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Advanced fluid dynamics.

RM

1.) Kinematics of fluid - deform^n / shear stress / substantial derivative / stress in line

⇒ Newton's 2nd Law → helps us define unit force (which when applied on unit mass performs unit acceleration)

2.) Viscous Incompressible flow → conservation eq^n

→ mass balance eq^n

→ conservation of M^o (Navier Stoke's Eq^n)

NS eq^n (Newton's 2nd Law for fluid system)

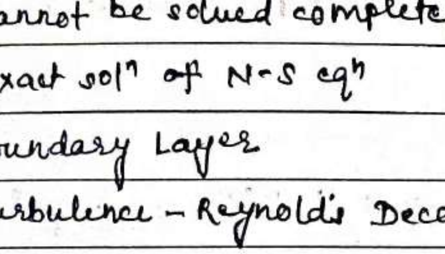
cannot be solved completely but for some cases it can provide analytical sol^n.

3.) Exact sol^n of N-S eq^n

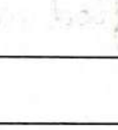
4.) Boundary Layer

5.) Turbulence - Reynold's Decomposition — (fluctuation) Turbulence

$\delta = f(x)$ fully developed flow.



The whole cross section is Boundary Layer flow.



→ inviscid flow

is it possible for a fluid which has $\mu \neq 0$?

Yes possible provided the u_∞ is lower than slip vel.

No otherwise bcoz even after everything there will be boundary layer even if it is very small.

v ↑y → x u

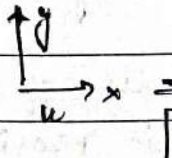
solid surface $u = 0$, $v = 0$ → zero for solid impermeable wall.

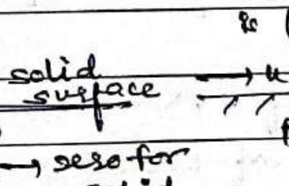
cause is no-slip cond^n

→ $u = 0$ ↑ $v \neq 0$ permeable surface or porous wall (membrane)

Why slip? → coeff. of friction f.

Slip cond^n is possible on surfaces where the friction is very very less.



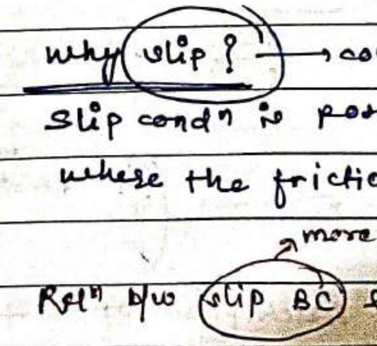


Rel^n b/w slip BC (more general) & no slip BC (special case of slip BC)

u_s = slip vel.

l_s = slip length

let surface vel. = u_s



same fluids in (a) and (b) surface s_1 is more slippery as slip vel. is higher

① Fox & McDonalds.

② Introdⁿ to fluid Mech. & fluid Machines
Som, Biswas, Chakraborty

③ Frank M. White – fluid Mech.

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<u>fluid</u> :– rel. position of molecules change (continuous deformation)
↳ deform subject to shear force however small that is.

<u>Solid</u>:– rel. position does not change continuously. They are held together by intermolecular inter^n (Vanderwaals)

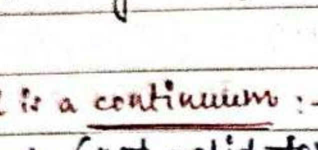
Strength of intermolecular interᵑ is stronger in solids, weaker in liqs and absent in gases. (Mean free path $\to \infty$)

fluid is a <u>continuum</u> :– we don't worry about the properties of each molecule (not valid for rarified gases)

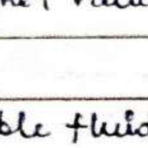
Incompressible fluid $\Rightarrow \rho = \text{const.}$ $\rho \neq f(x, y, z, t, P, T) \Rightarrow$ Liquids.
Compressible fluid $\Rightarrow$ gases.

<u>Flowfield</u> → area where we are trying to understand the flow → region of interest where flow is defined everywhere → either at every pt. or → for every fluid particle

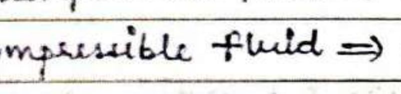
<u>fluid particles</u>



As a fⁿ of r, we know the value of vel. as a result of knowing parabolic profile.



– What is a fluid particle?
↳ a small entity which can change shape
imaginary — deform



<u>Method of Description of a flowfield</u> → 2 Approaches

<u>Lagrangian approach</u> → the flow/motion of each particle in the flow field is tracked.

$\overline{S}_{P_{n},t} = f(\overline{S}_{P_{n},t_0})$

We do not really use it bcoz the no. of particles is ∞.
When we are talking abt. every fluid particle, it is the Lagrangian approach.

Eulerian approach - what we use aka according to a frame of ref. we track our particles.

Types of flow :- based on what ?

compressible / incompressible → density const / not const.

closed / open → channel.

steady / unsteady → change with time

laminar / turbulent → flow type

uniform / non-uniform → uniformity

1-D / 2-D / 3-D → dimension.

$Re = \frac{\rho v D}{\mu} = \frac{\text{inertial force}}{\text{viscous force}} = \frac{\text{mass/vol} \times \text{Length} \times \text{vel}}{\text{viscosity}} = \frac{\text{mass/vol.} \times \text{vel.}}{}$ $\nu \to$ viscosity coeff.

$= \frac{\text{momentum}}{\nu V}$

inertia is nothing but Kinetic Energy.

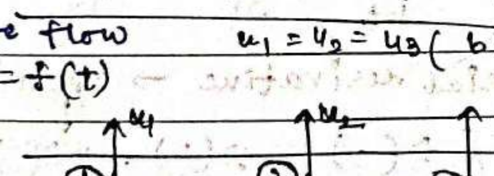
$Re = \frac{\rho v D \cdot v}{\mu \cdot v} = \frac{\rho v^2 D}{\mu v} = \frac{\rho v^2}{\left(\frac{\mu v}{D}\right)} = \frac{\rho v^2}{\mu\left(\frac{v}{D}\right)} \to$ Energy

vel. grad.

Steady State flow :- can only be defined based on Eulerian approach (depends on frame of reference)

Steady State flow uniform flow. $u \neq f(t)$ $u_1 = u_2 = u_3$ (bcoz it does not change as a f^n of time)

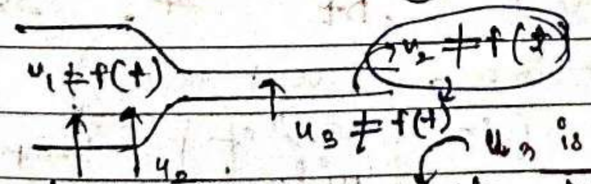
T_{xx}, T_{xy} } after ≈ 3hrs (will be same)



ρ = const.

$u_1 = u_2 \neq u_3$

steady non uniform flow



Infact $u_3 > u_1, u_2$ (Eq^n of continuity or conserv^n of mass)

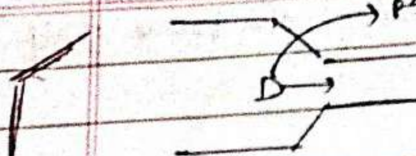
u_3 is still not a f^n of time.

we have time independent acceleration.

what is acceleration ? (dv/dt ?) there has to be a time dependence

How to handle time independent accl^n ?

→ Special abt. fluid particle → boundaries are deforming



→ particle has accl^n & deformative boundary

Initial position of a particle is given by (x, y, z) (acc. to Lagrangian approach)

3 vel. comp. (u, v, w)

3 – D flowfield (flow in 3 dirn's possible)

$\Rightarrow u = u(\underbrace{x, y, z}, \underbrace{t})$
x dirn — f^n of pos. & time.

$\Rightarrow v = v(x, y, z, t)$
y dirn

$\Rightarrow w = w(x, y, z, t)$
z dirn.

at time $t + \Delta t$

It was moved to $(x+\Delta x, y+\Delta y, z+\Delta z)$

velocity is $(u+\Delta u, v+\Delta v, w+\Delta w)$

$\Delta x = u\Delta t$

$\Delta y = v\Delta t$

$\Delta z = w\Delta t$

$$u + \Delta u = u(x+\Delta x, y+\Delta y, z+\Delta z, t+\Delta t)$$

Taylor Series expansion :- $\cancel{u} + \Delta u = \cancel{u(x, y, z, t)} + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z + \frac{\partial u}{\partial t}\Delta t + \underbrace{\ldots}_{0}$

$$\Rightarrow \frac{\Delta u}{\Delta t} = \frac{\Delta x}{\Delta t}\frac{\partial u}{\partial x} + \frac{\Delta y}{\Delta t}\frac{\partial u}{\partial y} + \frac{\Delta z}{\Delta t}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$\frac{\Delta x}{\Delta t} = u \qquad \frac{\Delta y}{\Delta t} = v \qquad \frac{\Delta z}{\Delta t} = w$$

$$\text{So} \Rightarrow \frac{\Delta u}{\Delta t} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$\lim_{\Delta t \to 0}\frac{\Delta u}{\Delta t} = \frac{Du}{Dt} \qquad \left(\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

Substantial / Material derivative → operator

multipliers u, v and w remain — (gives accl^n of fluid wrt particle)

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + \underbrace{u\frac{\partial(\ )}{\partial x} + v\frac{\partial(\ )}{\partial y} + w\frac{\partial(\ )}{\partial z}}$$

Time dependent term (total accl^n of fluid)

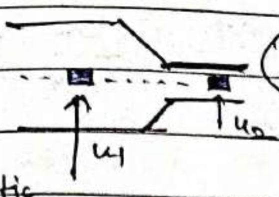
temporal accl^n of fluid

Time independent (space dependent spatial accl^n)

steady flow → $\left(\frac{\partial u}{\partial t} = 0\right)$
↳ does not guarantee no accl^n

uniform flow → (spatial accl^n = 0)
↳ can still have temporal accl^n

Momentum Balance Eqⁿ :— Whenever we are considering fluid accⁿ we are actually considering total accⁿ or substantial derivative (Du/Dt)



$u_1 = u_2$ → is it possible?

Yes, if it is not an incompressible flow bcoz we cannot say for sure (But if they follow certain specific geometry it is possible)

Why do we look only 2D flow? – (isometric deformⁿ becomes unmanageable)

Fluid particle in a 2D flow field :—

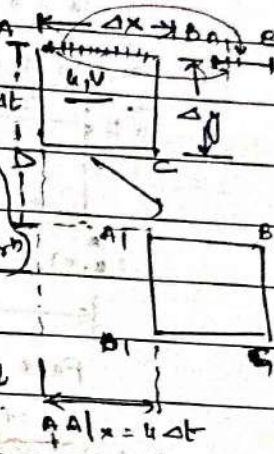
$u = f(y)$, $v = f(x)$	$u = f(x,y)$, $v = f(x,y)$	$u = 0$, $v = 0$	u = const, v = const $\neq f(x,y,z,t)$ $= f(x,y,z,t)$	$u = f(x)$, $v = f(y)$
special case	most generic case	particle does not move.	The fluid element is deformative. All pts. along the line AB in the x dirⁿ move with the same vel.	depends only on variable in that dirⁿ → Taylor Series linearised

Sitⁿ after time Δt :—

every pt. moves with same vel. $\Delta x = u\Delta t$

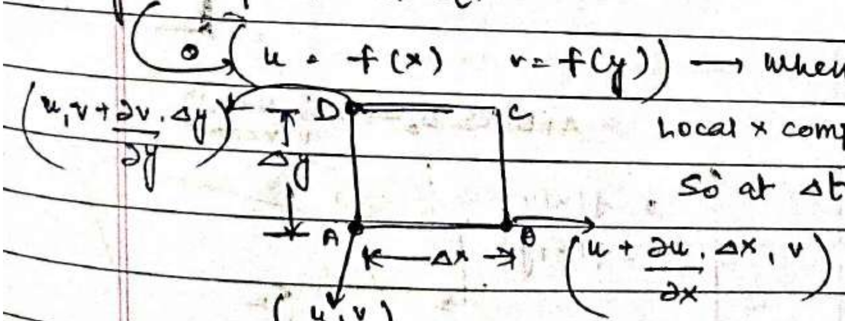
$AA_1|_y = v\Delta t$

Simple translational movement → Rigid body like behaviour (only this specific condⁿ)



Taylor swift linearised.

$u = f(x)$ $v = f(y)$ → When AB moves to A_1B_1 what happens?



Local x comp. vel. at A & B are different

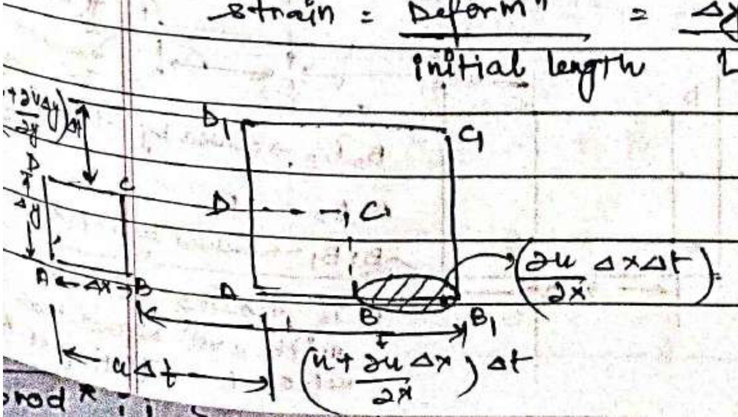
So at Δt, line segment will be of different sizes.

strain = $\frac{\text{Deformⁿ}}{\text{initial length}} = \frac{\Delta y}{L}$

Implicit Assumpⁿ → over the time Δt there is no change in vel.

How much length has increased for line AB?

$\left(\frac{\partial u}{\partial x}\cdot\Delta x\right)\Delta t$



$$\epsilon_{xx} = \frac{\left(\frac{\partial u}{\partial x}\right)\Delta x \Delta t}{\Delta x} = \left(\frac{\partial u}{\partial x}\right)\Delta t$$

change in strain in x dirn. — strain is a fn of time

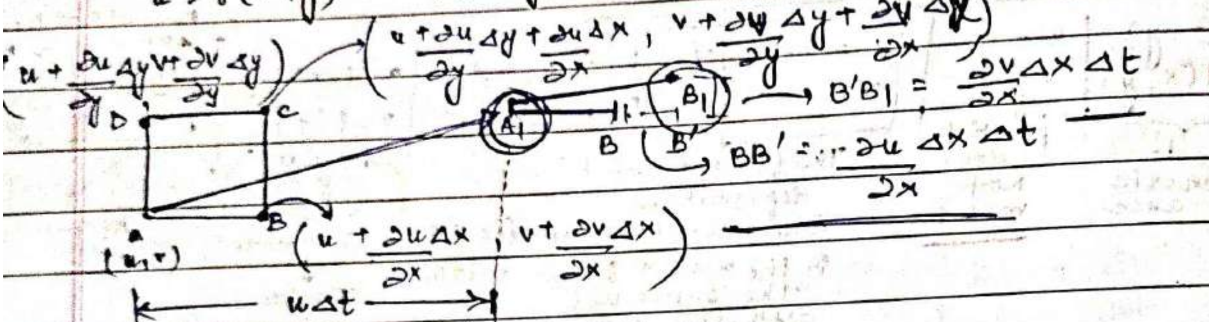
For a fluid element, strain is not imp. but for a solid strain is $\Delta x/L$. But for fluid strain rate is imp.

$$\frac{\epsilon_{xx}}{\Delta t} = \frac{\frac{\partial u}{\partial x}\Delta t}{\Delta t} = \frac{\partial u}{\partial x} = \dot{\epsilon}_{xx}$$

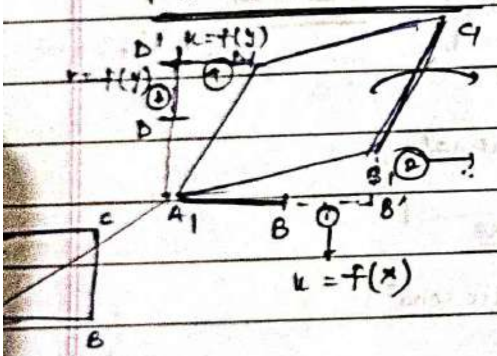
What is the fate of the fluid element?
→ Translation with Linear deformn where only $u = f(x)$ and $v = f(y)$

Generic Case

$u = f(x, y)$ $\quad v = f(x, y)$. Line not bent $(AB \to A_1B_1)$



Fate of line AD? : — 0



① → $u = f(x)$
② → $v = f(x)$
③ → $v = f(y)$
④ → $u = f(y)$

u & v not changing with time.

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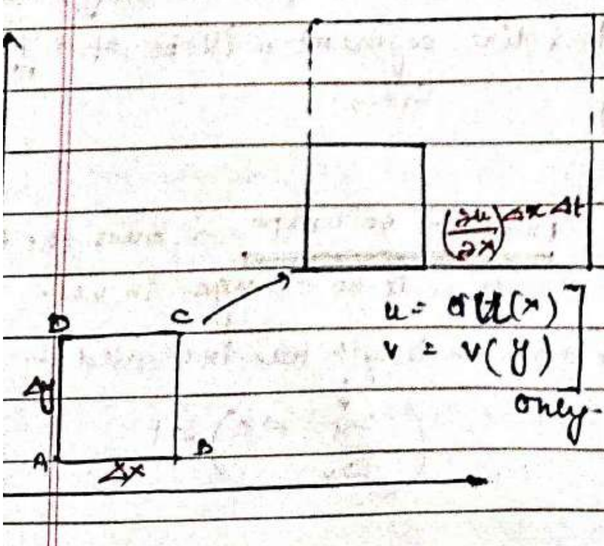
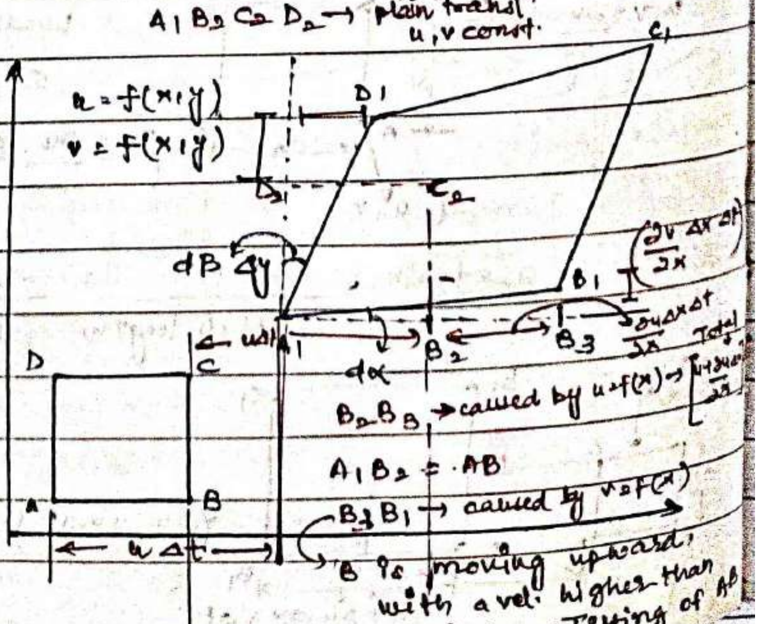
$A_1B_2C_2D_2$ → plain transln u, v const.



$u = u(x)$
$v = v(y)$ only.



B_2B_3 → caused by $u = f(x)$
$A_1B_2 = AB$
B_3B_1 → caused by $v = f(x)$
↳ B is moving upward with a vel. higher than vel. of A → Tilting of AB

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~~fluid particle~~ $\neq$ control vol.

not fixed. ← deformable — non-deformable. → fixed locn in space.
continuously moving → Lagrangian. — Eulerian.
(motion of each particle is tracked)

The movement of AB in anticlockwise & AD in clockwise. 1 movement in ↺ and AD in ↻ dirn.

$\angle DAB$ originally $= 90°$ → after $u = f(x,y)$, $v = f(x,y)$ → $\angle DAB$ becomes $< 90°$, and we get $d\alpha$ & $d\beta$

any shape possible (not necessarily this)

$$\tan(d\alpha) = \frac{B_1B_3}{A_1B_3} = \frac{\frac{\partial v}{\partial x}\Delta x\Delta t}{\left[\Delta x + \left(\frac{\partial u}{\partial x}\Delta x\right)\Delta t\right]} = \frac{\frac{\partial v}{\partial x}\Delta t}{\left[1+\frac{\partial u}{\partial x}\Delta t\right]} \approx \left(\frac{\partial v}{\partial x}\right)\Delta t$$

Assumpn is $\frac{\partial u}{\partial x}\Delta t <<< 1$

$d\alpha$ is very small as $\frac{\partial u}{\partial x}\Delta t$ is small so is $\frac{\partial v}{\partial x}\Delta t$

$$d\alpha = \left(\frac{\partial v}{\partial x}\right)\Delta t \rightarrow \frac{d\alpha}{\Delta t} = \left(\frac{\partial v}{\partial x}\right)$$

$$\rightarrow \boxed{\dot{\alpha} = \frac{\partial v}{\partial x}}$$ rate of angular deformn is a fn of (v & x)

$\dot{\gamma}_{xy}$ = rate of angular deformn ⇒ rate at which angle b/w AD & AB changes.

$$\tan(d\beta) = \frac{\frac{\partial u}{\partial y}\Delta y\,\Delta t}{\Delta y + \frac{\partial v}{\partial y}\Delta y\Delta t} = \frac{\frac{\partial u}{\partial y}\Delta t}{\left[1+\frac{\partial v}{\partial y}\Delta t\right]}$$

$<<1$

$$(d\beta) = \frac{\partial u}{\partial y}\Delta t \rightarrow \dot{\beta} = \frac{\partial u}{\partial y}$$

deformn is happening on an xy plane

$$\dot{\gamma}_{xy} = \dot{\alpha} + \dot{\beta} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.$$

$$\boxed{\dot{\epsilon}_{\gamma_{xy}} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}} \quad \boxed{\dot{\epsilon}_{xx} = \frac{\partial u}{\partial x} \quad \dot{\epsilon}_{yy} = \frac{\partial v}{\partial y}}$$

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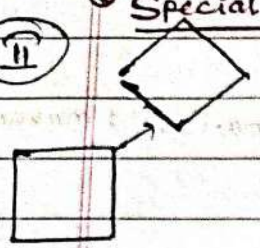
$$\dot{\gamma}_{xy} = \underbrace{\frac{\partial v}{\partial x}}_{\dot{\alpha}} + \underbrace{\frac{\partial u}{\partial y}}_{\dot{\beta}} \quad \leftarrow u = f(x,y), \quad v = f(x,y)$$

(II) Special Case ① where $\boxed{\frac{\partial v}{\partial x} = (-)\frac{\partial u}{\partial y}} \Longrightarrow \boxed{\dot{\gamma}_{xy} = 0}$

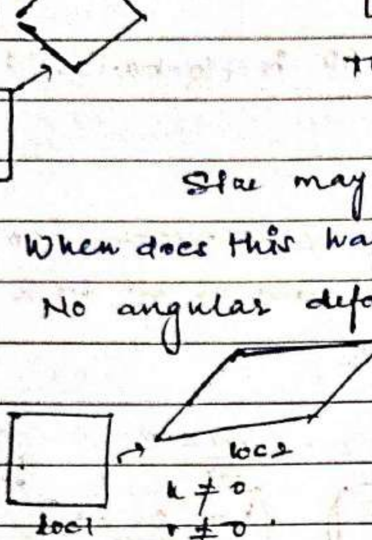


The rotⁿ of AB & AD in the same dirⁿ (bcoz when we were defining, we ~~assumed~~ considered them rotating in diff. dirⁿs)

Size may change depending on u is a fⁿ of x & v is a fⁿ of y.

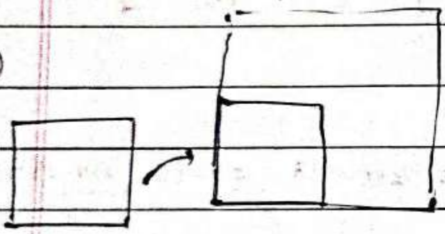
When does this happen?

No angular deformⁿ when $u = f(x)$, $v = f(y)$ only



particle moves
sides dilated
~~sides rotated~~

(I)



in (I) & (II) :- But mechanics is different

$\epsilon_{xx} = ✓$

$\epsilon_{yy} = ✓$

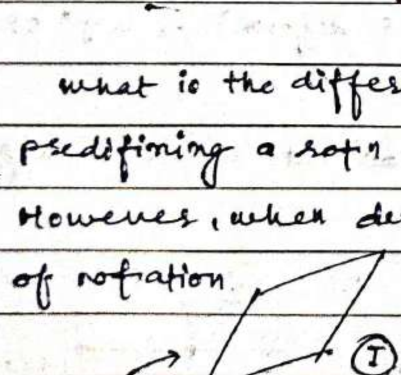
$\dot{\gamma}_{xy} = 0$

So in addⁿ to linear and angular deformⁿ, we have $\boxed{\text{fluid rot}^n}$

Fluid rotation is defined as $\boxed{\dot{\omega}_z = \frac{1}{2}(\dot{\alpha} - \dot{\beta})}$ $\boxed{\dot{\gamma}_{xy} = \dot{\alpha} + \dot{\beta}}$

what is the difference? → when we are using angular deformⁿ we are predifining a rotⁿ in clockwise & another in anticlockwise dirⁿ. However, when defining fluid rotⁿ, we ~~dif~~ ~~pref~~ predefine our axis of rotation.

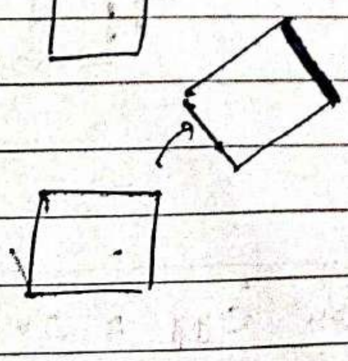
Special case 2 :-



(I)

Now if :- $\boxed{\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}} \rightarrow \dot{\omega}_z = 0$ rotational flow is zero.

Irrotational flow ↓ when numerically $\dot{\alpha} = \dot{\beta}$ ↓ there figures will not show difference bcoz its all abt magnitude



(II)

in (I) there may be rotⁿ bcoz we don't know ~~their ref. values bcoz if~~ $\dot{\alpha} \neq \dot{\beta}$ then $\dot{\alpha} - \dot{\beta} \neq 0$ and thus there is a rotational flow ($\omega_z \neq 0$)

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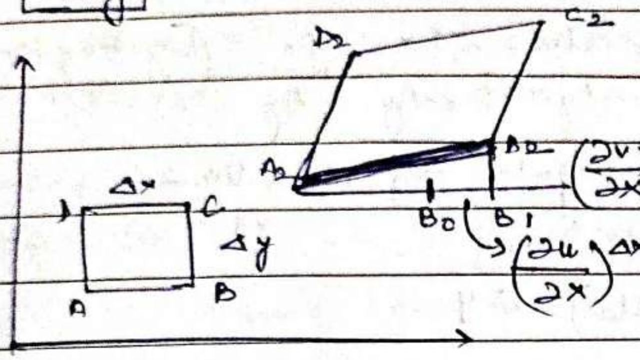
\# You can draw a figure and show your rotation by a schematic but an irrotational flow ~~bu~~ by a schematic.

\# $\dot{\gamma}_{xy}$ and ω_z come from cross dependence b/w velocities.

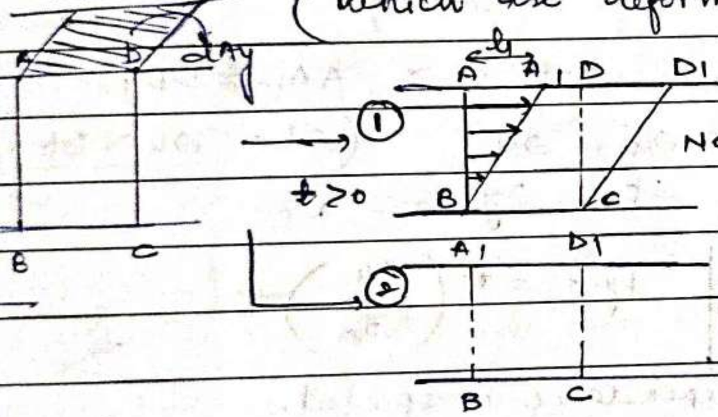
\# Case of pure rot^n → angular deform^n = 0 → can be shown

$\boxed{\begin{matrix}\tau_{yx}\\ \tau_{xy}\end{matrix}}$ $\quad \dot{\gamma}_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ $\quad$ | 28 | 08 | 23 |



τ_{xy} → x momentum transferred in y dir^n
τ_{yx} → y M^s transferred in x dir^n
→ Traditional → valid only under certain cond^{n}s.

Couette flow :— Layer b/w the 2 plates consists large no. of fluid particles (which are deformable)

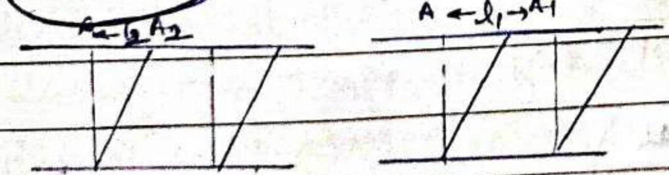


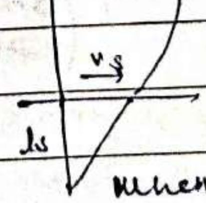
No slip BC → assumption of moving with same vel. (liq. layer with the wall boundary)

possible in case of slip BC → does not guarantee that the liq. layer with the wall boundary (adjoint) moves with same velocity.

is it possible to have cond^n ① with slip BC ? → Yes

difference in ① when :— slip BC and no slip BC





l_s → The extra length by which the surface slips (moved)

Slipping of liq. → $(l_1 - l_s)$ (l_2)

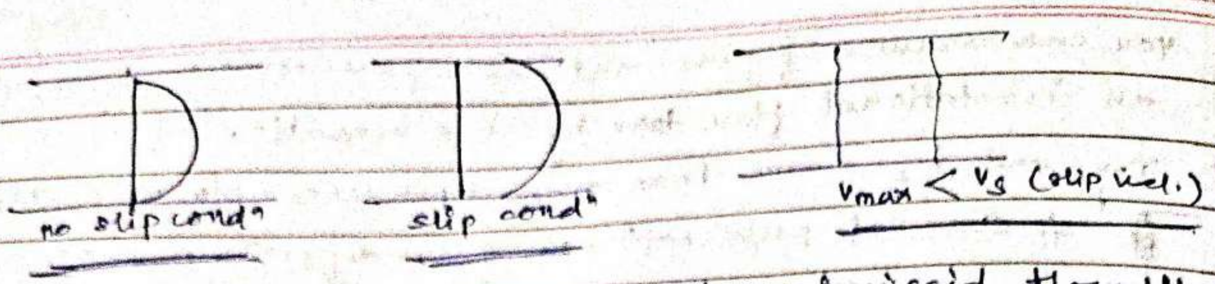
When the wall of the pipe moves the slipping of liq. may take place which is $= l_1 - l_s$ (l_2)

Plate has moved by an amt. l_2

$l_2 > l_s$; $l_2 < l_s$

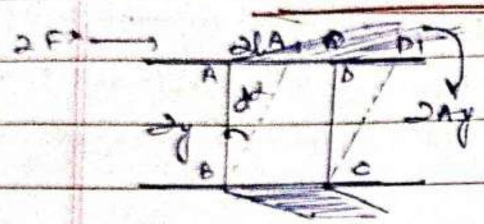
net deform^n $= l_1 - l_s$; net deform^n = 0 as ~~no~~ -ve slip not possible.

Plate has slipped by an amt. l_s.



if max. vel. at centre < slip vel. → inviscid flow like vel. profile

Classical Couette flow:–



$\tau_{yx} = \lim_{\partial A_y \to 0} \frac{\partial F_x}{\partial A_y} = \frac{F_x}{A_y}$ (Lagrangian approach)

Rate of angular deform^n $= \lim_{\partial t \to 0} \frac{\partial \alpha}{\partial t} = \frac{\partial \alpha}{\partial t}$

$\tan(d\alpha) = \frac{dl}{\partial y}$

$\Rightarrow d\alpha = \frac{dl}{\partial y}$ (1D flowfield)

⇒ only vel. at top plate is $\underline{\partial u}$. ∂u is const.

$\boxed{AA_1 = DD_1}$ (∂u or u is const)

if u would have been a f^n of x $AA_1 \neq DD_1$

$\partial\alpha = \frac{\partial u \times \partial t}{\partial y} \rightarrow \frac{\partial\alpha}{\partial t} = \frac{\partial u}{\partial y}$ $(\partial l = \partial u \times \partial t)$

spl. case $\boxed{\tau_{y,x} = \mu \frac{\partial u}{\partial y}}$ $\tau_{y,x} = f\left(\frac{\partial u}{\partial x}\right)$

Linear dependance is special.

What is the reason for angular deform^n → bcoz of stress (tangential)

$\tau_{yx} = f_1\left(\frac{\partial \alpha}{\partial t}\right)$ $\boxed{\frac{\partial \alpha}{\partial t} = f(\tau_{yx})}$

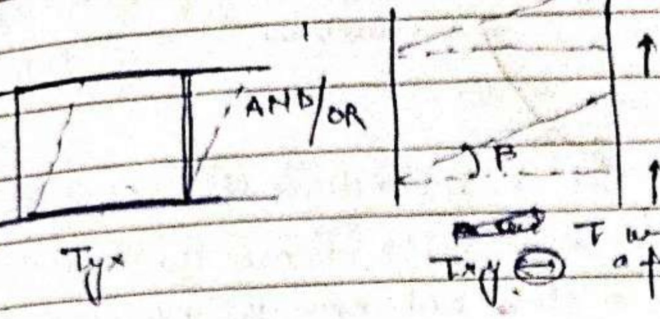
for small deform^n, $\frac{d\alpha}{\partial t} = \frac{\partial u}{\partial y}$

So, $\tau_{yx} = f_2\left(\frac{\partial u}{\partial y}\right)$

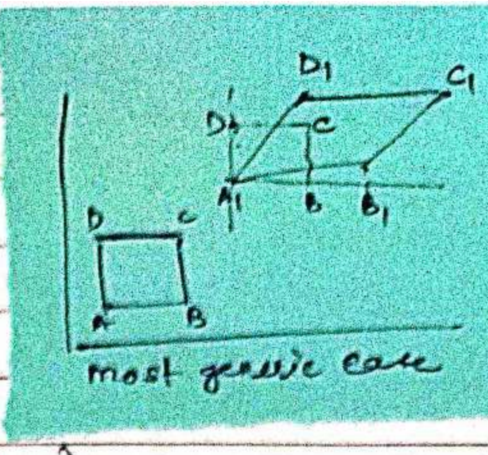
(1D) → $\tau_{yx} = f\left(\frac{\partial u}{\partial y}\right)$

Tyx and Txy diff. :-

Special case :-



Txy $\ominus$ τ will be a fn of $\frac{\partial v}{\partial x}$



* Shear stress = f(Angular deformation)

For a general 2D flow :- $u = f(x,y)$ & $v = f(x,y)$

angular deform^n :- $\dot{\gamma}_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

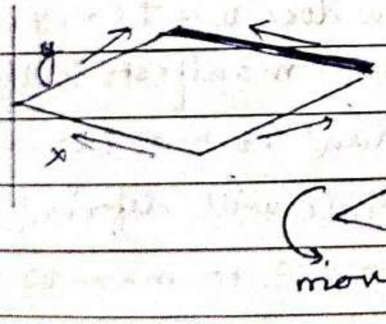
$$\text{Shear stress} = f(\dot{\gamma}_{xy}) = f\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\underset{(\tau_{xy} = \tau_{yx} = \tau)}{\tau \text{ acting on } x\text{-}y \text{ plane}} = f\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \text{(read sign)}$$

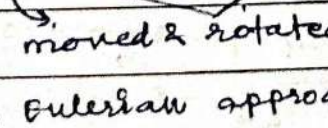
Shear stress of τ_{xy} → f^n of summation of 2 individual deformations.

in 3D flow → we are talking abt. shear stress at some pt. z.

$$\tau_{xy} = \tau_{yx} = f\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \text{(special case)}$$



We used the top view of this plane to derive our various deform^n & correl^{n}s.



moved & rotated.

Stream Lines :- Eulerian approach bcoz we are looking at a tangent at a fixed pt. (dir^n of v → pt of ref.). If we represent the flow as curves then the (path taken by particles of fluid, under SS) tangent at any pt. on the curve gives the dir^n of the fluid vel. at that pt.

Pathline :- Motion or path (trajectory) of an individual fluid particle.

↳ Lagrangian approach.

$$\vec{v} \times \vec{ds} = 0$$

$$\Rightarrow \frac{\partial x}{u} = \frac{\partial y}{v} = \frac{\partial w}{w} \quad (\text{Eq}^n \text{ of streamline})$$

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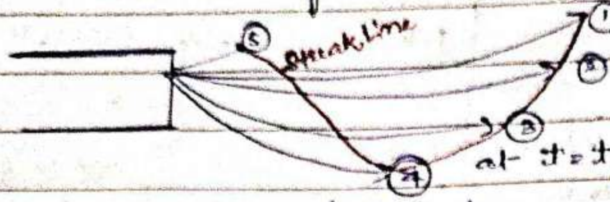
Streakline :— locus of the present locⁿ of particles which have all moved through the same pt. → combⁿ of both Lagrangian & Eulerian approaches



If streaklines do not change with time the it is a condⁿ of no flow. Sometime there was a flow but now the flow has stopped (diffusion has also stopped).

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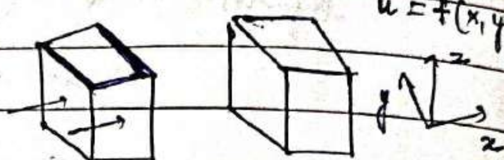
Streamfⁿ :—

Conservⁿ eqⁿs :— mass bal., M° bal.

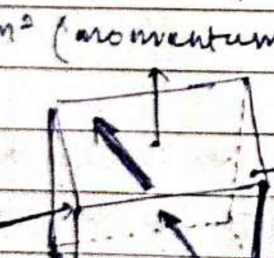
We are considering an isothermal system

$u = f(x, y, z)$

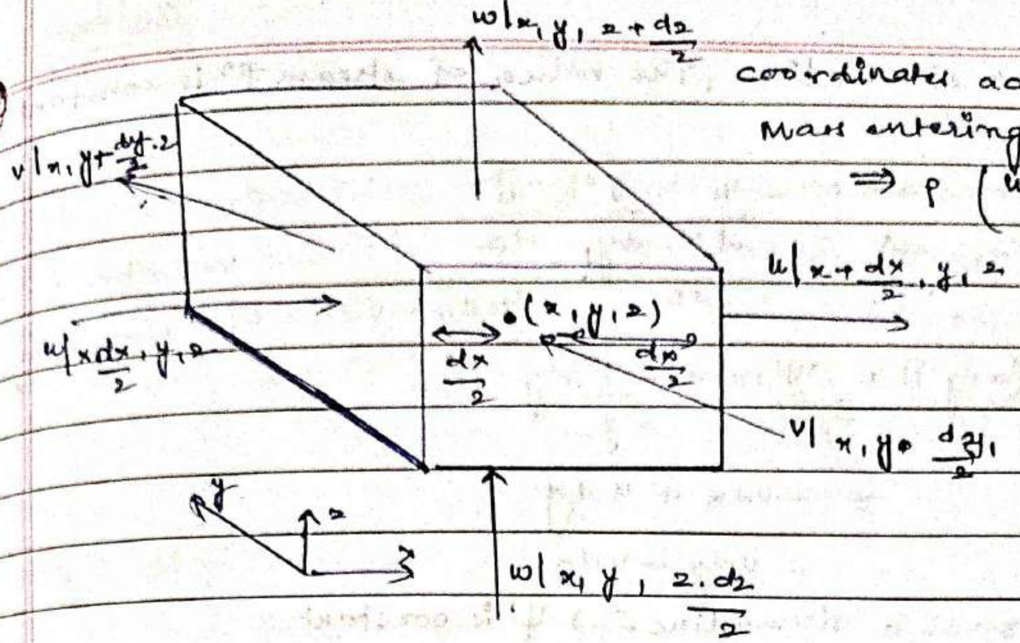


(1) Consv. of mass

→ Differential element
↓
control volume
↓
Eulerian approach

fixed locⁿ in space & there are no particles. 	* fluid particle will move if $v \neq 0$ on the other hand cv is stationary
	* how does $u = f(x, y, z)$ (spatial acc. of vel.) manifests itself on these 2? in addⁿ to moving away, fluid particle will deform. (linear & angular deformⁿ). It may not remain a cuboid.
Pu2 → M° (momentum) ~~tool~~  3D flowfield → centroid is (x, y, z) sides dx, dy, dz	In cv → fluid is causing shear & also entering cv bcoz it doesn't have a Boundary. The velocities are different u_1 and u_2. It must withstand the stress to prevent deformⁿ (fluid particle deforms.). For every comp. of vel The cv is going to experience 3 types of stress. 1 normal (τ_{xx}) and 2 shear (τ_{xy}, τ_{xz}). That leads to 9 comp. of stress.
If there is a flow, mass enters through 3 faces & leaves through 3 faces.	

#



coordinates acc. to centroid.

Mass entering through the left face

$\Rightarrow \rho \left(u|_{x-\frac{dx}{2}}\right) dy\, dz$

vel. is not const.

Mass entering through right face

$\rho\left(u|_{x+\frac{dx}{2}}\right) dy\, dz$

Net efflux due to x component vel. $= \rho\, dy\, dz \left[u|_{x+\frac{dx}{2}} - u|_{x-\frac{dx}{2}} \right]$

after Taylor expansion.

$$= \rho\, dx\, dy\, dz \left(\frac{\partial u}{\partial x}\right)$$

Net efflux due to all 3 components of vel. $= \rho\, dx\, dy\, dz \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$

flow is incompressible $\rightarrow \rho \neq \rho(x, y, z, t)$

rate (in − out + gen − acc) = 0

$$\Rightarrow \cancel{\rho\, dx\, dy\, dz} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \cancel{dx\, dy\, dz\, \frac{\partial \rho}{\partial t}} = 0.$$

$$\rightarrow \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0} \quad \text{continuity eq}^n.$$

Stream f^n :- valid only for a 2D flow.

Functional form of continuity for a 2D flow :- $\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$

$u = f(x, y, t) \quad v = f(x, y, t)$

Definition :- stream f^n is a pt. f^n $\psi(x, y, t)$ s.t. $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$.

Irrotational flow :- $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \rightarrow \boxed{\nabla^2 \psi = 0}$$

For an irrotational flow, the Laplacian of ψ f^n is 0.

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For a given streamline, the value of stream f^n is constant.

Eq^n of a streamline :– $\frac{u}{dx} = \frac{v}{dy} = \frac{w}{dz}$

$\psi = \psi(x, y, t)$ ← 0, SS

2D → $\boxed{u\,dy - v\,dx = 0}$ valid for stream line only.

valid? for any pt. → $d\psi = d(\psi(x,y)) = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}\cdot dy$

$= -v\,dx + u\,dy$

$= u\,dy - v\,dx$

$d\psi = 0$ only once a streamline $\Rightarrow \psi$ is constant

Fundamental difference :– streamlines are valid for 3D but stream f^n applicable only for 2D flow. for a 2D flow, stream f^n is const for a streamline.

Velocity potential (ϕ):–

$u = -\frac{\partial\phi}{\partial x}$ $v = -\frac{\partial\phi}{\partial y}$ $w = -\frac{\partial\phi}{\partial z}$

Incompressible flow :– $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \nabla u = 0$

$\boxed{\begin{array}{l}\nabla^2\psi = 0 \to \text{Irrotational flow} \\ \nabla^2\phi = 0 \to \text{Incompressible flow.}\end{array}}$

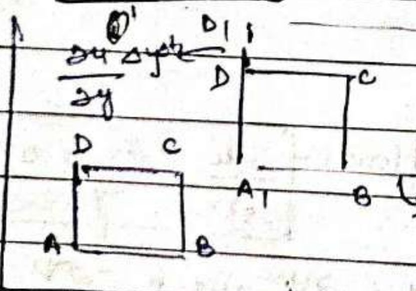
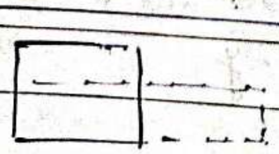


Figure is wrong for 2D incompressible flow

$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$



Can be right for :–

- compressible flow (red^n in density)
- in case of 3D flow.

Conservation of Linear M^m for a flowing system

Newton's second law $\vec{F} = \left.\frac{d\vec{P}}{dt}\right|_{system}$ $\vec{P}$ = linear momentum = $m \times v$

$\vec{P} = \int_{mass} \vec{V} dm$

For a system with infinitesimal mass dm, the eq^n can be written as $d\vec{F} = dm \frac{d\vec{V}}{dt}\Big|_{system}$ functional form :- F = ma.

$\boxed{d\vec{F} = dm \frac{D\vec{V}}{Dt}}$ → for all possible accl^{n}s, → $\underbrace{dF_x}_{\text{net force acting in x dir}^n} = dm\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right]$

$dF_x = dF_{Bx} + dF_{Sx}$

Body forces → electromagnetic forces, gravity

Surface forces → Surface Tension, stress.

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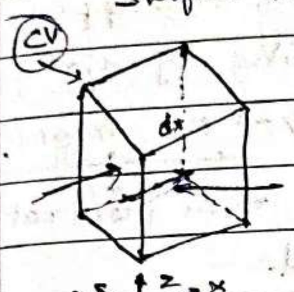


Continuity :- $\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$

For incompressible flow :- $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

Conserv^n of M^m :- F = ma
force ← $F_S + F_B$; ma → change in linear M^m.

M^m entering into faces → mass m carried by vel. v.

What is the mass flow rate over the left face due to x comp. vel. ?

mass entering due to x comp. of vel. → $\rho u \partial y \partial z$ → kg/s ; $\frac{kg}{m^3} \times \frac{m}{s} \times m^2$

$u \partial y \partial z \Big|_{x - \frac{dx}{2}}$ → volumetric flow rate entering through Left face.

What is the momentum entering throug the left face entering due to the x comp. of vel. ?

momentum entering = $\underbrace{\rho (u dy dz)}_{\text{mass flow rate } (\dot{m})}\Big|_{x-\frac{dx}{2}} \cdot (u)$ → carries velocity (vel. carrying the mass)

For a flowing sys, mass is mass flow rate

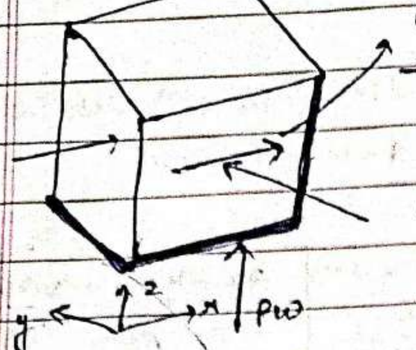
Front face :- $\boxed{(\rho v dx dz) v.}$ → not enough → To be added
↓
The effect of vel. and mass flowing in other directions

carrying some mass.

CV → 6 faces

for each vel. component M^o enters through 3 faces. When we are talking about x comp. vel. we are talking about any.

*. Due to x comp. vel., M^o will enter through 3 faces. (in the front face carries vel. is v, bottom face → w, side face → x)



x comp. → $\rho u^2 + \rho uv + \rho uw$

Left, front, Bottom

ρu → mass flow rate

Mass flowing

y comp. vel. hits it

1.) → too high magnitude of other vel.

2.) → too low magnitude (of other vel.

3.) → most likely to happen (changing of dirⁿ)

changing of Trajectory → M^o entering CV → M^o magnitude

mass flow rate × carrier vel

In all cases → mass flow due to x comp. vel.

The effect of 3 comp. → help in entering the mass into CV.

What are the terms due to x' component of vel.? (mass flow)

→ $\rho u u + \rho u v + \rho w$

Left face, right face, z comp. of mass entering due to z comp. vel, bottom face

$\rho u v$

$(\rho u) v$ — x component mass

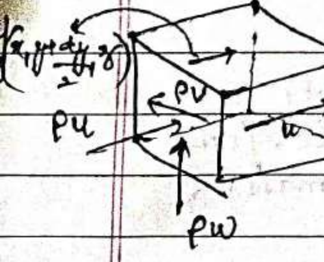
$(\rho v) u$ — y component mass

assumed that the flow happens in the dirⁿ of the axis.

y comp. → $(\rho v)u + \rho v^2 + (\rho v)w$

※ In a 6 phase CV, M^o enters through 3 faces and in each face, M^o enters using 3 vel.

9 terms → 6 distinct terms

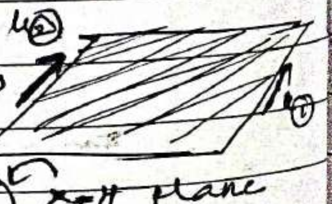


$(x, y+\frac{dy}{2}, z)$

ρv, ρu, w, ρw

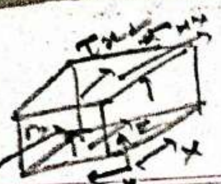
$u_{(1)}$ $(x, y-\frac{dy}{2}, z)$

$u_{(1)}$ and $u_{(2)}$ are diff → angular deformⁿ →



τ_{xy} → x-y plane deformⁿ

comp. Balance (1) – Momentum entering the left face due to x comp. Mass flow & x comp. carrier velocity $= (\rho u)_{x-\frac{dx}{2}, y, z}\, dy\, dz)\, u$



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(A) M^2 along front face due to x comp. mass flow and y comp. carries vel. $= (\rho u) v$.

(B) " ——— bottom - z - - $= (\rho u) w$

(A) $\left(\rho u\big|_{x, y-\frac{dy}{2}, z} \, dz \, dx\right) v$ (B) $\left(\rho u\big|_{x, y, z-\frac{dz}{2}} \, dx\,dy\right) w$

Net exchange of M^2 due to x comp. mass flow & x comp. carries vel.

$= \left(\rho u \frac{\partial u}{\partial x}\right) \partial x \, \partial y \, \partial z$ (carries vel: → centroid vel.)

y comp. carries vel. $= \rho v \left(\frac{\partial u}{\partial y}\right) \partial x \, \partial y \, \partial z$

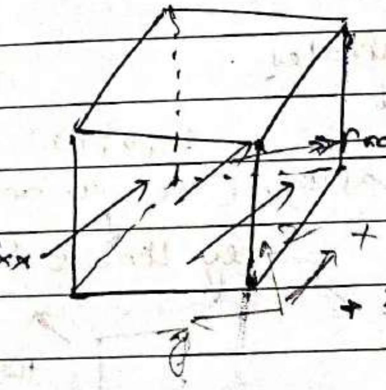
z comp. carries vel. $= \left(\rho w \frac{\partial u}{\partial z}\right) \partial x \, \partial y \, \partial z$

$\frac{\partial (\rho u)}{\partial t} \partial x \, \partial y \, \partial z$ → variation of u with time, time dependent accl^n.

we consider SS (gen) $m = \rho \, \partial x \, \partial y \, \partial z$

LMS:- $\underbrace{\rho \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) \partial x \, \partial y \, \partial z}_{\text{rate of change of } M^2 \text{ (Net Exchange of } M^2)}$

x comp :— $\boxed{F_x = F_{Sx} + F_{Bx}}$



mass flow rate due to x comp. vel. → 3 comp. of stress

σ_{xx}, τ_{xy}, τ_{xz}

left & right, front & back, bottom & top

$dF_{Sx} = \left\{ (\sigma_{xx})_{x+\frac{dx}{2}, y, z} - (\sigma_{xx})_{x-\frac{dx}{2}, y, z} \right\} dy \, dz$

$+ \left\{ (\tau_{xy})_{x, y+\frac{dy}{2}, z} \, dx \, dz - (\tau_{xy})_{x, y-\frac{dy}{2}, z} \, dx \, dz \right\} +$

$\left\{ (\tau_{xz})_{x, y, z+\frac{dz}{2}} \, dx \, dy - (\tau_{xz})_{x, y, z-\frac{dz}{2}} \, dx \, dy \right\}$

$= \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) dx \, dy \, dz$ $\cancel{\frac{d\sigma_{xx}}{dx}}$

Partial derivatives

* gravity $= -\rho g \hat{z}$

$\rho \left(\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}$ Cauchy's eq^n or (Navier's eq^n) (valid for any fluid)

SS if pr driven flow → it also adds normal stress → So nothing is missing. It is hidden in σ_{xx} term. $\sigma_{xx} = \sigma_{xx\,pr} + \sigma_{xx\,KE}$

[Important for Post Midsem]

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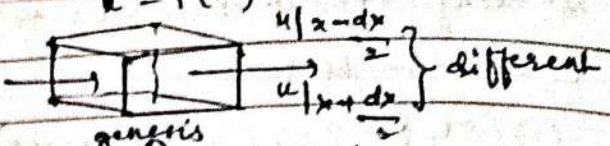
$u = f(x)$



3-D flowfield :-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

genesis

x: dirⁿ $$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho g_x$$

Objective :- To solve for flowfield. To fully define a flowfield what all parameters are reqd.? — (u, v, w, P)

That's why we are writing these eqⁿs. (4 eqⁿs)

We can't solve these eqⁿs. Why?

No. of variables in the eqⁿ in this form } (10) $\begin{pmatrix} \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{z} \end{pmatrix}$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho g_z$$

In order to make the sys. of eqⁿs solvable we'll have to make the no. of eqⁿs & no. of variables solvable.

(#) Relate the stress compⁿ to the primary variables.

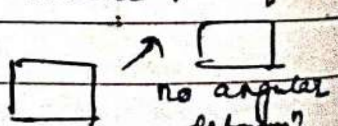
Stress is split up into 2 parts → Normal, → shear

(*) There can be stress even without deformation (u, v, w constant) $u, v, w = 0$

only linear transfⁿ.

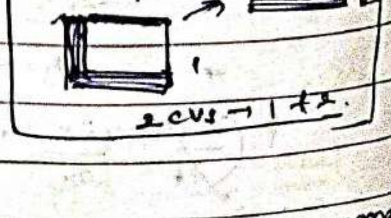
→ shear stress → fⁿ of angular deformⁿ

→ No shear stress



→ Is there stress even if no angular deformⁿ?

→ Stress happens because of spatial variations of the velocity components.



manifestⁿ of diff in Pr → ΔP → $P_1 = P_{atm} + \rho g h_1$, $P_2 = P_{atm} + \rho g h_2$

$P_1 > P_2$

Normal stress is different. It varies.

#. Even if the particles are moving with a const. vel. Not only normal stress acts on it but with time it varies

Buoyancy – Langrangian approach

Q. Is hydrostatic pr. the only pressure / stress ~~impos~~ acting in case of constant velocities acting? → No.

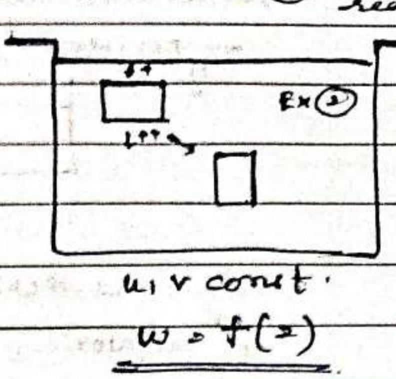
(diff in)

reasons for Normal stress → ① Difference in hydrostatic pressure ② other reasons (vel. variation)

$\left[\frac{\partial \sigma_{zz}}{\partial z} \neq 0\right]$ → Example ② → It is a particle → linear deform^n

① hydrostatic pr. → Normal stress on top face

② Normal stress in z dirn due to vel. variation



There can be stress that is not related to angular deformation.

stress → related to deform^n (Shear)
→ Not related to deform^n (Normal)
① part can be (hydrostatic pressure)
② other reasons (depending on physics of the flow as a result of vel. varn) → (Linear deform^n)

$u = f(z)$, $v = f(z)$ → shear stress

$w = f(z)$ → Normal stress in z dirn.

cross dependencies lead to shear stress

The net stress can be split up into 2 parts :—

① 1 being f^n of deformation (Deviatoric component) (not just angular deform^n)

② other part independent of deformation (Hydrostatic)

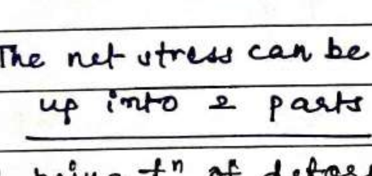
$$\boxed{\tau_{ij} = \tau_{ij}^{\text{Deviatoric}} + \tau_{ij}^{\text{hydrostatic}}}$$

τ_{ij} Deviatoric is related to velocity gradient

$\frac{\partial u_i}{\partial x_j}$ → Break it into symmetric and antisymmetric part.

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{angular deform}^n} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{fluid rot}^n} = \frac{1}{2}\dot{\gamma}_{ij} + \omega_{ij}$$

mathematical cond^n ← pure rot^n



$\boxed{\text{rot}^n \text{ does not contribute to stress}}$

Does rot^n lead to linear deform^n? → NO.

⟹ if the flow is in the nature of pure rot^n, mathematically it causes no angular deform^n. (pure rot^n: $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$)

⇓

$\dot{\gamma}_{xy} = 0$

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$\Rightarrow$ So the rotational term does not contribute to deviatoric component

Mathematically, we get

$\tau_{ij}^{\text{Deviatoric}} = f(\text{rate of deform}^n)$
- → related to the symmetric part.
- → Not a f^n of the antisymmetric part.

A rigorous derivation gives :-

$$\tau_{ij}^{\text{Deviatoric}} = \lambda \cdot \text{div}\, \vec{V}\, \delta_{ij} + \underset{\substack{\downarrow \\ \text{coeff of viscosity}}}{\mu} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$ (for a Newtonian fluid)

$$\text{div}\, \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

δ_{ij} = Kronecker Delta $= 1 \quad i = j$
$= 0 \quad i \neq j$

for an incompressible flow, $\boxed{\text{div}\, \vec{V} = 0}$

For a Newtonian fluid
$\Longrightarrow$ Stokes Hypothesis

* $k = \lambda + \frac{2}{3}\mu = 0$
↓
(For a Stokesian fluid)

λ = 2nd coeff of viscosity
k = coeff of Bulk viscosity.

Stokesian fluid → ~~ttt~~ Stokes fluid similar to ideal fluid. Stokes fluid is something for which $\boxed{P_T = P_M}$ P_T = Thermodynamic pressure and P_M = Mechanical pr. Ideal gas → Internal Energy of the sys is equal to Translational KE. ($U = U_T$) (intermolecular inter^n is zero rot^n and vibr^n = 0) (Monoatomic gases are the nearest candidates for ideal gas)
Thd. pr. → Total pr., P_M → due to KE.

Hydrostatic part :- $\boxed{\tau_{ij}^{\text{hydrostatic}} = -p\delta_{ij}}$

$$\tau_{ij} = \underbrace{-p\delta_{ij}}_{\tau_{ij\,H}} + \underbrace{\lambda \frac{\partial u_i}{\partial x_i}\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\tau_{ij\,D}}$$

The third term is always active.
for incompressible flow :— 2nd term gets cancelled out → active only when $i = j$ for δ_{ij}

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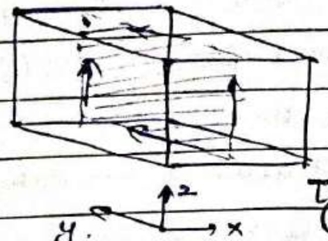
2) (a) 

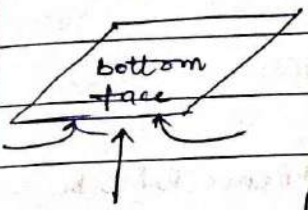
on which faces τ_{yz} will be active?

$\boxed{\text{front, back, top, bottom}}$

τ_{yz} → y comp. M° balance
→ z comp. M° balance

(b) What comp. of M° enters through the bottom?

3 components



Mass flow due to x comp. vel., y comp. vel., z comp. vel.

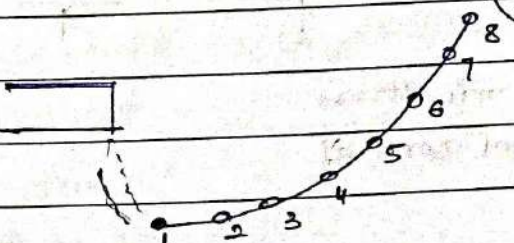
$\left(\rho u\Big|_{x,y,z-\frac{dz}{2}}\right), \left(\rho v\Big|_{x,y,z-\frac{dz}{2}}\right), \left(\rho w\Big|_{x,y,z-\frac{dz}{2}}\right)$

carries vel. $\Rightarrow \left(w\Big|_{x,y,z-\frac{dz}{2}}\right)$ dy dz

3) (a) angular deform^n ✓
rot^n ✓

(b) pure rot^n can be drawn. Irrot^n can be drawn. rectangular sys.

(c) Substantial Derivative. → Derivation.
$\lim_{\Delta t \to 0}$ → important.

(d) 

→ streakline
→ unsteady flow.
→ vel. ↑.
→ accl^n with time → vol. flow rate changing.

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Post MidSem Part

Exact Solution :-

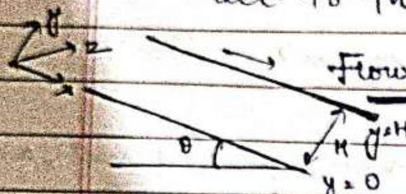
Shell is a CV. Depending on the sys. → we choose shell. Now, we consider the forces. Then, we use the eq^n → We get something integrable (v profile, τ profile). Forces → M^3 Bal.

Navier's Eq^n → 10 unknowns, 4 eq^{n}s.

Newtonian fluid → 4 unknowns, 4 eq^{n}s.

↳ Non Linear Terms make them unsolvable.

Write the eq^n → instead of shell balance → cancel the terms acc. to the physics of the sys.



→ Flow of liq. layer over an inclined plane

Mathematical consequence

Assump^n: ① Steady state, fully developed flow.

② z is ∞ wide, No variation along z. (all ∂z terms will be zero)

⇒ vel. profile ? shear force ? $\boxed{\text{Wall shear stress ?}}$

⇒ Volumetric flow rate (imp. from engg. approach)

Eq^{n}s :- ① Continuity Eq^n :-

$$\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

② x comp. balance :-

Hydrostatic comp. of Normal stress
stress Term

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}$$

Diff. b/w σ_{xx} & τ_{xx}

for normal stress → total normal stress, specific deviatoric stress component of normal.

③ y comp. balance :-

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}$$

Assumptions :-

(1) SS :- $\frac{\partial(\,)}{\partial t} = 0$

(2.) Incompressible flow :- $\rho \neq \rho(x, y, z, t)$
For a non newtonian fluid → μ change does not necessarily mean ρ changes.

(3) No flow in z dir^n :- ($w = 0$)

(4.) No variation in z dir^n. (z dir^n is ∞ wide) ⟹ $\frac{\partial(\,)}{\partial z} = 0$

(5.) Fully developed flow (something does not change) ⟹ $\boxed{\frac{\partial u}{\partial x} = 0}$
(only the vel. comp. in the dir^n of flow does not change)

(6) No slip boundary cond^n valid. At $y = 0$, $\forall\, x, z$ ⟹ $u = 0$, $w = 0$

(7.) Solid impermeable wall at $y = 0$, $\forall\, x, z$ ⟹ $\boxed{v = 0}$
(v is the vel. normal to the surface)

$$\Rightarrow \underbrace{\cancel{\frac{\partial \rho}{\partial t}}^{0}}_{\text{Assump}^n\,(2)} + \underbrace{\cancel{\frac{\partial u}{\partial x}}^{0}}_{\text{Assump}^n\,(5)} + \frac{\partial v}{\partial y} + \underbrace{\cancel{\frac{\partial w}{\partial z}}^{0}}_{\text{Assump}^n\,(4)} = 0 \quad \Rightarrow \frac{\partial v}{\partial y} = 0$$

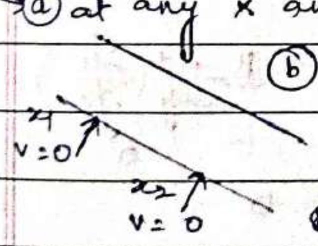
partial derivative = 0

$$\Longrightarrow v \neq f(y)$$

$$v = f(x, z)$$

Using assump^n (4) :- $\boxed{v = f(x)}$

from Assump^n (7) # (a) at any x and z, $v = 0$ at $y = 0$



(b) $\frac{\partial v}{\partial y} = 0$ means the value of v remains const. in the entire y.

(c) Since $\frac{dv}{dy} = 0$, So v = const. at all y, at any x.

Along the depth, v = const.
at $y = 0$, $v = 0$
So, $v = 0$ all along the depth at all points ($\forall\, x$)

$\forall\, x, y, z$ ⟹ $\boxed{v = 0}$

$v = 0$ in the flowfield.

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$\boxed{v=0}$ → now analytically derived using continuity eqⁿ.

z comp. Balance :-

O No variation along z axis; gravity does not act along z dirⁿ

$$\rho\left(\cancel{\frac{\partial w}{\partial t}} + u\cancel{\frac{\partial w}{\partial x}} + \cancel{v}\frac{\partial w}{\partial y} + \cancel{w}\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

0 SS. w = 0 w=0 w = 0; $\rho g_z = \boxed{0}$; assumpⁿ (4)

$$\Rightarrow \frac{\partial \tau_{xz}}{\partial x} = 0 \qquad \tau_{xz} = f\left(\cancel{\frac{\partial w}{\partial x}} + \cancel{\frac{\partial u}{\partial z}}\right)$$

assumpⁿ (3) w = 0

Had it been a viscous fluid:- $\tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$

$$\Rightarrow \frac{\partial \tau_{yz}}{\partial y} = 0 \Longrightarrow \tau_{yz} = f\left(\cancel{\frac{\partial v}{\partial z}} + \cancel{\frac{\partial w}{\partial y}}\right)$$

(v = 0) assumpⁿ (4) assumpⁿ (3)

$$\Rightarrow \frac{\partial \tau_{zz}}{\partial z} = 0 \rightarrow \text{assump}^n (4)$$

x comp. balance :-

$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + \cancel{u}\frac{\partial u}{\partial x} + \cancel{v}\frac{\partial u}{\partial y} + \cancel{w}\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \cancel{\frac{\partial \tau_{xx}}{\partial x}} + \frac{\partial \tau_{xy}}{\partial y} + \cancel{\frac{\partial \tau_{xz}}{\partial z}}$$

0 SS; 0 assumpⁿ (6) FDF; v = 0; w = 0 assumpⁿ (4); 0; 0 FDF; 0 assumpⁿ (4)

$$\tau_{yx} = \left(\frac{\partial u}{\partial y} + \cancel{\frac{\partial v}{\partial x}}\right) \to \text{for viscous fluid} \quad \frac{\partial}{\partial x}\left[\mu\frac{\partial u}{\partial y} + \cancel{\frac{\partial v}{\partial x}}\right] \quad \frac{\partial}{\partial x}\left(f\left(\cancel{\frac{\partial u}{\partial x}}\right)\right)$$

not 0

$$\text{but } \frac{\partial}{\partial x}\tau_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\cancel{\frac{\partial u}{\partial x}}\right) \quad \mu\left(\partial^2 u/\partial x\,\partial y\right) = \mu\frac{\partial f}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 0.$$

$$\Rightarrow -\partial P/\partial x + \rho g\sin\theta + \partial \tau_{xy}/\partial y = 0$$

y comp. balance :-

$$\rho\left(\cancel{\frac{\partial v}{\partial t}} + \cancel{u}\frac{\partial v}{\partial x} + \cancel{v}\frac{\partial v}{\partial y} + \cancel{w}\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho g_y + \frac{\partial \tau_{yx}}{\partial x} + \cancel{\frac{\partial \tau_{yz}}{\partial z}} + \cancel{\frac{\partial \tau_{yy}}{\partial y}}$$

0 SS; assumpⁿ (6); assumpⁿ (7) 0 continuity eqⁿ; assumⁿ (3); 0 assumpⁿ (4); 0; 0; 0 assumpⁿ (4) $f\left(\frac{\partial v}{\partial y}\right) = 0$

$$\Rightarrow \frac{\partial \tau_{yx}}{\partial x} = 0$$

$$\Rightarrow -\frac{\partial P}{\partial y} - \rho g\cos\theta = 0$$

$$\tau_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\frac{\partial}{\partial y}(\tau_{xy}) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \rightarrow \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} \rightarrow \underbrace{\frac{\partial^2 u}{\partial y^2}}_{\text{not zero}} + \frac{\partial}{\partial x}\underbrace{\left(\frac{\partial v}{\partial y}\right)}_{0}$$

So $\frac{\partial}{\partial y}(\tau_{xy}) \neq 0$.

$$\frac{\partial}{\partial x}(\tau_{xy}) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x \partial y} \rightarrow \frac{\partial}{\partial y}\underbrace{\left(\frac{\partial u}{\partial x}\right)}_{0} + \frac{\partial}{\partial x}\underbrace{\left(\frac{\partial v}{\partial y}\right)}_{0}$$

$$\frac{\partial}{\partial x}(\tau_{xy}) = 0$$

x comp. balance :- $\boxed{-\frac{\partial P}{\partial x} + \rho g \sin\theta + \frac{\partial \tau_{xy}}{\partial y} = 0}$

y comp. balance :- $\boxed{-\frac{\partial P}{\partial y} - \rho g \cos\theta = 0}$

2 PDEs after all the simplifications.

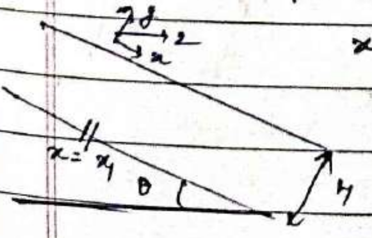
from x comp. balance :- $-\frac{\partial P}{\partial x} + \rho g \sin\theta + \frac{\partial \tau_{yx}}{\partial y} = 0$

from y comp. balance :- $-\frac{\partial P}{\partial y} - \rho g \cos\theta = 0$.

They are partial DE and they cannot be simply integrated.

$P = f(x, y)$

P is a f^n of x and y (similar to handling continuity eq^n)
We are focusing at one particular x → So at particular x $P = f(y)$ only.



$$\text{So } \frac{dP|_x}{dy} = -\rho g \cos\theta$$

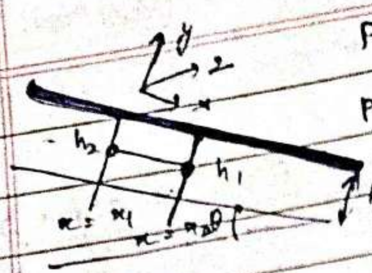
$$P|_x = -\rho g \cos\theta \, y + c_1$$

BCs :- at $y = H$, $P = P_{atm}$

$$c_1 = P_{atm} + \rho g \cos\theta \, H$$

$$P|_x = P_{atm} + \rho g \cos\theta (H - y)$$

P at any x



$P(x_1, h_1)$ | $P(x_1, h_2)$

$P(x_2, h_1)$ | $P(x_2, h_2)$ | $\dfrac{P(x_1,h_2) - P(x_2,h_2)}{x_1 - x_2}$

$\dfrac{P(x_1,h_1) - P(x_2,h_1)}{x_1 - x_2}$

$\dfrac{\partial v}{\partial x} = 0 \to P \neq f(x) \to P = f(y)$ only.

$P(x_1, h_1) = P_{atm} + \rho g \cos\theta\,(H - h_1)$ } for a particular value of x (at any x)

$P(x_2, h_1) = P_{atm} + \rho g \cos\theta\,(H - h_1)$

$P(x_1, h_2) = P_{atm} + \rho g \cos\theta\,(H - h_2)$

$P(x_2, h_2) = P_{atm} + \rho g \cos\theta\,(H - h_2)$

As $\dfrac{\partial P}{\partial x}$ is zero at any pt x. $\Rightarrow \dfrac{\partial P}{\partial x} \neq f(x)$

zero in the entire flowfield.

$$\boxed{\frac{\partial P}{\partial x} = 0}$$

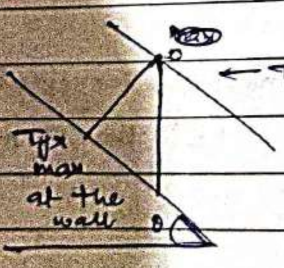
$$-\frac{\partial P}{\partial x} + \rho g \sin\theta + \frac{\partial \tau_{yx}}{\partial y} = 0$$

$$\to \frac{\partial \tau_{yx}}{\partial y} = -\rho g \sin\theta \to \tau_{yx} = -\rho g \sin\theta\, y + c_2$$

Zero shear free surface at $y = H$, $\tau_{yx} = 0 \to c_2 = \rho g \sin\theta\, H$

$$\boxed{\tau_{yx} = \rho g \sin\theta\,(H - y)}$$

nature of the fluid and dimensions are needed to proceed to further calculations. (Newtonian + 1D)



← This is independent of the type of fluid.

For a power law fluid :- $\tau_{yx} = m\left(\dfrac{\partial u}{\partial y}\right)^n = m\dfrac{du}{dy}$

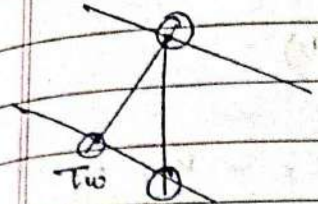
$\tau_{yx} = \rho g \sin\theta\,(H - y)$

Flow profile for a Bingham plastic fluid.

$\tau_{xy} = \tau_B + \mu_B\left(\dfrac{\partial u}{\partial y}\right)$ for $\tau_{xy} > \tau_B$

for $\tau_{xy} < \tau_B$, $\dfrac{\partial u}{\partial y} = 0$

Both the definitions (Bingham plastic and Power Law fluid) are valid only for ~~Power Law~~ 1D flow.



τ_w = wall shear stress = $\rho g \sin\theta H$ (max)

Should a Bingham fluid on inclined plane always flow? → may not
Power Law fluid & Newtonian fluid → always flow → bcoz of non zero body forces

$\underline{\tau_B}$

Any fluid on an inclined plane → whatever be the thickness of layer, θ, τ is max. at the wall. On ↑ tilt, shear stress ↑
if on same tilt, thickness ↑, then also shear stress ↑.

① if $\tau_w < \tau_B$, nowhere the flow is happening

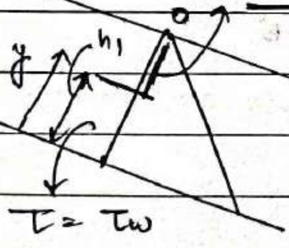
max. shear stress

② $\tau_w > \tau_B$ → flow is possible when $\tau_w > \tau_B$.

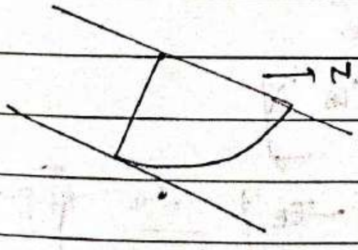
→ free surface $\tau = 0$ (at the top of the layer)

→ along the depth, τ is reducing in a linear ~~way~~ manner

→ <u>shear stress < τ_B</u>



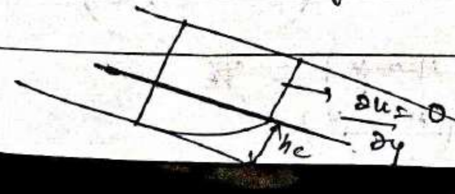
at $y = h$, $\tau = \tau_B = \rho g \sin\theta h_1$
at some pt.$_{(h_1)}$ local shear stress becomes equal to τ_B.



→ Newtonian fluid

parabolic vel. profile → Trend of τ is completely opposite (vel. keeps on ↑)

<u>vel. profile for Bingham plastic :-</u>



vel. gradient to the power 1 for Bingham.

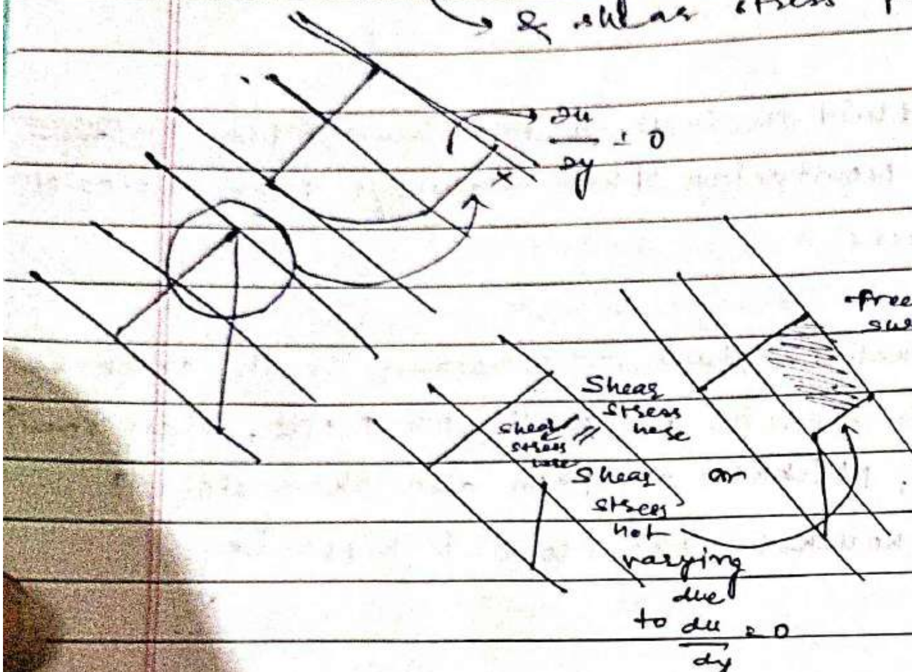
at the pt. $\tau_B = \tau_{xy}(\text{local})$ → beyond that $\partial u/\partial y = 0$ → linear

vel. profile for a Bingham plastic. (find out)

→ & shear stress profile

$$\tau_{xy} = \tau_B + \mu_B\left(\frac{\partial u}{\partial y}\right)$$

for $\tau_{xy} > \tau_B$



$$\tau_{yx} = \rho g \sin\theta (H - y)$$

$$\tau_B + \mu_B\left(\frac{\partial u}{\partial y}\right) = \rho g \sin\theta (H - y)$$

$$\Longrightarrow \frac{\partial u}{\partial y} = \frac{\rho g \sin\theta (H-y)}{\mu_B} - \frac{\tau_B}{\mu_B}$$

$$\rightarrow \int \partial u = \int \frac{(\rho g \sin\theta H - \tau_B)}{\mu_B} - y\frac{\rho g \sin\theta}{\mu_B}\,\partial y$$

$$\rightarrow u = \frac{\rho g \sin\theta H}{\mu_B}(y) - \frac{\tau_B y}{\mu_B} - \frac{y^2}{2}\frac{\rho g \sin\theta}{\mu_B} + c$$

$$\rightarrow u = \frac{\rho g \sin\theta}{\mu_B}\left(Hy - \frac{y^2}{2}\right) - \frac{\tau_B y}{\mu_B} + c$$

at $y = 0$, $u = 0$

$\rightarrow 0 = C$

$$\text{So } u = \frac{\rho g \sin\theta\, y}{\mu_B}\left(H - \frac{y}{2}\right) - \frac{\tau_B y}{\mu_B}$$

$$= \frac{y}{\mu_B}\left[\rho g \sin\theta\left(H - \frac{y}{2}\right) - \tau_B\right]$$

So $\cancel{\tau_{yx} = \tau_B + y\left[\rho g \sin\theta\left(H - \frac{y}{2}\right) - \tau_B\right]}$ $\frac{\partial u}{\partial y} = \frac{1}{\mu_B}\left[\rho g \sin\theta (H - y) - \tau_B\right]$

$$\cancel{= \tau_B(1 - y) + \rho g \sin\theta\left(Hy - \frac{y^2}{2}\right)}$$

$$\cancel{\tau_{yx} = \tau_B + \rho g \sin\theta (H - y) - \tau_B}$$

$$= \rho g \sin\theta (H - y)$$

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$\tau_B + \mu_B\left(\frac{\partial u}{\partial y}\right) = \rho g \sin\theta (H-y)$

$\frac{\partial u}{\partial y} = \frac{1}{\mu_B}\left[\rho g \sin\theta (H-y) - \tau_B\right]$

$\rightarrow \int \partial u = \int \frac{1}{\mu_B}\left[\rho g \sin\theta (H-y) - \tau_B\right] \partial y$

$\rightarrow u = \frac{1}{\mu_B}\left[\rho g \sin\theta \left(Hy - \frac{y^2}{2}\right) - \tau_B y\right] + c$

at $y = 0$; $u = 0$

$\rightarrow 0 = 0 + c \rightarrow c = 0$

$\rightarrow u = \frac{1}{\mu_B}\left[\rho g \sin\theta \left(Hy - \frac{y^2}{2}\right) - \tau_B y\right]$

$\tau_{yx} = \tau_B + \mu_B\left(\frac{\partial u}{\partial y}\right) \rightarrow$ The pt. where $\tau_B > \tau_{yx} \rightarrow \frac{\partial u}{\partial y} = 0$

$\frac{\partial u}{\partial y} = \frac{1}{\mu_B}\left[\rho g \sin\theta \left(H - \cancel{\frac{2y}{2}}\right) - \tau_B\right)$

$\rightarrow y = h_c \rightarrow 0 = \frac{1}{\mu_B}\left[\rho g \sin\theta \left(H - h_c\right) - \tau_B\right]$

$\rightarrow \tau_B = \rho g \sin\theta (H - h_c)$

$\rightarrow \frac{\tau_B}{\rho g \sin\theta} = H - h_c \rightarrow h_c = H - \frac{\tau_B}{\rho g \sin\theta}$.

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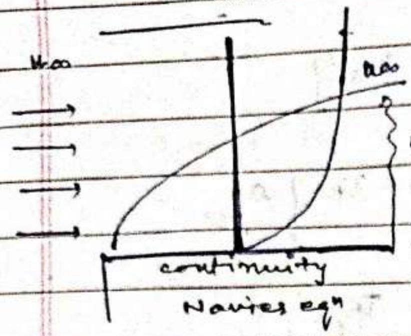
Prandalt Boundary Layer (1908)

Reasons ?



Flow over a flat plate :-

Q1. Why BL form ?

1.) viscosity

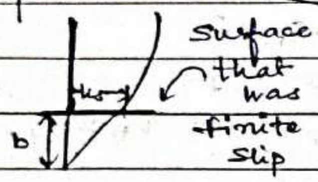
2.) no slip BC or (finite slippage or slip BC may lead to no BL form")

- BL → Governing eqn can be simplified & solved analytically.

⟹ Possible under 2 condns :- (a) inviscid flow

(b) slip vel. at the surface > u_∞

$\boxed{u_s > u_\infty}$

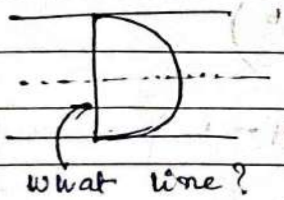


more likely is $\mu = 0$

also possible if b (slip length) = ∞

How do we get b ?

Tangent at the surface pt extrapolated further



Physically this line does not exist
↓
Just a geometric line

* We focus on BL because the modified form of Naviers eqn or NS eqn has an analytical soln within BL region.

* Bernoulli's Eqn is valid only for inviscid flow and streamlines ~~must~~ must exist.

Order of Magnitude Analysis → an idea abt. the max. possible value of a variable

* Prandtl found that the effect of viscosity is significant only within BL.

Flow is divided into 2 regions :- BL flow + Bulk flow
↓
Effect of viscosity comes out to be zero
$\mu \neq 0$ (necessarily not)
However vel. gradients are zero.

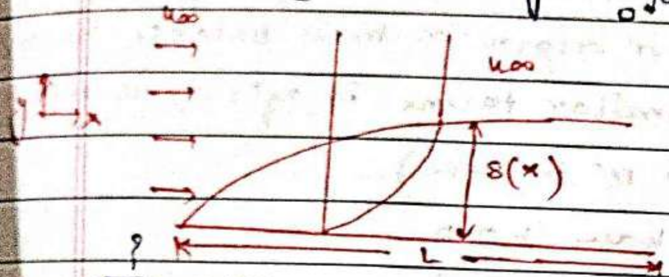
Assumptions for BL flow :-

1. SS
2. Incompressible
3. 2D flow
4. Solid Wall
5. No Slip

Continuity Eq^n :— $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

x component :— $\rho\left(\cancel{\frac{\partial u}{\partial t}}^{0} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \cancel{w\frac{\partial u}{\partial z}}^{0}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}^{0}\right) + \cancel{\rho g_x}^{0}$

y component :— $\rho\left(\cancel{\frac{\partial v}{\partial t}}^{0} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \cancel{w\frac{\partial v}{\partial z}}^{0}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \cancel{\frac{\partial^2 v}{\partial z^2}}^{0}\right) + \rho g_y$



$\delta = f(x)$

p = Non gravitational pr.

$P = p + \rho g y$

$\frac{\partial P}{\partial y} = \frac{\partial p}{\partial y} + \rho g \qquad \frac{\partial P}{\partial x} = \frac{\partial p}{\partial x}$

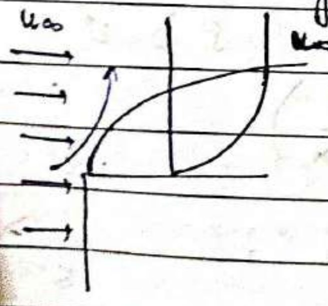
x comp. :— $\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

y comp :— $\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \rho g$

$O(u) \approx u_\infty$

$O(v) \approx$ not known

* L is the length over which BL seperation takes place.



What happens to the bottom part of the plate? Is it a thin sheet? or what is the bottom? For the sake of simplicity, we are restricting our discussion to the upper frame and the lower bottom part is not taken into consider

$O(x) \approx L \quad O(y) \approx \delta.$

Invoke physical understanding :—

* BL is always thin. $\rightarrow$ $\boxed{\delta \ll L}$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \rightarrow \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$

<u>Analysing order</u> :- $O\left(\frac{\partial u}{\partial x}\right) = O\left(\frac{\partial v}{\partial y}\right) \Rightarrow \frac{O(u)}{O(x)} = \frac{O(v)}{O(y)}$

-ve sign is not considered ~~whit~~ while taking the order.

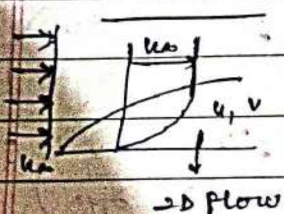
$$O\left(\frac{\partial^2 u}{\partial x^2}\right) = O\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right) = \frac{O(u)}{(O(x))^2}$$

$$O(v) = \frac{O(u)\times O(y)}{O(x)} = \frac{u_\infty \times \delta}{L} = \frac{u_\infty \delta}{L}$$

$$\boxed{O(v) = \frac{u_\infty \delta}{L}} \qquad \delta << L \rightarrow \left(\frac{\delta}{L}\right) \rightarrow \text{very small.}$$

* You cannot neglect any term based on the order of magnitude analysis if the equation has its genesis or origin in mass balance. However, smaller terms can be neglected wrt similar terms in eq^{n}s that refer to others balances. (in our case → m^e balance)

conclusion :- within BL, we have $\underline{v \neq 0}$.



$\underline{v << u_\infty}$ In BL, flow transforms to 2D flow from 1D flow

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\left(\underset{\text{neglect}}{\cancel{\frac{\partial^2 u}{\partial x^2}}} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\rightarrow \frac{O(u)\cdot O(u)}{O(x)} + \frac{O(v)\,O(u)}{O(y)} = -\frac{1}{\rho}\frac{O(P)}{O(x)} + \frac{\mu}{\rho}\left(\frac{O(u)}{(O(x))^2} + \frac{O(u)}{(O(y))^2}\right)$$

$$\rightarrow \underbrace{\frac{u_\infty^2}{L} + \frac{u_\infty \cancel{\delta}}{L}\times\frac{u_\infty}{\cancel{\delta}}}_{\left(\frac{u_\infty^2}{L}\right)} = \underbrace{-\frac{1}{\cancel{\rho}}\frac{O(P)}{L}}_{O\left(\frac{\partial P}{\partial x}\right)} + \frac{\cancel{\mu}}{\cancel{\rho}}\left[\frac{u_\infty}{L^2} + \frac{u_\infty}{\delta^2}\right]$$

$\delta << L$

$\delta^2 << L^2$

$\frac{1}{\delta^2} >> \frac{1}{L^2}$

$O\left(\partial^2 u/\partial x^2\right) << O\left(\partial^2 u/\partial y^2\right)$

$$\rightarrow \frac{u_\infty^2}{L} = -\frac{1}{\rho}\frac{O(P)}{L} + \frac{\mu u_\infty}{\rho\,\delta^2}$$

$$\cancel{\Rightarrow O(P) = \left(\frac{\mu u_\infty}{\rho \delta^2} - \frac{u_\infty^2}{L}\right)\rho L = \frac{\mu L u_\infty}{\delta^2} - u_\infty^2 \rho}$$

Both the advective terms will have the same order (LHS

$$\cancel{\frac{1}{\rho}\frac{O(P)}{L} = \frac{\mu u_\infty}{\rho\,\delta^2} \Rightarrow O(P) = \frac{\mu u_\infty L}{\delta^2}}$$

$\Rightarrow u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$ [Based on the physics of the system]

BL approxim^n :— $\partial^2 u/\partial x^2 \approx 0$ [we are just neglecting bcoz of a much larger similar term]

→ A sys. loses its generality as we plug in more assumptions.

y comp. bal. :—

$$\frac{u\partial v}{\partial x} + \frac{v\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\rightarrow \frac{O(u)\,O(v)}{O(x)} + \frac{O(v)\,O(v)}{O(y)} = O\left(\frac{\partial P}{\partial y}\right) + \left[\frac{O(v)}{(O(x))^2} + \frac{O(v)}{(O(y))^2}\right]$$

$$\rightarrow \frac{u_\infty \times \frac{u_\infty \delta}{L}}{L} + \frac{\left(\frac{u_\infty \delta}{L}\right)^2}{\delta} = O\left(\frac{\partial P}{\partial y}\right) + \left[\frac{u_\infty \delta/L}{L^2} + \frac{u_\infty \delta/L}{\delta^2}\right]$$

$\because \delta << L$

$\frac{1}{\delta^2} >> \frac{1}{L^2}$

$$\rightarrow \frac{u_\infty^2 \delta}{L^2} = O\left(\frac{\partial P}{\partial y}\right) + \frac{u_\infty \theta}{\delta L}$$

$\therefore$ So $\frac{\partial^2 v}{\partial x^2} \approx 0$

3 types of terms :—

$$\underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}_{\text{(advective terms)}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial x}}_{\text{(P. term)}} + \underbrace{\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial y^2}\right)}_{\text{(skin friction / shear stress)}}$$

due to the flow (inertial terms)

$A + B + C = 0$

if said In this eq^n all the 3 terms are acting/contribute :—

Conclusion :— $\boxed{O(A) \simeq O(B) \simeq O(C)}$

no term can be neglected.

In each of the componentwise eq^n → all 3 terms have almost same order

Simplified x component eq^n :—

$$\underbrace{\frac{u\partial u}{\partial x} + \frac{v\partial u}{\partial y}}_{\substack{\text{inertial} \\ (u_\infty^2/L)}} = \underbrace{-\frac{1}{\rho}\frac{\partial P}{\partial x}}_{\substack{\text{pressure} \\ 1/\rho\, O(\partial P/\partial x)}} + \underbrace{\nu\frac{\partial^2 u}{\partial y^2}}_{\substack{\text{friction / Stress} \\ (\nu u_\infty/\delta^2)}}$$

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y comp. eq^n :-

$$\underbrace{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}}_{\frac{\partial}{L}\left(\frac{u_\infty^2}{L}\right)} = \underbrace{-\frac{1}{\rho}\frac{\partial P}{\partial y}}_{\frac{1}{\rho}O\left(\frac{\partial P}{\partial y}\right)} + \underbrace{\nu\frac{\partial^2 v}{\partial y^2}}_{\nu\left(\frac{u_\infty}{\delta^2}\right)\left(\frac{\delta}{L}\right)}$$

* in either eq^{n}s, it is logical to conclude that the order of the inertial term = order of ps. term = order of friction term in the respective eq^{n}s

The only thing which is unknown is the pressure term.

$P = P(x, y)$

$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}\cdot dy \rightarrow O(dP) = O\left(\frac{\partial P}{\partial x}\right)O(x) + O\left(\frac{\partial P}{\partial y}\right)O(y)$

$\rightarrow \frac{dP}{dx} = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y}\cdot\frac{dy}{dx} \rightarrow O\left(\frac{dP}{dx}\right) = O\left(\frac{\partial P}{\partial x}\right) + O\left(\frac{\partial P}{\partial y}\right)\frac{O(y)^\nu}{O(x)^\nu}$

$L \qquad \checkmark$

(v) → known

$\rightarrow O\left(\frac{dP}{dx}\right) = \rho\nu\left(\frac{u_\infty}{\delta^2}\right) + \rho\nu\left(\frac{u_\infty}{\delta^2}\right)\frac{\delta}{L}\frac{\delta}{L}$

$\rightarrow O\left(\frac{dP}{dx}\right) = \mu\left(\frac{u_\infty}{\delta^2}\right) + \mu\left(\frac{u_\infty}{\delta^2}\right)\frac{\delta}{L}\cdot\frac{\delta}{L}$

$\rightarrow O\left(\frac{dP}{dx}\right) = \underbrace{\mu\left(\frac{u_\infty}{\delta^2}\right)}_{T_1} + \underbrace{\mu\left(\frac{u_\infty}{L^2}\right)}_{T_2}$ } doesn't has the origin in mass balance

$\simeq T_1$ as $T_1 >> T_2$

$\rightarrow \boxed{O\left(\frac{dP}{dx}\right) = \frac{\mu u_\infty}{\delta^2} = O\left(\frac{\partial P}{\partial x}\right)}$

→ P is a f^n of x only

~~$O(dP) = \rho\frac{u_\infty^2}{L}L + \rho\delta\frac{u_\infty^2}{L\,L}\cdot\delta = \rho u_\infty^2 + \rho u_\infty^2\left(\frac{\delta^2}{L^2}\right)$~~

$O(dP) = \nu\left(\frac{u_\infty L}{\delta^2}\right)L + \frac{\nu u_\infty}{\cancel{\delta^2}}\frac{\cancel{\delta}}{L}\delta$

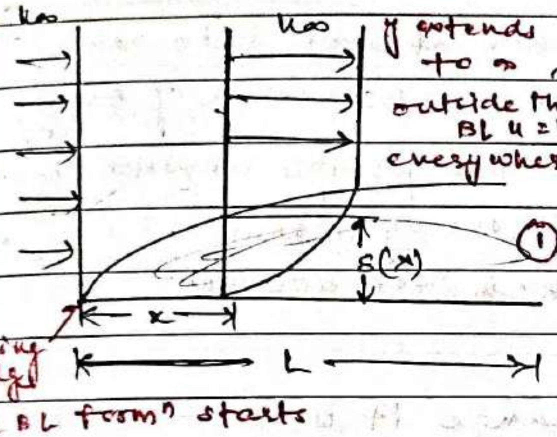
→ $\boxed{O\left(dP\right) = \rho\frac{\nu u_\infty}{(\delta/L)^2} + \rho\frac{\nu u_\infty}{L}}$ $\qquad \left(\frac{\delta}{L}\right)^2 << L.$

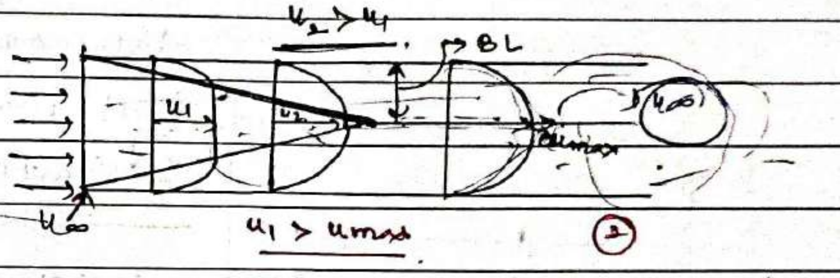
$$O\left(\frac{dP}{dy}\right) = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial x}\frac{dx}{dy} = O\left(\frac{\partial P}{\partial y}\right) + O\left(\frac{\partial P}{\partial x}\right)\frac{O(x)}{O(y)}$$

$$= \frac{\mu u_\infty}{\delta^2}\cdot\frac{\delta}{L} + \frac{\mu u_\infty}{\delta^2}\cdot\frac{L}{\delta} = \frac{\mu u_\infty}{\delta L} + \underbrace{\frac{\mu u_\infty L}{\delta^3}}_{>>}$$

$$O\left(\frac{dP}{dy}\right) = \frac{\mu u_\infty}{\delta^2}\left(\frac{L}{\delta}\right)$$

P is a fn of x only.

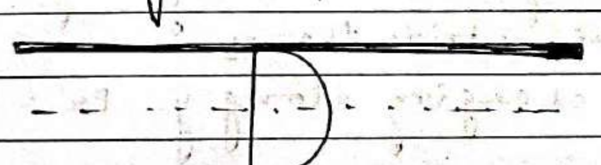




When the thickness of BL = the radius of the pipe → fully developed flow

$$u = u_{max}\left(1-\left(\frac{r}{R}\right)^2\right)$$

For a fully developed flow, det the BL.



nature of flow within a BL is 2D
some amt of fluid which cannot flow due to friction which is near to the wall accumulates to the center

Due to BL form^n in a tube:—

→ freestream vel. is remaining same through the entire length of the tube (outside BL)

→ The core vel. is ↑ till it reaches v_{max}.

Q. In the first case (1) the vel. is ↑ outside the BL but in case (2) it is increasing. Is it true/false? Explain the logic → True

Case (1) → only possible considering flow in y dir^n extends upto ∞

Case (2) → bounded y does not extend to ∞ and is finite

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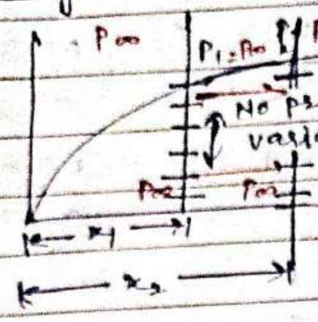
$P = f(x)$ only.

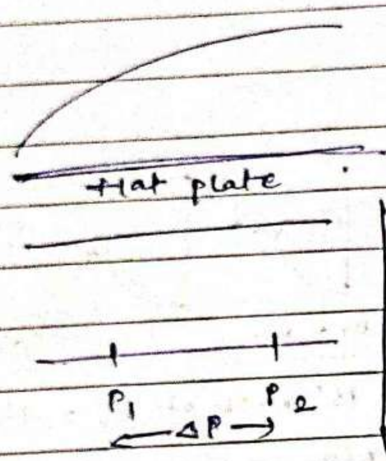
$\boxed{\frac{\partial P}{\partial y} = 0}$ → There is no pressure variation within the BL in y dirn.



If at the edge of the BL at $x = x_1$ $P = P_1$ then at every pt. within the BL $P = P_1$ along y dirn. Similarly $P = P_2$ for every pt. at $x = x_2$ along y dirn.

As soon as there is a pr. drop, it will show its manifestation on the dirn of flow.

Pr. drop means. dissipation of KE (Irreversible interconversion of work into heat) → drop in vel.

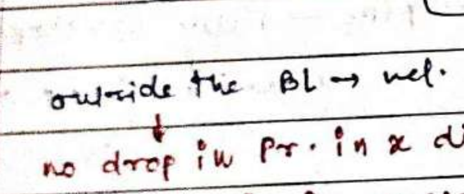


First Law of Thd. (conservn of Energy & mass) is limited. irreversible conversion of energy to heat not taken into account. continuity eq. nuclear rxns don't follow

* outside the BL → vel. does not change it is u_∞ → Pr. is constant no drop in Pr. in x dirn along the boundary layer outside (BL) pressure is P_∞ everywhere outside the BL

So $\frac{\partial P}{\partial x} = 0$. [∵ P is not changing along y $P_x = P_\infty$ at all pts

y stretches upto ∞ then only we can write u_∞ is same everywhere outside BL.



* Pr. at the edge of the BL is pressed along the depth of the BL.

* vel. not changing along x, so no change in pressure

* Therefore $P_1 = P_2 = P_\infty$

$\frac{\partial P}{\partial x} = 0$ $\frac{dP}{dy} = 0$

So Final Eqns :- $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$ | $\frac{\partial P}{\partial y} = 0$] $\frac{\partial P}{\partial x}$

2D flow :- $\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ x\text{-}M^L \\ y\ M^m \to \partial P/\partial y = 0 \end{cases}$ $\underbrace{u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y}}_{\text{Inertial}} = \underbrace{\nu\dfrac{\partial^2 u}{\partial y^2}}_{\text{friction}}$ } analytical sol^n available

$$O\left(\frac{u_\infty^2}{L}\right) \simeq O\left(\nu\frac{u_\infty}{\delta^2}\right) \qquad \left(\frac{u_\infty^2}{L}\right) \qquad \left(\frac{\nu u_\infty}{\delta^2}\right)$$

$$\frac{u_\infty^2}{L} = \nu\frac{u_\infty}{\delta^2} \to \nu = \frac{u_\infty\delta^2}{L} \to \boxed{\delta^2 = \frac{\nu L}{u_\infty}}$$

$$\to \delta^2 = \frac{\nu L}{u_\infty} = \left(\frac{\nu}{u_\infty L}\right)L^2 \to \left(\frac{\delta}{L}\right)^2 = \frac{\nu}{u_\infty L} \to \boxed{\frac{u_\infty L}{\nu} = \frac{u_\infty L\rho}{\mu}}$$

$$Re_L = \frac{u_\infty L\rho}{\mu} \to \boxed{Re_x = \frac{\rho u_\infty x}{\mu}} \to \text{at any pt}$$ Length Reynold's no.

2D flow inside a tube Re remains const if vel. const.
if vel. not const., Re_x keep on changing at every x.

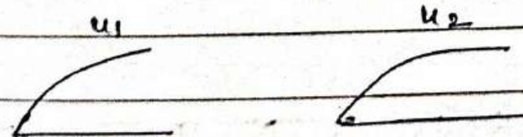
$$\to \left(\frac{\delta}{L}\right)^2 = (Re_L)^{-1} \to \frac{\delta}{L} = (Re_L)^{-1/2} \to \boxed{\delta = L(Re_L)^{-1/2}}$$

$$\boxed{\delta(x) = x(Re_x)^{-1/2}}$$

Utility of this? → so we can find the thickness of BL at any x

$$(Re_x)^{1/2} = \frac{x}{\delta(x)} \quad \text{or} \quad \boxed{(Re_L)^{1/2} = \frac{L}{\delta}} \to Re_L = \frac{L^2}{\delta^2}$$ → significance? → Dimensionless no.

① It is not the ratio of viscous to inertial terms.
② It is a geometrical parameter related to the geometry of the BL



$u_2 > u_1$
$Re_{L_2} > Re_{L_1}$ at same x. $\left(Re_L = \dfrac{\rho u_\infty L}{\mu}\right)$

$$\delta_2 << \delta_1 \leftarrow \left(\frac{L}{\delta_2}\right)^2 >> \left(\frac{L}{\delta_1}\right)^2$$

at $x = L$:- $Re_{L_1} = \dfrac{\rho u_1 L}{\mu}$, $Re_{L_2} = \dfrac{\rho u_2 L}{\mu}$ → as $u_2 > u_1$ $Re_{L_2} > Re_{L_1}$

$$Re_{L_1} = \left(\frac{L}{\delta_1}\right)^2 \qquad Re_{L_2} = \left(\frac{L}{\delta_2}\right)^2$$

δ_2 is smaller than δ_1
$u_\infty \uparrow$ BL thickness becomes lower

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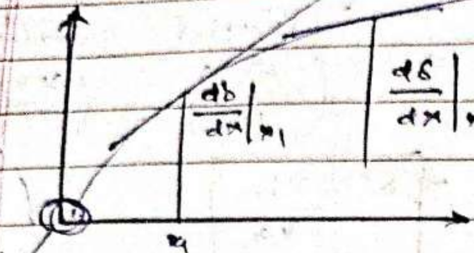
$$\frac{d\,\delta(x)}{dx} \qquad \delta(x) = x(Re_x)^{-1/2}$$

$$\therefore \delta = \left(\frac{\nu x}{u_\infty}\right)^{1/2}$$



$$\frac{d\delta}{dx} = \frac{1}{2}\left(\frac{\nu}{u_\infty}\,\frac{1}{x^{1/2}}\right)$$

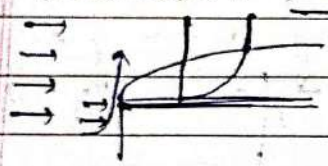
Slope at the leading edge $x=0$
↳ undefined

do not join it → joining means imposing a slope at BL at $x=0$

The local slope of the BL at the leading edge is undefined

10/11/23 Turbulence:- $\boxed{Re > 4000}$

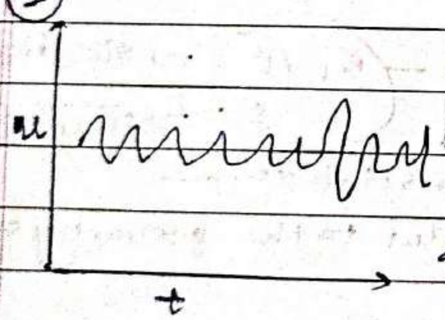
Turbulence is oscillatory motion. (Internal convergence leads to oscillation)



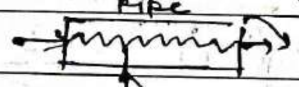
gust of flow from the bottom and everything starts to oscillate (due to the bottom solid surface)

These fluctuations can be sustained only if the sys. has high amt. of Energy. It is a manifest^n of high Re no. (Inertial / viscous forces → viscous forces are same. So KE high)

(I)

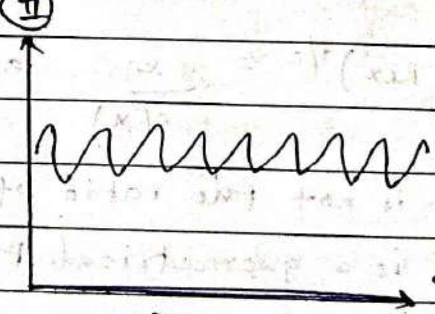


flow inside pipe



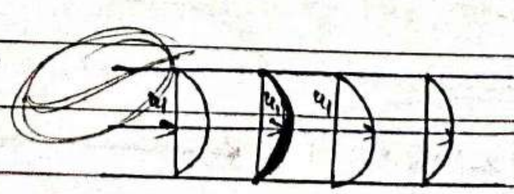
At a specific → time

(II)



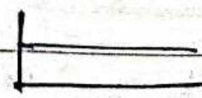
We are looking at a pt and tracking instantaneous velocities at different time.

At a particular loc^n, the instantaneous flow varies (vel. is random → randomness or oscillation means with time at a particular loc^n, instantaneous velocities differ)

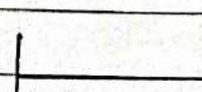


at x_0, $u \neq f(t)$

for a steady uniform flow



at $x = x_0$ (Steady)



at $t = t_0$ (Uniform)

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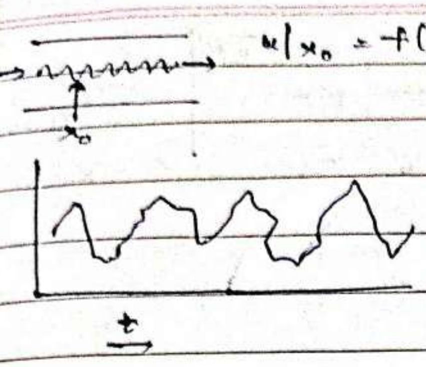


Reynolds' Decomposition of Turbulent flow.

mean → Time averaged mean

$$\boxed{u(y,t) = \bar{u}(y) + u'(\Gamma_1, t)}$$

fluctuation can happen in any possible plane

Γ → All possible spatial directions.

How are we going to handle these fluctuations mathematically?
We are going to use a time dependent acceleration and time independent mean.

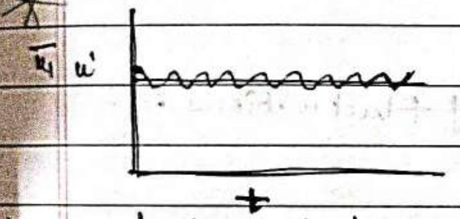
if both ① & ⑪ graphs are possible, there can be 2 types of means:–

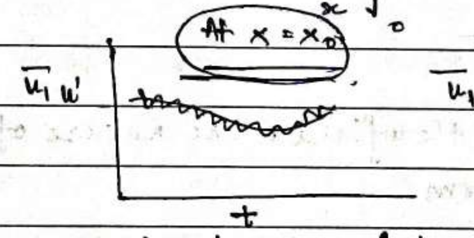
Time average:– $\bar{u}^t(x_0) = \lim_{t_1 \to \infty} \frac{1}{2t_1} \int_{-\infty}^{+\infty} u'(x_0, t)\, dt$

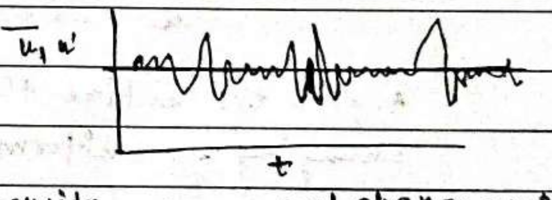
u' is fluctuations.

Space average:– $\bar{u}^s(t_0) = \lim_{x \to \infty} \frac{1}{x} \int_{0}^{\infty} u'(x_0, t_0)\, dx$

*



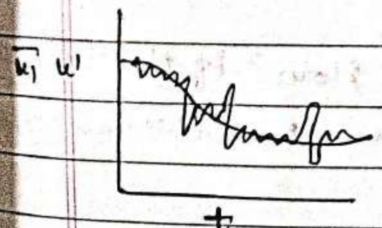




mean vel. does not change over time. & fluctuation amplitudes are constant.

fluctuation intensity does not change in time. mean vel. ~~does not~~ change

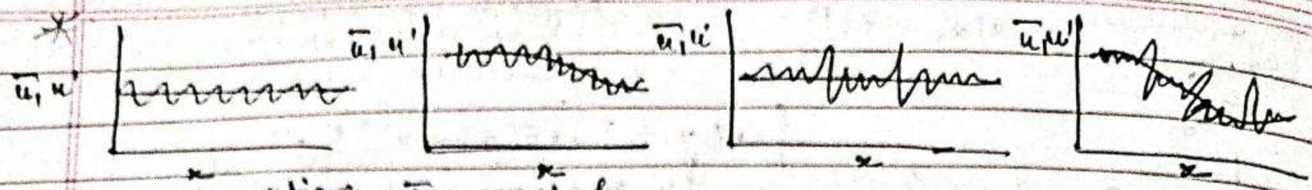
mean vel. changes with time. Turbulence is still stationary.



In case :– $\bar{u}^t(x_0) = 0$ → it is called stationary turbulence. Turbulent flowfield means vel. has fluctuation. ~~Also called~~ Homogenous turbulence → Incase = 0

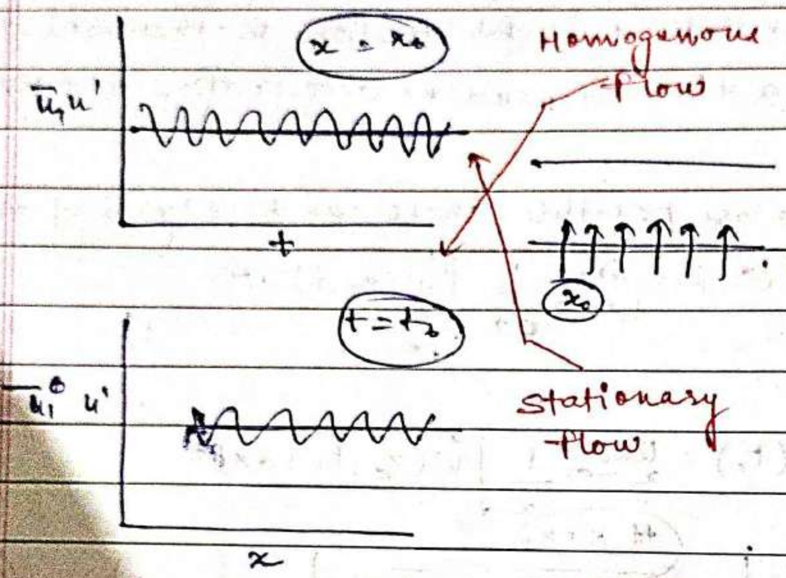
mean vel. as well as fluctuation intensity changing

at a particular loc^n over a period of time the fluctuations at $t = t_0$



along the entire length $\bar{u}$ = const & u' = uniform

$\bar{u}$ = const & fluctuations changes



Across the entire flowfield, the nature of fluctuations is not changing → uniform.

Is it possible to have steady state turbulence? → No bcoz turbulence is inherently time dependent. (vel. is a f^n of time)

Steady uniform flow

What is the closest analog to a steady uniform flow? If the turbulence is stationary and homogenous. but in the truest sense steady state turbulence is a wrong term.

For a 3D flowfield → the variables are (u, v, w, p) :-

$u = \bar{u} + u'$ $\quad w = \bar{w} + w'$

$v = \bar{v} + v'$ $\quad p = \bar{p} + p'$

$\bar{f}$ = Time mean

① $\overline{f + g} = \bar{f} + \bar{g}$ ② $\overline{f \cdot g} = \bar{f} \cdot \bar{g}$ ③ $\bar{\bar{f}} = \bar{f}$

continuity Eq^n:— $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ for a turbulent flow, we incorporate Reynold's Decomposition.

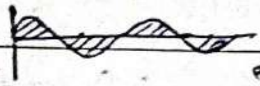
$$\frac{\partial(\bar{u}+u')}{\partial x} + \frac{\partial(\bar{v}+v')}{\partial y} + \frac{\partial(\bar{w}+w')}{\partial z} = 0 \quad \text{— E1}$$

we take time mean

$$\overline{\frac{\partial}{\partial x}(\bar{u}+u') + \frac{\partial}{\partial y}(\bar{v}+v') + \frac{\partial}{\partial z}(\bar{w}+w')} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\overline{(\bar{u}+u')} + \frac{\partial}{\partial y}\overline{(\bar{v}+v')} + \frac{\partial}{\partial z}\overline{(\bar{w}+w')} = 0$$

$$\Rightarrow \frac{\partial \bar{\bar{u}}}{\partial x} + \frac{\partial \bar{\bar{v}}}{\partial y} + \frac{\partial \bar{\bar{w}}}{\partial z} + \frac{\partial \bar{u}'}{\partial x} + \frac{\partial \bar{v}'}{\partial y} + \frac{\partial \bar{w}'}{\partial z} = 0$$



$$\Rightarrow \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} + \cancel{\frac{\partial \bar{u}'}{\partial x}}^{0} + \cancel{\frac{\partial \bar{v}'}{\partial y}}^{0} + \cancel{\frac{\partial \bar{w}'}{\partial z}}^{0} = 0 \quad \text{As } \underbrace{\bar{u}' = 0,\ \bar{v}' = 0,\ \bar{w}' = 0}_{\text{(stationary flow)}}$$

(area above = area below → so they cancel each other → on timed avg)

$$\Rightarrow \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \quad \text{— E2}$$ (The mean values of the 3 vel. follow the continuity eq^n)

E1 → continuity eq^n ~~with~~ after incorporating Reynold's Decompos^n

E2 → Continuity eq^n after incorporating Reynold's Decompos^n & time averaging.

E1 − E2 :— $\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$ — E3

The mean components E2 and the turbulent / fluctuating component E3 of the Reynold's Decomposition follow the continuity eq^n seperately.

Extra Class :—

$u(\gamma, t) = \bar{u}(\gamma) + u'(r, t)$

$$\boxed{u = \bar{u} + u'}$$

Time averaged mean.

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \qquad \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

We will not be using the above eq^n. we will be using the more conservative eq^n (interms of u^2, uv, uw)

M^2 balance

$$u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Conservative form:–

$$\underbrace{\frac{\partial u^2}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z}}_{\text{inertial terms}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial x}}_{\text{pr. grad}} + \underbrace{\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}_{\text{stress/friction terms}}$$

Incorporating Reynolds' Decomposition :–

$$\frac{\partial(\bar{u}+u')(\bar{u}+u')}{\partial x} + \frac{\partial(\bar{v}+v')(\bar{v}+v')}{\partial y} + \frac{\partial(\bar{w}+w')(\bar{w}+w')}{\partial z}$$

$$= -\frac{\partial(\bar{p}+p')}{\partial x} + \frac{\mu}{\rho}\left[\frac{\partial^2(\bar{u}+u')}{\partial x^2} + \frac{\partial^2(\bar{v}+v')}{\partial y^2} + \frac{\partial^2(\bar{w}+w')}{\partial z^2}\right]$$

$$\Rightarrow -\frac{\partial\overline{(\bar{p}+p')}}{\partial x} = -\frac{\partial \bar{p}}{\partial x} - \cancel{\frac{\partial \overline{p'}}{\partial x}}^{\,0} = -\frac{\partial \bar{p}}{\partial x}$$ ($\overline{u'}, \overline{v'}, \overline{w'} = 0$ due to stationary flow)

RHS :– $$-\frac{\partial \bar{p}}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2} + \frac{\partial^2 \bar{w}}{\partial z^2}\right)$$

Wrong Assumpⁿ → steady state → LHS → $\partial/\partial t$ term missing.
if we have turbulent flow we can't really say its a SS flow.

LHS :– $$\overline{(\bar{u}+u')(\bar{u}+u')} = \overline{\left((\bar{u})^2 + (u')^2 + 2\bar{u}u'\right)} = \overline{(\bar{u})^2} + \overline{(u')^2} + \overline{2\bar{u}\cdot u'}$$

$$= (\bar{\bar{u}})^2 + \overline{(u')^2} + 2\bar{\bar{u}}\,\cancel{\overline{u'}}^{\,0}$$

$$= \bar{u}^2 + \overline{u'^2}$$

$$\overline{(\bar{u}+u')(\bar{v}+v')} = \overline{\bar{u}\bar{v}} + \overline{\bar{u}v'} + \overline{u'\bar{v}} + \overline{u'v'}$$

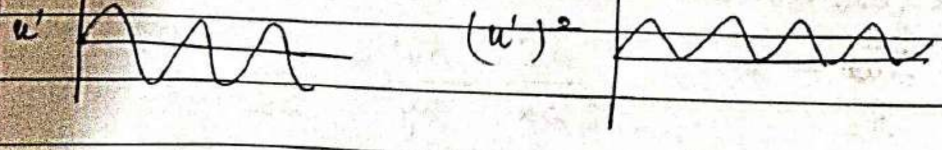
$$= \overline{\bar{u}\,\bar{v}}^{\,S} + \cancel{\overline{\bar{u}\,v'}}^{\,D} + \cancel{\overline{u'\,\bar{v}}}^{\,D} + \overline{u'\cdot v'}^{\,S}$$

$$\boxed{\overline{u'v'} \neq \overline{u'}\,\overline{v'}}$$

S → Similar
D → Dissimilar

$$= \bar{u}\,\bar{v} + \overline{u'\cdot v'}$$

$\overline{(u')^2}$ is different from $\overline{u'}$ → in case of $(u')^2$ we are only considering +ve part
it is not $\overline{u'}\cdot\overline{u'}$ ✗



* If you have similar terms, the multiplication precedes.
* u'v' is considered one entity.
→ We will not take the mean before performing the multiplication.

$$\frac{\partial(\bar{u}^2+\overline{u'^2})}{\partial x}+\frac{\partial(\bar{u}\bar{v}+\overline{u'v'})}{\partial y}+\frac{\partial}{\partial z}(\bar{u}\bar{w}+\overline{u'w'})=-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2\bar{u}}{\partial x^2}+\frac{\partial^2\bar{u}}{\partial y^2}+\frac{\partial^2\bar{u}}{\partial z^2}\right)$$

Eddy diffusivity → (?)

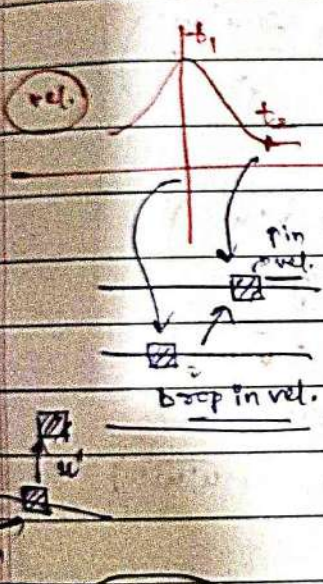
$$\underbrace{\left(\frac{\partial\bar{u}^2}{\partial x}+\frac{\partial\bar{u}\bar{v}}{\partial y}+\frac{\partial\bar{u}\bar{w}}{\partial z}\right)}_{\text{inertial terms}}+\underbrace{\left(\frac{\partial\overline{u'^2}}{\partial x}+\frac{\partial\overline{u'v'}}{\partial y}+\frac{\partial\overline{u'w'}}{\partial z}\right)}_{\text{I}}=\underbrace{-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x}}_{\text{p.s. terms}}+\underbrace{\frac{\mu}{\rho}\left(\frac{\partial^2\bar{u}}{\partial x^2}+\frac{\partial^2\bar{u}}{\partial y^2}+\frac{\partial^2\bar{u}}{\partial z^2}\right)}_{\text{stress terms}}$$

⇒ above Eqn looks similar to our initial eqn except the (I) set of terms

$$\Rightarrow\underbrace{\left(\frac{\partial\bar{u}^2}{\partial x}+\frac{\partial\bar{u}\bar{v}}{\partial y}+\frac{\partial\bar{u}\bar{w}}{\partial z}\right)}_{\text{inertial}}=\underbrace{-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x}}_{\text{p.s.}}+\underbrace{\frac{\mu}{\rho}\left(\frac{\partial^2\bar{u}}{\partial x^2}+\frac{\partial^2\bar{u}}{\partial y^2}+\frac{\partial^2\bar{u}}{\partial z^2}\right)}_{\text{stress due to viscosity}}-\boxed{\left(\frac{\partial\overline{u'^2}}{\partial x}+\frac{\partial\overline{u'v'}}{\partial y}+\frac{\partial\overline{u'w'}}{\partial z}\right)}$$

Stress due to fluctuation

+ve, -ve, -ve



$j=2$
$j=1$
$j=0$

at same pt. vel. is high & at same pt. vel. is low at different times
it is possible if fluid particle is changing levels.
redn in vel. → reduced mass flow rate → less no. of particles are passing. → if some particles are not passing from these → where are they? → The other particles are simply passing through another level. particles has changed its level (redn in vel or vel. drop)

(at j = 1) particle moves away so other particle fills it up. So, to fill up the ~~...~~ void, a particle from orthogonal dirn comes up. (fluctuation → change in instantaneous vel.) for our simplicity (any dirn possible)

Dirn of v' & u' – 1 coming close & other going away i.e. the signs of both the particles is opposite. (dirn opposite). Therefore $u'v'$ is a -ve quantity. Similarly $u'w'$, $w'v'$ are -ve. u'^2 is always +ve. ($u'u' = u'^2$)

We have used the conservative form bcoz the derivation is been done from the first principle.

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If a fluid particle is moving in a laminar flow, viscous resistance is due to the other particles. (Add^n to viscous stresses → fluctuations lead to additional resistance opposing the movement of the particles)

Whatever puts $\boxed{\text{resistance}}$ to flow → Stress

viscous stresses ← viscosity + fluctuation → additional stress.

stress

The physical nature of $\left(\frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}\right)$ → stress due to Turbulence or fluctuation

$$\text{So} \Rightarrow \left(\frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{u}\bar{v}}{\partial y} + \frac{\partial \bar{u}\bar{w}}{\partial z}\right) = \frac{-1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{\mu_E}{\rho}\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right)$$

{ μ_E = effective viscosity
$= \mu + \mu_T$ → Turbulent viscosity }

$$\frac{\mu_T}{\rho}\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right) = -\left(\frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}\right)$$

$$\text{RHS} := \mu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right) - \rho\left(\frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{u'w'}}{\partial y}\right)$$

$$= \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} - \rho\left(\frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\tau_{xx} - \rho\overline{u'^2}\right) + \frac{\partial}{\partial y}\left(\tau_{yx} - \rho\overline{u'v'}\right) + \frac{\partial}{\partial z}\left(\tau_{zz} - \rho\overline{u'w'}\right)$$

$$= \frac{\partial \tau_{xx_E}}{\partial x} + \frac{\partial \tau_{yx_E}}{\partial y} + \frac{\partial \tau_{xz_E}}{\partial z} \quad \{E = \text{Effective}\}$$

Effective shear stress > shear stress (for $u'v'$ & $u'w'$ being −ve) (for u'^2 it is less)

$\tau_E = \tau_v + \tau_T$

$$\tau_{xy_v} = \mu \frac{\partial \bar{u}}{\partial y}$$

$$\tau_{xy_T} = -\rho\overline{u'v'} = \mu_T \frac{\partial \bar{u}}{\partial y} \rightarrow \mu_T = \frac{-\rho\overline{u'v'}}{(\partial \bar{u}/\partial y)}$$

Turbulent μ

$\frac{\mu}{\rho}$: Kinematic ~~diffusivity~~ viscosity or M^2 diffusivity

$\frac{\mu_T}{\rho}$: Eddy diffusivity = turbulent μ → does not depend on ρ. system parameters. it depends on process parameters

higher fluctuations → higher turbulent stress

11/10/23

$\tau_E = \tau_V + \tau_T$

$$\Rightarrow \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{w}}{\partial z} = -\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial x}\left(\mu\frac{\partial \bar{u}}{\partial x} - \rho\overline{u'^2}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'}\right)$$
$$+ \frac{\partial}{\partial z}\left(\mu\frac{\partial \bar{u}}{\partial z} - \rho\overline{u'w'}\right)$$

5 eq^{n}s & 7 variables (3 mean comp. of vel. 3 fluctuating comp. → $\bar{p} + p'$))

(3 comp. bal. + 2 continuity eq^{n}s)

In the Naviers Eq^n, when we incorporated the stress → cannot be solved; To make it solvable → we correlated the stress terms to vel. gradients.

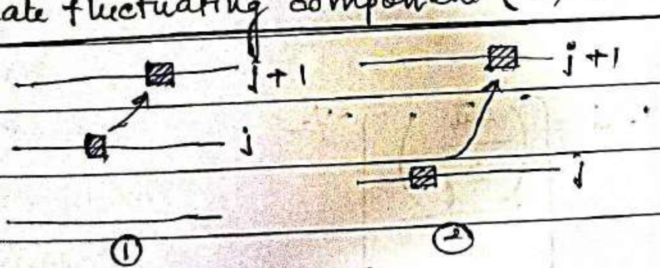
Correlate $\bar{u}$ & u' :—

Intensity of Turbulence (for 1-D flow predominantly) →

$$I = \frac{\sqrt{\frac{1}{3}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)}}{u_\infty} \quad ; \quad u_\infty = \text{freestream vel.}$$

special case :— $|u'| \simeq |v'| \simeq |w'| \Rightarrow$ extent of fluctuations is same in all 3-dir^{n}s → Isotropic Turbulence (mostly all natural turbulences are isotropic)

* Relate fluctuating component (u') to mean velocity ($\bar{u}$) :—

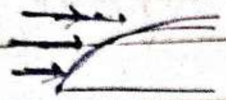


in 1st case the magnitude of fluctuation is less.

fluctuation is associated with vel. gradient.

$$\frac{\partial \bar{u}}{\partial y} = f(\bar{u}) \rightarrow \boxed{\bar{u} \propto \frac{\partial \bar{u}}{\partial y}} \Rightarrow \bar{u} = l\frac{\partial \bar{u}}{\partial y}.$$

bcoz of fluctuation a fluid particle goes from jth level to j+1 level. local vel. at jth level is dropping and local vel. at j+1 level is ↑. Thus this fluctuation is causing local vel. gradient. Thus fluctuation can be said to be a f^n of local vel. gradient & this proportionality is linear in case ② ∵ the intensity is higher in 2nd case the particle is moving by a higher distance. This distance is captured by l. It is called Prandtl Mixing Length. Higher is the l, higher are you ~~tak~~ taking the fluid particle from its original level (the extent of particle shift is more) (shift ↑ l ↑)



1D flow (Isotropic Turbulence)

$$u' = l\frac{\partial \bar{u}}{\partial y}$$

* All the analysis we are doing we are considering stationary & isotropic turbulence

Expression of v'? — $v' = -l\left(\frac{\partial \bar{u}}{\partial y}\right)$ → (isotropic, u & v always have opposite signs)

$$u'v' = -l^2\left(\frac{\partial \bar{u}}{\partial y}\right)^2$$

$$-\rho\overline{u'v'} = \tau_{xy_T} = -\rho l^2\left(\frac{\partial \bar{u}}{\partial y}\right)^2$$

$$\Rightarrow -\rho\overline{u'v'} = \mu_T\left(\frac{\partial \bar{u}}{\partial y}\right) = \rho l^2\left(\frac{\partial \bar{u}}{\partial y}\right)^2$$

$$\rho l^2\left(\frac{\partial \bar{u}}{\partial y}\right)^2 = \mu_T\left(\frac{\partial \bar{u}}{\partial y}\right)$$

$$l^2 = \frac{\left(\frac{\mu_T}{\rho}\right)}{\left(\frac{\partial \bar{u}}{\partial y}\right)} = \frac{\epsilon}{\left(\frac{\partial \bar{u}}{\partial y}\right)} \qquad \therefore l = \left[\frac{\left(\frac{\partial \bar{u}}{\partial y}\right)}{\epsilon}\right]^{-1/2}$$

Von Karman Hypothesis :- Turbulent flow over a plate (flat)

$l = K \cdot y$ where l → Prandtl mixing length.

K = Von Karman const

y = Distance from the wall

It says when you move away from the wall, Prandtl mixing length ↑

Governing eqn for a Turbulent B.L. (1D flow over flat plate) —

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = \frac{1}{\rho}\frac{\partial \tau_{xy_E}}{\partial y} = \frac{\mu_E}{\rho}\frac{\partial^2 \bar{u}}{\partial y^2}$$ (use of order of magnitude analysis)

$\tau_w \rightarrow$ wall stress $= \frac{F}{A} = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$

$\rho = ML^{-3}$

$\frac{\tau_w}{\rho} = \frac{ML^{-1}T^{-2}}{ML^{-3}} = L^2T^{-2}$

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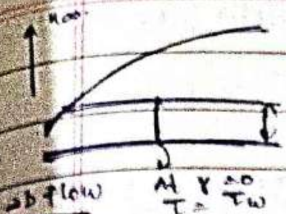

When a 1D flow hits flat plate, it becomes 2D inside the BL due to the presence of v along with u.

Very close to the wall :— $O(u) \simeq 0$, $O(v) \simeq 0$ — No slip + solid wall

$\frac{\mu_E}{\rho}\left(\frac{\partial^2 \bar{u}}{\partial y^2}\right) \simeq 0 \Rightarrow \frac{\partial}{\partial y}(\tau_{xy_E}) \simeq 0$ or within this zone $\rightarrow \tau_{xy_E} = $ const.

τ_{xy_E} does not change along y.

At $y=0$, $\tau = \tau_w$. shear stress across the entire layer will be $\tau_{xy_E} = \text{const} = \tau_w$ (very close to the wall)

$\tau_{xy_E} = \mu_E \frac{\partial \bar{u}}{\partial y} = (\mu + \mu_T)\frac{\partial \bar{u}}{\partial y} \Rightarrow (\mu + \mu_T)\frac{\partial \bar{u}}{\partial y} = \tau_w \Rightarrow (\nu + \epsilon)\frac{\partial \bar{u}}{\partial y} = \frac{\tau_w}{\rho}$

Dimension of $\frac{\tau_w}{\rho} = m^2/s^2 = \frac{kg_f/m \cdot s^2}{kg_m/m^3} = \frac{kg_f}{m \cdot s^2} \frac{m^3}{kg_m}$

kg_f = kg force
kg_m = kg mass

$\rho_{eq} \rightarrow$ body force acting (g) in opp dirn

Marks $\rightarrow$ capillary force acting.

g° From 2nd Law :— $F \propto ma$

$F = kma$.

For all problem solving we consider unit force k=1

However, $F = \frac{1}{g_c} ma$ $\quad [kg_f] = \frac{1}{g_c} \cdot \frac{kg_m m}{s^2} \rightarrow [g_c] = \frac{kg_m \cdot m}{kg_f \, s^2}$

interrelation b/w kg force & kg mass :— $[g_c] = \frac{kg_m \, m}{kg_f \, s^2}$

So $\rightarrow \frac{\tau_w}{\rho} = \frac{kg_m \cdot \cancel{m}}{\cancel{s^2}} \times \frac{\cancel{m^3}\, m^2}{\cancel{m^2} s^2 \, kg_m} = \frac{m^2}{s^2}$

u_* or $u_\tau = \left(\frac{\tau_w}{\rho}\right)^{1/2}$ friction velocity

$u^+ = \frac{\bar{u}}{u^*}$, $v^+ = \frac{\bar{v}}{v^*}$, $x^+ = \frac{x u^*}{\nu}$, $y^+ = \frac{y u^*}{\nu}$

consequently, $\left(1 + \frac{\epsilon}{\nu}\right)\frac{du^+}{dy^+} = 1$.

$\frac{\partial u^+}{\partial y^+} = \left(\frac{\partial u^+}{\partial y}\right)\left(\frac{\partial y}{\partial y^+}\right) = \frac{\partial}{\partial y}\left(\frac{\bar{u}}{u^*}\right)\frac{\nu}{u^*} = \frac{\partial \bar{u}}{\partial y} \cdot \frac{\nu}{u^{*2}}$

$\Rightarrow \frac{\partial \bar{u}}{\partial y} = \frac{u_*^2}{\nu}\frac{\partial u^+}{\partial y^+}$

Subst. in eqn :— $(\nu + \epsilon)\frac{\cancel{u_*^2}}{\nu}\frac{\partial u^+}{\partial y^+} = \frac{\tau_w}{\rho} = \cancel{u_*^2} \Rightarrow \left(1 + \frac{\epsilon}{\nu}\right)\frac{du^+}{dy^+} = 1$

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$\nu \rightarrow$ system parameter

$\epsilon \rightarrow$ process parameter

$\epsilon = f(y) \rightarrow$ so we cannot consider it a const. We can split the eqⁿ into 2 asymptotic parts. $(\epsilon/\nu) \rightarrow$ (Turbulent/shear) viscosity.

$\epsilon \rightarrow$ depends on extent of fluctuation.

Based on the assumpⁿ, $\bar{u}, \bar{v} = 0$ (close to the wall) (scaling analysis)

$\therefore$ No vel. very close to the wall, $(u', v' \simeq 0) \Rightarrow \dfrac{\epsilon}{\nu} \xrightarrow{\text{tends to}} 0$

2 asymptotic cases :- Discrepancy: considerⁿ within a region we have considered bulk flow = 0

$\epsilon << \nu$ (close to the wall)

$$\frac{\partial u^+}{\partial y^+} = 1 \Rightarrow \boxed{u^+ = y^+} \rightarrow \text{viscous sub layer}$$

(vel. profile close to the wall in wall coordinate system)

$\bar{u} = 0, \bar{v} = 0$

y^+_{VSL}, τ_w, $\epsilon << \nu$

$u' \simeq 0, v' \simeq 0$

$$\frac{du^+}{dy^+} = 1 \Rightarrow \int_0^{u^+} du^+ = \int_0^{y^+_{VSL}} dy^+ \quad \text{valid till } y^+ = y^+_{VSL}$$

$\epsilon >> \nu$ (away from the wall) → fluctuations dominating

y^+_{VSL} — $y \rightarrow \epsilon >> \nu$ — $\dfrac{\epsilon}{\nu} >> 1$

y viscous sublayer

$$\frac{\epsilon}{\nu}\frac{du^+}{dy^+} = 1$$

$$\epsilon = \underbrace{k^2 y^2}_{l^2}\left(\frac{\partial \bar{u}}{\partial y}\right)$$

$$\epsilon = k^2 y^{+2} \frac{\nu^2}{u^{*2}} \left(\frac{\partial \bar{u}}{\partial y}\right)$$

$$\frac{\epsilon}{\nu} = k^2 y^2 \frac{\partial u^+}{\partial y^+}$$

$$\epsilon = k^2 \left(\frac{y^+}{u^*}\right)^2 \frac{\nu^2 (u^*)^2}{\nu} \frac{\partial u^+}{\partial y^+}$$

$$\epsilon = k^2 \nu y^{+2} \left(\frac{\partial u^+}{\partial y^+}\right)$$

$$\Rightarrow k^2 y^{+2} \left(\frac{\partial u^+}{\partial y^+}\right)\left(\frac{\partial u^+}{\partial y^+}\right) = 1$$

$$\Rightarrow k y^+ \frac{\partial u^+}{\partial y^+} = 1 \rightarrow u^+ = \frac{1}{k} \ln y^+ + c$$

starts from edge of v_{SL} where $u^+ = y^+_{VSL}$

for $y^+ = y^+_{VSL}$, $u^+ = y^+_{VSL}$

$$c = y^+_{VSL} - \frac{1}{k} \ln y^+_{VSL}$$

$$u^+ = \frac{1}{k} \ln y^+ + y^+_{VSL} - \frac{1}{k} \ln y^+_{VSL}$$