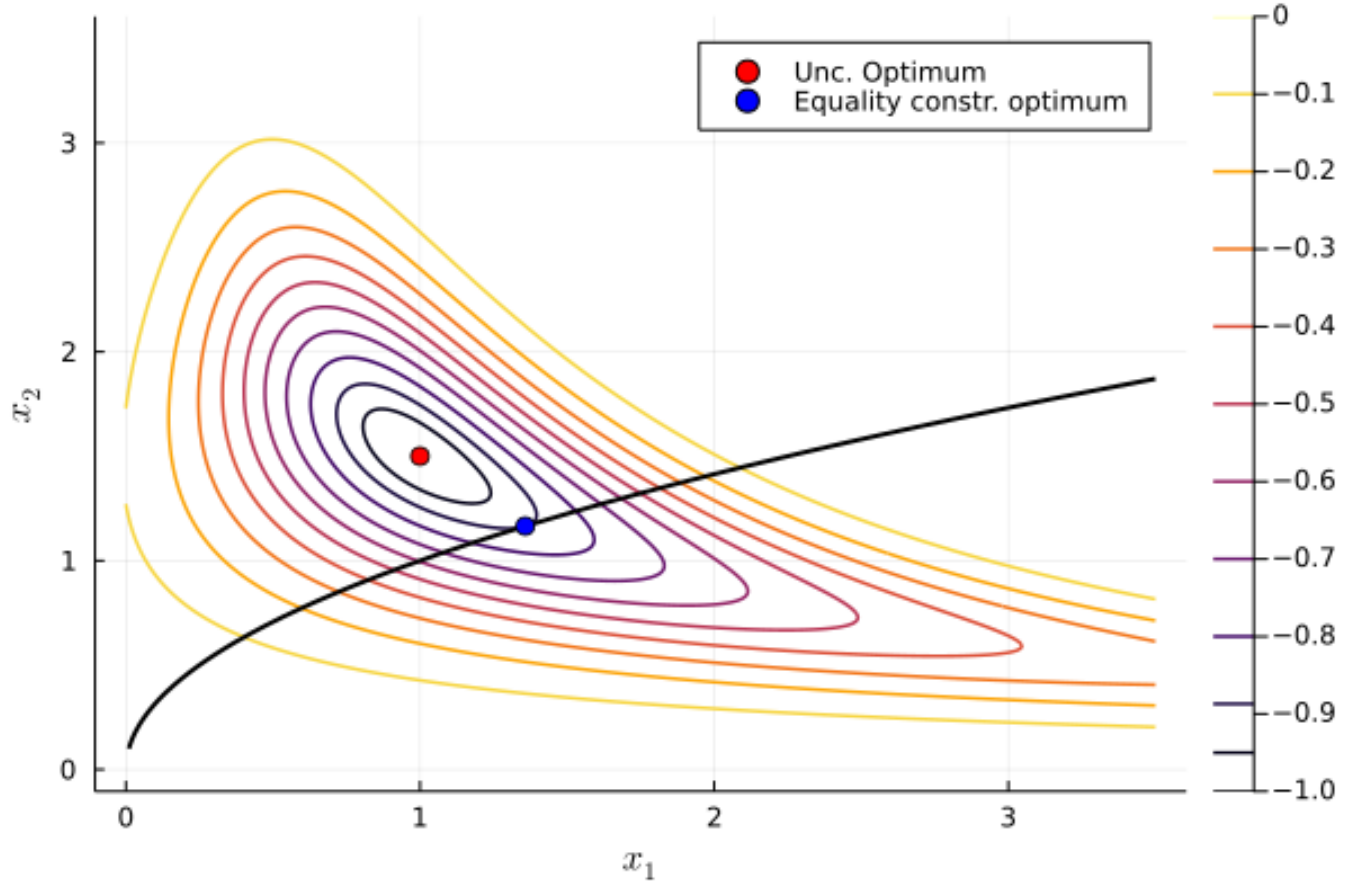
```
value(-x1 + x2^2) # We can evaluate expressions too
```

```
## 1.9877433032888803e-12
```

```
eqcon_obj = objective_value(model)
```

```
## -0.887974742266783
```

Follow along: solving the equality constrained model



Solving the inequality constrained model

I now initialize a new model with inequality constraint $-x_1 + x^2 \leq 0$

```
model2 = Model(Ipopt.Optimizer);
@variable(model2, x1 >=0);
@variable(model2, x2 >=0);
@NLobjective(model2, Min, f(x1, x2));
@NLconstraint(model2, -x1 + x2^2 <= 0);
optimize!(model2);
```

Solving the inequality constrained model

```
ineqcon_x1 = value(x1)
```

```
## 1.357804311747407
```

```
ineqcon_x2 = value(x2)
```

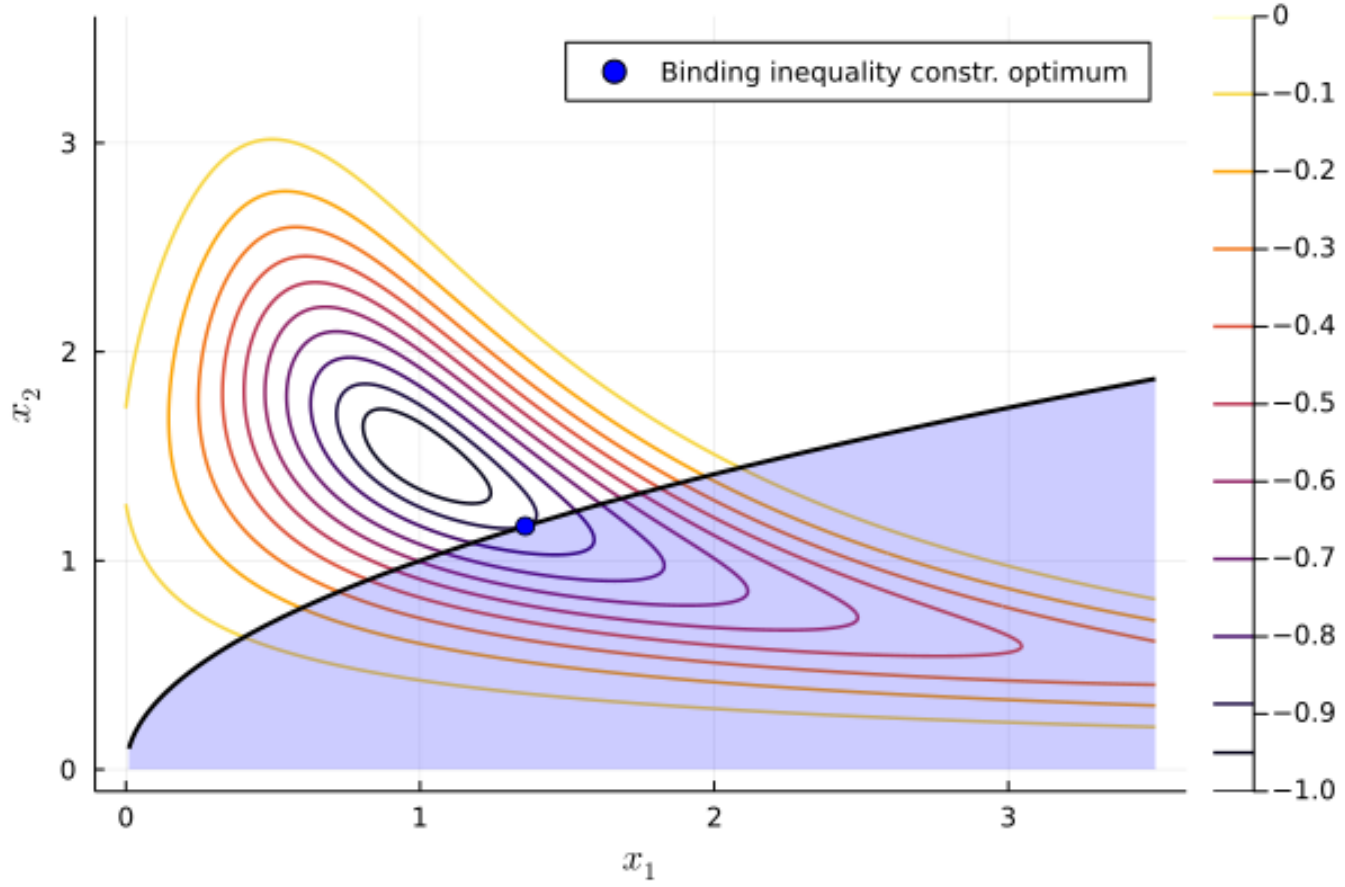
```
## 1.165248609391406
```

```
ineqcon_obj = objective_value(model2)
```

```
## -0.887974743957088
```

Same results as in the equality constraint: the constraint is binding

Solving the inequality constrained model



Relaxing the inequality constraint

What if instead we use inequality constraint $-x_1 + x^2 \leq 1.5$?

```
model3 = Model(Ipopt.Optimizer);
@variable(model3, x1 >=0);
@variable(model3, x2 >=0);
@NLobjective(model3, Min, f(x1, x2));
@NLconstraint(model3, c1, -x1 + x2^2 <= 1.5);
optimize!(model3);
```

Relaxing the inequality constraint

```
ineqcon2_obj = objective_value(model3)
```

```
## -1.0
```

```
ineqcon2_x1 = value(x1)
```

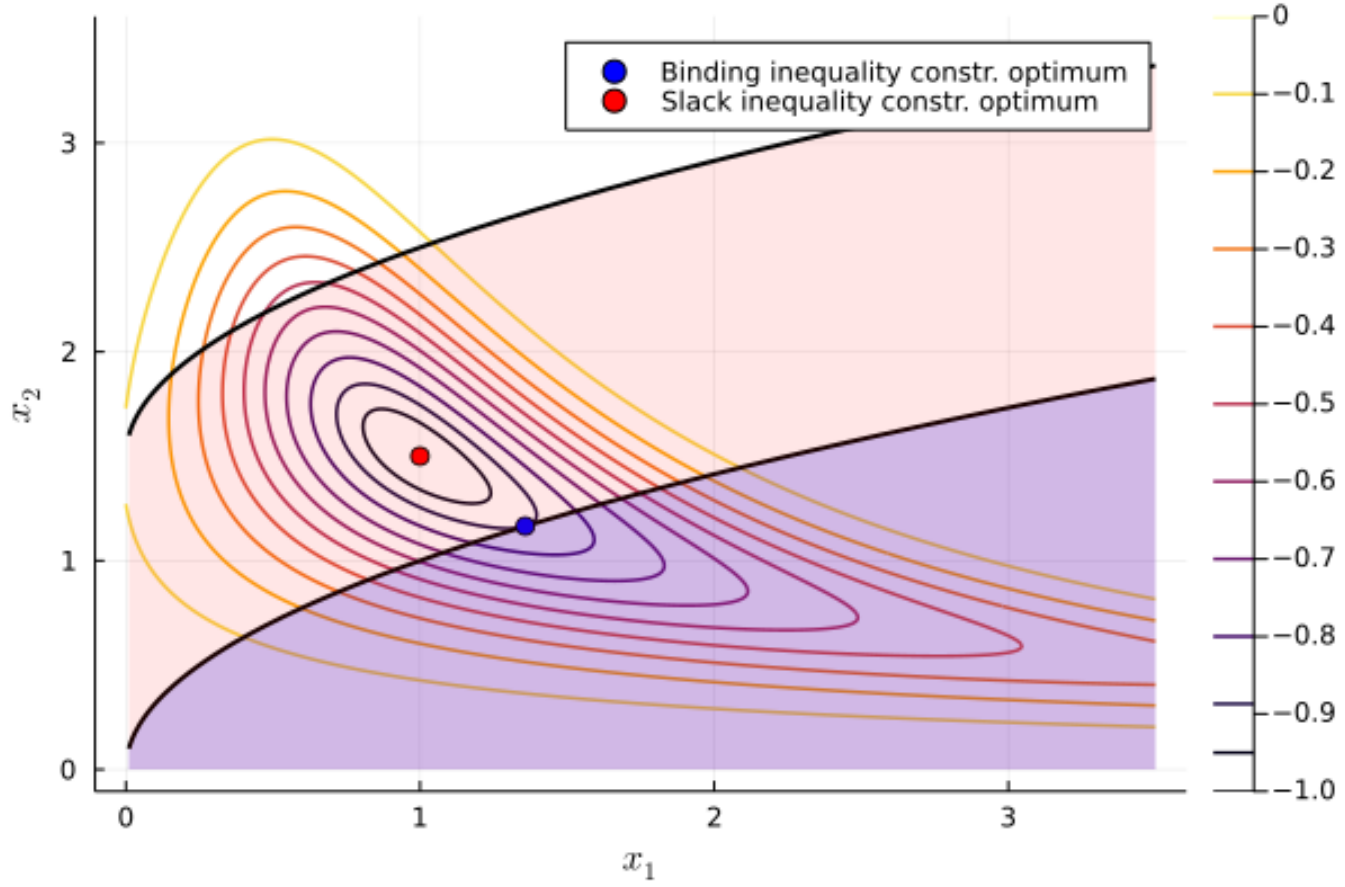
```
## 1.0000000035503052
```

```
ineqcon2_x2 = value(x2)
```

```
## 1.4999999931258383
```

We get the same results as in the unconstrained case

Solving the inequality constrained model



Practical advice for numerical optimization

Best practices for optimization

Plug in your guess, let the solver go, and you're done right?

WRONG!

These algorithms are not guaranteed to always find even a local solution, you need to test and make sure you are converging correctly

Check return codes

Return codes (or exit flags) tell you why the solver stopped

- There are all sorts of reasons why a solver ends execution
- Each solver has its own way of reporting errors
- In `JuMP` you can use `@show termination_status(mymodel)`

READ THE SOLVER DOCUMENTATION!

Use trace options to get a sense of what went wrong

- Did guesses grow unexpectedly?
- Did a gradient-based operation fail? (E.g., division by zero)

Check return codes

Examples from Ipopt.jl documentation

- `Solve_Succeeded`:

  Console Message: `EXIT: Optimal Solution Found.`

  This message indicates that Ipopt found a (locally) optimal point within the desired tolerances.

- `Solved_To_Acceptable_Level`:

  Console Message: `EXIT: Solved To Acceptable Level.`

  This indicates that the algorithm did not converge to the "desired" tolerances, but that it was able to obtain a point satisfying the "acceptable" tolerance level as specified by the **acceptable_tol** options. This may happen if the desired tolerances are too small for the current problem.

- `Feasible_Point_Found`:

  Console Message: `EXIT: Feasible point for square problem found.`

  This message is printed if the problem is "square" (i.e., it has as many equality constraints as free variables) and Ipopt found a point that is feasible w.r.t. constr_viol_tol. It may, however, not be feasible w.r.t. tol.

- `Infeasible_Problem_Detected`:

  Console Message: `EXIT: Converged to a point of local infeasibility. Problem may be infeasible.`

  The restoration phase converged to a point that is a minimizer for the constraint violation (in the ℓ_1-norm), but is not feasible for the original problem. This indicates that the problem may be infeasible (or at least that the algorithm is stuck at a locally infeasible point). The returned point (the minimizer of the constraint violation) might help you to find which constraint is causing the problem. If you believe that the NLP is feasible, it might help to start the optimization from a different point.

- `Search_Direction_Becomes_Too_Small`:

Try alternative algorithms

Optimization is approximately 53% art

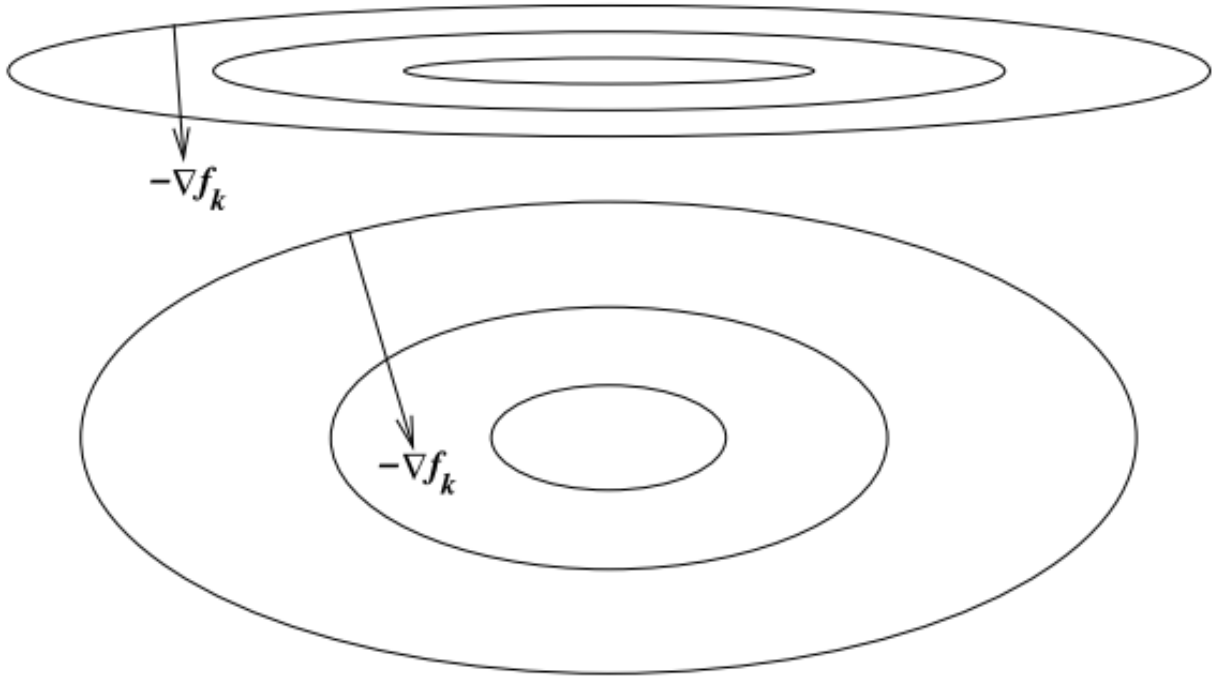
Not all algorithms are suited for every problem $\rightarrow$ it is useful to check how different algorithms perform

Interior-point is usually the default in constrained optimization solvers (low memory usage, fast), but try other algorithms and see if the solution generally remains the same

Problem scaling

The **scaling** of a problem matters for optimization performance

A problem is **poorly scaled** if changes to x in a certain direction produce much bigger changes in f than changes to in x in another direction



Problem scaling

Ex: $f(x) = 10^9 x_1^2 + x_2^2$ is poorly scaled

This happens when things change at different rates:

- Investment rates are between 0 and 1
- Consumption can be in trillions of dollars

How do we solve this issue?

Rescale the problem: put them in units that are generally within an order of magnitude of 1

- Investment rate in percentage terms: $0\% - 100\%$
- Consumption in units of trillion dollars instead of dollars

Be aware of tolerances

Two main tolerances in optimization:

1. `ftol` is the tolerance for the change in the function value (absolute and relative)
2. `xtol` is the tolerance for the change in the input values (absolute and relative)

What is a suitable tolerance?

Be aware of tolerances

It depends

Explore sensitivity to tolerance, typically pick a conservative (small) number

- Defaults in solvers are usually `1e-6`

If you are using simulation-based estimators or estimators that depend on successive optimizations, be even more conservative *because errors compound*

Be aware of tolerances

May be a substantial trade-off between accuracy of your solution and speed

Common bad practice is to pick a larger tolerance (e.g. `1e-3`) so the problem "works" (e.g. so your big MLE converges)

Issue is that `1e-3` might be pretty big for your problem if you haven't checked that your solution is not sensitive to the tolerance

Perturb your initial guesses

Initial guesses matter

Good ones can improve performance

- E.g. initial guess for next iteration of coefficient estimates should be current iteration estimates

Bad ones can give you terrible performance, or wrong answers if your problem isn't perfect

- E.g. bad scaling, not well-conditioned, multiple equilibria