PROPERTIES OF DETERMINANTS ALONG WITH PROOFS

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Abstract:

The determinant is useful for solving linear equations, capturing how linear transformation change area or volume, and changing variables in integrals.The determinant can be viewed as a function whose input is a square matrix and whose output is a number.

1. Introduction

In linear algebra, the determinant is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the linear transformation described by the matrix.

Generally, we define determinant of a square matrix as

2. DETERMINANT OF A SQUARE MATRIX:

The sum of the products of the elements of the first row(or column) of a square matrix A with their corresponding cofactors is called the determinant of the matrix A.

2.1. TERMINOLIGIES

2.1.1. MINOR OF A SQUARE MATRIX

If a_{ij} is an element which is in the ith row and jth column of a square matrix A, then the determinant of the matrix obtained by deleting the ith row and jth column of A is called MINOR of a_{ij}.It is denoted by M_{ij}

If

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{1}$$

then

$$M_{11} = Minor of a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23} \tag{2}$$

and

$$M_{12} = Minor of a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23}; and\ so\ on. \tag{3}$$

2.1.2. COFACTOR OF A SQUARE MATRIX

If a_{ij} is an element which is in the ith row and jth column of a square matrix A, then the product of $(-1)^{i+j}$ and the minor of a_{ij} is called COFACTOR of a_{ij}.It is denoted by A_{ij}.
General notation of cofactor is given by

$$A_{ij} = (-1)^{i+j} M_{ij} \tag{4}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{5}$$

then

$$A_{11} = Cofactorofa_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23} \tag{6}$$

and

$$A_{12} = Cofactorofa_{12} = (-1)^{1+2} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{31}a_{23}); and\ so\ on. \tag{7}$$

3. Properties of Determinants

3.1. $detA = detA^T$

THEOREM:If A is a square matrix,then $detA = detA^T$

PROOF:

$$LetA = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \tag{8}$$

now ,expanding determinant along the first column

$$detA = a_1[b_2c_3 - b_3c_2] - b_1[a_2c_3 - a_3c2] - c_1[a_2b_3 - a_3b_2] \tag{9}$$

$$And\ A^T = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \tag{10}$$

now, expanding determinant along the first row

$$detA^T = a_1[b_2c_3 - b_3c_2] - b_1[a_2c_3 - a_3c2] - c_1[a_2b_3 - a_3b_2] \tag{11}$$

Hence

$$detA = detA^T \tag{12}$$

3.2. If A is a triangular matrix then $detA = \prod_{i=1}^{n} a_{ij}$ for i=j.

THEOREM: If A is a triangular matrix then detA=

$$\prod_{i=1}^{n} a_{ij}\ ;\ for\ i = j \tag{13}$$

, where i=j represents diagonal elements

PROOF: Let us consider a lower triangular matrix

$$A = \begin{bmatrix} a_{11} & 0 \\ a_{12} & a_{22} \end{bmatrix} \tag{14}$$

Thus the $detA = a_{11}a_{22}$
Now, let us consider a 3x3 upper triangular matrix

$$LetA = \begin{bmatrix} a_1 & b_1 & b_3 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{bmatrix} \tag{15}$$

expanding determinant along first column

$$detA = a_1(b_2c_3) - 0(c_3b_1 - 0) + 0(c_2b_1 - b_2b_3) \tag{16}$$

And the

$$detA = a_1b_2c_3 \tag{17}$$

Thus for any triangular matrix the determinant of a matrix is given by product of its diagonal elements.

3.3. Determinant of a product of matrices

THEOREM: $det(AB) = det(A)det(B)$ $\forall$ nxn matrices

PROOF: Suppose A is a invertible. then there exist elementary row operations$E_k, ..., E_1$ such that

$$A = E_k...E_1. \tag{18}$$

Then

$$det(AB) = det(E_k...E_1B) \tag{19}$$
$$= det(E_k)det(E_{k-1}...E_1B) \tag{20}$$
$$= det(E_k)...det(E_1)det(B) \tag{21}$$
$$= det(E_k....E_1)det(B) \tag{22}$$
$$= det(A)det(B) \tag{23}$$

Thus A and B are two square matrices of same type, then det(AB)=det(A)det(B).

3.4. B is obtained by applying any one of row operation on A

3.4.1. $R_i \iff R_j$ then det(B)=det(A)

THEOREM: The value of the determinant remains unchanged if both rows and columns are interchanged.

PROOF:

$$LetA = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \tag{24}$$

now ,expanding determinant along the first column

$$detA = a_1[b_2c_3 - b_3c_2] - b_1[a_2c_3 - a_3c2] - c_1[a_2b_3 - a_3b_2] \tag{25}$$

and let B is the matrix formed from the matrix A by following the given operation $R_i \iff R_j$

$$And\ B = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \tag{26}$$

now, expanding determinant along the first row

$$detB = a_1[b_2c_3 - b_3c_2] - b_1[a_2c_3 - a_3c2] - c_1[a_2b_3 - a_3b_2] \tag{27}$$

Thus after the interchange of rows and column the determinant remains same.

3.4.2. $sR_i \longrightarrow R_i$ then det(B)=sdet(A)

THEOREM: The determinant is multiplied by the same scalar,if the row is transformed by that scalar times.

PROOF:

$$Let\ A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \tag{28}$$

$$detA = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{29}$$

now, expanding determinant along the first row

$$detA = a_1[b_2c_3 - b_3c_2] - b_1[a_2c_3 - a_3c2] - c_1[a_2b_3 - a_3b_2] \tag{30}$$

and let B is the matrix formed from the matrix A by following the given operation

$$R_1 \longrightarrow sR_1 \tag{31}$$

then,

$$B = \begin{bmatrix} sa_1 & sb_1 & sc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \tag{32}$$

$$B = \ s \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \tag{33}$$

applying det on both sides

$$detB = \ s \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{34}$$

$$Thus\ detB = sdetA \tag{35}$$

3.4.3. $sR_i + R_j \longrightarrow R_j$ then det(B)=det(A)

THEOREM: If the elements of a row(or a column) of a square matrix are added with k times the corresponding elements of another row(or column),then the value of the determinant of the matrix obtained is equal to the value of the determinant of the given matrix.

PROOF:

$$Let\ A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \tag{36}$$

$$detA = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \rightarrow detA = a_1[b_2c_3 - b_3c_2] - b_1[a_2c_3 - a_3c2] - c_1[a_2b_3 - a_3b_2] \tag{37}$$

and let B is the matrix formed from the matrix A by following the given column operation

$$C_1 \longrightarrow C_1 + sC_3 \tag{38}$$

$$B = \begin{bmatrix} a_1 + sc_1 & b_1 & c_1 \\ a_2 + sc_2 & b_2 & c_2 \\ a_3 + sc_3 & b_3 & c_3 \end{bmatrix} \tag{39}$$

$$B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} + \begin{bmatrix} sc_1 & b_1 & c_1 \\ sc_2 & b_2 & c_2 \\ sc_3 & b_3 & c_3 \end{bmatrix} \tag{40}$$

$$B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} + s \begin{bmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{bmatrix} \tag{41}$$

applying determinant on both sides

$$detB = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + s \begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix} \tag{42}$$

We know one of the determinants property which states as,If two rows or columns of a square matrix are identical, the value of the determinant of the matrix is 0

hence,we get

$$detB = detA + s(0) \tag{43}$$

Thus

$$detB = detA. \tag{44}$$

4. CONCLUSION:

Above are some of the properties of determinants and one of the main application of the determinants is to examine consistency of solutions for a set of system of linear equations.