Module I

Coordinate Systems, Its Transformation, and Vector Calculus

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Contents

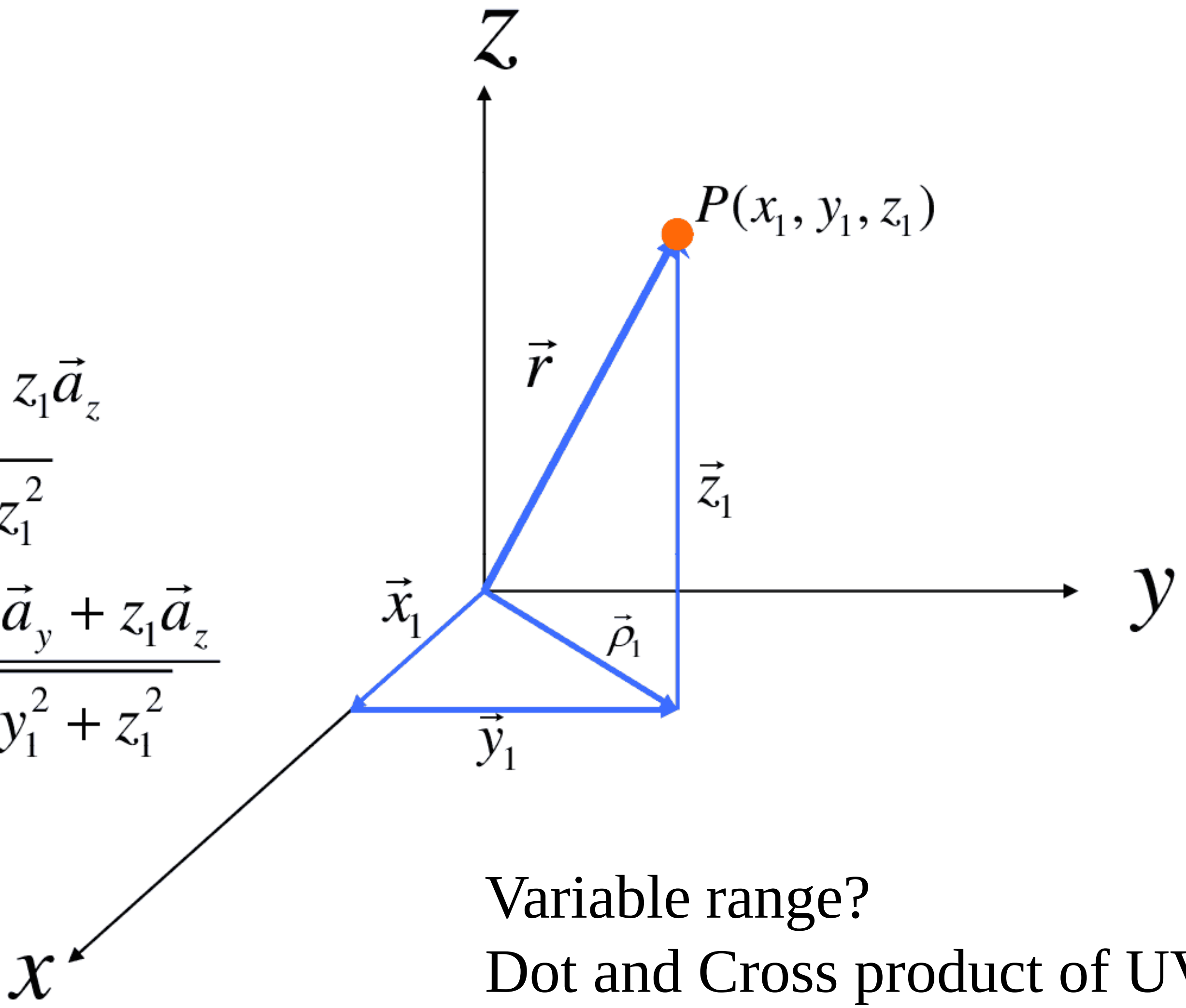
$$\vec{x}_1 = x_1\vec{a}_x$$
$$\vec{y}_1 = y_1\vec{a}_y$$
$$\vec{z}_1 = z_1\vec{a}_z$$

$$\vec{r} = x_1\vec{a}_x + y_1\vec{a}_y + z_1\vec{a}_z$$

$$|\vec{r}| = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

$$\vec{a}_r = \frac{\vec{r}}{|\vec{r}|} = \frac{x_1\vec{a}_x + y_1\vec{a}_y + z_1\vec{a}_z}{\sqrt{x_1^2 + y_1^2 + z_1^2}}$$

$$\vec{\rho}_1 = x_1\vec{a}_x + y_1\vec{a}_y$$



Variable range?
Dot and Cross product of UV?

Fig. 1 Rectangular Coordinate System

Variable range?
Dot and Cross product of UV?

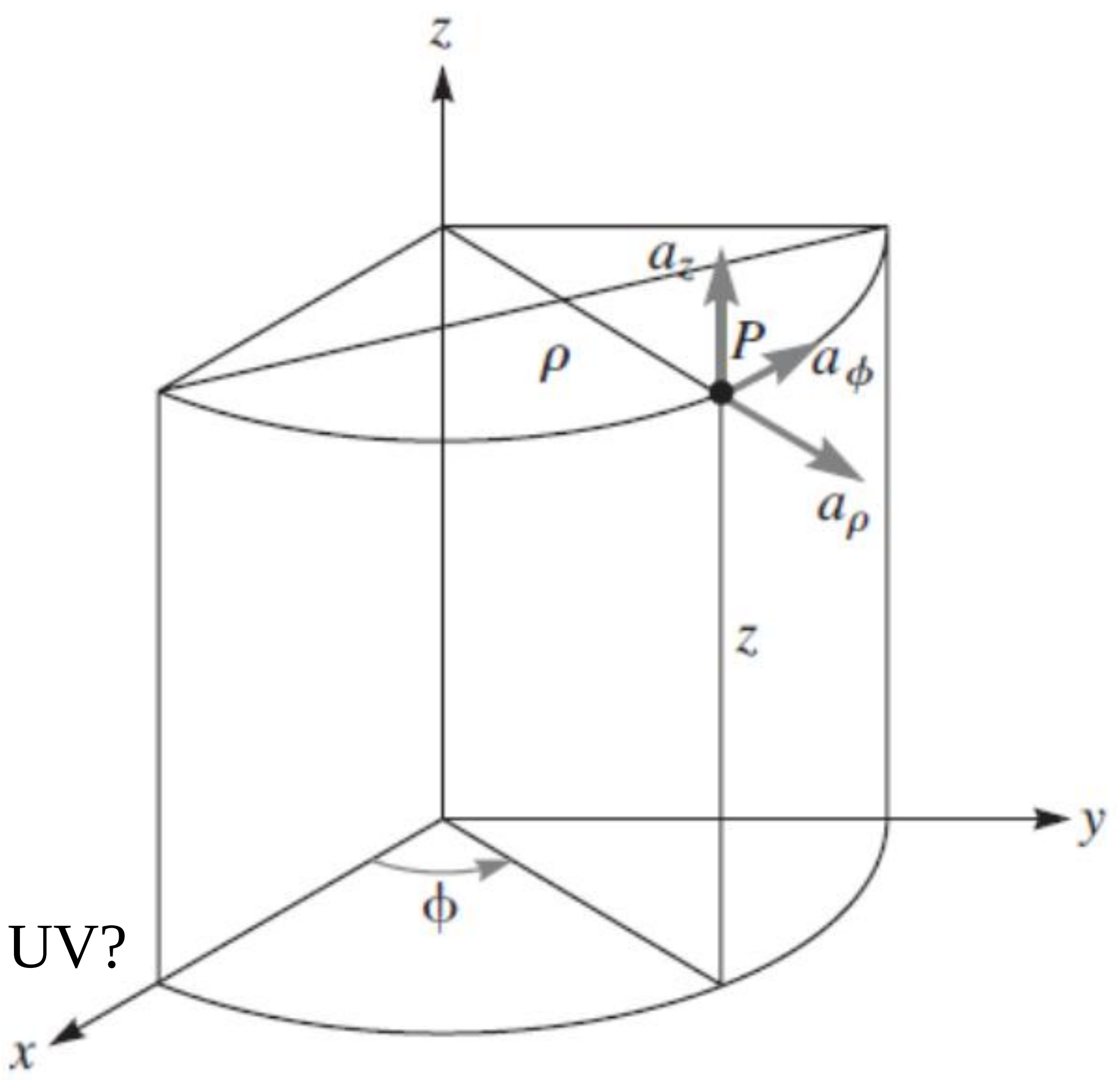


Fig. 2 Point *P* and unit vectors in the cylindrical coordinate system

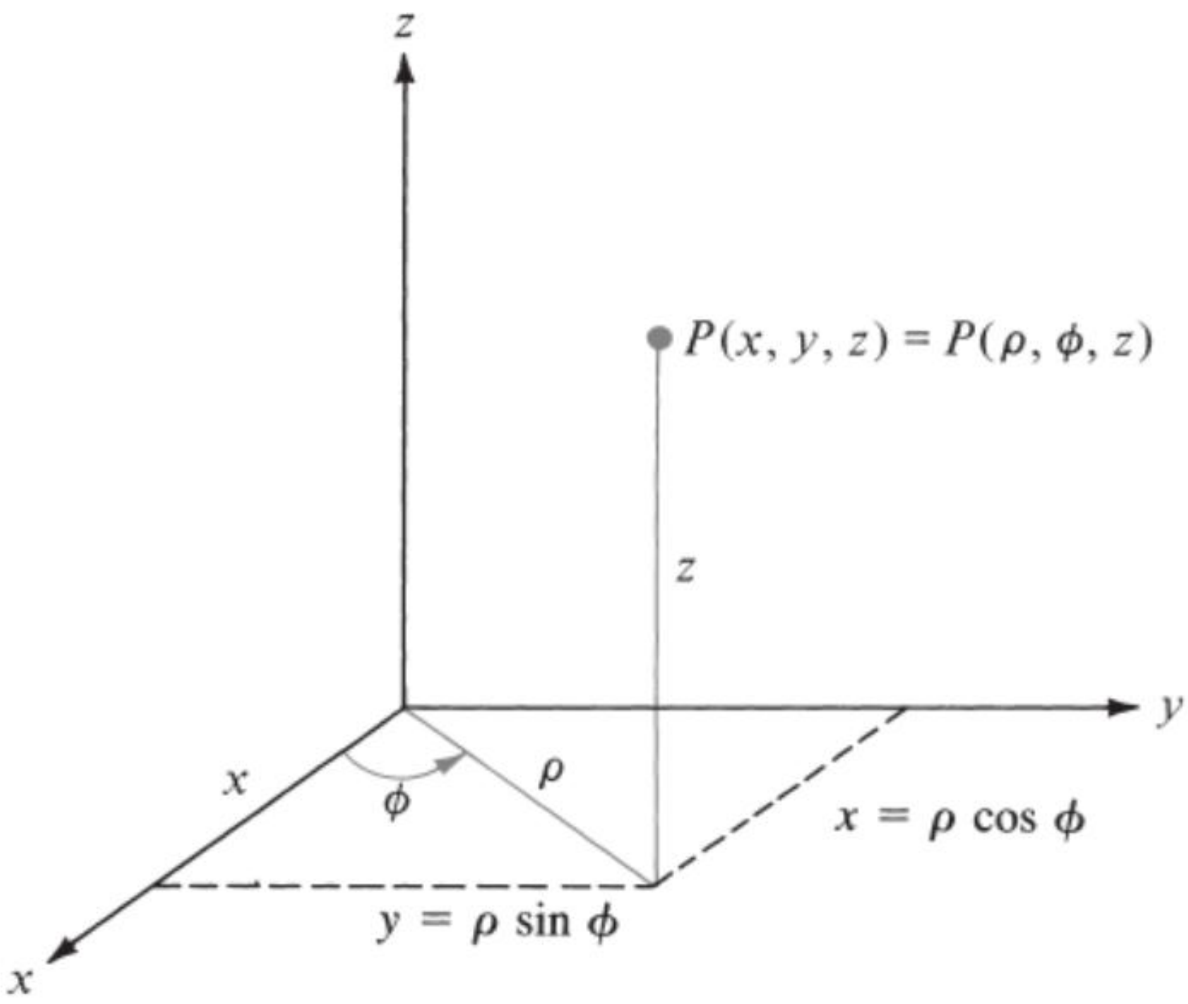


Fig. 3 Relationship between (x, y, z) and (ρ, ϕ, z)

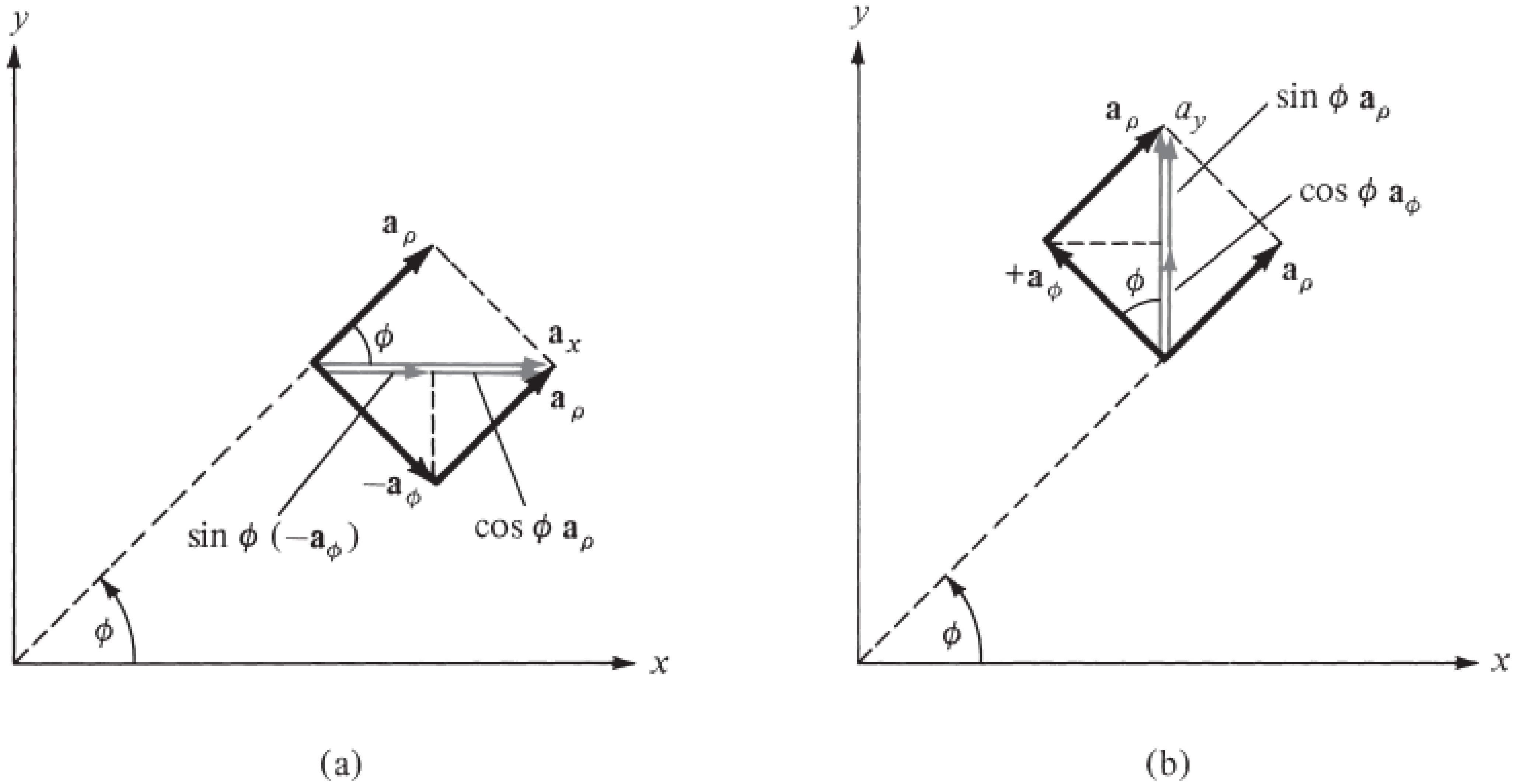


Fig. 4 Unit vector transformation: (a) Cylindrical component of a_x,

(b) Cylindrical component of a_y

$$\mathbf{a}_x = \cos\phi\,\mathbf{a}_\rho - \sin\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_y = \sin\phi\,\mathbf{a}_\rho + \cos\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_z = \mathbf{a}_z$$

$$\mathbf{a}_\rho = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y$$

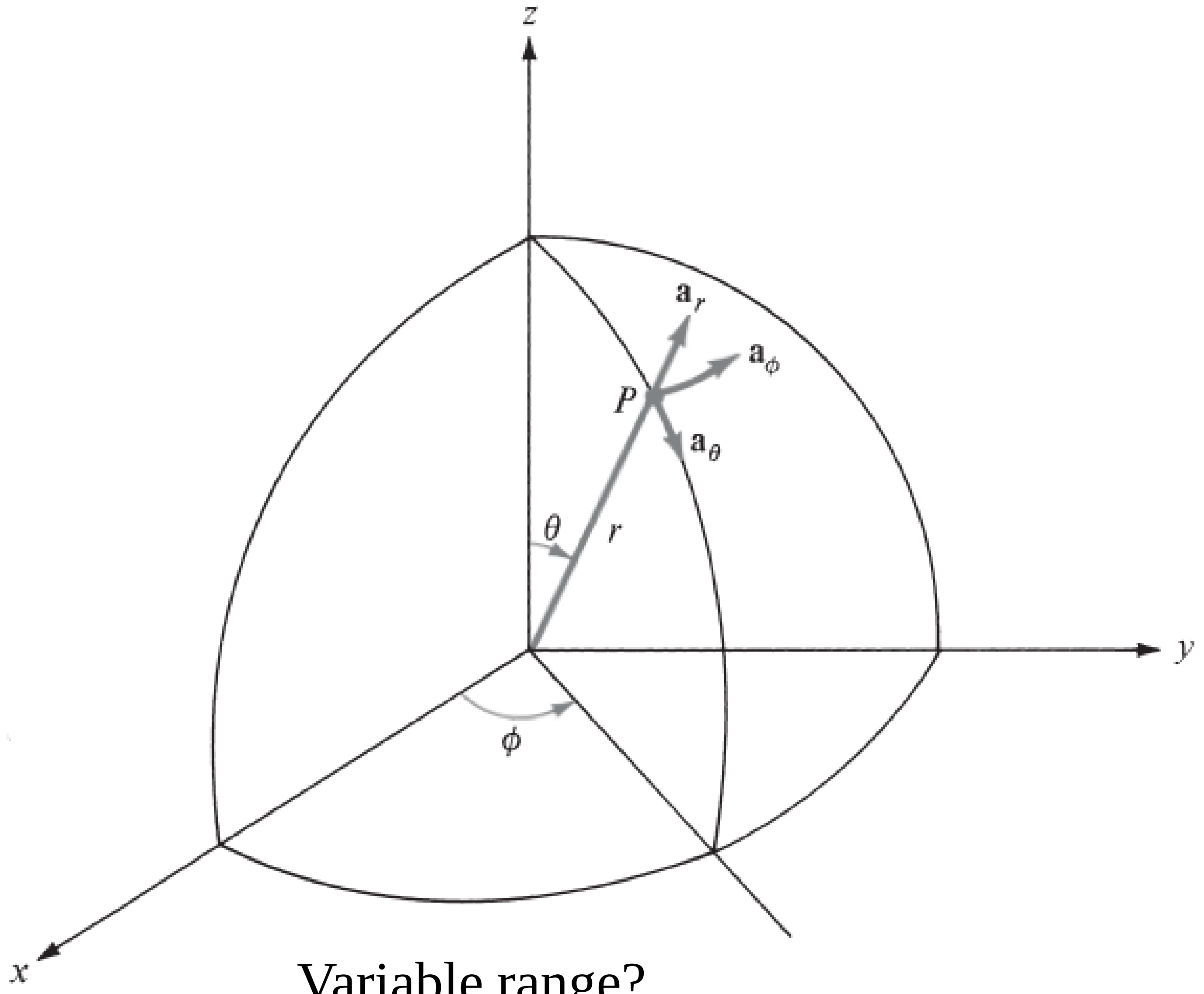
$$\mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$$

$$\mathbf{a}_z = \mathbf{a}_z$$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$



Variable range?
Dot and Cross product of UV?

Fig. 5 Point P and unit vectors in the Spherical coordinate system

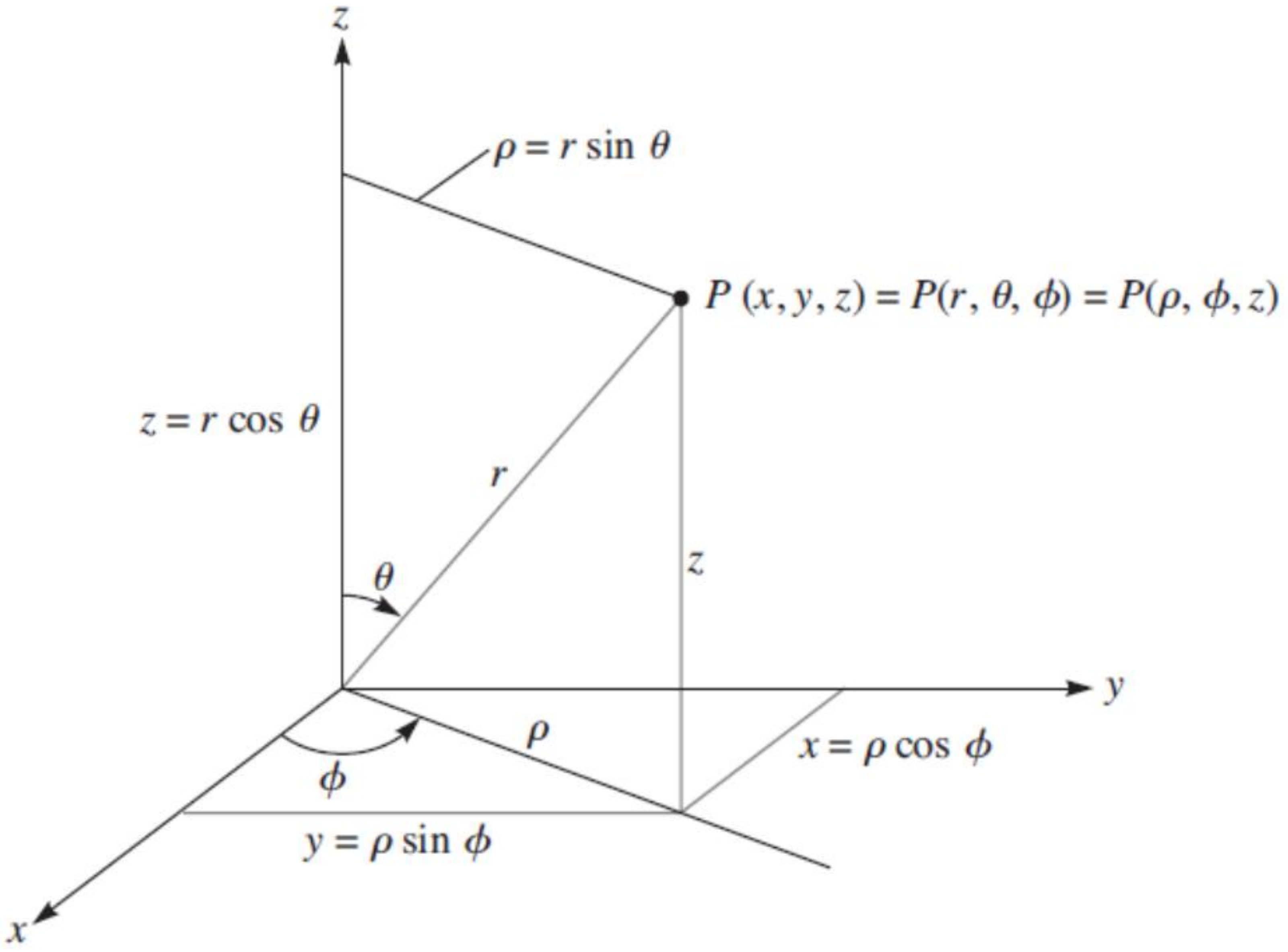


Fig. 6 Relationships between space variables (x, y, z), (ρ, ϕ, z), and (r, θ, ϕ)

$$\mathbf{a}_x = \sin\theta\cos\phi\,\mathbf{a}_r + \cos\theta\cos\phi\,\mathbf{a}_\theta - \sin\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_y = \sin\theta\sin\phi\,\mathbf{a}_r + \cos\theta\sin\phi\,\mathbf{a}_\theta + \cos\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_z = \cos\theta\,\mathbf{a}_r - \sin\theta\,\mathbf{a}_\theta$$

$$\mathbf{a}_r = \sin\theta\cos\phi\,\mathbf{a}_x + \sin\theta\sin\phi\,\mathbf{a}_y + \cos\theta\,\mathbf{a}_z$$

$$\mathbf{a}_\theta = \cos\theta\cos\phi\,\mathbf{a}_x + \cos\theta\sin\phi\,\mathbf{a}_y - \sin\theta\,\mathbf{a}_z$$

$$\mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \mathbf{a}_r \cdot \mathbf{a}_x & \mathbf{a}_r \cdot \mathbf{a}_y & \mathbf{a}_r \cdot \mathbf{a}_z \\ \mathbf{a}_\theta \cdot \mathbf{a}_x & \mathbf{a}_\theta \cdot \mathbf{a}_y & \mathbf{a}_\theta \cdot \mathbf{a}_z \\ \mathbf{a}_\phi \cdot \mathbf{a}_x & \mathbf{a}_\phi \cdot \mathbf{a}_y & \mathbf{a}_\phi \cdot \mathbf{a}_z \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

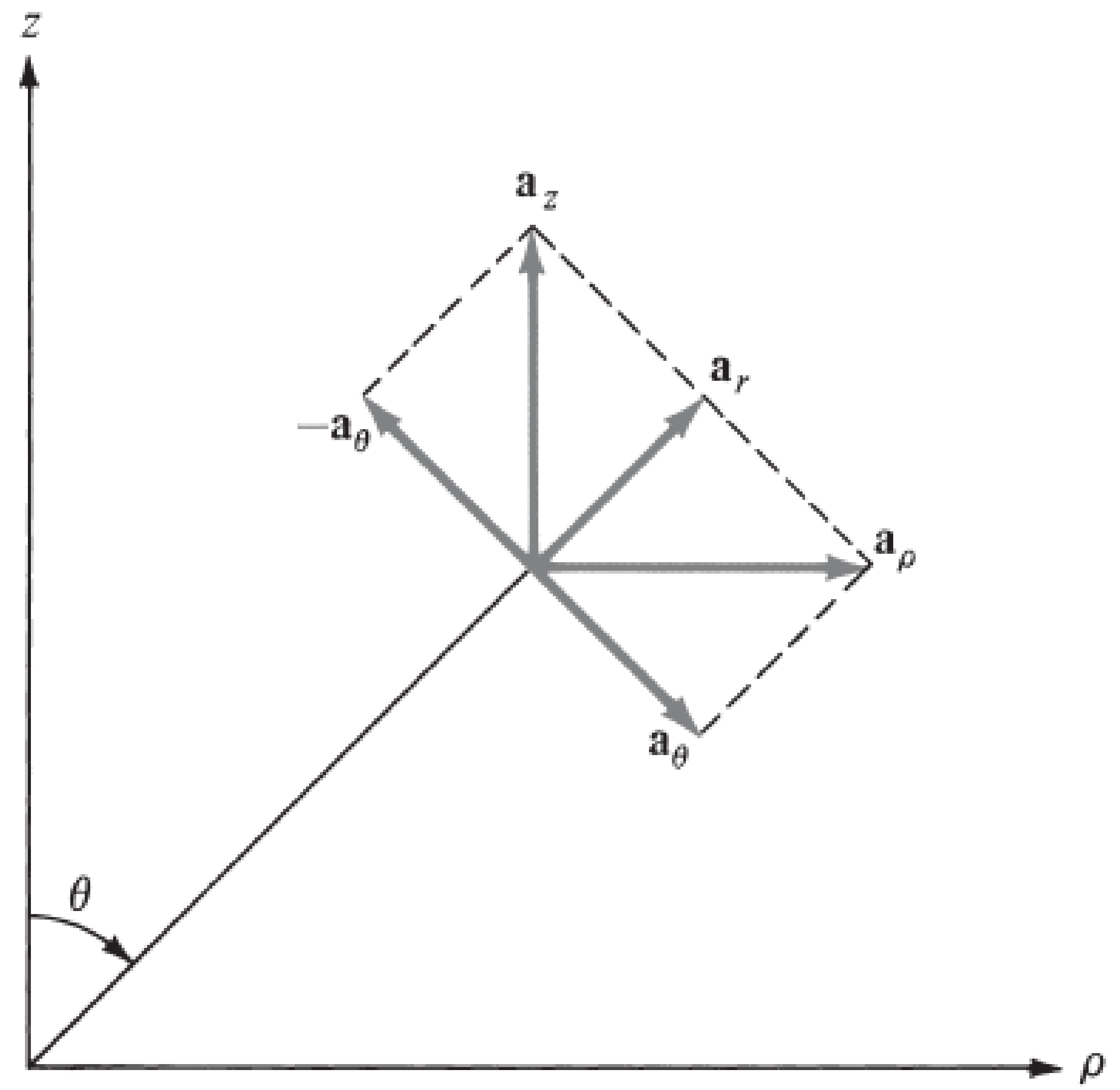


Fig. 7 Unit vector transformations for Cylindrical and Spherical coordinates

Find that the distance ‘d’ between two pints A and B in all the three coordinate systems.

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \text{ (Cartesian)}$$

$$d^2 = \rho_2^2 + \rho_1^2 - 2\rho_1\rho_2\cos(\phi_2 - \phi_1) + (z_2 - z_1)^2 \text{ (cylindrical)}$$

$$d^2 = r_2^2 + r_1^2 - 2r_1r_2\cos\theta_2\cos\theta_1 - 2r_1r_2\sin\theta_2\sin\theta_1\cos(\phi_2 - \phi_1) \text{ (spherical)}$$