Mid sem Solution
(CH 61015)

#1.

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

for non-trivial solution: $\lambda = \alpha^2 = +ve$

$$\frac{d^2y}{dx^2} + \alpha^2 y = 0$$

soln: $y(x) = c_1 \sin(\alpha x) + c_2 \cos(\alpha x)$

at $y=0$, $\frac{dy}{dx} + \beta_1 y = 0$

$$\Rightarrow c_1\alpha + \beta_1 c_2 = 0 \quad \ldots (1)$$

$$\Rightarrow c_2 = -c_1 \frac{\alpha}{\beta_1}$$

at $y=1$, $\frac{dy}{dx} + \beta_2 y = 0$

$$\Rightarrow c_1\alpha - c_2\alpha \tan\alpha + \beta_2 c_1 \tan\alpha + \beta_2 c_2 = 0 \quad \ldots (2)$$

From (1) & (2):

$$\boxed{\tan\alpha_n + \frac{\alpha_n(\beta_1 - \beta_2)}{\beta_1\beta_2 + \alpha_n^2} = 0}$$

Eigenvalues are roots of above transcendat equation

Eigenfunction: $y(x) = c_1 \sin(\alpha_n x) - \frac{c_1 \alpha_n}{\beta_1} \cos(\alpha_n x)$

$$= \frac{c_1}{\beta_1}\left(\beta_1 \sin(\alpha_n x) - \alpha_n \cos(\alpha_n x)\right)$$

$$= \frac{c_1}{\beta_1}\left[A\cos\gamma \sin(\alpha_n x) - A\sin\gamma \cos(\alpha_n x)\right]$$

$A\cos\gamma = \beta_1$

$A\sin\gamma = \alpha_n$

$A = \sqrt{\alpha_n^2 + \beta_1^2}$

$\gamma = \tan^{-1}(\alpha_n/\beta_1)$

$$= \frac{c_1}{\beta_1} A \sin(\alpha_n x - \gamma)$$

$$= c_1' A \sin(\alpha_n x - \gamma)$$

Where, $c_1' = c_1/\beta_1$; $A = \sqrt{\alpha_n^2 + \beta_1^2}$; $\gamma = \tan^{-1}(\alpha_n/\beta_1)$.

#2 $Lu = 5\dfrac{d^2u}{dx^2} + x\dfrac{du}{dx} + u$

at $x=0$, $\dfrac{du}{dx} + u = 0$; at $x=1$, $\dfrac{du}{dx} - 3u = 0$

$$\langle v, Lu\rangle = \int_0^1 5\frac{d^2u}{dx^2} v\,dx + \int_0^1 x\frac{du}{dx} v\,dx + \int_0^1 uv\,dx$$

$$= J(u,v) + \int_0^1 \left(5\frac{d^2v}{dx^2} - x\frac{dv}{dx}\right) u\,dx$$

$$= J(u,v) + \langle L^* v, u\rangle$$

$$\therefore \boxed{L^* = 5\frac{d^2}{dx^2} - x\frac{d}{dx}}$$

$L \neq L^*$

After simplification

$$J(u,v) = 5v(1)u'(1) - 5u(1)v'(1) - 5v(0)u'(0) + v(1)u(1) + 5u(0)v'(0)$$

using $u'(0) = -u(0)$; $u'(1) = 3u(1)$

$$J(u,v) = u(1)\left[16v(1) - 5v'(1)\right] + 5u(0)\left[v(0) + v'(0)\right]$$

B^*: at $x=0$, $\dfrac{dv}{dx} + v = 0$

$x=1$, $5\dfrac{dv}{dx} - 16v = 0$

$B \neq B^*$

#3 $\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$

$$\Rightarrow \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

Define $\theta = \frac{T - T_\infty}{T_0 - T_\infty}$; $x^* = x/L$

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial x^{*2}}$$

at $\tau = 0$, $\theta = 1$

at $x^* = 0$, $\frac{\partial \theta}{\partial x^*} + Bi\, \theta = 0$; $Bi = \frac{hL}{k}$ = Biot No.

at $x^* = 1$, $\frac{\partial \theta}{\partial x^*} = 0$

Define: $y = 1 - x^*$

$$\therefore \frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial y^2}$$

at $\tau = 0$, $\theta = 1$

at $y = 0$, $\frac{\partial \theta}{\partial y} = 0$; at $y = 1$, $\frac{\partial \theta}{\partial y} - Bi\,\theta = 0$

$$\theta = T(\tau)\, Y(y)$$

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{Y}\frac{d^2 Y}{dy^2} = -\alpha^2$$ for non-trivial solution

$\frac{d^2 Y}{dy^2} + \alpha^2 Y = 0$ subj to at $y = 0$, $\frac{dY}{dy} = 0$

at $y = 1$, $\frac{dY}{dy} - Bi\,Y = 0$

Eigenvalues: Roots of $\boxed{\alpha_n \tan \alpha_n + Bi = 0}$

Eigenfunctions: $Y_n = C_1 \cos(\alpha_n y)$

$$\theta(y, \tau) = \sum_{n=1}^{\infty} C_n \exp(-\alpha_n \tau) \cos(\alpha_n y)$$

at $\tau = 0$, $\theta = 1$ $\Rightarrow 1 = \sum_{n=1}^{\infty} C_n \cos(\alpha_n y)$

Using orthogonal properties of eigenfunctions:

$$C_n = \frac{\int_0^1 \cos \alpha_n y \, dy}{\int_0^1 \cos^2(\alpha_n y)\, dy} = 2\left(\frac{\sin \alpha_n}{\alpha_n}\right) \frac{\alpha_n^2 + Bi^2}{\alpha_n^2 - Bi + Bi^2}$$

$$\boxed{\therefore \theta(x^*, \tau) = 2 \sum_{n=1}^{\infty} \left(\frac{\sin \alpha_n}{\alpha_n}\right)\left(\frac{\alpha_n^2 + Bi^2}{\alpha_n^2 - Bi + Bi^2}\right) \exp(-\alpha_n \tau) \cos[\alpha_n (1 - x^*)]}$$