Structural estimation of a multi-agent model

Discrete choice within Nash-Bertrand competition

Agenda

We will return to the discrete choice model between car models A and B (from weekly problems) only this time without knowing the model parameters!

- You will estimate demand and supply models jointly using GMM
- Using estimated parameters, you will predict marginal costs
- Then, using marginal costs, you will quantify market power in this industry

Research problem

Question: **How much market power do car brands A and B have?**

We have data from a random sample from $N = 300$ local markets (indexed by j) for car models A and B

- Shares: s_k
- Prices: p_k
- Local steel prices: t_k
- Average local labor costs: l_i

We want to estimate the mean Lerner index for each firm

$$L_k = \frac{1}{N} \sum_{j=1}^{N} \frac{p_{k,j} - c_{k,j}}{p_{k,j}}$$

Demand: theoretical model

In a given market, consumers have 3 options:

- Buy one car of model a with price p_a
- Buy one car of model b with price p_b
- Not to buy a car, i.e., the outside option

If consumer i buys car k, they get utility

$$u_{i,k} = \beta_0 - \alpha p_k + \beta_1 B_k^D + \beta_2 \xi_k^D + \nu_{ik} \equiv V_k + \nu_{ik}$$

where

- α and βs are model parameters
- B_k^D is a vector of **observed** product characteristics that consumers care about
- ξ_k^D is a vector of **unobserved** product characteristics that consumers care about
- ν_{ik} is the (mean zero) idiosyncratic taste of consumer i for product k
- V_k is the mean utility for product k

If they do not buy a car, they get zero utility. Consumers are utility maximizers, so they pick the option that yields the highest utility.

Demand: theoretical model

Assuming ν_{ik} is distributed in the population following a Type I Extreme Value distribution, we can derive a closed-form expression for expected market shares of products a and b:

$$s_a = \frac{e^{V_a}}{1 + e^{V_a} + e^{V_b}}, \; s_b = \frac{e^{V_b}}{1 + e^{V_a} + e^{V_b}}$$

The demand for the outside option is given by $s_0 = 1 - s_A - s_B = \frac{1}{1+e^{V_a}+e^{V_b}}$

Useful trick. Note that $\frac{s_k}{s_0} = e^{V_k}$. Take logs on both sides:

$$\log(s_k) - \log(s_0) = V_k$$

Demand: statistical model

$$\log(s_k) - \log(s_0) = V_k$$

The expression above refers to expected shares and does not fit the data exactly. For this reason, we add a idiosyncratic demand shock $\epsilon_{k,j}^D$ to obtain

$$\log(s_{k,j}) - \log(s_{0,j}) = V_{k,j} + \epsilon_{k,j} = \beta_0 - \alpha p_{k,j} + \beta_1 B_{k,j}^D + \beta_2 \xi_{k,j}^D + \epsilon_{k,j}^D \quad (1)$$

We assume $\epsilon_{k,j}^D$ has mean zero and is uncorrelated with $p_{k,j}$, $B_{k,j}^D$, and $\xi_{k,j}^D$

Supply: theoretical model

Since this market is a duopoly, firms anticipate that their pricing decisions affect their demands mutually

- Firms then choose the price that maximizes their expected profits given the price their competitor has chosen: Bertrand competition

We assume that the outcome of this process is a Nash equilibrium: firms do the best they can given what others are doing and no firm has an incentive to unilaterally deviate

- Hence, each market is in a Nash-Bertrand equilibrium

Let c_k be the constant marginal cost. Then in each market each firm solves:

$$\max_{p_k} \pi(p_k, p_{-k}) = p_k s_k(p_k, p_{-k}) - c_k s_k(p_k, p_{-k})$$

The first-order condition is

$$\partial \pi_k / \partial p_k = 0 \Rightarrow s_k(p_k, p_{-k}) + (p_k - c_k) \frac{\partial s_k(p_k, p_{-k})}{\partial p_k} = 0$$

Supply: theoretical model

Using the closed-form expression for the derivative $\frac{\partial s_k(p_k, p_{-k})}{\partial p_k} = -\alpha s_k(1 - s_k)$, it follows that

$$s_k(p_k, p_{-k}) - \alpha(p_k - c_k) s_k(p_k, p_{-k})(1 - s_k(p_k, p_{-k})) = 0$$
$$c_k = p_k - \frac{1}{\alpha(1 - s_k)}$$

Thus, given parameter α, we can calculate marginal costs that rationalize equilibrium prices and shares

Supply: statistical model

With parameter estimate $\hat{\alpha}$ and using the expression for rationalized marginal costs based on the FOC, we can write predict expected marginal costs

$$\hat{c}_{k,j} = p_{k,j} - \frac{1}{\hat{\alpha}(1 - s_{k,j})} \quad (2)$$

We parameterize marginal cost using observable input costs and add an idiosyncratic cost shock $\epsilon^S_{k,j}$ to accomodate for differences from the expectation

$$\hat{c}_{k,j} = \gamma_0 + \gamma_1 B^S_{k,j} + \gamma_2 t_{k,j} + \gamma_3 l_{k,j} + \gamma_4 \xi^S_k + \epsilon^S_{k,j} \quad (3)$$

where

- γ s are parameters to be estimated
- $t_{k,j}$ and $l_{k,j}$ are local input costs for steel and labor
- B^S_k is a vector of **observed** product characteristics that affect costs
- ξ^S_k is a vector of **unobserved** product characteristics that affect costs

We assume $\epsilon^S_{k,j}$ to be uncorrelated with$t_{k,j}$, $l_{k,j}$, $B^S_{k,j}$, and ξ^S_k

Estimation: demand moment conditions

$$\log(s_{k,j}) - \log(s_{0,j}) = \beta_0 - \alpha p_{k,j} + \beta_1 B^D_{k,j} + \beta_2 \xi^D_{k,j} + \epsilon^D_{k,j} \quad (1)$$

Do we have a problem to estimate this equation?

Very likely, YES! Unobserved product characteristics affect willingness to pay: $E[p_{k,j}\xi^D_{k,j}] \neq 0$

Since we can't include $\xi^D_{k,j}$, we have that $E[p_{k,j}(\beta_2\xi^D_{k,j} + \epsilon^D_{k,j})] \neq 0 \Rightarrow$ ENDOGENEITY!

How can we solve this?

With instruments

- We can use input costs $t_{k,j}, l_{k,j}$ to instrument for price
- Key idea: with fixed product characteristics, higher costs lead to higher prices without affecting demand

Estimation: demand moment conditions

So, we rewrite the demand estimating equation as

$$\log(s_{k,j}) - \log(s_{0,j}) = \beta_0 - \alpha p_{k,j} + \beta_1 B^D_{k,j} + \epsilon^D_{k,j} = X^D_{k,j}\theta^D + \epsilon^D_{k,j} \quad (4)$$

where

- θ^D is the parameter column vector $[\beta_0 \ \ -\alpha \ \beta_1]^T$
- X^D is the row vector $[1 \ p_{k,j} \ B^D_{k,j}]$, with $B^D_{k,j}$ being a dummy variable receiving 1 for product b ($k = b$) or 0 otherwise
  - $B^D_{k,j}$ will pick up all the perceived product characteristics that are constant across markets

Denote the instrument vector $Z_{k,j} = [1 \ B^D_{k,j} \ t_{k,j} \ l_{k,j}]$. Then, we can form moment conditions

$$E\left[Z^T_{k,j}\epsilon^D_{k,j}\right] = 0$$

Q: How many moment conditions do we have here?

Estimation: demand moment conditions

A: We have **four moment** conditions from the demand side

$$E\left[\epsilon^D_{k,j}\right] = 0$$
$$E\left[B^D_{k,j}\epsilon^D_{k,j}\right] = 0$$
$$E\left[t_{k,j}\epsilon^D_{k,j}\right] = 0$$
$$E\left[l_{k,j}\epsilon^D_{k,j}\right] = 0$$

Estimation: supply moment conditions

$$\hat{c}_{k,j} = \gamma_0 + \gamma_1 B^S_{k,j} + \gamma_2 t_{k,j} + \gamma_3 l_{k,j} + \gamma_4 \xi^S_k + \epsilon^S_{k,j} \quad (3)$$

Again, we have unobserved characteristics that affect cost. However, endogeneity is less of a concern here if we consider this sector to be a small part of the economy:

- Prices for steel and labor clear on much bigger markets, thus decisions of product characteristics are unlikely to affect their prices

For this reason, we can more reasonably assume that $E[t_{k,j}\epsilon^S_{k,j}] = 0$, $E[l_{k,j}\epsilon^S_{k,j}] = 0$, and $E[B^S_{k,j}\epsilon^S_{k,j}] = 0$

Estimation: supply moment conditions

So we rewrite our supply estimating equation as

$$\hat{c}_{k,j} = \gamma_0 + \gamma_1 B^S_{k,j} + \gamma_2 t_{k,j} + \gamma_3 l_{k,j} + \epsilon^S_{k,j} = \theta^S X^S_{k,j} + \epsilon^S_{k,j} \quad (5)$$

where

- θ^S is the parameter column vector $[\gamma_0\ \gamma_1\ \gamma_2\ \gamma_3]^T$
- X^S is the row vector $[1\ B^S_{k,j}\ t_{k,j}\ l_{k,j}]$, with $B^S_{k,j}$ being a dummy variable receiving 1 for product b ($k = b$) or 0 otherwise

Since X^S are instruments for themselves (and note that $X^S_{k,j} = Z_{k,j}$, we can construct supply moment conditions

$$E\left[Z^T_{k,j}\epsilon^S_{k,j}\right] = 0$$

Q: How many moment conditions do we have here?

Estimation: demand moment conditions

A: Again, we have **four moment** conditions, now from the supply side

$$E\left[\epsilon^S_{k,j}\right] = 0$$
$$E\left[B^S_{k,j}\epsilon^S_{k,j}\right] = 0$$
$$E\left[t_{k,j}\epsilon^S_{k,j}\right] = 0$$
$$E\left[l_{k,j}\epsilon^S_{k,j}\right] = 0$$

Your turn!

Enough talk, let's work! The data for this exercise are in file `shares_data.csv`

1. Using these 8 moment conditions, joinly estimate parameter vector $\theta = [\theta^D, \theta^S]$ with GMM

2. Using your estimate $\hat{\alpha}$, predict marginal costs (equation 2) and calculate the average Lerner index for each model: $\hat{L}_k = \frac{1}{N}\sum_{j=1}^{N}\frac{p_{k,j}-\hat{c}_{k,j}}{p_{k,j}}$

Here are a few steps to get you started

Loading and preparing the data

In [ ]:

```
using DataFrames, CSV
df = CSV.read("shares_data.csv", DataFrame)
```

600 rows × 7 columns

	MarketID	s_k	p_k	s_0	steel	labor	product
	Int64	Float64	Float64	Float64	Float64	Float64	String1
1	1	0.0485396	33.96	0.878173	8.95	20.68	a
2	1	0.0732874	33.53	0.878173	8.95	20.68	b
3	2	0.0492018	34.06	0.866904	8.45	22.02	a
4	2	0.0838941	33.18	0.866904	8.45	22.02	b
5	3	0.0566908	33.2689	0.850551	7.43	21.96	a
6	3	0.0927581	33.6889	0.850551	7.43	21.96	b
7	4	0.0590058	33.8089	0.858772	7.93	21.73	a
8	4	0.0822226	33.5989	0.858772	7.93	21.73	b
9	5	0.0742836	31.94	0.831041	7.61	21.9	a
10	5	0.0946749	32.92	0.831041	7.61	21.9	b
11	6	0.0705296	32.79	0.845203	7.07	22.43	a
12	6	0.0842672	32.89	0.845203	7.07	22.43	b
13	7	0.0551974	32.49	0.88366	8.07	21.25	a
14	7	0.0611427	32.97	0.88366	8.07	21.25	b
15	8	0.0489004	34.26	0.872733	8.38	22.36	a
16	8	0.0783665	33.92	0.872733	8.38	22.36	b
17	9	0.0573168	33.7994	0.861865	7.97	22.49	a
18	9	0.0808183	34.5494	0.861865	7.97	22.49	b
19	10	0.0477127	34.03	0.894306	9.27	21.15	a
20	10	0.0579813	33.77	0.894306	9.27	21.15	b
21	11	0.0777428	34.65	0.87268	8.95	22.36	a
22	11	0.0495767	35.2	0.87268	8.95	22.36	b
23	12	0.0531182	32.74	0.875014	7.56	22.92	a
24	12	0.0718681	33.33	0.875014	7.56	22.92	b
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Loading and preparing the data

This data set already includes s_0 calculated for you. Let's get N and generate

- $log(s_{k,j}) - log(s_{0,j})$
- $B_{k,j}$

In [ ]:

```
N = nrow(df)
df.logsk_logs0 = log.(df.s_k) - log.(df.s_0);
df.is_B = (df.product .== "b");
df
```

600 rows × 9 columns (omitted printing of 1 columns)

	MarketID	s_k	p_k	s_0	steel	labor	product	logsk_logs0
	Int64	Float64	Float64	Float64	Float64	Float64	String1	Float64
1	1	0.0485396	33.96	0.878173	8.95	20.68	a	-2.89546
2	1	0.0732874	33.53	0.878173	8.95	20.68	b	-2.48346
3	2	0.0492018	34.06	0.866904	8.45	22.02	a	-2.869
4	2	0.0838941	33.18	0.866904	8.45	22.02	b	-2.33537
5	3	0.0566908	33.2689	0.850551	7.43	21.96	a	-2.70827
6	3	0.0927581	33.6889	0.850551	7.43	21.96	b	-2.21589
7	4	0.0590058	33.8089	0.858772	7.93	21.73	a	-2.67787
8	4	0.0822226	33.5989	0.858772	7.93	21.73	b	-2.34607
9	5	0.0742836	31.94	0.831041	7.61	21.9	a	-2.41479
10	5	0.0946749	32.92	0.831041	7.61	21.9	b	-2.17223
11	6	0.0705296	32.79	0.845203	7.07	22.43	a	-2.48354
12	6	0.0842672	32.89	0.845203	7.07	22.43	b	-2.30558
13	7	0.0551974	32.49	0.88366	8.07	21.25	a	-2.77316
14	7	0.0611427	32.97	0.88366	8.07	21.25	b	-2.67086
15	8	0.0489004	34.26	0.872733	8.38	22.36	a	-2.88184
16	8	0.0783665	33.92	0.872733	8.38	22.36	b	-2.41023
17	9	0.0573168	33.7994	0.861865	7.97	22.49	a	-2.7105
18	9	0.0808183	34.5494	0.861865	7.97	22.49	b	-2.36689
19	10	0.0477127	34.03	0.894306	9.27	21.15	a	-2.93085
20	10	0.0579813	33.77	0.894306	9.27	21.15	b	-2.73593
21	11	0.0777428	34.65	0.87268	8.95	22.36	a	-2.41816
22	11	0.0495767	35.2	0.87268	8.95	22.36	b	-2.86805
23	12	0.0531182	32.74	0.875014	7.56	22.92	a	-2.80172
24	12	0.0718681	33.33	0.875014	7.56	22.92	b	-2.49941
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Loading and preparing the data

It might also be useful to prepare matrices X and Z

In [ ]:

```
X = [ones(N) df.p_k df.is_B];
Z = [ones(N) df.is_B df.steel df.labor];
```

Moment conditions function

Program function `g_i(theta)` that receives parameter vector θ and returns the eight moment conditions for all observations ($N \times 8$ matrix). Here is a way to structure it

In [ ]:

```
function g_i(theta::AbstractVector{T}) where T
    # DEMAND SIDE
    theta_d = theta[1:3] # demand parameters
    # Epsilons for demand side
    e_d = df.logsk_logs0 - (X * theta_d)
    # Moment conditions for demand side
    # REPLACE THIS LINE <m_di = ... >

    # SUPPLY SIDE
    # Costs
    alpha = -theta[2] # we need it to estimate marginal costs
    c_k = df.p_k - 1/alpha .* 1 ./(1 .- df.s_k)

    theta_s = theta[4:7] # supply parameters
    # Epsilons for supply side
    e_s = c_k - (Z * theta_s)
    # Moment conditions for supply side
    # # REPLACE THIS LINE <m_si = ...>

    # Return matrices side by side (N x M)
    return([m_di m_si])
end
```

```
g_i (generic function with 1 method)
```

GMM estimation

With function `g_i(theta)`, you can use `Optim` package to minimize objective function `Q(theta)`

In [ ]:

```
using LinearAlgebra
M = 8
W = I(M) # Identity Matrix
function Q(theta)
    # Get moment vectors
    m_i = g_i(theta)
    # Take means of each column
    G = [sum(m_i[:, k]) for k in 1:M] ./ N
    # Calculate Q
    # REPLACE THIS LINE <...>
end
```

```
Q (generic function with 1 method)
```

GMM estimation

- For the first step, use W as the identity matrix and find estimate `theta_1`
- For the second step, $\hat{W} = (\hat{S})^{-1}$. So, you will need to calculate

$$\hat{S} = E[g_i(\hat{\theta}_1)^T g_i(\hat{\theta}_1)]$$

GMM estimation: initial guess

Given the dimensionality of the problem it's a good idea to start with a reasonable initial guess

- Let's use OLS to get an initial guess for θ^D
- Let's guess γ_0 as the minimum price and set 0.1 for the other parameters (we expect them to be positive)

In [ ]:

```
using GLM
ols_reg = lm(@formula(logsk_logs0 ~ 1 + p_k + is_B), df)
theta_0 = [coef(ols_reg); minimum(df.p_k); ones(3)./10]
```

```
7-element Vector{Float64}:
  0.31506305791962314
 -0.0928458769023904
  0.3630065729084741
 30.53116385673067
  0.1
  0.1
  0.1
```

GMM estimation: first step

In [ ]:

```
# Step 1
using Optim
res = Optim.optimize(Q, theta_0, Newton(), Optim.Options(f_abstol=1e-10, g_abstol=0.0
theta_1 =  res.minimizer
```

```
7-element Vector{Float64}:
  1.3167934612005487
 -0.1228198715837184
  0.3726111117323415
  6.90249208813232
  0.15417544082557208
  0.9851256773142594
  0.45457665443993567
```

In [ ]:

```
# Calculate W_hat
# REPLACE THIS LINE <W = inv(... >
```

```
8×8 Matrix{Float64}:
 22262.9       -9.34389   -315.713     …  -40.593     -1.72596    22.5192
    -9.34389  139.505       -6.33642       -5.44452    0.276477    0.435557
  -315.713     -6.33642     34.9919         1.64377   -0.623417    0.197342
  -893.464     -0.556555     1.81941        1.24632    0.298785   -1.09934
  -472.364    -11.7318       0.257378       2.28337   -5.31538   -29.8727
   -40.593     -5.44452      1.64377   …    5.88747   -0.14891    -0.166385
    -1.72596    0.276477    -0.623417      -0.14891    1.2495     -0.185561
    22.5192     0.435557     0.197342      -0.166385  -0.185561    1.4342
```

GMM estimation: second step

In [ ]:

```julia
# Use step 1 estimates as initial guess
res = Optim.optimize(Q, theta_1, Newton(), Optim.Options(f_abstol=1e-10, g_abstol=0.0
theta_GMM =  res.minimizer
```

```
7-element Vector{Float64}:
  1.0199256974273108
 -0.11397815604348306
  0.3710074786956938
  6.290279356119901
  0.14020424544797908
  0.9924727694197776
  0.4493242201423294
```

GMM estimation: standard errors

Recall

$$Var(\hat{\theta}_{GMM}) = \frac{1}{N}(\hat{D}'\hat{S}^{-1}\hat{D})^{-1}$$

where:

- $\hat{D} = E\left[\frac{\partial g_i(z_i;\hat{\theta})}{\partial \hat{\theta}'}\right]$
- $\hat{S} = E\left[g_i(z_i;\hat{\theta})g_i(z_i;\hat{\theta})'\right]$

To calculate $\hat{D}$, it will be useful to define function `Eg` for $E[g_i(z_i;\hat{\theta})]$

In [ ]:

```julia
function Eg(theta_GMM)
    gi = g_i(theta_GMM)
    Eg = [sum(gi[:, k]) for k in 1:M] ./ N; # This take means of each column
    return Eg
end;
```

GMM estimation: standard errors

So we calculate

- $\hat{D} = E\left[\frac{\partial g_i(z_i;\hat{\theta})}{\partial \hat{\theta}'}\right]$

- $\hat{S} = E\left[g_i(z_i;\hat{\theta})g_i(z_i;\hat{\theta})'\right]$
- $Var(\hat{\theta}_{GMM}) = \frac{1}{N}(\hat{D}'\hat{S}^{-1}\hat{D})^{-1}$

In [ ]:

```
using ForwardDiff
D_GMM = ForwardDiff.jacobian(Eg, theta_GMM);
S_GMM = g_i(theta_GMM)'  * g_i(theta_GMM) ./N;
V_GMM = inv(D_GMM' * inv(S_GMM) * D_GMM) ./N
```

Out[ ]:

```
7×7 Matrix{Float64}:
  0.0636808    -0.00190122    0.000430357  …  -0.00155039   -0.000665075
 -0.00190122    5.68441e-5   -1.56312e-5       4.65384e-5    2.00545e-5
  0.000430357  -1.56312e-5    0.000206462     -6.79124e-5   -4.0683e-5
  0.181612     -0.00543615    0.00274163      -0.0302212    -0.0483084
  0.00562975   -0.000167422   0.000175138      0.000148558   0.000160716
 -0.00155039    4.65384e-5   -6.79124e-5   …   0.00209994    0.000403963
 -0.000665075   2.00545e-5   -4.0683e-5        0.000403963   0.00195862
```

GMM estimation: standard errors

Having the estimated variance-covariance matrix $\hat{V}$, we can easily calculate the standard errors of our estimates and 95% confidence intervals

In [ ]:

```
SE_GMM = sqrt.(diag(V_GMM));
[theta_GMM SE_GMM (theta_GMM-1.96*SE_GMM) (theta_GMM+1.96*SE_GMM)]
```

Out[ ]:

```
7×4 Matrix{Float64}:
  1.01993   0.252351    0.525319     1.51453
 -0.113978  0.0075395  -0.128756    -0.0992007
  0.371007  0.0143688   0.342845     0.39917
  6.29028   1.32785     3.68769      8.89287
  0.140204  0.0741099  -0.00505119   0.28546
  0.992473  0.0458251   0.902656     1.08229
  0.449324  0.0442563   0.362582     0.536067
```

Marginal cost estimation

Now that we have estimated all model parameters, we can pick α to calculate implied marginal costs

In [ ]:

```
alpha_hat = -theta_GMM[2]
df.c_k = df.p_k - 1/alpha_hat .* 1 ./(1 .- df.s_k)
```

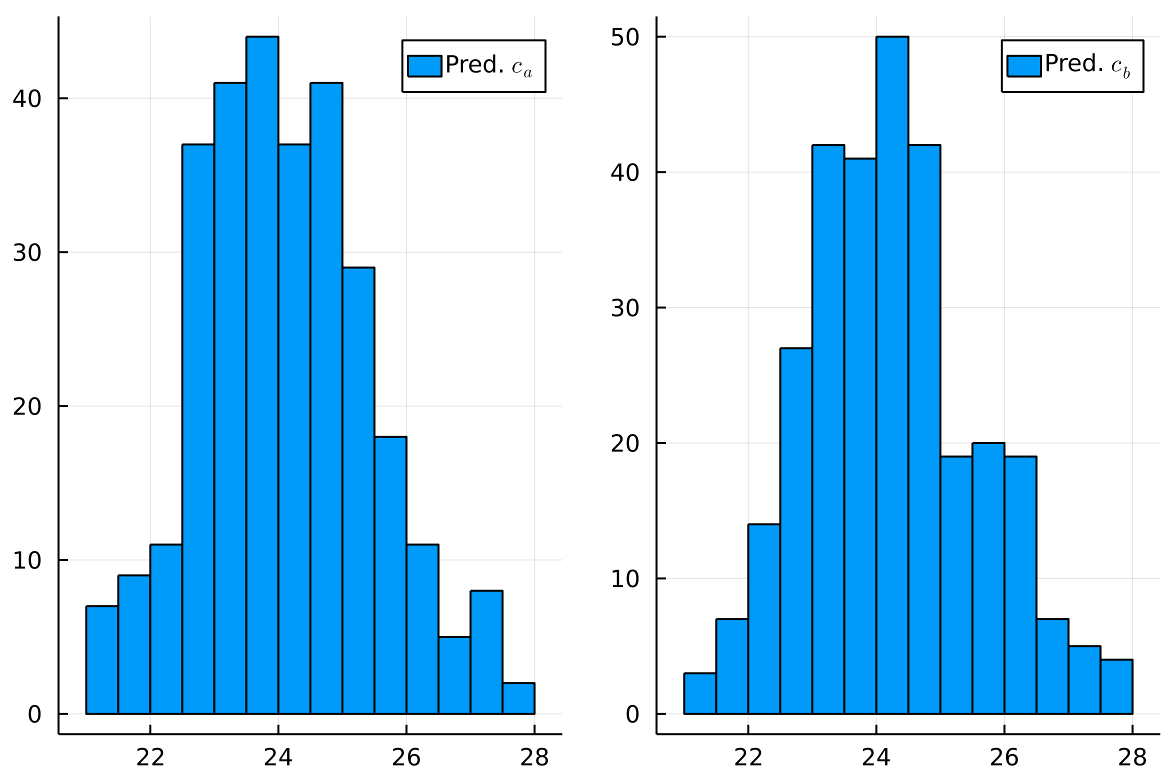
```
600-element Vector{Float64}:
 24.738795623135914
 24.062544181471655
 24.832373516777842
 23.60292920868719
 23.96804478334333
 24.01828910485132
 24.48508967353134
 24.039228021894246
 22.46235534622872
 23.228883020499666
  ⋮
 24.4522919411026
 24.9708247087463
 24.684056116326747
 23.57304222168553
 23.753233362550105
 23.03024376005476
 23.29227139103517
 23.255258601242673
 23.956042729733415
```

Marginal cost estimation

Let's take a look at the distribution of costs per firm

In [ ]:

```
using Plots, LaTeXStrings
plot(histogram(df.c_k[.!df.is_B], label=L"Pred. $c_a$", bins=30), histogram(df.c_k[df
```



Lerner index

We are now ready to calculate the average Lerner index per firm

In [ ]:
```
df.lerner_k = (df.p_k - df.c_k) ./ df.p_k;
# Firm A
sum(df.lerner_k[.!df.is_B])./(N/2)
```

```
0.2785174838631413
```

In [ ]:
```
# Firm B
sum(df.lerner_k[df.is_B])./(N/2)
```

```
0.2818733881583978
```

So about 28% of the price is due to market power

- Firm B has slightly higher market power due to its perceived higher quality ($\beta_1 = 0.37 > 0$)