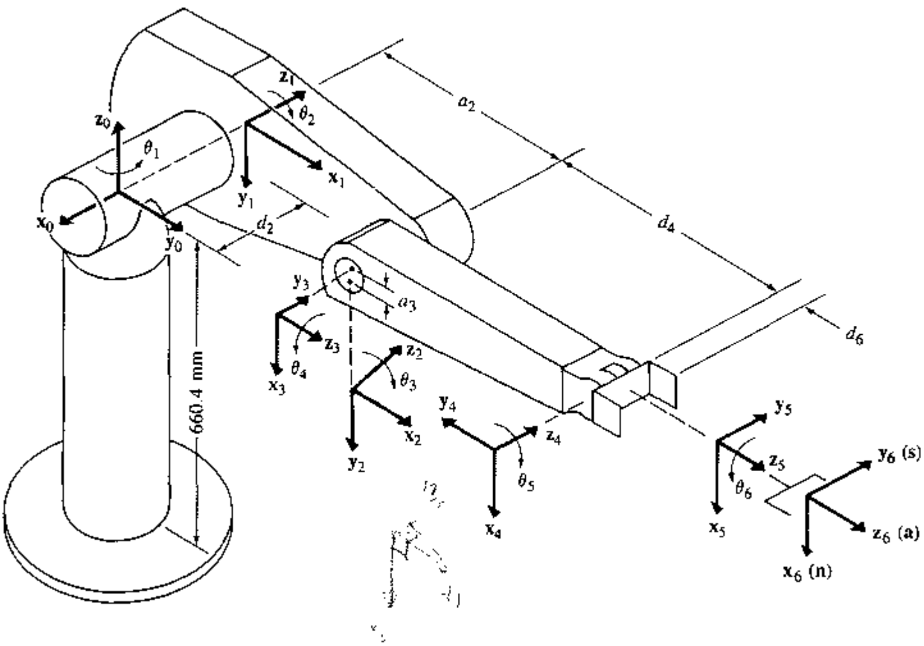
or prismatic joint. Referring to Fig. 2.10, these four parameters are defined as follows:

θ_i is the joint angle from the $\mathbf{x}_{i-1}$ axis to the $\mathbf{x}_i$ axis about the $\mathbf{z}_{i-1}$ axis (using the right-hand rule).

d_i is the distance from the origin of the $(i-1)$th coordinate frame to the intersection of the $\mathbf{z}_{i-1}$ axis with the $\mathbf{x}_i$ axis along the $\mathbf{z}_{i-1}$ axis.

a_i is the offset distance from the intersection of the $\mathbf{z}_{i-1}$ axis with the $\mathbf{x}_i$ axis to the origin of the ith frame along the $\mathbf{x}_i$ axis (or the shortest distance between the $\mathbf{z}_{i-1}$ and $\mathbf{z}_i$ axes).

α_i is the offset angle from the $\mathbf{z}_{i-1}$ axis to the $\mathbf{z}_i$ axis about the $\mathbf{x}_i$ axis (using the right-hand rule).



PUMA robot arm link coordinate parameters					
Joint i	θ_i	α_i	a_i	d_i	Joint range
1	90	−90	0	0	−160 to +160
2	0	0	431.8 mm	149.09 mm	−225 to 45
3	90	90	−20.32 mm	0	−45 to 225
4	0	−90	0	433.07 mm	−110 to 170
5	0	90	0	0	−100 to 100
6	0	0	0	56.25 mm	−266 to 266

Figure 2.11 Establishing link coordinate systems for a PUMA robot.

geometry, various arm configurations of a PUMA-like robot (Fig. 2.11) can be identified with the assistance of three configuration indicators (ARM, ELBOW, and WRIST)—two associated with the solution of the first three joints and the other with the last three joints. For a six-axis PUMA-like robot arm, there are four possible solutions to the first three joints and for each of these four solutions there are two possible solutions to the last three joints. The first two configuration indicators allow one to determine one solution from the possible four solutions for the first three joints. Similarly, the third indicator selects a solution from the possible two solutions for the last three joints. The arm configuration indicators are prespecified by a user for finding the inverse solution. The solution is calculated in two stages. First, a position vector pointing from the shoulder to the wrist is derived. This is used to derive the solution of each joint i ($i = 1, 2, 3$) for the first three joints by looking at the projection of the position vector onto the $\mathbf{x}_{i-1}\mathbf{y}_{i-1}$ plane. The last three joints are solved using the calculated joint solution from the first three joints, the orientation submatrices of ${}^0\mathbf{T}_i$ and ${}^{i-1}\mathbf{A}_i$ ($i = 4, 5, 6$), and the projection of the link coordinate frames onto the $\mathbf{x}_{i-1}\mathbf{y}_{i-1}$ plane. From the geometry, one can easily find the arm solution consistently. As a verification of the joint solution, the arm configuration indicators can be determined from the corresponding decision equations which are functions of the joint angles. With appropriate modification and adjustment, this approach can be generalized to solve the inverse kinematics problem of most present day industrial robots with rotary joints.

If we are given ${}^{\text{ref}}\mathbf{T}_{\text{tool}}$, then we can find ${}^0\mathbf{T}_6$ by premultiplying and postmultiplying ${}^{\text{ref}}\mathbf{T}_{\text{tool}}$ by $\mathbf{B}^{-1}$ and $\mathbf{H}^{-1}$, respectively, and the joint-angle solution can be applied to ${}^0\mathbf{T}_6$ as desired.

$$
{}^0\mathbf{T}_6 \equiv \mathbf{T} = \mathbf{B}^{-1}\,{}^{\text{ref}}\mathbf{T}_{\text{tool}}\,\mathbf{H}^{-1} = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.3-30}
$$

Definition of Various Arm Configurations. For the PUMA robot arm shown in Fig. 2.11 (and other rotary robot arms), various arm configurations are defined according to human arm geometry and the link coordinate systems which are established using Algorithm 2.1 as (Fig. 2.19)

RIGHT (shoulder) ARM: Positive θ_2 moves the wrist in the *positive* $\mathbf{z}_0$ direction while joint 3 is not activated.

LEFT (shoulder) ARM: Positive θ_2 moves the wrist in the *negative* $\mathbf{z}_0$ direction while joint 3 is not activated.

ABOVE ARM (elbow above wrist): Position of the wrist of the

$\left\{ \begin{matrix} \text{RIGHT} \\ \text{LEFT} \end{matrix} \right\}$ arm with respect to the shoulder coordinate system has

$\begin{Bmatrix} \text{negative} \\ \text{positive} \end{Bmatrix}$ coordinate value along the $\mathbf{y}_2$ axis.

BELOW ARM (elbow below wrist): Position of the wrist of the

$\begin{Bmatrix} \text{RIGHT} \\ \text{LEFT} \end{Bmatrix}$ arm with respect to the shoulder coordinate system has

$\begin{Bmatrix} \text{positive} \\ \text{negative} \end{Bmatrix}$ coordinate value along the $\mathbf{y}_2$ axis.

WRIST DOWN: The $\mathbf{s}$ unit vector of the hand coordinate system and the $\mathbf{y}_5$ unit vector of the $(\mathbf{x}_5, \mathbf{y}_5, \mathbf{z}_5)$ coordinate system have a positive dot product.

WRIST UP: The s unit vector of the hand coordinate system and the $\mathbf{y}_5$ unit vector of the $(\mathbf{x}_5, \mathbf{y}_5, \mathbf{z}_5)$ coordinate system have a negative dot product.

(Note that the definition of the arm configurations with respect to the link coordinate systems may have to be slightly modified if one uses different link coordinate systems.)

With respect to the above definition of various arm configurations, two arm configuration *indicators* (ARM and ELBOW) are defined for each arm configuration. These two indicators are combined to give one solution out of the possible four joint solutions for the first three joints. For each of the four arm configurations (Fig. 2.19) defined by these two indicators, the third indicator (WRIST) gives one of the two possible joint solutions for the last three joints. These three indicators can be defined as:

$$\text{ARM} = \begin{cases} +1 & \text{RIGHT arm} \\ -1 & \text{LEFT arm} \end{cases} \tag{2.3-31}$$

$$\text{ELBOW} = \begin{cases} +1 & \text{ABOVE arm} \\ -1 & \text{BELOW arm} \end{cases} \tag{2.3-32}$$

$$\text{WRIST} = \begin{cases} +1 & \text{WRIST DOWN} \\ -1 & \text{WRIST UP} \end{cases} \tag{2.3-33}$$

In addition to these indicators, the user can define a "FLIP" toggle as:

$$\text{FLIP} = \begin{cases} +1 & \text{Flip the wrist orientation} \\ -1 & \text{Do not flip the wrist orientation} \end{cases} \tag{2.3-34}$$

The signed values of these indicators and the toggle are prespecified by a user for finding the inverse kinematics solution. These indicators can also be set from the knowledge of the joint angles of the robot arm using the corresponding decision equations. We shall later give the decision equations that determine these indicator

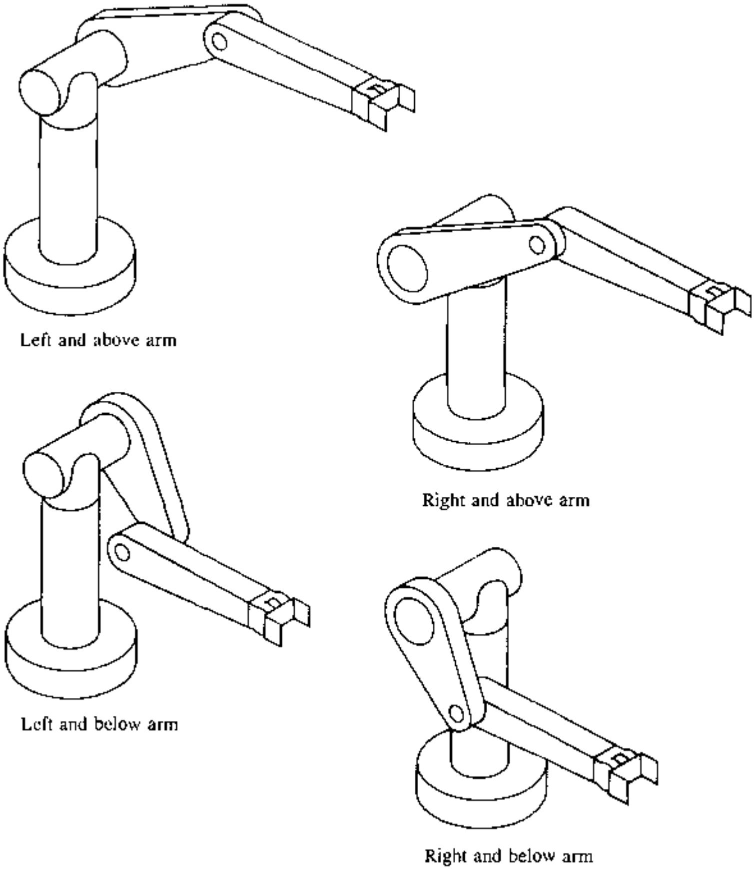


Figure 2.19 Definition of various arm configurations.

values. The decision equations can be used as a verification of the inverse kinematics solution.

Arm Solution for the First Three Joints. From the kinematics diagram of the PUMA robot arm in Fig. 2.11, we define a position vector $\mathbf{p}$ which points from the origin of the shoulder coordinate system ($\mathbf{x}_0$, $\mathbf{y}_0$, $\mathbf{z}_0$) to the point where the last three joint axes intersect as (see Fig. 2.14):

$$\mathbf{p} = \mathbf{p}_6 - d_6\mathbf{a} = (p_x, p_y, p_z)^T \tag{2.3-35}$$

which corresponds to the position vector of ${}^0\mathbf{T}_4$:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} C_1(a_2C_2 + a_3C_{23} + d_4S_{23}) - d_2S_1 \\ S_1(a_2C_2 + a_3C_{23} + d_4S_{23}) + d_2C_1 \\ d_4C_{23} - a_3S_{23} - a_2S_2 \end{bmatrix} \tag{2.3-36}$$

Joint 1 solution. If we project the position vector **p** onto the $\mathbf{x}_0\mathbf{y}_0$ plane as in Fig. 2.20, we obtain the following equations for solving θ_1:

$$\theta_1^L = \phi - \alpha \qquad \theta_1^R = \pi + \phi + \alpha \tag{2.3-37}$$

$$r = \sqrt{p_x^2 + p_y^2 - d_2^2} \qquad R = \sqrt{p_x^2 + p_y^2} \tag{2.3-38}$$

$$\sin\phi = \frac{p_y}{R} \qquad \cos\phi = \frac{p_x}{R} \tag{2.3-39}$$

$$\sin\alpha = \frac{d_2}{R} \qquad \cos\alpha = \frac{r}{R} \tag{2.3-40}$$

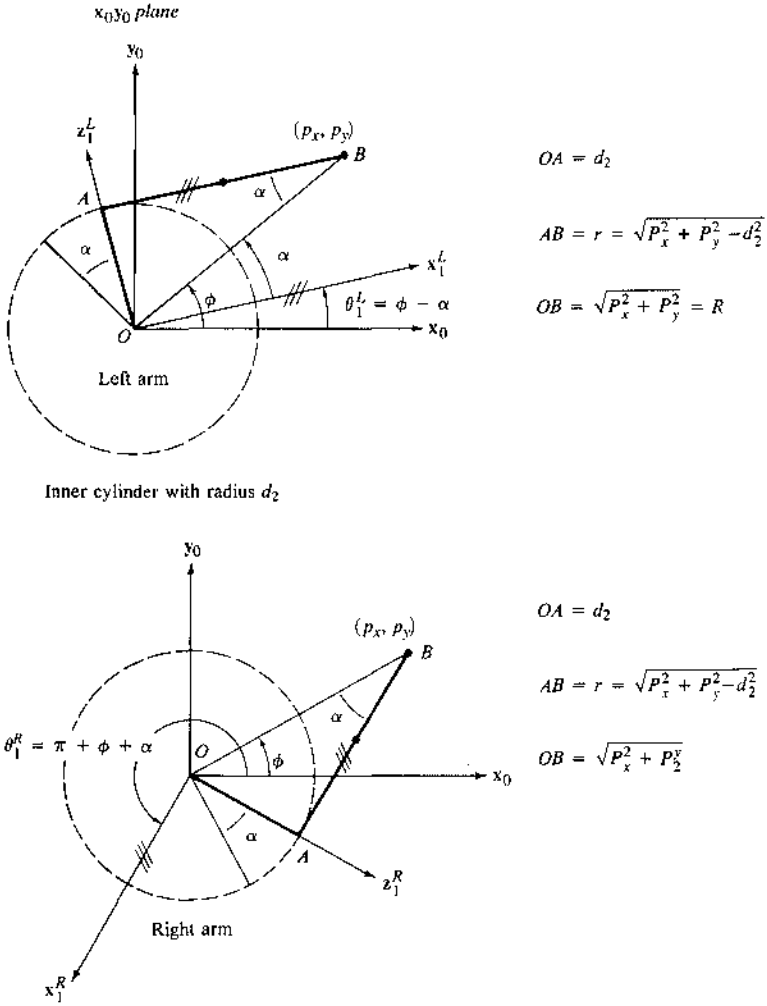


Figure 2.20 Solution for joint 1.

where the superscripts L and R on joint angles indicate the LEFT/RIGHT arm configurations. From Eqs. (2.3-37) to (2.3-40), we obtain the sine and cosine functions of θ_1 for LEFT/RIGHT arm configurations:

$$\sin\theta_1^L = \sin(\phi - \alpha) = \sin\phi\cos\alpha - \cos\phi\sin\alpha = \frac{p_y r - p_x d_2}{R^2} \tag{2.3-41}$$

$$\cos\theta_1^L = \cos(\phi - \alpha) = \cos\phi\cos\alpha + \sin\phi\sin\alpha = \frac{p_x r + p_y d_2}{R^2} \tag{2.3-42}$$

$$\sin\theta_1^R = \sin(\pi + \phi + \alpha) = \frac{-p_y r - p_x d_2}{R^2} \tag{2.3-43}$$

$$\cos\theta_1^R = \cos(\pi + \phi + \alpha) = \frac{-p_x r + p_y d_2}{R^2} \tag{2.3-44}$$

Combining Eqs. (2.3-41) to (2.3-44) and using the ARM indicator to indicate the LEFT/RIGHT arm configuration, we obtain the sine and cosine functions of θ_1, respectively:

$$\sin\theta_1 = \frac{-\text{ARM}\ p_y\sqrt{p_x^2 + p_y^2 - d_2^2} - p_x d_2}{p_x^2 + p_y^2} \tag{2.3-45}$$

$$\cos\theta_1 = \frac{-\text{ARM}\ p_x\sqrt{p_x^2 + p_y^2 - d_2^2} + p_y d_2}{p_x^2 + p_y^2} \tag{2.3-46}$$

where the positive square root is taken in these equations and ARM is defined as in Eq. (2.3-31). In order to evaluate θ_1 for $-\pi \leqslant \theta_1 \leqslant \pi$, an arc tangent function as defined in Eq. (2.3-7) will be used. From Eqs. (2.3-45) and (2.3-46), and using Eq. (2.3-7), θ_1 is found to be:

$$\begin{aligned}\theta_1 &= \tan^{-1}\left[\frac{\sin\theta_1}{\cos\theta_1}\right]\\ &= \tan^{-1}\left[\frac{-\text{ARM}\ p_y\sqrt{p_x^2 + p_y^2 - d_2^2} - p_x d_2}{-\text{ARM}\ p_x\sqrt{p_x^2 + p_y^2 - d_2^2} + p_y d_2}\right] \qquad -\pi \leqslant \theta_1 \leqslant \pi\end{aligned} \tag{2.3-47}$$

Joint 2 solution. To find joint 2, we project the position vector $\mathbf{p}$ onto the $\mathbf{x}_1\mathbf{y}_1$ plane as shown in Fig. 2.21. From Fig. 2.21, we have four different arm configurations. Each arm configuration corresponds to different values of joint 2 as shown in Table 2.3, where $0° \leqslant \alpha \leqslant 360°$ and $0° \leqslant \beta \leqslant 90°$.

Table 2.3 Various arm configurations for joint 2

Arm configurations	θ_2	ARM	ELBOW	ARM · ELBOW
LEFT and ABOVE arm	$\alpha - \beta$	−1	+1	−1
LEFT and BELOW arm	$\alpha + \beta$	−1	−1	+1
RIGHT and ABOVE arm	$\alpha + \beta$	+1	+1	+1
RIGHT and BELOW arm	$\alpha - \beta$	+1	−1	−1

From the above table, θ_2 can be expressed in one equation for different arm and elbow configurations using the ARM and ELBOW indicators as:

$$\theta_2 = \alpha + (\text{ARM} \cdot \text{ELBOW})\beta = \alpha + K \cdot \beta \tag{2.3-48}$$

where the combined arm configuration indicator $K = \text{ARM} \cdot \text{ELBOW}$ will give an appropriate signed value and the "dot" represents a multiplication operation on the indicators. From the arm geometry in Fig. 2.21, we obtain:

$$R = \sqrt{p_x^2 + p_y^2 + p_z^2 - d_2^2} \qquad r = \sqrt{p_x^2 + p_y^2 - d_2^2} \tag{2.3-49}$$

$$\sin\alpha = -\frac{p_z}{R} = -\frac{p_z}{\sqrt{p_x^2 + p_y^2 + p_z^2 - d_2^2}} \tag{2.3-50}$$

$$\cos\alpha = -\frac{\text{ARM} \cdot r}{R} = -\frac{\text{ARM} \cdot \sqrt{p_x^2 + p_y^2 - d_2^2}}{\sqrt{p_x^2 + p_y^2 + p_z^2 - d_2^2}} \tag{2.3-51}$$

$$\cos\beta = \frac{a_2^2 + R^2 - (d_4^2 + a_3^2)}{2a_2 R} \tag{2.3-52}$$

$$= \frac{p_x^2 + p_y^2 + p_z^2 + a_2^2 - d_2^2 - (d_4^2 + a_3^2)}{2a_2\sqrt{p_x^2 + p_y^2 + p_z^2 - d_2^2}}$$

$$\sin\beta = \sqrt{1 - \cos^2\beta} \tag{2.3-53}$$

From Eqs. (2.3-48) to (2.3-53), we can find the sine and cosine functions of θ_2:

$$\sin\theta_2 = \sin(\alpha + K \cdot \beta) = \sin\alpha\cos(K \cdot \beta) + \cos\alpha\sin(K \cdot \beta)$$

$$= \sin\alpha\cos\beta + (\text{ARM} \cdot \text{ELBOW})\cos\alpha\sin\beta \tag{2.3-54}$$

$$\cos\theta_2 = \cos(\alpha + K \cdot \beta)$$

$$= \cos\alpha\cos\beta - (\text{ARM} \cdot \text{ELBOW})\sin\alpha\sin\beta \tag{2.3-55}$$

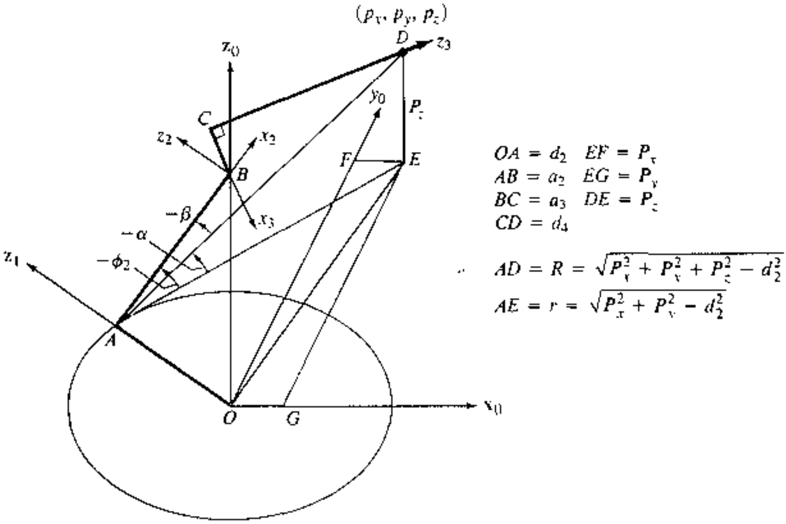


Figure 2.21 Solution for joint 2.

From Eqs. (2.3-54) and (2.3-55), we obtain the solution for θ_2:

$$\theta_2 = \tan^{-1}\left[\frac{\sin\theta_2}{\cos\theta_2}\right] \qquad -\pi \leqslant \theta_2 \leqslant \pi \tag{2.3-56}$$

Joint 3 solution. For joint 3, we project the position vector $\mathbf{p}$ onto the $\mathbf{x}_2\mathbf{y}_2$ plane as shown in Fig. 2.22. From Fig. 2.22, we again have four different arm configurations. Each arm configuration corresponds to different values of joint 3 as shown in Table 2.4, where $({}^2\mathbf{p}_4)_y$ is the y component of the position vector from the origin of $(\mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2)$ to the point where the last three joint axes intersect.

From the arm geometry in Fig. 2.22, we obtain the following equations for finding the solution for θ_3:

$$R = \sqrt{p_x^2 + p_y^2 + p_z^2 - d_2^2} \tag{2.3-57}$$

$$\cos\phi = \frac{a_2^2 + (d_4^2 + a_3^2) - R^2}{2a_2\sqrt{d_4^2 + a_3^2}} \tag{2.3-58}$$

$$\sin\phi = \text{ARM} \cdot \text{ELBOW}\sqrt{1 - \cos^2\phi}$$

$$\sin\beta = \frac{d_4}{\sqrt{d_4^2 + a_3^2}} \qquad \cos\beta = \frac{|a_3|}{\sqrt{d_4^2 + a_3^2}} \tag{2.3-59}$$

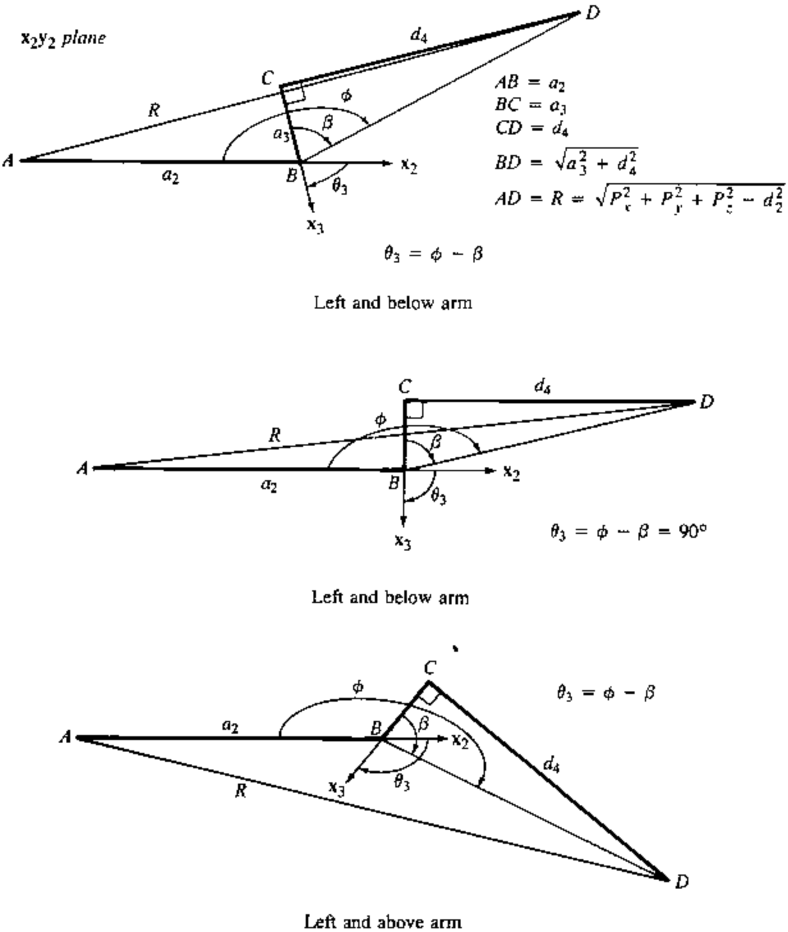


Figure 2.22 Solution for joint 3.

From Table 2.4, we can express θ_3 in one equation for different arm configurations:

$$\theta_3 = \phi - \beta \tag{2.3-60}$$

From Eq. (2.3-60), the sine and cosine functions of θ_3 are, respectively,

$$\sin\theta_3 = \sin(\phi - \beta) = \sin\phi\cos\beta - \cos\phi\sin\beta \tag{2.3-61}$$

$$\cos\theta_3 = \cos(\phi - \beta) = \cos\phi\cos\beta + \sin\phi\sin\beta \tag{2.3-62}$$

From Eqs. (2.3-61) and (2.3-62), and using Eqs. (2.3-57) to (2.3-59), we find the solution for θ_3:

Table 2.4 Various arm configurations for joint 3

Arm configurations	$({}^2\mathbf{p}_4)_y$	θ_3	ARM	ELBOW	ARM · ELBOW
LEFT and ABOVE arm	$\geqslant 0$	$\phi - \beta$	-1	$+1$	-1
LEFT and BELOW arm	$\leqslant 0$	$\phi - \beta$	-1	-1	$+1$
RIGHT and ABOVE arm	$\leqslant 0$	$\phi - \beta$	$+1$	$+1$	$+1$
RIGHT and BELOW arm	$\geqslant 0$	$\phi - \beta$	$+1$	-1	-1

$$\theta_3 = \tan^{-1}\left[\frac{\sin\theta_3}{\cos\theta_3}\right] \qquad -\pi \leqslant \theta_3 \leqslant \pi \tag{2.3-63}$$

Arm Solution for the Last Three Joints. Knowing the first three joint angles, we can evaluate the ${}^0\mathbf{T}_3$ matrix which is used extensively to find the solution of the last three joints. The solution of the last three joints of a PUMA robot arm can be found by setting these joints to meet the following criteria:

1. Set joint 4 such that a rotation about joint 5 will align the axis of motion of joint 6 with the given approach vector (**a** of **T**).
2. Set joint 5 to align the axis of motion of joint 6 with the approach vector.
3. Set joint 6 to align the given orientation vector (or sliding vector or $\mathbf{y}_6$) and normal vector.

Mathematically the above criteria respectively mean:

$$\mathbf{z}_4 = \frac{\pm(\mathbf{z}_3 \times \mathbf{a})}{\|\mathbf{z}_3 \times \mathbf{a}\|} \qquad \text{given } \mathbf{a} = (a_x, a_y, a_z)^T \tag{2.3-64}$$

$$\mathbf{a} = \mathbf{z}_5 \qquad \text{given } \mathbf{a} = (a_x, a_y, a_z)^T \tag{2.3-65}$$

$$\mathbf{s} = \mathbf{y}_6 \qquad \text{given } \mathbf{s} = (s_x, s_y, s_z)^T \text{ and } \mathbf{n} = (n_x, n_y, n_z)^T \tag{2.3-66}$$

In Eq. (2.3-64), the vector cross product may be taken to be positive or negative. As a result, there are two possible solutions for θ_4. If the vector cross product is zero (i.e., $\mathbf{z}_3$ is parallel to **a**), it indicates the degenerate case. This happens when the axes of rotation for joint 4 and joint 6 are parallel. It indicates that at this particular arm configuration, a five-axis robot arm rather than a six-axis one would suffice.

Joint 4 solution. Both orientations of the wrist (UP and DOWN) are defined by looking at the orientation of the hand coordinate frame (**n**, **s**, **a**) with respect to the $(\mathbf{x}_5, \mathbf{y}_5, \mathbf{z}_5)$ coordinate frame. The sign of the vector cross product in Eq. (2.3-64) cannot be determined without referring to the orientation of either the **n** or **s** unit vector with respect to the $\mathbf{x}_5$ or $\mathbf{y}_5$ unit vector, respectively, which have a fixed relation with respect to the $\mathbf{z}_4$ unit vector from the assignment of the link coordinate frames. (From Fig. 2.11, we have the $\mathbf{z}_4$ unit vector pointing at the same direction as the $\mathbf{y}_5$ unit vector.)

We shall start with the assumption that the vector cross product in Eq. (2.3-64) has a positive sign. This can be indicated by an orientation indicator Ω which is defined as:

$$\Omega = \begin{cases} 0 & \text{if in the degenerate case} \\ \mathbf{s} \cdot \mathbf{y}_5 & \text{if } \mathbf{s} \cdot \mathbf{y}_5 \neq 0 \\ \mathbf{n} \cdot \mathbf{y}_5 & \text{if } \mathbf{s} \cdot \mathbf{y}_5 = 0 \end{cases} \tag{2.3-67}$$

From Fig. 2.11, $\mathbf{y}_5 = \mathbf{z}_4$, and using Eq. (2.3-64), the orientation indicator Ω can be rewritten as:

$$\Omega = \begin{cases} 0 & \text{if in the degenerate case} \\ \mathbf{s} \cdot \dfrac{(\mathbf{z}_3 \times \mathbf{a})}{\|\mathbf{z}_3 \times \mathbf{a}\|} & \text{if } \mathbf{s} \cdot (\mathbf{z}_3 \times \mathbf{a}) \neq 0 \\ \mathbf{n} \cdot \dfrac{(\mathbf{z}_3 \times \mathbf{a})}{\|\mathbf{z}_3 \times \mathbf{a}\|}, & \text{if } \mathbf{s} \cdot (\mathbf{z}_3 \times \mathbf{a}) = 0 \end{cases} \tag{2.3-68}$$

If our assumption of the sign of the vector cross product in Eq. (2.3-64) is not correct, it will be corrected later using the combination of the WRIST indicator and the orientation indicator Ω. The Ω is used to indicate the initial orientation of the $\mathbf{z}_4$ unit vector (positive direction) from the link coordinate systems assignment, while the WRIST indicator specifies the user's preference of the orientation of the wrist subsystem according to the definition given in Eq. (2.3-33). If both the orientation Ω and the WRIST indicators have the same sign, then the assumption of the sign of the vector cross product in Eq. (2.3-64) is correct. Various wrist orientations resulting from the combination of the various values of the WRIST and orientation indicators are tabulated in Table 2.5.

Table 2.5 Various orientations for the wrist

Wrist orientation	$\Omega = \mathbf{s} \cdot \mathbf{y}_5$ or $\mathbf{n} \cdot \mathbf{y}_5$	WRIST	$M =$ WRIST sign (Ω)
DOWN	≥ 0	+1	+1
DOWN	< 0	+1	−1
UP	≥ 0	−1	−1
UP	< 0	−1	+1

Again looking at the projection of the coordinate frame $(\mathbf{x}_4, \mathbf{y}_4, \mathbf{z}_4)$ on the $\mathbf{x}_3\mathbf{y}_3$ plane and from Table 2.5 and Fig. 2.23, it can be shown that the following are true (see Fig. 2.23):

$$\sin\theta_4 = -M(\mathbf{z}_4 \cdot \mathbf{x}_3) \qquad \cos\theta_4 = M(\mathbf{z}_4 \cdot \mathbf{y}_3) \tag{2.3-69}$$

where $\mathbf{x}_3$ and $\mathbf{y}_3$ are the x and y column vectors of ${}^0\mathbf{T}_3$, respectively, $M =$ WRIST sign (Ω), and the sign function is defined as:

$$\operatorname{sign}(x) = \begin{cases} +1 & \text{if } x \geqslant 0 \\ -1 & \text{if } x < 0 \end{cases} \tag{2.3-70}$$

Thus, the solution for θ_4 with the orientation and WRIST indicators is:

$$\theta_4 = \tan^{-1}\left[\frac{\sin\theta_4}{\cos\theta_4}\right]$$

$$= \tan^{-1}\left[\frac{M(C_1 a_y - S_1 a_x)}{M(C_1 C_{23} a_x + S_1 C_{23} a_y - S_{23} a_z)}\right] \qquad -\pi \leqslant \theta_4 \leqslant \pi \tag{2.3-71}$$

If the degenerate case occurs, any convenient value may be chosen for θ_4 as long as the orientation of the wrist (UP/DOWN) is satisfied. This can always be ensured by setting θ_4 equals to the current value of θ_4. In addition to this, the user can turn on the FLIP toggle to obtain the other solution of θ_4, that is, $\theta_4 = \theta_4 + 180°$.

Joint 5 solution. To find θ_5, we use the criterion that aligns the axis of rotation of joint 6 with the approach vector (or $\mathbf{a} = \mathbf{z}_5$). Looking at the projection of the

$\sin\theta_4 = -(\mathbf{z}_4 \cdot \mathbf{x}_3)$

$\cos\theta_4 = \mathbf{z}_4 \cdot \mathbf{y}_3$

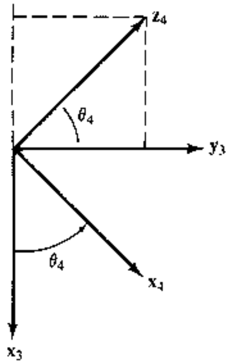


Figure 2.23 Solution for joint 4.

coordinate frame ($\mathbf{x}_5$, $\mathbf{y}_5$, $\mathbf{z}_5$) on the $\mathbf{x}_4\mathbf{y}_4$ plane, it can be shown that the following are true (see Fig. 2.24):

$$\sin\theta_5 = \mathbf{a} \cdot \mathbf{x}_4 \qquad \cos\theta_5 = -(\mathbf{a} \cdot \mathbf{y}_4) \tag{2.3-72}$$

where $\mathbf{x}_4$ and $\mathbf{y}_4$ are the x and y column vectors of ${}^0\mathbf{T}_4$, respectively, and $\mathbf{a}$ is the approach vector. Thus, the solution for θ_5 is:

$$\theta_5 = \tan^{-1}\left[\frac{\sin\theta_5}{\cos\theta_5}\right] \qquad -\pi \leq \theta_5 \leq \pi$$

$$= \tan^{-1}\left[\frac{(C_1C_{23}C_4 - S_1S_4)a_x + (S_1C_{23}C_4 + C_1S_4)a_y - C_4S_{23}a_z}{C_1S_{23}a_x + S_1S_{23}a_y + C_{23}a_z}\right] \tag{2.3-73}$$

If $\theta_5 \approx 0$, then the degenerate case occurs.

Joint 6 solution. Up to now, we have aligned the axis of joint 6 with the approach vector. Next, we need to align the orientation of the gripper to ease picking up the object. The criterion for doing this is to set $\mathbf{s} = \mathbf{y}_6$. Looking at the projection of the hand coordinate frame ($\mathbf{n}$, $\mathbf{s}$, $\mathbf{a}$) on the $\mathbf{x}_5\mathbf{y}_5$ plane, it can be shown that the following are true (see Fig. 2.25):

$$\sin\theta_6 = \mathbf{n} \cdot \mathbf{y}_5 \qquad \cos\theta_6 = \mathbf{s} \cdot \mathbf{y}_5 \tag{2.3-74}$$

where $\mathbf{y}_5$ is the y column vector of ${}^0\mathbf{T}_5$ and $\mathbf{n}$ and $\mathbf{s}$ are the normal and sliding vectors of ${}^0\mathbf{T}_6$, respectively. Thus, the solution for θ_6 is:

$$\theta_6 = \tan^{-1}\left[\frac{\sin\theta_6}{\cos\theta_6}\right] \qquad -\pi \leq \theta_6 \leq \pi$$

$$= \tan^{-1}\left[\frac{(-S_1C_4 - C_1C_{23}S_4)n_x + (C_1C_4 - S_1C_{23}S_4)n_y + (S_4S_{23})n_z}{(-S_1C_4 - C_1C_{23}S_4)s_x + (C_1C_4 - S_1C_{23}S_4)s_y + (S_4S_{23})s_z}\right] \tag{2.3-75}$$

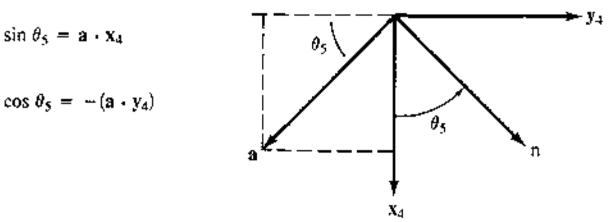


Figure 2.24 Solution for joint 5.

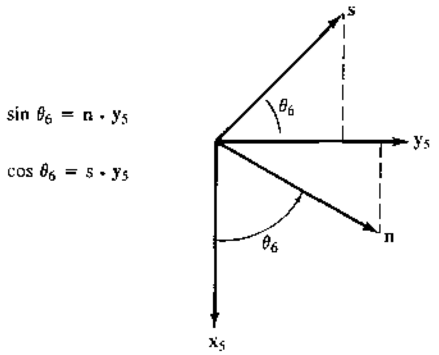


Figure 2.25 Solution for joint 6.

The above derivation of the inverse kinematics solution of a PUMA robot arm is based on the geometric interpretation of the position of the endpoint of link 3 and the hand (or tool) orientation requirement. There is one pitfall in the above derivation for θ_4, θ_5, and θ_6. The criterion for setting the axis of motion of joint 5 equal to the cross product of $\mathbf{z}_3$ and $\mathbf{a}$ may not be valid when $\sin\theta_5 \approx 0$, which means that $\theta_5 \approx 0$. In this case, the manipulator becomes *degenerate* with both the axes of motion of joints 4 and 6 aligned. In this state, only the sum of θ_4 and θ_6 is significant. If the degenerate case occurs, then we are free to choose any value for θ_4, and usually its current value is used and then we would like to have $\theta_4 + \theta_6$ equal to the total angle required to align the sliding vector $\mathbf{s}$ and the normal vector $\mathbf{n}$. If the FLIP toggle is on (i.e., FLIP = 1), then $\theta_4 = \theta_4 + \pi$, $\theta_5 = -\theta_5$, and $\theta_6 = \theta_6 + \pi$.

In summary, there are eight solutions to the inverse kinematics problem of a six-joint PUMA-like robot arm. The first three-joint solution (θ_1, θ_2, θ_3) positions the arm while the last three-joint solution, (θ_4, θ_5, θ_6), provides appropriate orientation for the hand. There are four solutions for the first three-joint solutions—two for the right shoulder arm configuration and two for the left shoulder arm configuration. For each arm configuration, Eqs. (2.3-47), (2.3-56), (2.3-63), (2.3-71), (2.3-73), and (2.3-75) give one set of solutions (θ_1, θ_2, θ_3, θ_4, θ_5, θ_6) and (θ_1, θ_2, θ_3, $\theta_4 + \pi$, $-\theta_5$, $\theta_6 + \pi$) (with the FLIP toggle on) gives another set of solutions.

Decision Equations for the Arm Configuration Indicators. The solution for the PUMA-like robot arm derived in the previous section is not unique and depends on the arm configuration indicators specified by the user. These arm configuration indicators (ARM, ELBOW, and WRIST) can also be determined from the joint angles. In this section, we derive the respective decision equation for each arm configuration indicator. The signed value of the decision equation (positive, zero, or negative) provides an indication of the arm configuration as defined in Eqs. (2.3-31) to (2.3-33).

For the ARM indicator, following the definition of the RIGHT/LEFT arm, a decision equation for the ARM indicator can be found to be:

$$g(\theta, \mathbf{p}) = \mathbf{z}_0 \cdot \frac{\mathbf{z}_1 \times \mathbf{p}'}{\|\mathbf{z}_1 \times \mathbf{p}'\|} = \mathbf{z}_0 \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ p_x & p_y & 0 \end{vmatrix} \frac{1}{\|\mathbf{z}_1 \times \mathbf{p}'\|}$$

$$= \frac{-p_y \sin\theta_1 - p_x \cos\theta_1}{\|\mathbf{z}_1 \times \mathbf{p}'\|} \tag{2.3-76}$$

where $\mathbf{p}' = (p_x, p_y, 0)^T$ is the projection of the position vector $\mathbf{p}$ [Eq. (2.3-36)] onto the $\mathbf{x}_0\mathbf{y}_0$ plane, $\mathbf{z}_1 = (-\sin\theta_1, \cos\theta_1, 0)^T$ from the third column vector of ${}^0\mathbf{T}_1$, and $\mathbf{z}_0 = (0, 0, 1)^T$. We have the following possibilities:

1. If $g(\theta, \mathbf{p}) > 0$, then the arm is in the RIGHT arm configuration.
2. If $g(\theta, \mathbf{p}) < 0$, then the arm is in the LEFT arm configuration.
3. If $g(\theta, \mathbf{p}) = 0$, then the criterion for finding the LEFT/RIGHT arm configuration cannot be uniquely determined. The arm is within the inner cylinder of radius d_2 in the workspace (see Fig. 2.19). In this case, it is default to the RIGHT arm (ARM = +1).

Since the denominator of the above decision equation is always positive, the determination of the LEFT/RIGHT arm configuration is reduced to checking the sign of the numerator of $g(\theta, \mathbf{p})$:

$$\text{ARM} = \text{sign}\,[g(\theta, \mathbf{p})] = \text{sign}\,(-p_x \cos\theta_1 - p_y \sin\theta_1) \tag{2.3-77}$$

where the sign function is defined in Eq. (2.3-70). Substituting the x and y components of $\mathbf{p}$ from Eq. (2.3-36), Eq. (2.3-77) becomes:

$$\text{ARM} = \text{sign}\,[g(\theta, \mathbf{p})] = \text{sign}\,[g(\theta)] = \text{sign}\,(-d_4 S_{23} - a_3 C_{23} - a_2 C_2) \tag{2.3-78}$$

Hence, from the decision equation in Eq. (2.3-78), one can relate its signed value to the ARM indicator for the RIGHT/LEFT arm configuration as:

$$\text{ARM} = \text{sign}\,(-d_4 S_{23} - a_3 C_{23} - a_2 C_2) = \begin{cases} +1 & \Rightarrow \text{RIGHT arm} \\ -1 & \Rightarrow \text{LEFT arm} \end{cases} \tag{2.3-79}$$

For the ELBOW arm indicator, we follow the definition of ABOVE/BELOW arm to formulate the corresponding decision equation. Using $({}^2\mathbf{p}_4)_y$ and the ARM indicator in Table 2.4, the decision equation for the ELBOW indicator is based on the sign of the y component of the position vector of ${}^2\mathbf{A}_3\,{}^3\mathbf{A}_4$ and the ARM indicator:

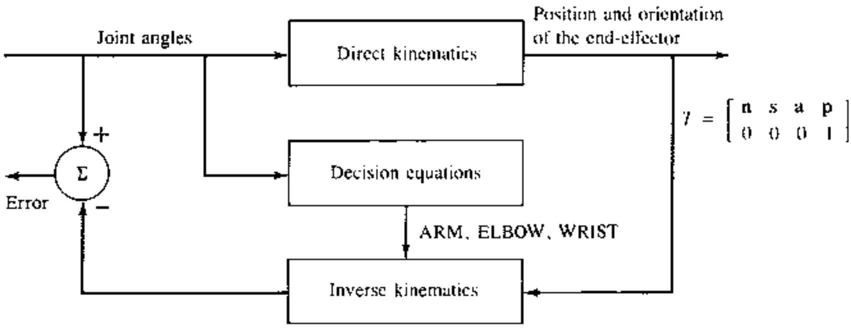


Figure 2.26 Computer simulation of joint solution.

$$\text{ELBOW} = \text{ARM} \cdot \operatorname{sign}(d_4 C_3 - a_3 S_3) = \begin{cases} +1 & \Rightarrow \text{ELBOW above wrist} \\ -1 & \Rightarrow \text{ELBOW below wrist} \end{cases} \tag{2.3-80}$$

For the WRIST indicator, we follow the definition of DOWN/UP wrist to obtain a positive dot product of the s and $\mathbf{y}_5$ (or $\mathbf{z}_4$) unit vectors:

$$\text{WRIST} = \begin{cases} +1 & \text{if } \mathbf{s} \cdot \mathbf{z}_4 > 0 \\ -1 & \text{if } \mathbf{s} \cdot \mathbf{z}_4 < 0 \end{cases} = \operatorname{sign}(\mathbf{s} \cdot \mathbf{z}_4) \tag{2.3-81}$$

If $\mathbf{s} \cdot \mathbf{z}_4 = 0$, then the WRIST indicator can be found from:

$$\text{WRIST} = \begin{cases} +1 & \text{if } \mathbf{n} \cdot \mathbf{z}_4 > 0 \\ -1 & \text{if } \mathbf{n} \cdot \mathbf{z}_4 < 0 \end{cases} = \operatorname{sign}(\mathbf{n} \cdot \mathbf{z}_4) \tag{2.3-82}$$

Combining Eqs. (2.3-81) and (2.3-82), we have

$$\text{WRIST} = \begin{cases} \operatorname{sign}(\mathbf{s} \cdot \mathbf{z}_4) & \text{if } \mathbf{s} \cdot \mathbf{z}_4 \neq 0 \\ \operatorname{sign}(\mathbf{n} \cdot \mathbf{z}_4) & \text{if } \mathbf{s} \cdot \mathbf{z}_4 = 0 \end{cases} = \begin{cases} +1 & \Rightarrow \text{WRIST DOWN} \\ -1 & \Rightarrow \text{WRIST UP} \end{cases} \tag{2.3-83}$$

These decision equations provide a verification of the arm solution. We use them to preset the arm configuration in the direct kinematics and then use the arm configuration indicators to find the inverse kinematics solution (see Fig. 2.26).

Computer Simulation. A computer program can be written to verify the validity of the inverse solution of the PUMA robot arm shown in Fig. 2.11. The software initially generates all the locations in the workspace of the robot within the joint angles limits. They are inputed into the direct kinematics routine to obtain the arm matrix **T**. These joint angles are also used to compute the decision equations to obtain the three arm configuration indicators. These indicators together with the arm matrix **T** are fed into the inverse solution routine to obtain the joint angle solution which should agree to the joint angles fed into the direct kinematics routine previously. A computer simulation block diagram is shown in Fig. 2.26.

2.4 CONCLUDING REMARKS

We have discussed both direct and inverse kinematics in this chapter. The parameters of robot arm links and joints are defined and a 4×4 homogeneous transformation matrix is introduced to describe the location of a link with respect to a fixed coordinate frame. The forward kinematic equations for a six-axis PUMA-like robot arm are derived.

The inverse kinematics problem is introduced and the inverse transform technique is used to determine the Euler angle solution. This technique can also be used to find the inverse solution of simple robots. However, it does not provide geometric insight to the problem. Thus, a geometric approach is introduced to find the inverse solution of a six-joint robot arm with rotary joints. The inverse solution is determined with the assistance of three arm configuration indicators (ARM, ELBOW, and WRIST). There are eight solutions to a six-joint PUMA-like robot arm—four solutions for the first three joints and for each arm configuration, two more solutions for the last three joints. The validity of the forward and inverse kinematics solution can be verified by computer simulation. The geometric approach, with appropriate modification and adjustment, can be generalized to other simple industrial robots with rotary joints. The kinematics concepts covered in this chapter will be used extensively in Chap. 3 for deriving the equations of motion that describe the dynamic behavior of a robot arm.

REFERENCES

Further reading on matrices can be found in Bellman [1970], Frazer et al. [1960], and Gantmacher [1959]. Utilization of matrices to describe the location of a rigid mechanical link can be found in the paper by Denavit and Hartenberg [1955] and in their book (Hartenberg and Denavit [1964]). Further reading about homogeneous coordinates can be found in Duda and Hart [1973] and Newman and Sproull [1979]. The discussion on kinematics is an extension of a paper by Lee [1982]. More discussion in kinematics can be found in Hartenberg and Denavit [1964] and Suh and Radcliffe [1978]. Although matrix representation of linkages presents a systematic approach to solving the forward kinematics problem, the vector approach to the kinematics problem presents a more concise representation of linkages. This is discussed in a paper by Chase [1963]. Other robotics books that discuss the kinematics problem are Paul [1981], Lee, Gonzalez, and Fu [1986], and Snyder [1985].

Pieper [1968] in his doctoral dissertation utilized an algebraic approach to solve the inverse kinematics problem. The discussion of the inverse transform technique in finding the arm solution was based on the paper by Paul et al. [1981]. The geometric approach to solving the inverse kinematics for a six-link manipula-