

约束优化问题

约束优化问题 (原问题)
primal problem

$$\begin{cases} \min_{x \in \mathbb{R}^p} f(x) \\ \text{s.t. } m_i(x) \leq 0, i=1, \dots, M \\ \quad n_j(x) = 0, j=1, \dots, N \end{cases}$$

拉格朗日函数:

$$L(x, \lambda, \eta) = f(x) + \sum_{i=1}^M \lambda_i m_i + \sum_{j=1}^N \eta_j n_j$$

$$\min_x \max_{\lambda, \eta} L(x, \lambda, \eta) \Rightarrow x \in \{\text{好的 } x \text{ 集合}\}$$

$$\begin{cases} \text{(原问题是关于 } x \text{ 的)} & \text{(原问题的无约束形式)} \\ \text{s.t. } \lambda_i \geq 0 \end{cases}$$

$$\begin{cases} \text{如果 } x \text{ 违反了约束 } m_i(x), m_i(x) > 0 & \max_{\lambda} L \rightarrow \infty \end{cases}$$

$$\begin{cases} \text{如果 } x \text{ 符合 } m_i(x) \leq 0, & \max_{\lambda} L \neq +\infty \end{cases}$$

$$\min_x \max_{\lambda} L = \min_x \left\{ \underbrace{\max_{\lambda} L}_{\text{好的 } x} \underbrace{, +\infty}_{\text{坏的 } x} \right\} = \min_x \left\{ \max_{\lambda} L \right\}$$

对偶性
dual problem

$$\begin{cases} \text{对偶问题:} \\ d \leftarrow \max_{\lambda, \eta} \min_x L(x, \lambda, \eta) \quad (\text{对偶问题是关于 } \lambda, \eta \text{ 的函数}) \\ \text{s.t. } \lambda_i \geq 0 \end{cases}$$

弱对偶性: 对偶问题 \leq 原问题

$$\begin{aligned} \max_{\lambda, \eta} \min_x L(x, \lambda, \eta) &\leq \min_x \max_{\lambda, \eta} L(x, \lambda, \eta) \\ \text{证明: } \max_{\lambda, \eta} \min_x L &\leq \min_x \max_{\lambda, \eta} L \end{aligned}$$

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$$\underbrace{\min_x L(x, \lambda, \eta)}_{A(\lambda, \eta)} \leq L(x, \lambda, \eta) \leq \underbrace{\max_{\lambda, \eta} L(x, \eta)}_{B(x)}$$

$$\begin{aligned} A(\lambda, \eta) &\leq B(x) \leq \max_{\lambda, \eta} B(x) \\ \Rightarrow A(\lambda, \eta) &\leq \min_x B(x) \\ \Rightarrow \max_{\lambda, \eta} A(\lambda, \eta) &\leq \min_x B(x) \end{aligned}$$

$$\boxed{\max_{\lambda, \eta} \min_x L \leq \min_x \max_{\lambda, \eta} L}$$

证毕.