Machine Element II Assignment-I

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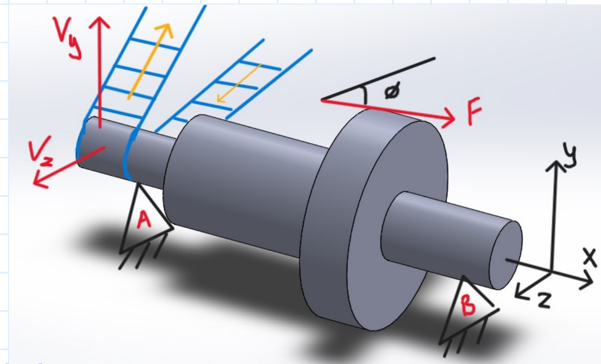
UGR/7646/14

The calculations were all done on PTC Mathcad Prime 10.0 for neatness and accuracy

1. Design a cold drawn AISI 1020 steel shaft with a safety factor of 2. The shaft is aimed to support a spur gear at the one end, with pitch diameter of 200mm and a belt at the other end. And the horizontal & the vertical force acting on the belt are 175N & 150N respectively. Two ball bearings are coupled, one in between the shaft and the belt and the second one is at the most end of the shaft. The force F from the drive gear acts at a pressure angle of 20°. The shaft transmits a torque of T = 450 N·m. Take D/d=1.25 and r/d=0.02.
(a) a static yield analysis using the distortion energy theory and
(b) a fatigue-failure analysis.
What will happen to the shaft if we consider the static analysis only?



Givens:

$n_{fs} := 2 \quad d_g := 200\ mm \quad V_y := 150\ N$

$V_z := 175\ N \quad \phi := 20\ deg \quad T := 450\ N \cdot m$

Step 1: Calculate for the unknown forces (Reactions and F)

Solve for F...

$$F \cdot \cos(\phi) \cdot \frac{d_g}{2} = T \xrightarrow[explicit]{solve, F} \frac{2 \cdot T}{d_g \cdot \cos(\phi)} = (4.789 \cdot 10^3)\ N \qquad F := (4.789 \cdot 10^3)\ N$$

In the x-y plane...

$$\sum F_y = 0 \qquad V_y + R_{Ay} - F \cdot sin(\phi) + R_{By} = 0$$

$$\sum M_A = 0 \qquad -V_y \cdot 50\ mm - F \cdot sin(\phi) \cdot 200\ mm + R_{By} \cdot 250\ mm = 0$$

$$\begin{bmatrix} V_y + R_{Ay} - F \cdot \sin(\phi) + R_{By} = 0 \\ -V_y \cdot 50\ mm - F \cdot \sin(\phi) \cdot 200\ mm + R_{By} \cdot 250\ mm = 0 \end{bmatrix} \xrightarrow[explicit]{solve, R_{Ay}, R_{By}} \begin{bmatrix} \frac{F \cdot \sin(\phi) - 6 \cdot V_y}{5} & \frac{4 \cdot F \cdot \sin(\phi) + V_y}{5} \end{bmatrix}$$

$$R_{Ay} := \frac{F \cdot \sin(\phi) - 6 \cdot V_y}{5} = 147.587\ N \qquad R_{By} := \frac{4 \cdot F \cdot \sin(\phi) + V_y}{5} = (1.34 \cdot 10^3)\ N$$

In the x-z plane, after substituting F by an equal force applied at the beam and not the gear with an added moment

$$\sum F_z = 0 \qquad >> \qquad -V_z + R_{Az} - F \cdot cos(\phi) + R_{Bz} = 0$$

$$\sum M_A = 0 \qquad >> \qquad -V_z \cdot 50\ mm + F \cdot cos(\phi) \cdot 200\ mm - R_{Bz} \cdot 250\ mm = 0$$

$$\begin{bmatrix} -V_z + R_{Az} - F \cdot \cos(\phi) + R_{Bz} = 0 \\ -V_z \cdot 50\ mm + F \cdot \cos(\phi) \cdot 200\ mm - R_{Bz} \cdot 250\ mm = 0 \end{bmatrix} \xrightarrow[explicit]{solve, R_{Az}, R_{Bz}} \begin{bmatrix} \frac{F \cdot \cos(\phi) + 6 \cdot V_z}{5} & \frac{4 \cdot F \cdot \cos(\phi) - V_z}{5} \end{bmatrix}$$

$$R_{Az} := \frac{F \cdot \cos(\phi) + 6 \cdot V_z}{5} = (1.11 \cdot 10^3)\ N \qquad R_{Bz} := \frac{4 \cdot F \cdot \cos(\phi) - V_z}{5} = (3.565 \cdot 10^3)\ N$$

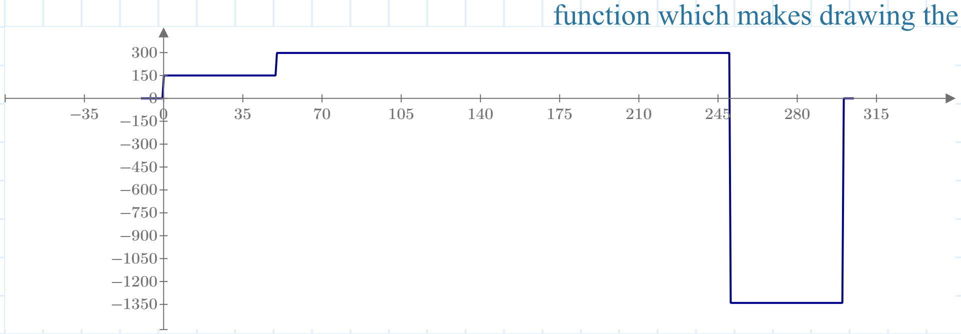
Step 2: Shear force and bending moment diagrams

Now to find the bending stresses in each plane, we draw the shear force and moment diagram

1. The x-y plane.

$$v_y(x) := 150 \cdot \Phi(x) + R_{Ay} \cdot \frac{1}{\boldsymbol{N}} \Phi(x-50) - F \cdot \sin(\phi) \cdot \frac{1}{\boldsymbol{N}} \Phi(x-250) + R_{By} \cdot \frac{1}{\boldsymbol{N}} \Phi(x-300)$$

*Nevermind the $\Phi(x)$ function, it's just Mathcad's unit-step function which makes drawing the shear force diagram easier

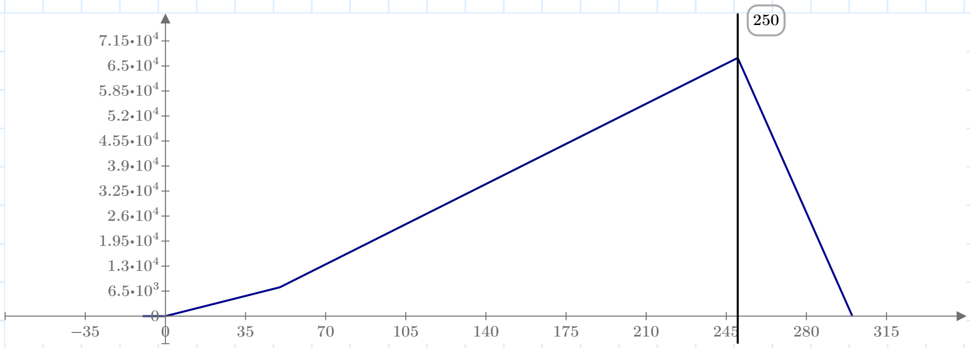


$v_y(x)$

x

Integrating the shear force to get bending moment...

$$M_z(x) := \int 148.0 \cdot \Phi(x-50.0) + (1340.0 \cdot \Phi(x-300.0) + 150.0 \cdot \Phi(x) - 4790.0 \cdot \sin(20.0 \cdot \boldsymbol{deg}) \cdot \Phi(x-250.0))\, \mathrm{d}x \rightarrow$$



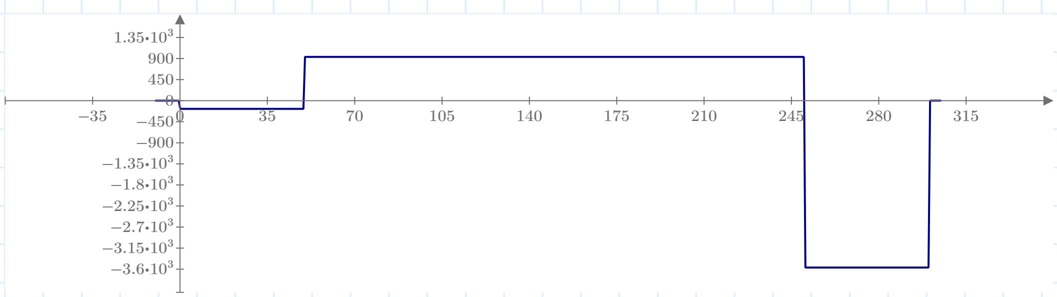
$M_z(x)$ in Newton-mm!

x

$$M\_max := M_z(250) \cdot \boldsymbol{N} \cdot \boldsymbol{mm} = 67.1\ \boldsymbol{N} \cdot \boldsymbol{m}$$

Similarly, in the x-z plane...

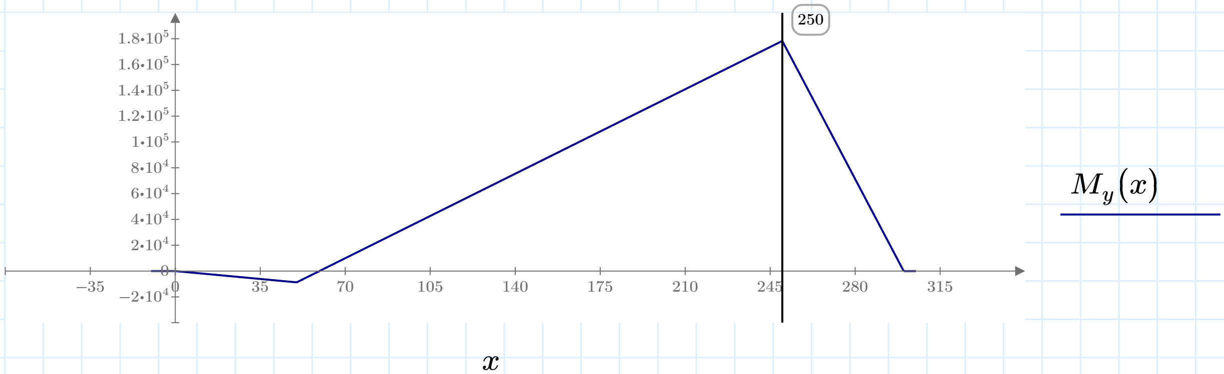
$$v_z(x) := -175 \cdot \Phi(x) + R_{Az} \frac{1}{\boldsymbol{N}} \Phi(x-50) - F \cdot \cos(\phi) \frac{1}{\boldsymbol{N}} \Phi(x-250) + R_{Bz} \cdot \frac{1}{\boldsymbol{N}} \Phi(x-300)$$



$v_z(x)$

x

$$M_y(x) := \int 1110.0 \cdot \Phi(x-50.0) + (3570.0 \cdot \Phi(x-300.0) - 175.0 \cdot \Phi(x) - 4790.0 \cdot \cos(20.0 \cdot deg) \cdot \Phi(x-250.0)) \, dx$$

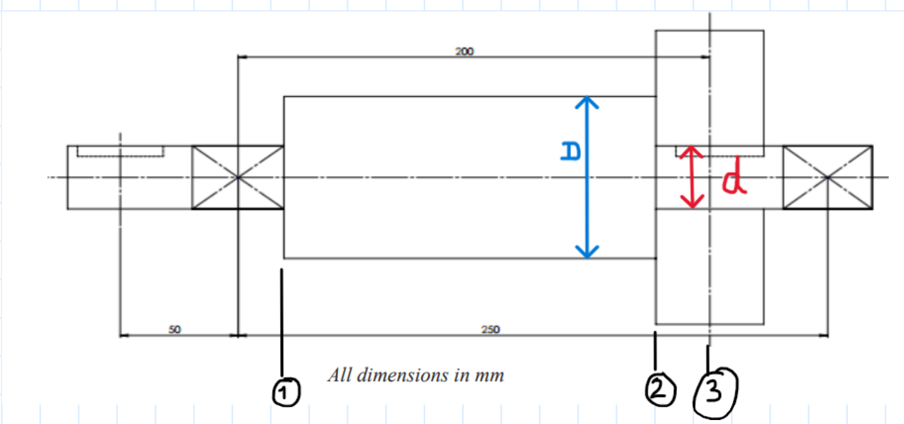


Again the critical point is at x=250 $M\_max := M_y(250) \cdot N \cdot mm = 178.25 \; N \cdot m$

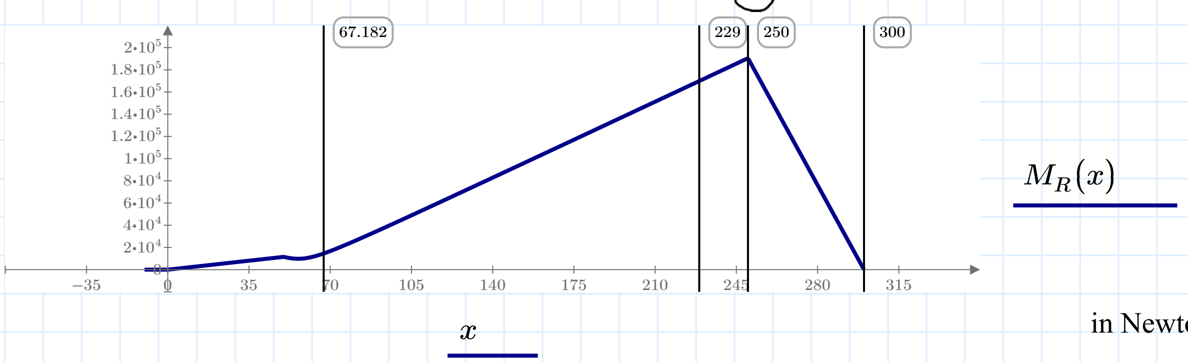
Now, the bending moments in the two orthogonal planes, x-y and x-z are are also orthogonal, (M_z in the x-y plane and M_x in the x-z plane). Since the moments are vectors, the resultant moment, M_R should be...

$$M_R = M_z \vec{k} + M_y \vec{j}$$ where $\vec{k}$ and $\vec{j}$ are the unit vectors along the z and y axis respectively.

$$|M_R|^2 = |M_z|^2 + |M_y|^2 \qquad M_R(x) := \sqrt{M_z(x)^2 + M_y(x)^2}$$



From the given shaft layout, we have two points along the shaft where stress concentrations arise at the shoulders (exact locations not given.)



in Newton-mm!

Since the moment at the left-most shoulder is much less than the others, it's clear that the maximum (critical) stress will be either at location 2 or 3.

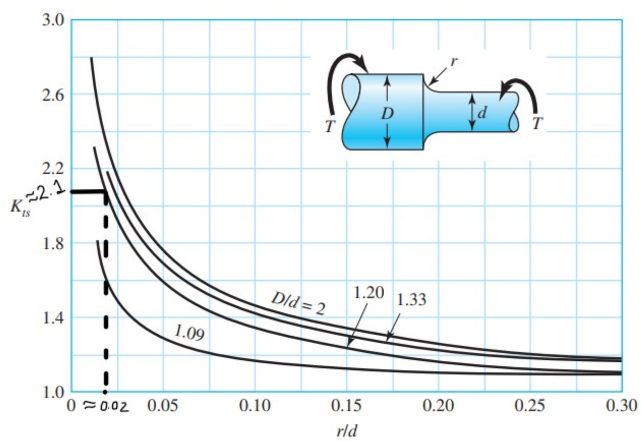
$M_2 := M_R(229) \; N \cdot mm = 169.884 \; N \cdot m$ $\qquad M_3 := M_R(250) \; N \cdot mm = 190.461 \; N \cdot m$

Step 3: Stress analysis

Since our shaft has a shoulder, we need to use a static stress concentration factor, k_t to account for the geometric discontinuity. We are given D/d= 1.25 and r/d= 0.02. From table A-15 of Shigley's 10th edition, we can get the stress concentration for a shaft in bending and the *shear* stress concentration for a shaft in torsion...

Figure A–15–8

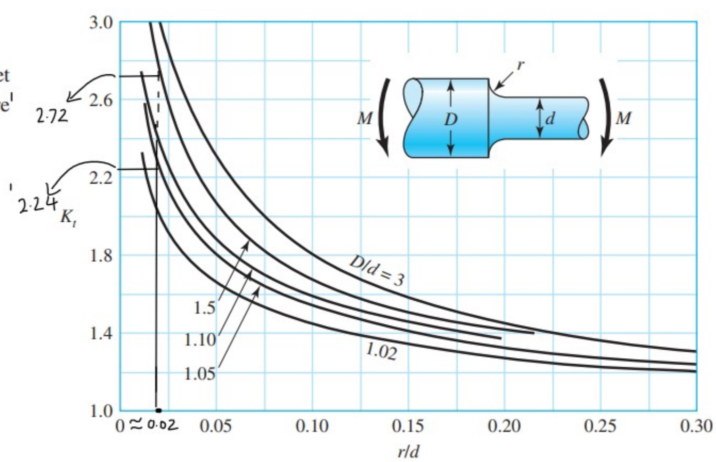
Round shaft with shoulder fillet in torsion. $\tau_0 = Tc/J$, where $c = d/2$ and $J = \pi d^4/32$.



Approximating D/d as 1.20 instead of 1.25 because the gap between the curves is small, we get $k_{ts} := 2.1$

Figure A–15–9

Round shaft with shoulder fillet in bending. $\sigma_0 = Mc/I$, where $c = d/2$ and $I = \pi d^4/64$.



For bending, we don't have a D/d=1.25 curve so we interpolate between the closes curves.
@ D/d= 1.10, we get $k_{t\_}1.1 := 2.24$ and
@D/d=1.5, we get $k_{t\_}1.5 := 2.72$

Interpolating, we have $\dfrac{k_t - k_{t\_}1.1}{1.25 - 1.10} = \dfrac{k_{t\_}1.5 - k_{t\_}1.1}{1.5 - 1.10} \xrightarrow{solve, k_t} 2.42$ $\qquad k_t := 2.42$

When calculating the stress at location 2, we do so for for both an element immediately left of the shoulder at 2 and an element immediately right of the shoulder. The difference in diameters will give different stresses.

$$\sigma_{2\_right} := k_t \cdot \frac{M_2 \cdot d}{2\ I} \xrightarrow{substitute, I = \frac{\pi \cdot d^4}{64}, float, 3} \frac{4190.0 \cdot N \cdot m}{d^3}$$

$$\sigma_{2\_left} := k_t \frac{M_2 \cdot D}{2\ I} \xrightarrow[\substack{substitute, I = \frac{\pi \cdot D^4}{64} \\ substitute, D = 1.25\ d \\ float, 3}]{} \frac{2140.0 \cdot N \cdot m}{d^3}$$

$$\sigma_3 := \frac{M_3 \cdot d}{2 \cdot I} \xrightarrow[\substack{substitute, I = \frac{\pi \cdot d^4}{64} \\ float, 3}]{} \frac{1940.0 \cdot N \cdot m}{d^3}$$

*no stress concentration at 3

$$\tau_{2\_left} := k_{ts} \frac{T \cdot D}{2 \cdot J} \xrightarrow[\text{float}, 3]{\substack{\text{substitute}, J = \frac{\pi \cdot D^4}{32} \\ \text{substitute}, D = 1.25\ d}} \frac{2460.0 \cdot N \cdot m}{d^3}$$

$$\tau_{2\_left} := \frac{2460.0 \cdot N \cdot m}{d^3} \rightarrow \frac{2460.0 \cdot N \cdot m}{d^3}$$

$$\tau_3 := \frac{T \cdot d}{2 \cdot J} \xrightarrow{\text{substitute}, J = \frac{\pi \cdot d^4}{32}} \frac{7200 \cdot N \cdot m}{d^3 \cdot \pi}$$

$$\tau_3 := \frac{7200 \cdot N \cdot m}{d^3 \cdot \pi} \xrightarrow{\text{float}, 3} \frac{2290.0 \cdot N \cdot m}{d^3}$$

$$\tau_{2\_right} := k_{ts} \cdot \tau_3 \rightarrow \frac{4809.0 \cdot N \cdot m}{d^3}$$

We see that both the bending stress and the shear stress are actually greater at location 2 (right) than 3 (max bending moment). So we now know the critical point is at location 2 (right or thinner section). Next we need to calculate the von-mises stress to get the critical diameter using the distortion energy failure theory.

Part I. Static failure analysis

Von-mises stress at location 2, right

Since there is only bending stress and shear stress substitute, $\sigma_x = \sigma_{2\_right}$ and $\sigma_y = 0$ and $\tau_{xy} = \tau_{2\_right}$ into Mohr's circle to get the principal stresses at the critical point. $\_\sigma_2$ is principal stress 2, not to be confused with σ_2 which is stress @ 2.

$$\_\sigma_1 := \frac{\sigma_{2\_right}}{2} + \sqrt{\left(\frac{\sigma_{2\_right}}{2}\right)^2 + \tau_{2\_right}^{\ 2}} \xrightarrow[\text{float}, 3]{\text{simplify}, \max} \frac{2100.0 \cdot N \cdot m}{d^3} + 5250.0 \cdot N \cdot m \cdot \left(\frac{1.0}{d^6}\right)^{0.5}$$

$$\_\sigma_2 := \frac{\sigma_{2\_right}}{2} - \sqrt{\left(\frac{\sigma_{2\_right}}{2}\right)^2 + \tau_{2\_right}^{\ 2}} \xrightarrow[\text{float}, 3]{\text{simplify}} -1.0 \cdot 5250.0 \cdot N \cdot m \cdot \left(\frac{1.0}{d^6}\right)^{0.5} + \frac{2100.0 \cdot N \cdot m}{d^3}$$

By von mises stress theory, for biaxial stress condition,

$$\sigma' := \sqrt{\_\sigma_1^{\ 2} + \_\sigma_2^{\ 2} - \_\sigma_1 \cdot \_\sigma_2} \xrightarrow{\text{simplify}, \text{float}, 5} \left(\frac{0.87098 \cdot 10^8 \cdot N^2 \cdot m^2}{d^6}\right)^{0.5}$$

And for cold drawn AISI 1020 steel, yield strength = 390MPa $\quad S_{yt} := 390\ MPa \quad \sigma_y := \frac{S_{yt}}{n_{fs}} = 195\ MPa$

Finally, we get the critical diameter ,d , by equating the von-mises stress to the allowable stress, σ_y

$$\sigma' = \sigma_y \rightarrow \left(\frac{87098000.0 \cdot N^2 \cdot m^2}{d^6}\right)^{0.5} = 195.0 \cdot MPa$$

Guess Values

$d := 1\ mm$

Constraints

$$\left(\frac{87098000.0 \cdot N^2 \cdot m^2}{d^6}\right)^{0.5} = 195.0 \cdot MPa$$

Solver

$\mathrm{d_{static}} := \mathrm{Find}(\mathrm{d}) = 36.307\ mm$

Solving for d using a solve-block, we get $\mathrm{d_{static}} = 36.307\ mm$

and since $\frac{D}{\mathrm{d}} = 1.25$, $D := 1.25 \cdot \mathrm{d_{static}} = 45.384\ mm$ So, the minimum allowable diameter should be 36.3 mm for the thinner sections and 45.4 mm for the thicker middle part.

Part II. Fatigue failure analysis

Factor of safety formula using the DE- Goodman criteria (DE means the stresses are combined using the DE theory). Goodman is chosen because it's more conservative and good for a first pass design.

$$d = \left(\frac{16 \cdot n_{fs}}{\pi}\left(\frac{2\left(k_f \cdot M_a\right)}{S_e} + \frac{\left(\left(3\left(k_{fs} \cdot T_m\right)\right)^2\right)^{0.5}}{S_{ut}}\right)\right)^{\frac{1}{3}}$$

...Equation 7-11, Shigley

Here, the M_a is going to be the max Moment M_max and M_m is going to be 0 because this is a case of completely reversed cycle. And the alternating torque, T_a, is zero assuming it's a constant motion shaft. So, the torque only has a midrange component (between the belt and the gear only)

$$d = \left(\frac{16 \cdot n_{fs}}{\pi}\left(\frac{2\left(k_f \cdot M_a\right)}{S_e} + \frac{\left(\left(3\left(k_{fs} \cdot T_m\right)\right)^2\right)^{0.5}}{S_{ut}}\right)\right)^{\frac{1}{3}} \xrightarrow[\text{explicit}, \text{all}]{\substack{\text{substitute}, M_a = M_{max} \\ \text{substitute}, T_m = T, \text{float}, 3}} d = \left(\frac{5.09 \cdot n_{fs} \cdot \left(9.0 \cdot k_{fs}^{\ 2} \cdot T^2\right)^{0.5}}{S_{ut}} + \frac{10.2 \cdot M_{max} \cdot k_f \cdot n_{fs}}{S_e}\right)^{0.333}$$

We already know that the critical point is at the shoulder at location 2. To calculate the diameter based on the Goodman criteria, we need the ultimate strength, S_{ut}, and the fully corrected endurance limit, S_e and the fatigue stress concentration factors for normal stress, k_f and for shear k_{fs}.

$M_{max} := M_3 = 190.461\ N \cdot m$ The max moment from the resultant moment diagram from page 3.

From Table A-20 Shigley, ultimate strength for a AISI 1020 steel, CD: $S_{ut} := 470\ MPa$

To find S_e we need to find the Marin factors first... $S_e = k_a \cdot k_b \cdot k_d \cdot k_e \cdot S_e'$

k_c (loading factor) is not used since it's already considered in the derivation of the formula for the safety factor from the Distortion energy theory

Since, we are not given the temperature, or any info about the reliability factor, we will neutralize $k_d \cdot k_e = 1$

From table 6-2, the size factor for a cold drawn surface finish is given as (in MPa): $k_a := 4.51 \cdot \left(S_{ut} \cdot \frac{1}{MPa}\right)^{-0.265} = 0.883$

From 6-20, k_b is given as: $k_b := 1.24 \cdot \left(d \cdot \frac{1}{mm}\right)^{-0.107} \rightarrow \frac{1.24}{\left(\frac{d}{mm}\right)^{0.107}}$ ...where d is in mm for 2.79 < d < 51 mm. (because we have seen that d is in that range from the static analysis.)

And from equation 6-8, uncorrected endurance limit is given as... $S_e' := 0.5\ S_{ut} = 235\ MPa$ ... for S_{ut}<1400MPa

So, the fully corrected endurance limit will be... $S_e := k_a \cdot k_b \cdot k_d \cdot k_e \cdot S_e' \xrightarrow[\text{float}, 5]{\text{substitute}, k_e = 1, k_d = 1} \frac{257.37 \cdot MPa}{\left(\frac{d}{mm}\right)^{0.107}}$

To find k_f and k_{fs} we need the notch sensitivity q and for that we need the notch radius, r. But we only have r/d and not r itself so, as is done in Example 7-2 in Shigley's 10th edition, "For quick, conservative first pass, assume Kf = Kt, Kfs = Kts."

$k_f := k_t = 2.42 \qquad k_{fs} := k_{ts} = 2.1$

Now we have all we need to solve for the diameter of the shaft using the DE- Goodman criteria...

$$d=\left(\frac{16\cdot n_{fs}}{\pi}\left(\frac{2\left(k_f\cdot M_a\right)}{S_e}+\frac{\left(\left(3\left(k_{fs}\cdot T_m\right)\right)^2\right)^{0.5}}{S_{ut}}\right)\right)^{\frac{1}{3}}\xrightarrow[\textit{explicit},ALL]{\substack{\textit{substitute},M_a=M_{max}\\ \textit{substitute},M_m=0,T_a=0\\ \textit{substitute},T_m=T,\textit{float},3}}d=\left(\frac{0.0217\cdot\left(0.804\cdot10^7\cdot N^2\cdot m^2\right)^{0.5}}{MPa}+\frac{36.5\cdot N\cdot m\cdot\left(\frac{d}{mm}\right)}{MPa}\right.$$

$$d=\left(\frac{0.0217\cdot\left(0.804\cdot10^7\cdot N^2\cdot m^2\right)^{0.5}}{MPa}+\frac{36.5\cdot N\cdot m\cdot\left(\frac{d}{mm}\right)^{0.107}}{MPa}\right)^{0.333}\xrightarrow{\substack{\textit{simplify}\\ \textit{float},5}}d=\left(\frac{36.5\cdot N\cdot m\cdot\left(\left(\frac{d}{mm}\right)^{0.107}+1.6858\right)}{MPa}\right)^{0.333}$$

Guess Values

$$d:=10\ mm$$

Constraints

$$d=\left(\frac{36.5\cdot N\cdot m\cdot\left(\left(\frac{d}{mm}\right)^{0.107}+1.6858\right)}{MPa}\right)^{\frac{1}{3}}$$

Solver

$$d_{fatigue}:=\mathrm{Find}\left(d\right)=48.892\ mm$$

Solving for d using a solve-block, we get

$d_{fatigue}=48.892\ mm$ and since $\frac{D}{d}=1.25$,

$D_{fatigue}:=1.25\cdot d_{fatigue}=61.115\ mm$

Comparing this with the diameter we designed using only static analysis, $\mathbf{d_{static}}=36.307\ mm$ which is less than the minimum allowable value we got using the fatigue analysis. This means that had we only considered the static analysis and set the diameter to 37 mm, the shaft would have failed by fatigue failure.