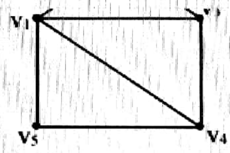


G has a Hamiltonian circuit C: v_1 - v_5 - v_2 - v_3 - v_4 - v_1. Hence G is a Hamiltonian graph.

Remember:

(i) Let G be a graph with n≥3 vertices. If deg(v)≥ n/2 for all vertices of G, then G is Hamiltonian.

(ii) A graph with n vertices and with no loops or parallel edges which has at least $\frac{1}{2}(n-1)(n-2)+2$ edges is Hamiltonian.

(iii) The complete graph K_n (n ≥ 3) is Hamiltonian.

(iv) The complete bipartite graph $K_{m,n}$ is Hamiltonian if and only if m = n and n > 1.

(v) For a simple connected graph G with n (≥ 3) vertices has a Hamiltonian circuit if the degree of each vertex is at least $\frac{n}{2}$.

11.8 Vertex and Edge connectivity:

11.8.1 Cut vertex:

Let v be a vertex of a connected graph G such that G – v is not connected. Then the vertex v is called cut vertex.

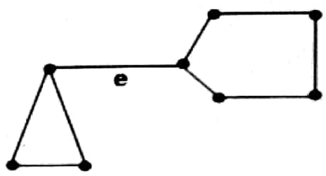
In other words, if the removal of a vertex v from a connected graph G, makes the graph disconnected, then the vertex v is called a cut vertex.

v_1 (≠v) and v_2 (≠v) such that every path connecting the vertices v_1 and v_2 contains the vertex v.

11.8.2 Cut edge (bridge):

Let 'e' be an edge of connected graph G such that G – e is disconnected. Then the edge 'e' is called cut edge. In other words if the removal of an edge 'e' from a connected graph G, makes the graph disconnected, then the edge 'e' is called a cut edge.

Example:

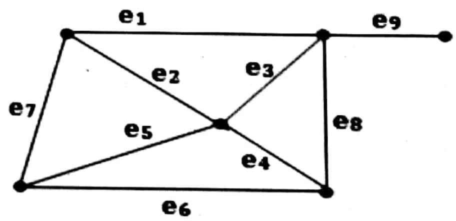


Here e is a cut edge.

11.8.3 Cut set:

A cut set in a graph G is a set of edges whose removal from the graph; makes the graph disconnected, provided no proper subset of it disconnects the graph G.

Example:



Here some of cut set are {e_1, e_2, e_3, e_6} and {e_1, e_2, e_7}

Note:

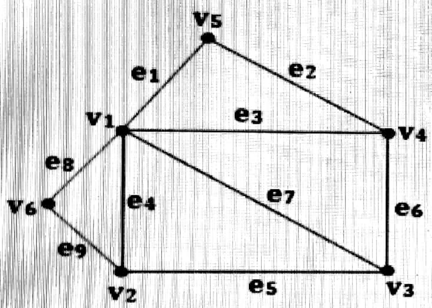
colour theorem).

(xi) A planar graph can be coloured with 5 colours **(Five colour theorem).**

11.9.3 Matching:

A matching in a graph G is a subset of edges in which no two edges are adjacent i.e; no two edges in G have a common end vertex.

Example:



Here $M = \{e_2, e_7, e_9\}$ is a matching.

✓ **Remember:**

(i) If M is a matching in a graph G with edges set E, then M is called a maximal matching if $M \cup \{e\}$ is not a matching for arbitrary $e \in E - M$ i.e. a maximal matching is a matching for which no edge in the graph can be added.

(ii) The maximal matching with the largest number of edges are called maximum matching.

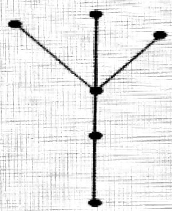
and only if E contains no path of length three or more.

11.10 Tree

11.10.1 Definition of a Tree:

A tree is a connected graph without any circuits.

Example:



✓ **Remember:**

(i) A tree is a connected and cyclic graph

(ii) A directed tree is a connected , cyclic and directed graph.

(iii) A simple non- directed graph G is a tree if and only if G is connected and has no cycles.

(iv) Any non-trivial tree has exactly one vertex of degree 1.

(v) A tree has n vertices exactly $n-1$ edges.

(vi) Every non-trivial tree has at least 2 vertices of degree 1.

(vii) There is one and only one path between every pair of vertices in a tree T.

(viii) If in a graph there is one and only one path between every pair of vertices, then G is a tree.

(ix) Any connected graph with n vertices and n-1 edges is a tree.

(x) A graph is a tree if and only if it is minimally connected.

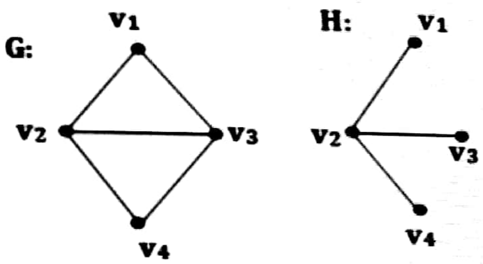
11.10.2 Spanning tree:

Let G be a connected graph. Then a sub graphs H of G is called a spanning tree of G if

(i) H is a tree

(ii) H contains all the vertices of G.

Example:



✓ **Remember:**

(i) A non- directed graph G is connected if and only G contains a spanning tree.

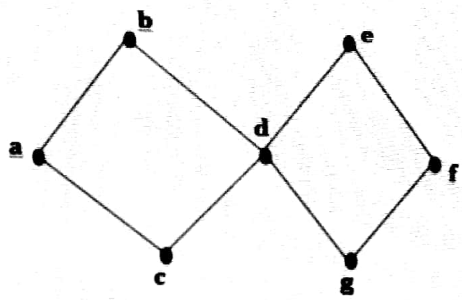
(ii) The complete graph K_n has n^{n-2} different spanning trees.

❖ **Construction of spanning trees:**

(I) BFS (breath first search) algorithm:

Step-I: Chose a vertex arbitrarily and designate it as a root.

Step-II: Add all the edges incident to this



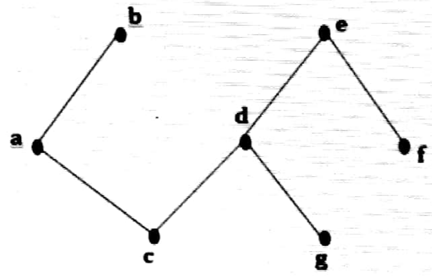
We chose 'a' as the root. Then add edges (a, c) and (a, b) as they are incident with 'a'. The vertices 'b' and 'c' are at level 1.

Next we add the edge (c, d). The vertex 'd' is at level 2.

Next we consider the edges (d, b), (d, g) and (d, e) as they are incident with 'd'. Out of these only two edges (d, g) and (d, e) can be added [since the edge (d, b) forms a cycle]. Then the vertices 'g' and 'e' are at level 3.

Finally we add the edge (e, f) at level 3.

The spanning tree so obtained is given by:



(II) DFS (depth first search) algorithm:

Step-I: Chose a vertex arbitrarily and designate it as a root.

Step-II: Form a path starting from this root by successively adding edges as long as possible where each new edge is incident with the last

v_4 v_4

✓ **Remember:**

(i) A non- directed graph G is connected if and only G contains a spanning tree.

(ii) The complete graph K_n has n^{n-2} different spanning trees.

❖ Construction of spanning trees:

(I) BFS (breath first search) algorithm:

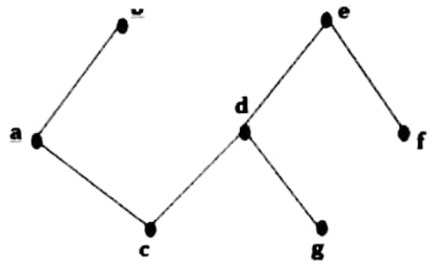
Step-I: Chose a vertex arbitrarily and designate it as a root.

Step-II: Add all the edges incident to this vertex so that no loop is formed. The new vertices added at this stage become the vertices at level 1 in the spanning tree.

Step-III: For each vertex at level 1, add each edge incident to this vertex as long as it doesn't produce any cycle. This produces the vertices at level 2 in the spanning tree.

Step-IV: Continue the same process until all the vertices of the given tree are added.

Example:



(II) DFS (depth first search) algorithm:

Step-I: Chose a vertex arbitrarily and designate it as a root.

Step-II: Form a path starting from this root by successively adding edges as long as possible where each new edge is incident with the last vertex in the path and no cycle is formed.

Step-III: If the path goes through all vertices of the graph, the tree so obtained becomes a spanning tree. Otherwise go to step IV.

Step-IV: Move back to the next to last vertex in the path and if possible, form a new path starting from this vertex passing through the vertices that were not already visited. If this can't be done, move back to another vertex in the path and repeat the process.

Step-V: Repeat the procedure until no more edges can be added.

11.10.5 Rooted tree:

A rooted tree is a tree in which a particular vertex is distinguished from the others.

Remember:

(i) The level of a vertex is the number of edges along the unique path between it and the root. The level of the root is defined as '0'.

(ii) The height of a rooted tree is the maximum level to any vertex of the tree.

(iii) The depth of a vertex 'v' in a tree is the length of the path from the root to 'v'.

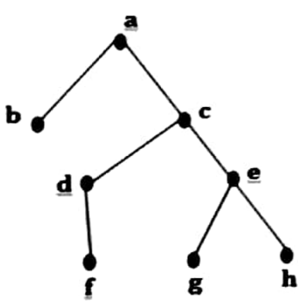
(iv) Given any internal vertex 'v' of a rooted tree, the children of 'v' are all those vertices that are adjacent to 'v' and are one level farther away from the root than 'v'.

(v) If a vertex 'v' has one or two children, then 'v' is called an internal vertex.

(vi) The descendants of the vertex 'v' is the set consisting of all the children of 'v' together with the descents of those children.

Example:

Consider the rooted tree given below:



Here,
'a' is the root of the tree.
Height of the rooted tree is 3.
The root 'a' is at level '0'; 'c' are at level '1'; the vertic level '2'; the vertices 'f', 'g', 'h' are at level '3'.

(iii) The maximum number of vertices in a binary tree of depth 'd' is 2^{d}-1.

10.5.6 TRAVERSAL OF A TREE:

Traversal of a tree is a process to traverse a tree in a systematic way so that each vertex is visited exactly once.

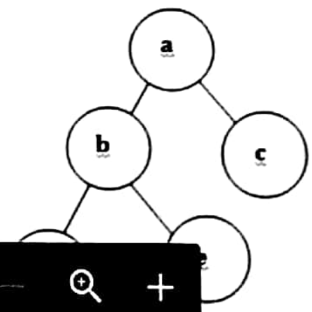
To implement this, a tree is considered to have three components: root, left sub-tree and right sub-tree. These three components can be arranged in six different ways: (left, root, right), (root, left, right), (left, right, root), (right, left, root), (right, root, left) and (root, right, left). The first three are used where as the last three combinations are of no use as it alternates the positions of a node in a positional tree.

In-order traversal: In this form of traversal, a tree is traversed in the sequence: left sub-tree, root, right sub-tree.

Pre-order traversal: In this form of traversal, a tree is traversed in the sequence: root, left sub-tree, right sub-tree.

Post-order traversal: In this form of traversal, a tree is traversed in the sequence: left sub-tree, right sub-tree, root.

Consider the following tree:



The children of 'c' are d and e.
The descendants of 'a' are b, c, d, e, f, g, h.

Remark:

(i) A rooted tree is an m-ary if every internal vertex has at most m children. A m-ary tree is called a full m-ary tree if every internal vertex has exactly m children. Thus in particular, a full binary tree is a binary tree in which each internal vertex has exactly two children.

(ii) A full m-ary tree with ' i' internal vertex has n=mi+1 vertices.

c, Pre-order traversal is a b d e c and Post-order traversal is d e b c a.

11.11 Fully solved MCQs:

01. Which of the following is a planar graph?

(a) a complete graph with 4 vertices

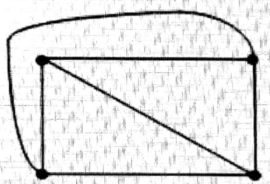
(b) $K_{2,2}$

(c) $K_{2,3}$

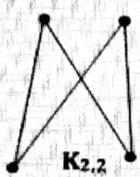
(d) all of these

Ans: (d)

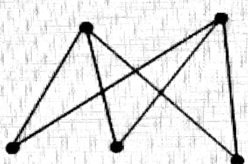
Explanation:



K4



K2,2



K2,3

Each of these graphs can be drawn without crossover of edges .

02. If f denotes the number of region in a graph G, then

(a) $n = f$ (b) $n \geq$

$\sum_i \deg(V_i) = 2|E| = 2 \times 14 = 28$

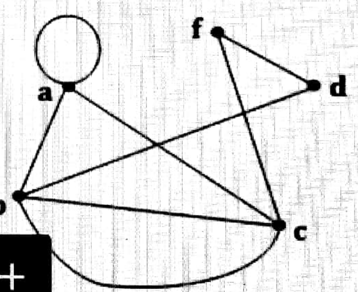
Let x be the no . of vertices in the graph G.

Again

$\sum_i \deg(V_i) < 5 \times 3 + 3x$

$\Rightarrow 28 < 15 + 3x \Rightarrow 3x > 13 \Rightarrow x > \frac{13}{3}$.

05. The incidence matrix of the following graph is



Each of these graphs can be drawn without crossover of edges .

02. If f denotes the number of region in a graph G, then

(a) $n = f$ (b) $n \geq 2 + \frac{f}{2}$

(c) $n \leq 2 + \frac{f}{2}$ (d) none of these

Ans: (b)

Explanation:

$3f \geq 2|E| \Rightarrow |E| \geq \frac{3}{2} f$

Now by Euler formula ,

$|V| = |E| - |R| + 2 \geq \frac{3}{2} f - f + 2$

$\Rightarrow n \geq \frac{f}{2} + 2$

03. A non directed graph has 10 edges. If the degree of each vertex is 2, then the number of vertex will be

(a) 10 (b) 5 (c) 15 (d) 8

Ans: (a)

Explanation:

$\sum_i \deg(V_i) = 2|E|$

$\Rightarrow 2|V| = 2 \times 10$

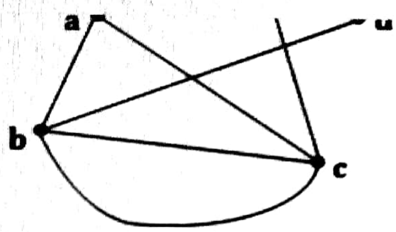
$\Rightarrow |V| = 10$

04. Let G be a non directed graph with 14 edges . If G has 5 vertices each of degree 3 and rest have degree less than , then the minimum number of vertices of G is

(a) 5 (b) (c) 7 (d) 8

Ans: (a)

Explanation:



(a) $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

(d) none of these

Ans: (a)

Explanation:

Use the definition of incidence matrix.

06. The adjacency matrix of the following graph is

$$\begin{matrix} 0 & 1 & 1 & 1 & 0 \end{matrix}$$

(c) $\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

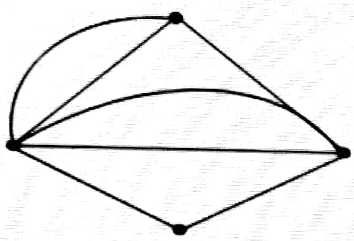
(d) none of these

Ans: (a)

Explanation:

Use the definition of adjacent matrix.

07. The order and size of the following graph are respectively



(a) 6, 6 (b) 6,7 (c) 4, 7 (d) 8,8

Ans: (c)

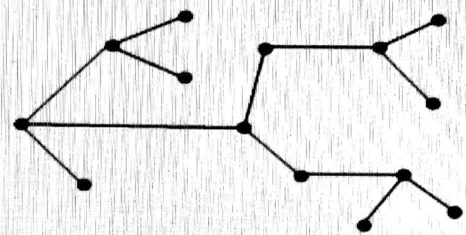
Explanation:

Order of the graph=no. of vertices=4 and size of the graph=no. of edges=7.

08. The chromatic number of the following graph is

the chromatic number of the graph is '3'.

09. The chromatic number of the following graph is



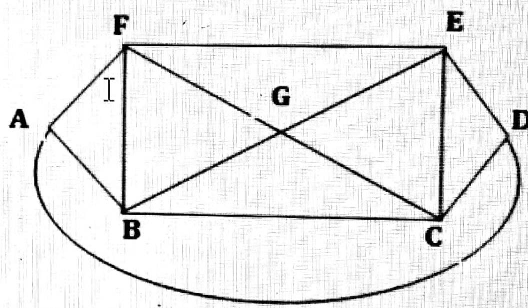
(a) 4 (b) 5 (c) 2 (d) 3

Ans: (c)

Explanation:

Here the graph is a tree. So the chromatic number of the graph is '2'.

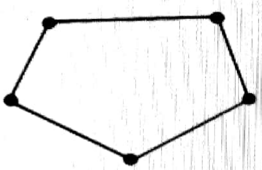
10. The chromatic number of the following graph is



(a) 4 (b) 5 (c) 2 (d) 3

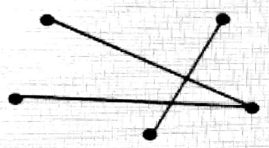
Ans: (c)

11. The complement of the following graph is

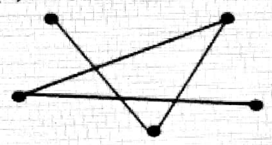


(a)

(b)



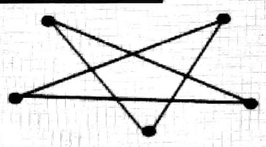
(c)



(d) none of these

Ans: (c)

Explanation:



Remember:

The *complement* of a graph G is simple graph with the same vertex set as G and where two vertices 'u' and 'v' are adjacent only when they are not adjacent in G.

12. Which of the following is not true

(a) K_n has n vertices and $\frac{n(n-1)}{2}$ edges.

13. The size of a simple graph of order n cannot exceed

(a) ${}^nC_2$ (b) ${}^nC_3$ (c) ${}^nC_4$ (d) n

Ans: (a)

Explanation:

Let G=G(V, E) be a graph of order n. Then V contains 'n' elements as vertices and elements of E are distinct two elements subsets of V. But the number of ways we can choose two elements from V is ${}^nC_2$. Thus E contains maximum ${}^nC_2$ edges.

14. Which of the following is possible?

(a) a simple graph with 4 vertices and 8 edges

(b) a simple graph with 4 vertices and 6 edges

(c) a simple graph with 4 vertices and 7 edges

(d) a simple graph with 4 vertices and 9 edges

Ans: (b)

Explanation:

We know that if a simple graph G has 'n' vertices, then it has $\frac{n(n-1)}{2}$ edges.

Thus a simple graph with 4 vertices has $\frac{4(4-1)}{2}$ i.e, 6 edges.

15. What is the size of a k-regular (n, q) graph?

(a) $\frac{nk}{2}$ (b) $\frac{(n-1)k}{2}$

(c) $\frac{n(k-1)}{2}$ (d) nk

Ans: (a)

Hence size of the graph= $q = \frac{nk}{2}$.

16. Which of the following is not possible?
(a) a 4-regular graph with 6 vertices
(b) a 3-regular graph with 14 vertices
(c) a 3-regular graph with 17 vertices
(d) a 5-regular graph with 8 vertices

Ans: (c)

Explanation:

We know that a 'k-regular' graph has $\frac{nk}{2}$ edges if 'n' is the number of vertices.

- Case-I: If n=6, k=4; then

total number of vertices $= \frac{6\times4}{2} = 12$, which is a positive integer.

Hence a 4-regular graph with 6 vertices exist.

- Case-II: If n=14, k=3; then

total number of vertices $= \frac{14\times3}{2} = 21$, which is a positive integer.

Hence 3-regular graph with 14 vertices exist.

- Case-III: If n=17, k=3; then

total number of vertices $= \frac{17\times3}{2} = \frac{51}{2}$, which is not a positive integer.

Hence 3-regular graph with 17 vertices doesn't exist.

- Case-III: If n=8, k=5; then

total number of vertices $= \frac{8\times5}{2} = 20$, which is a positive integer.

Hence 5-regular graph with 8 vertices exist.

17. If a graph G with 'n' vertices is isomorphic

of edges in G = total number of edges ...

This gives total number of edges in G= $\frac{n(n-1)}{4}$

$\left[\because \text{total number of edges in } K_n \text{ is } \frac{n(n-1)}{2}\right]$.

Since $\frac{n(n-1)}{4}$ must be a positive integer, so either 'n' or 'n-1' must be a multiple of 4.

18. A simple graph with 6 vertices is connected if it has
(a) more than 10 edges
(b) more than 20 edges
(c) less than 10 edges
(d) less than 20 edges

Ans: (a)

Explanation:

We know that a simple graph with 'n' vertices is connected if it has more than $\frac{(n-1)\times(n-2)}{2}$ edges

Therefore a simple graph with 6 vertices is connected if it has more than $\frac{(6-1)\times(6-2)}{2}$ *i.e*, 10 edges.

19. The rank and nullity of the complete graph K_n are respectively
(a) $n-1$ and $\frac{(n-1)(n-2)}{2}$
(b) n and $\frac{n-1}{2}$
(c) $n-1$ and $\frac{n-2}{2}$
(d) n and $\frac{n(n-1)}{2}$

Ans: (a)

27. Given T is a graph with 'n' vertices. If T is connected and T has (n-1) edges, then
(a) T is a tree
(b) T contains no cycles
(c) Every pair of vertices in T is connected by exactly one path
(d) All the above
Ans: (d)
Explanation:
We know that a connected graph with 'n' vertices and n-1 edges edges is always a tree. Hence T is a tree.
Since a tree contains no cycles, so T has no cycles.
As T has no cycles, so only one path is possible between each pair of vertices in T.

28. Consider a rooted tree in which every node has at least 3 children. What will be the minimum number of nodes at level 'i' (i>0) of this tree? (assume that root is at level zero)
(a) 3^i (b) 3i (c) 3 (d) 3i+1
Ans: (a)
Explanation:
We know that the number of nodes at level 'i' of a n-ary tree is n^i. Here n=3. Hence minimum number of nodes at level 'i' (i>0) of this tree=3^i.

29. The maximum number of nodes in a binary tree of depth 5 is
(a) 28 (b) 29 (c) 30 (d) 31
Ans: (d)
Explanation:
Maximum number of nodes in a binary tree of depth 5=2^5-1=31.

30. How many internal nodes are there in

Ans: (b)
Explanation:
Number of different trees with 'n' nodes $= n^{n-2}$. Here n=5 and so Number of different trees with '5' nodes is 125.

32. Which of the following does not define a tree?
(a) A tree is a graph with no cycles
(b) A tree is a connected acyclic graph
(c) A tree is a connected graph with n-1 edges where n is the number of vertices
(d) A tree is an acyclic graph with n-1 edges where n is the number of vertices
Ans: (a)
Explanation:
Consider a graph G with three isolated vertices. Then G has no cycles and G is not a tree.

33. Which two of the following statements are equivalent for an undirected graph G?
(i) G is a tree
(ii) There is at least one path between any two distinct vertices of G
(iii) G contains no cycles and has (n-1) edges.
(iv) G has n edges
(a) (i) and (ii)
(b) (i) and (iii)
(c) (i) and (iv)
(d) (ii) and (iii)
Ans: (b)
Explanation:

$$= \frac{{}^{2n}C_n}{n+1} = \frac{{}^{8}C_4}{4+1} \text{ (here } n=4)$$

$$= \frac{8\times7\times6\times5}{5\times4\times3\times2\times1} = 14.$$

35. Consider the following statements:
(i) A graph in which there is a unique path between every pair of vertices is a tree.
(ii) A connected graph with $|e| = |v| - 1$ edges is a tree.
(iii) A graph with $|e| = |v| - 1$ edges that has no circuit is a tree.
Which of the above statements is/are true?
(a) (i) and (iii) (b) (ii) and (iii)
(b) (i) and (ii) (d) all of these
Ans: (d)
Explanation:
It follows from the definition of a tree.

36. The number of colours required to properly colour the vertices of every planar graph is
(a) 2 (b) 3 (c) 4 (d) 5
Ans: (c)
Explanation:
Result follows from Four colour's theorem.

37. To find the shortest path in a weighted graph, which of the following algorithms is not used?
(a) Warshall's algorithm

Ans: (d)
Explanation:
In case of strongly connected graph, each vertex must be reachable from any other vertex.

39. A full binary tree with n leaves contains
(a) n nodes (b) $\log_2 n$ nodes
(c) $2n$-1 nodes (d) 2^n nodes
Ans: (c)
Explanation:
Total number of nodes
=total number of leaves + number of non-leaves
=n+(n-1)=2n-1.

40. The number of edges which must be removed from a connected graph with 'n' vertices and 'm' edges to produce a spanning tree is:
(a) m-n (b) m-n+1
(c) n-m (d) n-m-1
Ans: (b)
Explanation:
Number of edges in spanning tree=n-1.
Hence number of edges to be removed= m-(n-1)=m-n+1.

11.12 PREVIOUS YEARS SOLVED QUESTION PAPERS (2000-2018):

01. The minimum number of colours required