AM5650 - Nonlinear Dynamics

Course Project

Coupling of structure and wake oscillators in vortex-induced vibrations

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1 Summary of the Selected Paper

1.1 Introduction: Motivation to study vortex-induced vibrations

Vortex-induced vibrations (VIV) are critical to the design of various engineering structures exposed to fluid flows, such as chimneys, cables, risers, and offshore components, due to the risk of fatigue damage. Early studies modeled VIV as a two-dimensional problem using wake oscillator models—particularly van der Pol and Rayleigh-type equations—to represent the interaction between vortex shedding and structural motion. While direct numerical simulations (DNS) have provided detailed insights into these flow fields, extending them to realistic three-dimensional (3D) domains remains computationally intensive. To address this, phenomenological models have evolved to capture 3D effects through distributed wake oscillators and other low-order systems. This paper critically examines such models by focusing on a simplified case: transverse VIV of a one-degree-of-freedom structure in uniform flow, using a linearly coupled van der Pol-type oscillator. By comparing this new coupling approach with existing models, the study shows that a linear inertial coupling is more effective in describing, both qualitatively and in some aspects quantitatively, the key features of 2D VIV behavior.

1.2 Vortex-Induced Vibration Model

The VIV model considers a rigid circular cylinder of diameter D constrained to oscillate transversely in a uniform flow of velocity U. The structural motion is described by a one-degree-of-freedom (1-DOF) linear oscillator including fluid-added mass and damping effects. The dimensional equation of motion is

$$m\ddot{Y} + r\dot{Y} + hY = S, \tag{1}$$

where $m = m_s + m_f$ is the total mass per unit length, including the structural mass m_s and added fluid mass $m_f = C_M \rho \pi D^2/4$. The damping is $r = r_s + r_f$ with $r_f = \gamma \omega \rho D^2$, and ω is typically the vortex shedding frequency $\omega = 2\pi St U/D$. The structural equation can be rewritten in dimensionless form by defining $y = Y/D$, $t = T\omega_f$, and reduced frequency $\delta = \omega_s/\omega_f$:

$$\ddot{y} + 2\zeta\delta\dot{y} + \left(\frac{\gamma}{m}\right)\dot{y} + \delta^2 y = s, \tag{2}$$

where $\zeta = r_s/(2m\omega_s)$ is the structural damping ratio. The wake dynamics are modeled by a Van der Pol oscillator representing the fluctuating lift coefficient:

$$\ddot{q} + \varepsilon(q^2 - 1)\dot{q} + q = f, \tag{3}$$

where q is a dimensionless wake variable related to the vortex lift coefficient. The system is coupled via forcing terms s and f, which represent the influence of the wake on the structure and vice versa. In dimensionless form, the coupled system becomes:

$$\ddot{y} + \left(2\zeta\delta + \frac{\gamma}{m}\right)\dot{y} + \delta^2 y = s, \tag{4}$$

$$\ddot{q} + \varepsilon(q^2 - 1)\dot{q} + q = f. \tag{5}$$

The coupling term from the wake to the structure is typically taken as:

$$s = Mq, \quad \text{where } M = \frac{C_{L0}}{2} \cdot \frac{1}{8\pi^2 St^2 m}. \tag{6}$$

Three types of linear coupling from the structure to the wake are considered:

$$f = Ay \quad \text{(displacement coupling)}, \tag{7}$$

$$f = A\dot{y} \quad \text{(velocity coupling)}, \tag{8}$$

$$f = A\ddot{y} \quad \text{(acceleration coupling)}. \tag{9}$$

These equations form the core of the low-order model used to study vortex-induced vibrations and analyse the effect of different fluid–structure coupling mechanisms.

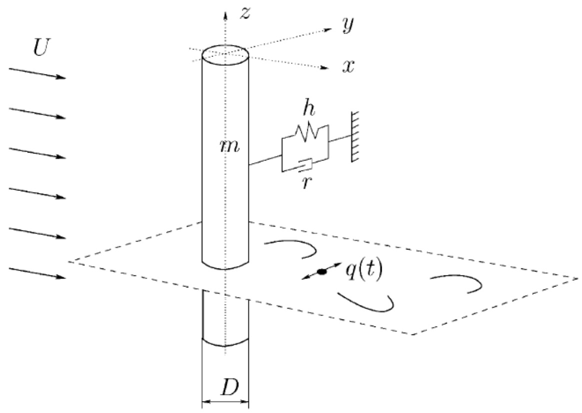


Figure 1: Model of coupled structure and wake oscillators for 2-D vortex-induced vibrations.

1.3 Values of Model Parameters

In this study, the Strouhal number is taken as $St = 0.2$, and the reference lift coefficient is set to $C_{L0} = 0.3$. The added mass coefficient is assumed to be $C_M = 1$, corresponding to a circular cylinder in potential flow. The drag coefficient is amplified to $C_D = 2.0$ to account for wake effects. The values of ε and A are chosen such that $A/\varepsilon = 40$, based on experimental data fitting of lift magnification and lock-in boundaries. The following table summarises the dimensionless parameters used in the VIV model, along with their meanings and chosen values:

Parameter	Meaning	Value
δ	Reduced natural frequency of structure	$\delta = 1/St \cdot U_r$
μ	Dimensionless mass ratio	$\mu = 0.05/M$
M	Mass number, scales effect of wake on structure	$M = 2 \times 10^{-4}$
γ	Fluid-added damping coefficient	$\gamma = 0.8$
ξ	Structural damping ratio	$\xi = 3.1 \times 10^{-3}$
ε	van der Pol nonlinearity parameter	$\varepsilon = 0.3$
s	Forcing from wake on structure	$s = Mq$
A	Scaling of coupling force	$A = 12$

Table 1: Dimensionless model parameters and values used in the VIV system.

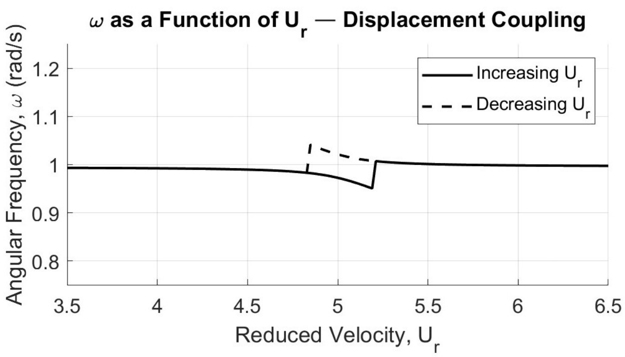
1.4 Outcomes

The coupled VIV model was analysed to evaluate the dynamic response under different fluid–structure coupling schemes. In forced vibration, only the acceleration coupling model accurately captures the phase shift between wake and structural oscillations. For the fully coupled system, all three models exhibit lock-in behavior—defined as frequency synchronisation between structure and wake—but differ in response characteristics. Displacement coupling results in weak oscillations and no lift magnification, while velocity and acceleration couplings reproduce experimentally observed lock-in with large oscillation amplitudes. Hysteresis was observed prominently: the acceleration model displays it at both lock-in boundaries, while the velocity model shows it asymmetrically. In Griffin plots, only the velocity and acceleration models correctly reflect the dependence of maximum amplitude on the Skop–Griffin parameter. At low mass ratios, only the acceleration coupling predicts persistent lock-in and frequency deviation with increasing flow velocity, consistent with experimental trends. Overall, the acceleration coupling emerges as the most robust model, qualitatively capturing all key VIV features, including lift magnification, phase behaviour, hysteresis, and extended lock-in domains. [1]

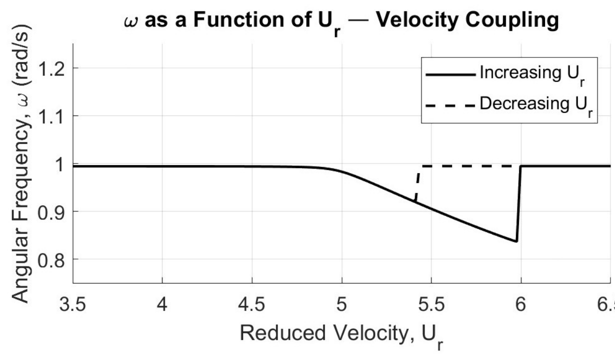
2 Reproduction of the Results Presented in the Paper

2.1 Frequency Response (Figure 5)

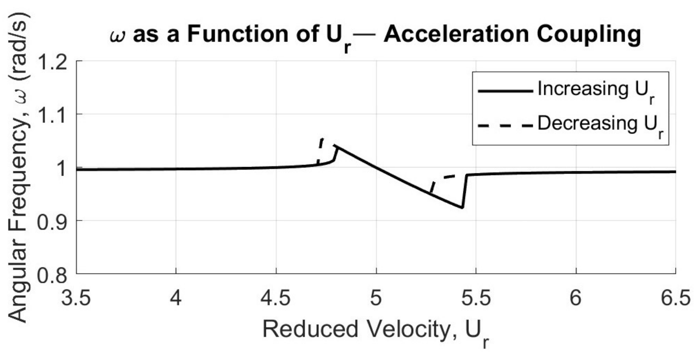
Figure 2 shows the variation of the system's angular frequency ω with reduced velocity U_r for the three types of fluid–structure coupling. All three models exhibit lock-in behaviour near $U_r \approx 1/St$, where the vortex shedding and structural frequencies synchronise.



(a) Displacement coupling



(b) Velocity coupling



(c) Acceleration coupling

Figure 2: Angular frequency ω vs. reduced velocity U_r.

The reduced velocity is defined as

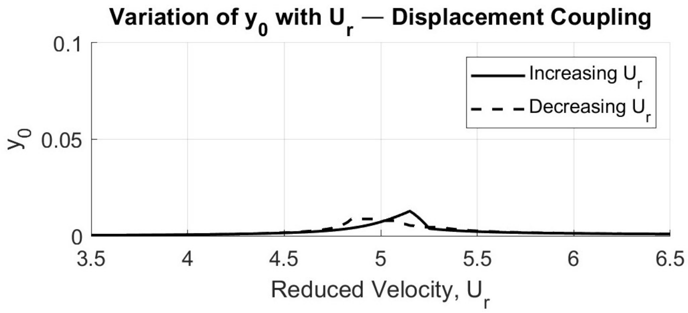
$$U_r = \frac{2\pi}{\omega \Omega_f} \cdot \frac{U}{D} = \frac{1}{\omega \, \mathrm{St}},$$

where U is the free-stream velocity, D is the cylinder diameter, Ω_f is the vortex shedding angular frequency, and St is the Strouhal number. In the displacement coupling model, the lock-in region is narrow and shows weak hysteresis. The velocity model has an asymmetric lock-in region—developing only for $U_r > 1/St$—while the acceleration model exhibits symmetric lock-in with clear hysteresis on both sides. This hysteresis is a hallmark of nonlinear coupled dynamics and matches experimental observations well, especially in the acceleration case. [1]

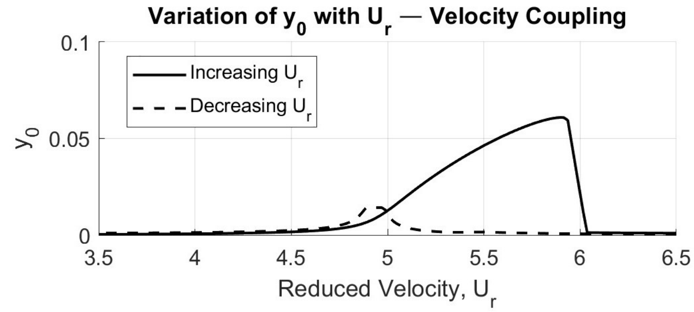
2.2 Structural Oscillation Amplitude (Figure 6)

Figure 3 compares the amplitude y_0 of the structural oscillator as a function of U_r. The displacement model fails to capture significant motion, with amplitudes an order of magnitude smaller than in the other two models. The authors attribute this to its "adiabatic" nature: energy transferred to the structure causes a decrease in wake oscillation amplitude, rather than an amplification. In contrast, both velocity and acceleration couplings exhibit pronounced peaks during lock-in, consistent with large-amplitude vibrations observed in experiments.

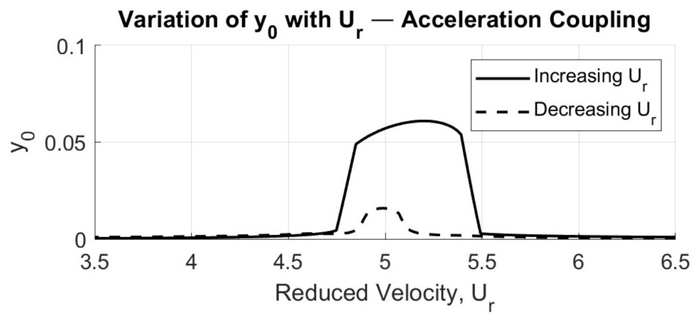
[1] Figures 5,6,7 refer to the corresponding figure numbers in the paper.



(a) Displacement coupling



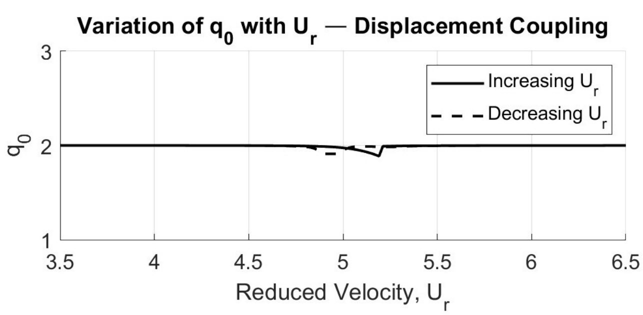
(b) Velocity coupling



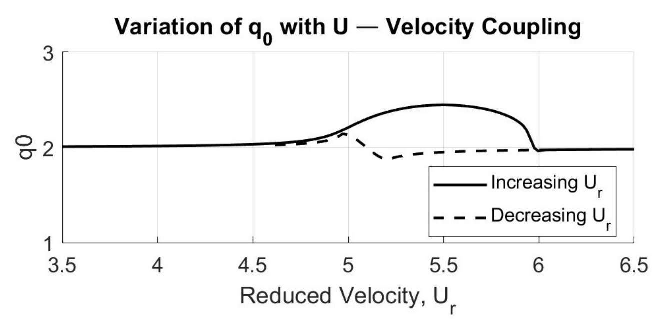
(c) Acceleration coupling

Figure 3: Amplitude of structure oscillator y_0 vs. reduced velocity U_r.

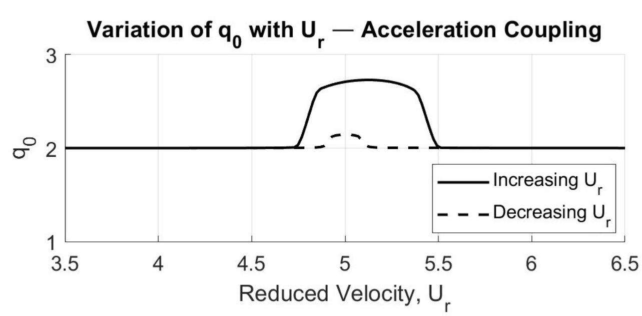
2.3 Wake Oscillator Amplitude (Figure 7)



(a) Displacement coupling



(b) Velocity coupling



(c) Acceleration coupling

Figure 4: Amplitude of wake oscillator q_0 vs. reduced velocity U_r.

Figure 4 presents the amplitude q_0 of the wake oscillator across the same U_r range. The displacement model again underperforms, showing a reduction in wake amplitude during lock-in—opposite to experimental trends. The velocity and acceleration models correctly show magnified wake amplitudes in the lock-in region.

3 Novel Result: Nonlinear coupling of structure and wake oscillators

3.1 Modifications to the Model

The classical wake oscillator model is modified to incorporate nonlinear coupling terms between the structural displacement and the wake oscillator. Unlike traditional linear acceleration and velocity-based couplings, this formulation includes polynomial nonlinearities in displacement, velocity, and acceleration up to the cubic order. The forcing term f acting on the wake oscillator is defined as

$$f = \sum_{n=0}^{3} \left[A_n |y|^n \ddot{y} + B_n |y|^n \dot{y} \right], \tag{10}$$

where A_n and B_n are tuning parameters selected to match experimental forced vibration data. The chosen parameters are: $A_0 = 0.330$, $A_1 = 0$, $A_2 = -0.060$, $A_3 = 0.002$, and $B_0 = -0.005$, $B_1 = 0.003$, $B_2 = 0$, $B_3 = 0$. [2]. This approach allows the model to capture amplitude-dependent wake effects—including the negative damping region associated with lock-in.

3.2 Result Obtained

The effects of nonlinear coupling were examined by analysing the system response under varying reduced velocity (U_r). As shown in Figure 5, the structural oscillation frequency ω exhibits a significantly different trend compared to linearly coupled models. Specifically, in the lock-in region, the frequency "blow-up" is suppressed. This indicates that nonlinear coupling dampens the abrupt energy transfer that causes wide hysteresis loops in classical VIV models. Figure 6 plots the steady-state structural amplitude y_0 versus U_r. Unlike linear (velocity and acceleration) models, where amplitude increases steeply over a broad lock-in range, the nonlinear coupling delays the onset of amplitude growth until near $U_r = 5$, the nominal synchronisation condition based on natural frequency matching. Moreover, the amplitude rise is limited in magnitude, indicating reduced sensitivity to vortex forcing. This behaviour confirms that nonlinear coupling not only preserves the qualitative features of lock-in but also suppresses the bistability responsible for hysteresis.

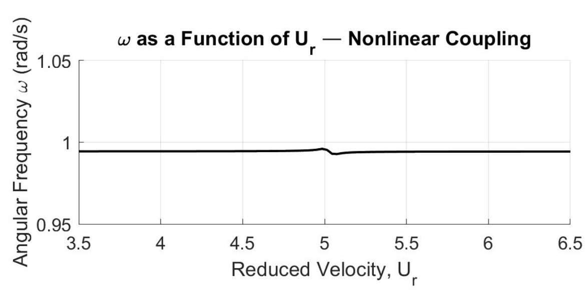


Figure 5: Oscillation frequency ω vs. reduced velocity U_r for nonlinear coupling.

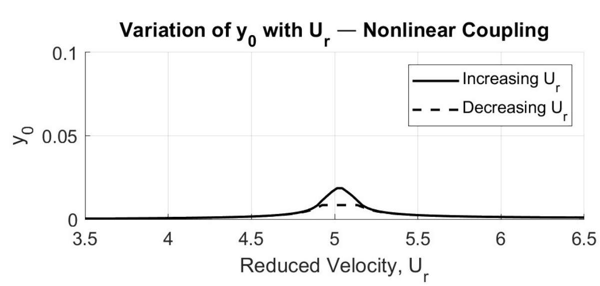


Figure 6: Structural amplitude y_0 vs. reduced velocity U_r for nonlinear coupling.

4 Discussions and Conclusions

4.1 Nonlinearity in the Model

The nonlinearity in the model arises from the use of the Van der Pol oscillator to represent the self-limiting, amplitude-saturated behavior of vortex-induced vibrations. Specifically, the nonlinearity is introduced through the term $\epsilon(1 - q^2)\dot{q}$ where, ϵ is the nonlinearity parameter. This term causes the system to exhibit a limit-cycle behavior, allowing for sustained oscillations with a bounded amplitude even in the absence of external forcing. The nonlinear damping effectively destabilizes small oscillations and stabilizes large ones, capturing the essential physics of vortex shedding and lock-in phenomena.

4.2 Explanation for Hysteresis

Hysteresis in the vortex-induced vibration (VIV) model arises due to the coupling between the structure and wake dynamics, specifically through the nonlinear interaction captured by the Van der Pol-type oscillator governing the wake variable. The system exhibits memory because the state of the wake oscillator depends not only on the instantaneous displacement and velocity of the structure, but also on its past evolution. Because the wake has its own dynamics — with inertia, damping, and nonlinearity — its response to the structure's motion is not instantaneous, but evolves over time. This time-lag in response means that for the same instantaneous velocity and position of the structure, the force from the wake can be different depending on the history of motion — i.e., whether the system reached that state by increasing or decreasing flow velocity. Thus, as the reduced velocity is varied quasi-statically, the system can follow different stable solution branches depending on the direction of parameter change, thereby exhibiting hysteresis. This multistability and memory effect are rooted in the nonlinear feedback between the structure's motion and the wake dynamics.

4.3 Conclusions

In this study, the dynamics of vortex-induced vibrations (VIV) were explored using a coupled structure and wake oscillator model, incorporating both linear and nonlinear coupling schemes. The results highlight the significance of the coupling mechanism in accurately predicting the system's behaviour, particularly in terms of lock-in, hysteresis, and amplitude response. The linear acceleration coupling model proved to be the most effective in capturing the key features of VIV, including lift magnification and phase behaviour. The nonlinear coupling, in particular, suppressed the excessive energy transfer typically observed in classical models, mitigating the abrupt frequency blow-up and reducing the sensitivity of the structure's amplitude to vortex forcing. Nonlinearities play a crucial role in modelling more realistic VIV behaviour, especially in scenarios involving large oscillations and complex wake-structure interactions.

References

[1] M. L. Facchinetti, E. de Langre, and F. Biolley. Coupling of structure and wake oscillators in vortex-induced vibrations. *Journal of Fluids and Structures*, 19(2):123–140, 2004.

[2] Yang Qu. *A single wake oscillator model for coupled cross-flow and in-line vortex-induced vibrations of marine structures*. PhD thesis, Delft University of Technology, 2019.

Code

The github page with the code can be found here.