Euler's Eq$^n$ Associated Legendre polynomial

(1) $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}=0$

BC :- at $r=1$, $u=f(\theta,\phi)$

at $r=0$, $u$ = finite

at $\theta=0$, $\theta=\pi$ } $u$ = finite } Physical BCs

periodic BCs:- $u|_{\phi=\pi} = u|_{\phi=-\pi}$

$\frac{\partial u}{\partial\phi}\Big|_{\phi=\pi} = \frac{\partial u}{\partial\phi}\Big|_{\phi=-\pi}$

Euler's Eq$^n$ Legendre Eq$^n$

(2) $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=0$

BC at $r=0$, $u$ = finite

$r=1$, $u=u_0$ or $u_0(\theta)$ } Physical BCs

$\theta=0,\pi \rightarrow u$ = finite

(3) $\frac{\partial u}{\partial t}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) \rightarrow v=ur$

BC :- at $t=0$, $u=f(r)$ / const.

at $r=1$, $u=0$ (const T at surface)

at $r=0$, $u$ = finite (Physical BC)

$y=\lambda R$ Bessel Eq$^n$

(4) $\frac{\partial u}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$ 3 cases (the -ve, 0) trivial soln

at $t=0$, $u=u_0(r,\theta)$

at $r=1$, $u=0$

at $r=0$, $u$ = finite } Physical BC

& Periodic BCs :- $u|_{\theta=\pi}=u|_{\theta=-\pi}$

$\frac{\partial u}{\partial\theta}\Big|_{\theta=\pi}=\frac{\partial u}{\partial\theta}\Big|_{\theta=-\pi}$

(5) $\frac{\partial u}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$

at $t=0$, $u=u_0$

at $r=1$, $u=0$

at $r=0$, $u$ = finite (Physical BC)

(6) $\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}$

$t=0$ $u=u_0$

$\frac{\partial u}{\partial t}=0$

at $x=0$, $u=u_1$

$x=1$, $u=0$.

Legendre Eq$^n$ :- orthogonal to $\sin\theta$

$(1-x^2)y''-2xy'+n(n+1)y$

at $x=\pm 1$ $y=0$.

$y(x)=c_1P_n(x)+c_2Q_n(x)$

$Q$ is unbounded at $x=\pm1 \rightarrow c_2=0$

Euler's Eq$^n$ :- $\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-n(n+1)R=0$

$x^2y''+axy'+by=0$

$y=x^n$

$y(x)=c_1x^{n_1}+c_2x^{n_2}$

Bessel eq$^n$ :- $y^2\frac{d^2R}{dy^2}+y\frac{dR}{dy}-(n^2-y^2)R=0 \rightarrow$ of order $n$.

$R(y)=c_1J_n(y)+c_2Y_n(y)$

if $n=0 \rightarrow$ Bessel eq$^n$ of order 0.

$y^2\frac{d^2R}{dy^2}+y\frac{dR}{dy}+(n^2-y^2)R=0$ modified Bessel eq$^n$ of order

$R(y)=c_1I_n(y)+c_2K_n(y)$

$y$ & $K$ as $\infty$ at $y=0$

## Solution of non homogenous PDE

① We replace the src/sink term by pt. src/sink term → unit impulse (dirac delta $f^n$) and at the same time we make all the BCs to be homogenous by ...

② So, dependent vas → green's $f^n$. $Lg = \delta(x - x_0)$ $Bg = 0$

③ Next → find Adj operator and adj. green's $f^n$. Then rel. it to initially given ...

### Checking of operator

| | |
|---|---|
| operator is $\nabla^2$ (Laplacian) | operator is parabolic Ex: $L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$ (Transient state op.) |
| SS operator | ↓ find out $L^*$ |
| Self adjoint $g = g^*$ | ↓ construct governing eq^n of adj. green's $f^n$ $(g^*)$ |
| | $L^* g^* = \delta(x - x_1)$ $Bg^* = 0$ |
| | (need not solve for $g^*$) soln expression of $g^*$ is obtd. by simply changing indices $x_0, x_1$ |

Prop. of dirac delta $f^n$ → $\delta(x - x_0) = 0 \ \forall \ x \neq x_0$
$= 1 \ \forall \ x = x_0$

rel^n b/w $g$ & $g^*$ :- $\boxed{g(x_1/x_0) = g^*(x_0/x_1)}$

$Lg(x/x_0) = \delta(x - x_0)$ — ①  $\langle ①, g^* \rangle - \langle ②, g \rangle = \int \delta(x - x_0) g^*(x/x_1) dx - \int \delta(x - x_1) g(x/x_0)$

$L^* g^*(x/x_1) = \delta(x - x_1)$ — ② ..

connect $g^*$ with $u$ :- $u(x) = m(x) - \underbrace{J(g^*, u)}$

$Lu = f$ — ①  not be present in case of self adj operator

$L^* g^* = \delta(x - x_1)$ — ②

$\langle ①, g^* \rangle - \langle ②, u \rangle = \int f(x) g(x/x_1) dx - \int u(x) \delta(x - x_1) dx$

Ex ① :- $\frac{d^2 u}{dx^2} = x$ at $x = 0$, $u = 1$
$x = 1$, $u = 2$

Ex ② :- $\frac{d^2 u}{dx^2} = x$ at $x = 0$, $u = 1$
$x = 1$, $\frac{du}{dx} + u = 2$

Ex ③ :- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x) \mid f(y) \mid f(x, y) \mid$ constant

Subj. to :- $x = 0$, $u = a$  $y = 0$, $u = c$
$x = 1$; $u = b$.  $y = 1$, $u = d$.

Step 1 :- Construction of causal green $f^n$ :- $L = \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$

$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = \delta(x - x_0)\delta(y - y_0)$  BC :- $x = 0, 1$ $g = 0$  $y = 0, 1$ $g = 0$

**Step 2 :- Corresponding eigenvalue problem**

$L\phi + \lambda r\phi = 0$ where $\phi$, eigenf$^n$, $r$, wt f$^n$ = 1, $\lambda$ = corresp. eigenvalues

$\Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \lambda\phi = 0$ $\qquad g(x,y|x_0,y_0) = \sum_m \sum_n a_{mn}\phi_{mn}(x,y)$

since we get 2 independent eigen value problem in both x and y dir$^n$, we can express $g$ in terms of eigenf$^n$ $\phi_n(x,y)$

at $x=0$, $x=1$, $y=0$, $y=1$: $\phi = 0$

$\phi_i \rightarrow$ eigenf$^n$ of corresponding eigenvalues problem.

Full eigenf$^n$ expansion method.

Step 3 $\phi = X(x)Y(y) \Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \lambda = 0 \Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda - \frac{1}{Y}\frac{d^2Y}{dy^2} = -\alpha^2 \Rightarrow \frac{d^2X}{dx^2} + \alpha^2 X = 0$

at $x=0$, $x=1$: $X=0$

$X = A_n \sin\alpha_n x + B_n \cos\alpha_n x \rightarrow X_n = A_n \sin(n\pi x)$, $\alpha_n = n\pi$, $n = 1,2,\dots\infty$

$-\lambda - \frac{1}{Y}\frac{d^2Y}{dy^2} = -\alpha^2 \rightarrow \frac{1}{Y}\frac{d^2Y}{dy^2} = -\lambda + \alpha^2 = -\beta^2 \rightarrow \frac{d^2Y}{dy^2} + \beta^2 Y = 0$ subj to $y=0$, $y=1$: $Y=0$

$\beta_m = m\pi$, $m = 1,2,\dots\infty$, $Y_m = B_m \sin(m\pi y)$ $\qquad \lambda_{mn} = \alpha_n^2 + \beta_m^2 = \pi^2(n^2+m^2)$

$$\boxed{\phi_{mn}(x,y) = c_{mn}\sin(n\pi x)\sin(m\pi y)}$$

Expression of $g$ in terms of eigenf$^n$s :- $g = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\phi_{mn}(x,y)$

$a_{mn}$ is obtained by using orthogonal property of eigenfunctions.

$a_{mn} = \langle g, \phi_{mn}\rangle$ $\qquad \int_x\int_y c_{mn}^2 \sin^2(n\pi x)\sin^2(m\pi y)\,dx\,dy = 1$ (making $\phi_{mn}$ orthonormal)

$\|\phi_{mn}\|^2 = \langle\phi_{mn},\phi_{mn}\rangle = 1$ $\qquad c_{mn}^2 = 4 \rightarrow c_{mn} = 2$

$a_{mn} = \langle g,\phi_{mn}\rangle$ $\qquad \phi_{mn} = 2\sin(n\pi x)\sin(m\pi y)$

**Step 4 :- Expression of $a_{mn}$ $\rightarrow$ estim$^n$ of $\langle g,\phi_{mn}\rangle$**

① $\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = \delta(x-x_0)\delta(y-y_0)$

② $\frac{\partial^2\phi_{mn}}{\partial x^2} + \frac{\partial^2\phi_{mn}}{\partial y^2} + \lambda_{mn}\phi_{mn} = 0$

$\langle ①,\phi_{mn}\rangle - \langle ②, g\rangle$

$\iint_{x\,y} \phi_{mn}\frac{\partial^2 g}{\partial x^2}\,dx\,dy + \iint_{x\,y}\phi_{mn}\frac{\partial^2 g}{\partial y^2}\,dx\,dy - \iint \lambda_{mn}\phi_{mn} g\,dx\,dy - \iint g\frac{\partial^2\phi_{mn}}{\partial x^2}\,dx\,dy - \iint g\frac{\partial^2\phi_{mn}}{\partial y^2}\,dx\,dy = \underbrace{\iint \delta(x-x_0)\delta(y-y_0)\phi_{mn}\,dx\,dy}_{\phi_{mn}(x_0,y_0)} - 0$

$\Rightarrow \int_y \left[\phi_{mn}\frac{\partial g}{\partial x}\Big|_0^1 dy - \int_x \cancel{\frac{\partial\phi_{mn}}{\partial x}\frac{\partial g}{\partial x}}\,dy\,dx\right] + \int_x\left[\phi_{mn}\frac{\partial g}{\partial y}\Big|_0^1 dx - \int_y \cancel{\frac{\partial\phi_{mn}}{\partial y}\frac{\partial g}{\partial y}}\,dy\,dx\right]$

$-\int_y\left[g\frac{\partial\phi_{mn}}{\partial x}\Big|_0^1 dy - \int_x \cancel{\frac{\partial g}{\partial x}\frac{\partial\phi_{mn}}{\partial x}}\,dx\,dy\right] - \int_x\left[g\frac{\partial\phi_{mn}}{\partial y}\Big|_0^1 dx - \int_y \cancel{\frac{\partial g}{\partial y}\frac{\partial\phi_{mn}}{\partial y}}\,dx\,dy\right] = \phi_{mn}(x_0,y_0) + \iint \lambda_{mn}\phi_{mn} g\,dx\,dy$

$\Rightarrow \phi_{mn}(x_0,y_0) + \lambda_{mn}\langle g,\phi_{mn}\rangle = \int_y\left[\cancel{\phi_{mn}(x=1)}\frac{\partial g}{\partial x}\Big|_1 - \cancel{\phi_{mn}(x=0)}\frac{\partial g}{\partial x}\Big|_0\right]dy + \int\left[\phi_{mn}\frac{\partial g}{\partial y}\Big|_0^1\right]$

$- \iint \cancel{g}\frac{\partial\phi_{mn}}{\partial x}\Big|_0^1 dy - \int \cancel{g}\frac{\partial\phi_{mn}}{\partial y}\Big|_0^1 dx$

$\rightarrow \langle g,\phi_{mn}\rangle = \frac{-\phi_{mn}(x_0,y_0)}{\lambda_{mn}} = \frac{-2\sin(n\pi x_0)\sin(m\pi y_0)}{\lambda_{mn}}$

$\Rightarrow g = \sum\sum a_{mn}\phi_{mn}(x,y) = -\sum\sum \phi_{mn}(x_0,y_0)\,\phi_{mn}(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{-4\sin(n\pi x_0)\sin(m\pi y_0)\sin(n\pi x)\sin(m\pi y)}{\pi^2(m^2+n^2)}$

**Step 5 :-** get the expression of $g$ $\quad \lambda_{mn}$

$L = \nabla^2 =$ Self adjoint operator $\quad L = L^*,\ B = B^*$

**Step 6 :-** Connect $g$ with $u$ :- ① $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f$ ② $\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = \delta(x-x_0)\,\delta(y-y_0)$

② $\langle ①, g\rangle - \langle ②, u\rangle$ :-

$$\iint g\frac{\partial^2 u}{\partial x^2}\,dx\,dy + \iint g\frac{\partial^2 u}{\partial y^2}\,dx\,dy - \iint u\frac{\partial^2 g}{\partial x^2}\,dx\,dy - \iint u\frac{\partial^2 g}{\partial y^2}\,dx\,dy = \iint fg\,dx\,dy - u(x_0,y_0)$$

$$\Rightarrow \int\left[g\frac{\partial u}{\partial x}\Big|_0^1 dy - \int \cancel{\frac{\partial g}{\partial x}\frac{\partial u}{\partial x}\,dx\,dy}\right] + \int\left[g\frac{\partial u}{\partial y}\Big|_0^1 dx - \int \cancel{\frac{\partial g}{\partial y}\frac{\partial u}{\partial y}\,dx\,dy}\right] - \int\left[u\frac{\partial g}{\partial x}\Big|_0^1 dy - \int \cancel{\frac{\partial u}{\partial x}\frac{\partial g}{\partial x}\,dx\,dy}\right]$$

$$-\int\left[u\frac{\partial g}{\partial y}\Big|_0^1 dx - \int \cancel{\frac{\partial u}{\partial y}\frac{\partial g}{\partial y}\,dx\,dy}\right] = \iint fg\,dx\,dy - u(x_0,y_0)$$

$$\Rightarrow -\int\left[u(x=1)\frac{\partial g}{\partial x}\Big|_{x=1} - u(x=0)\frac{\partial g}{\partial x}\Big|_{x=0}\right]dy - \int\left[u(y=1)\frac{\partial g}{\partial y}\Big|_{y=1} - u(y=0)\frac{\partial g}{\partial y}\Big|_{y=0}\right]dx$$

$$= \iint fg\,dx\,dy - u(x_0,y_0)$$

$$\Rightarrow \int\left[-b\frac{\partial g}{\partial x}\Big|_{x=1} + a\frac{\partial g}{\partial x}\Big|_{x=0}\right]dy + \int\left[-d\frac{\partial g}{\partial y}\Big|_{y=1} + c\frac{\partial g}{\partial y}\Big|_{y=0}\right]dx = \iint fg\,dx\,dy - u(x_0,y_0)$$

**LHS :-** $a\int_{y=0}^{y=1}\frac{\partial g}{\partial x}\Big|_{x=0}dy - b\int_{y=0}^{y=1}\frac{\partial g}{\partial x}\Big|_{x=1}dy + c\int_{x=0}^{x=1}\frac{\partial g}{\partial y}\Big|_{y=0}dx - d\int_{x=0}^{x=1}\frac{\partial g}{\partial y}\Big|_{y=1}dx$

**RHS :-** $\iint fg\,dx\,dy - u(x_0,y_0)$

$$u(x_0,y_0) = \iint fg\,dx\,dy - a\int_{y=0}^{1}\frac{\partial g}{\partial x}\Big|_{x=0}dy + b\int_{y=0}^{1}\frac{\partial g}{\partial x}\Big|_{x=1}dy - c\int_{x=0}^{1}\frac{\partial g}{\partial y}\Big|_{y=0}dx + d\int_{x=0}^{1}\frac{\partial g}{\partial y}\Big|_{y=1}dx$$

$a\int\frac{\partial g}{\partial x}\Big|_{x=0}dy \qquad g = -4\sum\sum\frac{\sin(n\pi x_0)\sin(m\pi y_0)\sin(n\pi x)\sin(m\pi y)}{\lambda_{mn}}$

$$\frac{\partial g}{\partial x} = -4\sum\sum\frac{\sin(n\pi x_0)\sin(m\pi y_0)(n\pi)\cos(n\pi x)\sin(m\pi y)}{\lambda_{mn}}$$

$$a\int\frac{\partial g}{\partial x}\Big|_{x=0}dy = -4a\sum\sum\frac{\sin(n\pi x_0)\sin(m\pi y_0)(n\pi)}{\lambda_{mn}}\int_0^1\sin(m\pi y)\,dy$$

$$= -4a\sum\sum\frac{\sin(n\pi x_0)\sin(m\pi y_0)(n\pi)}{\lambda_{mn}}\left(\frac{1-\cos m\pi}{m\pi}\right)$$

$$d\int\frac{\partial g}{\partial y}\Big|_{y=1}dx = -4d\sum\sum\frac{m\pi\cos m\pi\,\sin(n\pi x_0)\sin(m\pi y_0)}{\lambda_{mn}}\int\sin(n\pi x)\,dx$$

$$= -4d\sum\sum\frac{m\pi\cos m\pi}{\lambda_{mn}}\left(\frac{1-\cos n\pi}{n\pi}\right)\sin(n\pi x_0)\sin(m\pi y_0)$$

$\iint fg\,dx\,dy \rightarrow$ if $f$ is const $= f\iint g\,dx\,dy = -4f\sum\sum\frac{\sin n\pi x_0\sin m\pi y_0}{\lambda_{mn}}\int_0^1\sin n\pi x\,dx\int\sin m\pi y$

$$= -4f\sum\sum\left(\frac{1-\cos n\pi}{n\pi}\right)\left(\frac{1-\cos m\pi}{m\pi}\right)\frac{\sin n\pi x_0\,\sin m\pi y_0}{\lambda_{mn}}$$

$u(x_0,y_0) =$ expression in $x_0, y_0$ only $\quad u(x,y) =$ soln expression.

$x_0, y_0 \rightarrow x, y$

Q4 $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t)$ subj to BCs :— at $t=0,\ u=u_0$   $L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$

at $x=0,\ u=a$

$x=1,\ u=b$

Step 1 :- Construction of Causal green $f^n$.

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial x^2} + \delta(x-x_0)\,\delta(t-t_0) \qquad \text{at } t=0,\ g=0 \quad \text{at } x=0,1 \rightarrow g=0$$

Step 2 :- Expression of $g$ :- Using partial eigen $f^n$ expansion method.

$$g(x,t/x_0,t_0) = \sum_n a_n(t)\phi_n(x)$$ $\phi_n$ is the eigen $f^n$ in $x$ dir$^n$, $a_n = f(t)$

Step 3 :- Evaluation of $\phi_n(x)$ :- $L' = \partial^2/\partial x$ is the operator that defines eigenvalue problem in $x$ dir$^n$.

$L'\phi_n + \lambda_n \sigma\phi_n = 0$ (Governing eq$^n$ of the eigen $f^n$ problem) :- $\frac{d^2\phi_n}{dx^2} + \alpha_n^2\phi_n = 0$

at $x=0,1 \rightarrow \phi_n = 0$ $\} \Rightarrow \alpha_n = n\pi \quad n=1,2\ldots\infty \quad \phi_n = c_1\sin(n\pi x)$

Using property of orthogonal $f^n$ :- $a_n = \langle g, \phi_n\rangle$

$$c_1^2\int_0^1 \sin^2(n\pi x)\,dx = 1 \qquad \langle\phi_n,\phi_n\rangle \Rightarrow \text{making it orthonormal} \rightarrow \|\phi_n\|^2$$

$$\Rightarrow c_1 = \sqrt{2}. \quad a_n = \langle g, \phi_n\rangle$$

$$\boxed{\phi_n = \sqrt{2}\sin(n\pi x)}$$

Step 4 :- Expression of $a_n$ :- ① $\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial x^2} + \delta(x-x_0)\cdot\delta(t-t_0)$   $a_n = \langle g, \phi_n\rangle$

Take inner prod. of $g$ with $\phi_n$

$$\int \phi_n \frac{\partial g}{\partial t}\,dx - \int \phi_n \frac{\partial^2 g}{\partial x^2}\,dx = \int \delta(x-x_0)\,\delta(t-t_0)\,\phi_n(x)\,dx$$

$\downarrow$ only $f(x)$

$$\Rightarrow \frac{\partial}{\partial t}\int_0^1 g\phi_n\,dx - \phi_n\frac{\partial g}{\partial x}\Big|_0^1 + \int_0^1 \frac{\partial g}{\partial x}\cdot\frac{\partial\phi_n}{\partial x}\,dx = \delta(t-t_0)\phi_n(x_0).$$

(0)

$$\Rightarrow \delta(t-t_0)\phi_n(x_0) = \frac{d}{dt}(\langle g,\phi_n\rangle) + g\frac{\partial\phi_n}{\partial x}\Big|_0^1 - \int g\frac{\partial^2\phi_n}{\partial x^2}\,dx$$

(0)

$$\rightarrow \delta(t-t_0)\phi_n(x_0) = \frac{da_n}{dt} - \alpha_n^2\int \phi_n g\,dx$$

$$\Rightarrow \boxed{\frac{da_n}{dt} + n^2\pi^2 a_n = \delta(t-t_0)\,\phi_n(x_0)}$$ governing eq$^n$ of $a_n$

$$\frac{da_n}{dt} + \alpha_n^2 a_n = 0 \quad \forall\ t \neq t_0 \text{ (prop. of dirac delta } f^n)$$

$\Rightarrow a_n = A\exp(-\alpha_n^2 t)$

for $t > t_0$ & $t < t_0$ → only constants change, the nature of the sol$^n$ remains same.

$a_n^- = A\exp(-\alpha_n^2 t)\ \forall\ t < t_0^-$

$a_n^+ = B\exp(-\alpha_n^2 t)\ \forall\ t > t_0^+$

at $t=0,\ g=0$ → $A=0$ → $a_n = 0$ for $t < t_0^-$   $\Big]$ $a_n = 0$ for $t < t_0^-$

$B\exp(-\alpha_n^2 t)$ for $t > t_0$   $= B_n\exp(-\alpha_n^2 t)$ for $t > t_0$

$g = 0$ for $t < t_0$   $= \sum_n a_n\phi_n = \sum B\exp(-\alpha_n^2 t)\,\phi_n(x)$ for $t > t_0$

Jump discontinuity cond$^n$ :- $\int_{t_0-\epsilon}^{t_0+\epsilon}\frac{da_n}{dt}\,dt + \alpha_n^2\int_{t_0-\epsilon}^{t_0+\epsilon} a_n\,dt = \int_{t_0-\epsilon}^{t_0+\epsilon}\delta(t-t_0)\,\phi_n(x_0)\,dt$

$$\Rightarrow (a_n)_{t_0-\epsilon}^{t_0+\epsilon} + \alpha_n^2(0) = \phi_n(x_0) \Rightarrow a_n(t_0+\epsilon) - a_n(t_0-\epsilon) = \phi_n(x_0) \rightarrow a_n(t_0) = \phi_n(x_0)$$

$B_n = \exp(\alpha_n^2 t_0)\,\phi_n(x_0) = \exp(\alpha_n^2 t_0)\sqrt{2}\sin(n\pi x_0)$

$g(x,t/x_0,t_0) = 0 \;\forall\; t < t_0$

$= \sum_{n=1}^{\infty} \exp(-\alpha_n^2 (t-t_0))\,\phi_n(x)\,\phi_n(x_0)$ for $t > t_0$

$= 2\sum_{n=1}^{\infty} \exp(-\alpha_n^2 (t-t_0))\sin(n\pi x)\sin(n\pi x_0)$

$g(x,t/x_0,t_0) = H(t-t_0)\sum \exp(-\alpha_n^2 (t-t_0))\,\phi_n(x)\,\phi_n(x_0)$

$H(t-t_0)$ = HEAVYSIDE Function $= 0 \;\forall\; t < t_0$

$1 \;\forall\; t > t_0$

Step 4 :- Adjoint operator ($L^*$)

$Lg = \frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2}$ $\quad \langle g^*, Lg\rangle = \int_0^1\int_0^t g^* \frac{\partial g}{\partial t}\,dx\,dt + \int_0^t\int_0^1 g^* \frac{\partial^2 g}{\partial x^2}\,dx\,dt$

$= \int_0^1 \left[(g^* g)_0^t\,dx - \int_0^t \frac{dg^*}{\partial t}\frac{dg}{dt}\,dt\right] + \int_0^t \left[\cancel{(g^* g)}_0^1\,dt - \int_0^1 \frac{dg^*}{dx}\frac{dg}{dx}\,dx\,dt\right]$

$\int_x \left[(g^* g|_0^t) - \int \frac{\partial g^*}{\partial t} g\,dt\right]dx - \int_t\int_x \frac{\partial g^*}{\partial x}\frac{\partial g}{\partial x}\,dx\,dt$ $\quad\Big|\quad$ $B^*$ :- at $x = 0$ & $x = 1$ $g^* = 0$; at $t > t_1 \rightarrow g^* = 0$

$= \int_x \left[\cancel{g^* g}\,\big|_t - g^*(0)\,\cancel{g(t=0)}\right] - \int \frac{\partial g^*}{\partial t} g\,dt\,dx - \int_t \left[\cancel{\frac{\partial g^*}{\partial x} g}\,\Big|_0^1 - \int_0 \frac{\partial^2 g^*}{\partial x^2} g\,dx\right]dt$

$= -\iint g\left(\frac{\partial g^*}{\partial t} + \frac{\partial^2 g}{\partial x^2}\right)dx\,dt = \int_x\int_t (L^* g^*)\,g\,dt\,dt = \langle g, L^* g^*\rangle$

$L^* = -\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$ $\rightarrow$ Not a self adjoint operator

Step 5 :- governing eq$^n$ of $g^*$ :- $L^* g^* = \delta(t-t_1)\,\delta(x-x_1) \Rightarrow -\frac{\partial g^*}{\partial t} - \frac{\partial^2 g^*}{\partial x^2} = \delta(t-t_1)\,\delta(x-x_1)$

$B^*$ at :- $t > t_1 \Rightarrow g^* = 0$ at $x = 0, 1 \rightarrow g^* = 0$

Step 6 :- Connect $g$ & $g^*$ :-

$Lg(x,t/x_0,t_0) = \delta(x-x_0)\,\delta(t-t_0)$ ① $L^* g^*(x,t/x_1,t_1) = \delta(x-x_1)\,\delta(t-t_1)$ ②

$\langle ①, g^*\rangle - \langle ②, g\rangle = \iint g^*\left(\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2}\right)dx\,dt + \iint g\left(\frac{\partial g^*}{\partial t} + \frac{\partial^2 g^*}{\partial x^2}\right)dx\,dt = g^*(x_0,t_0/x_1,t_1) - g(x_1,t_1/x_0,t_0)$

$\rightarrow \int_x\int_t \frac{d}{dt}(g g^*)\,dt\,dx = g^*(x_0,t_0/x_1,t_1) - g(x_1,t_1/x_0,t_0)$.

for $t > t_1 \rightarrow g^* = 0$ (for $t > t_1$)

So $\rightarrow g^*(x_0,t_0/x_1,t_1) = g(x_1,t_1/x_0,t_0)$

$g(x,t/x_0,t_0) = H(t-t_0)\sum 2\sin(n\pi x)\sin(n\pi x_0)\exp(-\alpha_n^2(t-t_0))$

change $x,t \rightarrow (x_1,t_1)$

$g(x_1,t_1/x_0,t_0) = H(t_1-t_0)\sum 2\sin(n\pi x_1)\sin(n\pi x_0)\exp[-\alpha_n^2(t_1-t_0)]$

$g^*(x_0,t_0/x_1,t_1) = H(t_1-t_0)\sum_n 2\sin(n\pi x_1)\sin(n\pi x_0)\exp[-\alpha_n^2(t_1-t_0)]$

Change $x_0, t_0 \rightarrow x_1, t_1$

$$g^*(x_0,t_0/x_1,t_1) = H(t_1-t)\sum_n 2\sin(n\pi x_1)\sin(n\pi x)\exp[-\alpha_n^2(t_1-t)]$$

$H(t_1-t) = 0 \;\forall\; t_1<t \rightarrow$ ~~H(t-t_1)~~ So, $g^* = 0$ for $t>t_1$

$1 \;\forall\; t_1>t$ $\quad = \sum_{n=1} 2\sin(n\pi x_1)\sin(n\pi x)\exp(-\alpha_n^2(t_1-t))$ for $t<t_1$

(1) $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t)$

(2) $-\frac{\partial g^*}{\partial t} - \frac{\partial^2 g^*}{\partial x^2} = \delta(x-x_1)\delta(t-t_1)$

$\langle (1), g^*\rangle - \langle (2), u\rangle$ :-

$$\iint g^*\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right)dt\,dx + \iint u\left(\frac{\partial g^*}{\partial t} + \frac{\partial^2 g^*}{\partial x^2}\right)dx\,dt$$

$$= \iint f g^* \,dx\,dt - u(x_1,t_1)$$

$$\int_0^1\int_0^{t_1}\frac{d(ug^*)}{dt}dt\,dx - \int_t\left[g^*\frac{\partial u}{\partial x}\Big|_0^1 - \int \cancel{\frac{\partial g^*}{\partial x}\frac{\partial u}{\partial x}}dx\right]dt + \int_t\left[u\frac{\partial g^*}{\partial x}\Big|_0^1 - \int\cancel{\frac{\partial u}{\partial x}\frac{\partial g^*}{\partial x}}dx\right]dt = \text{RHS}$$

$$\rightarrow \int_x ug^*\Big|_0^t dx + \int_t\left[b\frac{\partial g^*}{\partial x}\Big|_{x=1} - a\frac{\partial g^*}{\partial x}\Big|_{x=0}\right]dt = \iint f g^* dx\,dt - u(x_1,t_1)$$

$ug^*\Big|_0^t = [u(t)\cancel{g^*(t)} - u(0)g^*(0)] = -u(0)g^*(0)$

$g^*$ does not exist beyond $t_1$ therefore limit of $t$ is $t_1$ otherwise no exist. at $t>t_1$ $g^*=0$

$$\Longrightarrow u(x_1,t_1) = \underbrace{\int_0^{t_1}\int_0^1 f g^* dx\,dt}_{T_1} + \underbrace{u_0\int_{x=0}^{x=1} g^*(\text{at } t=0)\,dx}_{T_2} - \underbrace{b\int_0^{t_1}\frac{\partial g^*}{\partial x}\Big|_{x=1}dt}_{T_3} + \underbrace{a\int\frac{\partial g^*}{\partial x}\Big|_{x=0}}_{T_4}$$

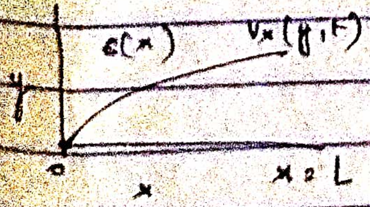
$$T_1 = f\,(\text{const})\iint g^* dx = 2f\int_0^{t_1}\int_0^1 \sum\sin(n\pi x_1)\sin(n\pi x)\exp[-n^2\pi^2(t_1-t)]\,dt\,dx$$

$$= 2f\sum\sin(n\pi x_1)\exp(-n^2\pi^2(t_1))\int_0^1\sin(n\pi x)dx\int_0^{t_1}\exp(n^2\pi^2 t)\,dt$$

$$= 2f\sum_n\left(\frac{1-\cos n\pi}{(n\pi)^3}\right)\sin(n\pi x_1)[1-\exp(-n^2\pi^2 t_1)]$$

Partial eigen $f^n$ expansion. So only $t$ limits are broken. The domain of $x$ remains intact

**Stoke's first Problem** :- similarity sol$^n$ method or combined variable method.



Eq$^n$ of continuity :- $\cancel{\frac{\partial\rho}{\partial t}} + \frac{\partial(\rho v_x)}{\partial x} + \cancel{\frac{\partial(\rho v_y)}{\partial y}} + \cancel{\frac{\partial(\rho v_z)}{\partial z}} = 0$

BOC

(i) $\rho \neq f(x,y,z,t) \rightarrow$ water incompressible

(ii) Motion in $x$ dir$^n$ only. $v_y = 0$

(iii) $z$ dir$^n$ too long. $v_z \sim 0$ $\quad \partial/\partial z \sim 0$.

From (EOC) :- $\quad \partial v_x/\partial x = 0$

Eq$^n$ of motion in $x$ dir$^n$ :- $\rho\left[\frac{\partial u}{\partial t} + \cancel{v_x\frac{\partial v_x}{\partial x}} + \cancel{v_y\frac{\partial v_x}{\partial y}} + \cancel{v_z\frac{\partial v_x}{\partial z}}\right] = -\cancel{\frac{\partial p}{\partial x}} + \mu\left[\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_x}{\partial z^2}}\right] + \cancel{\rho g_x}$

not zero $\qquad$ no pr grad. $\qquad$ not 0

$$\boxed{\rho\frac{\partial v_x}{\partial t} = \mu\frac{\partial^2 v_x}{\partial y^2}}$$

at the edge of BL :- at $t=0$, $v_x = 0$

$y = 0$, $v_x = v_0$

$y = \infty$, $v_x = 0$

Similarity sol$^n$ applicable when eq$^n$ parabolic & 1 of BCs tend to $\infty$

Non dimensionalis$^n$ : $v_x^* = \frac{v}{u_0}$ , $y^* = \frac{y}{L}$

$$\frac{\partial v_x^*}{\partial t} = \frac{\nu}{L^2}\frac{\partial^2 v_x^*}{\partial y^{*2}} \rightarrow \frac{L^2}{\nu}\frac{\partial v_x^*}{\partial t} = \frac{\partial^2 v_x^*}{\partial y^{*2}} \rightarrow \tau = \frac{t\nu}{L^2} \rightarrow \boxed{\frac{\partial v_x^*}{\partial \tau} = \frac{\partial^2 v_x^*}{\partial y^{*2}}}$$

at $\tau = 0$, $v_x^* = 0$
$y^* = 0$, $v_x^* = 1$
$= \infty$, $v_x^* = 0$

at the edge of BL , $\frac{\Delta v_x^*}{\tau - 0} = \frac{\Delta v_x^*}{\delta^{*2} - 0^2} \Rightarrow \delta^{*2} \simeq \tau$
$\delta^* = \sqrt{\tau}$

it varies with $t$ as $t^{1/2}$ , $\eta = y^*/\delta^* = y^*/\sqrt{\tau}$ ($v_x^*$ is a sole f$^n$ of $\eta$)

$$\Rightarrow \frac{\partial v_x^*}{\partial \tau} = \frac{\partial v_x^*}{\partial \eta}\cdot\frac{\partial \eta}{\partial \tau} = -\frac{1}{2\tau}\frac{y^*}{\sqrt{\tau}}\frac{\partial v_x^*}{\partial \eta} \Rightarrow \frac{\partial v_x^*}{\partial y^*} = \frac{\partial v_x^*}{\partial \eta}\frac{\partial \eta}{\partial y^*} = \frac{1}{\sqrt{\tau}}\frac{\partial v_x^*}{\partial \eta}$$

$$\Rightarrow \frac{\partial^2 v_x^*}{\partial y^{*2}} = \frac{\partial}{\partial \eta}\left(\frac{1}{\tau}\frac{\partial v_x^*}{\partial \eta}\right)\times\frac{\partial \eta}{\partial y^*} = \frac{1}{\tau}\frac{\partial^2 v_x^*}{\partial \eta^2} \Rightarrow -\frac{y^*}{2\tau\sqrt{\tau}}\frac{\partial v_x^*}{\partial \eta} = \frac{1}{\tau}\frac{\partial^2 v_x^*}{\partial \eta^2} \rightarrow \boxed{\frac{\partial^2 v_x^*}{\partial \eta^2} = -\frac{\eta}{2}\frac{\partial v_x^*}{\partial \eta}}$$

$$\Rightarrow \left.\begin{array}{l} \text{at } \eta = 0,\ v_x^* = 1 \\ \eta = \infty,\ v_x^* = 0 \end{array}\right\} \frac{\partial v_x^*}{\partial \eta} = z \text{ (let)} \Rightarrow \frac{\partial z}{\partial \eta} = -\frac{\eta z}{2} \rightarrow \ln z = -\frac{\eta^2}{4} + \ln k_1$$

$$\Rightarrow z = k_1 \exp\left(\frac{-\eta^2}{4}\right) \rightarrow v_x^* = k_1\int \exp\left(\frac{-\eta^2}{4}\right)d\eta + k_2 \Rightarrow (k_2 = 1)$$

$$0 = k_1\int_0^\infty \exp\left(\frac{-\eta^2}{4}\right)d\eta + 1 \rightarrow k_1 = \frac{-1}{\int_0^\infty \exp\left(-\eta^2/4\right)d\eta}$$

$$\boxed{v_x^* = 1 - \frac{\int_0^\eta \exp\left(-\eta^2/4\right)d\eta}{\int_0^\infty \exp\left(-\eta^2/4\right)d\eta}}$$