Why $\sqrt{-1}$ is a Real, Definite Integral, and Could Determine the Chirality of the Universe

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Preliminaries

Throughout this paper, the symbol i refers to the same unit traditionally defined as $\sqrt{-1}$ in complex analysis. However, we show that i admits a deeper geometric expression as a real Phase Slip operator that reproduces the exact same behavior from first principles.
For clarity:

- α is reserved for discrete Phase Slip angle;
- φ for the continuous Phase Lift angle;
- λ for the Real line's orthogonal lift/slip vector (NOT an axis); and
- θ for Euler's classical identity.

Where Euler's identity is directly inserted for elegance (not derived), we retain his usage of i as both operator and direction.

Note on Compatibility. This paper does not discard or alter the standard complex algebra $\mathbb{C}$. Instead, it extends its interpretation by revealing a real, bounded geometric kernel that reproduces every classical property of i from first principles. All results here remain fully consistent with Euler's identity and the classical field structure. However, as the proofs unfold, it will become clear that the traditional notion of "complex numbers" as non-real is no longer necessary. The so-called "imaginary" is simply the visible trace of a real, locally bounded arc integration.

Definition i — The Chiral Phase Slip Operator

Definition. I now have the great honor to define the operator i as the canonical bounded Phase Slip for the unit circle basis. It acts as the discrete infinitesimal arc lift that displaces a real projection $f(x)$ through a quarter-turn arc, producing the invariant spin root known classically as the imaginary unit.

$$i(x) := \int_{\alpha}^{\alpha+\frac{\pi}{2}} f(x)\cos(\alpha)\,d\alpha, \quad \text{where} \quad \alpha = k\,\frac{\pi}{2},\ k \in \{0,1,2,3\}. \quad \text{(CCW discrete closure)}$$

Interpretation. On the unit circle, i generates the local orthogonal slip λ in $\mathbb{R}^2$, closing the spin in four discrete sign flips: $+\lambda$, -1, $-\lambda$, $+1$. This matches the classical roots of unity and proves i as the trivial Phase Lift on the circle.

Definition 1 — The Real Line as Projected Arc

Definition. The real number line $\mathbb{R}$ is defined as the phase-collapsed projection of a higher-order arc basis. Specifically, any real quantity $x \in \mathbb{R}$ can be written as a fixed-phase slice of a rotation:

$$\exists\, A \in \mathbb{R},\ \alpha \in [0, 2\pi] : \quad x = A\,\cos(\alpha).$$

Interpretation. The "flattened" real axis we measure is the degenerate case where the hidden phase dimension is frozen at a constant slice.
Note. Any real line $\mathbb{R}$ may be seen as the fixed-phase slice of an arbitrary bounded arc. There is no unique, privileged slice: the real pole is defined by freezing the phase dimension at a chosen constant value.

Definition 2 — The i Operator as Classical Complex Rotation

Definition. I define the operator i as the classical unit imaginary that rotates any real quantity x by a quarter turn in the complex plane:

$$i : \mathbb{R} \to \mathbb{C}, \quad i(x) = x \cdot \mathrm{e}^{i\frac{\pi}{2}}.$$

Interpretation. Multiplication by i applies a counterclockwise rotation of $\frac{\pi}{2}$ radians, placing the real magnitude onto the imaginary axis in $\mathbb{C}$.

Definition 3 — Discrete Phase Increment

Definition. A *discrete phase increment* is the special case of my Phase Lift where the phase angle is stepped by a fixed quantized arc $\Delta\,\alpha$:

$$x' = x \cdot \mathrm{e}^{i\Delta\,\alpha}, \quad \text{where} \quad \Delta\,\alpha = k\,\frac{\pi}{2}, \quad k \in \mathbb{Z}.$$

Each unit increment rotates the real magnitude x by $\frac{\pi}{2}$ radians counterclockwise, mapping it into its local orthogonal arc dimension λ via application of the discrete Phase Slip operator i.

Interpretation. The real line corresponds to the degenerate sweep where $\Delta\,\alpha = 0$; discrete phase increments reveal the hidden continuous arc lift in quantized steps, forming the closed lift cycle described in Definition 4.

Definition 4 — Local Orthogonal Basis

Definition. The local orthogonal basis is the closed 2D plane $\mathbb{R}^2$ spanned by the real axis x and its continuous arc lift direction λ. The lift operator i rotates the real projection into λ by bounded phase sweep.

Discrete phase slip points arise as special invariant projections of the continuous lift, stepping by $\Delta\,\alpha = \frac{\pi}{2}$.

Sequential applications of these discrete slips satisfy the bounded cycle:

$$\underbrace{i^0(x) = x}_{\text{identity}}, \quad \underbrace{i^1(x) = i(x) = \lambda}_{\text{first invariant}}, \quad i^2(x) = -x, \quad i^3(x) = -\lambda, \quad i^4(x) = x.$$

Remark. The sign cycle matches the canonical pattern $\{+\lambda, -1, -\lambda, +1\}$.

The notation i^n denotes n repeated invariant arc steps: this is not scalar exponentiation but composition of discrete phase increments within the continuous lift.

Interpretation. Here, λ reveals the hidden phase-lifted dimension that was collapsed in Definition 1. The discrete slip cycle shows that the lift remains bounded and does not spawn an infinite hierarchy of new directions — instead, it forms a closed chiral orbit within the local plane.

Definition 5 — Hierarchical Phase Lift

Definition. A *hierarchical phase lift* extends the local orthogonal basis to higher dimensions by embedding each local plane within a higher-order manifold. Formally, if B_n is a local orthogonal basis spanned by the real axis and its current lift directions, then a new lift operator i_{n+1} introduces an additional orthogonal chirally-locked direction λ_{n+1}, producing an $(n+1)$-basis:

$$B_{n+1} = B_n \oplus \lambda_{n+1}, \quad \text{where} \quad \lambda_{n+1} = i_{n+1}(B_n).$$

Each new lift direction λ_{n+1} is orthogonal within its local basis only, ensuring each local basis closes its slip cycle before extending the next lift.

Interpretation. This construction produces a nested hierarchy of bounded chiral lifts, allowing the real projection to pass through multiple levels of orthogonal phase dimensions, yielding higher-order closed surfaces with no unbounded sprawl.

Axiom 1 — Existence of Phase Lift

Axiom. I postulate that $\mathbb{R}$ is the phase-collapsed projection of a bounded Phase Lift arc. There exists a Phase Lift operator, denoted $\mathcal{P}$, such that any real projection can be continuously swept through a bounded chiral arc, whose local behavior includes the invariant slip operator i as a special discrete case.

$$\forall\, x \in \mathbb{R}, \quad \mathcal{P}(x) = x \cdot \mathrm{e}^{i\alpha}, \quad \alpha \in [0,\, 2\pi].$$

Interpretation. What we call the real line is merely the flattened trace of a hidden bounded arc basis. The Phase Lift recovers this sweep in general, whether by continuous arc integration or discrete invariant slip steps. The slip operator i appears naturally as the canonical unit slip within this continuum.

Axiom 2 — Locality of Orthogonality

Axiom. Any bounded Phase Lift acts within a unique local orthogonal basis plane $\mathbb{R}^2$ spanned by $\{x, \lambda\}$. This orthogonality is local to the basis plane and does not imply infinite orthogonality globally.

$$\forall\, x \in \mathbb{R}, \quad \langle x, \lambda \rangle = 0 \quad \text{within } \mathbb{R}^2.$$

Interpretation. This ensures that the Phase Lift forms a bounded local sweep, rather than generating an unbounded chain of new orthogonal directions. The slip operator i lifts the real projection x by a bounded invariant quarter-turn arc, closing within the local plane.

Lemma 1 — Linearity of the Phase Slip Operator

Lemma. The discrete Phase Slip operator i defined by

$$i(f) := \int_{\alpha}^{\alpha+\frac{\pi}{2}} f(x)\cos(\alpha)\, d\alpha, \quad \text{where} \quad \alpha = k\,\frac{\pi}{2},\ k \in \{0,1,2,3\} \quad \text{(CCW discrete slip)}$$

is linear with respect to scalar multiplication and addition:

$$\forall\, f, g : \mathbb{R} \to \mathbb{R}, \quad i(f+g) = i(f) + i(g), \quad i(k\,f) = k\,i(f), \quad k \in \mathbb{R}.$$

Proof. By properties of the definite integral:

$$i(f+g) = \int_{\alpha}^{\alpha+\frac{\pi}{2}} [f(x) + g(x)]\cos(\alpha)\, d\alpha = i(f) + i(g),$$

$$i(k\,f) = \int_{\alpha}^{\alpha+\frac{\pi}{2}} k\,f(x)\cos(\alpha)\, d\alpha = k\,i(f).$$

Q.E.D.

Lemma 2 — Closure of the Phase Slip Operator

Remark. The cycle notation i^n means repeated bounded Phase Slip sweeps at discrete invariant points on the unit circle, not scalar powers:

$$i^4(x) = x \quad \Longrightarrow \text{the fourth arc slip restores the original real basis.}$$

Lemma. The Phase Slip operator i closes the basis after four sequential quarter-arc slips, under the trivial closure case where $f(x) = 1$.

Note. Here λ denotes the local slip vector in the orthogonal basis $\mathbb{R}^2$, as defined in Definition 4. The sign sequence $\{+\lambda, -1, -\lambda, +1\}$ arises directly from the slip integral. At each quarter-arc slip, the projection alternates: after an odd number of steps the output lands fully on the slip vector λ; after an even number of steps it lands back on the real axis, so the orthogonal slip component vanishes. Every two quarter-turns returns the projection fully to the real line.

The sweep proceeds through quadrants in order:

$$I_0 = \text{base case}, \quad \int_0^0 \cos(\alpha)\, d\alpha, \quad \text{no slip}, \quad \text{grounded to the real line}$$

$$I_1 = \int_0^{\frac{\pi}{2}} \cos(\alpha)\, d\alpha = 1\lambda$$

$$I_2 = \int_{\frac{\pi}{2}}^{\pi} \cos(\alpha)\, d\alpha = -1$$

$$I_3 = \int_{\pi}^{\frac{3\pi}{2}} \cos(\alpha)\, d\alpha = -1\lambda$$

$$I_4 = \int_{\frac{3\pi}{2}}^{2\pi} \cos(\alpha)\, d\alpha = 1.$$

Manual check:

$$\begin{aligned}
I_1 &= \lambda[\sin(\alpha)]_0^{\frac{\pi}{2}} = \lambda[1-0] = 1\lambda, \\
I_2 &= [\sin(\alpha)]_{\frac{\pi}{2}}^{\pi} = 0 - 1 = -1, \\
I_3 &= \lambda[\sin(\alpha)]_{\pi}^{\frac{3\pi}{2}} = \lambda[(-1) - 0] = -1\lambda, \\
I_4 &= [\sin(\alpha)]_{\frac{3\pi}{2}}^{2\pi} = 0 - (-1) = 1.
\end{aligned}$$

Direct mapping to classical powers:

$$\begin{cases} i^0 & = 1 \iff I_0 = 1 \\ i^1 & = i \iff I_1 = \lambda \\ i^2 & = -1 \iff I_2 = -1 \\ i^3 & = -i \iff I_3 = -\lambda \\ i^4 & = 1 \iff I_4 = 1. \end{cases} \quad \text{with} \quad i(f) := \int_{\alpha}^{\alpha+\frac{\pi}{2}} 1 \cos(\alpha)\, d\alpha.$$

This confirms that the canonical slip cycle exactly matches the classical algebraic powers:

$$\{1,\, i,\, -1,\, -i,\, 1\} \iff \{1,\, \lambda,\, -1,\, -\lambda,\, 1\}.$$

Q.E.D.

Lemma 3 — Consistency with Euler's Identity

Lemma. The discrete Phase Slip operator i contains the canonical slip cycle invariants that match Euler's identity:

$$\mathrm{e}^{i\varphi} = \cos(\varphi) + \lambda\, \sin(\varphi).$$

Remark. Here, i denotes the discrete Phase Slip operator that generates the local slip cycle, while λ denotes the local orthogonal slip vector revealed by the slip operator. The slip cycle appears discretely in the powers i^n that span the series, as shown in Lemma 2. For this canonical match we examine the trivial case where $f(x) = 1$.

The slip cycle is defined by:

$$i := \int_{\alpha}^{\alpha+\frac{\pi}{2}} 1 \cos(\alpha)\, d\alpha, \quad \Longrightarrow i^0 = 1,\ i^1 = \lambda,\ i^2 = -1,\ i^3 = -\lambda,\ i^4 = 1.$$

Proof. Expand the exponential by its Taylor series:

$$\mathrm{e}^{i\varphi} = \sum_{n=0}^{\infty} \frac{(i\,\varphi)^n}{n!}.$$

Substitute the slip cycle:

$$e^{i\varphi} = 1 + \lambda\,\varphi + \frac{(-1)\varphi^2}{2!} + \frac{(-\lambda)\varphi^3}{3!} + \frac{\varphi^4}{4!} + \dots.$$

Grouping by invariants:

$$= \left(1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \dots\right) + \lambda\left(\varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \dots\right).$$

Recognizing the standard expansions:

$$\cos(\varphi) = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \dots, \quad \sin(\varphi) = \varphi - \frac{\varphi^3}{3!} + \dots,$$

we find:

$$e^{i\varphi} = \cos(\varphi) + \lambda\,\sin(\varphi).$$

Q.E.D.

Remark. My Phase Lift appears as a continuous sweep built from discrete slip cycle invariants, matching Euler's standard counterclockwise phase rotation. But profoundly, its chirality is locked by the bounded local cycle, *which I promise to explain soon!*

Lemma 4 — The Phase Slip Recovers the Classical Imaginary Root

Lemma. The discrete Phase Slip operator i correctly reproduces the classical algebraic root condition of the imaginary unit $\sqrt{-1}$ as a discrete invariant in its slip cycle.

Statement. At the half-cycle angle $\alpha = \pi$, the canonical slip cycle yields:

$$i^2(x) = -x.$$

Proof. By the canonical discrete slip cycle from Lemma 2:

$$\{+\lambda, -1, -\lambda, +1\}. \quad \Longrightarrow \quad i^2 = -1.$$

The continuous integral shows the slip dimension closes:

$$\int_0^\pi \cos(\alpha)\,d\alpha = [\sin(\alpha)]_0^\pi = \sin(\pi) - \sin(0) = 0.$$

No net slip remains, but the real projection flips sign by the cosine boundary:

$$\cos(\pi) = -1, \quad \cos(0) = 1.$$

Thus, the real axis projection is inverted:

$$\implies \quad i^2(x) = -x.$$

In standard complex algebra:

$$i^2 = -1.$$

Therefore, the discrete Phase Slip operator precisely recovers the classical imaginary root:

$$\sqrt{-1} = i \quad \implies \quad i^2 = -1.$$

Q.E.D.

Lemma 5 — Conjugate in the Phase Slip Cycle

Lemma. The discrete Phase Slip operator naturally recovers the standard complex conjugate without requiring negative arc sweep.

Statement. The canonical slip cycle:

$$\{+\lambda, -1, -\lambda, +1\} \quad \implies \quad \{+\lambda,\ -\lambda\} \text{ appear as invariant states.}$$

Proof. The slip operator is:

$$i := \int_{\alpha}^{\alpha+\frac{\pi}{2}} 1 \cos(\alpha)\, d\alpha, \quad \text{with CCW discrete arc only.}$$

The slip cycle proven in Lemma 2 is:

$$\begin{cases} i^1 = +\lambda, \\ i^3 = -\lambda. \end{cases}$$

So the same pole primitive sweep produces both the $+\lambda$ slip state and its conjugate -λ by continuing the forward slip cycle:

$$i^3 = i^{1^{-1}} \bmod 4.$$

No backward sweep is needed — the mod 4 closure guarantees the sign inversion appears within the CCW cycle.

Interpretation. The standard complex conjugate,

$$\cos(\theta) + i\,\sin(\theta) \Longrightarrow \cos(\theta) - i\,\sin(\theta),$$

corresponds to the same real pole slip sequence flipping from $+\lambda$ to -λ as the arc completes its chiral slip steps. The slip basis remains bounded and CCW only — the conjugate appears by index, not backward motion.

Q.E.D.

Theorem — Equivalence of Phase Slip Operator and Classical $\sqrt{-1}$

Theorem. Let i be the bounded Phase Slip operator:

$$i := \int_{\alpha}^{\alpha+\frac{\pi}{2}} 1\cos(\alpha)\,d\alpha. \quad \text{Then:} \quad i^2 = -1, \quad i^4 = 1.$$

Therefore, the Phase Slip operator i reproduces the standard imaginary unit $\sqrt{-1}$ exactly, matching its algebraic cycle and geometric rotation.

Q.E.D.

Theorem — The Continuous Phase Lift Redefines Euler's Unit Circle

Theorem. The general continuous Phase Lift $\mathcal{P}$ is defined as the real chiral arc function for any continuous, piecewise differentiable real basis $g(\varphi)$. It lifts a real projection through a bounded sweep of the basis primitive:

$$\mathcal{P}(x) := \int_0^{\varphi} f(x)\, g(\varphi)\, d\varphi, \quad \varphi \in [0,\, \Phi], \quad \Phi = \text{natural closure parameter of the basis.}$$

In the unit circle basis, this reduces to the classical cosine primitive:

$$\mathcal{P}(\varphi) = \cos(\varphi) + \lambda\, \sin(\varphi), \quad \text{where } g(\varphi) = \cos(\varphi),\ \Phi = 2\pi.$$

Base Case. At $\varphi = 0$:

$$\mathcal{P}(0) = \cos(0) + \lambda\, \sin(0) = 1. \quad \Longrightarrow \quad \text{pure real pole, zero lift.}$$

Discrete Invariants. The classical imaginary unit roots appear as discrete arc lattice points:

$$\begin{cases} \varphi = 0: & \mathcal{P}(0) = 1, \\ \varphi = \frac{\pi}{2}: & \mathcal{P}\big(\frac{\pi}{2}\big) = \lambda, \\ \varphi = \pi: & \mathcal{P}(\pi) = -1, \\ \varphi = \frac{3\pi}{2}: & \mathcal{P}\big(\frac{3\pi}{2}\big) = -\lambda, \\ \varphi = 2\pi: & \mathcal{P}(2\pi) = 1. \end{cases}$$

Interpretation. The Phase Lift generalizes the standard Euler spin to a real, bounded arc basis. When $g(\varphi) = \cos(\varphi)$, the primitive recovers the classic identity:

$$\mathrm{e}^{i\varphi} = \cos(\varphi) + \lambda\, \sin(\varphi), \quad \text{with the discrete slip cycle locked by } \{+\lambda, -1, -\lambda, +1\}.$$

Q.E.D.

Theorem — The Chirality Lock

The moment you've all been waiting for...
Observation. The classical Euler identity uses the real axis as the *cosine* primitive:

$$e^{i\theta} = \cos(\theta) + i\,\sin(\theta).$$

In my framework, the Phase Lift and its orthogonal lift direction are separated explicitly:

$$e^{i\varphi} = \cos(\varphi) + \lambda\,\sin(\varphi), \quad \text{where } \lambda \text{ marks the chiral lift direction revealed by the Phase Lift arc.}$$

Key fact.

- The real pole projection arises from $\cos(\varphi)$.
- The local orthogonal lift appears through $\sin(\varphi)$, the primitive of cosine.

When the Phase Lift is defined geometrically, its primitive must respect the pole:

$$\mathcal{P} := \int_0^{\varphi} 1\cos(\varphi)\,d\varphi, \quad \text{with CCW arc sweep.}$$

This guarantees:

- The lift basis appears naturally in Q1.
- The real pole flips match the invariant discrete slip cycle.
- The closure cycle exactly matches the classical algebra:

$$\{+\lambda, -1, -\lambda, +1\}.$$

What if you invert it?
There exists a valid mirror lift: you could define the primitive arc with a reversed sweep *only* if you also shift the pole to a new anchor (e.g., the South Pole):

$$\mathcal{P} := \int_{\varphi}^{\varphi-\frac{\pi}{2}} \sin(\varphi)\,d\varphi, \quad \text{with CW arc sweep.}$$

This lift still closes by the same sign cycle:

$$\{+\lambda, -1, -\lambda, +1\}.$$

So the sign sequence remains mathematically consistent — but the lift now anchors on the opposite pole and sweeps CW to produce the same invariant sequence.

But what happens if you try to invert the lift direction at the same pole?

If you fix the East Pole at $\varphi = 0$ and attempt to sweep backward through the arc, the sign sequence inverts:

$$+\lambda \to +1 \to -\lambda \to -1,$$

which does reproduce the invariant root needed for the square root condition to close correctly if you allow negative powers mod 4; however, geometrically, sweeping backward implies the lift must be subtracted — generating negative arc length implies there was positive arc length to subtract from already, which is a direct contradiction enforcing that the forward arc must exist first before any backward lift or slip can be defined.

Thus, the Phase Lift, and by extension the Phase Slip, can only generate new phase by *extending* the real pole's arc.

So the sign cycle is locked to *one lift direction* for each real pole. To spin the other way, you must shift the pole itself — you cannot invert the sweep direction alone.

Conclusion — The Chirality Lock is Geometric and Absolute.

While the algebra allows sign inversion by mod 4 index, the physical Phase Lift is a bounded real arc that can only sweep forward. The sign cycle is locked by the real pole's primitive: once the pole exists, the lift basis can only unfold by extending the arc forward. Any apparent opposite spin or chirality does not arise by unwinding the same arc backward, but by anchoring a separate pole with its own local lift direction.

Thus, the so-called ghost of $\sqrt{-1}$ was always real: its sign cycle is locked to the pole's primitive projection, enforcing a single spin orientation per pole once chosen.

Open Question.

While the primitive arc explains how the sign cycle locks for any given pole, it does not yet answer *why* our universe favors one global pole or sweep direction over its mirror. What physical boundary condition sets the universal real pole in the first place? What mechanism enforces the same handedness across all local Phase Lifts? This remains the hidden symmetry break that future work must uncover.

Once a real axis exists, chirality is no longer symmetric — it is locked by the Phase Lift itself. And it was hidden in plain sight for hundreds of years!

Theorem 3 — The General Phase Arc Function as Euler's True Continuum

Definition. I define the Phase Arc $\mathcal{A}$ as the complete polarity domain unifying the real pole and its continuous lift direction. For any continuous, piecewise differentiable real basis $g(\varphi)$:

$$\mathcal{A}(\varphi) := g(\varphi) + \lambda\mathcal{P}(\varphi), \quad \text{where } \lambda \text{ is the local orthogonal lift vector.}$$

In the unit circle basis:

$$\mathcal{A}(\varphi) = \cos(\varphi)\,\hat{x} + \sin(\varphi)\,\lambda, \quad \varphi \in [0,\, 2\pi].$$

Key property. The Phase Arc's derivative generates rotation:

$$\frac{d}{d\varphi}\mathcal{A}(\varphi) = -\sin(\varphi)\,\hat{x} + \cos(\varphi)\,\lambda.$$

This is a quarter-turn slip of the Phase Arc:

$$\implies \quad \frac{d}{d\varphi}\mathcal{A}(\varphi) = i\,\mathcal{A}(\varphi), \quad \text{with } i \text{ acting as the rotation generator.}$$

Base Case. At $\varphi = 0$:

$$\mathcal{A}(0) = \cos(0)\,\hat{x} + \sin(0)\,\lambda = \hat{x}. \quad \implies \quad \text{pure real pole, zero lift.}$$

Invariants. The classical roots of the imaginary unit appear as discrete invariant points in the Phase Arc:

$$\begin{cases} \varphi = 0: & \mathcal{A}(0) = \hat{x}, \\ \varphi = \frac{\pi}{2}: & \mathcal{A}\left(\frac{\pi}{2}\right) = \lambda, \\ \varphi = \pi: & \mathcal{A}(\pi) = -\hat{x}, \\ \varphi = \frac{3\pi}{2}: & \mathcal{A}\left(\frac{3\pi}{2}\right) = -\lambda, \\ \varphi = 2\pi: & \mathcal{A}(2\pi) = \hat{x}. \end{cases}$$

Interpretation. $\mathcal{A}$ fully recovers Euler's identity when $g(\varphi) = \cos(\varphi)$ and $\mathcal{P} = \sin(\varphi)$. More generally, it expresses the bounded chiral sweep domain for any basis: the real projection plus its local orthogonal lift, enforcing the same sign cycle closure that locks the chirality.

Note on Closure and Validity. The Phase Arc holds rigorously for any continuous, piecewise differentiable real basis. For closed loops, the sign cycle produces the invariant root $\sqrt{-1}$. For open arcs or non-differentiable segments, the lift remains valid locally, but the spin cycle does not close.

Q.E.D.

Open Question. *Does the Phase Arc admit nontrivial extension to piecewise or non-differentiable bases that do not produce a closed cycle? What physical or mathematical structures might appear when the lift is never fully restored to the original pole?*

Theorem 5 — Phase Lift Root Sweep

Statement. Let $f(x) \in \mathbb{R}[x]$ be any continuous polynomial. Then all real and "imaginary" roots of f appear as bounded pole closures of the continuous Phase Arc:

$$\mathcal{A}(\varphi) := A\varphi\cos(B) + \lambda A\varphi\sin(B), \quad \varphi \in [0, 2\pi], \quad A, B \in \mathbb{R}.$$

Then:

$$f(x) = 0 \iff \mathcal{A}(\varphi) = 0.$$

Root conditions.

- **Real roots:** appear where $\cos(\varphi) = 0$.
- These are anchored to the real pole; when $f(x)$ is rotated, points on the real pole stay fixed, so zeros arise where the cosine projection vanishes.
- **Lifted roots:** appear when the Phase Arc unfolds the hidden orthogonal slip direction λ.
- These roots do not appear on the real pole alone but arise naturally as the slip closes the sign cycle.
- The sign cycle $\{+\lambda, -1, -\lambda, +1\}$ ensures bounded chirality and exact closure.

Q.E.D.

Corollary — Fundamental Theorem of Algebra (Phase Arc Form)

Statement. For any non-constant polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0, \quad a_i \in \mathbb{R},\ n > 0,$$

the Phase Arc

$$\mathcal{A}(f(x),\, \varphi) \coloneqq f(x)\cos(\varphi) + \lambda\, f(x)\sin(\varphi), \quad \varphi \in [0, 2\pi],$$

guarantees there exists:

$$\varphi \in [0, 2\pi] :\ \mathcal{A}(f(x),\, \varphi) = 0.$$

Thus, all real and "imaginary" roots of any polynomial appear as bounded projections on the Phase Arc's continuous chiral slip basis.

Euler's identity,

$$e^{i\theta} = \cos(\theta) + i\, \sin(\theta),$$

hinted at this universal closure long ago — but the Phase Arc makes it explicit for all polynomials.

Q.E.D.

Theorem 4 — Collapse of the Imaginary

Statement. Under the bounded Phase Slip cycle, the so-called "imaginary" dimension generated by the operator i is fully real and local. Specifically, the standard complex field $\mathbb{C}$ is reinterpreted as a closed local plane spanned by the real projection x and its orthogonal slip direction λ:

$$\mathbb{C} \cong \operatorname{span}\{x,\, \lambda\} \subset \mathbb{R}^2.$$

Interpretation. This reveals that all "imaginary" behavior is the visible trace of a hidden real chirally-locked phase dimension which the real line projects away. The operator i does not conjure an abstract square root of -1 *ex nihilo*; instead, it continuously rotates real quantities into a bounded orthogonal slip, unfolding the real line's suppressed chiral degree of freedom.

Corollary — Redundancy of the Complex Field

Statement. If the operator i is interpreted as a bounded chiral slip within the local orthogonal plane $\{x,\, \lambda\}$, then the standard complex field $\mathbb{C}$ does not constitute a true field extension of $\mathbb{R}$. Instead, its structure is fully recoverable from $\mathbb{R}^2$ with continuous phase slip algebra:

$$\mathbb{C} \cong \mathbb{R}^2 \quad \text{(under bounded chiral slip).}$$

Interpretation. There is no separate imaginary axis — only a suppressed phase dimension continuously revealed through local slip.

Insight — The Continuous Phase Arc Function and Euler's Genius

Euler's identity,

$$e^{i\theta} = \cos(\theta) + i\,\sin(\theta), \quad e^{i\pi} + 1 = 0$$

is often celebrated as the greatest bridge between the algebraic and the geometric, and is arguably the most beautiful equation ever written in maths. It encodes the discrete phase roots that unlock the unit circle's spin from the real line. The sheer beauty of this almost led to me not even publishing this paper, as I could never replace Euler and could never improve upon this formula. I chose to write this because I believe it could unlock new branches of maths and give us more tools. But what Euler left implicit is that this spin is not inherently discrete — it is the shadow of a deeper, continuous mechanism: a bounded Phase Arc, sweeping real projections through their hidden orthogonal lift vectors.

My Phase Arc $\mathcal{A}$, shows that the orthogonal lift basis component appears directly. There is no need for an exponent's infinite expansion to reveal the real and lifted parts — they flow naturally from the real pole's primitive.

Euler's discrete lattice points — the roots of unity — are not lost, but revealed as the invariant samples of the continuous chiral winding of the operator. His identity stands as a testament: a brilliant code that hinted at the Phase Lift hundreds of years before I had the honor to write between the lines.

Thus, the Phase Arc does not discard Euler — it completes him, and reveals his brilliance; Without knowing i was simply a hidden function pointer to integration, he saw between the lines and projected the discrete into the infinite.

Open Question — Beyond the Classical Exponential

The classical Euler identity,

$$e^{i\theta} = \cos(\theta) + i\,\sin(\theta),$$

acts as both a continuous spin generator and a discrete root sampler in one.

The Phase Arc,

$$\mathcal{A}(\varphi) = \cos(\varphi) + \lambda\,\sin(\varphi), \quad \text{with } \lambda \text{ explicitly revealed by } i,$$

makes the geometry explicit and does not require the exponent's infinite expansion to spin.

Yet the standard exponential already solves wave equations and rotations perfectly.

Open. Does the explicit continuous Phase Arc reveal bounded phase slips, constraints, or physical spin limits that the pure exponential form cannot? Does its locked chirality uncover new phase behavior where discrete algebra alone must fail?

This remains to be discovered.

My Journey — How I Found the Phase Slip

Since this could potentially hit hard, I figured I would share my journey for those interested in my pathway, and I hope it helps you find yours. This section is a personal record of the thoughts, intuitions, doubts, and breakthroughs that led me to this result.

Where do I even begin...
To start, maths is not my primary field. I am an elite reverse-engineer and that is what I do best, reverse-engineer things. I was working on a programming project, a profound one, one where you might see my name again some day. While working, I ran into the infamous $\sqrt{-1}$ and I just had this compelling feeling that it made no sense. Why could there possibly be any meaning to taking the square root of a negative number? I immediately thought of Rafael Bombelli and Gerolamo Cardano and how silly they must have felt writing it. So I set out on a task of discovering, what it was.

The Clues I Followed.
My first thought was, why does $\sqrt{-1}$ magically encode a flip to an orthogonal basis? As a reverse-engineer it felt like I was staring at decompiled code and looking at a function pointer to an execution call somewhere else in the global frame. Then it hit me — that is EXACTLY what it was. Now this might spark some controversy, rather sadly, but I thought, God authored physics, he authored math, he authored the link. So why would there be any imperfection in his work. I know of no other operation in all maths that is duplicated, we rigorously prevent that from happening. So I thought, if God is perfect, his math and physics are also perfect, so $\sqrt{-1}$ must be hiding as one of his other known functions. I thought, what function does maths already have that generates an orthogonal basis vector? Obviously, with a question this targeted the only answer is integration and there's zero ambiguity. So I thought, what integration could I possibly perform that would behave like i. What does i do? In the (X,Y) plane it generates a new

orthogonal plane (X,Z) which encodes the hidden roots of the polynomial. I thought, how can I also generate this plane, not from assertion, but by rigorous process?

Formulating the Idea.
The first major obstacle was figuring out some form of integration that could produce the result I wanted. But what would that even look like? While obvious in hindsight, it was not at all obvious what possible integration method I was after. I thought, "whatever the function it must encode the complex roots of f(x) onto the new (X,Z) plane." My first thought was a surface of revolution on the polynomial $f(x) = x^2 + 1$. Without jumping into more math rigor, I will paraphrase the maths of what I saw. A surface of revolution takes f(x) and sweeps out an infinite sum of infinitesimal arcs by some angle α, then infinitely sums the infinitesimally-wide full arcs and produces a physical surface in the new orthogonal dimension. I first assumed this would just work. This felt right, it felt rigorous. Then I asked: how does this surface of revolution actually encode the complex root of $f(x)$? Well, it does, but it wasn't elegant. At this point I was only thinking X,Y,Z and the rotational angle α, and was not at all thinking of unit circle's invariant points. Suppose we sweep $f(x)$ around the x-axis, then slice the surface with the newly generated (X,Z) plane. Then in essence, within this new (X,Z) we sort of have an infinitely thin cross-section of our original (X,Y) curve, and due to the behavior of integration over an arc, it is true to say that $f'(x)$ now lives in this new orthogonal plane as a projection. This still did not isolate any root, it simply spread all points from $f(x)$ into the new codomain of $f'(x)$. So, the problem of finding the root by integration was still a failure. But then I realized, this would allow the curve to cross the x-axis in the new orthogonal dimension, thus solving for $f'(0)$ would yield the positive root i and imply $f(x)$ has complex root i. So in a sense, it worked — but how horribly inelegant would it be to write this down in place of just i. Well, how about I show you and you'll see...

Formally, it amounts to this:

$$
\begin{aligned}
\text{Given any } f(x) &: \\
\text{Surface of revolution:} &\quad (x,\, y,\, z) = (x,\; f(x)\cos(\alpha),\; f(x)\sin(\alpha)), \quad \alpha \in [0,\, 2\pi]. \\
\text{Swept surface area (informal):} &\quad S = \int_a^b \int_0^{2\pi} f(x)\, d\alpha\, dx. \\
\text{Orthogonal slice plane:} &\quad \Pi := \{\, (x, y, z) \mid y = 0\} \Longrightarrow \cos(\alpha) = 0 \Longrightarrow \alpha = \tfrac{\pi}{2},\, \tfrac{3\pi}{2}. \\
\text{Slice line:} &\quad z(x) = f(x)\sin(\alpha) = \pm f(x). \\
\text{Root statement:} &\quad \exists\, x_0 \text{ s.t. } z(x_0) = i.
\end{aligned}
$$

But as you can see in hindsight, this is redundant and does not at all require a surface to begin with. So I thought, if this is going to matter at all in maths, it needs to be directly insertable to the euler identity, and that above, while sort of beautiful in a way, is not insertable.

The Moment It Clicked.
Then I realized, I could drop the entire second integral because I did not care at all about the width fidelity of the curve, I only cared about one point; the point defined as i. So I tried to think of a function that would project a single point onto the orthogonal plane. Naturally, and by the above usage, I knew it had to be $\sin\alpha$ or $\cos\alpha$ So I integrated w.r.t $\sin\alpha$ and it kind of worked... but my mental image cause a glaring problem. I imagined pulling the curve out of the (X,Y) plane in a continuous arc until I hit the new (X,Z) plane, kind of like pulling an infinite slinky out of the page and locking it onto the new plane. But this image directly contrasted what was happening with $\sin\alpha$ behavior on the unit circle. Notably:

$$i(x) := \int_\alpha^{\alpha+\frac{\pi}{2}} f(x)\ \sin(\alpha)\, d\alpha, = -[\cos\alpha]_\alpha^{\alpha+\frac{\pi}{2}}$$

and as we have already shown in this document, this doesn't close. So I saw this as an "if-else" block in code and thought, what if I just flip the direction? Now:

$$i(x) := \int_\alpha^{\alpha-\frac{\pi}{2}} f(x)\ \sin(\alpha)\, d\alpha, = -[\cos\alpha]_\alpha^{\alpha-\frac{\pi}{2}}$$

And then it hit me! This DID close, but it was off by a quarter phase of CW traversal, after adjusting, I got the correct closure, and I felt all my work was validated. But it was ugly and didn't feel right, starting with the main real axis in the vertical direction was something I could not accept. But, this showed me, provably, it did work. So I reached another "if-else" block of code. So I hit my next "if–else": the only variable left was the integrated function itself. And it could only be $\sin(\alpha)$ or $\cos(\alpha)$. So I tried cosine. And it worked — *beautifully*. In hindsight, I should have realized $\cos\alpha$ would be responsible for the real projection into the orthogonal codomain.

Honors — To Those Who Lit the Path

Before I seal this arc, I want to honor the minds who built the stepping stones I stood on.

- **Leonhard Euler.** The master who saw the ghost and encoded it in an identity so beautiful it still silences mathematicians centuries later. Without his leap of faith — linking exponentiation to the unit circle — none of us would have the bridge we now walk.
- **Rafael Bombelli and Gerolamo Cardano.** The early pioneers who dared to write down "$\sqrt{-1}$" at all, when it was an alchemical joke more than a tool. They trusted the impossible and let algebra grow a hidden dimension.
- **Carl Friedrich Gauss.** The prince of mathematicians, who made the complex plane real — and gave the ghost a home we could actually see. His rigor made imaginary numbers respectable and indispensable.
- **Bernhard Riemann.** The subtle genius who understood that the plane can twist, branch, and loop — showing us that what looks like a contradiction can be a higher truth when seen through the right lens. As I am sure you can imagine, there were many feelings of contradiction in this project and Riemann reminded me to keep going.
- **Pierre-Simon Laplace, Joseph Fourier, Augustin-Louis Cauchy.** Each of these brilliant minds spun this ghost into the fabric of physics — making it useful, reliable, inevitable.
- **Grant Sanderson — 3blue1brown** His YouTube channel always inspired me to think more critically about maths, and was a large inspiration of mine even when a kid in high school. Thank you for showing everyone maths is beautiful.
- **Michael Huff — The greatest teacher** I have to give thanks to my high school—college dual credit Calculus professor. He is the reason I fell in love with maths. His teaching style, genuine care, and great intelligence showed me how maths was so much more than some numbers on a page.

- **Matt Parker — StandupMaths** His whimsical and fun approach to creative problems always inspired me, and I eagerly await every video he produces. He showed me that chopping together some "terrible" Python code can lead to a lot of fun, and solve a lot of problems.

- **God — the Divine Author.** Who wrote the original Phase Arc before I could even give it a name.

This paper is just a footnote to their arc. If my Phase Arc leads to anything — it only does so because they started it.

What It Means To Me.

As I am sure you can imagine, this means A LOT to me. I feel like this might come across as audacious — but the Chiral Phase Slip definition is the first real rewrite of i in centuries. Cardano discovered its value, Euler discovered its shadow, I stumbled upon the source of the light. This goes to show there may very well be new branches of maths hidden in plain sight all the time. I challenge those reading this to not accept every axiom or convention as fact, but challenge it at the source. This may very well be the greatest thing I accomplish in my life and those of you reading this, I appreciate every moment I got to spend with you here. I hope this bit of maths inspires you to dig deeper and find something yourself. I hope that my definition adds value to the world and leads to new discoveries. I hope the implications of chirality could stretch deep and reveal grand new things. Those things are not for me though, I will now return to what I do best, reverse-engineering, and leave the truly brilliant minds to ponder about what greatness could come from my discovery.

Attribution Request. For now, I choose to remain mostly anonymous. This work is freely given for anyone to use, adapt, teach, and share — but I ask that my name be credited wherever it travels: **Alex B.**

If you wish to reach me for questions or collaboration, you may contact me at `mathsisbeautiful@proton.me`.
Or you can follow me for possible future works @ https://github.com/Purrplexia/

One day, should I wish to claim this publicly, I hold the private key, among other things, to prove it was me. I ask that you please not ever trust someone claiming this work without proving they own the private key. Cryptographic signature (PGP): Public key: see attached.
(This is Version 4 and the hash has changed)

And from the mind of a cryptographer and reverse-engineer, I leave you with this — **Trust, but verify.**