INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR

Department of Chemical Engineering

End-semester (Autumn) Examination 2018-2019

Subject: Advanced Mathematical Techniques in Chemical Engineering (CH61015)

Remarks:

1. This question paper contains two parts: **Part A** and **Part B**. Attempt both parts.
2. Write all the answers of a part together.
3. Unless otherwise stated, usual mathematical notations apply.
4. Time = 3 h; maximum marks = 100; total number of printed pages = 3.

Part A: Differential equations

1. Solve the following equation using separation of variables.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial z^2}=0$$

At $r=1, u=0$; at $z=0, u=u_0$; at $z=1, u=0$. Assume other physical boundary conditions.

. . . 10 marks

2. Solve the following equation.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{1}{sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial u}{\partial \theta}\right)+\frac{1}{r^2}\frac{1}{sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}=0$$

At $r=1, u=1$. Assume other physical boundary conditions associated with spherical coordinates.

. . . 10 marks

3. Solve the following equation completely using Green's function method.

$$\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}+x$$

At $t=0, T=T_0$; at $x=0, T=0$; At $x=1, \dfrac{\partial T}{\partial x}+T=0$.

. . . 20 marks

4. Find eigenvalues and eigenfunctions of the following problem.

$$(1+x)^2\frac{d^2u}{dx^2}+\lambda u=0$$

subject to $u(x=0)=u(x=1)=0$.
Hint: Use $e^{\pm i\theta}=cos\theta \pm isin\theta$.

. . . 10 marks

Part B: Linear algebra

5. For the following two systems of equations in unknowns x_i's and y_i's, determine whether the corresponding range spaces are identical using Fredholm's alternative theorem.

System 1

$$\begin{aligned} x_1 + 3x_2 + 3x_3 + 2x_4 &= a_1 \\ 2x_1 + 6x_2 + 9x_3 + 7x_4 &= a_2 \\ -x_1 - 3x_2 + 3x_3 + 4x_4 &= a_3 \end{aligned}$$

System 2

$$\begin{aligned} y_1 + 2y_3 + 3y_4 &= b_1 \\ y_2 + 4y_3 + 5y_4 &= b_2 \\ 3y_1 + y_2 + 10y_3 + 14y_4 &= b_3 \end{aligned}$$

. . . 10 marks

6. Determine an orthonormal set from the following set.

$$T = \{\underline{u}_1, \underline{u}_2, \underline{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 1+i \\ 1 \end{bmatrix}, \begin{bmatrix} -2-3i \\ 1-i \\ 2+5i \end{bmatrix}, \begin{bmatrix} -3-i \\ 1+3i \\ -1-i \end{bmatrix} \right\}$$

. . . 10 marks

7. Consider the following non-autonomous, non-separable, non-linear ODE.

$$\frac{dy}{dt} = y^2 + 16t^2 - 8yt + 2y - 8t + \lambda$$

(a) Verify whether you obtain an autonomous equation by the following transformation:
$u(t) = y(t) - 4t$.

(b) For the new equation, sketch a well-labelled bifurcation diagram.

(c) For $\lambda = 5$, sketch a well-labelled phase portrait for the autonomous system.

. . . 15 marks

8. Consider the nonhomogeneous equation

$$\underline{X}' = \underline{\underline{A}}X + \underline{G}(t)$$

where $\underline{\underline{A}}$ is $n \times n$ matrix and $\underline{G}(t)$ is a continuous function of t. The solution of this matrix equation satisfying $\underline{X}(0) = \underline{X}_0$ is given by

$$\underline{X}(t) = exp(t\underline{\underline{A}})\left(\underline{X}_0 + \int_0^t exp(-s\underline{\underline{A}})\underline{G}(s)ds\right)$$

Using the above result, determine the general solution of the following equation.

$$\frac{d^2x}{dt^2} + x = cos(t)$$

. . . 15 marks

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