2/8/23

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* Adv. Mathematical Techniques in Chemical Engg

which is that part of maths that allows to deal with collections of $\boxed{\text{Set Theory}}$

$A = \{1, 2\}$ } same book / element is written again in C
$B = \{1, 2, 1, 2\}$ } That's it.

$C = \{1, 2, 3\}$ → To get set C from set A. We cannot add 1+2 in A because it is not allowed in Set Theory. We need a set and an operator

$A = \{1, 2\}$ + → ~~to generate elements. This is what is an Algebraic structure~~

Algebraic structures : $(G, \otimes)$ → $\boxed{\text{set + operator}}$

Binary operators :- if G is a non empty set and ⊗ is defined on G then ⊗ is said to be a binary operation on G if $\boxed{\otimes : G \times G \longrightarrow G}$ o/p

Cross maps G cross G to G (reading the b̲line) 1st input → 2nd i/p

$G \equiv \mathbb{R}$, $\otimes \equiv +$ set of real nos. addition is a binary opn defined on a set of R	$G = \mathbb{I}^+$ $\otimes \equiv -$ set of +ve ints. subtraction is not a binary opn becoz it can give -ve ints. in return.	$G = \mathbb{I}$ $\otimes \equiv -$ now subtraction is a binary opn becoz subtraction of an int with int returns int.

Commutativity :- A binary operator ⊗ defined on a non empty set G is said to be commutative if $a \otimes b = b \otimes a \quad \forall a, b \in G$. (must be true for every element)

$G = \mathbb{R}$ $\otimes = +$	$G = \mathbb{R}$ $\otimes = \times$	G = matrices (M) $\otimes = \times$
$\boxed{\text{yes}}$	$\boxed{\text{yes}}$	$\boxed{\text{No}}$ No guarantee.

Associativity :- $a \otimes (b \otimes c) = (a \otimes b) \otimes c \quad \forall a, b, c \in G$.

addition & multiplication of real nos.

addition & multiplication of 3 matrices.

→ more prominent than commutativity.

Distributivity :- A binary operator ⊗ is said to be "left distributive over ⊙"

if $a \otimes (b \odot c) = (a \otimes b) \odot (a \otimes c)$

Eg. ⊗ → multiplication ⊙ → addition → True

right distribution if :- $(a \odot b) \otimes c = (a \otimes c) \odot (b \otimes c) \quad \forall a, b, c \in G$

Identity Element :- An element $e \in G$ is said to be an identity element if

$e \otimes a = a \quad \forall a \in G$

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$a \otimes e = a \; \forall a \in G$.

$a \otimes e = e \otimes a = a \; \forall a \in G$.

$G = \mathbb{R}$, $+ = \otimes$ → 0 is the identity operator

$G = \mathbb{R}$, $\otimes = \times$ → 1 is the identity operator

$G = M$, $\otimes = \times$ → Identity Matrix — identity operator

$G = M$, $\otimes = +$ → zero matrix — identity operator

Inverse of an element :—

b is the left inverse of a if b operating a gives identity element

→ $b \otimes a = e$

$a \otimes b = e$

$G = \mathbb{R}$, $\otimes = +$ → identity element = zero. = inverse element zero

$a \to -a$.

$G = \mathbb{R}$, $\otimes = \times$

$a \to 1/a$. ← inverse

group :— A group is an algebraic structure $(G, \otimes)$ which satisfies the following

(a) $a \otimes (b \otimes c) = (a \otimes b) \otimes c \;\; \forall a, b, c \in G$ ($\exists$ → there exists)

(b) $\forall a \in G \; \exists e$ s.t. $a \otimes e = e \otimes a = a$ → identity element must exist

(c) $\forall a \in G \; \exists b$ s.t. $a \otimes b = b \otimes a = e$ → inverse element must exist

Q. If G is a group such that $(ab)^2 = a^2b^2 \; \forall a, b \in G$ then prove that G is abelian

proof :— 3 methods.

$(ab)^2 = a^2b^2$

$(a \otimes b) \otimes (a \otimes b) = (a \otimes a) \otimes (b \otimes b)$

An abelian group is a group in which the operators are commutative

A general group does not requires the operators to be commutative

groups show associativity

$(a \otimes b) \otimes (a \otimes b) = a \otimes (a \otimes b) \mid c = a \otimes b \in G$.

$= (c \otimes a) \otimes b$

$= [(a \otimes b) \otimes a] \otimes b$

$= [a \otimes (b \otimes a)] \otimes b$

If $b \otimes a = a \otimes b$. ⟸

$= [a \otimes (a \otimes b)] \otimes b$

$= [(a \otimes a) \otimes b] \otimes b$

$= (d \otimes b) \otimes b, \; d = a \otimes a \in G$

$= d \otimes (b \otimes b)$

$= (a \otimes a) \otimes (b \otimes b)$

$= a^2 b^2$

2nd method:— $(ab)^2 = a^2b^2$

$(a \otimes b) \otimes (a \otimes b) = (a \otimes a) \otimes (b \otimes b)$

$c \otimes (a \otimes b) = d \otimes (b \otimes b)$; $c = a \otimes b \in G$, $d = a \otimes a \in G$

$\Rightarrow (c \otimes a) \otimes b = (d \otimes b) \otimes b$

We can multiply both sides of the group with an element f that creates an identity element on both sides (as it is a group)

$(c \otimes a) \otimes (b \otimes f) = (d \otimes b) \otimes (b \otimes f)$

$(c \otimes a) \otimes e = (d \otimes b) \otimes e$

$c \otimes a = d \otimes b$

$\Rightarrow (a \otimes b) \otimes a = (a \otimes a) \otimes b$

$\Rightarrow \cancel{(a \otimes b) \otimes a = (a \otimes b) \otimes a}$

$a \otimes (b \otimes a) = a \otimes (a \otimes b)$ → commutativity proved.

3rd method:— using right side values.

$a^2b^2 = (a \otimes a) \otimes (b \otimes b)$

Let $b \otimes b = c \in G \Rightarrow a^2b^2 = (a \otimes a) \otimes c$

$= (a \otimes c) \otimes a$ (associativity)

$= (a \otimes (b \otimes b)) \otimes a$

$= (b \otimes (a \otimes b)) \otimes a$

$= (b \otimes a) \otimes (a \otimes b)$

$= (a \otimes b) \otimes (a \otimes b)$

LHS = RHS

Groups, Rings & fields

With only sets we can't do anything. We need to be able to generate and manipulate the elements in order to make use of it. Thus, we need generators.

Q. Verify whether set of $n \times n$ non singular matrices with real elements forms a group under matrix addition.

Cond's which any group must satisfy :— A group $(G, \circledast)$ must satisfy

1) $a \circledast (b \circledast c) = (a \circledast b) \circledast c$
2) $\forall a \in G \ \exists \ e$ st $a \circledast e = e \circledast a = a$. ($e \to$ identity element and it must be a member of the set).
3) $\forall a \in G \ \exists \ b \in G$ s.t $a \circledast b = b \circledast a = e \ (e \in G)$

$$[A]_{n\times n} + [B]_{n\times n} + \cdots\cdots [N]_{n\times n}$$

$(G, +)$

$$G = \left\{ \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & \ddots & & \\ & & & r_{nn} \end{bmatrix} : r_{ij} \in \mathbb{R},\ n \in \mathbb{N} \right\}$$

Test of associativity :—

Consider $\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{C}} \in G$

$\underbrace{=}$ → denotes matrix.

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \ddots & & \vdots \\ & & & a_{nn} \end{bmatrix} ; \ \underline{\underline{B}} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ & \ddots & & \vdots \\ & & & b_{nn} \end{bmatrix} ; \ \underline{\underline{C}} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ & \ddots & & \vdots \\ & & & c_{nn} \end{bmatrix}$$

$a_{ij}, b_{ij}, c_{ij} \in \mathbb{R}$

$$B + C = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ & \ddots & & \\ & & & b_{nn} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ & \ddots & & \\ & & & c_{nn} \end{bmatrix} = \begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} & \cdots & b_{1n}+c_{1n} \\ & \ddots & & \vdots \\ & & & b_{nn}+c_{nn} \end{bmatrix}$$

$$A + (B + C) = \begin{bmatrix} a_{11}+(b_{11}+c_{11}) & a_{12}+(b_{12}+c_{12}) & + \cdots & a_{1n}+(b_{1n}+c_{1n}) \\ & \ddots & & \\ & & & a_{nn}+(b_{nn}+c_{nn}) \end{bmatrix}$$

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consider the ij^{th} element of $A+(B+C)$

$$a_{ij} + (b_{ij} + c_{ij}) = (a_{ij} + b_{ij}) + c_{ij} \quad \text{as } a_{ij}, b_{ij}, c_{ij} \in \mathbb{R}$$

$$A+(B+C) = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots \\ & \ddots & \vdots \\ & & a_{nn}+b_{nn} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ & \ddots & & \vdots \\ & & & c_{nn} \end{bmatrix}$$

$$\Rightarrow \underline{\underline{A}} + (\underline{\underline{B}} + \underline{\underline{C}}) = (\underline{\underline{A}} + \underline{\underline{B}}) + \underline{\underline{C}}$$

Test of identity element :-

Let $\underline{\underline{E}} \in G$ s.t. $\underline{\underline{A}} + \underline{\underline{E}} = \underline{\underline{E}} + \underline{\underline{A}} = \underline{\underline{A}}$ and $E = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ & \ddots & & \\ & & & e_{nn} \end{bmatrix}$; $e_i \in \mathbb{R}$

$\underline{A} + \underline{E} = \underline{A}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \ddots & & \vdots \\ & & & a_{nn} \end{bmatrix} + \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ & \ddots & & \vdots \\ & & & e_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \ddots & & \\ & & & a_{nn} \end{bmatrix}$$

consider the ij^{th} element of $A+E$

$a_{ij} + e_{ij}$

$\Rightarrow a_{ij} + e_{ij} = a_{ij} \Rightarrow e_{ij} = 0 \; \forall \; i,j$

$$E = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ & \ddots & & \\ & & & 0 \end{bmatrix}_{n \times n} \notin G. \quad \text{cond}^n \text{ not satisfied.}$$

Now consider matrices (no cond^n of non singularity) :-

so identity element matrix available in this case

Test of Inverse of an element :-

Let $\underline{B} \in G$ s.t. $\underline{A} + \underline{B} = \underline{\underline{E}}$ and $\underline{\underline{B}} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ & \ddots & & \vdots \\ & & & b_{nn} \end{bmatrix}$; $b_{ij} \in \mathbb{R}$

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12}+ & \cdots & a_{1n}+b_{1n} \\ & \ddots & & \vdots \\ & & & a_{nn}+b_{nn} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ & \ddots & & \\ & & & 0 \end{bmatrix}$$

ij^{th} element :- Test of inverse of an element :-

$a_{ij} + b_{ij} = 0$

$b_{ij} = -a_{ij}$

$$\Rightarrow B = \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ & \ddots & & \vdots \\ & & & -a_{nn} \end{bmatrix}$$

$\forall a \in \mathbb{R}, \; -a \in \mathbb{R}$

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$\Rightarrow A \in G$ inverse exists for all non singular matrix elements
so all the 3 condns for being a group are satisfied.

A ring is an algebraic structure $(R, +, *)$ which satisfies the following
1) $(R, +)$ is an Abelian group.
1a.) $(a+b)+c = a+(b+c) \ \forall a,b,c \in R$
1b.) $\forall a \in R, \exists e \in R$ s.t. $a+e = e+a = a$
1c.) $\forall a \in R \ \exists b \in R$ s.t. $a+b = b+a = e$
1d.) $\forall a,b \in R \quad a+b = b+a$
2) $(R, *)$ is a monoid
$a \circledast (b \circledast c) = (a \circledast b) \circledast c \rightarrow$ semi group } monoid.
$e \in G$ (inverse may not exist)
$a^{-1} \in G.$ → Group
$a \circledast b = b \circledast a$ → Abelian Group.

$(a * b) * c = a*(b*c) \ \forall a,b,c \in \mathbb{R}$ | $e' = 1, e = 0$
$a * e' = e' * a = a \ \forall a \in \mathbb{R}, e' \in \mathbb{R}$

Subgroup → $H \subset G$

3.) Distributivity of $*$ over $+$
3. a.) $a * (b+c) = (a*b) + (a*c)$ } $\forall a,b,c \in R$
$(a+b) * c = (a*c) + (b*c)$

Not included in the definition:
(a.) Multiplicative inverse
(b.) commutativity of multiplication.

} when these 2 properties get satisfied by a ring it is called field.

Ques. verify whether a set of polynomials with real coeffs. is a ring.
$(R, +, *)$, $R = \{ \sum_{i=0}^{n} a_i x^i : a_i \in \mathbb{R}, n \in \mathbb{N} \cup 0 \}$

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots a_n x^n$
Let us consider $f(x), g(x), h(x) \in R$
$g(x) = b_0 + b_1 x + b_2 x^2 + \dots b_n x^n$
$h(x) = c_0 + c_1 x + c_2 x^2 + \dots c_n x^n$

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$a_i, b_i, d \in \mathbb{R}$

for $f(x) + g(x) \xrightarrow[\text{coeff}]{} i^{th}$ term $(a_i + b_i) = (b_i + a_i)$

For $(R, +)$:—

(i) Associativity Test :—

$f(x) \circledast (g(x) \circledast h(x)) = (f(x) \circledast g(x)) \circledast h(x)$

$f(x) + (g(x) + h(x)) = (f(x) + g(x)) + h(x)$

LHS $\Rightarrow a_0 + a_1 x + a_2 x^2 \dots + (b_0 + b_1 x + b_2 x^2 + \dots c_0 + c_1 x + \dots)$

$(a_0 + b_0 + c_0) + (a_1 + b_1 + c_1)x + \dots$

LHS = RHS $\because$ real nos. add^n.

(ii) Identity element test :—

$f(x) \circledast e = e \circledast f(x) = f(x)$ or $e + f(x) = f(x) + e = f(x)$

$\boxed{e = 0}$

(iii) inverse test :—

$f(x) \circledast d = d \circledast f(x) = y$ or $f(x) + d = d + f(x) = y$

$y = 0$ $d = -f(x)$

(iv) commutativity :—

$f(x) + g(x) = g(x) + f(x)$ real add^n of coeffs.

for $(R, *)$:—

(i) Associativity test :—

$f(x) * (g(x) * h(x)) = (f(x) * g(x)) * h(x)$

holds true for ~~add~~ multiplic^n of polynomials with real coeffs.

(ii) Multiplicative identity :—

$f(x) \odot e = e \odot f(x) = f(x)$

$f(x) \times e = e \times f(x) = f(x)$

$e = 1$

(iii) distributivity of $*$ over $+$:—

$f(x) * (g(x) + h(x)) = (f(x) * g(x)) + (f(x) * h(x))$

Left & right distributive ✓

"Linear Vector Spaces" :-

Let V be a set of ordered pairs (a,b), $a,b \in \mathbb{R}$. The following opⁿ's are defined :- $(a,b) \otimes (c,d) = (a+c, b+d)$

$$k \odot (a,b) = (ka, 0), \ k \in \mathbb{R}$$

Identify whether V is a vectorspace over $\mathbb{R}$.

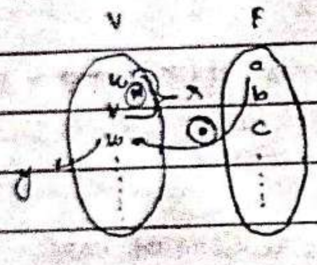
(2 sets & operations)

An algebraic structure $(V, F, \otimes, \odot)$ equipped with a non empty set V, a field F, a binary operation $\otimes : V \times V \to V$ and an external mapping $\odot : F \times V \to V$ (first element from F mapped on V returns element in V) which satisfies :

(1.) $(V, \otimes)$ is an Abelian group

(2.) $a \odot (u \otimes v) = (a \odot u) \otimes (a \odot v)$ } $u,v \in V$

$(u \otimes v) \odot b = (u \odot b) \otimes (v \odot b)$ } $a,b \in F$



why is F considered ext. mapping?

- output in V. set of interest is V.

A space is a collection of elements that are associated with each other by a collection of rules.

$$V = \left\{ \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} ; r_i \in \mathbb{R}, n \in \mathbb{Z}^+ \right\}$$

$\otimes = +$, $\odot = *$

$F = \mathbb{R}$

Test of associativity :- $u, v, w \in V$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} ; v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} ; w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} ; \ u_i, v_i, w_i \in \mathbb{R}$$

$$u + v = u_i + v_i = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} \qquad (u+v) + w = \begin{bmatrix} u_1 + v_1 + w_1 \\ u_2 + v_2 + w_2 \\ \vdots \\ u_n + v_n + w_n \end{bmatrix}$$

as all are real nos.

$$(u_i + v_i) + w_i = u_i + (v_i + w_i)$$

Test of identity element :- let $[e_1, \dots, e_n]$ or $\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$ is an identity element

$$e \otimes u = u \otimes e = u$$

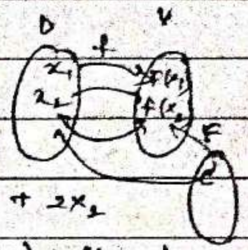
$$\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \rightarrow \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \Rightarrow \begin{bmatrix} e_1 \\ \vdots \\ e_{n-1} \\ e_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

coordinates of an element can be considered to be an ordered n-tuple

vector V → column matrix $(n \times 1)$

matrix M



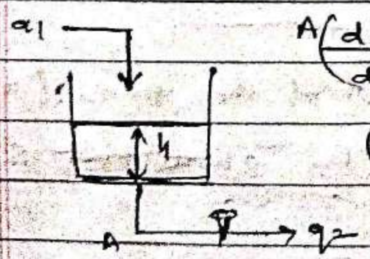
$f(x) = 2x$	Linear fⁿ / Eqⁿ	$f(x_1) = 2x_1$
$g(x) = \sin x$	Non Linear fⁿ / Eqⁿ	$f(x_2) = 2x_2$
$\frac{dy}{dx} = 2y \quad y = ce^{2x}$	Non Linear fⁿ	$f(x_1 + x_2) = 2x_1 + 2x_2$
		$= f(x_1) + f(x_2)$
$\frac{dy}{dx} - y^2 \sin x$		$f(cx_1) = cf(x_1)$

$$\frac{dy}{dx} = 2y \Rightarrow \frac{d}{dx}(y_1 + y_2) = 2(y_1 + y_2) \Rightarrow \frac{dy_1}{dx} + \frac{dy_2}{dx}$$

$$\frac{d}{dx}(cy_1) = 2cy_1 = c(2y_1) = c\frac{dy}{dx}$$

Linearity is defined for operations / operators. The elements themselves can be non linear.

Ex



$$A\left(\frac{dh}{dt}\right) = q_1 - q_2 \Longrightarrow \frac{dh}{dt} = \frac{1}{A} f(h)$$

(solⁿˢ to the eqⁿ lie in a linear vector space)

Let $u = (a_1, b_1)$, $v = (a_2, b_2)$, $w = (a_3, b_3) \in V$.

$a, b \in \mathbb{R}$

① Test of Associativity:—

$$(u + v) = (a_1 + a_2, \ b_1 + b_2)$$
$$(u + v) + w = (a_1 + a_2 + a_3, \ b_1 + b_2 + b_3)$$
$$(v + w) = (a_2 + a_3, \ b_2 + b_3)$$
$$u + (v + w) = (a_1 + a_2 + a_3, \ b_1 + b_2 + b_3)$$
$$(u + v) + w = u + (v + w)$$
$$\rightarrow (u \circledast v) \circledast w = u \circledast (v \circledast w)$$

(ii) Test of identity element :— let e be the identity element $e = (e_1, e_2)$

$u + e = e + u = u$

$(a_1, b_1) \circledast (e_1, e_2) = (a_1, b_1)$

$(e_1, e_2) = (0, 0)$

(iii) Test of inverse operation :— let i be the inverse op^n, $i = (i_1, i_2)$

$u + i = i + u = 0$

$(a_1, b_1) + (i_1, i_2) = (0, 0)$

$i_1 = -a_1 \quad i_2 = -b_1$

(iv) Test of commutativity :—

$u + v = (a_1 + a_2, b_1 + b_2)$

$v + u = (a_2 + a_1, b_2 + b_1)$

$u + v = v + u \implies \boxed{u \circledast v = v \circledast u}$

(v) Test of Distributivity :—

$k \odot [(a_1 + a_2), (b_1 + b_2)] = (k(a_1 + a_2), 0)$

$k \odot (a_1, b_1) = (ka_1, 0)$

$k \odot (a_2, b_2) = (ka_2, 0)$

$[k \odot (a_1, b_1)] \circledast [k \odot (a_2, b_2)]$

$[k(a_1 + a_2), \underbrace{\cancel{k(b_1 + b_2)}}_{0}] = LHS$

(vi) vector add^n & multiplication of a vector by a scalar v is a vector space.

Subspace let S be the set of all elements of the form $(x + y, y, -x + 3y) \in \mathbb{R}^3$, $x, y \in \mathbb{R}$. Show that S is a subspace of $\mathbb{R}^3$.

subspace

Sol^n :— (1.) $\underline{u} + \underline{u} \in S$

(2.) $c\underline{u} \in S$

Let $\underline{u} = (x_1 + y_1, y_1, -x_1 + 3y_1)$ $\Big\}$ $x_i, y_i \in \mathbb{R}$

$v = (x_2 + y_2, y_2, -x_2 + 3y_2)$

$u + v = ((x_1 + x_2) + (y_1 + y_2), y_1 + y_2, -(x_1 + x_2) + 3(y_1 + y_2))$

let $x_1 + x_2 = x_3$, $y_1 + y_2 = y_3$; $x_3, y_3 \in \mathbb{R}$

$= (x_3 + y_3, y_3, -x_3 + 3y_3)$

since $x_3, y_3 \in \mathbb{R}$

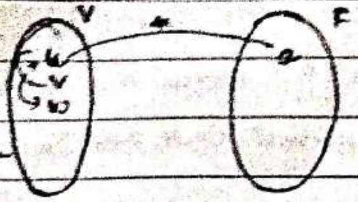
$x_3 + 2y_3 \in \mathbb{R}$ $\quad -x_3 + 3y_3 \in \mathbb{R}$

1st cond^n is satisfied.

$$cu = (cx_1 + c(2y_1),\ cy_1,\ c(-x_1) + 3(cy_1))$$

$$= (c(x_1 + 2y_1),\ cy_1,\ c(-x_1 + 3y_1))$$

Linear Combination Operation :-



$+ \quad V \times V \to V$

$\cdot \quad F \times V \to V$

$w = au + bv$

Q Express the polynomial $f(x)$ as a linear combination of $\phi_i(x)$'s

$f(x) = x^2 + 4x - 3 \quad \phi_1(x) = x^2 - 2x + 5 \quad \phi_2(x) = 2x^2 - 3x \quad \phi_3(x) = x + 3$

$f(x) = a\phi_1(x) + b\phi_2(x) + c\phi_3(x) \quad a, b, c \in F$

$x^2 + 4x - 3 = a(x^2 - 2x + 5) + b(2x^2 - 3x) + c(x + 3)$

$x^2 + 4x - 3 = x^2(a + 2b) + x(-2a - 3b + c) + (5a + 3c)$

$$\left.\begin{aligned} a + 2b &= 1 \\ -2a - 3b + c &= 4 \\ 5a + 3c &= -3 \end{aligned}\right\} \quad \begin{aligned} b + c &= 6. \\ 5a + 3c &= -3. \end{aligned}$$

$-\cancel{2a} - 3b + c + \cancel{2a} + 4b = 2 + 4$

$\Rightarrow b + c = 6$

$\cancel{2b} + 2c - a - \cancel{2b} = 12 - 1 \quad \rightarrow 2c - a = 11$

$\rightarrow 10c - \cancel{5a} = 55$

$\cancel{+5a} + 3c = -3$

$13c = 52 \rightarrow c = 4$

$\rightarrow a = -3 \quad b = 2.$

$\boxed{a = -3 \quad b = 2 \quad c = 4.}$

$$\begin{bmatrix} 1 & 2 & 0 \\ -2 & -3 & 1 \\ 5 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} \qquad [A|b] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ -2 & -3 & 1 & 4 \\ 5 & 0 & 3 & -2 \end{array}\right]$$

$Ax = b$

$$[A \mid b] : \begin{array}{l} R_2 \to 2R_1 + R_2 \\ R_3 \to 5R_1 - R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 6 \\ 0 & 10 & -3 & 8 \end{array}\right] \quad R_3 \to R_3 - 10R_2 \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -13 & -52 \end{array}\right]$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -52 \end{bmatrix} \to \begin{array}{l} a + 2b = 1 \to a = -3 \\ b + c = 6 \to b = 2 \\ -13c = -52 \to c = 4 \end{array}$$

8. Consider $v_1 = [1\ 2\ 3]^T$, $v_2 = [2\ 3\ 1]^T$. Find the condn on a,b,c so that v can be written as a linear combn on v_1 and v_2.

$v = [a\ b\ c]^T$

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \quad v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$v = v_1 + v_2$

$2x + y = a \to x = \frac{a-y}{2}$

$3x + 2y = b \to x = \frac{b-2y}{3}$

$x + 3y = c \to x = c - 3y$

$\frac{a-y}{2} = c - 3y = \frac{b-2y}{3}$

$\to a - y = 2c - 6y$

$\to \underline{a - 2c} = -5y \to y = \frac{a-2c}{-5}$

$\to \dfrac{b + 2\left(\frac{a-2c}{5}\right)}{3} = c$

$\to \dfrac{5b + 2a - 4c}{15} = \dfrac{a + a - 2c}{5} = 2c + 6\left(\frac{a-2c}{5}\right)$

$V = \alpha V_1 + \beta V_2 \quad \alpha, \beta \in F$

$$\alpha\begin{bmatrix}1\\2\\3\end{bmatrix} + \beta\begin{bmatrix}2\\3\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix} \to \left[\begin{array}{cc|c}1&2&a\\2&3&b\\3&1&c\end{array}\right] \to \left[\begin{array}{cc|c}1&2&a\\0&-1&b-2a\\0&+5&3a-c\end{array}\right]$$

$$\left[\begin{array}{cc|c}1&2&a\\0&-1&b-2a\\0&0&-c+3a+5b-10a\end{array}\right] \to \left[\begin{array}{cc|c}1&2&a\\0&-1&b-2a\\0&0&-7a+5b-c\end{array}\right]$$

$\alpha + 2\beta = a \qquad -\beta = b - 2a$

$-7a + 5b - c = 0 \to \boxed{7a - 5b + c = 0}$

Q show that the vectors $v_1 = [1 \ \ 1]^T$, $v_2 = [1 \ \ 2]^T$ span $\mathbb{R}^2$ over $\mathbb{R}$

span means every single vector on V can be generated by linear comb^n of v_1 & v_2.

Let $\underline{v} = [a, b]^T \in \mathbb{R}^2$

$$\alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1&1&a\\1&2&b\end{array}\right] \Rightarrow \left[\begin{array}{cc|c}1&1&a\\0&1&b-a\end{array}\right]$$

$\alpha + \beta = a \qquad \beta = b - a \in \mathbb{R}$

$\to \alpha = a - \beta = a - b + a = 2a - b \to \alpha = 2a - b \in \mathbb{R}$

Since $a, b \in \mathbb{R}$ $2a - b \in \mathbb{R}$ and $b - a \in \mathbb{R}$ which mean $\alpha, \beta \in \mathbb{R}$ (α and β belongs to the field $\mathbb{R}$)

Linear independence, basis, dimension :–

$f(x) = 2x^3 + x^2 + x + 1 \qquad af(x) + bg(x) + ch(x) = 0$

$g(x) = x^3 + 3x^2 + x - 2 \qquad a = b = c = 0$

$h(x) = x^3 + 2x^2 - x + 3 \qquad 2a + b + c = 0$

$a + 3b + 2c = 0$

$a + b - c = 0$

$a - 2b + 3c = 0$

$$\left[\begin{array}{ccc|c}2&1&1&a\\1&3&2&b\\1&1&-1&c\end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c}2&1&1&a\\0&5&3&2b-a\\0&1&-3&2c-a\end{array}\right] \Rightarrow \left[\begin{array}{ccc|c}2&1&1&a\\0&5&3&2b-a\\0&0&-18&10c-5a-2b+a\end{array}\right]$$

$a = b = c = 0$

Linearly independent.

Q. Check the linear independence of $f(x)$ and $g(x)$

$f(x) = x \quad g(x) = e^{2x}$

$a f(x) + b(g(x)) = 0$

$ax + be^{2x} = 0$

What is linear dependence?

$g = af \rightarrow$ dependence g depends on f. (f, g)

one cannot be written in terms of another if they are independent

$$\Rightarrow \begin{vmatrix} f & af \\ \frac{df}{dx} & a\frac{df}{dx} \end{vmatrix} = 0 \rightarrow \text{f}^{\text{ns}} \text{ are linearly dependent}$$

$\boxed{\text{Wronskian determinant}}$

$$\begin{vmatrix} x & e^{2x} \\ 1 & 2e^{2x} \end{vmatrix} \rightarrow \text{~~dependent~~}$$

$2xe^{2x} - e^{2x} = (x-1)e^{2x}$

$h(x) = \sin x$

$$|W| = \begin{vmatrix} f & g & h \\ \frac{df}{dx} & \frac{dg}{dx} & \frac{dh}{dx} \\ \frac{d^2f}{dx^2} & \frac{d^2g}{dx^2} & \frac{d^2h}{dx^2} \end{vmatrix}$$

if $\boxed{h = af + bg}$

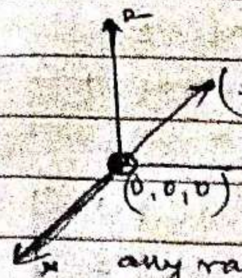
$$\begin{vmatrix} f & g & a\frac{df}{dx} + b\frac{dg}{dx} \\ \frac{df}{dx} & \frac{dg}{dx} & a\frac{d^2f}{dx^2} + b\frac{d^2g}{dx^2} \\ \frac{d^2f}{dx^2} & \frac{d^2g}{dx^2} & \end{vmatrix} \rightarrow \begin{vmatrix} f & g & af + bg \\ \frac{df}{dx} & \frac{dg}{dx} & a\frac{d^2f}{dx^2} + b\frac{d^2g}{dx^2} \\ \frac{d^2f}{dx^2} & \frac{d^2g}{dx^2} & a\frac{d^2f}{dx^2} + b\frac{d^2g}{dx^2} \end{vmatrix}$$

Basis & Dimension :-

Q. Det. whether $v_1 = [1 \; 1 \; 1 \; 1]^T$; $v_2 = [1 \; 2 \; 3 \; 2]^T$; $v_3 = [2 \; 5 \; 6 \; 4]$; $v_4 = [2 \; 6 \; 8 \; 5]^T$, these 4 vectors form a basis of $\mathbb{R}^4(\mathbb{R})$ ($\mathbb{R}^4$ over $\mathbb{R}$) field if not then det. dimension of the subspace they span

Euclidean space - 3D Cartesian coordinate system

Real & complex vector ~~field~~ Space are det. by the vectors not the field



$(3,1,2) \rightarrow 3(1,0,0) + 1(0,1,0) + 2(0,0,1)$

(linear combn)

any random vector (x_i, y_i, z_i) can be obtained by

$x_i(1,0,0) + y_i(0,1,0) + z_i(0,0,1)$

$\mathbb{R}^3$ bcoz all these 3 are real nos. & $\mathbb{R}$ because the coeff. are also $\mathbb{R}$

→ They are linearly independent form the Basis

* Basis are linearly independent set of vectors, no. of which is equal to linear vector space dimension and span the ~~field~~ linear vector space

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a_1 \\ 1 & 2 & 3 & 2 & a_2 \\ 2 & 5 & 6 & 4 & a_3 \\ 2 & 6 & 8 & 5 & a_4 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a_1 \\ 0 & 1 & 2 & 1 & a_2 - a_1 \\ 0 & 3 & 4 & 2 & a_3 - 2a_1 \\ 0 & 4 & 6 & 3 & a_4 - 2a_1 \end{array}\right]$$

$\downarrow$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a_1 \\ 0 & 1 & 2 & 1 & a_2 - a_1 \\ 0 & 0 & -2 & -1 & a_1 - 3a_2 + a_3 \\ 0 & 0 & -2 & -1 & 2a_1 - 4a_2 + a_4 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a_1 \\ 0 & 1 & 2 & 1 & a_2 - a_1 \\ 0 & 0 & -2 & -1 & a_1 - 3a_2 + a_3 \\ 0 & 0 & 0 & 0 & a_1 - a_2 - a_3 + a_4 \end{array}\right]$$

① $aV_1 + bV_2 + cV_3 + dV_4 = [0,0,0,0]$

② No. of vectors = dimension

$a + b + 2c + 2d = 0$

$a + 2b + 5c + 6d = 0$

$a + 3b + 6c + 8d = 0$

$a + 2b + 4c + 5d = 0$

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 5 & 6 \\ 1 & 3 & 6 & 8 \\ 1 & 2 & 4 & 5 \end{bmatrix}$$

$$\begin{array}{c} \\ \\ R_3 \\ R_4 \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & 1 & 3 & 4 & 0 \\ 0 & 2 & 4 & 6 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right]$$

$R_3 \rightarrow R_3 - 2 \times R_4$

$R_3 \rightarrow R_3 - 2R_2$

$R_4 \rightarrow R_4 - R_3$

$\downarrow$

$$\left[\begin{array}{cccc|c} 0 & 1 & 2 & 2 & 0 \\ 0 & 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right]$$

$a_1 - a_2 - a_3 + a_4 = 0$

Var = 4 Eqn = 1 DOF = 3.

$a_1 = \alpha \quad a_2 = \beta \quad a_3 = \gamma$

$a_4 = \beta + \gamma - \alpha$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

So Dimensions = 3

therefore these 4 vectors do not form the basis as the dimension $\neq$ no of vectors

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Null space and range space

S. (1) $x+y=2$

$x+2y+z=4$

$y+2z+t=4$

$z+2t=3$

$$=\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 1 & 2 & 1 & 0 & 4 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right]$$

$$=\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right]$$

$$=\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right]$$

$$=\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$x+y=2$

$y+z=2$

$3z+t=4$

$t=1.$

$z=1 \quad y=1 \quad x=1$

$x=y=z=t=1.$

$$\left[\begin{array}{cccc|c|c|c} 1 & 1 & 0 & 0 & 0 & 2 & b_1 \\ 1 & 2 & 1 & 0 & 0 & 4 & b_2 \\ 0 & 1 & 2 & 1 & 0 & 4 & b_3 \\ 0 & 0 & 1 & 2 & 0 & 3 & b_4 \end{array}\right]$$

(2) $x_1+x_2+x_3=1$

$2x_1+3x_2+x_3=2$

$5x_1+6x_2+4x_3=1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 2 \\ 5 & 6 & 4 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & -4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{array}\right]$$

No sol^n

~~$x_1 + x_2 + x_3 = 1$~~

~~$x_2 - x_3 = 0$~~

~~$0 x_3 = -4$~~

~~$x_3 = -4/5$~~

~~$x_2 = -4/5$~~

is it solvable?

$$\left[\begin{array}{ccc|c|c|c} 1 & 1 & 1 & 0 & 1 & b_1 \\ 2 & 3 & 1 & 0 & 2 & b_2 \\ 5 & 6 & 4 & 0 & 1 & b_3 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 1 & 1 & 0 & 1 & b_1 \\ 0 & 1 & -1 & 0 & 0 & b_2-2b_1 \\ 0 & 1 & -1 & 0 & -4 & b_3-5b_1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 1 & 1 & 0 & 1 & b_1 \\ 0 & 1 & -1 & 0 & 0 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & -4 & b_3-5b_1-b_2+2b_1 \end{array}\right]$$

$\quad\quad = b_3-b_2-3b_1$

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$$\left[\begin{array}{cccc|c|c|c} 1 & 1 & 0 & 0 & 0 & 2 & b_1 \\ 0 & 1 & 1 & 0 & 0 & 2 & b_2-b_1 \\ 0 & 1 & 2 & 1 & 0 & 4 & b_3 \\ 0 & 0 & 1 & 2 & 0 & 3 & b_4 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c|c|c} 1 & 1 & 0 & 0 & 0 & 2 & b_1 \\ 0 & 1 & 1 & 0 & 0 & 2 & b_2-b_1 \\ 0 & 0 & 1 & 1 & 0 & 2 & b_3-b_2+b_1 \\ 0 & 0 & 1 & 2 & 0 & 3 & b_4 \end{array}\right]$$

$$\downarrow$$

$$\begin{array}{c} \quad A \qquad\qquad\qquad 0 \text{ vector} \quad B \\ \left[\begin{array}{cccc|c|c|c} 1 & 1 & 0 & 0 & 0 & 2 & b_1 \\ 0 & 1 & 1 & 0 & 0 & 2 & b_2-b_1 \\ 0 & 0 & 1 & 1 & 0 & 2 & b_3-b_2+b_1 \\ 0 & 0 & 0 & 1 & 0 & 1 & b_4-b_3+b_2-b_1 \end{array}\right] \end{array}$$

$Ax=0$

if we want to solve a homogenous eq^n

i.e. $Ax=0$, $t=0 \Rightarrow y+z=0 \to x+y=0$

$x=y=z=t=0$

$$\boxed{\text{if } Ax=0 \text{ has a trivial sol}^n \text{ only then } Ax=B \text{ will always have a unique sol}^n. \text{ (always valid for } N\times N \text{ sys)}}$$

Null space is the sol^n space of homogenous problems; **Range space** is the space for vector (b) which will render your sol^n solvable.

Now, non homogenous eq^n will ~~be~~ ~~many sol~~ solvable for range space.

in $N\times N$ → homogenous → sol^n always exist
→ non-homogenous

→ sol^{n}s will be ∞ many

→ $x_2 - x_3 = 0$
$x_1 + x_2 + x_3 = 0$
Eq^{n}s = 2 var = 3 DOF = 1 } Degree of freedom analysis

out of these 3 var, 1 can be set arbitarily.

let $x_2 = \alpha$, $x_3 = \alpha$, $x_1 = -2\alpha$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2\alpha \\ \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

∞ sol^n for homogenous case.

These ∞ sol^n belong to a linear vector space.

Dimension of that vector space = 1

Basis of that sol^n space =

$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ → 1 of the possible bases

sol^{n}s belong to NULL space

1 vector makes it 1D

Rank of the matrix ⇒
→ no of non zero rows = 2 (for this matrix) (3×3)

condition for solvability :-
$b_3 - b_2 - 3b_1 = 0$
$\rho(A) = 2$, $\rho(A|b) = 2$
rank

→ Degree of freedom analysis

Eq^n = 1 var = 3 DOF = 2

Then 2 var can be set arbitarily

$b_1 = \beta$ $b_2 = \gamma$

$b_3 = 3\beta + \gamma$ dim = 2

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \beta \\ \gamma \\ 3\beta+\gamma \end{bmatrix} = \beta \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Q2) $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 2 \\ 4 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 3 & 0 & 0 & 2 \end{bmatrix}^T$

$3b_4 - 3b_2 + b_3 - 2b_1 - b_2$

$$\left[\begin{array}{ccc|c|c|c} 1 & -1 & 2 & 0 & 3 & b_1 \\ 2 & 1 & 2 & 0 & 0 & b_2 \\ 4 & -1 & 3 & 0 & 0 & b_3 \\ 2 & 1 & 1 & 0 & 2 & b_4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c|c|c} 1 & -1 & 2 & 0 & 3 & b_1 \\ 0 & 3 & -2 & 0 & -6 & b_2 - 2b_1 \\ 0 & 3 & 1 & 0 & -4 & b_3 - 4b_1 \\ 0 & 3 & -3 & 0 & -4 & b_4 - 2b_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c|c|c} 1 & -1 & 2 & 0 & 3 & b_1 \\ 0 & 3 & -2 & 0 & -6 & b_2 - 2b_1 \\ 0 & 0 & 3 & 0 & 2 & b_3 - 2b_1 - b_2 \\ 0 & 0 & -1 & 0 & 2 & b_4 - b_2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c|c|c} 1 & -1 & 2 & 0 & 3 & b_1 \\ 0 & 3 & -2 & 0 & -6 & b_2 - 2b_1 \\ 0 & 0 & 3 & 0 & 2 & b_3 - b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & 8 & -2b_1 - 4b_2 + b_3 + 3b_4 \end{array}\right]$$

$Ax = 0 \Rightarrow \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = 0$

cond^n for solvability :-

null space

① — $x_1 - x_2 + 2x_3 = 0$

② — $x_2 - 2x_3 = 0$

③ — $x_3 = 0$

Eq^n = 3 $\quad$ Var = 3 $\quad$ DOF = 0

$\boxed{2b_1 - 4b_2 + b_3 + 3b_4 = 0}$

Eq^n = 1 $\quad$ Var = 4 $\quad$ DOF = 3

$b_1 = \alpha \quad b_2 = \beta \quad b_4 = \gamma$

$b_3 = 2\alpha + 4\beta - 3\gamma$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ 2\alpha + 4\beta - 3\gamma \\ \gamma \end{bmatrix}$$

$$= \underbrace{\alpha \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ -3 \\ 1 \end{bmatrix}}_{\text{dimension} = 3 \text{ range space}}$$

※ Does not guarantee sol^n

if the sol^n exists for non homogenous eq^n $\Rightarrow$ it will be unique

Sum of dimension of null space and range space is equal to dimension of space

From a zero vector we cannot generate any vector so, $\vec{0}$ can never be our basis

affine space $\rightarrow \vec{0}$ does not necessarily exist

M×N

M = N

$Ax = 0$ trivial $\rightarrow$ $Ax = b$ unique

$Ax = 0 \rightarrow \infty$ $\rightarrow$ $Ax = b \rightarrow \infty$

N ≠ M

$Ax = 0$ trivial $\rightarrow$ $Ax = b$ unique if sol^n exists

$Ax = 0 \rightarrow \infty$ $\rightarrow$ $Ax = b$ ∞ many if sol^n exist.

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Linear Transformations :-

Why call transform^n & not simply f^n ? → bcoz we move from one domain to another domain.

$v_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ $v_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, $a, b \in F$ $v_3 = a\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ $v_4 = a\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + b\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

v_1, v_2, v_3 and v_4 are all in the same linear vector space.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

connec^n or rel^n b/w 2 vector spaces or more is → using Linear transform

$t : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 1}$

Operator → maps elements of A with A (resultant belongs to the set)

Transform^n → maps elements of A with elements of B.

$S = \{ax : a \in \mathbb{R}, a > 0\}$ → Linear Transform^n

→ ~~not identical~~

$V = \{a : a \in \mathbb{R}, a > 0\}$ → ~~Not a transform^n~~

v_1, v_2, α

$t(v_1 + v_2) = t(v_1) + t(v_2)$

$t(\alpha v_1) = \alpha t(v_1)$

Q Let V be a vector space of all continuous f^ns with the following operator defined :-

$T : V \to V$

$T[f(x)] = \int_0^x f(t)\,dt \;\; \forall \;\; f(x) \in V, x \in \mathbb{R}$

Identify if this is a linear transform^n.

sol^n for being a linear transform^n :-

$$\left.\begin{aligned} t(v_1 + v_2) &= t(v_1) + t(v_2) \\ t(\alpha v_1) &= \alpha t(v_1) \end{aligned}\right\} \text{ or } t(\alpha v_1 + \beta v_2) = \alpha t(v_1) + \beta t(v_2)$$

$T[f(x) + g(x)] = \int_0^x f(t)\,dt + \int_0^x g(t)\,dt$

$T[\underbrace{\alpha f(x)}_{g(x)}] = \int_0^x g(t)\,dt = \int_0^x \alpha f(t)\,dt = \alpha \int_0^x f(t)\,dt$

$g(t) = \alpha f(t)$ $= \alpha T[f(x)]$

$T[f(x) + g(x)]$

let $f(x) + g(x) = P(x)$

$T[P(x)] = \int_0^x P(t)\, dt$

since, $P(x) = f(x) + g(x)$

$\Rightarrow P(t) = f(t) + g(t)$

$T[P(x)] = \int_0^x f(t) + g(t)\, dt$

$= \int_0^x f(t)\,dt + \int_0^x g(t)\, dt$

$T[f(x) + g(x)] = \int_0^x f(t)\,dt + \int_0^x g(t)\,dt$

$= T[f(x)] + T[g(x)]$

integration is a linear transformⁿ.

laplace operator is a linear transformⁿ

Q identify where the following ~~operation~~ are linear transformations.

(a.) $t : \mathbb{R}^3 \to \mathbb{R}^2$

$t(x, y, z) = (x + y - 2z,\ 4x - 5y + 6z)$

(b.) $t : \mathbb{R}^3 \to \mathbb{R}^2$

$t(x, y, z) = (x + 1, y + 1)$

(a.) $x + y - 2z = f_1$

$4x - 5y + 6z = f_2$

$(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$

$t(x_1 + x_2, y_1 + y_2, z_1 + z_2)$

let $x_1 + x_2 = x_3$ $y_1 + y_2 = y_3$ $z_1 + z_2 = z_3$

where $(x_3, y_3, z_3) \in \mathbb{R}^3$

so, $t(x_3, y_3, z_3) = (x_3 + y_3 - 2z_3,\ 4x_3 - 5y_3 + 6z_3)$

$t(x_1 + x_2, y_1 + y_2, z_1 + z_2) = ((x_1 + x_2) + (y_1 + y_2) - 2(z_1 + z_2),$

$4(x_1 + x_2) - 5(y_1 + y_2) + 6(z_1 + z_2))$

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$\rightarrow \; t(x_1+x_2, y_1+y_2, z_1+z_2) = ((x_1+y_1-2z_1)+(x_2+y_2-2z_2), (4x_1-5y_1+6z_1)$
$(4x_2-5y_2+6z_2))$

$\Rightarrow \; t(\alpha x_1, \alpha y_1, \alpha z_1) = (\alpha(x_1+y_1-2z_1), \alpha(4x_1-5y_1+6z_1))$

$= \alpha((x_1+y_1-2z_1), (4x_1-5y_1+6z_1))$.

(b) Let $v_1 = (x_1, y_1, z_1)$, $v_2 = (x_2, y_2, z_2)$ $v_1, v_2 \in \mathbb{R}^3$

$t(\alpha v_1 + \beta v_2) = \alpha t(v_1) + \beta t(v_2)$

$t(v_1) = (x_1+1, y_1+1)$ $\quad t(v_2) = (x_2+1, y_2+1)$

$t(\alpha v_1 + \beta v_2)$:-

$\alpha v_1 = (\alpha x_1, \alpha y_1, \alpha z_1)$ $\quad \alpha v_2 = (\beta x_2, \beta y_2, \beta z_2)$

$(\alpha v_1 + \beta v_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$

$t(\alpha v_1 + \beta v_2) = (\alpha x_1 + \beta x_2 + 1, \alpha y_1 + \beta y_2 + 1)$

$t(\alpha v_1) = (\alpha x_1 + 1, \alpha y_1 + 1)$ $\quad t(\beta v_2) = (\beta x_2 + 1, \beta y_2 + 1)$

Q Identify whether the 2 operators are linear

(1) $\hat{L} = \frac{d^2}{dx^2} - m^2$ $\qquad$ (2) $\hat{L} = \frac{\partial}{\partial t} - \alpha^2 \frac{\partial}{\partial z^2} + \beta$

$\hat{L} = \frac{d^2}{dx^2} - m^2$

Q Let $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbb{R}^3(\mathbb{R}) \right\}$ be a basis for $t: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

be a linear transform^n such that :-

$t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad t\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Determine $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

range
kernel
nullity
ordered basis
matrix of transformⁿ.

$$a\begin{bmatrix}-1\\0\\1\end{bmatrix} + b\begin{bmatrix}0\\1\\-1\end{bmatrix} + c\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}-1 & 0 & 1 & x\\0 & 1 & -1 & y\\1 & -1 & 1 & z\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}-1 & 0 & 1 & x\\0 & 1 & -1 & y\\0 & -1 & 2 & z+x\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}-1 & 0 & 1 & x\\0 & 1 & -1 & y\\0 & 0 & 1 & x+y+z\end{array}\right]$$

$$\boxed{\begin{aligned}c &= x+y+z\\ b &= x+2y+z\\ a &= y+z\end{aligned}} \quad \begin{bmatrix}x\\y\\z\end{bmatrix} = a\begin{bmatrix}-1\\0\\1\end{bmatrix} + b\begin{bmatrix}0\\1\\-1\end{bmatrix} + c\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

if the ~~transformⁿ~~ is linear then t of any vector will be equal to the t of its components

$$t\begin{bmatrix}x\\y\\z\end{bmatrix} = t\left\{a\begin{bmatrix}-1\\0\\1\end{bmatrix} + b\begin{bmatrix}0\\1\\-1\end{bmatrix} + c\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$$

$$\rightarrow t\begin{bmatrix}x\\y\\z\end{bmatrix} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$t\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}y+z\\x+2y+z\\x+y+z\end{bmatrix}$$

Q. How to det the linearity of 2 vector spaces ?

∞ elements

1. basis → same
2. field → same
3. dimension → same

Q. are these 2 vector spaces same

$$\begin{cases}[1\ \ 0]^T & [1\ \ 1]^T & \mathbb{R}^2(\mathbb{R})\\ [1\ \ 2]^T & [3\ \ -9]^T & \mathbb{R}^2(\mathbb{R})\end{cases}$$

1. Field same
2. Dimension → 2 for both → same
3. Basis ?

any 2 vectors linearly independent and have dimension = 2 can span the entire space. So, it is not

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Inner Product spaces :—

Prove R^n is an inner product upon R^n over R (vector space)

Std. inner prod. :— $\langle u, v \rangle = a_1 b_1 + a_2 b_2 + \cdots\cdots a_n b_n = \sum_{i=1}^{n} a_i b_i$

where $\underline{u} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, $\underline{v} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, $a_i, b_i \in \mathbb{R}$

① Linearity :— (i) $\langle \underline{u} + \underline{v}, \underline{w} \rangle = \langle \underline{u} + \underline{w} \rangle + \langle v, w \rangle$

let $\underline{w} = [c_1\ c_2 \cdots c_n]^T$, $c_i \in \mathbb{R}$

~~$u+v = (a_1+b_1)c_1 + (a_2+b_2)c_2 + \cdots + (a_n+b_n)c_n$~~

$u_i, v_i, w_i \in \mathbb{R}$

$\underline{u} = [u_1\ u_2\ u_3 \cdots u_n]^T$

$\underline{v} = [v_1\ v_2\ v_3 \cdots v_n]^T$

$\underline{u} + \underline{v} = [u_1+v_1\quad u_2+v_2 \cdots\quad u_n+v_n]^T$

$\langle \underline{u}+\underline{v}, \underline{w} \rangle = [(u_1+v_1)w_1\quad (u_2+v_2)w_2 \cdots\quad (u_n+v_n)w_n]^T$

$= [u_1w_1 + v_1w_1\quad u_2w_2 + v_2w_2 \cdots\cdots u_nw_n + v_nw_n]^T$

$= [u_1w_1\ \ u_2w_2\ \ u_3w_3 \cdots u_nw_n]^T + [v_1w_1\ \ v_2w_2 \cdots v_nw_n]$

$= \langle \underline{u}, \underline{w} \rangle + \langle \underline{v}, \underline{w} \rangle$

(ii) $\langle \alpha \underline{u}, \underline{v} \rangle = \alpha \langle \underline{u}, \underline{v} \rangle$

$\langle \alpha \underline{u}, \underline{v} \rangle = [\alpha u_1\ \alpha u_2\ \alpha u_3 \cdots \alpha u_n]^T [v_1\ v_2\ v_3 \cdots v_n]^T$

$= \alpha [u_1\ u_2\ u_3 \cdots u_n]^T [v_1\ v_2\ v_3 \cdots v_n]^T$

$= \alpha \langle \underline{u}, \underline{v} \rangle$

(iii) $\langle \alpha \underline{u} + \beta \underline{v}, \underline{w} \rangle = \alpha \langle \underline{u}, \underline{w} \rangle + \beta \langle \underline{v}, \underline{w} \rangle$

② Symmetry :— $\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle$

$\langle \underline{u}, \underline{v} \rangle = [u_1v_1\ \ u_2v_2 \cdots u_nv_n]^T$

$\langle \underline{v}, \underline{u} \rangle = [v_1u_1\ \ v_2u_2 \cdots v_nu_n]^T$

$\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle$

if nothing is mentioned in the ques. consider std. inner prod. case i.e. $\sum_{i=1} u_i v_i$

③ Positive definiteness :— $\langle \underline{u}, \underline{v} \rangle \geq 0$; 0 iff $\underline{u} = 0$

$\langle \underline{u}, \underline{u} \rangle = u_1^2 + u_2^2 + u_3^2 + \cdots u_n^2 \quad \forall u_i \in \mathbb{R}$

$u_i^2 \geq 0 \rightarrow 0$; $\sum u_i^2 \geq 0$

$\therefore \langle \underline{u}, \underline{u} \rangle \geq 0$

Ques $\underline{u} = \begin{bmatrix} 2+2i \\ 3 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} 3i \\ 5 \end{bmatrix}$, $\underline{w} = \begin{bmatrix} 3 \\ 5i \end{bmatrix}$. Check whether these 3 vectors form inner prod. space

Sol^n We cannot compare complex nos. So we multiply it with its conjugate in order to compare.

$\langle \underline{u}, \underline{u} \rangle = \left[(2+2i)(2+2i) + 3\times 3 \right]$

$= 8i + 9 = \underline{9 + 8i} \rightarrow$ can't comment on the definiteness

$\langle \underline{v}, \underline{v} \rangle = -9 + 25 = 16$

above definition does not satisfy the 3rd criterial property

Std. inner product : $\langle \underline{u}, \underline{v} \rangle = \sum_{i=1}^{2} u_i \bar{v}_i$

conjugate multiplic^n $\Rightarrow \langle \underline{u}, \bar{\underline{u}} \rangle = (2+2i)(2-2i) + 3\times 3$

conjugate $= 4 + 4 + 9 = 17 > 0$

but notation same

1.) Linearity $- \langle \alpha \underline{u} + \beta \underline{v}, \underline{w} \rangle = \alpha \langle \underline{u}, \underline{w} \rangle + \beta \langle \underline{v}, \underline{w} \rangle$

2.) Symmetry $- \langle \underline{u}, \underline{v} \rangle = \overline{\langle \underline{v}, \underline{u} \rangle}$ (first exchange $\underline{u}, \underline{v}$ and after getting result multiply with conjugate)

3.) Positive definiteness : $\langle \underline{u}, \bar{\underline{u}} \rangle \geq 0$

conjugate

$\langle \underline{u}, \alpha \underline{v} \rangle = \overline{\langle \alpha \underline{v}, \underline{u} \rangle} = \bar{\alpha} \, \overline{\langle \underline{v}, \underline{u} \rangle}$

$\langle \underline{u}, \alpha \underline{v} \rangle = \bar{\alpha} \langle \underline{u}, \underline{v} \rangle$

Ques Det the value of λ which makes the following op^n of inner prod

$\langle \underline{x}, \underline{y} \rangle = x_1 y_1 - 3x_1 y_2 - 3x_2 y_1 + \lambda x_2 y_2$

$x = [x_1 \;\; x_2]^T \quad y = [y_1 \;\; y_2]^T \quad x, y \in \mathbb{R}^2$

$\langle \underline{x}, \underline{x} \rangle > 0$

$= x_1 x_1 - 3x_1 x_2 - 3x_2 x_1 + \lambda x_2 x_2 = x_1^2 - 6x_1 x_2 + \lambda x_2^2 > 0$

Positive definiteness should hold true.

$\Rightarrow \frac{x_1^2}{x_2^2} - \frac{6x_1 x_2}{x_2^2} + \frac{\lambda x_2^2}{x_2^2} > 0 \rightarrow \left(\frac{x_1}{x_2}\right)^2 - 6\left(\frac{x_1}{x_2}\right) + \lambda > 0$

let $x_1/x_2 = a \Rightarrow a^2 - 6a + \lambda > 0$

$a = \frac{6 \pm \sqrt{36 - 4\lambda}}{2}$

$D \leq 0$

$\frac{6 + \sqrt{36 - 4\lambda}}{2} \leq 0 \rightarrow 3 + \sqrt{9 - \lambda} \leq 0 \rightarrow 36 - 4\lambda \leq 0$

$9 - \lambda \leq 0 \rightarrow \lambda \geq 9 \rightarrow \lambda = 9.$

Ques. Det. a non zero vector in $\mathbb{R}^3$ which is orthogonal to vector

$v_1 = [1\ 1\ 2]^T \quad v_2 = [2\ 1\ 3]^T \quad v_3 = [1\ 2\ 3]^T$ (inner prod. = 0)

Let $v = [a\ b\ c]^T \in \mathbb{R}^3$

$$\left.\begin{aligned} \langle \underline{v_1}, \underline{v} \rangle &= a + b + 2c = 0 \\ \langle \underline{v_2}, \underline{v} \rangle &= 2a + b + 3c = 0 \\ \langle \underline{v_3}, \underline{v} \rangle &= a + 2b + 3c = 0 \end{aligned}\right\} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow c = \alpha \quad -b - c = 0 \rightarrow b = -\alpha$

$a + b + 2c = 0 \rightarrow a = -\alpha$

$$\underline{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -\alpha \\ -\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Ques. Verify whether the functions are orthogonal to each other

a) $\sin x, \cos x$ in $[-\pi, \pi]$

Std. inner prod. for continuous f^n's: $\langle f(x), g(x) \rangle = \int_a^b f(x) g(x)\,dx$

So. $\langle f(x), g(x) \rangle = \int_{-\pi}^{\pi} \sin x \cos x\,dx$

$= \frac{1}{2}\left(-\cos 2x\right)_{-\pi}^{\pi} = 0$

$\because$ sin is an odd f^n = 0

c) $\sin(n\pi x), \cos(m\pi x)$ in $[-1, 1]$

$\langle f(x), g(x) \rangle = \int_{-1}^{1} \sin(n\pi x) \cos(m\pi x)\,dx$

$= \frac{1}{2}\int_{-1}^{1} \sin(n+m)\pi x + \sin(n-m)\pi x\,dx$

$= \frac{1}{2}\left[\frac{-\cos(n+m)\pi x}{(n+m)\pi} - \frac{\cos(n-m)\pi x}{(n-m)\pi}\right]_0^1$

$= \frac{1}{2}\left[\frac{-\cos(n+m)\pi}{(n+m)\pi} - \frac{\cos(n-m)\pi}{(n-m)\pi} + \frac{2n}{(n^2-m^2)\pi}\right]$

b.) $\sin(n\pi x), \sin(m\pi x)$ in $[-1, 1]$

$\langle f(x), g(x) \rangle = \int_{-1}^{1} \sin(n\pi x) \sin(m\pi x)\,dx$

$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$

$= \int_{-1}^{1} \frac{1}{2}\left[\cos(n\pi x - m\pi x) - \cos(n\pi x + m\pi x)\right] dx$

$= \frac{2 \times 1}{2} \int_0^1 \cos(n-m)\pi x - \cos(n+m)\pi x\,dx$

$= \left[\frac{\sin(n-m)\pi x}{(n-m)\pi} - \frac{\sin(n+m)\pi x}{(n+m)\pi}\right]_0^1$

$= \frac{\sin(n-m)\pi}{(n-m)\pi} - \frac{\sin(n+m)\pi}{(n+m)\pi} = 0$

d) $\cos(n\pi x), \cos(m\pi x)$ in $[-1, 1]$

Q. Let B be the vector space of all polynomials over $\mathbb{R}^2$, det a polynomial which is orthogonal to $f(t) = 2t+1$, if degree ≤ 2

$\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt \qquad f(t) = 2t+1$

$g(t) = a_0 + a_1 t + a_2 t^2$

Det. a_0, a_1, a_2 such that $\int_0^1 (a_0 + a_1 t + a_2 t^2)(2t+1)\, dt = 0$

soln

$$\int_0^1 2a_0 t + 2a_1 t^2 + 2a_2 t^3 + a_0 + a_1 t + a_2 t^2 \, dt = 0$$

$$\rightarrow \int_0^1 2a_2 t^3 + (2a_1 + a_2) t^2 + (2a_0 + a_1) t + a_0\, dt = 0$$

$$\rightarrow \left(\frac{2a_2 t^4}{4} + (2a_1 + a_2)\frac{t^3}{3} + (2a_0 + a_1)\frac{t^2}{2} + a_0 t \right)\Big|_0^1 = 0$$

$$\rightarrow \frac{2a_2}{4} + \frac{2a_1 + a_2}{3} + \frac{2a_0 + a_1}{2} + a_0 = 0$$

$$\Rightarrow 2a_0 + \left(\frac{2a_1}{3} + \frac{a_1}{2}\right) + \left(\frac{\cancel{2}a_2}{\cancel{4}_2} + \frac{a_2}{3}\right) = 0$$

$$\Longrightarrow 2a_0 + \frac{7a_1}{6} + \frac{5a_2}{6} = 0 \rightarrow 12a_0 + 7a_1 + 5a_2 = 0$$

DOF analysis :- No. of eqns = 1 Var. = 3 DOF = 3 − 1 = 2

~~$a_1 = \alpha$ $a_2 = \beta$ $12\alpha + 7\beta + 5a_2 = 0$ $12a_0 +$~~

~~$\rightarrow a_2 = -\left(\frac{12\alpha + 7\beta}{5}\right)$~~ $a_0 = \alpha \quad a_1 = \beta$

$$a_2 = -\left(\frac{12\alpha + 7\beta}{5}\right)$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ -\left(\frac{12\alpha + 7\beta}{5}\right) \end{bmatrix}$$

(angle between fns should not be zero to be orthogonal)

$$= \alpha \begin{bmatrix} 5 \\ 0 \\ -12 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 5 \\ -7 \end{bmatrix}$$

→ These 2 vectors dimension & the 2 given in the problem

dimension atleast greater than 2 → cannot be 2 higher than 2 is needed.

Normed and Metric Spaces :—

Set of cartesian coordinate spaces → Is it an inner product space?

2 operators → +, *.

→ Linearity will hold true (addⁿ of corresponding elements is defined. similarly for multiplicⁿ)

→ Symmetry holds true.

→ Positive definiteness for a cartesian coord. sys. is going to be the inner product itself

3-Dimensional or n-dimension vector space has inner prod. spaces

What is the significance of 90°? (orthogonality matters?)

(1, 3, 2) → Do we need something more? Is it unique? or can a given pt be represented by more set of references

Requirements :— ① Reference (0, 0, 0) needs to be set

② we need to define the direction (position) of coordinate axes.

When our coordinate axes are (1, 0, 0), (0, 1, 0) and (0, 0, 1) then our pt holds true.

③ only condⁿ to define our axes is that they should be linearly independent. What did we achieve by setting our axes orthogonal to each other? (Simplicity) we can't simply write the pt. as (1, 3, 2) if the coordinate axes were not orthogonal.

Q1. There are vectors u_1 and u_2 in a 2 dim. linear vector space.

$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. (a) Are u_i's linearly independent? – Yes

(b.) Do they form an orthogonal basis? – No

$u_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ (c.) If not, then obtain an orthonormal basis from them.

(a) Condⁿ for linear independence :— $\sum c_i u_i = 0$ should have only solⁿ as $c_i = 0$ for all u_i

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$c_1 + c_2 i = 0$

$c_2 = 0$

So $c_1 = 0$

$c_1 = c_2 = 0 \Rightarrow$ linear independence

(b) So we have two vectors → dim = 2 → they form a basis

$\langle u_1, u_2 \rangle$ =

no, the 2 vectors don't form orthogonal basis

(c) If we have a basis which is not orthogonal. Then orthogonal basis can be formed using the former basis
Reorientation is not possible in → this case (linear dependence) and both the vectors move parallely in the same dirⁿ.

Gram Schmidt orthogonalisation

Orthonormal → magnitude (1)

How to get orthonormal vectors ? → simply divide by total magnitude

$\underline{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\underbrace{\underline{v}_1}_{\text{reference vector}}^{\text{orthogonal}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ norm of the vector $\sqrt{\langle u, u \rangle} = \|u\| = \sqrt{u_1^2 + u_2^2}$

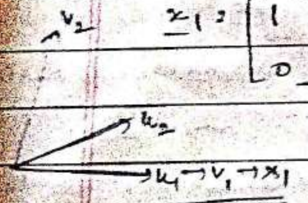
First reorientⁿ → Even before is there a need to reorient ?

Length of a vector = Norm of the vector

$\underline{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\underline{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $v_2 = u_2 - \alpha x_1$: $\langle v_2, x_1 \rangle = 0$

x_1 → first vector in the orthogonal basis

v_2 → second vector in the orthogonal basis



$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-\alpha \\ 1 \end{bmatrix}$

$\left\langle \begin{bmatrix} 1-\alpha \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = 0 \to 1 - \alpha = 0 \to \boxed{\alpha = 1}$

$v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

u_i → original vector v_i → orthogonal vector x_i → orthonormal vector

(c) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Q2 $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}; \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}; \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$ → ~~Eith~~ find the orthonormal basis for the given vectors.

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$u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$; $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\left\{ \begin{matrix} \text{Keeping one} \\ \text{of the vectors} \\ \text{as reference} \end{matrix} \right\} \Rightarrow x_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ Norm $= \|v_1\| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$

$v_2 = u_2 - \alpha x_1$: $\langle \underline{v_2}, \underline{x_1} \rangle = 0$

$\underline{u_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\underline{v_2} = \begin{bmatrix} \\ \\ \end{bmatrix}$, $\underline{x_2} : \begin{bmatrix} \\ \\ \end{bmatrix}$

$$\left\langle \begin{bmatrix} 1 - \alpha/\sqrt{2} \\ 0 \\ -1 - \alpha/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\rangle = 0$$

$\Rightarrow \cancel{1 - \alpha/\sqrt{2}} \; \cancel{- \frac{1-\alpha}{\sqrt{2}}} \rightarrow \alpha = 0$

$x_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$ $u_3 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$; $v_3 = u_3 - \beta x_1 - \gamma x_2$

$\langle v_3, x_1 \rangle = 0$

$\langle v_3, x_2 \rangle = 0$

$$\left\langle \begin{bmatrix} -\beta/\sqrt{2} - \gamma/\sqrt{2} \\ 3 \\ 4 - \beta/\sqrt{2} + \gamma/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\rangle = 0$$

$$\left\langle \begin{bmatrix} -\beta/\sqrt{2} - \gamma/\sqrt{2} \\ 3 \\ 4 - \beta/\sqrt{2} + \gamma/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \right\rangle = 0$$

$-\frac{\beta}{2} - \cancel{\frac{\gamma}{2}} + 0 + \frac{4}{\sqrt{2}} - \frac{\beta}{2} + \cancel{\frac{\gamma}{2}} = 0 \rightarrow \cancel{+}\beta = \cancel{+}\frac{4}{\sqrt{2}} \rightarrow \beta = 2\sqrt{2}$

$-\cancel{\frac{\beta}{2}} - \frac{\gamma}{2} - \frac{4}{\sqrt{2}} + \cancel{\frac{\beta}{2}} - \frac{\gamma}{2} = 0 \rightarrow -\gamma = \frac{4}{\sqrt{2}} \rightarrow \gamma = -2\sqrt{2}$

So. $\underline{v_3} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$; $\underline{x_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

When $\alpha = \beta = 0$ becomes zero, then we know that the vectors are orthogonal

Q3 A 3 dim. space of polynomial ~~coeff~~ functions of the form x^n $\forall$ $n \in \mathbb{Z}^+ \cup 0$ is to be expanded using an orthonormal basis. Det. the basis if $\langle f_1(x), f_2(x) \rangle = \int_{-1}^{1} f_1(x) f_2(x) \, dx$

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3 dim $\rightarrow$ 3 f^n

$f_1(x) = 1$ | 1 ; $g_1(x) = 1$; $h_1(x) = 1/\sqrt{2}$

$f_2(x) = x$ | x ; $g_2(x) = f_2(x) - \alpha h_1(x)$: $\langle g_2(x), h_1(x) \rangle = 0$

$f_3(x) = x^2$ | x^2 ; $g_3(x) = f_3(x) - \beta h_1(x) - \gamma (h_2(x))$: $\langle g_3(x), h_1(x) \rangle$

$\langle g_3(x), h_2(x) \rangle$

much easier

$\| f_1(x) \| = \sqrt{\int_{-1}^{1} f_1(x) f_1(x)\, dx} = \sqrt{\int_{-1}^{1} (f_1(x))^2\, dx}$

$= \sqrt{(x)\Big|_{-1}^{1}} = \sqrt{2}$

$\| f_2(x) \| = \sqrt{\int_{-1}^{1} x^2\, dx} = \sqrt{\frac{2}{3}}$ $\quad \| f_3(x) \| = \sqrt{\int_{-1}^{1} x^4\, dx} = \sqrt{\frac{2}{5}}$

$g_2(x) = x - \frac{\alpha}{\sqrt{2}}$

$\langle g_2(x), h_1(x) \rangle = \int_{-1}^{1} \left(x - \frac{\alpha}{\sqrt{2}} \right)\left(\frac{1}{\sqrt{2}} \right) dx = \int_{-1}^{1} \frac{x}{\sqrt{2}} - \frac{\alpha}{2}\, dx$

$\Rightarrow \left(\frac{x^2}{2\sqrt{2}} - \frac{x\alpha}{2} \right)_{-1}^{1} = \cancel{\frac{1}{2\sqrt{2}}} - \frac{\alpha}{2} - \cancel{\frac{1}{2\sqrt{2}}} - \frac{\alpha}{2} = 0$

$\boxed{\alpha = 0}$

So $g_2(x) = x$ $\quad h_2(x) = \frac{x}{\sqrt{2/3}} = \sqrt{\frac{3}{2}}\, x$

$g_3(x) = x^2 - \frac{\beta}{\sqrt{2}} - \gamma \sqrt{\frac{3}{2}}\, x$

$\int_{-1}^{1} \frac{1}{\sqrt{2}} g_3(x)\, dx = 0 \rightarrow \int_{-1}^{1} \sqrt{\frac{3}{2}}\, x \left(g_3(x) \right) dx = 0$

$= \int_{-1}^{1} \sqrt{\frac{3}{2}} x^3 - \frac{\sqrt{3}\beta x}{2} - \frac{3\gamma x^2}{2}\, dx \quad \left(\sqrt{\frac{3}{2}} \frac{x^4}{4} - \frac{\sqrt{3}\beta x^2}{2 \times 2} - \frac{\gamma x^3}{2} \right)_{-1}^{1}$

$\beta = \frac{2}{3\sqrt{2}}$; $\gamma = 0$ $\quad = \frac{1}{4}\sqrt{\frac{3}{2}} - \cancel{\frac{\sqrt{3}\beta}{4}} - \frac{\gamma}{2} - \sqrt{\frac{3}{2}}\left(\cancel{\frac{1}{4}}\right) - \cancel{\frac{\sqrt{3}\beta}{4}} + \frac{\gamma}{2}$

$\rightarrow \gamma = 0$

what can cause the entire process to crash?

1. Norm = 0 $\rightarrow$ undefined behaviours

We cannot have zero vector as our basis as it does not create something new.

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(2) Norm $= \infty \rightarrow$ Now, we cannot normalise the f^n.

So, this procedure is only applicable for subsets which are finite and $\|f_1(x)\| = \sqrt{\int_a^b f_1(x) f_1(x)\, dx}$ is finite

$(<\infty) \leftarrow$ L^2 - space $\rightarrow$ A space in which the integration of square of a f^n would be a finite quantity. (also called space of sq. integrable f^ns)
$\rightarrow$ Norm spaces $\rightarrow$ Norm is defined. $\rightarrow \|x+y\| \leq \|x\| + \|y\|$

$d(x,y) > 0$ if $x \neq y$ — Metric spaces :— A metric space is a space where we can do measurements. All those spaces where a quantity which indicates measurement is defined are metric spaces.

$d(x,x) = 0$

$d(x,y) = d(y,x)$ $\quad d(x,y) \leq d(x,z) + d(z,y)$

Q4 $L^2_0(0, a) \rightarrow$ find if the f^ns belong to this space.

(a) $f(x) = \sin x ; a = \pi$	(b) $f(x) = \cos x ; a = \pi$	(c) $f(x) = 1/(1+x)$, $a = \infty$
$\int_0^{\pi} \sin^2 x\, dx$	$\int_0^{\pi} \cos^2 x\, dx$	$\int_0^{\infty} \frac{1}{(1+x)^2} dx$
$\frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)_0^{\pi}$	$\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)_0^{\pi}$	$\left[\frac{(1+x)^{-1}}{-1}\right]_0^{\infty}$
$= \frac{\pi}{2} \rightarrow$ Yes.	$= \frac{\pi}{2} \rightarrow$ Yes.	$\left(\frac{-1}{1+x}\right)_0^{\infty} = 1 \rightarrow$ Yes.

Q.5 Verify whether set $V = \{e^{inx} ; n \in \mathbb{R}\}$ belongs to $L^2(-\pi, \pi)$

sol^n $\|V\| = \int_{-\pi}^{\pi} e^{inx} \cdot e^{inx}\, dx = \int_{-\pi}^{\pi} e^{2inx} dx = \left(\frac{e^{2inx}}{2in}\right)_{-\pi}^{\pi}$

$= \left(\frac{e^{2i\pi n}}{2in} - \frac{e^{-2i\pi n}}{2in}\right) = \frac{1}{2in}\left(e^{2i\pi n} - e^{-2i\pi n}\right)$

$= \frac{1}{2in}\left(\cancel{\cos(2\pi n)} + i\sin(2\pi n) - \cancel{\cos(2\pi n)} + i\sin(2\pi n)\right)$

$= 0.$

Test of orthogonality :— $e^{inx}, e^{imx} \rightarrow 2$ f^ns.

$e^{inx} \cdot e^{imx}$

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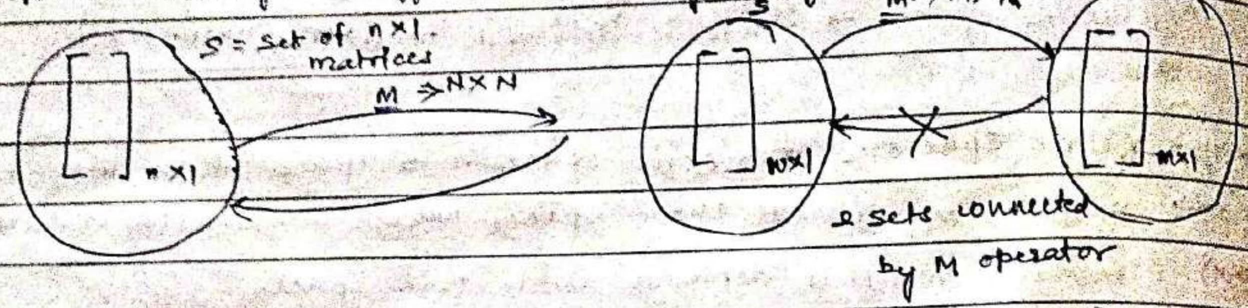
Adjoint of an operator :—

3. If $L = \frac{d^2}{dx^2} + x\frac{d}{dx}$; $x = 0$, $\frac{du}{dx} = 0$

$x = 1$, $\frac{du}{dx} + u = 0$

How can we get a different quantity using matrix ?



for an operator there is corresponding operator which is called the adjoint of an operator .

$\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\underline{Au} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} a_{11}u_1 + a_{12}u_2 \\ a_{21}u_1 + a_{22}u_2 \end{bmatrix}$

nature of elements of the set $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$\langle \underline{Au}, \underline{v} \rangle = \begin{bmatrix} a_{11}u_1 + a_{12}u_2 \\ a_{21}u_1 + a_{22}u_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = a_{11}u_1v_1 + a_{12}u_2v_1 + a_{21}u_1v_2 + a_{22}u_2v_2$$

$$= u_1(a_{11}v_1 + a_{21}v_2) + u_2(a_{12}v_1 + a_{22}v_2)$$

$$\langle \underline{Au}, \underline{v} \rangle = \left\langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} a_{11}v_1 + a_{21}v_2 \\ a_{12}v_1 + a_{22}v_2 \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle$$

$$\langle \underline{Au}, \underline{v} \rangle = \langle \underline{u}, \underline{Bv} \rangle$$

$\underline{B}$ is called the adjoint of operator $\underline{A}$ (not the matrix A)

for 2 × 2 matrix $\underline{B}$ = Transpose of $\underline{A}$

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} a_{11}u_1 + a_{12}u_2 + a_{13}u_3 \\ a_{21}u_1 + a_{22}u_2 + a_{23}u_3 \end{bmatrix}_{2\times 1}$$

$$\langle Au, v \rangle = \begin{bmatrix} a_{11}u_1 + a_{12}u_2 + a_{13}u_3 \\ a_{21}u_1 + a_{22}u_2 + a_{23}u_3 \end{bmatrix}_{2\times 1} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_{3\times 1} = \times$$

$\Rightarrow$ can't be proved like the previous question.

only for real systems → the transpose as adjoint is possible

$\langle a\underline{u}, \underline{v} \rangle = a \langle \underline{u}, \underline{v} \rangle$

$\langle \underline{u}, a\underline{v} \rangle = \bar{a} \langle \underline{u}, \underline{v} \rangle$

complex conjugation → Take the conjugate also for complex operⁿ sys.

Complex conjugate Transpose

$\Rightarrow$ $\langle \underline{A}\underline{u}, \underline{v} \rangle = \langle \underline{u}, \underline{A}\underline{v} \rangle$

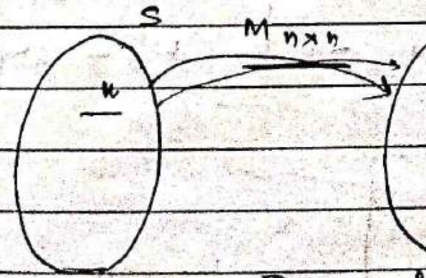
$$\underline{A} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

$\underline{A}$ is the adjoint of itself $\Rightarrow$ Self adjoint operator

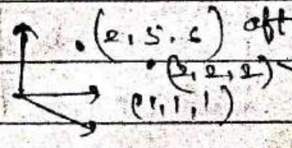
Imp. 1) The eigenvalues of self adjoint operator are always real.

Proof:— for non square matrices, eigen values are not defined.

$\langle \underline{A}\underline{u}, \underline{u} \rangle = \langle \lambda \underline{u}, \underline{u} \rangle$.



$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$



after matrix multiplicⁿ

Scaling up

not rotating the vector

$\underline{A}\,u = \lambda u$ ← eigen-vector

eigenvalue of $\underline{A}$ for corresponding eigenvector $\underline{u}$

special (german word)

only fⁿs & vectors for which dirⁿ is conserved

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$\langle A\underline{u}, \underline{u}\rangle = \langle \lambda \underline{u}, \underline{u}\rangle = \lambda \langle \underline{u}, \underline{u}\rangle$ — ①

$\langle \underline{u}, A\underline{u}\rangle = \langle u, \lambda u\rangle = \bar{\lambda}\langle \underline{u}, \underline{u}\rangle$ — ②

if $\underline{A}$ is ~~co~~ self adjoint, it will be same for ① & ②

Thus conjugate in ① = ② and thus, λ is always real

possible only for real sys. $(\bar{\lambda} = \lambda)$

⇒ 2.) The eigenvectors of self adjoint operator corresponding to distinct eigen values are always orthogonal.

Proof :- $\lambda_1 \to v_1$ $\langle \underline{A}\,\underline{v_1}, \underline{v_2}\rangle = \lambda_1\langle v_1, v_2\rangle$ — ③

$\lambda_2 \to v_2$ $\langle \underline{v_1}, \underline{A}\,\underline{v_2}\rangle = \bar{\lambda}_2\langle v_1, v_2\rangle$ — ④

③ − ④ :- $(\lambda_1 - \lambda_2)(\langle \underline{v_1}, \underline{v_2}\rangle) = 0$

$\lambda_1 \neq \lambda_2$

$\langle \underline{v_1}, \underline{v_2}\rangle = 0$ (orthogonal property)

if this is true for eigenvalues & eigenvectors it is will also be true for eigenvalues & eigenfunctions

eigenvalues correspond to physical quantity in our system.

Fredholm's Alternative Theorem

Q. Det. the range space of the following system of eqns.

$x_1 + x_2 + x_3 = 1$

$2x_1 + 3x_2 + x_3 = 2$

$5x_1 + 6x_2 + 4x_3 = 1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 2 & 3 & 1 & b_2 \\ 5 & 6 & 4 & b_3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & -1 & b_2 - 2b_1 \\ 0 & 1 & -1 & b_3 - 5b_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & -1 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 3b_1 - b_2 \end{array}\right]$$

$b_3 - 3b_1 - b_2 = 0$

DOF = 2 eqn = 1 var = 3.

So $b_1 = \alpha$ $b_2 = \beta$

$b_3 = 3\alpha + \beta$

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$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ 3\alpha + \beta \end{bmatrix} \Rightarrow \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The dimension of range space is 2

Possible bases $[3 \; 0 \; 1]^T$ or $[0 \; 1 \; 1]^T$

<u>alternative :-</u>

Theorem :- $\underline{A}\,\underline{x} = \underline{b}$ is solvable

if $\langle \underline{b}, \underline{y} \rangle = 0$! $\underline{A}^* \underline{y} = 0$

$$\left\langle \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right\rangle$$

$\underline{y}$ belongs to the null space of the adjoint of A

$A^* = A^T$ for real (3×3) matrix

$$[A^* | \underline{y}] = \left[\begin{array}{ccc|c} 1 & 2 & 5 & y_1 \\ 1 & 3 & 6 & y_2 \\ 1 & 1 & 4 & y_3 \end{array}\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \left[\begin{array}{ccc|c|c} 1 & 2 & 5 & y_1 & 0 \\ 1 & 3 & 6 & y_2 & 0 \\ 1 & 1 & 4 & y_3 & 0 \end{array}\right] = 0$$

$$\rightarrow \left[\begin{array}{ccc|c|c} 1 & 2 & 5 & y_1 & 0 \\ 0 & 1 & 1 & y_2 - y_1 & 0 \\ 0 & -1 & -1 & y_3 - y_1 & 0 \end{array}\right] = 0$$

$$\rightarrow \left[\begin{array}{ccc|c|c} 1 & 2 & 5 & y_1 & 0 \\ 0 & 1 & 1 & y_2 - y_1 & 0 \\ 0 & 0 & 0 & y_3 + y_2 - 2y_1 & 0 \end{array}\right] = 0$$

~~$y_3 + y_2 - 2y_1 = 0$~~

~~Var = 3, Eq^n = 1, DOF = 2~~ ~~$y_1 = \alpha$ $y_2 = \beta$~~

~~$y_3 = 2\alpha - \beta$~~

$$\cancel{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}$$

$$x_1 + 2x_2 + 5x_3 = 0$$
$$x_2 + x_3 = 0$$

Eq^n = 2 Var = 3 DOF = 1.

$x_3 = -\alpha$

$x_2 = \alpha$

$x_1 = -2x_2 - 5x_3 = 5\alpha - 2\alpha = 3\alpha$

$$y = \alpha \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

$$\langle \underline{b}, y \rangle = 0 \rightarrow \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = 3b_1 + b_2 - b_3 = 0$$

$b_1 = \beta$; $b_2 = \gamma$, $b_3 = 3\beta + \gamma$.

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \beta \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Question papes :—

(3)
$$x + ay + (a^2 - bc)z = \cancel{b_1}$$
$$x + by + (b^2 - ac)z = b_2$$
$$x + cy + (c^2 - ab)z = b_3$$

- Dimension of LVS in which these eq^{n}s hold true.

→ Range space is the one in which this will have a sol^n.

Dimension of LVS = Dimension of (range space + null space)

(4) $f(x)$, $g(x)$, $h(x)$ ⇒ verify whether they make a basis.

$f(x) = x^2 - 2x + 5$

$g(x) = 2x^2 - 3x$ ~~[illegible]~~

$h(x) = x + 3$

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*. Needed to find the dimension.

Test for span was ~~done~~ needed to be done.

$\alpha f(x) + \beta(g(x)) + \gamma h(x) = 0.$

$\sum a_i f_i = 0$

(5) $V = \left\{ \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} : x_{ij} \in \{0,1\} \right\}$

purpose to stay in the V.S. (not get 2)

$V = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$V \times V \longrightarrow V$

2 inputs from same set. in the same set

if we can prove that it is true for any 2 combn's then its true → summing up of all combns

ij-th element of 1.

4 such combns.

1 → 1
2 → 0
3 → 0
4 → 2

→ only 1 matrix.

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

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Eigenvalue problems

Q. Solve the simultaneous equations:

(a) $\frac{dx}{dt} = 2x$, $\frac{dy}{dt} = -3y$

(b) $\frac{dx}{dt} = x + 2y$, $\frac{dy}{dt} = y$

(a)

$\Rightarrow \int \frac{dx}{x} = \int 2dt$

$\rightarrow \ln x = 2t + \ln c_1$

$\int \frac{dy}{y} = -\int 3dt$

$\rightarrow \ln y = -3t + \ln c_2$

$\rightarrow x(t) = c_1 e^{2t}$

$\rightarrow y(t) = c_2 e^{-3t}$

although these are simultaneous eqns but they are independent of each other.

(b)

$\frac{dy}{dt} = y$

$\rightarrow y = c_1 e^{t}$.

$\frac{dx}{dt} = x + 2c_1 e^{t} \rightarrow \frac{dx}{x} = dt + \frac{2c_1 e^{t}}{x} dt$

$\rightarrow \frac{dx}{dt} - x = 2c_1 e^{t}$

$\rightarrow e^{-t}\frac{dx}{dt} - e^{-t}x = 2c_1$

$\rightarrow \frac{d}{dt}(x e^{-t}) = 2c_1 \rightarrow \int d(x e^{-t}) = \int 2c_1 dt$

$\rightarrow x e^{-t} = 2c_1 t + c_2$

$\rightarrow \boxed{x = 2c_1 t e^{t} + c_2 e^{t}}$

$\frac{dx}{dt}, 2y \quad \frac{dy}{dt} = -3x$

~~For above problem~~ $\Rightarrow \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$x(t) = c_1 e^{2t}$

$y(t) = c_2 e^{-3t}$

$(2-\lambda)(-3-\lambda) = 0$

$(\lambda+3)(\lambda-2) = 0$

$\lambda = 2, \ \underline{v_1} = [1 \ 0]^T$

$\lambda = -3, \ \underline{v_2} = [0 \ 1]^T$

$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$x = c_1 e^{2t} \quad y = c_2 e^{-3t}$

$\frac{dx}{dt} = x + 2y \quad \frac{dy}{dt} = x$

$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$\frac{dx}{dt} = Ax$.

Eigen $\lambda_1 = -1, \ \underline{v_1} = [1 \ -1]^T$

Eigen $\lambda_2 = 2, \ \underline{v_2} = [2 \ 1]^T$

$\begin{bmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{bmatrix} \rightarrow -\lambda + \lambda^2 - 2$

$\lambda^2 - \lambda - 2$

$\rightarrow \lambda^2 - 2\lambda + \lambda - 2$

$(\lambda+1)(\lambda-2)$

$\underline{A}\,\underline{v_1} = \lambda \underline{v_1}$

$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

(b) for ~~$v_1 + 2v_2 = \lambda v_1 \rightarrow v_1(\lambda - 1) = 2v_2$~~

~~$v_1 = \lambda v_2 \rightarrow v_2 = \frac{v_1}{\lambda}$~~ ~~$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} =$~~

~~$\lambda(\lambda - 1) = 0 \rightarrow \lambda = 0, 1$~~

~~$v_2 = -v_1$~~

~~$v_2 = 0$~~ $\langle \underline{A}\,\underline{u}, \underline{v} \rangle$

$$\underline{A} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -1\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$\rightarrow v_1 + 2v_2 = -v_1$

$v_1 = -v_2$.

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$\rightarrow v_1 + 2v_2 = 2v_1$

$v_1 = 2v_2$

for λ

$$x(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$x(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} v_i$$

corresponding eigenvalue.

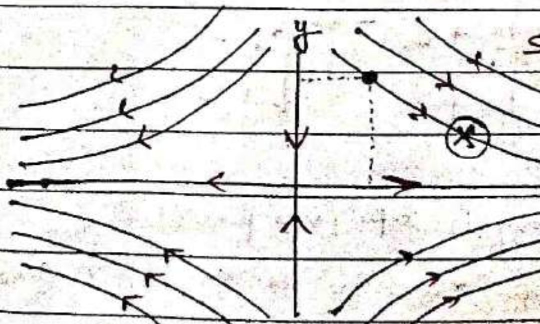
independent variable, corresponding eigenvector

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + 2c_2 e^{2t} \\ -c_1 e^{-t} + c_2 e^{2t} \end{bmatrix}$$

What is the importance of the word "simultaneous"? → We have put a restriction that both of our eqns should be true at that time. So, we need a condn that proves that yes, both of them are going to be satisfied.

Dependent variables → x, y should be satisfied at the same instant of time → common values only → only those which satisfy both will be a part of the soln.

(c)



Saddle soln

As c_1 & c_2 change curves change. These curves are the solution. c_1 & c_2 → from initial condns.

IVP → Every IVP will give the time evolution of a system.

Is x axis an eigenvector for this system → Yes (y axis also)

$\lambda_1 = 2$; $v_1 = [1 \ 0]^T$

$\lambda_2 = -3$; $v_2 = [0 \ 1]^T$

What is the difference b/w the nature of x axis & y axis eigenve

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \underbrace{e^{2t}}_{+ve} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \underbrace{e^{-3t}}_{-ve} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

As time progresses it ↑ as time ↑ it will asymptotically move towards 0.

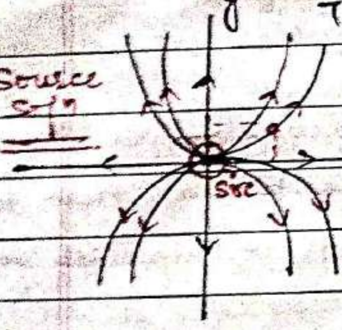
* Depending on c_1 & c_2 we can end up in any quadrant. The evolution for my IVP problem should be such that it should conform to the graph → x variable moving towards ∞ & y variable moving towards 0. Such sol^n's are called saddle solutions.

Can some change be done such that both the variables become either zero or ∞.

if both exponentials become +ve → our sol^n becomes ∞ with time; −ve → our sol^n becomes 0 with time

The nature of the lines will change

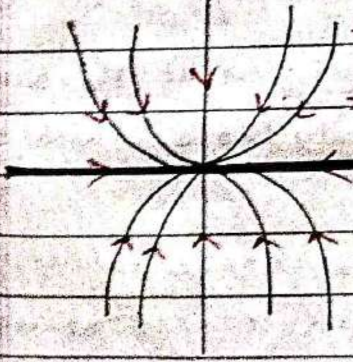
Source sol^n



$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{+3t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

if we wait long enough our variable is going to blow upto ∞. Such sol^n's are called src sol^n. Sol^n's are emanating from the src.

Sink sol^n



(zero, zero) is the eqlbm sol^n as nothing changes at that pt. in both the cases

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Linear stability analysis → Analysis of IVP problems

Src sol^n → reactor will blow up due to ∞ ↑ in T (not practically)

Sink sol^n → reactor will shut down (become cold) because T decreases ↓

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Saddle sol^n → along 1 dir^n the variable ↑ and in other dir^n it ↓ses.

Q. Non simultaneous eq^{n}s – let dy/dx be z.

(a) $\frac{d^2y}{dx^2} - y = 0$

$\frac{dz}{dx} = (1)y + (0)z$

$\frac{dy}{dx} = (0)y + (1)z$

$\frac{d}{dx}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$

$\lambda_1 = 1$; $\underline{v_1} = [1 \quad 1]^T$

$\lambda_2 = -1$; $\underline{v_2} = [1 \quad -1]^T$

$\begin{bmatrix} y \\ z \end{bmatrix} = c_1 e^{x}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-x}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$y = c_1 e^{x} + c_2 e^{-x}$ (saddle sol^n)

(b) $\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} (c_1+c_2)\sin x \\ (c_1+c_2)\cos x \end{bmatrix} + i\begin{bmatrix} (c_2-c_1)\cos x \\ (c_1+c_2)\sin x \end{bmatrix}$

$= z_1 + iz_2$

$\frac{d^2}{dx^2}(z_1 + iz_2) + z_1 + iz_2 = 0$

$\boxed{\frac{d^2z_1}{dx^2} + z_1} + \boxed{i\left(\frac{d^2z_2}{dx^2} + z_2\right)} = 0.$

$\frac{d^2z_1}{dx^2} + z_1 = 0$ → z_1 is a sol^n

$\frac{d^2z_2}{dx^2} + z_2 = 0$ → z_2 is also a sol^n

Overall sol^n: – If we have 2 sol^{n}s the general sol^n is the linear combin^n of those 2 sol^{n}s.

(b) $\frac{d^2y}{dx^2} + y = 0$

$\frac{dz}{dx} = (-1)y + (0)(z)$

$\frac{dy}{dx} = (0)(y) + (1)(z)$

$\frac{d}{dx}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$

$\lambda_1 = i$; $\underline{v_1} = [-i \quad 1]^T$

$\lambda_2 = -i$; $\underline{v_2} = [i \quad 1]^T$

$\begin{bmatrix} y \\ z \end{bmatrix} = c_1 e^{ix}\begin{bmatrix} -i \\ 1 \end{bmatrix} + c_2 e^{-ix}\begin{bmatrix} i \\ 1 \end{bmatrix}$

~~$y = c_1 e^{ix}$~~ if we have a real system described by eq^{n}s consisting of imaginary nos., is these a sys. of real eq^{n}s solving the same? → yes

$= c_1(\cos x + i\sin x)\begin{bmatrix} -i \\ 1 \end{bmatrix}$

$+ c_2(\cos x - i\sin x)\begin{bmatrix} i \\ 1 \end{bmatrix}$

~~$+ c_2\cos$~~ $\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \; \\ \; \end{bmatrix} + i\begin{bmatrix} \; \\ \; \end{bmatrix}$

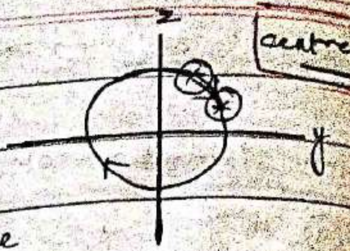
$\begin{bmatrix} y \\ z \end{bmatrix} = c_1\cos x\begin{bmatrix} -i \\ 1 \end{bmatrix} + c_2\cos x\begin{bmatrix} i \\ 1 \end{bmatrix}$

$+ i\sin x\left(c_1\begin{bmatrix} -i \\ 1 \end{bmatrix} - c_2\begin{bmatrix} i \\ 1 \end{bmatrix}\right)$

$\begin{bmatrix} y \\ z \end{bmatrix} = [c_1 i\cos x + c_1\sin x \quad c_1\cos x + c_1 i\sin x]^T$

$\begin{bmatrix} c_2 i\cos x + c_2\sin x \\ c_2\cos x - c_2 i\sin x \end{bmatrix}$

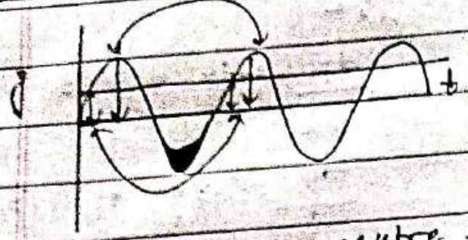
centre sol^n



(8) $\begin{bmatrix} y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \sin x \\ \cos x \end{bmatrix} + \beta \begin{bmatrix} \cos x \\ -\sin x \end{bmatrix}$

$y = \alpha \sin x + \beta \cos x$

difference Now, we have an imaginary eigenvalue



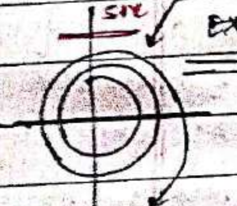
at regular intervals we get same value of y. However t values are different.

difference b/w centre sol^ns of IVP & other 3 → centre sol^ns are periodic. They correspond to systems whose sol^ns are periodic and oscillatory. However, for the other 3, the system of sol^ns are monotonous in nature.

Periodic and oscillatory are a result of imaginary eigenvalues

Imaginary eigenvalues have no real part. (Not complex)

spiral src



Ex. $\boxed{\begin{matrix} 2+i \\ 2-i \end{matrix}}$ will lead to oscillations. However $e^{(2+i)t} \to e^{2t} e^{it} \to$ sin cos → oscillation

scalar with value changing

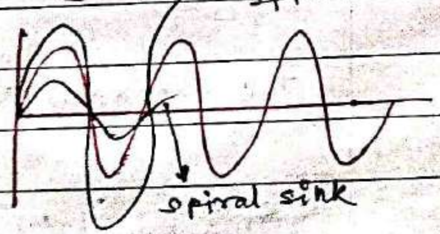
$\boxed{\begin{matrix} -2+i \\ -2-i \end{matrix}}$ — $e^{-2t} e^{it}$ → oscill^n

spiral sink

$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dt} + a_0 y = 0$

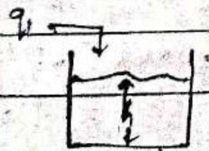
Eigenvalues are complex nos.

for centre sol^ns → spiral src.



spiral sink

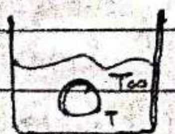
Ex



$\frac{dh}{dt} = \frac{1}{A}(q_1 - q_2)$ $\quad \frac{dh}{dt} = -\frac{1}{A} q_2$

$\frac{dh}{dt} = -\frac{1}{A} f(h)$ $\quad$ gravity drainage :- $-\frac{1}{A} C_d \sqrt{2gh} = \frac{dh}{dt}$

discharge coeff depends on the walls of the pipe



$\frac{dT}{dt} = \alpha (T - T_\infty)$

after putting the pump :- $\frac{dh}{dt} = -\frac{1}{A} q_2 - f_2(t)$

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$$\frac{dx}{dt} = Ax$$
$$x(t) = \sum_{i=1}^{2} c_i e^{\lambda_i t} v_i$$
} autonomous eqn (presence of only dependent vars)

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(1) $\frac{dx}{dt} = x + y - t - 1$ ← new → autonomous eqns

$\frac{dy}{dt} = 4x + y - 4t - 2$

Similarity Transformⁿ

$$\frac{dx}{dt} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} x + \begin{bmatrix} -t & -1 \\ -4t & -2 \end{bmatrix}$$

$$\frac{dx}{dt} = Ax + B(t)$$

In absence of B this solⁿ is trivial

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

$v_1 = [1 \ -2]$; $\lambda_1 = -1$

$v_2 = [1 \ 2]$; $\lambda_2 = 3$

$$P = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}, \quad P^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$$

$$P^{-1}A = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2 & 1 \\ 6 & 3 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{4}\begin{bmatrix} -2 & 1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -4 & 0 \\ 0 & 12 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

- The diagonals have the eigen values and it is a ~~identity~~ diagonal matrix.
- relⁿ b/w A & $P^{-1}AP$ → similar matrices (eigenvalues are same eigen-functions/vectors are different)

$$\frac{dx}{dt} = A\vec{x} + B(t)$$

(A: matrix; x: vector)

$$\frac{dx}{dt} = ax + b(t) \quad \text{(Scalar)}$$

$$\int \frac{dx}{ax+b} = \int dt$$

(2) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0, \ n \geq 0$

Sturm Liouville Theory - Bessel Eqⁿ

const coeff. made the eqⁿ solvable

Frobenius Method
or
The Series Solution Method.

$$y = \sum_{i=0}^{\infty} a_i x^{i+k}$$

$$\rightarrow \frac{dy}{dx} = \sum_{i=0}^{\infty} a_i x^{i+k-1}(i+k)$$

$$\rightarrow \frac{d^2y}{dx^2} = \sum_{i=0}^{\infty} a_i (i+k)(i+k-1) x^{i+k-2}$$

$$\Rightarrow \sum_{i=0} a_i (i+k)(i+k-1) x^{i+k} + \sum_{i=0} a_i (i+k) x^{i+k} - n^2 \sum_{i=0} a_i x^{i+k} + \sum_{i=0} a_i x^{i+k+2} = 0$$

$$\Rightarrow \sum_{i=0} a_i \{(i+k)(i+k-1) + (i+k) - n^2\} x^{i+k} + \sum_{i=0} a_i x^{i+k+2} = 0$$

$$\Rightarrow x^k \ x^{k+1} \ x^{k+2} \ x^{k+3} \ldots \ | \ x^{k+2} \ x^{k+3} \ldots$$

$$a_0\{k(k-1) + k - n^2\} = 0$$

$$\Rightarrow k = \pm n$$

$$a_1\{(1+k)(k) + (1+k) - n^2\} = 0$$

$$\Rightarrow \{k^2 + 2k + 1 - n^2\} = 0$$

$$\rightarrow \{(k+1)^2 - n^2\} = 0$$

$$\rightarrow (k+1-n)(k+1+n) = 0$$

$$\rightarrow k = -(1-n), -(1+n)$$

$$\rightarrow k = -1 \pm n \text{ or } a_1 = 0$$

→ Solving for $k = +n$:-

$$\rightarrow \sum_{i=0}^{\infty} a_i \{(i+n)(i+n-1) + (i+n) - n^2\} x^{i+n} + \sum_{i=0} a_i x^{i+n+2} = 0$$

to get coeff of x^{n+2} → $a_2\{(n+2)(n+1) + (n+2) - n^2\} + a_0 = 0$

$$\rightarrow a_2 = \frac{-a_0}{2(2n+2)}$$

1) ~~$x = ax + b$~~

~~$\ln(ax+b) = t + \ln c$~~

~~$\to ax + b(t)$~~

$$\frac{dx}{dt} = ax + b(t)$$

$$\to \frac{dx}{dt} - ax = b(t)$$

$$e^{-at}\frac{dx}{dt} - a e^{-at} = e^{-at} b$$

$$\to \frac{d}{dt}\left(e^{-at} x\right) = e^{-at} b(t)$$

$$\to \int d\left(e^{-at} x\right) = \int e^{-at} b(t)\, dt$$

$$\to e^{-at} x = \int e^{-at} b(t)\, dt + c$$

$$\to x(t) = e^{at}\int e^{-at} b(t)\, dt + c e^{at}$$

$$\to \frac{d\underline{x}}{dt} - \underline{\underline{A}}\,\underline{x} = \underline{B}(t)$$

$$\Longrightarrow \frac{d}{dt}\left(\underline{\underline{P}}^{-1}\underline{x}\right) - \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{x} = \underline{\underline{P}}^{-1}\underline{B}(t)$$

$$\Rightarrow \frac{d}{dt}\left(\underline{\underline{P}}^{-1}\underline{x}\right) - \left(\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}\right)\left(\underline{\underline{P}}^{-1}\underline{x}\right) = \underline{\underline{P}}^{-1}\underline{B}(t)$$

$$\to \frac{d}{dt}\underline{y} - \underline{\underline{\Delta}}\,\underline{y} = \underline{G}(t)$$

$$\underline{y} = \underline{\underline{P}}^{-1}\underline{x}, \quad \underline{G}(t) = \underline{\underline{P}}^{-1}\underline{B}(t)$$

$$\underline{\underline{\Delta}} = P^{-1}AP$$

$$\Rightarrow \frac{d\underline{x}}{dt} - \underline{\underline{A}}\,\underline{x} = \underline{B}(t)$$

$$\to \left(\frac{d\underline{y}}{dt}\right)e^{-\underline{\underline{\Delta}}t} - \underline{\underline{\Delta}}\, e^{-\underline{\underline{\Delta}}t}\underline{y} = e^{-\underline{\underline{\Delta}}t} G(t)$$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{-\underline{\underline{\Delta}}t} = \underline{I} - (\underline{\underline{\Delta}}t) + \frac{1}{2!}(\underline{\underline{\Delta}}t)^2 + \dots$$

2) $\to a_3\left\{(3+n)(2+n) + (3+n) - n^2\right\} + a_1 = 0$

$$\to a_3\left\{(n+3)^2 - n^2\right\} + a_1 = 0$$

$$\to a_3 = \frac{-a_1}{(2n+3)(3)} = \frac{-a_1}{3(2n+3)}$$

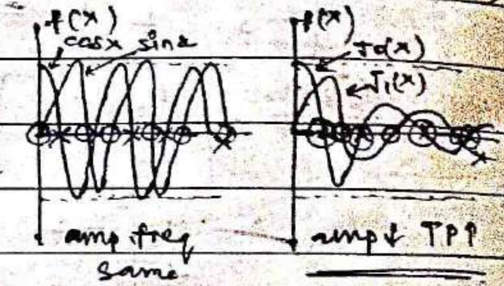
$$\to a_4 = \frac{-a_2}{4(2n+4)} \qquad a_5 = \frac{-a_3}{5(2n+5)}$$

$$a_1 = a_3 = a_5 = \dots = 0$$

$$a_2 = \frac{a_0}{4\cdot 2\cdot 2\cdot(n+1)(n+2)} = \frac{a_0}{16(n+1)(n+2)}$$

$$a_6 = \frac{-a_4}{6(2n+6)} = \frac{-a_0}{2\cdot 2\cdot 2\cdot 4\cdot 6(n+1)(n+2)(n+3)}$$

$$= \frac{-a_0}{2^3\cdot 4\cdot 6\,(n+1)(n+2)(n+3)}$$



$$a_0 = \frac{1}{2^n n!}$$

$J_0, J_1 \dots$ are all linearly independent and are orthogonal to each other

all 2nd order eqns belong to Strum Louville prob.

S-L operator :- self adj operator with real eigenvalues & orthogonal eigenvectors

$$\hat{L} = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + r(x)$$

- Legendre eqn
- Leguerre eqn
- Hermite eqn

1) $-\underline{\Lambda}t = \begin{bmatrix} -t & 0 \\ 0 & -3t \end{bmatrix}$

$(-\underline{\Lambda}t)^2 = \begin{bmatrix} (-t)^2 & 0 \\ 0 & (-3t)^2 \end{bmatrix}$

$(-\underline{\Lambda}t)^3 = \begin{bmatrix} (-t)^3 & 0 \\ 0 & (-3t)^3 \end{bmatrix}$

$\vdots$

$e^{-\underline{\Lambda}t} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} (-t)^2 & 0 \\ 0 & (-3t) \end{bmatrix} + \cdots$

$= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix}$

$\frac{d}{dt}(e^{-\underline{\Lambda}t}\underline{y}) = e^{-\underline{\Lambda}t} G(\underline{t})$

$d(e^{-\underline{\Lambda}t}\underline{y}) = e^{-\underline{\Lambda}t} G(t)\, dt$

$\Rightarrow \underline{y} = e^{\underline{\Lambda}t} \int e^{-\underline{\Lambda}t} G(t)\, dt + e^{\underline{\Lambda}t}\underline{c}$

$\underline{x} = P\underline{y}$