PROPERTIES OF EIGEN VALUES & EIGEN VECTORS ALONG WITH PROOFS

Purnima Tadipathri

Abstract:

We can utilise Eigenvalues and Eigenvectors to reduce the dimension space. To elaborate, one of the key methodologies to improve efficiency in computationally intensive tasks is to reduce the dimensions after ensuring most of the key information is maintained.So understanding of eigen values & eigen vectors and their properties is important.

1. Introduction

Eigenvalues and Eigenvectors have their importance in linear differential equations to find a rate of change .

Generally, we define eigen values and eigen vectors as

EIGEN VALUE:

If A=$[a_{ij}]_{nxn}$ be any equation of order n, we can form the matrix $A - \lambda I$ where I is n^{th} order unit matrix.The determinant of this matrix is equal to zero i.e $|A - \lambda I| = 0$ is called characteristic equation of matrix A and roots of this equation are called Eigen values of matrix A.

- Eigenvalue can also be simply stated as scalar that is used to transform (stretch) an Eigenvector.

EIGEN VECTOR:

If λ is a eigen value of a nxn matrix A then non-zero vector X; $AX = \lambda$X is called eigen vector of matrix A corresponding to eigen value λ.

- Can also be defined as , If the new transformed vector is just a scaled form of the original vector then the original vector is known to be an eigenvector of the original matrix
.
- Vectors that have this characteristic are special vectors and they are known as eigenvectors. Eigenvectors can be used to represent a large dimensional matrix.
- And also a vector that undergoes pure scaling with the valueλ known as eigenvalue, without any rotation is known as the eigenvector.

2. Derivation of Eigen values and Eigen vectors

We use eigen values and eigen vectors to solve system of linear system of differential equations. let y be a vector and A is any nxn matrix, where

$$\frac{dy}{dt} = Ay; \tag{1}$$

and a solution that satisfies the differential equation is given as

$$y = e^{\lambda t}\overline{X}; \tag{2}$$

where $\overline{X}$ is a vector which is independent of time

from 1 and 2

$$\frac{d(e^{\lambda t})}{dt}\overline{X} = Ae^{\lambda t}\overline{X} \tag{3}$$

$$\lambda e^{\lambda t}\overline{X} = A\overline{X}e^{\lambda t} \tag{4}$$

divinding both sides with $e^{\lambda t}$

$$A\overline{X} = \lambda\overline{X} \tag{5}$$

where A is nxn matrix ,$\overline{X}$ is a eigenvector and λ is a eigenvalue.

3. PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS

3.1. 1.A and A^T will have the same eigen value

THEOREM: Let A be nxn square matrix, then A and A^T will have the same Eigen value

PROOF: As we know that the eigen values of a matrix are roots of its characteristic polynomial,

$$P(A) = det(A - \lambda I); \tag{6}$$

We have

$$P(A^T) = det(A^T - \lambda I) \qquad for A^T; \tag{7}$$

$$P(A^T) = det(A^T - \lambda I^T) \quad since : I^T = I; \tag{8}$$

$$P(A^T) = det[(A - \lambda I)]^T \tag{9}$$

$$P(A^T) = det(A - \lambda I) \tag{10}$$

since the determinant of A and A^T are same

Thus We obtain $P(A^T) = P(A)$ and we conclude that the eigenvalues of A and A^T are the same.

3.2. Eigenvalues of a scalar multiple matrix(ESMM)

THEOREM: Suppose A is a square matrix and λ is an eigenvalue of A, Then $\alpha\lambda$ is an eigenvalue of αA.

PROOF:we know that $A\overline{X} = \lambda\overline{X}$,Let us multiply a scalar α with A$\overline{X}$

$weget = \alpha A\overline{X}$(11) where: $A\overline{X} = \lambda\overline{X}$

$$= \alpha\lambda\overline{X} \tag{12}$$

$$= (\alpha\lambda)\overline{X} \tag{13}$$

Thus for an eigenvector $\overline{X}$,$\alpha\lambda$ is an eigenvalue of αA.

3.3. Eigen values of matrix powers(EOPM)

THEOREM: Suppose A is a square matrix,λ is an eigenvalue of A and $k \geq 0$ is an integer. Then λ^k is an eigenvalue of A^k.

PROOF:

3.3.1. METHOD 1:

If $A\overline{X} = \lambda\overline{X}$ then multipling by A yields

$$A(A\overline{X}) = A\lambda\overline{X} \tag{14}$$

$$A^2\overline{X} = \lambda(A\overline{X}) \tag{15}$$

$$A^2\overline{X} = \lambda(\lambda\overline{X}) \tag{16}$$

$$A^2\overline{X} = (\lambda)^2\overline{X} \tag{17}$$

Similarly for A^k,the eigenvalue will be λ^k.

3.3.2. METHOD 2:

now, let us multiply both sides with $A^{(k-1)}$ yeilds

$$A^{k-1}A\overline{X} = A^{k-1}(\lambda)\overline{X} \tag{18}$$

$$A^k\overline{X} = (\lambda)(\lambda^{k-1}\overline{X}) \tag{19}$$

$$A^k\overline{X} = (\lambda)^k\overline{X} \tag{20}$$

Thus,for every A that multiplied to both sides,the right side gains a factor λ,where A$\overline{X}$ can be substituted by $\lambda\overline{X}$ while the eigenvector remains the same.

3.4. Eigenvalues of the Inverse of a matrix(EIM)

THEOREM: Suppose A is a square non singular matrix and λ is an eigenvalue of A.Then λ^{-1} is an eigenvalue of the matrix A^{-1}

PROOF: For the existence of inverse of a matrix we need to assume that,'any of the eigenvalue of the given matrix A, the eigenvalue should be not equal to zero $\lambda \neq 0$'.

considering

$$A\overline{X} = (\lambda)\overline{X} \tag{21}$$

multiplying both sides with A^{-1}

$$A^{-1}A\overline{X} = A^{-1}(\lambda)\overline{X} \tag{22}$$

$$\overline{X} = (\lambda)A^{-1}\overline{X} \tag{23}$$

divide both sides by λ,since λ is a non-zero value, if we assume that matrix has inverse and rewriting it as

$$(\frac{1}{\lambda})\overline{X} = A^{-1}\overline{X} \tag{24}$$

$$A^{-1}\overline{X} = (\frac{1}{\lambda})\overline{X} \tag{25}$$

Thus for a inverse matrix A^{-1}, the eigenvector$\overline{X}$ remains same and λ^{-1} is the corresponding eigenvalue of the inverse matrix A^{-1}.

3.5. Determinant of a Matrix using Eigenvalues

THEOREM: For a square matrix A of the order nxn ,let λ be the eigenvalue, then the determinant of A is given by

$$detA = \prod_{i=1}^{n} \lambda_i \tag{26}$$

PROOF: Let us assume that there exits an nxn invertible matrix P such that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & a_{12} & \cdots & a_{1n} \\ 0 & \lambda_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{27}$$

this forms an upper triangular matrix and diagonal entries are eigenvalues

Since the determinant of triangular matrix is product of its diagonal entries,we have

$$\prod_{i=1}^{n} \lambda_i = det(P^{-1}AP) \tag{28}$$

$$= det(P^{-1)}det(A)det(P) \tag{29}$$

$$= det(P)^{-1}det(A)det(P) \tag{30}$$

$$= detA \tag{31}$$

Thus determinant of A matrix is given by product of its eigenvalues.

Note1:-The determinant of the inverse of an invertible matrix is the inverse of the determinant: det(A-1) = 1 / det(A) .

Note2:-Similar matrices have the same determinant; that is, if S is invertible and of the same size as A then det(S A S-1) = det(A).

3.6. Trace of a matrix using Eigenvalues

THEOREM: For a square matrix A of the order nxn ,let λ be the eigenvalue, then the trace of A is given by

$$tr(A) = \sum_{i=1}^{n} \lambda_i \tag{32}$$

PROOF: Let us assume that there exits an nxn invertible matrix P such that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & a_{12} & \cdots & a_{1n} \\ 0 & \lambda_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{33}$$

we apply trace on both sides for above equation

$$tra = \sum_{i=1}^{n} \lambda_i = tr(P^{-1}AP) \tag{34}$$

$$= tr(A) \tag{35}$$

Thus we obtained the result

$$tr(A) = \sum_{i=1}^{n} \lambda_i \tag{36}$$

where the trace of a matrix is given by sum of its eigen values.

4. USECASES OF EIGEN VALUES AND EIGEN VECTORS:

1) We can represent a large set of information in a matrix. Performing computations on a large matrix is a very slow process. To elaborate, one of the key methodologies to improve efficiency in computationally intensive tasks is to reduce the dimensions after ensuring most of the key information is maintained.

2) Hence, one eigenvalue and eigenvector are used to capture key information that is stored in a large matrix. This technique can also be used to improve the performance of data churning components.

3) Component analysis is one of the key strategies that is utilised to reduce dimension space without losing valuable information. The core of component analysis (PCA) is built on the concept of eigenvalues and eigenvectors.

4) Additionally, eigenvectors and eigenvalues are used in facial recognition techniques such as EigenFaces.

5. CONCLUSION:

Eigenvalues and Eigenvectors are one of core concepts to understand in data science.The technique of Eigenvectors and Eigenvalues are used to compress the data and they can help us improve efficiency in computationally intensive tasks.