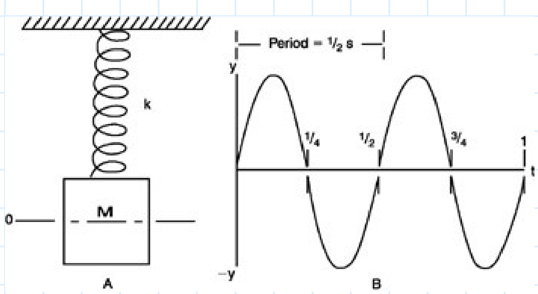
Mechanical Vibrations

Modelling real life objects as mathematical objects:

Simply put, the world is too complex! Even when we know all the governing equations that describe some phenomena, those equations are usually hard to solve entirely (analytically). This is because real life problems require solving partial differential equations of many continuous and independent variables over all of space and time! Consider for example a deceptively simple problem of calculating the trajectory of projectiles. One only needs to apply Newton's Laws of Motion and the Universal Law of Gravitation in order to get the trajectory of a ball thrown from the surface of the earth. But to do this one has to also make a number of assumptions like; assume the mass is a point-mass, assume there is no air resistance, ignore the effect of the balls rolling motion, assume the ground is perfectly flat and that the change in gravity of the earth along the trajectory is negligible, and that the gravitation of the moon is also negligible (to get a uniform gravitational field)... you get the point. It's not that the problem is unsolvable without these assumptions, it's just too hard and the extra accuracy gained by our loyalty to reality is not worth all the computational cost. So that's why we assume that the mass is a particle at the objects center of mass. We make assumptions to make our lives easier.
That said, here's what Prof. Stephen Hawking said about 'model-dependent-realism':

"According to model-dependent realism, it is pointless to ask whether a model is real, only whether it agrees with observation. If there are two models that both agree with observation, like the goldfish's picture [distorted by the bowl] and ours, then one cannot say that one is more real than the other. One can use whichever model is more convenient in the situation under consideration. For example, if one were inside the bowl, the goldfish's picture would be useful, but for those outside, it would be very awkward to describe events from a distant galaxy in the frame of a bowl on earth...". So, what matters for a model is how successful it is and not how much 'real' we think it is. With that said , let's now try to model a few real life examples...

Example: Try and find a mathematical equation to describe the motion of the mass in the following spring-mass system.



If we try to tackle the problem as it is, we will find that the external force, the weight, is actually distributed so will require to integrate the density over volume. Furthermore, each section of the spring will 'feel' a different weight as the spring itself has weight. But if we **model** the system as a point-mass attached to a massless spring with stiffness k, we will arrive at a solution much quickly and readily without sacrificing too much accuracy. So, starting with a governing equation...

$$\sum F_{external} = m \overrightarrow{a}$$

$$-m \cdot g - k \cdot x + c \cdot x' = m \cdot x''$$

where k is the stiffness constant and c is some dampness scaling factor...

This is a 2nd order, linear ODE which we then solve to find the solution, x(t).

Notice there are 3 major constants attached to the three orders of derivation of x. Mass, stiffness and damping. This turns out to be an over-arching theme in vibrations. To model complex systems, we need to find the equivalent mass, stiffness and damping.

Lumping of Parameters: In every vibration, there's an exchange between kinetic and potential energies. The potential energy is maximum whenever the spring is compressed or stretched and the kinetic energy is maximum when the mass zooms by the equilibrium position at maximum speed. Like we said before, there are three major elements of every vibration system: mass, stiffness and damping. The kinetic energy of the system is due to the mass and how fast it goes, the potential energy due the stiffness of the 'spring' and how far it's stretched and the damping constant is due to lost energy from friction or viscosity. When modeling problems, we often 'lump' parameters together to get an equivalent and simplified system. In mechanical vibrations specifically, we want to simplify complex problems into 3 elements only or at least combinations of plenty of these three elements. The elements are, a mass equivalent to the total mass of the system, a stiffness constant equivalent to the total stiffness of the system and a damping constant equivalent to the 'damping effect' of the whole system.

Equivalent Stiffness: Stiffness gives rise to potential energy. The equivalent stiffness constant is that which stores the same potential energy as all the springs in a system combined. That is...

$$\frac{1}{2}\, k_{eq} \cdot {x_{eq}}^2 = \sum \frac{1}{2}\, k_i \cdot {x_i}^2$$

For springs connected in parallel, the deflection (x) is the same on all springs...

$$\frac{1}{2}\, k_{eq} \cdot x^2 = \sum \frac{1}{2}\, k_i \cdot x^2 \quad >> \quad k_{eq} = \sum k_i$$

For springs connected in series, the force experienced by all springs is the same.

$$\frac{1}{2}\, F \cdot x_{eq} = \sum \frac{1}{2}\, F \cdot x_i \quad >> \quad x_{eq} = \sum x_i \quad >> \quad \frac{F}{k_{eq}} = \sum \frac{F}{k_i}$$

$$>> \quad \frac{1}{k_{eq}} = \sum \frac{1}{k_i}$$

Equivalent Mass: Mass gives rise to kinetic energy, so the equivalent mass is defined as...

$$\frac{1}{2}\, m_{eq} \cdot {v_{eq}}^2 = \sum \frac{1}{2}\, m_i \cdot {v_i}^2$$

...were v_i means each element velocity

$$m_{eq} \cdot {x'_{eq}}^2 = \sum m_i \cdot x'_i$$

...where x_i is the measure of displacement for each moving members of the system

To solve for m_{eq}, we should first relate x_{eq} to all x_i through what are called 'constraint' equations...

Equivalent Damping More on this later...

Free Vibration of a single degree of freedom

Consider a simple point mass attached to the end of a spring. When the mass is given some push, two forces are at play, the elastic resistive force of the spring (acting against the direction of acceleration), and the mass's own inertial force (F=ma) in the direction of acceleration. Both forces are functions of the position of the mass only.

Spring-mass system

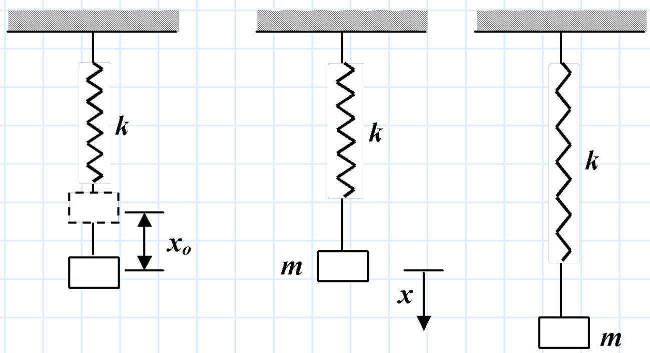


Fig.(13.7) A block of mass m attached to a spring of stiffness k oscillates in a vertical plane.

elastic force $F = -k \cdot x(t)$

inertial force $F = m \cdot a = m \cdot \left(\frac{d^2}{dt^2} x(t)\right)$

So >> $-k \cdot x(t) = m \cdot x''(t)$

...and the elastic force is always opposite (restorative to the motion), hence the minus

or $x''(t) + \frac{k}{m} x(t) = 0$

The above is only true when there is no external load acting on the spring-mass system, i.e. at the natural frequency of the spring-mass. It's a 2nd order, linear and homogenous O.D.E with constant coefficients so its general solution will have the form:

$$x''(t) + \frac{k}{m} x(t) = 0 \xrightarrow[\text{}]{\substack{solve, x \\ rewrite, \cos \\ assume, k>0, m>0}} (\_z1 - \_z) \cdot 1\mathrm{i} \cdot \sin\left(\frac{t \cdot \sqrt{k}}{\sqrt{m}}\right) + (\_z1 + \_z) \cdot \cos\left(\frac{t \cdot \sqrt{k}}{\sqrt{m}}\right)$$

the $\sqrt{\frac{k}{m}}$ term is the natural resonance frequency of the spring-mass. If we add a damping constant, c, and some external force; $x''(t) + c \cdot x'(t) + \frac{k}{m} x(t) = F(t)$ now it becomes forced oscillation and it's no longer homogenous O.D.E.

A better way of writing the general solution is as follows:

$$A \cdot cos(\omega \cdot t) + B \cdot sin(\omega \cdot t) = C \cdot cos(\omega \cdot t - \gamma)$$

where $C^2 = A^2 + B^2$ and $\gamma = atan\left(\frac{B}{A}\right)$

Here C is the amplitude of motion (range of motion), ω is the angular frequency and γ is the phase shift of the sine wave.

Example:

Suppose that m=2kg and k=8N/m.
The whole mass and spring setup is sitting on a truck that was traveling at 1 m/s.
The truck crashes and hence stops. The mass was held in place 0.5 meters forward from the rest position. During the crash the mass gets loose. That is, the mass is now moving forward at 1 m/s, while the other end of the spring is held in place. The mass therefore starts oscillating. What is the frequency of the resulting oscillation? What is the amplitude? The units are the mks units (meters-kilograms-seconds).

Given: $m := 2\ kg$ $\quad k := 8\ \frac{N}{m}$ $\quad$ I.C.: $x_0 := 0.5\ m$ $\quad x'_0 := 1\ \frac{m}{s}$

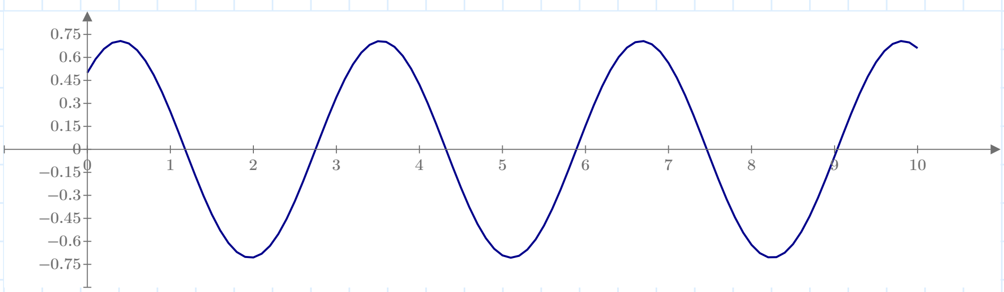
Assuming an undamped, free oscillation: $x''(t) + \frac{k}{m}\, x(t) = 0$

General solution: $\omega := \sqrt{\frac{k}{m}} = 2\ \frac{rad}{s}$ $\quad x(t) := A \cdot \cos(\omega \cdot t) + B \cdot \sin(\omega \cdot t) \rightarrow B \cdot \sin\left(\frac{2 \cdot rad \cdot t}{s}\right) + A \cdot \cos\left(\right.$

To find A and B:

$$[A\ \ B] := \begin{bmatrix} x(0) = x_0 \\ x'(0) = x'_0 \end{bmatrix} \xrightarrow{solve, A, B} \begin{bmatrix} 0.5 \cdot m & \frac{0.5 \cdot m}{rad} \end{bmatrix} \qquad C := \sqrt{A^2 + B^2} = 0.707\ m \qquad \gamma := \operatorname{atan}\left(\frac{B}{A}\right) = 0.785$$

$t := 0\ s, .1\ s .. 10\ s$ $\qquad$ So the specific answer is: $x(t) := C \cdot \cos(\omega \cdot t - \gamma)$ $\qquad T := \frac{2\ \pi}{\omega} = 3.142\ s$



$x(t)\ \ (m)$

$t\ \ (s)$

You can see from the plot that the period is close to 3s which is also what we get from the calculation.

clear $(x, t, T, C, \gamma, A, B, k, m, x_0, x'_0, \omega)$

Damped Oscillations: Most spring-mass systems don't oscillate forever as in the ideal case. There is friction, which opposes motion and has a *damping* effect on the oscillation. It's proportional to the speed with a constant of proportionality, C. So, now the general equation for spring mass system becomes:

$m \cdot x'' + c \cdot x' + k \cdot x = F$ $\qquad$ where c is the damping constant and F is the external load. For now, let F=0.

$m \cdot x'' + c \cdot x' + k \cdot x = 0$ $\qquad$ This is the general equation for *free*, damped oscillation at natural frequency, a 2nd order, linear, homogeneous O.D.E

Plugging in $e^{r \cdot t}$ to get the characteristic equation, we get what's called the *characteristic* equation of the differential equation.

$x(t) := e^{r \cdot t} \rightarrow e^{r \cdot t}$

$m \cdot x''(t) + c \cdot x'(t) + k \cdot x(t) = 0 \rightarrow \left(m \cdot r^2 + c \cdot r + k\right) \cdot e^{r \cdot t} = 0$

$$\begin{bmatrix} r1 \\ r2 \end{bmatrix} := m \cdot r^2 + c \cdot r + k = 0 \xrightarrow{solve, r} \begin{bmatrix} \dfrac{\sqrt{-(4 \cdot k \cdot m) + c^2} - c}{2 \cdot m} \\ \dfrac{-\sqrt{-(4 \cdot k \cdot m) + c^2} - c}{2 \cdot m} \end{bmatrix}$$

We can see that the $-(4 \cdot k \cdot m) + c^2$ factor determines whether we get a real or complex root.

Case 1). $-(4 \cdot k \cdot m) + c^2 > 0$:**2 real and distinct roots**.

$$\begin{bmatrix} r1 \\ r2 \end{bmatrix} \rightarrow \begin{bmatrix} \dfrac{\sqrt{-(4 \cdot k \cdot m) + c^2} - c}{2 \cdot m} \\ \dfrac{-\sqrt{-(4 \cdot k \cdot m) + c^2} - c}{2 \cdot m} \end{bmatrix} \xrightarrow{expand} \begin{bmatrix} \dfrac{\sqrt{-(4 \cdot k \cdot m) + c^2}}{2 \cdot m} - \dfrac{c}{2 \cdot m} \\ -\dfrac{\sqrt{-(4 \cdot k \cdot m) + c^2}}{2 \cdot m} - \dfrac{c}{2 \cdot m} \end{bmatrix}$$

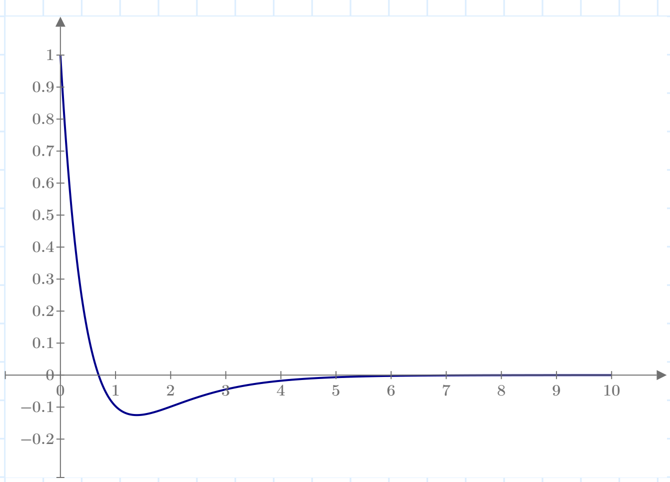
This is equivalent to: $\begin{bmatrix} r1 \\ r2 \end{bmatrix} := \begin{bmatrix} \sqrt{-\left(\dfrac{k}{m}\right) + (p)^2} - p \\ -\sqrt{-\left(\dfrac{k}{m}\right) + (p)^2} - p \end{bmatrix} \rightarrow \begin{bmatrix} \sqrt{-\dfrac{k}{m} + p^2} - p \\ -\sqrt{-\dfrac{k}{m} + p^2} - p \end{bmatrix}$ where p= $\dfrac{c}{2 \cdot m}$

Now since r1 and r2 are always real, $-\dfrac{k}{m} + p^2 < p^2$ because k and m are positive, so, $|p| > \sqrt{-\dfrac{k}{m} + p^2}$ and the roots r1 and r2 will always be negative, i.e. the general solution given as $C1 \cdot e^{r1 \cdot t} + C2 \cdot e^{r2 \cdot t}$ will be a decaying function with time.

let $r1 := -1$ $r2 := -2$

$$x(t) := C1 \cdot e^{r1 \cdot t} + C2 \cdot e^{r2 \cdot t} \rightarrow C1 \cdot e^{-t} + C2 \cdot e^{-(2 \cdot t)} \qquad [C1 \;\; C2] := \begin{bmatrix} x(0) = 1 \\ x'(0) = -3 \end{bmatrix} \xrightarrow{solve, C1, C2} [-1 \;\; 2]$$

$$x(t) := C1 \cdot e^{r1 \cdot t} + C2 \cdot e^{r2 \cdot t} \rightarrow -e^{-t} + 2 \cdot e^{-(2 \cdot t)}$$



$x(t)$

t

In this case, the mass-spring system is said to be '**over-damped**' because the damping factor is more than the elastic force (bouncy factor). The graph always decays to zero, which is the rest position of the mass. Note that changing the initial conditions changes the graph for the first few seconds but as t approaches ∞, x always goes to zero. If x' is big enough in the negative direction, the graph 'overshoots' and starts decaying back to zero from the negative side.

clear $(x, r1, r2, C1, C2, C)$

Case 2). $-(4 \cdot k \cdot m)+c^2 <0$: **complex roots**. Or, $-\frac{k}{m}+p^2$ <0 or $p<\sqrt{\frac{k}{m}}$

$$\begin{bmatrix} r1 \\ r2 \end{bmatrix} := \begin{bmatrix} \sqrt{-1} \cdot \sqrt{\frac{k}{m}-p^2} - p \\ -\sqrt{-1} \cdot \sqrt{\frac{k}{m}-p^2} - p \end{bmatrix} \xrightarrow{substitute, \frac{k}{m} = \omega_0{}^2} \begin{bmatrix} 1\mathrm{i} \cdot \sqrt{-p^2+\omega_0{}^2} - p \\ -\left(1\mathrm{i} \cdot \sqrt{-p^2+\omega_0{}^2}\right) - p \end{bmatrix} \xrightarrow{substitute, \sqrt{-p^2+\omega_0{}^2} = \omega_1} \begin{bmatrix} -p+1\mathrm{i} \cdot \omega_1 \\ -p-1\mathrm{i} \cdot \omega_1 \end{bmatrix}$$

where ω_0 is the angular frequency if the system were undamped and ω_1 is always positive

Now general solution will be: $r1 \rightarrow -p+1\mathrm{i} \cdot \omega_1$ $\quad r2 \rightarrow -p-1\mathrm{i} \cdot \omega_1$

$$C1 \cdot e^{r1 \cdot t}+C2 \cdot e^{r2 \cdot t} \xrightarrow[]{\substack{expand \\ collect, e^{-(p \cdot t)}}} e^{-(p \cdot t)} \cdot \left(C1 \cdot e^{1\mathrm{i} \cdot \omega_1 \cdot t}+C2 \cdot e^{-(1\mathrm{i} \cdot \omega_1 \cdot t)}\right)$$

$$C1 \cdot e^{1\mathrm{i} \cdot \omega_1 \cdot t}+C2 \cdot e^{-(1\mathrm{i} \cdot \omega_1 \cdot t)} \xrightarrow{rewrite, sincos} (-(1\mathrm{i} \cdot C2)+1\mathrm{i} \cdot C1) \cdot \sin(\omega_1 \cdot t)+(C2+C1) \cdot \cos(\omega_1 \cdot t)$$

*where C1 and C2 are any complex numbers...

Replacing this with new const. A and B and substituting...

general solution: $x(t) = e^{-(p \cdot t)} \cdot \left(A \cdot \sin(\omega_1 \cdot t)+B \cdot \cos(\omega_1 \cdot t)\right)$

let... $\quad p:=.3 \quad \omega_0:=2 \quad \omega_1:=\sqrt{\omega_0{}^2-p^2} \xrightarrow{float, 3} 1.98$

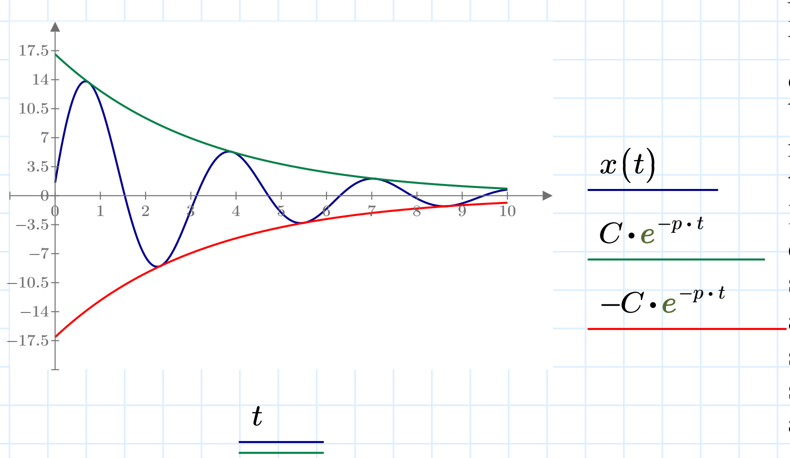
* ω_1 is the **damped** natural frequency of vibration

$$x(t):=e^{-(p \cdot t)} \cdot \left(A \cdot \sin(\omega_1 \cdot t)+B \cdot \cos(\omega_1 \cdot t)\right) \rightarrow e^{-0.3 \cdot t} \cdot (A \cdot \sin(1.98 \cdot t)+B \cdot \cos(1.98 \cdot t))$$

$$[A \;\; B] := \begin{bmatrix} x(0) = 17 \\ x'(0) = -2 \end{bmatrix} \rightarrow \begin{bmatrix} B = 17 \\ -0.3 \cdot B+1.98 \cdot A = -2.0 \end{bmatrix} \xrightarrow[]{\substack{solve, A, B \\ float, 3}} [1.57 \;\; 17.0]$$

We can further simplify this as: $x(t):=e^{-(p \cdot t)} \cdot \left(C \cdot \cos(\omega_1 \cdot t-\gamma)\right)$ where $C:=\sqrt{A^2+B^2}$ and $\gamma:=\mathrm{atan}\left(\frac{B}{A}\right)$

$t:=0,.1..10$



Now, the spring mass is said to be **under-damped**. You can see that the mass oscillates back and forth, overshooting the equilibrium mark multiple times before decaying to zero eventually. The two traces in red and green are the envelope equations that bound the oscillation,i.e. the mass will never go beyond this lines. Like before, changing the initial conditions changes the graph, same way increasing the initial stretch on the spring-mass makes it oscillate longer, and so on. When $-(4 \cdot k \cdot m)+c^2 = 0$ the system is said to be **critically damped** and the behavior is similar to the over-damped case. However, this is almost never the case since it's unstable.

$$\mathbf{clear}\left(A, B, C, \gamma, \omega_1, \omega_0, p, x, t\right)$$

Forced Oscillations:

is when the R.H.S is a non-zero fun_ of time that 'forces' the spring-mass to oscillate at a frequency other than its natural frequency.

$$m \cdot x'' + c \cdot x' + k \cdot x = F$$

This is now a linear, 2nd order, non-homogeneous equation. Suppose our external load is a periodic fun_ given as: $F = F_0 \cdot sin\left(\omega \cdot t\right)$ at some known frequency ω.

$$m \cdot x'' + c \cdot x' + k \cdot x = F_0 \cdot sin\left(\omega \cdot t\right)$$

Ignoring the damping for a moment and solving for the solution of this nonhomogeneous equation, we see that it will have a general solution of the form:

$$r^2 + \frac{k}{m} = 0 \qquad r = \sqrt{\frac{-k}{m}} = \begin{bmatrix} 1\mathrm{i} \cdot \omega_0 \\ -1\mathrm{i} \cdot \omega_0 \end{bmatrix} \qquad x_n(t) := A \cdot \cos\left(\omega_0\right) + B \cdot \sin\left(\omega_0\right) \rightarrow B \cdot \sin\left(\omega_0\right) + A \cdot \cos\left(\omega_0\right)$$

*where x_n denotes the null space...

$$x'' + \frac{k}{m} x = \frac{F_0 \cdot sin\left(\omega \cdot t\right)}{m} \qquad \text{try } x(t) := C \cdot \sin\left(\omega \cdot t\right) \rightarrow C \cdot \sin\left(\omega \cdot t\right)$$

$$x''(t) + {\omega_0}^2 \; x(t) \rightarrow \left({\omega_0}^2 - \omega^2\right) \cdot C \cdot \sin\left(\omega \cdot t\right)$$

$$C := \left({\omega_0}^2 - \omega^2\right) \cdot C \cdot \sin\left(\omega \cdot t\right) = \frac{F_0 \cdot \sin\left(\omega \cdot t\right)}{m} \xrightarrow{solve, C} \frac{F_0}{\left({\omega_0}^2 - \omega^2\right) \cdot m}$$

So, the particular solution: $\quad x_p := C \cdot \sin\left(\omega \cdot t\right) \rightarrow \dfrac{F_0 \cdot \sin\left(\omega \cdot t\right)}{m \cdot \left({\omega_0}^2 - \omega^2\right)}$

So, general solution:

$$x_p + x_n(t) \rightarrow \frac{F_0 \cdot \sin\left(\omega \cdot t\right)}{m \cdot \left({\omega_0}^2 - \omega^2\right)} + B \cdot \sin\left(\omega_0\right) + A \cdot \cos\left(\omega_0\right)$$

This is only true when $\omega_0 \neq \omega$!

$$\mathbf{clear}\left(A, B, C\right)$$

But now, suppose that our external periodic load is oscillating at the natural frequency, ω_0, the null space will still be the same, but an Xp that's any multiple of sine will not work because that is included in the null space.

$$\text{try} \quad x(t) := A \cdot t \cdot \sin\left(\omega_0 \cdot t\right) + B \cdot t \cdot \cos\left(\omega_0 \cdot t\right) \rightarrow A \cdot t \cdot \sin\left(\omega_0 \cdot t\right) + B \cdot t \cdot \cos\left($$

$$x''(t) + {\omega_0}^2 \; x(t) \xrightarrow{simplify} -\left(2 \cdot \omega_0 \cdot \left(B \cdot \sin\left(\omega_0 \cdot t\right) - A \cdot \cos\left(\omega_0 \cdot t\right)\right)\right)$$

$$-\left(2 \cdot \omega_0 \cdot \left(B \cdot \sin\left(\omega_0 \cdot t\right) - A \cdot \cos\left(\omega_0 \cdot t\right)\right)\right) = \frac{F_0 \cdot \sin\left(\omega_0 \cdot t\right)}{m} \qquad B := \frac{F_0}{2 \cdot \omega_0 \cdot m} \rightarrow \frac{F_0}{2 \cdot \omega_0 \cdot m} \qquad A := 0$$

*Treating sin() and cos() are independent basis, so now Xp becomes...

$$x_p := A \cdot t \cdot \sin\left(\omega_0 \cdot t\right) + B \cdot t \cdot \cos\left(\omega_0 \cdot t\right) \rightarrow \frac{F_0 \cdot t \cdot \cos\left(\omega_0 \cdot t\right)}{2 \cdot \omega_0 \cdot m}$$

and the general solution:

$$x_p + x_n(t) \rightarrow \frac{F_0 \cdot t \cdot \cos\left(\omega_0 \cdot t\right)}{2 \cdot \omega_0 \cdot m} + B \cdot \sin\left(\omega_0\right) + A \cdot \cos\left(\omega_0\right)$$

This is only true when $\omega_0 = \omega$!

$$\text{clear}\,(A, B)$$

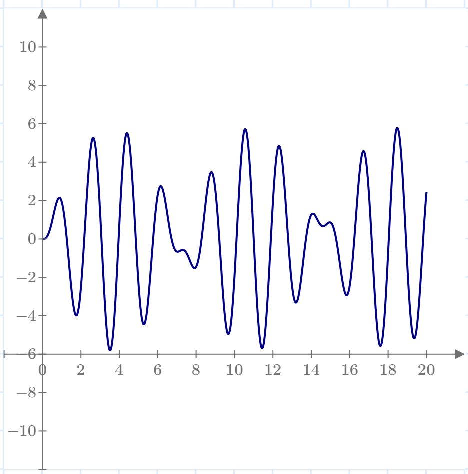
Lets plot each case to better understand what's going on...

let ... $\omega := \pi \quad \omega_0 := 4 \quad m := 0.5 \quad F_0 := 10$

$$x1(t) := \frac{F_0 \cdot \sin(\omega \cdot t)}{m \cdot (\omega_0{}^2 - \omega^2)} + B \cdot \sin(\omega_0 \cdot t) + A \cdot \cos(\omega_0 \cdot t) \xrightarrow{float, 3} B \cdot \sin(4.0 \cdot t) + 3.26 \cdot \sin(3.14 \cdot t) + A \cdot \cos(4.0 \cdot t)$$

$$\begin{bmatrix} A & B \end{bmatrix} := \begin{bmatrix} x1(0) = 0 \\ x1'(0) = 0 \end{bmatrix} \rightarrow \begin{bmatrix} A = 0 \\ 4.0 \cdot B + 10.2364 = 0.0 \end{bmatrix} \xrightarrow{solve, A, B} \begin{bmatrix} 0.0 & -2.5591 \end{bmatrix}$$

$$x1(t) := \frac{F_0 \cdot \sin(\omega \cdot t)}{m \cdot (\omega_0{}^2 - \omega^2)} + B \cdot \sin(\omega_0 \cdot t) + A \cdot \cos(\omega_0 \cdot t) \xrightarrow{float, 3} -2.56 \cdot \sin(4.0 \cdot t) + 3.26 \cdot \sin(3.14 \cdot t)$$



$x1(t)$

t

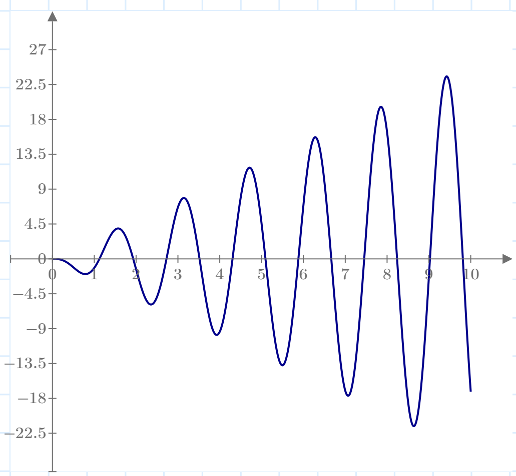
Undamped forced oscillation at $\omega \neq \omega_0$

$$\text{clear}\,(A, B)$$

$$x2(t) := \frac{F_0 \cdot t \cdot \cos(\omega_0 \cdot t)}{2 \cdot \omega_0 \cdot m} + B \cdot \sin(\omega_0 \cdot t) + A \cdot \cos(\omega_0 \cdot t) \xrightarrow{floa}$$

$$\begin{bmatrix} A & B \end{bmatrix} := \begin{bmatrix} x2(0) = 0 \\ x2'(0) = 0 \end{bmatrix} \xrightarrow{solve, A, B} \begin{bmatrix} 0.0 & -0.625 \end{bmatrix}$$

$$x2(t) := \frac{F_0 \cdot t \cdot \cos(\omega_0 \cdot t)}{2 \cdot \omega_0 \cdot m} + B \cdot \sin(\omega_0 \cdot t) + A \cdot \cos(\omega_0 \cdot t)$$



$x2(t)$

t

Undamped forced oscillation at $\omega = \omega_0$

From the plot above, you can see the oscillation itself is sort of periodic at T roughly 7 sec. This is due to the different frequencies, natural frequency ω_0 and forced frequency ω. There's both constructive and destructive interference. However, from the plot to the right, we see that the amplitude of the oscillation increases non-stop with time. Mathematically, this is due to the 't' in the first term of x2(t). But more importantly, the physics interpretation of this is that the external load *resonates* with the material undergoing the oscillation. This is just a fancy way of saying that the forced oscillation is at the natural frequency of the material.

$$\text{clear}\,(x1, x2, A, B, \omega, \omega_0, F_0, m)$$

Practical Resonance: (Because real life is almost never ideal!)

In real life, there's always some sort of damping that brings the oscillation to a stop. To see the effect of damping on forced oscillations, solve for the general solution of the linear, 2nd order, nonhomogeneous O.D.E for mechanical vibration: $m \cdot x'' + c \cdot x' + k \cdot x = F_0 \cdot cos(\omega \cdot t)$

First, for the null-space: $m \cdot x'' + c \cdot x' + k \cdot x = 0$

This is the same as for free, damped oscillations that we saw before with $p = \dfrac{c}{2\ m}$, $\omega_0 = \sqrt{\dfrac{k}{m}}$ and $\omega_1 = \sqrt{{\omega_0}^2 - p^2}$

$$x_n = \begin{bmatrix} C1 \cdot e^{r1 \cdot t} + C2 \cdot e^{r2 \cdot t} \\ e^{-(p \cdot t)} \cdot \left(A \cdot \sin(\omega_1 \cdot t) + B \cdot \cos(\omega_1 \cdot t)\right) \\ C1 \cdot t \cdot e^{-p \cdot t} + C2 \cdot e^{-p \cdot t} \end{bmatrix}$$

If it's over-damped ($p^2 > {\omega_0}^2$)
If it's under-damped ($p^2 < {\omega_0}^2$)
If it's critically-damped ($p^2 = {\omega_0}^2$), with r1 and r2 being the roots of the characteristic equation.

Now, to find the X_P for the case with $\omega \neq \omega_0$ plugin... $x_p(t) := A \cdot \cos(\omega \cdot t) + B \cdot \sin(\omega \cdot t) \rightarrow B \cdot \sin(\omega \cdot t) + A \cdot \cos(\omega$

$$x_p''(t) + 2\ p \cdot x_p'(t) + {\omega_0}^2\ x_p(t) = \frac{F_0}{m}\ cos(\omega \cdot t)$$

rearranging the equation...

$$x_p''(t) + 2\ p \cdot x_p'(t) + {\omega_0}^2\ x_p(t) \xrightarrow{collect, \cos, \sin} \left(-(2 \cdot \omega \cdot A \cdot p) + \left({\omega_0}^2 - \omega^2\right) \cdot B\right) \cdot \sin(\omega \cdot t) + \left(2 \cdot \omega \cdot B \cdot p + \left({\omega_0}^2 - \omega\right.\right.$$

$$\left(-(2 \cdot \omega \cdot A \cdot p) + \left({\omega_0}^2 - \omega^2\right) \cdot B\right) \cdot \sin(\omega \cdot t) + \left(2 \cdot \omega \cdot B \cdot p + \left({\omega_0}^2 - \omega^2\right) \cdot A\right) \cdot \cos(\omega \cdot t) = \frac{F_0}{m}\ cos(\omega \cdot t)$$

$$[A\ \ B] := \begin{bmatrix} -(2 \cdot \omega \cdot A \cdot p) + \left({\omega_0}^2 - \omega^2\right) \cdot B = 0 \\ 2 \cdot \omega \cdot B \cdot p + \left({\omega_0}^2 - \omega^2\right) \cdot A = \dfrac{F_0}{m} \end{bmatrix} \xrightarrow[simplify]{solve, A, B} \left[\frac{F_0 \cdot (\omega_0 - \omega) \cdot (\omega_0 + \omega)}{m \cdot \left(4 \cdot \omega^2 \cdot p^2 + {\omega_0}^4 - 2 \cdot \omega^2 \cdot {\omega_0}^2 + \omega^4\right)} \quad \frac{}{m \cdot (4 \cdot \omega^2 \cdot} \right.$$

$$A \rightarrow \frac{F_0 \cdot (\omega_0 - \omega) \cdot (\omega_0 + \omega)}{m \cdot \left(4 \cdot \omega^2 \cdot p^2 + {\omega_0}^4 - 2 \cdot \omega^2 \cdot {\omega_0}^2 + \omega^4\right)} \qquad B \rightarrow \frac{2 \cdot F_0 \cdot \omega \cdot p}{m \cdot \left(4 \cdot \omega^2 \cdot p^2 + {\omega_0}^4 - 2 \cdot \omega^2 \cdot {\omega_0}^2 + \omega^4\right)}$$

$$C := \sqrt{A^2 + B^2} \xrightarrow{simplify} \sqrt{\frac{{F_0}^2}{m^2 \cdot \left(4 \cdot \omega^2 \cdot p^2 + {\omega_0}^4 - 2 \cdot \omega^2 \cdot {\omega_0}^2 + \omega^4\right)}}$$

$$x_p(t) := A \cdot \cos(\omega \cdot t) + B \cdot \sin(\omega \cdot t) \rightarrow \frac{2 \cdot F_0 \cdot \omega \cdot p \cdot \sin(\omega \cdot t)}{m \cdot \left(4 \cdot \omega^2 \cdot p^2 + {\omega_0}^4 - 2 \cdot \omega^2 \cdot {\omega_0}^2 + \omega^4\right)} + \frac{F_0 \cdot (\omega_0 - \omega) \cdot (\omega_0 + \omega) \cdot \cos(\omega \cdot t)}{m \cdot \left(4 \cdot \omega^2 \cdot p^2 + {\omega_0}^4 - 2 \cdot \omega^2 \cdot {\omega_0}^2 + \omega\right.}$$

$$x_p = C \cdot \cos(\omega \cdot t - \gamma) \rightarrow function = \cos(\omega \cdot t - \gamma) \cdot \sqrt{\frac{{F_0}^2}{m^2 \cdot \left(4 \cdot \omega^2 \cdot p^2 + {\omega_0}^4 - 2 \cdot \omega^2 \cdot {\omega_0}^2 + \omega^4\right)}}$$

*where γ such that $tan(\gamma) = B \div A$

Now lets see what happens at resonance (when $\omega = \omega_0$)

$$A \xrightarrow{assume, \omega = \omega_0} 0 \qquad B \xrightarrow{assume, \omega = \omega_0} \frac{F_0}{2 \cdot \omega \cdot m \cdot p} \qquad C \xrightarrow{assume, \omega = \omega_0} \frac{|F_0|}{2 \cdot \sqrt{\omega^2 \cdot m^2 \cdot p^2}}$$

$$\tan(\gamma) = \frac{B}{A} \xrightarrow{assume, \omega = \omega_0} undefined$$ because A=0 and so tan() blows up to ∞. that happens at $\gamma = \frac{\pi}{2}$

Now lets make some plots and try to understand things visually. (The best way to learn!)
Suppose:

$k := 1 \quad m := 1 \quad F_0 := 1 \quad c := .7 \quad \omega := 1.1 \quad p := \frac{c}{2\ m} = 0.35 \quad \omega_0 := \sqrt{\frac{k}{m}} = 1 \quad \omega_1 := \sqrt{{\omega_0}^2 - p^2} = 0.937$

$$A := \frac{F_0 \cdot (\omega_0 - \omega) \cdot (\omega_0 + \omega)}{m \cdot \left(4 \cdot \omega^2 \cdot p^2 + {\omega_0}^4 - 2 \cdot \omega^2 \cdot {\omega_0}^2 + \omega^4\right)} = -0.33 \qquad B := \frac{2 \cdot F_0 \cdot \omega \cdot p}{m \cdot \left(4 \cdot \omega^2 \cdot p^2 + {\omega_0}^4 - 2 \cdot \omega^2 \cdot {\omega_0}^2 + \omega^4\right)} = 1.209$$

$x_p(t) := A \cdot \cos(\omega \cdot t) + B \cdot \sin(\omega \cdot t)$ To find the null space, compare p^2 and ${\omega_0}^2$ $\quad p^2 > {\omega_0}^2 = 0$ $\quad p^2 < {\omega_0}^2 = 1$

So, we have a case of underdamped oscillation!

clear.sym (A, B)

$x_n(t) := e^{-(p \cdot t)} \cdot \left(A \cdot \sin(\omega_1 \cdot t) + B \cdot \cos(\omega_1 \cdot t)\right) \qquad x(t) := x_p(t) + x_n(t)$

To find arbitrary constants A and B, we need some I.C.

$$\begin{bmatrix} \_A_0 & \_B_0 \end{bmatrix} := \begin{bmatrix} x(0) = 0 \\ x'(0) = 0 \end{bmatrix} \xrightarrow[float, 4]{solve, A, B} \begin{bmatrix} -1.296 & 0.3297 \end{bmatrix} \qquad \begin{bmatrix} \_A_1 & \_B_1 \end{bmatrix} := \begin{bmatrix} x(0) = 1 \\ x'(0) = 0 \end{bmatrix} \xrightarrow[float, 3]{solve, A, B} \begin{bmatrix} -0.923 & 1.33 \end{bmatrix}$$

$$\begin{bmatrix} \_A_2 & \_B_2 \end{bmatrix} := \begin{bmatrix} x(0) = 1 \\ x'(0) = 1 \end{bmatrix} \xrightarrow[float, 3]{solve, A, B} \begin{bmatrix} 0.145 & 1.33 \end{bmatrix} \qquad \begin{bmatrix} \_A_3 & \_B_3 \end{bmatrix} := \begin{bmatrix} x(0) = 1 \\ x'(0) = -1 \end{bmatrix} \xrightarrow[float, 3]{solve, A, B} \begin{bmatrix} -1.99 & 1.33 \end{bmatrix}$$

$$x_n1(t) := e^{-(p \cdot t)} \cdot \left(\_A_0 \cdot \sin(\omega_1 \cdot t) + \_B_0 \cdot \cos(\omega_1 \cdot t)\right) \xrightarrow{float, 3} e^{-0.35 \cdot t} \cdot (-1.3 \cdot \sin(0.937 \cdot t) + 0.33 \cdot \cos(0.937 \cdot t))$$

$$x_n2(t) := e^{-(p \cdot t)} \cdot \left(\_A_1 \cdot \sin(\omega_1 \cdot t) + \_B_1 \cdot \cos(\omega_1 \cdot t)\right) \xrightarrow{float, 3} e^{-0.35 \cdot t} \cdot (-0.923 \cdot \sin(0.937 \cdot t) + 1.33 \cdot \cos(0.937 \cdot t))$$

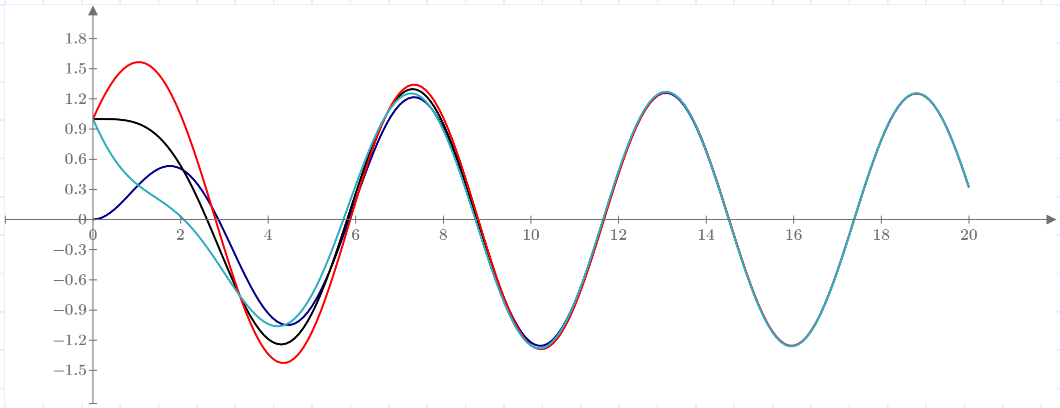
$$x_n3(t) := e^{-(p \cdot t)} \cdot \left(\_A_2 \cdot \sin(\omega_1 \cdot t) + \_B_2 \cdot \cos(\omega_1 \cdot t)\right) \xrightarrow{float, 3} e^{-0.35 \cdot t} \cdot (0.145 \cdot \sin(0.937 \cdot t) + 1.33 \cdot \cos(0.937 \cdot t))$$

$$x_n4(t) := e^{-(p \cdot t)} \cdot \left(\_A_3 \cdot \sin(\omega_1 \cdot t) + \_B_3 \cdot \cos(\omega_1 \cdot t)\right) \xrightarrow{float, 3} e^{-0.35 \cdot t} \cdot (-1.99 \cdot \sin(0.937 \cdot t) + 1.33 \cdot \cos(0.937 \cdot t))$$

What I did here is not as complex as it looks. I just made up 4 different I.C. to get 4 different null spaces which I then added to the Xp to get 4 different general solutions for each initial condition. Each of these general solution is represented by a different color in the plot below. Note how they all seem to converge to the same sine wave as time progresses...

$x1(t) := x_p(t) + x_n1(t) \qquad x2(t) := x_p(t) + x_n2(t) \qquad x3(t) := x_p(t) + x_n3(t) \qquad x4(t) := x_p(t) + x_n4(t)$

$$t := 0, .1..20$$



$x1(t)$

$x2(t)$

$x3(t)$

$x4(t)$

t

Important Observations!

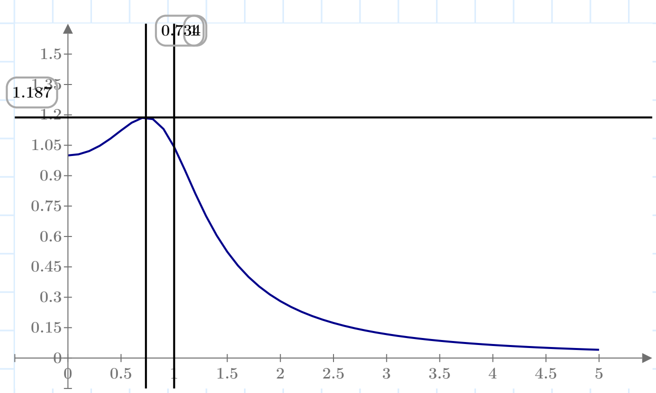
- The null space decays to zero with time (because of damping) but the particular solution oscillates at the forced frequency ω with an amplitude of $C = A^2 + B^2$. So as t increases, Xp becomes dominant and the null space vanishes. For this reason. the null space is termed the *transient* solution whereas the particular solution is termed the *steady-periodic* solution.
- The initial condition only affects the start region of the graph and as t approaches ∞, the effects of the I.C. vanish. This is because the effect of I.C. (arbitrary constants) is only present in the transient solution, Xtr, and not present in the steady-periodic Xsp solution.

clear (ω)

$$C(\omega) := \sqrt{\frac{F_0^{\ 2}}{m^2 \cdot \left(4 \cdot \omega^2 \cdot p^2 + \omega_0^{\ 4} - 2 \cdot \omega^2 \cdot \omega_0^{\ 2} + \omega^4\right)}} \qquad s := \qquad p := \frac{s}{50} \qquad p = 0.48$$

$$\omega_{cr} := C'(\omega) = 0 \xrightarrow{solve, \omega} \begin{bmatrix} 0 \\ \frac{\sqrt{337}}{25} \\ \frac{-\sqrt{337}}{25} \end{bmatrix} \qquad \omega_{Cr} := \omega_{cr_1} = 0.734 \qquad C_{Cr} := C\left(\omega_{Cr}\right) = 1.187$$

$$\omega := 0, 0.1..5$$



$C(\omega)$

ω

- Critical amplitude decreases as damping (p) increases and vice versa. And as damping decreases, ω_{cr} approaches ω_0 as can be seen in the plot below.
- Finally, as ω approaches ∞, C approaches zero, meaning that for large frequencies, the interference is mostly destructive. Play with plot below!