A

Differential Geometry[1]

Our objective in the text is to describe momentum, energy, and mass transfer in real multiphase materials. Upon a little thought, it quickly becomes apparent that one of our first requirements is a mathematical representation for the space in which the physical world around us is situated. There are at least two features of our physical space that we wish to retain: length and relative direction. A Euclidean space incorporates both of these properties.

We assume here that the reader is familiar with tensor analysis. If he is not, we suggest that he first stop and read the introduction given by Slattery [42], by Leigh [665], or by Coleman et al. [447]. We also have found helpful the discussions by Ericksen [40] and by McConnell [73].

In this introduction to differential geometry, we shall confine our attention to scalar, vector, and tensor fields on or evaluated at surfaces in a Euclidean space. Much of this material is drawn from McConnell [73]. My primary modification has been to explicitly recognize the role of various bases in the vector space. We have employed wherever possible a coordinate-free notation, which we feel is a little easier on the eyes.

A.1 Physical Space

A.1.1 Euclidean Space

Let E^n be a set of elements a, b, ..., which we will refer to as **points**. Let V^n be a real, n-dimensional inner product space (linear vector space upon which an inner product is defined). We can define a relation between points and vectors in the following way.

1. To every ordered pair (a, b) of points in E^n there is assigned a vector of V^n called the **difference vector** and denoted by

[1]Based in part on work by Lifshutz [664].

$$\mathbf{ab}$$

2. If o is an arbitrary point in E^n, then to every vector $\mathbf{a}$ of V^n there corresponds a unique point such that

$$\mathbf{oa} = \mathbf{a}$$

3. If a, b and c are three arbitrary points in E^n, then

$$\mathbf{ab} + \mathbf{bc} = \mathbf{ac} \tag{A.1.1-1}$$

We refer to V^n as the **translation space** corresponding to E^n; the couple (E^n, V^n) is a **Euclidean space.**

The distance between any two points a and b is defined by

$$d\,(a, b) \equiv |\mathbf{ab}| \equiv (\mathbf{ab} \cdot \mathbf{ab})^{1/2} \tag{A.1.1-2}$$

We use the notation $(\mathbf{a} \cdot \mathbf{b})$ for the inner product of two vectors $\mathbf{a}$ and $\mathbf{b}$ that belong to V^n.

We represent the physical space in which we find ourselves by the three-dimensional Euclidean space (E^3, V^3). This is the only Euclidean space with which we shall be concerned.

Our primary concern in the text is with surfaces in this Euclidean space. Many of our discussions will focus on $V^2(\mathbf{z})$, the subspace of V^3 composed of the vectors in V^3 that are tangent to a surface at some point $\mathbf{z}$. We would like to emphasize that V^2 will in general be a function of position on the surface. If the surface in question is moving and deforming as a function of time, V^2 will be a function of both time and position.

One's first thought might be that these surfaces can themselves be thought of as Euclidean spaces of the form (E^2, V^3), where E^2 is a subset of E^3 corresponding to points on the surface. This certainly works well for planes; planes are Euclidean spaces. But curved surfaces are nonEuclidean, because the difference vectors are not tangent to the surface and do not belong to V^2 evaluated at either of the points defining the difference. The distance between two points measured along a curved surface is in general not equal to the distance between these same two points measured in the three-dimensional Euclidean space (E^3, V^3).

We could associate a two-dimensional Euclidean space (plane) with a curved surface in much the same manner as we prepare maps of the world. Such a space has the unpleasant feature that distance measured in the plane does not correspond to distance measured either along the curved surface or to distance measured in (E^3, V^3). This objection outweighs the potential usefulness of such a map in our discussion.

This viewpoint of a Euclidean space is largely drawn from Greub [666, p. 282] and Lichnerowicz [667, p. 24], both of which we recommend for further reading.

A.1.2 Notation in (E^2, V^3)

This discussion of differential geometry is based upon a previous treatment of tensor analysis [42, Appendix A]. The definitions and notation introduced there for (E^3, V^3) are certainly necessary here. While we must refer the reader to the original development for many points, we thought it might be helpful to review some of the most basic notation before going any further.

The inner product space V^3 will be referred to as the space of **spatial vectors**.

We find it convenient to choose some point O in E^3 as a reference point or **origin** and to locate all points in E^3 relative to O. Instead of referring to the point z, we will use the **position vector**

$$\mathbf{z} \equiv \mathbf{Oz}$$

Temperature, concentration, and pressure are examples of real numerically valued functions of position. We call any real numerically valued function of position a **real scalar field**.

When we think of water flowing through a pipe or in a river, we recognize that the velocity of water is a function of position. At the wall of the pipe, the velocity of the water is zero; at the center, it is a maximum. the velocity of the water in the pipe is an example of a spatial vector-valued function of position. We shall term any spatial vector-valued function a **spatial vector field**.

As another example, consider the **position vector field** $\mathbf{p}(z)$. It maps every point z of E^3 into the corresponding position vector $\mathbf{z}$ measured with respect to a previously chosen origin O:

$$\mathbf{z} = \mathbf{p}(z) \tag{A.1.2-1}$$

Every spatial vector field $\mathbf{u}$ may be written as a linear combination of rectangular cartesian basis field $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ [42, p. 624]:

$$\begin{aligned} \mathbf{u} &= u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 \\ &= \sum_{i=1}^{3} u_i\mathbf{e}_i \\ &= u_i\mathbf{e}_i \end{aligned} \tag{A.1.2-2}$$

A special case is the position vector field

$$\begin{aligned} \mathbf{p} &= z_1\mathbf{e}_1 + z_2\mathbf{e}_2 + z_3\mathbf{e}_3 \\ &= \sum_{i=1}^{3} z_i\mathbf{e}_i \\ &= z_i\mathbf{e}_i \end{aligned} \tag{A.1.2-3}$$

The rectangular cartesian components $\{z_1, z_2, z_3\}$ of the position vector field $\mathbf{p}$ are called the **rectangular cartesian coordinates** with respect to the previously chosen origin O. They are one-to-one functions of position z in E^3:

$$z_i = z_i(z) \qquad \text{for } i = 1, 2, 3 \tag{A.1.2-4}$$

For this reason we will often find it convenient to think of $\mathbf{p}$ as being a function of the rectangular cartesian coordinates:

$$\mathbf{z} = \mathbf{p}(z_1, z_2, z_3) \tag{A.1.2-5}$$

Note that (A.1.2-2$_3$) and (A.1.2-3$_3$) we have employed the summation convention [42, p. 626]. We find it convenient to use this convention hereafter.

Let us assume that each $z_i (i = 1, 2, 3)$ may be regarded as a function of three parameters $\{x^1, x^2, x^3\}$ called **curvilinear coordinates**:

$$z_i = z_i\left(x^1, x^2, x^3\right) \qquad \text{for } i = 1, 2, 3 \tag{A.1.2-6}$$

Here we use the common notation preserving device of employing the same symbol for both the function z_i and its value $z_i(x^1, x^2, x^3)$. So long as

$$\det\left(\frac{\partial z_i}{\partial x^j}\right) = \begin{vmatrix} \frac{\partial z_1}{\partial x^1} & \frac{\partial z_1}{\partial x^2} & \frac{\partial z_1}{\partial x^3} \\ \frac{\partial z_2}{\partial x^1} & \frac{\partial z_2}{\partial x^2} & \frac{\partial z_2}{\partial x^3} \\ \frac{\partial z_3}{\partial x^1} & \frac{\partial z_3}{\partial x^2} & \frac{\partial z_3}{\partial x^3} \end{vmatrix} \neq 0 \tag{A.1.2-7}$$

we can solve (A.1.2-6) for the x^i to find

$$x^i = x^i(z_1, z_2, z_3) \qquad \text{for } i = 1, 2, 3 \tag{A.1.2-8}$$

This means that for each set $\{x^1, x^2, x^3\}$ there is a unique set $\{z_1, z_2, z_3\}$ and vice versa. Consequently, each set $\{x^1, x^2, x^3\}$ determines a point in space.

Setting $x^1 =$ constant in (A.1.2-6) gives us a family of surfaces, one member corresponding to each value of the constant. Similarly, $x^2 =$ constant and x^3 $=$ constant define two other families of surfaces. We will refer to all of these surfaces as coordinate surfaces. The line of intersection of two coordinate surfaces defines a coordinate curve. Because of (A.1.2-7), three coordinate surfaces obtained by taking a member from each family intersect in one and only one point.

Every spatial vector field $\mathbf{u}$ may be written as

$$\mathbf{u} = u^i \mathbf{g}_i \tag{A.1.2-9}$$

$$\mathbf{u} = u_i \mathbf{g}^i \tag{A.1.2-10}$$

or in the case of an orthogonal coordinate system

$$\mathbf{u} = u_{<i>}\mathbf{g}_{<i>} \tag{A.1.2-11}$$

Here $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ are the **natural** basis fields

$$\mathbf{g}_i \equiv \frac{\partial \mathbf{p}}{\partial x^i} \tag{A.1.2-12}$$

We refer to $\mathbf{g}^1$, $\mathbf{g}^2$, $\mathbf{g}^3$ as the **dual** basis fields,

$$\mathbf{g}^i \equiv \nabla x^i \tag{A.1.2-13}$$

For orthogonal coordinate systems, we define the **physical** basis fields $\{\mathbf{g}_{<1>}, \mathbf{g}_{<2>}, \mathbf{g}_{<3>}\}$ in this way:

$$\mathbf{g}_{<i>} \equiv \frac{\mathbf{g}_i}{\sqrt{g_{ii}}} = \sqrt{g_{ii}}\mathbf{g}^i \quad \text{(no sum on i)} \tag{A.1.2-14}$$

We say that the u^i ($i = 1, 2, 3$) are the **contravariant** components of $\mathbf{u}$; u_i the **covariant** components of $\mathbf{u}$; $u_{<i>}$ the **physical** components of $\mathbf{u}$. The scalar fields

$$g_{ij} \equiv \mathbf{g}_i \cdot \mathbf{g}_j \tag{A.1.2-15}$$

are the covariant components of the identity tensor [42, p. 646]; $\sqrt{g_{ii}}$ (no sum on i) is the magnitude of $\mathbf{g}_i$.

A **second-order spatial tensor** field $\mathbf{T}$ is a transformation (or mapping or rule) that assigns to each given spatial vector field $\mathbf{v}$ another spatial vector field $\mathbf{T} \cdot \mathbf{v}$ such that the rules

$$\mathbf{T} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{T} \cdot \mathbf{v} + \mathbf{T} \cdot \mathbf{w} \tag{A.1.2-16}$$

$$\mathbf{T} \cdot (\alpha \mathbf{v}) = \alpha (\mathbf{T} \cdot \mathbf{v}) \tag{A.1.2-17}$$

hold. By α we mean here a real scalar field. [Here the dot denotes that the tensor $\mathbf{T}$ operates on or transforms the vector $\mathbf{v}$. It does not indicate a scalar product. Our choice of notation is suggestive, however, of the rules for transformation in (A.1.2-18) and (A.1.2-21).] If two spatial vector fields $\mathbf{a}$ and $\mathbf{b}$ are given, we can define a second-order tensor field $\mathbf{ab}$ by the requirement that it transform every vector field $\mathbf{v}$ into another field $(\mathbf{ab}) \cdot \mathbf{v}$ according to the rule

$$(\mathbf{ab}) \cdot \mathbf{v} \equiv \mathbf{a} (\mathbf{b} \cdot \mathbf{v}) \tag{A.1.2-18}$$

(On the other side, the dot indicates the transformation of a vector by a tensor; on the right, the dot denotes the scalar product of $\mathbf{b}$ with $\mathbf{v}$.) this tensor field $\mathbf{ab}$ is called the **tensor product** or dyadic product of the spatial vector fields **a** and **b**. Every second-order spatial tensor field **T** can be written as a linear combination of tensor products:

$$\begin{aligned}\mathbf{T} &= T^{ij}\mathbf{g}_i\mathbf{g}_j = T_{ij}\mathbf{g}^i\mathbf{g}^j \\ &= T^i{}_j\mathbf{g}_i\mathbf{g}^j = \mathbf{T}_i{}^j\mathbf{g}^i\mathbf{g}_j\end{aligned} \tag{A.1.2-19}$$

For an orthogonal coordinate system, we can write

$$\mathbf{T} = T_{<ij>}\mathbf{g}_{<i>}\mathbf{g}_{<j>} \tag{A.1.2-20}$$

We refer to the T^{ij} $(i,\ j = 1,\ 2,\ 3)$ as the **contravariant** components of **T**; T_{ij} the **covariant** components of **T**; $T^i{}_j$ and $T_i{}^j$ the **mixed** components of **T**; $T_{<ij>}$ the **physical** components of **T**. In terms of these components, the vector $\mathbf{T} \cdot \mathbf{v}$ takes the form

$$\begin{aligned}\mathbf{T}\cdot\mathbf{v} &= \left(T^{ij}\mathbf{g}_i\mathbf{g}_j\right)\cdot\mathbf{v} \\ &= T^{ij}\mathbf{g}_i\left(\mathbf{g}_j\cdot\mathbf{v}\right) \\ &= T^{ij}v_j\mathbf{g}_i \\ &= T_{ij}v^j\mathbf{g}^i \\ &= T^i{}_jv^j\mathbf{g}_i \\ &= T_i{}^jv_i\mathbf{g}^i\end{aligned} \tag{A.1.2-21}$$

A **third-order tensor** field $\boldsymbol{\beta}$ is a transformation (or mapping or rule) that assigns to each given spatial vector field **v** a second-order tensor field $\boldsymbol{\beta}\cdot\mathbf{v}$ such that the rules

$$\boldsymbol{\beta}\cdot(\mathbf{v}+\mathbf{w}) = \boldsymbol{\beta}\cdot\mathbf{v} + \boldsymbol{\beta}\cdot\mathbf{w} \tag{A.1.2-22}$$

$$\boldsymbol{\beta}\cdot(\alpha\mathbf{v}) = \alpha\left(\boldsymbol{\beta}\cdot\mathbf{v}\right) \tag{A.1.2-23}$$

hold. Again α is a real scalar field. (The dot denotes that the tensor $\boldsymbol{\beta}$ operates on, or transforms, the vector **v**. This is not a scalar product between two vectors.) In this text, we deal with only one third-order spatial tensor field

$$\boldsymbol{\varepsilon} = \varepsilon^{ijk}\mathbf{g}_i\mathbf{g}_j\mathbf{g}_k = \varepsilon_{ijk}\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k \tag{A.1.2-24}$$

where

$$\varepsilon^{ijk} \equiv \frac{1}{\sqrt{g}}e^{ijk} \tag{A.1.2-25}$$

$$\varepsilon_{ijk} \equiv \sqrt{g}\,e_{ijk} \tag{A.1.2-26}$$

$$g \equiv \det\left(g_{ij}\right) \tag{A.1.2-27}$$

and $e^{ijk} = e_{ijk}$ have only three distinct values:

0, when any two of the indices are equal

$+1$, when ijk is an even permutation of 1 2 3

-1, when ijk is an odd permutation of 1 2 3

This tensor is used in forming the **vector product** of two spatial vector fields $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{a} \times \mathbf{b} \equiv \boldsymbol{\varepsilon}{:}\mathbf{ba} \equiv (\boldsymbol{\varepsilon} \cdot \mathbf{b}) \cdot \mathbf{a} = \varepsilon^{ijk} a_j b_k \mathbf{g}_i \tag{A.1.2-28}$$

It is also employed in the definition for the **curl** of a spatial vector field $\mathbf{v}$,

$$\operatorname{curl} \mathbf{v} \equiv \boldsymbol{\varepsilon}{:}\nabla \mathbf{v} = \varepsilon^{ijk} v_{k,j} \mathbf{g}_i \tag{A.1.2-29}$$

where $v_{k,j}$ is referred to as the **covariant derivative** of v_k, the covariant component of the spatial vector field $\mathbf{v}$.

For more details, we suggest the treatments by Slattery [42], Leigh [665], Coleman et al. [447], Ericksen [40], and McConnell [73].

A.1.3 Surface in (E^3, V^3)

As illustrated in Fig. A.1.3-1, a surface in (E^3, V^3) is the locus of a point whose position is a function of two parameters y^1 and y^2:

$$\mathbf{z} = \mathbf{p}^{(\sigma)} \left(y^1, y^2\right) \tag{A.1.3-1}$$

Since the two numbers y^1 and y^2 uniquely determine a point on the surface, we call them the **surface coordinates**. A y^1 coordinate curve is a line in the surface along which y^1 varies while y^2 takes a fixed value. Similarly, a y^2 coordinate curve is one along which y^2 varies while y^1 assumes a constant value.

For any surface, there is an infinite number of surface coordinate systems that might be used. Any two families of lines may be chosen as coordinate curves, so long as each member of one family intersects each member of the other in one and only one point.

Equation (A.1.3-1) represents three scalar equations. If we eliminate the surface coordinates y^1 and y^2 among the three components of (A.1.3-1), we are left with one scalar equation of the form

$$f(\mathbf{z}) = 0 \tag{A.1.3-2}$$

It is sometimes more convenient to think in terms of this single scalar equation for the surface.

A.2 Vector Fields

A.2.1 Natural Basis

A spatial vector field has been previously defined as any spatial vector-valued function of position [42, p. 621]. In the present context, we are concerned with spatial vector-valued functions of position on a surface.

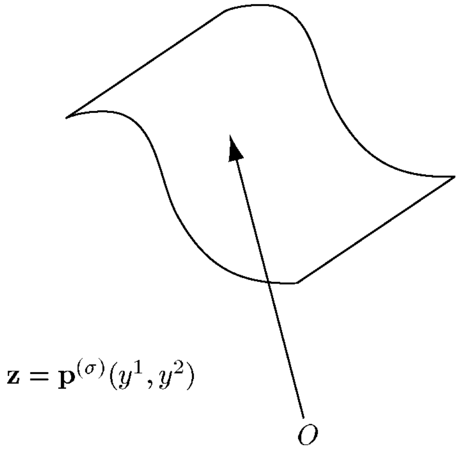

Fig. A.1.3-1. A surface in (E^3, V^3) is the locus of a point whose position is a function of two parameters y^1 and y^2

Referring to the parametric representation for a surface in Sect. A.1.3, we define

$$\mathbf{a}_\alpha \equiv \frac{\partial \mathbf{p}^{(\sigma)}}{\partial y^\alpha} = \frac{\partial x^i}{\partial y^\alpha} \mathbf{g}_i \tag{A.2.1-1}$$

At every point on the surface, the values of these spatial vector fields are tangent to the y^α coordinate curves and therefore tangent to the surface. These two spatial vector fields are also linearly independent; at no point is a member of one family of surface coordinate curves allowed to be tangent to a member of the other family. Note that the definition for $\mathbf{a}_\alpha$ parallels that for the natural basis vector field $\mathbf{g}_k$ for V^3 [42, p. 633; see also Sect. A.1.2].

Let $\boldsymbol{\xi}$ be the unit normal to the surface [42, p. 632]. Since the vector fields $\mathbf{a}_1$, $\mathbf{a}_2$, and $\boldsymbol{\xi}$ are linearly independent, they form a basis for the spatial vector fields on the surface. We are particularly interested in the **tangential vector fields**, tangent vector-valued functions of position on a surface. The tangential vector fields are a two-dimensional subspace of the spatial vector fields on a surface in the sense that every tangential vector field $\mathbf{c}$ can be expressed as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$:

$$\begin{aligned} \mathbf{c} &= c^1 \mathbf{a}_1 + c^2 \mathbf{a}_2 \\ &= \sum_{\alpha=1}^{2} c^\alpha \mathbf{a}_\alpha \\ &= c^\alpha \mathbf{a}_\alpha \end{aligned} \tag{A.2.1-2}$$

(Note that we have introduced here a modification of the summation convention. A repeated Greek index in any term is to be summed from 1 to 2. A repeated italic index in any term will continue to indicate a summation from 1 to 3.) In what follows, $\{\mathbf{a}_1, \mathbf{a}_2\}$ are known as the **natural basis** for the surface coordinate system $\{y^1, y^2\}$.

Since it occurs quite frequently, let us define

$$\begin{aligned} a_{\alpha\beta} &\equiv \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \\ &= \frac{\partial \mathbf{p}^{(\sigma)}}{\partial y^\alpha} \cdot \frac{\partial \mathbf{p}^{(\sigma)}}{\partial y^\beta} \\ &= g_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} \end{aligned} \tag{A.2.1-3}$$

We also will be concerned with the determinant

$$a \equiv \det\left(a_{\alpha\beta}\right) \tag{A.2.1-4}$$

which has as its typical entry $a_{\alpha\beta}$. Expanding by columns, we find [73, p. 10; 42, p. 627]

$$e^{\alpha\beta} a_{\alpha\mu} a_{\beta\nu} = a\, e_{\mu\nu} \tag{A.2.1-5}$$

An expansion by rows gives

$$e^{\alpha\beta} a_{\mu\alpha} a_{\nu\beta} = a e_{\mu\nu} \tag{A.2.1-6}$$

the symbols $e_{\alpha\beta}$ and $e^{\alpha\beta}$ have only three distinct values:

0, when the indices are equal,

$+1$, when $\alpha\beta$ is 12

-1, when $\alpha\beta$ is 21

If we define the normalized cofactor $a^{\alpha\beta}$ as

$$a^{\alpha\beta} \equiv \frac{1}{a} e^{\alpha\nu} e^{\beta\gamma} a_{\gamma\nu} \tag{A.2.1-7}$$

we see

$$a^{\alpha\beta} a_{\beta\gamma} = a_{\gamma\beta} a^{\beta\alpha} = \delta^\alpha_\gamma \tag{A.2.1-8}$$

Here δ^α_γ is the Kronecker delta.

Exercise A.2.1-1. *Transformation of surface coordinates* Let the $\mathbf{a}_\alpha$ ($\alpha = 1, 2$) be the natural basis vector fields associated with one surface coordinate system $\{y^1, y^2\}$. A change of surface coordinates is suggested:

$$y^\alpha = y^\alpha\left(\bar{y}^1, \bar{y}^2\right) \qquad \alpha = 1, 2$$

i) Prove that

$$\bar{\mathbf{a}}_\alpha = \frac{\partial y^\beta}{\partial \bar{y}^\alpha} \mathbf{a}_\beta$$

ii) Given one surface coordinate system $\{y^1, y^2\}$, we will admit $\{\bar{y}^1, \bar{y}^2\}$ as a new surface coordinate system, only if the corresponding natural basis vector fields $\{\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2\}$ actually form a basis for the tangential vector fields. They can be said to form a basis, if every tangential vector field can be represented as a linear combination of them [41, p. 14]. In particular, $\{\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2\}$ form a basis for the tangential vector fields, if the natural basis vector fields $\{\mathbf{a}_1, \mathbf{a}_2\}$ can be represented as a linear combination of them:

$$\mathbf{a}_\alpha = \frac{\partial \bar{y}^\beta}{\partial y^\alpha} \bar{\mathbf{a}}_\beta$$

Conclude that $\{\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2\}$ form a basis for the tangential vector fields, if everywhere

$$\det\left(\frac{\partial y^\alpha}{\partial \bar{y}^\beta}\right) \neq 0$$

For another point of view leading to the same conclusion, see a discussion of the implicit function theorem [668, p. 150].

Exercise A.2.1-2. If $\mathbf{c}$ is a tangential vector field, determine that

$$\mathbf{c} = c^\alpha \frac{\partial x^i}{\partial y^\alpha} \mathbf{g}_i$$

and that the contravariant spatial components are

$$c^i = \frac{\partial x^i}{\partial y^\alpha} c^\alpha$$

Exercise A.2.1-3. If the $\mathbf{a}_\alpha$ $(\alpha = 1, 2)$ are the natural basis vector fields associated with one surface coordinate system $\{y^1, y^2\}$ and if the $\bar{\mathbf{a}}_\beta$ $(\beta = 1, 2)$ are the natural basis vector fields associated with another surface coordinate system $\{\bar{y}^1, \bar{y}^2\}$, prove that

$$\det\left(\bar{a}_{\alpha\beta}\right) = \det\left(a_{\mu\nu}\right)\left[\det\left(\frac{\partial y^\gamma}{\partial \bar{y}^\delta}\right)\right]^2$$

Exercise A.2.1-4. *Plane surface* Given a rectangular cartesian coordinate system and a plane surface z_3 = a constant, let us choose

$$y^1 \equiv z_1$$
$$y^2 \equiv z_2$$

Prove that

$$a_{11} = \frac{1}{a^{11}} = a_{22} = \frac{1}{a^{22}} = 1$$
$$a_{12} = a_{21} = 0$$

and

$$a = 1$$

Exercise A.2.1-5. *Plane surface in polar coordinates* Given a cylindrical coordinate system

$$z_1 = x^1 \cos x^2 = r \cos\theta$$
$$z_2 = x^1 \sin x^2 = r \sin\theta$$
$$z_3 = x^3 \equiv z$$

and a plane surface $z_3 =$ a constant, let us choose

$$y^1 \equiv x^1 \equiv r$$
$$y^2 \equiv x^2 \equiv \theta$$

Prove that

$$a_{11} = \frac{1}{a^{11}} = 1$$
$$a_{22} = \frac{1}{a^{22}} = r^2$$
$$a_{12} = a_{21} = 0$$

and

$$a = r^2$$

Exercise A.2.1-6. *Alternative form for plane surface in polar coordinates [78]* Given the cylindrical coordinate system described in Exercise A.2.1-5 and a plane surface $\theta =$ a constant, let us choose

$$y^1 \equiv x^1 \equiv r$$
$$y^2 \equiv x^3 \equiv z$$

Prove that

$$a_{11} = \frac{1}{a^{11}} = 1$$
$$a_{22} = \frac{1}{a^{22}} = 1$$
$$a_{12} = a_{21} = 0$$

and

$$a = 1$$

Exercise A.2.1-7. *Cylindrical surface* Given the above cylindrical coordinate system described in Exercise A.2.1-5 and a cylindrical surface of radius R, let us choose

$$y^1 \equiv x^2 \equiv \theta$$
$$y^2 \equiv x^3 \equiv z$$

Prove that

$$a_{11} = \frac{1}{a^{11}} = R^2$$
$$a_{22} = \frac{1}{a^{22}} = 1$$
$$a_{12} = a_{21} = 0$$

and

$$a = R^2$$

Exercise A.2.1-8. *Spherical surface* Given a spherical coordinate system

$$z_1 = x^1 \sin x^2 \cos x^3 = r \sin\theta \cos\phi$$
$$z_2 = x^1 \sin x^2 \sin x^3 = r \sin\theta \sin\phi$$
$$z_3 = x^1 \cos x^2 = r \cos\theta$$

and a spherical surface of radius R, let us choose

$$y^1 \equiv x^2 \equiv \theta$$
$$y^2 \equiv x^3 \equiv \phi$$

Prove that

$$a_{11} = \frac{1}{a^{11}} = R^2$$
$$a_{22} = \frac{1}{a^{22}} = R^2 \sin^2\theta$$
$$a_{12} = a_{21} = 0$$

and

$$a = R^4 \sin^2\theta$$

Exercise A.2.1-9. *Two-dimensional waves* Given a rectangular cartesian coordinate system and a surface

$$z_3 = h(z_1, t)$$

let us choose

$$y_1 \equiv z_1$$
$$y^2 \equiv z_2$$

Prove that

$$a_{11} = 1 + \left(\frac{\partial h}{\partial z_1}\right)^2$$
$$a_{22} = 1$$
$$a_{12} = a_{21} = 0$$

and

$$a = 1 + \left(\frac{\partial h}{\partial z_1}\right)^2$$

(See also Exercises A.2.6-1 and A.5.3-10.)

Exercise A.2.1-10. *Axially symmetric surface in cylindrical coordinates* Given the cylindrical coordinate system described in Exercise A.2.1-5 and an axially symmetric surface

$$z = h(r)$$

let us choose

$$y^1 \equiv r$$
$$y^2 \equiv \theta$$

Prove that

$$a_{11} = 1 + \left(\frac{\partial h}{\partial r}\right)^2$$
$$a_{22} = r^2$$
$$a_{12} = a_{21} = 0$$

and

$$a = r^2 \left[1 + \left(\frac{\partial h}{\partial r}\right)^2\right]$$

(See also Exercises A.2.6-2 and A.5.3-11.)

Exercise A.2.1-11. *Alternative form for axially symmetric surface in cylindrical coordinates* Given the cylindrical coordinate system described in Exercise A.2.1-5 and an axially symmetric surface

$$r = c(z)$$

let us choose

$$y^1 = x^3 = z$$
$$y^2 = x^2 = \theta$$

Prove that

$$a_{11} = 1 + \left(\frac{\mathrm{d}c}{\mathrm{d}z}\right)^2$$
$$a_{22} = c^2$$
$$a_{12} = a_{21} = 0$$

and

$$a = c^2 \left[1 + \left(\frac{\mathrm{d}c}{\mathrm{d}z}\right)^2\right]$$

(See also Exercises A.2.6-3 and A.5.3-12.)

A.2.2 Surface Gradient of Scalar Field

We will frequently be concerned with scalar fields on a surface. It is easy, for example, to visualize temperature as a function of position on a phase interface. It is for this reason that we must be concerned with derivatives on a surface.

By analogy with a spatial gradient [42, p. 630], the **surface gradient** of a scalar field ϕ is a tangential vector field denoted by $\nabla_{(\sigma)}\phi$ and specified by defining its inner product with an arbitrary tangential vector field $\mathbf{c}$:

$$\begin{aligned}&\nabla_{(\sigma)}\phi\left(y^1,y^2\right)\cdot\mathbf{c}\\&\qquad\equiv \text{limit s} \rightarrow 0: \frac{1}{s}\{\phi\left(y^1+sc^1,y^2+sc^2\right)-\phi\left(y^1,y^2\right)\}\end{aligned}\tag{A.2.2-1}$$

Equation (A.2.2-1) may be rearranged into a more easily applied expression,

$$\begin{aligned}&\nabla_{(\sigma)}\phi\left(y^1,y^2\right)\cdot\mathbf{c}\\&\quad= \text{limit s} \rightarrow 0: \frac{1}{s}\{\phi\left(y^1+sc^1,y^2+sc^2\right)-\phi\left(y^1,y^2+sc^2\right)\}\\&\qquad+ \text{limit s} \rightarrow 0: \frac{1}{s}\{\phi\left(y^1,y^2+sc^2\right)-\phi\left(y^1,y^2\right)\}\\&\quad= c^1\,\text{limit}\, sc^1 \rightarrow 0: \frac{1}{sc^1}\{\phi\left(y^1+sc^1,y^2+sc^2\right)-\phi\left(y^1,y^2+sc^2\right)\}\\&\qquad+ c^2\,\text{limit}\, sc^2 \rightarrow 0: \frac{1}{sc^2}\{\phi\left(y^1,y^1+sc^2\right)-\phi\left(y^1,y^2\right)\}\\&\quad= c^1\frac{\partial\phi}{\partial y^1}\left(y^1,y^2\right)+c^2\frac{\partial\phi}{\partial y^2}\left(y^1,y^2\right)\\&\quad= c^\alpha\frac{\partial\phi}{\partial y^\alpha}\left(y^1,y^2\right)\end{aligned}\tag{A.2.2-2}$$

Since $\mathbf{c}$ is an arbitrary tangential vector field, take $\mathbf{c} = \mathbf{a}_\beta$:

$$\nabla_{(\sigma)}\phi\cdot\mathbf{a}_\beta = \frac{\partial\phi}{\partial y^\beta}\tag{A.2.2-3}$$

Since $\nabla_{(\sigma)}\phi$ is defined to be a tangential vector field, we conclude that

$$\nabla_{(\sigma)}\phi = \frac{\partial\phi}{\partial y^\alpha}a^{\alpha\beta}\mathbf{a}_\beta\tag{A.2.2-4}$$

In arriving at this result, you will find (A.2.1-8) helpful.

A.2.3 Dual Basis

The **dual** tangential vector fields $\mathbf{a}^\alpha$ ($\alpha = 1, 2$) are defined as the surface gradients of the surface coordinates,

$$\mathbf{a}^\alpha \equiv \nabla_{(\sigma)} y^\alpha \tag{A.2.3-1}$$

Applying the expression for the surface gradient developed in Sect. A.2.2, we find

$$\begin{aligned}\mathbf{a}^\alpha &= \frac{\partial y^\alpha}{\partial y^\beta} a^{\beta\gamma}\mathbf{a}_\gamma \\ &= a^{\alpha\gamma}\mathbf{a}_\gamma\end{aligned} \tag{A.2.3-2}$$

Observe that

$$\begin{aligned}a_{\beta\alpha}\mathbf{a}^\alpha &= a_{\beta\alpha}a^{\alpha\gamma}\mathbf{a}_\gamma \\ &= \delta^\gamma_\beta \mathbf{a}_\gamma \\ &= \mathbf{a}_\beta\end{aligned} \tag{A.2.3-3}$$

or that each natural basis vector may be written as a linear combination of the dual vectors:

$$\mathbf{a}_\alpha = a_{\alpha\beta}\mathbf{a}^\beta \tag{A.2.3-4}$$

Since $\mathbf{a}_1$ and $\mathbf{a}_2$ form a basis for the tangential vector fields, it follows that the two dual vectors also comprise a basis.

Exercise A.2.3-1. If the $\mathbf{a}^\alpha$ ($\alpha = 1, 2$) are the dual basis vector fields associated with one surface coordinate system $\{y^1, y^2\}$ and if the $\bar{\mathbf{a}}^\alpha$ ($\alpha = 1, 2$) are the dual basis vector fields associated with another surface coordinate system $\{\bar{y}^1, \bar{y}^2\}$, prove that

$$\bar{\mathbf{a}}^\alpha = \frac{\partial \bar{y}^\alpha}{\partial y^\beta}\mathbf{a}^\beta$$

Exercise A.2.3-2. Prove that

$$\mathbf{a}^\alpha \cdot \mathbf{a}^\beta = a^{\alpha\beta}$$

Exercise A.2.3-3. Prove that

$$\mathbf{a}^\alpha \cdot \mathbf{a}_\beta = \delta^\alpha_\beta$$

A.2.4 Covariant and Contravariant Components

By definition, every tangential vector field may be written as a linear combination of the two natural basis vectors,

$$\mathbf{c} = c^\alpha \mathbf{a}_\alpha \tag{A.2.4-1}$$

In the last section, we concluded that every tangential field may also be written as a linear combination of the dual basis fields,

$$\mathbf{c} = c_\alpha \mathbf{a}^\alpha \tag{A.2.4-2}$$

the c^α and c_α are referred to respectively as the **contravariant** and **covariant** surface components of the tangential vector field $\mathbf{c}$.

Since for any tangential vector field

$$\mathbf{c} = c_\alpha \mathbf{a}^\alpha = c_\alpha a^{\alpha\beta} \mathbf{a}_\beta = c^\beta \mathbf{a}_\beta \tag{A.2.4-3}$$

we may write

$$\left(c^\beta - c_\alpha a^{\alpha\beta}\right) \mathbf{a}_\beta = 0 \tag{A.2.4-4}$$

The dual basis fields are linearly independent and (A.2.4-4) implies that

$$c^\beta = a^{\alpha\beta} c_\alpha \tag{A.2.4-5}$$

In the same way,

$$\mathbf{c} = c^\alpha \mathbf{a}_\alpha = c^\alpha a_{\alpha\beta} \mathbf{a}^\beta = c_\beta \mathbf{a}^\beta \tag{A.2.4-6}$$

so that we may identify

$$c_\beta = a_{\alpha\beta} c^\alpha \tag{A.2.4-7}$$

We find in this way that the $a_{\alpha\beta}$ and the $a^{\alpha\beta}$ may be used to *raise and lower* indices.

Exercise A.2.4-1. i) Let $\mathbf{c}$ be some tangential vector field. If the c^α ($\alpha = 1, 2$) are the contravariant components of $\mathbf{c}$ with respect to one surface coordinate system $\{y^1, y^2\}$ and if the $\bar{c}^\beta$ ($\beta = 1, 2$) are the contravariant components of $\mathbf{c}$ with respect to another surface coordinate system $\{\bar{y}^1, \bar{y}^2\}$, determine that

$$c^\alpha = \frac{\partial y^\alpha}{\partial \bar{y}^\beta} \bar{c}^\beta$$

ii) Similarly, show that

$$c_\alpha = \frac{\partial \bar{y}^\beta}{\partial y^\alpha} \bar{c}_\beta$$

A.2.5 Physical Components

The natural basis fields are **orthogonal**, if

$$\mathbf{a}_\alpha \cdot \mathbf{a}_\beta = g_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} = 0 \qquad \text{for } \alpha \neq \beta \tag{A.2.5-1}$$

When the natural basis fields are orthogonal to one another, we say that they correspond to an orthogonal surface coordinate system. Four examples of orthogonal surface coordinate systems are given in Exercises A.2.1-4 through A.2.1-7.

When possible, it is usually more convenient to work in terms of an orthonormal basis, one consisting of orthogonal unit vectors. For an orthogonal surface coordinate system, the natural basis fields defined in (A.2.1-1) may be normalized to form an orthonormal basis $\{\mathbf{a}_{<1>}, \mathbf{a}_{<2>}\}$

$$\mathbf{a}_{<\alpha>} \equiv \frac{\mathbf{a}_\alpha}{\sqrt{a_{\alpha\alpha}}} \quad \text{(no summation on } \alpha) \tag{A.2.5-2}$$

The basis is referred to as the **physical basis** for the surface coordinate system.

In this text, we will employ a normalized natural basis only for orthogonal surface coordinate systems. For these coordinate systems, the physical basis has a particularly convenient relation to the normalized dual basis:

$$\begin{aligned}\mathbf{a}_{<\alpha>} &= \frac{\mathbf{a}^\alpha}{\sqrt{a^{\alpha\alpha}}} \\ &= \sqrt{a_{\alpha\alpha}}\,\mathbf{a}^\alpha \qquad \text{(no summation on } \alpha)\end{aligned} \tag{A.2.5-3}$$

Any tangential vector field $\mathbf{c}$ may consequently by expressed as a linear combination of the two physical basis vector fields associated with an orthogonal surface coordinate system:

$$\mathbf{c} = c_{<\alpha>}\mathbf{a}_{<\alpha>} \tag{A.2.5-4}$$

The two coefficients $\{c_{<1>}, c_{<2>}\}$

$$\begin{aligned}c_{<\alpha>} &\equiv \sqrt{a_{\alpha\alpha}}\,c^\alpha \\ &= \frac{c_\alpha}{\sqrt{a_{\alpha\alpha}}} \quad \text{(no summation on } \alpha)\end{aligned} \tag{A.2.5-5}$$

are known as the **physical** surface components of $\mathbf{c}$ with respect to this particular surface coordinate system.

A.2.6 Tangential and Normal Components

In the text, spatial vector fields defined on a surface play an important role. It is often convenient to think in terms of their tangential and normal components.

If as suggested in Sect. A.1.3 we describe our surface by the single scalar equation

$$f(\mathbf{z}) = 0 \tag{A.2.6-1}$$

then the unit normal to the surface is given by [42, p. 632]

$$\boldsymbol{\xi} \equiv \frac{\nabla f}{|\nabla f|} \tag{A.2.6-2}$$

We require that the sign of the function f in (A.2.6-1) is such that $\boldsymbol{\xi} \cdot \mathbf{a}_1 \times \mathbf{a}_2$ is positive. This means that, with respect to "right-handed" spatial coordinate systems, the spatial vector fields $\mathbf{a}_1$, $\mathbf{a}_2$ and $\boldsymbol{\xi}$ have the same orientation as the index finger, middle finger and thumb on the right hand [42, p. 676].

Since $\mathbf{a}_1$, $\mathbf{a}_2$ and $\boldsymbol{\xi}$ are linearly independent, they form a basis for the spatial vector fields on the surface. If $\mathbf{v}$ is any spatial vector field on the surface, we can write

$$\mathbf{v} = v^{\alpha}\mathbf{a}_{\alpha} + v_{(\xi)}\boldsymbol{\xi} \tag{A.2.6-3}$$

By the same argument, $\mathbf{v}$ can also be expressed as

$$\mathbf{v} = v_{\alpha}\mathbf{a}^{\alpha} + v_{(\xi)}\boldsymbol{\xi} \tag{A.2.6-4}$$

Here $v_{(\xi)}$ is known as the normal component of the spatial vector field $\mathbf{v}$.

Exercise A.2.6-1. *Two-dimensional waves* Given a rectangular cartesian coordinate system and a surface

$$z_3 = h\left(z_1, t\right)$$

determine that the rectangular cartesian components of $\boldsymbol{\xi}$ are

$$\xi_1 = -\frac{\partial h}{\partial z_1}\left[1 + \left(\frac{\partial h}{\partial z_1}\right)^2\right]^{-1/2}$$

$$\xi_2 = 0$$

$$\xi_3 = \left[1 + \left(\frac{\partial h}{\partial z_1}\right)^2\right]^{-1/2}$$

(See also Exercises A.2.1-9 and A.5.3-10.)

Exercise A.2.6-2. *Axially symmetric surface in cylindrical coordinates* Given the cylindrical coordinate system described in Exercise A.2.1-5 and an axially symmetric surface

$$z = h\left(r\right)$$

determine that the cylindrical components of $\boldsymbol{\xi}$ are

$$\xi_r = -\frac{\mathrm{d}h}{\mathrm{d}r}\left[1 + \left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^2\right]^{-1/2}$$

$$\xi_\theta = 0$$

$$\xi_z = \left[1 + \left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^2\right]^{-1/2}$$

(See also Exercises A.2.1-10 and A.5.3-11.)

Exercise A.2.6-3. *Alternative form for axially symmetric surface in cylindrical coordinates* Given the cylindrical coordinate system described in Exercise A.2.1-5 and an axially symmetric surface

$$r = c\,(z)$$

determine that the cylindrical components of $\boldsymbol{\xi}$ are

$$\begin{aligned}
\xi_r &= \left[1 + \left(\frac{\mathrm{d}c}{\mathrm{d}z}\right)^2\right]^{-1/2} \\
\xi_\theta &= 0 \\
\xi_z &= -\frac{\mathrm{d}c}{\mathrm{d}z}\left[1 + \left(\frac{\mathrm{d}c}{\mathrm{d}z}\right)^2\right]^{-1/2}
\end{aligned}$$

(See also Exercises A.2.1-11 and A.5.3-12.)

A.3 Second-Order Tensor Fields

A.3.1 Tangential Transformations and Surface Tensors

A second-order tensor field $\mathbf{A}$ is a transformation (or mapping or rule) that assigns to each given spatial vector field $\mathbf{v}$ another spatial vector field $\mathbf{A} \cdot \mathbf{v}$ such that the rules

$$\begin{aligned}
\mathbf{A} \cdot (\mathbf{v} + \mathbf{w}) &= \mathbf{A} \cdot \mathbf{v} + \mathbf{A} \cdot \mathbf{w} \\
\mathbf{A} \cdot (\alpha \mathbf{v}) &= \alpha\,(\mathbf{A} \cdot \mathbf{v})
\end{aligned} \tag{A.3.1-1}$$

hold. By α we mean a real scalar field.

A **tangential transformation** $\boldsymbol{\mathcal{T}}$ is a particular type of second-order tensor field that is defined only on the surface, that assigns to each given tangential vector field $\mathbf{c}$ on a surface Σ another tangential vector field $\boldsymbol{\mathcal{T}} \cdot \mathbf{c}$ on a surface $\bar{\Sigma}$, and that transforms every spatial vector field normal to Σ into the zero vector. Let $\{\mathbf{a}_1, \mathbf{a}_2\}$ and $\{\mathbf{a}^1, \mathbf{a}^2\}$ be the natural and dual basis field on Σ; $\{\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2\}$ and $\{\bar{\mathbf{a}}^1, \mathbf{a}^2\}$ denote the natural and dual basis field on $\bar{\Sigma}$. A tangential transformation $\boldsymbol{\mathcal{T}}$ can be defined by the way in which it transforms the natural basis fields $\{\mathbf{a}_1, \mathbf{a}_2\}$:

$$\boldsymbol{\mathcal{T}} \cdot \mathbf{a}_\alpha = \mathcal{T}^A_{.\alpha} \bar{\mathbf{a}}_A \tag{A.3.1-2}$$

By an argument which is similar to that given for general second-order tensor fields [42, p. 645], we conclude[2]

$$\boldsymbol{\mathcal{T}} = \mathcal{T}^A_{.\alpha} \bar{\mathbf{a}}_A \mathbf{a}^\alpha \tag{A.3.1-3}$$

[2] As individual indices are raised, their order will be preserved with a dot in the proper subscript position.