SOLUTIONS TO THE EXERCISES

FOR

STURM-LIOUVILLE THEORY AND ITS APPLICATIONS

M.A. AL-GWAIZ

Chapter 1

1.1 (a) $0 \cdot \mathbf{x} = 0 \cdot \mathbf{x} + [0 \cdot \mathbf{x} + (-0 \cdot \mathbf{x})] = (0+0) \cdot \mathbf{x} + (-0 \cdot \mathbf{x}) = 0 \cdot \mathbf{x} + (-0.\mathbf{x}) = \mathbf{0}$.

(b) $a \cdot \mathbf{0} = a \cdot \mathbf{0} + [a \cdot \mathbf{0} + (-a \cdot \mathbf{0})] = a \cdot (\mathbf{0} + \mathbf{0}) + (-a \cdot \mathbf{0}) = a \cdot \mathbf{0} + (-a \cdot \mathbf{0}) = \mathbf{0}$.

(c) $(-1) \cdot \mathbf{x} + \mathbf{x} = (-1+1) \cdot \mathbf{x} = 0 \cdot \mathbf{x} = \mathbf{0}$.

(d) If $a \neq 0$ then $a \cdot \mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = a^{-1} \cdot (a \cdot \mathbf{x}) = a^{-1} \cdot \mathbf{0} = \mathbf{0}$.

1.2 (a) Complex vector space, (b) real vector space, (c) not a vector space, (d) real vector space.

1.3 If $\mathbf{x}_1, ..., \mathbf{x}_n$ are linearly dependent then there are scalars $b_1, ..., b_n$, not all zeros, such that $\sum_{i=1}^{n} b_i \mathbf{x}_i = \mathbf{0}$. Assuming $b_k \neq 0$ for some $k \in \{1, ..., n\}$, we can multiply by b_k^{-1} to obtain $\mathbf{x}_k = \sum_{i \neq k} a_i \mathbf{x}_i$, where $a_i = -b_k^{-1} b_i$. The converse is obvious. If the set of vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ...$ is infinite , then it is linearly dependent if, and only if, it has a finite subset which is linearly dependent, and the desired conclusion follows.

1.4 Assume that $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_m\}$ are bases of the same vector space with $n \neq m$ and show that this leads to a contradiction. If $m > n$, express each y_i, $0 \leq i \leq n$, as a linear combination of $x_1, \cdots, x_n$. The resulting system of n linear equations can be solved uniquely for each x_i, $0 \leq i \leq n$, as a linear combination of y_i, $0 \leq i \leq n$ (why?). Since each vector $y_{n+1}, ..., y_m$ is also a linear combination of $x_1, \cdots, x_n$ (and hence of $y_1, \cdots, y_n$), this contradicts the linear independence of $\{y_1, \cdots, y_m\}$. Similarly, If $m < n$ then $\{x_1, ..., x_n\}$ is linearly dependent. Hence $m = n$.

1.5 Assume $a_n x^n + ... + a_1 x + a_0 = 0$ for all x in the interval I. We can differentiate both sides of this identity n times to conclude that $a_n = 0$, then $n-1$ times to obtain $a_{n-1} = 0$, etc. Therefore all the coefficients a_k are zeros, and so $\{1, x, ..., x^n : x \in I\}$ is linearly independent for every n. It follows that the infinite set $\{1, x, ... : x \in I\}$ is linearly independent.

1.6 It suffices to consider the case where both $\dim X$ and $\dim Y$ are finite. If $\mathcal{B}$ is a basis of Y, then $\mathcal{B}$ lies in X. Since the vectors in $\mathcal{B}$ are linearly independent, $\dim X$ cannot be less than the number of vectors in $\mathcal{B}$, namely $\dim Y$.

1.7 Recall that a determinant is zero if, and only if, one of its rows (or columns) is a linear combination of the other rows (or columns).

1.8 Since $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2 \operatorname{Re} \langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2$, the equality holds if, and only if, $\operatorname{Re} \langle \mathbf{x}, \mathbf{y} \rangle = 0$. Consider $\mathbf{x} = (1, 1)$ and $\mathbf{y} = (i, i)$ in $\mathbb{C}^2$.

1.9 (a) Let $a(\mathbf{x}+\mathbf{y})+b(\mathbf{x}-\mathbf{y})=(a+b)\mathbf{x}+(a-b)\mathbf{y}=0$. Because $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, it follows that $a+b=0$ and $a-b=0$. But this implies $a=b=0$, hence the linear independence of $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$.

(b) $\langle \mathbf{x}+\mathbf{y},\mathbf{x}-\mathbf{y}\rangle=\|\mathbf{x}\|^2+\langle \mathbf{y},\mathbf{x}\rangle-\langle \mathbf{x},\mathbf{y}\rangle-\|\mathbf{y}\|^{\mathbf{2}}=\|\mathbf{x}\|^2-\|\mathbf{y}\|^2$. Therefore $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$ are orthogonal if $\|\mathbf{x}\|=\|\mathbf{y}\|$.

1.10 (a) 0, (b) 2/3, (c) 8/3, (d) $\sqrt{14}$.

1.11 $\langle \varphi_1,\varphi_3\rangle=\langle \varphi_1,\varphi_4\rangle=\langle \varphi_2,\varphi_4\rangle=\langle \varphi_3,\varphi_4\rangle=0$. Thus the largest orthogonal subset is $\{\varphi_1,\varphi_3,\varphi_4\}$.

1.12 $\langle f,f_1\rangle / \|f_1\|=\sqrt{\pi/2}$, $\langle f,f_2\rangle / \|f_2\|=0$, $\langle f,f_3\rangle / \|f_3\|=\sqrt{\pi}/2$.

1.13 Let $a+bx+cx^2=0$ for all $x\in[-1,1]$. Setting $x=0$, $x=1$, and $x=-1$, the resulting three equations yield the only solution $a=b=c=0$. The corresponding orthogonal functions are given by

$$\begin{aligned}
f_1(x)&=1,\\
f_2(x)&=x-\frac{\langle x,1\rangle}{\|1\|^2}=x,\\
f_3(x)&=x^2-\frac{\langle x^2,1\rangle}{\|1\|^2}-\frac{\langle x^2,x\rangle}{\|x\|^2}x=x^2-\frac{1}{3}.
\end{aligned}$$

1.14 Let $a+bx+c\,|x|=0$ for all $x\in[-1,1]$. Setting $x=0$, $x=1$, and $x=-1$ yields $a=b=c=0$. The corresponding orthogonal set is

$$\begin{aligned}
f_1(x)&=1,\\
f_2(x)&=x-\frac{\langle x,1\rangle}{\|1\|^2}=x\\
f_3(x)&=|x|-\frac{\langle |x|,1\rangle}{\|1\|^2}-\frac{\langle |x|,x\rangle}{\|x\|^2}=|x|-\frac{1}{2},
\end{aligned}$$

and the normalized set is

$$\begin{aligned}
\frac{f_1(x)}{\|f_1\|}&=\frac{1}{\sqrt{2}},\\
\frac{f_2(x)}{\|f_2\|}&=\frac{x}{\sqrt{2/3}},\\
\frac{f_3(x)}{\|f_3\|}&=\frac{1}{\sqrt{6}}\left(|x|-1/2\right).
\end{aligned}$$

The set $\{1,x,|x|\}$ is not linearly independent on $[0,1]$ because $|x|=x$ on $[0,1]$.

1.15 From the result of Exercise 1.3 we know that $f_1, ..., f_n$ are linearly dependent if, and only if, there is a number $k \in \{1, ..., n\}$ such that $f_k = \sum_{i \neq k} a_i f_i$ on I. By differentiating this identity up to order $n-1$, we arrive at the system of equations $f_k^{(j)} = \sum_{i \neq k} a_i f_i^{(j)}$, $0 \leq j \leq n-1$. Writing this system in matrix form, and using the properties of determinants, we conclude that the system is equivalent to the single equation $\det(f_i^{(j)}) = 0$ on I, where $1 \leq i \leq n$ and $0 \leq j \leq n-1$.

1.16 Noting that both φ_1 and φ_2 are even whereas φ_3 is odd, we conclude that $\langle \varphi_1, \varphi_3 \rangle = \langle \varphi_2, \varphi_3 \rangle = 0$. Moreover, $\langle \varphi_1, \varphi_2 \rangle = 0$. The corresponding orthonormal set is

$$\begin{aligned} \frac{\varphi_1(x)}{\|\varphi_1\|} &= \frac{1}{\sqrt{2}}, \\ \frac{\varphi_2(x)}{\|\varphi_2\|} &= \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3} \right), \\ \frac{\varphi_3(x)}{\|\varphi_3\|} &= \frac{1}{\sqrt{2}} \varphi_3(x). \end{aligned}$$

1.17 Solving the pair of equations $\langle x^2+ax+b, x+1 \rangle = 0$ and $\langle x^2+ax+b, x-1 \rangle = 0$ gives $a = -1$, $b = 1/6$.

1.18 If $f : [a,b] \to \mathbb{C}$ is a continuous function and $\|f\| = 0$, then it follows that $\int_a^b |f(x)|^2 \, dx = 0$ and $|f(x)|^2$ is continuous and nonnegative on $[a,b]$. But this implies $f(x) = 0$ for all $x \in [a,b]$. On the other hand, the noncontinuous function

$$f(x) = \begin{cases} 0, & x \in [0,1] \backslash \{1/2\} \\ 1, & x = 1/2 \end{cases}$$

clearly satisfies $\|f\| = 0$, but f is not identically 0 on $[0,1]$.

1.19 From the CBS inequality, $\|f+g\|^2 = \|f\|^2 + 2\,\mathrm{Re}\langle f, g \rangle + \|g\|^2 \leq \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 = (\|f\| + \|g\|)^2$. The triangle inequality follows by taking the square root of each side.

1.20 $\langle 1, x \rangle = 1/2$, $\|1\| = 1$, and $\|x\| = 1/\sqrt{3}$. Clearly $\langle 1, x \rangle < \|1\| \|x\|$.

1.21 Use the definition of the Riemann integral, based on Riemann sums, to show that f, and hence fg, is not integrable on $[0,1]$, whereas $f^2 = 1$ and $g^2 = 1$ are both integrable.

1.22 (i) $1/\sqrt{2}$, (ii) not in $\mathcal{L}^2(0,\infty)$, (iii) 1, (iv) not in $\mathcal{L}^2(0,\infty)$.

1.23 If $\|f\| = 0$ then f, being continuous, is identically 0 and the pair f, g is linearly dependent. The same is true if $\|g\| = 0$. Hence we assume $\|f\| \neq 0$ and $\|g\| \neq 0$. Now

$$\begin{aligned}\left\| \frac{f}{\|f\|} - \frac{g}{\|g\|} \right\|^2 &= \int_a^b \frac{f^2}{\|f\|^2} + \int_a^b \frac{g^2}{\|g\|^2} - 2\int_a^b \frac{fg}{\|f\| \, \|g\|} \\ &= 1 + 1 - 2 = 0,\end{aligned}$$

where we used $\langle f, g\rangle = \|f\| \, \|g\|$ in the second equality. The implies $g = \lambda f$ with $\lambda = \|g\| \,/\, \|f\|$.

Conversely, if $g = \lambda f$ for some positive number λ, then $\langle f, g\rangle = \lambda \|f\|^2 = \|f\| \, \|g\|$.

1.24 In general $\|f + g\|^2 = \|f\|^2 + \|g\|^2 + 2\,\mathrm{Re}\langle f, g\rangle$. If $\|f + g\| = \|f\| + \|g\|$ then we must have $\mathrm{Re}\langle f, g\rangle = \|f\| \, \|g\|$, and if, furthermore, the functions f and g are positive and continuous, then $\mathrm{Re}\langle f, g\rangle = \langle f, g\rangle = \|f\| \, \|g\|$ and (by Exercise 1.23) f and g are linearly dependent.

Conversely, if the functions are linearly dependent then $g = \alpha f$ for some number α, and $\|f + g\| = \|(1+\alpha) f\| = |1 + \alpha| \, \|f\|$. For the equality $\|f + g\| = \|f\| + \|g\|$ to hold we must therefore have $|1 + \alpha| = 1 + |\alpha|$, which implies $\alpha \geq 0$.

1.25 The norm $\|x^\alpha\|^2 = \int_0^1 x^{2\alpha} dx$ is finite if, and only if, $2\alpha > -1$, that is, $\alpha > -1/2$.

1.26 $\alpha < -1/2$.

1.27 Suppose $\lim_{x\to\infty} f(x) = \ell \neq 0$, then $\lim_{x\to\infty} |f(x)| = |\ell| > 0$ and there is a positive integer n such that, for all $x \geq n$,

$$\begin{gathered}||f| - |\ell|| < |\ell|/2 \\ 0 < |\ell|/2 < |f(x)| < 3|\ell|/2.\end{gathered}$$

But this implies $\int_n^\infty |f(x)|^2 \, dx = \infty$, which contradicts the integrability of $|f|^2$ on $(0, \infty)$.

1.28 $\int_a^b |f(x)| \, dx = \langle |f(x)|, 1\rangle \leq \|f\| \, \|1\| = \sqrt{b-a}\, \|f\|$ by the CBS inequality. $f(x) = 1/\sqrt{x}$ is integrable on $(0, 1)$ but $f^2(x) = 1/x$ is not.

1.29 Suppose $|f(x)| \leq M$ for all $x \geq 0$. Then

$$\int_0^\infty f^2(x) dx \leq M \int_0^\infty |f(x)| \, dx < \infty.$$

The function $f(x) = (1+x)^{-1}$ is bounded on $[0, \infty)$, lies in $\mathcal{L}^2(0, \infty)$, but is not integrable on $[0, \infty)$.

1.30 Using familiar trigonometric identities,

$$\begin{aligned}\sin^3 x &= \sin x(1-\cos^2 x)\\ &= \sin x - \frac{1}{2}\cos x \sin 2x\\ &= \sin x - \frac{1}{4}(\sin 3x + \sin x)\\ &= \frac{3}{4}\sin x - \frac{1}{4}\sin 3x.\end{aligned}$$

1.31 There are many answers to this exercise. Any odd function, such as $f(x) = ax$, where a is a (non-zero) constant satisfies $\langle f, x^2+1\rangle = 0$ since x^2+1 is even. For this choice of f, the constant a has to satisfy $\|ax\| = |a|\sqrt{2/3} = 2$, that is, $|a| = \sqrt{6}$. Thus $f(x) = \sqrt{6}x$ is one possible answer.

1.32 The $\mathcal{L}^2_\rho(0,\infty)$ norm of a polynomial p is given by $\left[\int_0^\infty p^2(x)\, e^{-x}dx\right]^{1/2}$. It is therefore sufficient to show that the integral $I = \int_0^\infty q(x)e^{-x}dx$ is finite for any polynomial q. This follows from integrating by parts, noting that $q(x)e^{-x} \to 0$ as $x \to \infty$, to obtain $I = q(0) + \int_0^\infty q'(x)e^{-x}dx$. Similarly, $\int_0^\infty q'(x)e^{-x}dx = q'(0) + \int_0^\infty q''(x)e^{-x}dx$. If q has degree n, we can use induction to show that, in the n-th step, the integral is reduced to a constant multiple of $\int_0^\infty e^{-x}dx = 1$.

1.33 Using the monotonic property of the integral,

$$\|f\|_\sigma^2 = \int_a^b |f(x)|^2\, \sigma(x)dx \leq \int_a^b |f(x)|^2\, \rho(x)dx = \|f\|_\rho^2.$$

Therefore, if $f \in \mathcal{L}^2_\rho(a,b)$ then $f \in \mathcal{L}^2_\sigma(a,b)$.

1.34 (a) The limit is the discontinuous function

$$\lim_{n\to\infty}\frac{x^n}{1+x^n} = \begin{cases} 1, & |x|>1\\ 0, & |x|<1\\ 1/2, & x=1\\ \text{undefined}, & x=-1.\end{cases}$$

(b) $\lim_{n\to\infty}\sqrt[n]{x} = \begin{cases} 0, & x=0\\ 1, & x>0.\end{cases}$

(c) $\lim_{n\to\infty}\sin nx$ does not exist, except when x is an integral multiple of π.

1.35 (a) Pointwise (not uniform), by Theorem 1.17(i), since the limit (Exercise 1.34(a)) is discontinuous.

(b) Uniform, since $x^{1/n} \to 1$ for all $x \in [1/2, 1]$ and $\left|x^{1/n} - 1\right| \leq 1 - (1/2)^{1/n} \to 0$.

(c) Pointwise (not uniform), since the limit is discontinuous at $x = 0$ (see Exercise 1.34(b)).

1.36 f_n is continuous for every n, whereas $\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & x = 0 \\ 1, & 0 < x \leq 1. \end{cases}$ is discontinuous, hence the convergence $f_n \to f$ is not uniform. Clearly,

$$\lim \int_0^1 f_n(x)dx = 1 = \int_0^1 \lim f_n(x)dx.$$

1.37 At $x = 0$, $f_n(0) = 0$ for every n. When $x > 0$, $f_n(x) = n(1-x)/(n-1) \to 1$. Therefore the sequence f_n converges pointwise to

$$f(x) = \begin{cases} 0, & x = 0 \\ 1 - x, & 0 < x \leq 1. \end{cases}$$

Since f is not continuous the convergence is not uniform.

1.38 $f_n(0) = f_n(1) = 0$ and, for all $x \in (0, 1)$, $|f_n(x)| = nx(1 - x^2)^n \leq n(1 - x^2)^n \to 0$ as $n \to \infty$. Hence $\lim_{n\to\infty} f_n(x) = 0$ on $[0, 1]$. Since

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} \frac{n}{2n+2} = \frac{1}{2},$$

whereas $\int_0^1 \lim_{n\to\infty} f_n(x)dx = 0$, the convergence $f_n \to 0$ is not uniform (by Theorem 1.17(ii)).

1.39 $0 \leq \dfrac{x}{n+x} \leq \dfrac{a}{n} \to 0$, hence $\dfrac{x}{n+x} \xrightarrow{u} 0$ on $[0, a]$.

For $x \geq 0$, we also have $\lim_{n\to\infty} \dfrac{x}{n+x} = 0$ pointwise. In this case, assuming $0 < \varepsilon < 1$, the inequality $\dfrac{x}{n+x} < \varepsilon$ cannot be satisfied when $x \geq n\varepsilon/(1-\varepsilon)$, hence the convergence is not uniform.

Another approach: Since the statement $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in I$ is equivalent to the requirement that $\sup_{x\in I} |f_n(x) - f(x)| \leq \varepsilon$, we see that $f_n \xrightarrow{u} f$ on I if, and only if, $\sup_{x\in I} |f_n(x) - f(x)| \to 0$ as $n \to \infty$. When $x \in [0, a]$, $\sup f_n(x) = a/n \to 0$; but when $x \in [0, \infty)$ we have $f_n(n) = 1/2$, hence $\sup f_n(x) \geq 1/2 \nrightarrow 0$.

1.40 The sequence f_n is defined by

$$f_n(x) = \begin{cases} 1/n, & |x| \leq n \\ 0, & |x| > n. \end{cases}$$

Since $0 \leq f_n(x) \leq 1/n \to 0$ for all $x \in \mathbb{R}$, the convergence $f_n \to 0$ is uniform. $\int_{-\infty}^{\infty} f_n(x)dx = 2$, therefore

$$\lim \int_{-\infty}^{\infty} f_n(x)dx \neq \int_{-\infty}^{\infty} \lim f_n(x)dx,$$

the reason being that the domain of definition of f_n is not bounded.

1.41 Suppose $f_n \xrightarrow{u} f$ on $[a,b]$. Given any $\varepsilon > 0$, it then follows that there is an integer N_1 such that

$$n \geq N_1 \ \Rightarrow \ |f_n(x) - f(x)| < \varepsilon \text{ for all } x \in [a,b],$$

which implies $|f_n(x) - f(x)| \xrightarrow{u} 0$. We can also find an integer N_2 such that

$$n \geq N_2 \ \Rightarrow \ |f_n(x) - f(x)| < 1 \text{ for all } x \in [a,b].$$

If $N = \max\{N_1, N_2\}$, then

$$n \geq N \ \Rightarrow \ |f_n(x) - f(x)|^2 < \varepsilon \text{ for all } x \in [a,b],$$

which implies $|f_n(x) - f(x)|^2 \xrightarrow{u} 0$.

1.42 (a) $|f_n(x)| \leq 1/n^2$ for all $x \in \mathbb{R}$. Since $\sum 1/n^2$ converges, $\sum f_n(x)$ converges uniformly on $\mathbb{R}$ by the Weierstrasse M-test.

(b) If $x \in (-1,1)$ then there is an integer N such that $|x|^n < 1/2$ for all $n \geq N$, and hence

$$\left| \frac{x^n}{1+x^n} \right| \leq \frac{|x|^n}{1-1/2} = 2\,|x|^n \ \text{ for all } n \geq N,$$

from which we conclude that the series converges by comparison with the geometric series. It diverges on $(-\infty,-1] \cup [1,\infty)$ where $f_n(x) \nrightarrow 0$ (see Exercise 1.34(a)).

1.43 $|a_n \sin nx| \leq |a_n|$ for all $x \in \mathbb{R}$. Since $\sum |a_n|$ converges, $\sum a_n \sin nx$ converges uniformly.

1.44

$$\begin{aligned}
\int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x} dx &\leq \frac{1}{n\pi} \int_{n\pi}^{(n+1)\pi} |\sin x|\, dx \\
&= \frac{1}{n\pi} \int_0^{\pi} \sin x dx \\
&= \frac{2}{n\pi} \to 0 \text{ as } n \to 0.
\end{aligned}$$

Let $A_n = \int_{n\pi}^{(n+1)\pi} x^{-1} |\sin x|\, dx$. Because $x^{-1} |\sin x| > (x+\pi)^{-1} |\sin(x+\pi)|$ for every $x > 0$, we see that $A_n \geq A_{n+1}$ for all n and $A_n \to 0$. Moreover,

$$\begin{aligned}\int_0^{(n+1)\pi} \frac{\sin x}{x} dx &= \sum_{k=0}^{k=n} \int_{k\pi}^{(k+1)\pi} (-1)^k \frac{|\sin x|}{x} dx \\ &= \sum_{k=0}^{k=n} (-1)^k A_k.\end{aligned}$$

Hence $\int_0^\infty x^{-1} \sin x dx = \sum_{k=0}^\infty (-1)^k A_k$, which converges by the alternating series test.

On the other hand,

$$\begin{aligned}&\int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x} dx \geq \frac{1}{(n+1)\pi} \int_0^\pi \sin x dx = \frac{2}{(n+1)\pi} \\ \Rightarrow &\int_0^\infty \frac{|\sin x|}{x} dx = \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x} dx \geq \sum_{k=0}^\infty \frac{2}{(n+1)\pi} = \infty.\end{aligned}$$

1.45 Let $r = R - \varepsilon$. Then $|a_n x^n| \leq |a_n| r^n$ for all $x \in [-r, r]$. Since $0 < r < R$, the numerical series $\sum |a_n| r^n$ is convergent. By the M-test, with $M_n = |a_n| r^n$, the series $\sum a_n x^n$ is uniformly convergent in $[-r, r]$.

1.46 Being uniformly convergent on $[R-\varepsilon, R+\varepsilon]$, the series $\sum a_n x^n$ represents a continuous function f on $[R-\varepsilon, R+\varepsilon]$. Since this is true for every $\varepsilon > 0$, the series is continuous on $(-R, R)$. Each term $f_n(x) = a_n x^n$ is differentiable, $f_n'(x) = n a_n x^{n-1}$, and the series $\sum_{n=1}^\infty n a_n x^{n-1}$ has the same radius of convergence R as the original series (by the root test). Now the power series $\sum_{n=1}^\infty n a_n x^{n-1}$, by the result of Exercise 1.45, is uniformly convergent on $[R - \varepsilon, R + \varepsilon]$. Therefore Theorem 1.17(iii) applies and we conclude that $f'(x) = \left(\sum_{n=0}^\infty a_n x^n\right)' = \sum_{n=1}^\infty n a_n x^{n-1}$.

1.47 With $b_n = (n+1)a_{n+1}$ we can write $f'(x) = \sum_{n=0}^\infty b_n x^n$ and repeat the argument in Exercise 1.46 to conclude that $f''(x) = \sum_{n=1}^\infty n b_n x^{n-1} = \sum_{n=2}^\infty n(n-1) a_n x^{n-2}$, and so on to any order of differentiation. By induction we clearly have

$$f^{(k)}(x) = \sum_{n=k}^\infty \frac{n!}{(n-k)!} a_n x^{n-k} = k! a_k + \frac{(k+1)!}{1!} a_{k+1} x + \frac{(k+2)!}{2!} a_{k+2} x^2 + \cdots .$$

At $x = 0$ this yields $a_k = f^{(k)}(0)/k!,\ k \in \mathbb{N}$.

1.48 Let the function $f(x)$ be continuous and differentiable up to any order on $\mathbb{R}$. According to Taylor's theorem we can represent such a function at any $x \in \mathbb{R}$ by

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1},$$

where c lies between 0 and x. If $f(x) = e^x$, then $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$ for all n. Thus we obtain

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^c.$$

Since, for any $x \in \mathbb{R}$, $x^{n+1}/(n+1)! \to 0$ as $n \to \infty$, we arrive at the desired power series representation of e^x by taking the limit of the right-hand side as $n \to \infty$.

If $f(x) = \cos x$, then $f^{(n)}(0) = 0$ if n is odd and $f^{(n)}(0) = (-1)^{n/2}$ if n is even. The remainder term is bounded by $x^{n+1}/(n+1)!$ which tends to 0 as $n \to \infty$, and we obtain $\cos x = \sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)!$. Similarly we arrive at the given representation for $\sin x$.

1.49 Euler's formula is obtained by replacing x by ix in the power series which represent e^x, $\cos x$, and $\sin x$, and using the equations $i^{2n} = (-1)^n$ and $i^{2n+1} = (-1)^n i$.

1.50 (a) 1.

(b) 1.

(c) $f_n(0) = f_n(1) = 0$. For every $x \in (0,1)$, $f_n(x) = nx(1-x)^n \to 0$ as $n \to \infty$. Therefore $f_n(x) \to 0$ pointwise.

$$\begin{aligned}
\|f_n - 0\|^2 &= n^2 \int_0^1 x^2(1-x)^{2n}dx \\
&= \frac{2n^2}{2n+1}\int_0^1 x(1-x)^{2n+1}dx \\
&= \frac{n^2}{(2n+1)(n+1)}\int_0^1 (1-x)^{2n+2}dx \\
&= \frac{n^2}{(2n+1)(n+1)(2n+3)} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence $f_n \xrightarrow{\mathcal{L}^2} 0$.

1.51 (a) convergent, since $\left\|\sum k^{-2/3}\sin kx\right\|^2 = \|\sin kx\|^2 \sum k^{-4/3} = \pi \sum k^{-4/3} < \infty$.

(b) convergent, since $\left\|\sum k^{-1}e^{ikx}\right\|^2 = \sum k^{-2} < \infty$.

(c) divergent, since $\left\|\sum (k+1)^{-1/2}\cos kx\right\|^2 = \sum (k+1)^{-1}\left\|\cos kx\right\|^2 = \pi \sum (k+1)^{-1} = \infty$.

1.52 By the CBS inequality, $|\langle f_n, g\rangle - \langle f, g\rangle| = |\langle f_n - f, g\rangle| \leq \|f_n - f\| \|g\| \to 0$.

1.53 By the triangle inequality, $\|f\| = \|f - g + g\| \leq \|f - g\| + \|g\|$. Therefore $\|f\| - \|g\| \leq \|f - g\|$. Interchanging f and g, we obtain $\|g\| - \|f\| \leq \|g - f\| = \|f - g\|$, hence $|\|f\| - \|g\|| \leq \|f - g\|$.

1.54 If $\sum |a_n|$ converges then $|a_n| \to 0$, so there is an integer N such that $|a_n| < 1$ for all $n \geq N$. It then follows that $|a_n|^2 \leq a_n$ for all $n \geq N$. By the comparison test, $\sum |a_n|^2$ converges. The series $\sum a_n \sin nx$ and $\sum a_n \cos nx$ are both dominated by the convergent numerical series $\sum |a_n|$, and are therefore uniformly convergent on $[-\pi, \pi]$ by the M-test. By Corollary 1.19(i), they represent continuous functions on $[-\pi, \pi]$. In fact, the same argument implies $\sum |a_n| \sin nx$ and $\sum |a_n| \cos nx$ are continuous on $\mathbb{R}$.

1.55 This follows from the inequality $\|f_n - f\|_\sigma \leq \|f_n - f\|_\rho$ (see Exercise 1.33).

1.56 For all $n \neq m$,

$$\begin{aligned}
\langle \cos(n\pi x/l), \cos(m\pi x/l)\rangle &= \int_0^l \cos(n\pi x/l)\cos(m\pi x/l)\ dx \\
&= \frac{l}{\pi}\int_0^\pi \cos n\xi \cos m\xi\ d\xi \\
&= \frac{l}{2\pi}\left[\frac{1}{n-m}\sin(n-m)\xi + \frac{1}{n+m}\sin(n+m)\xi\right]_0^\pi = 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle \sin(n\pi x/l), \sin(m\pi x/l)\rangle &= \frac{l}{\pi}\int_0^\pi \sin n\xi \sin m\xi\ d\xi \\
&= \frac{l}{2\pi}\left[\frac{1}{n-m}\sin(n-m)\xi - \frac{1}{n+m}\sin(n+m)\xi\right]_0^\pi = 0 \text{ for all } n \neq m.
\end{aligned}$$

Using the formulas $\|1\| = \sqrt{l}$, $\|\cos(n\pi x/l)\| = \sqrt{l/2}$ and $\|\sin(n\pi x/l)\| = \sqrt{l/2}$, the corresponding orthonormal sets are

$$\left\{\frac{1}{\sqrt{l}}, \sqrt{\frac{2}{l}}\cos\left(\frac{n\pi x}{l}\right) : n \in \mathbb{N}\right\} \text{ and } \left\{\sqrt{\frac{2}{l}}\sin\left(\frac{n\pi x}{l}\right) : n \in \mathbb{N}\right\}.$$

1.57 The functions $f_1(x) = 1,\ f_2(x) = \sin \pi x$, and $f_3(x) = \sin 2\pi x$ are orthogonal in $\mathcal{L}^2(0,2)$, therefore the coefficient c_i is the projections of $f(x) = x$ on $f_i(x)$ in $\mathcal{L}^2(0,2)$.

$$c_1 = \frac{\langle x, 1\rangle}{\|1\|^2} = \frac{1}{2}\int_0^2 x\ dx = 1.$$

Similarly, $c_2 = \langle x, \sin \pi x\rangle / \|\sin \pi x\|^2 = -2/\pi$, $c_3 = \langle x, \sin 2\pi x\rangle / \|\sin 2\pi x\|^2 = -1/\pi$.

1.58 $a_0 = \langle |x|, 1\rangle / \|1\| = \pi/2,\ a_1 = \langle |x|, \cos x\rangle / \|\cos x\|^2 = -4/\pi,\ a_2 = 0$, $b_1 = 0,\ b_2 = 0$.

1.59 $p_1(x) = 1,\ p_2(x) = x - \langle x, 1\rangle / \|1\|^2 = x,\ p_3(x) = x^2 - \langle x^2, 1\rangle / \|1\|^2 - \langle x^2, x\rangle / \|x\|^2 = x^2 - 1/3$.

$$a_1 = \frac{1}{\|p_1\|^2}\langle e^x, p_1(x)\rangle = \frac{1}{2}\int_{-1}^1 e^x dx = \frac{1}{2}\left(e - e^{-1}\right).$$

Similarly,

$$a_2 = \frac{\langle e^x, x\rangle}{\|x\|^2} = 3e^{-1},\ a_3 = \frac{\langle e^x, x^2 - 1/3\rangle}{\|x^2 - 1/3\|^2} = \frac{15}{4}\left(e - 7e^{-1}\right).$$

The resulting second-degree polynomial which gives the best approximation of e^x in $\mathcal{L}^2(-1,1)$ is

$$\sinh 1 + \frac{3}{e}x + \frac{5}{4}\left(e - \frac{7}{e}\right)\left(3x^2 - 1\right).$$

The first three terms of the Taylor series expansion of e^x about $x = 0$ (see Exercise 1.48),

$$1 + x + \frac{1}{2}x^2,$$

provide a pointwise approximation to e^x in the neighbourhood of $x = 0$, which becomes sharper as $|x|$ gets smaller.

1.60 By direct computation, it is a simple matter to verify that the set of functions $\{\cos(2n-1)\pi x/2 : n \in \mathbb{N}\}$ is orthogonal in $\mathcal{L}^2(0,2)$. Therefore, by Parseval's relation (1.23),

$$\|1 - x\|^2 = \sum_{n=1}^{\infty} \frac{\left|\langle 1 - x, \cos \frac{1}{2}(2n-1)\pi x\rangle\right|^2}{\left\|\cos \frac{1}{2}(2n-1)\pi x\right\|^2}.$$

Since

$$\|1-x\|^2 = \int_0^2 (1-x)^2 dx = 2/3,$$
$$\langle (1-x, \cos\frac{1}{2}(2n-1)\pi x\rangle = \frac{8}{(2n-1)^2\pi^2},$$
$$\left\|\cos\frac{1}{2}(2n-1)\pi x\right\|^2 = 1,$$

the equality $\pi^4 = 96\sum_{n=1}^{\infty}(2n-1)^{-4}$ follows by direct substitution.

1.61 $a_k = 1/n$. $\mathcal{L}^2$ convergence.

1.62 To prove the orthogonality of the set $\{\varphi_n\} \cup \{\psi_n\}$ in $\mathcal{L}^2(-l, l)$ we first note that $\langle \varphi_n, \psi_m \rangle = 0$ for all $n, m \in \mathbb{N}$, because φ_n is an even function whereas ψ_m is an odd function over the symmetric interval $(-l, l)$. Furthermore,

$$\begin{aligned}\langle \varphi_n, \varphi_m \rangle &= \int_{-l}^{l} \varphi_n(x)\bar{\varphi}_m(x)dx \\ &= \int_{-l}^{0} f_n(-x)\bar{f}_m(-x)dx + \int_0^l f_n(x)\bar{f}_m(x)dx \\ &= 2\int_0^l f_n(x)\bar{f}_m(x)dx = 0.\end{aligned}$$

Similarly, $\langle \psi_n, \psi_m \rangle = 2\int_0^l \psi_n(x)\bar{\psi}_m(x)dx = 0$.

$\|\varphi_n\|^2 = \int_{-l}^0 |f_n(-x)|^2 dx + \int_0^l |f_n(x)|^2 dx = 2 = \|\psi_n\|^2$. Hence $\{\varphi_n/\sqrt{2}\} \cup \{\psi_n/\sqrt{2}\}$ is the corresponding orthonormal set in $\mathcal{L}^2(-l, l)$.

Chapter 2

2.1 (a) $y = e^{2x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x) + e^x/4$.

(b) $y = c_1x^2 + c_2 + x^3$.

(c) $y = x^{-1}(c_1 + c_2 \log x) + \dfrac{1}{4}x - 1, \quad x > 0$.

2.2 Assuming $y(x) = y(t^2) = z(t)$, we have, by the chain rule,

$$\frac{dy}{dx} = \frac{dz}{dt}\frac{1}{2\sqrt{x}}, \quad \frac{d^2y}{dx^2} = \frac{dz^2}{dt^2}\frac{1}{4x} - \frac{dz}{dx}\frac{1}{4\sqrt{x^3}}.$$

The transformed equation $z'' - 4z = 0$ has the general solution $z = c_1e^{2t} + c_2e^{-2t}$, or $y = c_1e^{2\sqrt{x}} + c_2e^{-2\sqrt{x}}$.

2.3 Assuming $y = \sum_{n=0}^{\infty} c_n x^n$ and substituting into the differential equation yields the recursion relation

$$c_{n+2} = -\frac{2}{n+1} c_n, \ n \in \mathbb{N}_0.$$

With c_0 and c_1 arbitrary, the solution is therefore

$$y = c_0 \left(1 - 2x^2 + \frac{4}{3}x^4 + \cdots\right) + c_1 \left(x - x^3 + \frac{1}{2}x^5 + \cdots\right),$$

which converges for all $x \in \mathbb{R}$, since q and r (see Equation (2.4)) are analytic in $\mathbb{R}$.

2.4 The differential equation may be put in the form

$$y'' + \frac{xy' - y}{1 - x} = (y' - y)' + \frac{y' - y}{1 - x} = 0,$$

whose general solution is $y = c_1 x + c_2 e^x$. The initial conditions imply $c_2 = 0$ and $c_1 = 1$, hence $y = x$.

2.5 A second-order equation has at most two linearly independent solutions.

2.6 Solving the pair of equations

$$\begin{aligned} y_1'' + q y_1' + r y_1 &= 0 \\ y_2'' + q y_2' + r y_2 &= 0 \end{aligned}$$

for q and r, we obtain

$$q = -\frac{y_2 y_1'' - y_1 y_2''}{y_2 y_1' - y_1 y_2'} = \frac{y_2 y_1'' - y_1 y_2''}{W(y_1, y_2)}, \ r = \frac{y_1' y_2'' - y_2' y_1''}{W(y_1, y_2)}.$$

2.7 Writing $y = c_1 y_1 + c_2 y_2$, and eliminating c_1 and c_2 by differentiation, we obtain (a) $y'' + 2y' + 5y = 0$, (b) $x^2 y'' + xy' - y = 0$, (c) $xy'' + y' = 0$.

2.8 Use Lemma 2.7 and the fact that a bounded infinite set of real numbers has at least one cluster (or limit) point. This property of the real (as well as the complex) numbers is known as the *Bolzano-Weierstrass theorem* (see [1]). Being isolated, the zeros of any (nontrivial) solution cannot therefore have a finite cluster point.

2.9 With $x = e^t$, the function $y(x) = y(e^t) = z(t)$, as a function of t, satisfies the differential equation $z'' - z' + kz = 0$, whose solution is $z(t) = e^{t/2}\left(c_1 e^{\sqrt{1-4k}t/2} + c_2 e^{-\sqrt{1-4k}t/2}\right)$. Thus

$$y(x) = \sqrt{x}\left(c_1 e^{\sqrt{1-4k}\log\sqrt{x}} + c_2 e^{-\sqrt{1-4k}\log\sqrt{x}}\right),$$

which is oscillatory if, and only if, $1-4k<0$. The given equation is actually of the Cauchy-Euler type and can be solved directly to yield $y(x)=\sqrt{x}\left(c_1x^{\sqrt{1-4k}/2}+c_2x^{-\sqrt{1-4k}/2}\right)$.

2.10 Use Theorem 2.10.

2.11 If φ has no zeros on (x_0,∞) then, being a continuous, $\varphi(x)>0$ for all $x>0$. Since $\varphi''(x)=-r(x)\varphi(x)<0$ for all $x>0$, we conclude that φ' is strictly decreasing on $(0,\infty)$. In particular

$$\varphi'(x)<\varphi'(x_0) \quad \text{for all } x>x_0.$$

Integrating this inequality over $[x_0,x]$, we obtain

$$\varphi(x)\leq\varphi(x_0)+(x-x_0)\varphi'(x_0) \quad \text{for all } x>x_0.$$

Since $\varphi'(x_0)<0$, this implies $\lim_{x\to\infty}\varphi(x)=-\infty$, which contradicts the assumption that $\varphi(x)>0$ for all $x>0$.

2.12 The solutions of (a) and (c) are oscillatory.

2.13 When $\nu=1/2$, Equation (2.20) becomes $u''+u=0$, whose solution is $u(x)=c_1\cos x+c_2\sin x$. Therefore $y(x)=x^{-1/2}(c_1\cos x+c_2\sin x)$. The zeros of $x^{-1/2}\cos x$ are $\{\frac{\pi}{2}+n\pi\}$, and those of $x^{-1/2}\sin x$ are $\{n\pi\}$, $n\in\mathbb{N}_0$.

2.14 Since $\lim_{x\to\infty}f(x)=0$ there is a number a such that $|f(x)|\leq 1/2$ for all $x\geq a$. Consequently $1/2\leq 1+f(x)\leq 3/2$ on $[a,\infty)$, and we conclude that the solutions of $y''+(1+f(x))y=0$ oscillate on $[a,\infty)$ by comparison with those of $y''+y/2=0$.

2.15 For any $\varepsilon>0$, the solutions of $y''+xy=0$ oscillate on $[\varepsilon,\infty)$ by comparison with $y''+\varepsilon y=0$. On $(-\infty,-\varepsilon]$ they cannot oscillate by comparison with $y''-\varepsilon y=0$. Since $\varepsilon>0$ is arbitrary, the number of zeros for such solutions is infinite on $(0,\infty)$ and finite on $(-\infty,0)$.

2.16

$$\begin{aligned} uLv-vLu &= u\left[(pv')'+rv\right]-v\left[(pu')'+ru\right] \\ &= p'(uv'-vu')+p(uv''-vu'') \\ &= p'(uv'-vu')+p(uv'-vu')' \\ &= [p(uv'-vu')]'. \end{aligned}$$

2.17 The linearly independent solutions of $y''=\lambda y$ in $C^2(0,\infty)$ are $e^{\pm\sqrt{\lambda}x}$ when $\lambda\in\mathbb{C}\backslash\{0\}$, and $\{1,x\}$ when $\lambda=0$. Hence the eigenfunctions and corresponding eigenvalues are : (a) $e^{\pm\sqrt{\lambda}x}$ for $\lambda\in\mathbb{C}\backslash\{0\}$, and $\{1,x\}$ for $\lambda=0$, (b) $e^{-\sqrt{\lambda}x}$, $\operatorname{Re}\sqrt{\lambda}>0$.

2.18 Since $p = -1$, $q = 0 = p'$, and $r = 0$, the operator $-d^2/dx^2$ is formally self-adjoint. The given boundary conditions ensure that Equation (2.26) holds, hence it is also self-adjoint. If $\lambda \leq 0$ the only solution of $u'' + \lambda u = 0$ which satisfies the boundary conditions is the trivial solution (see Example 2.17). If $\lambda > 0$ the solution is

$$u(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x.$$

The boundary condition $u(0) = 0$ implies $c_1 = 0$, and from $u'(\pi) = 0$ we conclude that $\sqrt{\lambda} = \frac{1}{2}(2n+1)$. Therefore we obtain the following sequence of eigenvalues and their corresponding eigenfunctions

$$\lambda_n = \left(\frac{2n+1}{2}\right)^2, \; u_n(x) = \sin\left(\frac{2n+1}{2}\right) x, \; n \in \mathbb{N}_0.$$

The eigenvalues are clearly real and it is a simple matter to verify that $\langle u_n, u_m \rangle = \int_0^\pi u_n(x) u_m(x) dx = 0$ whenever $m \neq n$.

2.19 (a) $\rho = 1/x^2$, (b) $e^{-x^2/2}$, (c) $\rho = e^{-x^3/3}$, (d) $1/x$.

2.20 The solutions of $u'' + \lambda u = 0$, where $\lambda > 0$, under the boundary condition $u(0) = 0$ is $u(x) = c \sin \sqrt{\lambda} x$.

If $h = 0$ then the second boundary condition $u'(l) = 0$ yields the sequence of eigenvalues

$$\lambda_n = \left(\frac{2n+1}{2l}\pi\right)^2, \quad n \in \mathbb{N}_0,$$

and the corresponding sequence of eigenfunctions

$$u_n(x) = \sin \sqrt{\lambda_n} x, \quad x \in [0, l].$$

If $h < 0$ then we follow the method of Example 2.17, except that the line $y = -\alpha/hl$ now has positive, instead of negative, slope. Its points of intersection with the curve $y = \tan \alpha$ (which determine α_n) will therefore be in the upper half plane. It intersects the first branch of $\tan \alpha$ in the first interval $(0, \pi/2)$ only if its slope is greater than 1 (the slope of $\tan \alpha$ at 0), that is, only if $-1/hl > 1$. From Figure 2.2 we would expect a shift to the right in the value of α_n as a result, so that $n\pi < \alpha_n < (n + \frac{1}{2})\pi$. Hence the eigenvalues $\lambda_n = \alpha_n^2/l^2$ in this case will be greater than the corresponding eigenvalues of Example 2.17, and α_n behaves like $\left(n + \frac{1}{2}\right)\pi$ for large values of n.

2.21 $\rho = e^{2x}$, $\lambda_n = n^2\pi^2 + 1$, $u_n(x) = e^{-x} \sin \lambda_n x$.

2.22 The equation is of the Cauchy-Euler, so we assume a solution of the form x^m, where m satisfies $m(m-1)-\lambda=0$. Therefore $m=\frac{1}{2}\pm\frac{1}{2}\sqrt{1-4\lambda}=\frac{1}{2}\pm\beta$. If $\lambda<1/4$ then β is positive and the solution is $\sqrt{x}(c_1x^{\beta}+c_2x^{-\beta})$. The boundary conditions give $c_1+c_2=0$ and $c_1e^{\beta}+c_2e^{-\beta}=0$, which implies $c_1=c_2=0$. If $\lambda=1/4$, the solution takes the form $u(x)=\sqrt{x}(c_1+c_2\log x)$, and again the boundary conditions imply $c_1=c_2=0$. If $\lambda>1/4$, then $m=\frac{1}{2}\pm i\sqrt{\lambda-1/4}$, and the solution of $x^2u''+\lambda u=0$ is given by

$$u(x)=\sqrt{x}\left[c_1\cos\left(\sqrt{\lambda-1/4}\log x\right)+c_2\sin\left(\sqrt{\lambda-1/4}\log x\right)\right].$$

The condition $u(0)=0$ implies $c_1=0$, and $u(e)=0$ implies $\sqrt{\lambda-1/4}=n\pi$. Thus the eigenvalues and eigenfunctions are

$$\lambda_n=n^2\pi^2+\frac{1}{4},\quad u_n(x)=\sqrt{x}\sin(n\pi\log x),\quad n\in\mathbb{N}.$$

2.23 If $\lambda\leq 0$ the boundary-value problem has only the trivial solution, as shown in Example 2.16. If $\lambda>0$, the solution of the differential equation can be represented by $c_1\cos\sqrt{\lambda}x+c_2\sin\sqrt{\lambda}x$ as we have been doing, or, more conveniently in this case, by $c\sin(\sqrt{\lambda}x+\phi)$, where c is a non-zero arbitrary constant and ϕ is an arbitrary phase angle. Applying the boundary conditions to the second expression yields

$$(b-a)\sqrt{\lambda}=n\pi,\quad n\in\mathbb{N}.$$

Hence the eigenvalues and eigenfunctions are

$$\lambda_n=\frac{n^2\pi^2}{(b-a)^2},\quad u_n(x)=\sin\left(\frac{n\pi(x-a)}{b-a}\right).$$

2.24 This is a Cauchy-Euler equation, hence we solve $m(m-1)-m+\lambda=m^2-2m+\lambda=0$ to obtain

$$m=\frac{2\pm\sqrt{4-4\lambda}}{2}=1\pm\sqrt{1-\lambda}.$$

If $\lambda\leq 1$ the only solution of the equation which satisfies $u(1)=u(e)=0$ is the trivial solution (see Exercise 2.22), so we assume $\lambda>1$. In this case the solution is

$$u(x)=x\left[c_1\cos(\sqrt{\lambda-1}\log x)+c_2\sin(\sqrt{\lambda-1}\log x)\right]$$

and the boundary conditions imply $c_1=0$ and $\sqrt{\lambda-1}=n\pi$. Hence the eigenvalues and eigenfunctions are

$$\lambda_n=n^2\pi^2+1,\quad u_n(x)=x\sin(n\pi\log x),\quad n\in\mathbb{N}.$$

To write the orthogonality relation between the eigenfunctions, we first have to put the differential equation in standard Sturm-Liouville form by dividing by x^3 (see Equation (2.29)),

$$\frac{1}{x}u'' - \frac{1}{x^2}u' + \lambda\frac{1}{x^3}u = 0,$$

from which we conclude that the weight function is $1/x$. Hence

$$\begin{aligned}\langle u_n, u_m\rangle &= \int_1^e x^2 \sin(n\pi \log x)\sin(m\pi\log x)\frac{dx}{x^3} \\ &= \frac{1}{\pi}\int_0^\pi \sin n\xi \sin n\xi \; d\xi \\ &= 0 \quad \text{for all } n \neq m.\end{aligned}$$

2.25 If $\alpha_2 = 0$ then $f(a) = g(a) = 0$ and $(f'\bar{g} - f\bar{g}')(a) = 0$. If $\alpha_2 \neq 0$ then $f'(a) = \alpha_1 f(a)/\alpha_2$ and $g'(a) = \alpha_1 g(a)/\alpha_2$. Therefore $(f'\bar{g} - f\bar{g}')(a) = \alpha_1\alpha_2^{-1}[f(a)\bar{g}(a) - f(a)\bar{g}(a)] = 0$. Similarly $(f'\bar{g} - f\bar{g}')(b) = 0$.

2.26 This follows from the equality

$$\begin{aligned}p(b)\,[f'(b)\bar{g}(b) - f(b)\bar{g}'(b)] - p(a)[f'(a)\bar{g}(a) - f(a)\bar{g}'(a)] = \\ [p(b) - p(a)][f'(a)\bar{g}(a) - f(a)\bar{g}'(a)].\end{aligned}$$

2.27 (a), (b), (c) and (f).

2.28 Boundary conditions (c), (e), and (f) define singular problems because $p(0) = 0$ in each case.

Condition (d) does not satisfy Equation (2.26), so it does not define an SL problem.

2.29 Change the independent variable to $\xi = x + 3$ and solve the SL problem

$$\begin{aligned}\left[\xi^2 z'(\xi)\right]' + \lambda z = 0, \quad \xi \in [1, 4], \\ z(1) = z(4) = 0.\end{aligned}$$

The roots of the equation $m^2 + m + \lambda = 0$ are $-\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4\lambda}$. As in Exercise 2.22, only when $\lambda > 1/4$ do we get a non-trivial solution, given by

$$z(\xi) = \xi^{-1/2}\left[c_1 \cos\left(\sqrt{\lambda - 1/4}\log\xi\right) + c_2 \sin\left(\sqrt{\lambda - 1/4}\log\xi\right)\right].$$

Now the boundary conditions at $\xi = 1$ implies $c_1 = 0$ and from the condition at $\xi = 4$ we arrive at the sequence of eigenvalues

$$\lambda_n = \left(\frac{n\pi}{\log 4}\right)^2 + \frac{1}{4}, \quad n \in \mathbb{N}.$$

In terms of the original variables, the corresponding eigenfunctions are

$$y_n(x) = (x+3)^{-1/2} \sin\left(\frac{n\pi}{\log 4} \log(x+3)\right).$$

2.30 (a) This SL problem is a special case of Exercise 2.20 in which $l = \pi$ and $h = -2$. Therefore the positive eigenvalues λ are determined by the solutions of the transcendental equation $\tan\alpha = \alpha/2\pi$, where $\alpha = \pi\sqrt{\lambda} > 0$. These may be obtained graphically, as in Example 2.17, and form a positive increasing sequence $\alpha_1, \alpha_2, \alpha_3, ...$ which tends to ∞, and $\lambda_n = \alpha_n^2/\pi^2$.

If $\lambda = 0$ the solution of $u'' = 0$ is $c_1 x + c_2$, which does not satisfy the boundary conditions $u(0) = 0$ and $2u(\pi) - u'(\pi) = 0$ unless $c_1 = c_2 = 0$.

If $\lambda < 0$, the solution is expressed in terms $\cosh\sqrt{-\lambda}x$ and $\sinh\sqrt{-\lambda}x$ and the resulting transcendental equation which determine the eigenvalues is $\tanh\alpha = \alpha/2\pi$ with $\alpha = \pi\sqrt{-\lambda} > 0$. The function $f(\alpha) = \tanh\alpha - \alpha/2\pi$ satisfies $f(0) = 0,\ f(\alpha) \to -\infty,\ f'(0) = 1 - 1/2\pi > 0$, and $f'(\alpha) = 0$ has a unique positive solution where f attains its maximum value. Hence f has a single zero on $(0, \infty)$ which we denote α_0, and the problem therefore has a single negative eigenvalue $\lambda_0 = -\alpha_0^2/\pi^2$.

(b) The eigenfunctions corresponding to the positive eigenvalues are

$$u_n(x) = \sin\sqrt{\lambda_n}x = \sin(\alpha_n x/\pi), \quad n \in \mathbb{N},$$

and the eigenfunction corresponding to $\lambda_0 = -\alpha_0^2/\pi^2$ is

$$u_0(x) = \sinh(\alpha_0 x/\pi).$$

2.31 Imposing the boundary conditions on the solution $u(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$ leads to the pair of equations

$$\sqrt{\lambda}c_2 = c_1, \quad \sqrt{\lambda}\left(c_2 \cos\sqrt{\lambda}l - c_1 \sin\sqrt{\lambda}l\right) = 0.$$

Together these imply

$$\frac{1}{\sqrt{\lambda}} = \tan\sqrt{\lambda}l.$$

The solutions of $\tan\alpha = l/\alpha$ in $(0, \infty)$ is an increasing sequence $\alpha_1, \alpha_2, \alpha_3, ...$ such that $(n-1)\pi < \alpha_n < (n - \frac{1}{2})\pi$, which is defined by the points of intersection of $\tan\alpha$ and the hyperbola l/α. The eigenvalues and eigenfunctions are

$$\lambda_n = \alpha_n^2/l^2, \quad u_n(x) = \frac{\alpha_n}{l}\cos\left(\frac{\alpha_n}{l}x\right) + \sin\left(\frac{\alpha_n}{l}x\right), \quad n \in \mathbb{N}.$$

$\lambda = 0$ is not an eigenvalue.

If $\lambda < 0$ we follow the procedure of the solution to Exercise 2.30 to conclude that $\tanh \alpha = -l/\alpha$ has no solution on $\alpha = \sqrt{-\lambda} > 0$.

2.32 Multiply the differential equation by $\bar{u}$ and integrate over $[a, b]$ to obtain

$$\begin{aligned} \int_a^b (pu')'\bar{u}\ dx + \int_a^b r\,|u|^2\ dx + \lambda \int_a^b |u|^2\ dx &= 0 \\ -\int_a^b p\,|u'|^2\ dx + \int_a^b r\,|u|^2\ dx + \lambda \int_a^b |u|^2\ dx &= 0, \end{aligned}$$

where we integrated by parts and used the boundary conditions to arrive at the second equation. Therefore

$$\begin{aligned} \lambda \int_a^b |u|^2\ dx &= \int_a^b p\,|u'|^2\ dx - \int_a^b r\,|u|^2\ dx \\ &\geq -\int_a^b r\,|u|^2\ dx \\ &\geq -c\int_a^b |u|^2\ dx. \end{aligned}$$

Dividing by $\int_a^b |u|^2\ dx$ yields the desired result.

2.33 By the CBS inequality and Equation (2.42) $|\langle Tu, u\rangle| \leq \|Tu\| \leq \|T\|$ for all $u \in C([a, b])$, $\|u\| = 1$. Therefore

$$\sup_{\|u\|=1} |\langle Tu, u\rangle| \leq \|T\|.$$

On the other hand,

$$\begin{aligned} \langle T(u+v), u+v\rangle &= \langle Tu, u\rangle + \langle Tv, v\rangle + 2\,\mathrm{Re}\langle Tu, v\rangle \\ \langle T(u-v), u-v\rangle &= \langle Tu, u\rangle + \langle Tv, v\rangle - 2\,\mathrm{Re}\langle Tv, u\rangle \end{aligned}$$

Let $N(T)$ denote $\sup_{\|u\|=1} |\langle Tu, u\rangle|$. For any $w \neq 0$ we can then write $|\langle Tw, w\rangle| = \|w\|^2 |\langle Tw/\|w\|, w/\|w\|\rangle| \leq \|w\|^2 N(T)$. Consequently,

$$\begin{aligned} 4\,\mathrm{Re}\langle Tu, v\rangle &= \langle T(u+v), u+v\rangle - \langle T(u-v), u-v\rangle \\ &\leq |\langle T(u+v), u+v\rangle| + |\langle T(u-v), u-v\rangle| \\ &\leq \|u+v\|^2 N(T) + \|u-v\|^2 N(T) \\ &\leq 2\left(\|u\|^2 + \|v\|^2\right) N(T). \end{aligned}$$

Setting $\|u\| = 1$ and $v = Tu/\|Tu\|$, we obtain $\|T\| = \sup \|Tu\| \leq N(T)$.

Chapter 3

3.1 $\langle e^{in\pi x/l}, e^{im\pi x/l}\rangle = \int_{-l}^{l} e^{i(n-m)\pi x/l}dx = \frac{l}{i(n-m)\pi}\left[e^{i(n-m)\pi} - e^{-i(n-m)\pi}\right] = 0$ for all $m \neq n$.

$e^{in\pi x/l}/\left\|e^{in\pi x/l}\right\| = e^{in\pi x/l}/\sqrt{2l}, \quad n \in \mathbb{Z}.$

3.2 No, because its sum is discontinuous at $x = 0$ (Example 3.4).

3.3

$$\begin{aligned}
a_0 &= \frac{1}{2}\int_{-1}^{1} f(x)dx = \frac{1}{2}\int_0^1 dx = \frac{1}{2},\\
a_n &= \int_{-1}^{1} f(x)\cos n\pi x\ dx = \int_0^1 \cos n\pi x\ dx = \frac{1}{n\pi}\sin n\pi = 0,\\
b_n &= \int_0^1 \sin n\pi x\ dx = \frac{1}{n\pi}\left(1-\cos n\pi\right) = \frac{1}{n\pi}[1-(-1)^n], \quad n \in \mathbb{N}.
\end{aligned}$$

Hence

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x + \ \cdots\right).$$

3.4 $\pi - |x| = \dfrac{\pi}{2} + \dfrac{4}{\pi}\sum_{n=0}^{\infty} \dfrac{1}{(2n+1)^2}\cos(2n+1)x$. Uniformly convergent by the Weierstrass M-test with $M_n = (4/\pi)(2n+1)^{-2}$.

3.5

$$\begin{aligned}
a_0 &= \frac{1}{4}\int_{-2}^{2}(x^2+x)dx = \frac{4}{3},\\
a_n &= \frac{1}{2}\int_{-2}^{2}(x^2+x)\cos\frac{n\pi}{2}x\ dx = \frac{16}{n^2\pi^2}(-1)^n,\\
b_n &= \frac{1}{2}\int_{-2}^{2}(x^2+x)\sin\frac{n\pi}{2}x\ dx = -\frac{4}{n\pi}(-1)^n, \quad n \in \mathbb{N}.\\
x^2 + x &= \frac{4}{3} + \frac{4}{n\pi}\sum_{n=1}^{\infty}(-1)^n\left(\frac{4}{n\pi}\cos\frac{n\pi}{2}x - \sin\frac{n\pi}{2}x\right)
\end{aligned}$$

The convergence of the Fourier series to $f(x) = x^2 + x$ on the interval $[-2, 2]$ is not uniform for the following reason. Since f is piecewise smooth, its Fourier series S_n converges pointwise on $[-2, 2]$ to some function S. If the convergence were uniform S would be continuous on $[-2, 2]$. But, f being in $\mathcal{L}^2(-2, 2)$, we also have $\|f - S\| = 0$ by Theorem 3.2. Because both f and S are continuous, $f = S$ pointwise on $[-2, 2]$, because otherwise equality would not hold in $\mathcal{L}^2$. But $S(-2) = S(2)$ whereas $f(2) \neq f(-2)$, so S cannot be continuous, and this contradicts the assumption that the convergence of S_n to f is uniform. It turns out (see Corollary 3.15) that the periodicity of f, as well as its continuity, are necessary for the convergence to be uniform.

3.6 Use the M-test.

3.7 The series $S_n(x) = \sum_{k=1}^n (a_k \cos kx + b_k \sin kx)$ satisfies

$$\begin{aligned} \|S_n - S_m\|^2 &= \left\| \sum_{k=m+1}^n (a_k \cos kx + b_k \sin kx) \right\|^2 \\ &= \pi \sum_{k=m+1}^n (a_k^2 + b_k^2), \end{aligned}$$

for all $n > m$, due to the orthogonality of $\{\cos kx, \sin kx\}$ in$\mathcal{L}^2(-\pi, \pi)$. Therefore (S_n) is a Cauchy sequence in $\mathcal{L}^2(-\pi, \pi)$, and hence convergent, if and only if $\sum_{k=m+1}^n (a_k^2 + b_k^2)$ converges.

3.8 Since f is periodic in p,

$$\begin{aligned} \int_0^x f(t)dt &= \int_0^x f(t+p)dt \\ &= \int_p^{x+p} f(t)dt \\ &= \int_p^x f(t)dt + \int_x^{x+p} f(t)dt, \end{aligned}$$

from which Equation (3.14) follows.

3.9 (a) and (c) are piecewise continuous. (b), (d) and (e) are piecewise smooth.

3.10 f is continuous on $\mathbb{R}$. $f'(x) = 1$ when $x > 0$, and $f'(x) = -1$ when $x < 0$, hence f is piecewise smooth on $\mathbb{R}$. At $x = 0$,

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

does not exist, hence f is not differentiable at $x = 0$.

The function $f(x) = [x]$ is piecewise smooth on $\mathbb{R}$ but not differentiable at any $n \in \mathbb{Z}$.

3.11 f is clearly smooth on $\mathbb{R}\backslash\{0\}$. At $x = 0$, we have

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0} h \sin(1/h) \\ &= 0, \end{aligned}$$

hence f is differentiable at 0. On the other hand, for all $x \neq 0$,

$$f'(x) = 2x \sin(1/x) - \cos(1/x)$$

does not approach a limit as $x \to 0^+$ or $x \to 0^-$, so f is not piecewise smooth on $\mathbb{R}$.

3.12 According to Definition 3.6, if f is piecewise smooth on $(-l, l)$ it is also piecewise smooth on $[-l, l]$. Being periodic in $2l$, f is therefore piecewise smooth on any interval of the form $[kl, (k+2)l]$, where k is any integer, and hence on any finite union of such intervals. If I is any finite interval in $\mathbb{R}$, then there is a positive integer n such that $I \subseteq [-nl, nl] = \cup_{k=-n}^{n-1}[kl, (k+2)l]$, from which we conclude that f is piecewise smooth on I. By Definition 3.6, f is therefore piecewise smooth on I.

3.13 Let I be a finite interval and f be piecewise smooth on I. Then both f and f' are continuous on I except at a finite number of points $\{x_1, x_2, ..., x_n\}$, where the left-hand and right-hand limits exist, and the one-sided limits at the endpoints of I also exist. Similarly for g. If $\{\xi_1, \xi_2, ..., \xi_m\}$ are the points of (jump) discontinuity of g and g', then it follows that the functions $f + g$ and $(f+g)' = f'+g'$ are continuous on I except at $\{x_1, ..., x_n\} \cup \{\xi_1, ..., \xi_m\}$, where the left-hand and right-hand limits exist, as well as the one-sided limits at the endpoints of I. If I is unbounded, then f and g, and therefore $f+g$, are smooth on any bounded subinterval. f/g is also piecewise smooth on I provided g has no zeros in I. If $g(x) = 0$ on some subset A of I, then f/g is only defined on $I \backslash A$. It is piecewise smooth on $I \backslash N(A)$, where $N(A)$ is any neighborhood of A, that is, an open set in I which contains A. This is to ensure the existence of the limit of $f(x)/g(x)$ as x approaches a zero of g either from the left or from the right.

3.14 From the expression

$$D_n(\alpha) = \frac{\sin\left(n + \frac{1}{2}\right)\alpha}{2\pi \sin \frac{1}{2}\alpha}, \quad \alpha \neq 0, \pm 2\pi, \pm 4\pi, ...$$

it is clear that $D_n(\alpha) = 0$ when $\left(n + \frac{1}{2}\right)\alpha = k\pi$, that is, when $\alpha = 2k\pi/(2n + 1)$ for $k = \pm 1, \pm 2, ...$. Therefore the zeros of D_n are located at

$$\frac{\pm 2\pi}{2n+1}, \frac{\pm 4\pi}{2n+1}, \frac{\pm 8\pi}{2n+1}, \dots$$

Its maximum value is $\frac{1}{2\pi}(2n + 1)$, located at the points $\alpha = 2k\pi,\ k \in \mathbb{Z}$.

3.15 (a) $2\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n} \sin nx$.

(b) $\dfrac{1}{2} - \dfrac{4}{\pi^2} \sum_{k=0}^{\infty} \dfrac{1}{(2k+1)^2} \cos(2k+1)\pi x$.

(c) $\dfrac{1}{2} - \dfrac{1}{2}\cos 2x$.

(d) $\cos^3 x = \dfrac{3}{4}\cos x + \dfrac{1}{4}\cos 3x$.

(e) $\frac{1}{2}\sinh 2 + \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\frac{n^2\pi^2}{4}} \left[\sinh 2 \cos\left(\tfrac{n\pi}{2}x\right) - \tfrac{n\pi}{2}\sinh 2 \sin\left(\tfrac{n\pi}{2}x\right)\right]$.

(f) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \left(\tfrac{12l^3}{n^3\pi^3} - \tfrac{2l^3}{n\pi}\right)\sin\left(\tfrac{n\pi}{l}x\right)$.

3.16 The convergence is uniform where f is continuous on $\mathbb{R}$, hence in (b), (c), and (d).

3.17 In Exercise 3.15 (e), $S(\pm 2) = \frac{1}{2}[f(2^+) + f(2^-)] = \frac{1}{2}(e^2 + e^{-2})$, and in (f), $S(\pm l) = 0$.

3.18 Using the fact that $x = \pi/2$ is a point of continuity for f, we have $f(\pi/2) = S(\pi/2)$, where $S(x)$ is the Fourier series obtained in Exercise 3.15(a). Therefore

$$\frac{\pi}{2} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi}{2}.$$

Since $\sin(n\pi/2)$ is 0 when n is even and $(-1)^k$ when $n = 2k+1,\ k \in \mathbb{N}_0$, we obtain

$$\pi = 4\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$

3.19 Setting $x = \pi$ in the Fourier expansion of $|x|$ (Exercise 3.15(b)) yields

$$\pi^2 = 8\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

3.20 Setting $t = 0$ in the Fourier series representation of $f(t)$, we obtain

$$\pi = 2 - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}.$$

3.21 $x^2 = \dfrac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2}\cos nx$. Evaluating at $x = \pi$ and $x = 0$, we arrive at

(a) $\sum_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$.

(b) $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} = -\dfrac{\pi^2}{12}$.

By adding and subtracting these results, we obtain

(c) $\sum_{n=1}^{\infty} \dfrac{1}{(2n)^2} = \dfrac{\pi^2}{24}$.

(d) $\sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^2} = \dfrac{\pi^2}{8}$.

3.22 $|\sin x|$ is an even function whose Fourier coefficients are $a_0 = 2/\pi$, $a_n = 0$ when n is odd, and $a_n = -4/(n^2-1)\pi$ when n is even. Its series representation is therefore

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)} \cos 2nx.$$

This series is uniformly convergent by the M-test with $M_n = (4n^2-1)^{-1}$. Setting $x = 0$ and $x = \pi/2$ gives

$$\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)} = \frac{1}{2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(4n^2-1)} = \frac{\pi-2}{4}.$$

3.23 Since $\sin x$ is a C^∞ function, $f(x) = |\sin x|$ is continuous on $\mathbb{R}$ whereas $f'(x)$ and $f''(x)$ are continuous except at the points $x = n\pi$, $n \in \mathbb{Z}$, where f vanishes. Since $f'(n\pi^{\pm}) = \pm 1$ and $f''(n\pi^{\pm}) = 0$, the functions f' and f'' are piecewise continuous, hence f' is piecewise smooth.

f' is an odd function which is periodic in π with $f'(x) = \cos x$ on $[0, \pi]$. Its Fourier coefficients are $b_n = \frac{2n}{\pi}\left[\frac{1+(-1)^n}{n^2-1}\right]$, $n > 1$, $b_1 = 0$, and its series representation is

$$S(x) = \frac{8}{\pi}\sum_{k=1}^{\infty} \frac{k}{4k^2-1} \sin 2kx.$$

$S(n\pi) = \frac{1}{2}[f'(n\pi^+) + f'(n\pi^-)] = \frac{1}{2}[1 + (-1)] = 0$ (points of discontinuity of f'). $S(\frac{\pi}{2} + n\pi) = f'(\frac{\pi}{2} + n\pi) = 0$ (points of continuity).

3.24 (a) Since f is continuous on $[0, l]$ and periodic in $2l$, we need only check continuity at $x = 0$ and $x = l$. Being an even function, $\tilde{f}_e$ has the same left-hand and right-hand limits at $x = 0$ and, being periodic as well, it satisfies $\lim_{x\to l^+} \tilde{f}_e(x) = \lim_{x\to(-l)^+} \tilde{f}_e(x) = \lim_{x\to l^-} \tilde{f}_e(x)$. Hence $\tilde{f}_e$ is continuous on $\mathbb{R}$.

$\tilde{f}_o$, on the other hand, is continuous at 0 if, and only if, $\lim_{x\to 0^-} \tilde{f}_o(x) = \lim_{x\to 0^+} \tilde{f}_o(x)$. But since

$$\lim_{x\to 0^-} \tilde{f}_o(x) = -\lim_{x\to 0^-} \tilde{f}_o(-x) = -\lim_{x\to 0^+} \tilde{f}_o(x),$$

it is clear that we have continuity at $x = 0$ if, and only if $\lim_{x\to 0^+} \tilde{f}_o(x) = f(0) = 0$. A similar argument shows that continuity at $x = l$ is achieved if, and only if, $\lim_{x\to l^-} \tilde{f}_o(x) = f(l) = 0$.

(b)
$$\tilde{f}_e(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{l}x, \quad \tilde{f}_o(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l}x,$$

where

$$\begin{aligned} a_0 &= \frac{1}{l}\int_0^l f(x)dx, \\ a_n &= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi}{l}x\ dx, \\ b_n &= \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x\ dx. \end{aligned}$$

(c) $a_0 = \int_0^1 x\ dx = 1/2$, $a_n = 2\int_0^1 x\cos n\pi x\ dx = -2[1-(-1)^n]/n^2\pi^2$, $b_n = 2\int_0^1 x\sin n\pi x\ dx = -2(-1)^n/n\pi$. Therefore

$$\begin{aligned} \tilde{f}_e(x) &= \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos(2k+1)\pi x, \\ \tilde{f}_0(x) &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi x. \end{aligned}$$

3.25 (a) The even periodic extension of $\sin x$ from $[0,\pi]$ to $\mathbb{R}$ is a continuous function which coincides with its Fourier (cosine) series expansion at every point in $\mathbb{R}$. At $x = 0$ and $x = \pi$, the function is 0.

(b) The odd periodic extension of $f(x) = \cos x$ has jump discontinuities at integral multiples of π, where the left-hand and right-hand limits have opposite signs. $f(0) = 1$, $f(\pi) = -1$, $S(0) = S(\pi) = 0$.

(c) As in (b), the periodic extension here is also not continuous at integral multiples of π, and we have $f(0) = 1$, $f(\pi) = e^\pi$, $S(0) = S(\pi) = 0$.

3.26 The function $f(x) = |x|^{1/2}$ being even and continuous on $[-\pi,\pi]$, it suffices to show that its cosine series $\sum a_n \cos nx$ converges uniformly on $[-\pi,\pi]$.

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi x^{1/2}\cos nx\,dx, \quad n \in \mathbb{N} < \\
&= \frac{2}{\pi}\left[\frac{1}{n}x^{1/2}\sin nx\Big|_0^\pi - \frac{1}{2n}\int_0^\pi x^{-1/2}\sin nx\,dx\right] \\
&= -\frac{1}{\pi n}\int_0^\pi x^{-1/2}\sin nx\,dx \\
&= -\frac{1}{\pi n^{3/2}}\int_0^{n\pi} y^{-1/2}\sin y\,dy.
\end{aligned}$$

The integral

$$\int_0^{n\pi} y^{-1/2}\sin y\,dy = \sum_{k=0}^{n-1}\int_{k\pi}^{(k+1)\pi} y^{-1/2}\sin y\,dy = \sum_{k=0}^{n-1}(-1)^k A_k$$

converges by the alternating series test, where $A_k = \int_{k\pi}^{(k+1)\pi} y^{-1/2}|\sin y|\,dy$ is a decreasing sequence which tends to 0. Hence $\left|\int_0^{n\pi} y^{-1/2}\sin y\,dy\right|$ is bounded by some positive constant M, and it follows that the numerical series $\sum |a_n|$ converges by comparison with $\sum M/\pi n^{3/2}$. $\sum a_n \cos nx$ therefore converges uniformly in $[-\pi,\pi]$ to a continuous function $S(x)$. With $|x|^{1/2}$ in $\mathcal{L}^2(-\pi,\pi)$ we also know from Theorem 3.2 that $\sum a_n \cos nx$ converges to $|x|^{1/2}$ in $\mathcal{L}^2(-\pi,\pi)$. Since both $S(x)$ and $|x|^{1/2}$ are continuous, they coincide and the convergence is pointwise in $[-\pi,\pi]$.

3.27 (a) Solving the heat equation by separation of variables, and applying the boundary conditions at $x=0$ and $x=\pi$, yields the sequence of solutions $u_n(x,t) = b_n e^{-n^2 t}\sin nx$, $n \in \mathbb{N}$. Applying the initial condition to the general solution $u(x,t) = \sum b_n e^{-n^2 t}\sin nx$, and using the identity

$$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$$

leads to

$$u(x,t) = \frac{3}{4}e^{-t}\sin x - \frac{1}{4}e^{-9t}\sin 3x.$$

(b) $u(x,t) = e^{-k\pi^2 t/4}\sin\dfrac{\pi x}{2} - e^{-25k\pi^2 t/36}\sin\dfrac{5\pi x}{6}$.

3.28 Suppose $v(x,t)$ is a solution of the heat equation and satisfies the homogeneous boundary conditions $v(0,t) = v(l,t) = 0$. Then we know (see Equation (3.30)) that it has the form

$$v(x,t) = \sum_{n=0}^{\infty} b_n e^{-(n\pi/l)^2 t}\sin\frac{n\pi}{l}x.$$

Assume $u(x,t) = v(x,t) + \psi(x)$, where ψ is the correction on v which is needed for u to satisfy the nonhomogeneous conditions. To determine ψ, substitute into the heat equation and use the fact that both u and v are solutions, to obtain $\psi''(x) = 0$. Thus $\psi(x) = ax + b$. Now apply the nonhomogeneous boundary conditions to $u = v + \psi$ to conclude that $\psi(0) = T_0$ and $\psi(l) = T_1$, and therefore

$$\psi(x) = \frac{T_1 - T_0}{l}x + T_0.$$

Finally, apply the initial condition at $t = 0$ to $u = v + \psi$ in order to determine b_n,

$$\begin{aligned} u(x,0) &= v(x,0) + \psi(x) \\ f(x) &= \sum_{n=0}^{\infty} b_n \sin\frac{n\pi}{l}x + \frac{T_1 - T_0}{l}x + T_0. \\ \Rightarrow\ b_n &= \frac{2}{l}\int_0^l \left[f(x) - \frac{T_1 - T_0}{l}x - T_0\right]\sin\frac{n\pi}{l}x\ dx. \end{aligned}$$

This determines the solution $v(x,t) + \psi(x)$ completely.

3.29 Use the procedure outlined in Section 3.3.2 to obtain

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{l}x\ \cos\frac{n\pi}{l}t,$$

where $a_n = \dfrac{2}{l}\displaystyle\int_0^l x(l-x)\sin\frac{n\pi}{l}x\ dx$.

3.30 The general solution of the wave equation under the boundary conditions at $x = 0$ and $x = l$ is given by Equation (3.40), with

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x\ dx, \quad b_n = \frac{2}{cn\pi}\int_0^l g(x)\sin\frac{cn\pi}{l}x\ dx.$$

D'Alembert's formula can be proved by substituting the above expressions for a_n and b_n back into Equation (3.40),

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{l}t + b_n\sin\frac{n\pi}{l}t\right)\sin\frac{n\pi}{l}x,$$

and using the periodicity properties of $\tilde{f}$ and $\tilde{g}$.

3.31 Assume $u(x,t) = v(x,t) + \psi(x)$ where v satisfies the homogeneous wave equation with homogeneous boundary conditions at $x = 0$ and $x = l$. This leads to $\psi(x) = \frac{g}{2c^2}(x^2 - lx)$, and $v(x,t) = \sum_{n=1}^{\infty} a_n \sin\dfrac{n\pi}{l}x\ \cos\dfrac{cn\pi}{l}t$, with $a_n = -\dfrac{2}{l}\displaystyle\int_0^l \psi(x)\sin\frac{n\pi}{l}x\ dx$.

3.32 Let $u(x,t) = v(x)w(t)$ and substitute into the differential equation to arrive at the pair of equations $v'' + \lambda^2 v = 0$ and $w'' + kw' + \lambda^2 c^2 w = 0$. Under homogeneous boundary conditions, the first equation is solved by

$$v_n(x) = a_n \sin \frac{n\pi}{10} x, \quad n \in \mathbb{N}.$$

The general solution of the second equation is

$$w_n(t) = e^{-kt/2}(b_n \cosh \mu_n t + c_n \sinh \mu_n t),$$

where $\mu_n = \frac{1}{2}\sqrt{k^2 - 4(cn\pi/10)^2}$, assuming $k^2 \neq 4(cn\pi/10)^2$. Since the string is released from rest, $u_t(x,0) = v(x)w'(0) = 0$ for all $x \in (0,10)$, hence $w'(0) = 0$. This implies $c_n = kb_n/2\mu_n$ and

$$w_n(t) = b_n e^{-kt/2}\left(\cosh \mu_n t + \frac{k}{2\mu_n}\sinh \mu_n t\right).$$

Before applying the initial (nonhomogeneous) condition we use superposition to form the solution

$$u(x,t) = \sum_{n=1}^{\infty} v_n(x) w_n(t).$$

Now $u(x,0) = \sum_{n=1}^{\infty} v_n(x) w_n(0) = \sum_{n=1}^{\infty} d_n \sin(n\pi x/10) = 1$ implies

$$d_n = \frac{2}{10}\int_0^{10} \sin \frac{n\pi x}{10} dx = \frac{2}{n\pi}(1 - \cos n\pi),$$

$$u(x,t) = \sum_{n=1}^{\infty} d_n \sin \frac{n\pi x}{10} e^{-kt/2}\left(\cosh \mu_n t + \frac{k}{2\mu_n}\sinh \mu_n t\right).$$

It should be noted that, as n increases, there is a positive integer N such that $k^2 - 4(cn\pi/10)^2 < 0$ and $\mu_n = \frac{i}{2}\sqrt{4(cn\pi/10)^2 - k^2} = i\alpha_n$ for all $n \geq N$, where α_n is a positive number. In the above infinite series representation for u, the sum over $n \geq N$ converges because $\cosh i\alpha_n t = \cos \alpha_n t$ and $\frac{1}{i\alpha_n}\sinh i\alpha_n t = \frac{1}{\alpha_n}\sin \alpha_n t$.

3.33 Assume $u(x,y,t) = v(x,y)w(t)$ and conclude that $w''/w = c^2 \Delta v/v = -\lambda^2$ (separation constant). Hence $w(t) = A\cos \lambda t + B \sin \lambda t$. Assume $v(x,y) = X(x)Y(y)$, and use the given boundary conditions to conclude that

$$\lambda = \lambda_{mn} = \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\pi, \quad m, n \in \mathbb{N},$$

$$X(x) = \sin \frac{n\pi}{a} x, \quad Y(y) = \sin \frac{m\pi}{b} y,$$

$$u_{mn}(x,y,t) = (A_{mn} \cos \lambda_{mn} ct + B_{mn} \sin \lambda_{mn} ct) \sin \frac{n\pi}{a} x \ \sin \frac{m\pi}{b} y.$$

Apply the initial conditions to the solution

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}u_{mn}(x,y,t)$$

to evaluate the coefficients. This yields

$$A_{mn}=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{n\pi}{a}x\ \sin\frac{m\pi}{b}y\ dxdy,$$
$$B_{mn}=\frac{4}{\lambda_{mn}ab}\int_0^a\int_0^b g(x,y)\sin\frac{n\pi}{a}x\ \sin\frac{m\pi}{b}y\ dxdy.$$

For the series representing u to converge both f and g are assumed to be piecewise smooth in x and y.

3.34 Let $u(x,y)=v(x)w(y)$ and substitute into $u_{xx}+u_{yy}=0$ to arrive at

$$\frac{v''}{v}=-\frac{w''}{w}=-\lambda^2.$$

The sign of the separation constant determines whether v is a linear combination of $\sin\lambda x$ and $\cos\lambda x$ and w is a linear combination of the corresponding hyperbolic functions $\sinh\lambda y$ and $\cosh\lambda y$ (corresponding to $-\lambda^2$), or vice-versa. But since, eventually, we shall need to expand $f(x)$ and $g(x)$ in Fourier series, we should obviously opt for $-\lambda^2$. In any case, choosing the right sign for λ^2 at the beginning is a question of convenience, for if we choose the wrong sign the worst that can happen is that λ will turn out to be imaginary. Thus

$$v_\lambda(x)=\alpha_\lambda\cos\lambda x+\beta_\lambda\sin\lambda x,\quad w(y)=\gamma_\lambda\cosh\lambda y+\delta_\lambda\sinh\lambda y.$$

The boundary conditions $u(0,y)=u(a,y)=0$ imply $v_n(x)=\beta_n\sin\lambda_n x$, where $\lambda_n=n\pi/a$. Before imposing the nonhomogeneous boundary conditions, we express u as the formal sum

$$u(x,y)=\sum_{n=1}^{\infty}(c_n\cosh\lambda_n y+d_n\sinh\lambda_n y)\sin\lambda_n x.$$

The remaining boundary conditions yield

$$f(x)=\sum_{n=1}^{\infty}c_n\sin\lambda_n x,$$

from which $c_n=\frac{2}{a}\int_0^a f(x)\sin(n\pi x/a)\ dx$, and

$$g(x)=\sum_{n=1}^{\infty}\lambda_n(d_n\cosh\lambda_n b+c_n\sinh\lambda_n b)\sin\lambda_n x,$$

from which $\lambda_n(d_n \cosh \lambda_n b + c_n \sinh \lambda_n b) = \frac{2}{a}\int_0^a g(x)\sin(n\pi x/a)\,dx$. This completely determines the coefficients c_n and d_n in the series representation of u. To ensure the convergence of the series, the functions f and g obviously have to be piecewise smooth.

3.35 Here again, as in Exercise 3.34, the separated solution of Laplace's equation is a product of trigonometric and hyperbolic functions. But in this case the homogeneous boundary conditions $u(0,y) = u(x,0) = 0$ annihilate the coefficients of $\cosh \lambda y$ and $\cos \lambda x$, the condition at $x = 1$ determines the admissible discrete values of λ, and the nonhomogeneous boundary condition $u(x,1) = \sin \frac{3\pi}{2}x$ reduces the infinite formal sum for $u(x,y)$ to the single term

$$u(x,y) = \left(\sinh \frac{3\pi}{2}\right)^{-1} \sin \frac{3\pi}{2}x \ \sinh \frac{3\pi}{2}y.$$

3.36 Starting with the solution of Laplace's equation $u(x,y) = (a\cos\lambda x + b\sin\lambda x)(c\cosh\lambda y + d\sinh\lambda y)$, the conditions at $x = 0$ and $x = \pi$ imply $\lambda b = 0$ and $\lambda = n$. If $\lambda = 0$, the solution is simply a constant. If $\lambda \neq 0$, then $b = 0$ and $\lambda = n \in \mathbb{N}$, and we can take $a = 1$ without loss of generality. The condition at $y = 0$ implies $\lambda d \cos nx = \cos x$, hence $n = d = 1$. The condition at $y = \pi$ implies $(\cosh\pi + c\sinh\pi)\cos x = 0$, hence $c = -\coth\pi$. Thus the solution is

$$u(x,y) = \cos x(\sinh y - \coth\pi\cosh y) + c_0,$$

where c_0 is an arbitrary constant. Thus the solution is determined up to an additive constant. This is consistent with the fact that this is a Neumann problem for Laplace's equation, where the boundary conditions are all on the *derivatives* of u.

3.37 (a) Substituting $u(r,\theta) = v(r)w(\theta)$ into Laplace's equation yields the two equations

$$w'' + \lambda^2 w = 0, \quad r^2 v'' + rv' - \lambda^2 v = 0,$$

whose solutions are, respectively,

$$w(\theta) = \begin{cases} a_0 + b_0\theta, & \lambda = 0 \\ a_\lambda \cos\lambda\theta + b_\lambda \sin\lambda\theta, & \lambda \neq 0, \end{cases}$$

$$v(r) = \begin{cases} c_0 + d_0 \log r, & \lambda = 0 \\ c_\lambda r^\lambda + d_\lambda r^{-\lambda}, & \lambda \neq 0. \end{cases}$$

Since $u(r,\theta + 2\pi) = u(r,\theta)$ for all θ in $(-\pi,\pi]$, it follows that $b_0 = 0$ when $\lambda = 0$, and $a_\lambda \cos\lambda(\theta+2\pi) + b_\lambda \sin\lambda(\theta+2\pi) = a_\lambda\cos\lambda\theta + b_\lambda\sin\lambda\theta$

when $\lambda \neq 0$. This last equation implies $\lambda = n \in \mathbb{N}$, and hence the sequence of solutions

$$u_n(r,\theta) = \begin{cases} c_0 + d_0 \log r, & n = 0 \\ (c_n r^n + d_n r^{-n})\,(a_n \cos n\theta + b_n \sin n\theta), & n \in \mathbb{N}. \end{cases}$$

(b) Use the fact that u must be bounded at $r = 0$ to eliminate the coefficients d_n for all $n = 0, 1, 2, \cdots$, then apply the boundary condition. The series $A_0 + \sum_{n=1}^{\infty} (r/R)^n\,(A_n \cos n\theta + B_n \sin n\theta)$ is clearly convergent for all $r < R$ by the M-test.

(c) Here $\log r$ and the positive powers of r are unbounded outside the circle $r = R$. The solution is therefore $u(r,\theta) = A_0 + \sum_{n=1}^{\infty} (R/r)^n\,(A_n \cos n\theta + B_n \sin n\theta)$, which converges in $r > R$.

Chapter 4

4.1 With $u(x) = P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$, $u'(x) = \frac{3}{2}(5x^2 - 1)$, $u''(x) = 15x$, and $\lambda = 3(3+1) = 12$, we have

$$\begin{aligned}(1-x^2)P_3''(x) - 2xP_3'(x) + 6P_3(x) &= 15x - 15x^3 - 15x^3 + 3x + 30x^3 - 18x \\ &= 0,\end{aligned}$$

hence Equation (4.4) is satisfied. It is also satisfied if $u = P_4$ and $\lambda = 4(4+1) = 20$.

4.2 $y_1 = 1$, $y_2(x) = x$, $y_3 = x^2 - \frac{1}{3}$, $y_4 = x^3 - \frac{3}{5}x$, $y_5 = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$, $y_6 = x^5 - \frac{10}{9}x^3 + \frac{5}{21}x$. Clearly, we have $y_1 = P_0$, $y_2 = P_1$, $y_3 = \frac{2}{3}P_2$, $y_4 = \frac{2}{5}P_3$, $y_5 = \frac{8}{35}P_4$, $y_6 = \frac{8}{63}P_5$.

4.3 From the recursion formula (4.8) with $k = 2j$, it follows that

$$\lim_{j\to\infty} \frac{\left|c_{2(j+1)}x^{2(j+1)}\right|}{\left|c_{2j}x^{2j}\right|} = x^2 < 1 \quad \text{for all } x \in (-1,1).$$

The same conclusion holds if $k = 2j+1$. Since $Q_0'(x) = (1-x^2)^{-1}$,

$$\lim_{x\to\pm 1}(1-x^2)Q_0'(x) = 1,$$

whereas

$$\lim_{x\to\pm 1}(1-x^2)Q_0(x) = 0.$$

4.4 From the recursion formula (4.8), with $n = 1$ and $k = 0, 2, 4, ...,$

$$\begin{aligned}
Q_1(x) &= 1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots \\
&= 1 - x\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots\right) \\
&= 1 - \frac{x}{2}\log\left(\frac{1+x}{1-x}\right).
\end{aligned}$$

4.5 The first two formulas follow from the fact that P_n is an even function when n is even, and odd when n is odd.

$$P_{2n}(0) = a_0 = \frac{(-1)^n(2n)!}{2^{2n}n!n!} = (-1)^n\frac{(2n-1)\cdots(3)(1)}{(2n)\cdots(4)(2)}.$$

4.6 Let $u(x) = u(\cos\theta) = y(\theta)$. Then

$$\begin{aligned}
u'(x) &= \frac{dy}{d\theta}\frac{d\theta}{dx} = -\frac{1}{\sin\theta}\frac{dy}{d\theta} \\
u''(x) &= -\frac{1}{\sin\theta}\frac{d^2y}{d\theta^2}\left(-\frac{1}{\sin\theta}\right) + \frac{\cos\theta}{\sin^2\theta}\left(-\frac{1}{\sin\theta}\right)\frac{dy}{dx}.
\end{aligned}$$

Substituting into Equation (4.4) with $\lambda = n(n+1)$ yields the transformed equation.

4.7 Differentiate Rodrigues' formula (4.13) for P_n and replace n by $n+1$ to obtain

$$P'_{n+1} = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[\left((2n+1)x^2 - 1\right)(x^2-1)^{n-1}\right],$$

then differentiate P_{n-1} and subtract to arrive at (4.14). The first integral formula follows directly from (4.14) and the equality $P_n(\pm 1) = (\pm 1)^n$. The second formula is a special case of the first in which $x = 1$.

4.8 The polynomial $(n+1)P_{n+1}(x) - (2n+1)xP_n(x)$ is of degree $n+1$. According to the Rodrigues formula, the coefficient of x^{n+1} is given by

$$(n+1)\frac{1}{2^{n+1}(n+1)!}\frac{(2n+2)!}{(n+1)!} - (2n+1)\frac{1}{2^n n!}\frac{(2n)!}{n!} = 0,$$

Hence we can write

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) = \sum_{k=0}^{n-1} c_k P_k(x).$$

Let m be a non-negative integer such that $m \leq n-2$, and take the inner product of $P_m(x)$ with the equation above. Because $\langle P_m, P_k\rangle = 0$ for all $k \neq m$, we obtain

$$-(n+1)\langle P_m, xP_n\rangle = -(2n+1)\langle xP_m, P_n\rangle = c_m \left\|P_m\right\|^2.$$

But xP_n is a polynomial of degree $m+1 \leq n-1$ for all $m = 0, 1, ..., n-2$. Therefore the left-hand side of the equation is zero and $c_m = 0$ for all $m = 0, 1, ..., n-2$. Thus

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) = c_{n-1}P_{n-1}(x).$$

Taking the limit as $x \to 1$ in this equation yields $c_{n-1} = (n+1) - (2n+1) = -n$.

4.9

$$(1 - 2xt + t^2)\frac{\partial f}{\partial t}(x) = (1 - 2xt + t^2)\frac{x - t}{(1 - 2xt + t^2)^{3/2}} = (x - t)f(x).$$

Using the series representation of f (Equation (4.21)) in this equation, we arrive at

$$\sum_{n=1}^{\infty} \left(na_n t^{n-1} - 2xna_n t^n + na_n t^{n+1}\right) = \sum_{n=0}^{\infty} \left(xa_n t^n - a_n t^{n+1}\right)$$

$$a_1 + \sum_{n=1}^{\infty} \left[(n+1)a_{n+1} - 2nxa_n + (n-1)a_{n-1}\right] t^n = xa_0 + \sum_{n=1}^{\infty} (xa_n - a_{n-1})t^n.$$

Equating corresponding coefficients of the powers of t gives the recursion relation for a_n.

4.10

$$\begin{aligned}
\frac{1}{|1 - re^{i\theta}|} &= \frac{1}{\sqrt{(1 - r\cos\theta)^2 + (r\sin\theta)^2}} \\
&= \frac{1}{\sqrt{1 - 2r\cos\theta + r^2}} \\
&= \sum_{n=0}^{\infty} P_n(\cos\theta)t^n, \quad |t| < 1,
\end{aligned}$$

by the formula (4.20). We can write

$$\frac{1}{\|\mathbf{y} - \mathbf{x}\|} = \frac{1}{\|\mathbf{y}\|}\frac{1}{\left\| \|\mathbf{y}\|^{-1}\mathbf{y} - \|\mathbf{y}\|^{-1}\mathbf{x} \right\|}.$$

In the plane defined by the vectors $\mathbf{x}$ and $\mathbf{y}$, let

$$\frac{\mathbf{y}}{\|\mathbf{y}\|} = (\cos\alpha, \sin\alpha), \quad \frac{\mathbf{x}}{\|\mathbf{y}\|} = (r\cos\beta, r\sin\beta),$$

where α and β are the polar angles of $\mathbf{x}$ and $\mathbf{y}$, respectively, and $r = \|\mathbf{x}\| / \|\mathbf{y}\| < 1$. Consequently

$$\begin{aligned}\frac{1}{\|\mathbf{y}-\mathbf{x}\|} &= \frac{1}{\|\mathbf{y}\|\sqrt{(\cos\alpha - r\cos\beta)^2 + (\sin\alpha - r\sin\beta)^2}} \\ &= \frac{1}{\|\mathbf{y}\|\sqrt{(1-2r\cos(\alpha-\beta)+r^2}} \\ &= \frac{1}{\|\mathbf{y}\|}\sum_{n=0}^{\infty} P_n(\cos(\alpha-\beta))r^n.\end{aligned}$$

4.11 (a) $1 - x^3 = P_0(x) - \frac{3}{5}P_1(x) - \frac{2}{5}P_3(x)$.

(b) $|x| = \frac{1}{2}P_0(x) + \frac{5}{8}P_2(x) - \frac{3}{16}P_4(x) + \cdots$.

4.12 The formula $\|P_n\|^2 = 2/(2n+1)$ can be verified by direct computation for $n = 0$ and $n = 1$. For $n \geq 2$ we have, by Equation (4.15),

$$\begin{aligned}(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0, \\ nP_n(x) - (2n-1)xP_{n-1}(x) + (n-1)P_{n-2}(x) = 0.\end{aligned}$$

Multiply the first equation by P_{n-1} and the second equation by P_n and integrate over $[-1, 1]$ to obtain

$$\begin{aligned}n\|P_{n-1}\|^2 &= (2n+1)\langle xP_{n-1}, P_n\rangle, \\ n\|P_n\|^2 &= (2n-1)\langle xP_n, P_{n-1}\rangle.\end{aligned}$$

Since $\langle xP_{n-1}, P_n\rangle = \langle xP_n, P_{n-1}\rangle$, this implies

$$(2n+1)\|P_n\|^2 = (2n-1)\|P_{n-1}\|^2 = (2n-3)\|P_{n-2}\|^2 = \cdots = 3\|P_1\|^2 = 2.$$

4.13 $f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$, where $c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx$. Since f is odd, $c_n = 0$ for all even values of n. For $n = 2k+1$,

$$\begin{aligned}c_{2k+1} &= (4k+3)\int_0^1 P_{2k+1}(x)dx \\ &= (4k+3)\frac{1}{4k+3}[P_{2k}(0) - P_{2k+2}(0)] \\ &= (-1)^k\left[\frac{(2k)!}{2^{2k}k!k!} + \frac{(2k+2)!}{2^{2k+2}(k+1)!(k+1)!}\right] \\ &= (-1)^k\frac{(2k)!}{2^{2k}k!k!}\frac{(4k+3)}{(2k+2)}, \quad k \in \mathbb{N}_0.\end{aligned}$$

Hence $f(x) = \frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) + \cdots$. At $x = 0$,

$$\sum_{n=0}^{\infty} c_n P_n(0) = 0 = \frac{1}{2}[f(0^+) + f(0^-)].$$

4.14 In even polynomials, we have $x = \sum_{n=0}^{\infty} c_{2n} P_{2n}(x)$ for all $x \in (0, 1)$, where

$$c_{2n} = \frac{\langle |x|, P_{2n} \rangle}{\|P_{2n}\|^2} = (4n + 1) \int_0^1 x P_{2n}(x) dx.$$

Hence

$$x = \frac{1}{2} P_0(x) + \frac{5}{8} P_2(x) - \frac{3}{16} P_4(x) + \cdots .$$

Since 0 is a point of continuity for the even extension of x to $(-1, 1)$, the expansion is valid at $x = 0$. In odd degree polynomials, $c_{2n+1} = \langle x, P_{2n+1} \rangle / \|P_{2n+1}\|^2 = 1$ if $n = 0$ and 0 otherwise. Therefore $x = P_1(x)$, as would be expected. The equality is obviously valid on $[0, 1]$.

For the function $f(x) = 1$, the expansion in even polynomials is simply $1 = P_0(x)$, which is valid on $[0, 1]$. The odd expansion is

$$1 = \sum_{n=0}^{\infty} c_{2n+1} P_{2n+1}(x) = \frac{3}{2} P_1(x) - \frac{7}{8} P_3(x) + \frac{11}{16} P_5(x) + \cdots ,$$

which is valid on $(0, 1)$ but not at $x = 0$, where the right-hand side is 0, being the average of $f(0^+) = 1$ and $f(0^-) = -1$. Here $f(0^-)$ is obtained from the odd extension of f to $(-1, 1)$.

4.15 $\left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 = 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy = 4 \int_0^{\infty} \int_0^{\pi/2} e^{r^2} r \, dr d\theta = \pi.$

4.16 Replace t by $-t$ in Equation (4.25) to obtain

$$\sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)(-t)^n = e^{-2xt-t^2} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(-x) t^n,$$

which implies $H_n(-x) = (-1)^n H_n(x)$.

4.17 Setting $x = 0$ in (4.25) yields

$$\sum_{k=0}^{\infty} \frac{1}{k!} H_k(0) t^k = e^{-t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} t^{2n}.$$

By equating corresponding coefficients of the powers of t we obtain the desired formulas.

4.18 The equality is true for $n = 0$. Assume its validity for any $n \in \mathbb{N}$. Replace n by $n + 1$ in Equation (4.29) and integrate both sides to obtain

$$H_{n+1}(x) = 2(n + 1) \int_0^x H_n(t) dt + c_n,$$

where

$$
\begin{aligned}
2(n+1)\int_0^x H_n(t)dt &= 2(n+1)n!\sum_{k=0}^{[n/2]}(-1)^k\frac{(2x)^{n+1-2k}}{k!(n+1-2k)!2}\\
&= (n+1)!\sum_{k=0}^{[n/2]}(-1)^k\frac{(2x)^{n+1-2k}}{k!(n+1-2k)!}
\end{aligned}
$$

and c_n is a constant of integration. At $x=0$, $H_{n+1}(0)=0+c_n$. Therefore $c_n=0$ if n is even, and (from Exercise 4.17),

$$
c_n = (-1)^{(n+1)/2}\frac{(n+1)!}{((n+1)/2)!}
$$

if n is odd. Consequently,

$$
\begin{aligned}
H_{n+1}(x) &= (n+1)!\sum_{k=0}^{[n/2]}(-1)^k\frac{(2x)^{n+1-2k}}{k!(n+1-2k)!}\\
&= (n+1)!\sum_{k=0}^{[(n+1)/2]}(-1)^k\frac{(2x)^{n+1-2k}}{k!(n+1-2k)!}
\end{aligned}
$$

when n is even, and

$$
\begin{aligned}
H_{n+1}(x) &= (n+1)!\sum_{k=0}^{[n/2]}(-1)^k\frac{(2x)^{n+1-2k}}{k!(n+1-2k)!}+c_n\\
&= (n+1)!\sum_{k=0}^{[(n+1)/2]}(-1)^k\frac{(2x)^{n+1-2k}}{k!(n+1-2k)!}
\end{aligned}
$$

when n is odd.

4.19 If $m=2n$,

$$
x^{2n}=\frac{(2n)!}{2^{2n}}\sum_{k=0}^{n}\frac{H_{2k}(x)}{(2k)!(n-k)!}.
$$

If $m=2n+1$,

$$
x^{2n+1}=\frac{(2n+1)!}{2^{2n+1}}\sum_{k=0}^{n}\frac{H_{2k+1}(x)}{(2k+1)!(n-k)!},\quad x\in\mathbb{R},\ n\in\mathbb{N}_0.
$$

4.20 The coefficients in the Hermite expansion are given by

$$
\frac{\langle f,H_n\rangle}{\|H_n\|^2}=\frac{1}{2^n n!\sqrt{\pi}}\int_0^\infty H_n(x)e^{-x^2}dx,
$$

the first of which is $1/2$. Now the function $f(x) - (1/2)H_0(x) = f(x) - 1/2$ is odd, hence it has an expansion of the form $\sum_{k=0}^{\infty} c_{2k+1}H_{2k+1}(x)$, and we can write

$$f(x) = \frac{1}{2}H_0(x) + \sum_{k=0}^{\infty} c_{2k+1}H_{2k+1}(x),$$

where

$$c_{2k+1} = \frac{1}{2^{2k+1}(2k+1)!\sqrt{\pi}} \int_0^{\infty} H_{2k+1}(x)e^{-x^2}dx$$

From (4.23) we have $e^{-x^2}H_n(x) = -[e^{-x^2}H_{n-1}(x)]'$. Using this in the above formula for c_{2k+1}, we obtain

$$\begin{aligned} c_{2k+1} &= \frac{H_{2k}(0)}{2^{2k+1}(2k+1)!\sqrt{\pi}} \\ &= \frac{(-1)^k}{2^{2k+1}(2k+1)k!\sqrt{\pi}}. \end{aligned}$$

Therefore,

$$\begin{aligned} f(x) &= \frac{1}{2}H_0(x) + \frac{1}{2\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(2k+1)k!} H_{2k+1}(x), \\ &= \frac{1}{2}H_0(x) + \frac{1}{2\sqrt{\pi}} \left[H_1(x) - \frac{1}{12}H_3(x) + \frac{1}{160}H_5(x) + \cdots \right], \quad x \in \mathbb{R}. \end{aligned}$$

4.21 (a) Substituting $y = \sum_{k=0}^{\infty} c_k x^k$ into the differential equation yields

$$\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - 2\sum_{k=0}^{\infty} kc_k x^k + 2\lambda \sum_{k=0}^{\infty} c_k x^k = 0,$$

from which it follows that

$$c_{k+2} = \frac{2(k-\lambda)}{(k+2)(k+1)}c_k, \quad k \in \mathbb{N}_0,$$

with c_0 and c_1 arbitrary.

(b) The power series $\sum c_k x^k$ converges for all $x \in \mathbb{R}$ by the ratio test. The general solution of Hermite's equation can therefore be represented by the sum $u = c_0u_0 + c_1u_1$, where

$$\begin{aligned} u_0(x) &= 1 - \frac{2\lambda}{2!}x^2 + \frac{2^2\lambda(\lambda-2)}{4!}x^4 - \frac{2^3\lambda(\lambda-2)(\lambda-4)}{6!}x^6 + \cdots, \\ u_1(x) &= x - \frac{2(\lambda-1)}{3!}x^3 + \frac{2^2(\lambda-1)(\lambda-3)}{5!}x^5 \\ &\quad - \frac{2^3\lambda(\lambda-1)(\lambda-3)(\lambda-5)}{7!}x^7 + \cdots, \end{aligned}$$

are linearly independent (even and odd) solutions.

(c) When $\lambda = n$ is a nonnegative integer, either u_0 or u_1 becomes a polynomial, depending on whether n is even or odd, which coincides with a constant multiple of H_n.

4.22 If the solution u_i ($i = 1$ or 2) is a polynomial, the product $e^{-x^2}u_i(x)$ will clearly tend to 0 as $|x| \to \infty$. If u_i is an infinite series, we cannot give an analytical proof that $e^{-x^2}u_i(x)$ will tend to a nonzero constant, but we present the following argument towards that conclusion. If u_i is an infinite series then

$$c_{k+2} = \frac{2(k-\lambda)}{(k+2)(k+1)}c_k \sim \frac{2}{k}c_k \quad \text{as } k \to \infty,$$

in the sense that $kc_{k+2}/2c_k$ tends to a constant as $k \to \infty$. The power series expansion for e^{x^2} is given by

$$\begin{aligned} e^{x^2} &= \sum_{k=0}^{\infty} a_{2k}x^{2k} \\ &= 1 + \frac{1}{1!}x^2 + \frac{1}{2!}x^4 + \cdots + \frac{1}{k!}x^{2k} + \frac{1}{(k+1)!}x^{2(k+1)} + \cdots . \end{aligned}$$

Therefore, with $n = 2k$,

$$a_{n+2} = \frac{1}{\frac{n}{2}+1}a_n \sim \frac{2}{n}a_n \quad \text{as } n \to \infty.$$

This implies that there is a constant γ such that $c_n/a_n \sim \gamma$ as $n \to \infty$. γ is not zero because none of the coefficients in the series for u_i can be zero as long as $\lambda \notin \mathbb{N}_0$. We can therefore argue that the higher terms of the series for u_0 and u_1 differ from the corresponding terms of e^{x^2} only by multiplicative constants, say γ_0 and γ_1, respectively; and that, for large values of $|x|$,

$$u_0(x) \sim \gamma_0 e^{x^2}, \quad u_1(x) \sim \gamma_1 e^{x^2} \quad \text{as } |x| \to \infty,$$

since for such values the higher terms dominate the series. It then follows that $u_i e^{-x^2/2}$ is not square integrable on $\mathbb{R}$, so u_i cannot belong to $\mathcal{L}^2_{e^{x^2}}(\mathbb{R})$. The interested reader may wish to consult *Special Functions and Their Applications* by N.N. Lebedev (Dover, 1972) on the asymptotic behaviour of Hermite funtions.

4.23 Use Leibnitz' rule for the derivative of a product,

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}g^{(k)},$$

with $f(x) = x^n$ and $g(x) = e^{-x}$.

4.24 $x^3 - x = 5L_0(x) - 17L_1(x) + 18L_2(x) - 6L_3(x)$.

4.25 $x^m = \sum_{n=0}^{m} c_n L_n(x)$, where $c_n = \int_0^\infty e^{-x} x^m L_n(x) dx = (-1)^n \dfrac{m!m!}{n!(m-n)!}$.

4.26 $\langle e^{-x/2}, L_n \rangle_{e^{-x}} = \dfrac{1}{n!}\dfrac{1}{2^n}\int_0^\infty x^n e^{-3x/2} dx = \dfrac{2}{3^{n+1}}$. Therefore

$$e^{-x/2} = 2\sum_{n=0}^{\infty} 3^{-n-1} L_n(x).$$

4.27 First we establish two important identities,

$$L_n'(x) = L_{n-1}'(x) - L_{n-1}(x), \quad xL_n'(x) = n[L_n(x) - L_{n-1}(x)],$$

both of which may be proved by using the definition (4.34). In the case of the first identity,

$$\begin{aligned}
L_n'(x) &= \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right) + \frac{e^x}{n!}\frac{d^n}{dx^n}\left(nx^{n-1}e^{-x} - x^n e^{-x}\right) \\
&= \frac{e^x}{n!}\frac{d^n}{dx^n}(nx^{n-1}e^{-x}) \\
&= \frac{e^x}{(n-1)!}\frac{d}{dx}\frac{d^{n-1}}{dx^{n-1}}(x^{n-1}e^{-x}) \\
&= \frac{d}{dx}\left[\frac{e^x}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}(x^{n-1}e^{-x})\right] - \frac{e^x}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}(x^{n-1}e^{-x}) \\
&= L_{n-1}'(x) - L_{n-1}(x).
\end{aligned}$$

For the second identity,

$$\begin{aligned}
xL_n'(x) &= x\frac{e^x}{n!}\frac{d^n}{dx^n}(nx^{n-1}e^{-x}) \\
&= \frac{e^x}{(n-1)!}\left[\frac{d^n}{dx^n}(x^n e^{-x}) - n\frac{d^{n-1}}{dx^{n-1}}(x^{n-1}e^{-x})\right] \\
&= nL_n(x) - nL_{n-1}(x).
\end{aligned}$$

Use the equality $L_{n-1}(x) = L_n(x) - \frac{1}{n}xL_n'(x)$ in the first identity to arrive at Equation (4.32).

4.28 $u(x) = c_1 + c_2\displaystyle\int \frac{e^x}{x}dx = c_1 + c_2\left(\log x + x + \frac{1}{2}\frac{x^2}{2!} + \cdots\right)$.

4.29 The surface $\varphi = \pi/2$, corresponding to the xy-plane.

4.30 The bounded solution outside the sphere, according to Equation (4.41), is $\sum_{n=0}^{\infty} b_n r^{-n-1} P_n(\cos\varphi)$, where the coefficients can evaluated from the boundary condition $u(1,\varphi) = f(\varphi) = \sum_{n=0}^{\infty} b_n P_n(\cos\varphi)$ as

$$b_n = \frac{2n+1}{2} \int_0^{\pi} f(\varphi) P_n(\cos\varphi) \sin\varphi \, d\varphi, \quad n \in \mathbb{N}_0.$$

4.31 The solution of Laplace's equation in the spherical coordinates (r,φ) is given by Equation (4.42) as $u(r,\varphi) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\varphi)$. Using the given boundary condition in Equation (4.43),

$$\begin{aligned} a_n &= \frac{2n+1}{2R^n} \int_0^{\pi/2} 10 P_n(\cos\varphi) \sin\varphi \, d\varphi \\ &= \frac{5(2n+1)}{R^n} \int_0^1 P_n(x) dx \\ &= \frac{5}{R^n} [P_{n-1}(0) - P_{n+1}(0)], \quad n \in \mathbb{N}, \end{aligned}$$

where the result of Exercise 4.7 is used in the last equality. We therefore arrive at the solution

$$\begin{aligned} u(r,\varphi) &= 5 + 5 \sum_{n=1}^{\infty} \left[P_{n-1}(0) - P_{n+1}(0) \right] \left(\frac{r}{R} \right)^n P_n(\cos\varphi) \Big] \\ &= 5 \left[1 + \frac{3}{2} \frac{r}{R} P_1(\cos\varphi) - \frac{7}{8} \left(\frac{r}{R} \right)^3 P_3(\cos\varphi) + \cdots \right]. \end{aligned}$$

Note that $u(R,\varphi) - 5$ is an odd function of φ, hence the summation (starting with $n = 1$) is over odd values of n.

4.32 Inside the hemisphere $u(r,\varphi)$ is represented by a series of the form (4.42). If $u(1,\varphi)$ is extended as an odd function from $[0,\pi/2]$ to $[0,\pi]$, then $u(1,\varphi)$ has a series expansion in odd Legendre polynomials in which the coefficients are

$$\begin{aligned} a_{2n+1} &= \frac{4n+3}{2} \int_0^{\pi} u(1,\varphi) P_{2n+1}(\cos\varphi) \sin\varphi \, d\varphi \\ &= (4n+3) \int_0^1 P_{2n+1}(x) dx \\ &= [P_{2n}(0) - P_{2n+2}(0)] \\ &= (-1)^n \left[\frac{(2n)!}{2^{2n}(n!)^2} + \frac{(2n+2)!}{2^{2n+2}((n+1)!)^2} \right] = (-1)^n \frac{4n+3}{2n+2} \frac{(2n)!}{2^{2n}(n!)^2}. \end{aligned}$$

4.33 In view of the boundary condition $u_\varphi(r,\pi/2) = 0$, f may be extended as an even function of φ from $[0,\pi/2]$ to $[0,\pi]$. By symmetry the solution is even about $\varphi = \pi/2$, hence the summation in Equation (4.42) is over even orders of the Legendre polynomials.

Chapter 5

5.1 For all $n \in \mathbb{N}$, the integral $I_n(x) = \int_0^n e^{-t}t^{x-1}dt$ is a continuous function of $x \in [a,b]$, where $0 < a < b < \infty$. Since

$$0 \le \int_n^\infty e^{-t}t^{x-1}dt \le \int_n^\infty e^{-t}t^{b-1}dt \xrightarrow{u} 0 \quad \text{as } n \to \infty,$$

it follows that $I_n(x)$ converges uniformly to $\Gamma(x)$ on $[a,b]$. Therefore $\Gamma(x)$ is continuous on $[a,b]$ for any $0 < a < b < \infty$, and hence on $(0,\infty)$. By a similar procedure we can also show that its derivatives $\Gamma'(x) = \int_0^\infty e^{-t}t^{x-1}\log t \; dt$, $\Gamma''(x) = \int_n^\infty e^{-t}t^{x-1}(\log t)^2 dt$, ... , are all continuous on $(0,\infty)$.

5.2 $\Gamma(1/2) = \int_0^\infty e^{-t}t^{-1/2}dt = 2\int_0^\infty e^{-y^2}dy = \int_{-\infty}^\infty e^{-y^2}dy = \sqrt{\pi}$ (see Exercise 4.15).

5.3 $\Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\cdots\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \dfrac{(2n)!}{n!2^{2n}}\Gamma\left(\frac{1}{2}\right)$. From Exercise 5.2 we know that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

5.4 (a) Replace t by $u/(1+u)$ and dt by $du/(1+u)^2$ with appropriate limits on u.

(b) Replace t in (5.1) by st.

(c) Using the result of part (b),

$$\begin{aligned}\frac{1}{(u+1)^{x+y}} &= \frac{1}{s^z} \\ &= \frac{1}{\Gamma(z)}\int_0^\infty e^{-st}t^{z-1}dt \\ &= \frac{1}{\Gamma(x+y)}\int_0^\infty e^{-(u+1)t}t^{x+y-1}dt.\end{aligned}$$

Using this in the integral expression in (a),

$$\begin{aligned}\beta(x,y) &= \frac{1}{\Gamma(x+y)}\int_0^\infty u^{x-1}\left(\int_0^\infty e^{-(u+1)t}t^{x+y-1}dt\right)du \\ &= \frac{1}{\Gamma(x+y)}\int_0^\infty e^{-t}t^{x+y-1}\int_0^\infty e^{-ut}u^x du dt \\ &= \frac{1}{\Gamma(x+y)}\int_0^\infty e^{-t}t^{y-1}\left(\int_0^\infty e^{-v}v^{x-1}dv\right)dt \\ &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.\end{aligned}$$

5.5 Use the integral definition of the gamma function to write

$$\begin{aligned}
2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) &= \int_0^\infty \int_0^\infty e^{-(s+t)}\left(2\sqrt{st}\right)^{2x-1} t^{-1/2} ds dt \\
&= 4\int_0^\infty \int_0^\infty e^{-(\alpha^2+\beta^2)}(2\alpha\beta)^{2x-1}\alpha \; d\alpha d\beta \\
&= 4\int_0^\infty \int_0^\infty e^{-(\alpha^2+\beta^2)}(2\alpha\beta)^{2x-1}\beta \; d\alpha d\beta.
\end{aligned}$$

Interchange α and β to obtain a similar formula, then add and divide by 2 to arrive at

$$\begin{aligned}
2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) &= 2\int_0^\infty \int_0^\infty e^{-(\alpha^2+\beta^2)}(2\alpha\beta)^{2x-1}(\alpha+\beta) d\alpha d\beta \\
&= 4\int\int e^{-(\alpha^2+\beta^2)}(2\alpha\beta)^{2x-1}(\alpha+\beta) d\alpha d\beta,
\end{aligned}$$

where the last double integral is over the sector $0 \leq \beta < \alpha < \infty$. By changing the variables of integration to $\xi = \alpha^2 + \beta^2$, $\eta = 2\alpha\beta$, we see that $\alpha \pm \beta = \sqrt{\xi \pm \eta}$, and the Jacobian of the transformation $(\alpha, \beta) \mapsto (\xi, \eta)$ is $1/\sqrt{\xi^2 - \eta^2}$. Therefore

$$\begin{aligned}
2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) &= \int_0^\infty \eta^{2x-1} d\eta \int_0^\infty \frac{e^{-\xi}}{\sqrt{\xi-\eta}} d\xi \\
&= 2\int_0^\infty e^{-\eta}\eta^{2x-1} d\eta \int_0^\infty e^{-y^2} dy \\
&= \sqrt{\pi}\,\Gamma(2x).
\end{aligned}$$

5.6 (a) Change the variable of integration from t to $-t$.

(b) Use the result of Exercise 4.15.

(c) The exponential function e^{-t^2} has a power series expansion which converges everywhere in $\mathbb{R}$, so the same is true of $\int_0^x e^{-t^2} dt$.

5.7 Apply the ratio test.

5.8 Using (5.12),

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \sum_{m=0}^\infty \frac{(-1)^m}{2^{2m} m!\,\Gamma\left(m+1+\frac{1}{2}\right)} x^{2m}.$$

Since $2^{2m} m!\,\Gamma\left(m+1+\frac{1}{2}\right) = (2m+1)!\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}(2m+1)!$, we obtain

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sum_{m=0}^\infty (-1)^m \frac{x^{2m+1}}{(2m+1)!} = \sqrt{\frac{2}{\pi x}} \sin x.$$

Similarly

$$
\begin{aligned}
J_{-1/2}(x) &= \sqrt{\frac{2}{x}} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m} m! \Gamma\left(m+\frac{1}{2}\right)} x^{2m} \\
&= \sqrt{\frac{2}{x}} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m} m! \Gamma\left(m+\frac{1}{2}\right)} x^{2m} \\
&= \sqrt{\frac{2}{x}} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m} m! \Gamma\left(m+\frac{1}{2}\right)} x^{2m} \\
&= \sqrt{\frac{2}{x}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)! \Gamma(1/2)} x^{2m} = \sqrt{\frac{2}{\pi x}} \cos x.
\end{aligned}
$$

5.9 Differentiating Equation (5.12) and multiplying by x,

$$
\begin{aligned}
x J_\nu'(x) &= \nu \left(\frac{x}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{m! \Gamma(m+\nu+1)} + \left(\frac{x}{2}\right)^\nu \sum_{m=1}^{\infty} \frac{(-1)^m 2m (x/2)^{2m}}{m! \Gamma(m+\nu+1)} \\
&= \nu J_\nu(x) + \left(\frac{x}{2}\right)^\nu \sum_{m=1}^{\infty} \frac{(-1)^m 2}{(m-1)! \Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m} \\
&= \nu J_\nu(x) + \left(\frac{x}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^{m+1} 2}{m! \Gamma(m+\nu+2)} \left(\frac{x}{2}\right)^{2m+2} \\
&= \nu J_\nu(x) - x \left(\frac{x}{2}\right)^{\nu+1} \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{m! \Gamma(m+\nu+2)} \left(\frac{x}{2}\right)^{2m} \\
&= \nu J_\nu(x) - x J_{\nu+1}(x).
\end{aligned}
$$

5.10 From the identity in Exercise 5.9, $J_{3/2}(x) = \frac{\nu}{x} J_{1/2}(x) - J_{1/2}'(x)$. Use the representation of $J_{1/2}(x)$ obtained in Exercise 5.8.

5.11 From Equation (5.12) we can write

$$
\begin{aligned}
x^\nu J_\nu(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+\nu+1)} \frac{x^{2m+2\nu}}{2^{2m+\nu}} \\
\left[x^\nu J_\nu(x)\right]' &= \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2\nu)}{m! \Gamma(m+\nu+1)} \frac{x^{2m+2\nu-1}}{2^{2m+\nu}} \\
&= x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+\nu)} \left(\frac{x}{2}\right)^{2m+\nu-1} \\
&= x^\nu J_{\nu-1}(x).
\end{aligned}
$$

Substitute $\nu = -1/2$ into this identity and use Exercise 5.8.

5.12 From Example 5.2 and Exercise 5.11, we have

$$[x^{-\nu}J_\nu(x)]' = -\nu x^{-\nu-1}J_\nu(x) + x^{-\nu}J'_\nu(x) = -x^{-\nu}J_{\nu+1}(x)$$
$$[x^{\nu}J_\nu(x)]' = \nu x^{\nu-1}J_\nu(x) + x^{\nu}J'_\nu(x) = x^{\nu}J_{\nu+1}(x).$$

Multiply the first equation by x^ν and the second by $x^{-\nu}$ and add.

5.13 Subtract, instead of adding, the equations in Exercise 5.12.

5.14 (a) Integrating by parts, and using Example 5.4,

$$\begin{aligned}\int_0^x t^2 J_1(t)dt &= -\int_0^x t^2 J'_0(t)dt \\ &= -x^2 J_0(x) + 2\int_0^x tJ_0(t)dt \\ &= -x^2 J_0(x) + 2xJ_1(x).\end{aligned}$$

(b) From Exercise 5.12 we have the relation

$$J_3(t) = J_1(t) - 2J'_2(t) = -J'_0(t) - 2J'_2(t).$$

Therefore

$$\int_0^x J_3(t)dt = 1 - J_0(x) - 2J_2(x) = 1 - J_2(x) - 2J_1(x)/x,$$

where the second equality follows from Exercise 5.13.

5.15

$$\begin{aligned}\int_0^x t^n J_0(t)dt &= \int_0^x t^{n-1}tJ_0(t)dt \\ &= \int_0^x t^{n-1}[tJ_1(t)]'dt \\ &= t^{n-1}tJ_1(t)\Big|_0^x - (n-1)\int_0^x t^{n-2}tJ_1(t)dt \\ &= x^n J_1(x) + (n-1)\int_0^x t^{n-1}J'_0(t)dt \\ &= x^n J_1(x) + (n-1)\left[t^{n-1}J_0(t)\Big|_0^x - (n-1)\int_0^x t^{n-2}J_0(t)dt\right] \\ &= x^n J_1(x) + (n-1)x^{n-1}J_0(x) - (n-1)^2\int_0^x t^{n-2}J_0(t)dt,\end{aligned}$$

for all $n = 2, 3, 4, ...$

5.16 J_ν and $J_{-\nu}$ satisfy the same Bessel equation, that is,

$$x^2 J''_{\pm\nu}(x) + xJ'_{\pm\nu}(x) + (x^2 - \nu^2)J_{\pm\nu}(x) = 0,$$
$$J'_{\pm}(x) + xJ''_{\pm}(x) + \left(x - \frac{\nu^2}{x}\right) J_{\pm}(x) = 0.$$

Multiplying one equation by J_ν and the other by $J_{-\nu}$ and subtracting, we obtain

$$xW'(x) + W(x) = (xW(x))' = 0,$$

where $W(x) = J_\nu(x)J'_{-\nu}(x) - J'_\nu(x)J_{-\nu}(x)$. Integrating the above equation,

$$W(x) = \frac{c}{x},$$

where the constant c can be evaluated from

$$c = \lim_{x\to 0} xW(x).$$

Using the representation (5.12), this limit is given by

$$\frac{-2\nu}{\Gamma(1+\nu)\Gamma(1-\nu)} = \frac{-2}{\Gamma(\nu)\Gamma(1-\nu)}, \quad \nu \notin \mathbb{N}_0,$$

hence the desired result.

5.17 Substitute directly into Bessel's equation. $J_n(x)$ is bounded at $x = 0$ for all $n \in \mathbb{N}_0$, whereas $y_n(x)$ is not . Hence the two functions cannot be linearly dependent.

5.18 Follows directly from the series representation of Y_n.

5.19 For nonintegral values of ν the function $Y_\nu(x)$ is a linear combination of $J_\nu(x)$ and $J_{-\nu}(x)$. Since the formula $[x^\nu J_\nu(x)]' = x^\nu J_{\nu+1}(x)$ holds for all real values of ν, it also holds for $Y_\nu(x)$ provided $\nu \notin \mathbb{Z}$. The relation

$$Y_\nu(x) = \frac{1}{\sin \nu\pi}[J_\nu(x)\cos \nu\pi - J_{-\nu}(x)]$$

shows that Y_ν is analytic in ν, except possibly at integer values. But since the limit $Y_n(x) = \lim_{\nu\to n} Y_\nu(x)$ exists at every $x > 0$, $Y_\nu(x)$ is analytic (and hence continuous) as a function of ν on $\mathbb{R}$, so the same formula applies when ν is an integer.

5.20 Use the same argument as in Exercise 5.19.

5.21

$$\begin{aligned} Y_{-n}(x) &= \frac{1}{\pi}\left[\left.\frac{\partial J_\nu(x)}{\partial\nu}\right|_{\nu=-n} - (-1)^{-n}\left.\frac{\partial J_{-\nu}(x)}{\partial\nu}\right|_{\nu=-n}\right] \\ &= \frac{1}{\pi}\left[-\left.\frac{\partial J_{-\nu}(x)}{\partial\nu}\right|_{\nu=n} + (-1)^{n}\left.\frac{\partial J_{\nu}(x)}{\partial\nu}\right|_{\nu=n}\right] \\ &= (-1)^n\frac{1}{\pi}\left[\left.\frac{\partial J_\nu(x)}{\partial\nu}\right|_{\nu=n} - (-1)^{n}\left.\frac{\partial J_{-\nu}(x)}{\partial\nu}\right|_{\nu=n}\right] \\ &= (-1)^n Y_n(x), \quad n \in \mathbb{N}_0. \end{aligned}$$

5.22 Under the transformation $x \mapsto ix$ Bessel's equation takes the form

$$i^2x^2J_\nu''(ix) + ixJ_\nu(ix) + (i^2x^2 - \nu^2)J_\nu(ix) = 0.$$

Using the relation $J_\nu(ix) = i^\nu I_\nu(x)$ yields the equation $x^2I_\nu''(x) + xI_\nu(x) - (x^2+\nu^2)J_\nu(x) = 0$.

5.23 From Equation (5.12) we have

$$\begin{aligned} I_\nu(x) &= i^{-\nu}J_\nu(ix) \\ &= i^{-\nu}\left(\frac{ix}{2}\right)^\nu \sum_{m=0}^\infty \frac{(-1)^m}{m!\Gamma(\nu+m+1)}\left(\frac{ix}{2}\right)^{2m} \\ &= \sum_{m=0}^\infty \frac{1}{m!\Gamma(\nu+m+1)}\left(\frac{x}{2}\right)^{2m+\nu}. \end{aligned}$$

5.24 In the series representation of $I_\nu(x)$ in Exercise 5.23 it is clear that, with $x > 0$, the series is positive provided the coefficients are positive. This is the case if $\nu > 0$. Moreover,

$$I_{-n}(x) = \sum_{m=0}^\infty \frac{1}{m!\Gamma(m-n+1)}\left(\frac{x}{2}\right)^{2m-n}.$$

Since $1/\Gamma(m-n+1) = 0$ for all $m-n+1 \le 0$, it is clear that the sum may be taken over $m \ge n$. Therefore

$$\begin{aligned} I_{-n}(x) &= \sum_{m=n}^\infty \frac{1}{m!\Gamma(m-n+1)}\left(\frac{x}{2}\right)^{2m-n} \\ &= \sum_{m=0}^\infty \frac{1}{(m+n)!\Gamma(m+1)}\left(\frac{x}{2}\right)^{2m+n} \\ &= I_n(x). \end{aligned}$$

5.25 The definition of I_ν, as given in Exercise 5.22, extends to negative values of ν. Equation (5.18) is invariant under a change of sign of ν, hence it is satisfied by both I_ν and $I_{-\nu}$.

5.26 Differentiation can be carried across the integral sign because the integrand is differentiable and the interval of integration is finite. The integral representation of $J_n^{(k)}$ will be proved by induction: When $k = 1$ the formula is true because $\cos(n\theta - x\sin\theta - \pi/2) = \sin(n\theta - x\sin\theta)$. Moreover

$$\begin{aligned} J_n^{(k+1)}(x) &= \frac{d}{dx}\frac{1}{\pi}\int_0^\pi \sin^k\theta\cos(n\theta - x\sin\theta - k\pi/2)d\theta \\ &= \frac{1}{\pi}\int_0^\pi \sin^{k+1}\theta\sin(n\theta - x\sin\theta - k\pi/2)d\theta \\ &= \frac{1}{\pi}\int_0^\pi \sin^{k+1}\theta\cos[n\theta - x\sin\theta - (k+1)\pi/2]d\theta, \end{aligned}$$

hence the formula holds for all $k = 1, 2, 3, \dots$.

5.27 Follows from the bounds on the sine and cosine functions.

5.28 (a) Set $\theta = 0$ in Equation (5.20) and use the relation $J_{-n}(x) = (-1)^n J_n(x)$.

(b) Set $\theta = \pi/2$ in the imaginary part of Equation (5.20), that is, $\sin(x\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(x)\sin n\theta$.

(c) Set $\theta = \pi/2$ in the imaginary part of Equation (5.20).

(d) Differentiate Equation (5.20) with respect to θ and set $\theta = 0$ to obtain

$$x = \sum_{n=-\infty}^{\infty} nJ_n(x) = \sum_{n=1}^{\infty}[nJ_n(x) - nJ_{-n}(x)] = 2\sum_{m=1}^{\infty}(2m-1)J_{2m-1}(x).$$

5.29 Applying Parseval's relation to Equations (5.22) and (5.23), we obtain

$$\begin{aligned} \int_{-\pi}^{\pi}\cos^2(x\sin\theta)d\theta &= 2\pi J_0^2(x) + 4\pi\sum_{m=1}^{\infty} J_{2m}^2(x) \\ \int_{-\pi}^{\pi}\sin^2(x\sin\theta)d\theta &= 4\pi\sum_{m=1}^{\infty} J_{2m-1}^2(x). \end{aligned}$$

By adding these two equations we arrive at the desired identity.

5.30 Noting that the integrand in Equation (5.24) is symmetric with respect to $\theta = \pi/2$, we can write

$$\begin{aligned}
J_{2m}(x) &= \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\cos 2m\theta \; d\theta \\
&= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos[x\sin(\theta+\pi/2)]\cos 2m(\theta+\pi/2) \; d\theta \\
&= \frac{(-1)^m}{\pi}\int_{-\pi/2}^{\pi/2} \cos(x\cos\theta)\cos 2m\theta \; d\theta \\
&= 2\frac{(-1)^m}{\pi}\int_{-\pi/2}^{0} \cos(x\cos\theta)\cos 2m\theta \; d\theta \\
&= \frac{2}{\pi}\int_0^{\pi/2} \cos(x\sin\theta)\cos 2m\theta \; d\theta.
\end{aligned}$$

Similarly, using the symmetry of the integrand in Equation (5.25) with respect to $\theta = \pi/2$, we arrive at

$$\begin{aligned}
J_{2m-1}(x) &= \frac{1}{\pi}\int_0^{\pi} \sin(x\sin\theta)\sin(2m-1)\theta \; d\theta \\
&= \frac{2}{\pi}\int_0^{\pi/2} \sin(x\sin\theta)\sin(2m-1)\theta \; d\theta.
\end{aligned}$$

5.31 Apply Lemma 3.7 to Equations (5.24) and (5.25).

5.32 For any $n \in \mathbb{N}_0$, we conclude from the formulas for J_{2m} and J_{2m-1} that

$$|J_n(x)| \leq \frac{2}{\pi}\left|\int_0^{\pi/2} \cos(x\sin\theta)d\theta\right|.$$

We shall show that the integral on the right-hand side may be made arbitrarily small by taking x large enough. Under the change of variable $y = \sin\theta$, this integral takes the form

$$\int_0^1 \cos(xy)(1-y^2)^{-1/2}dy,$$

and may be expressed as a sum of two integrals,

$$\int_0^{1-\delta} \cos(xy)(1-y^2)^{-1/2}dy + \int_{1-\delta}^1 \cos(xy)(1-y^2)^{-1/2}dy,$$

where $\delta > 0$. Let ε be any positive number. The second integral is improper but bounded by

$$\frac{2}{\pi}\int_{1-\delta}^1 (1-y^2)^{-1/2}dy$$

which is convergent, so we can choose $\delta > 0$ small enough to obtain

$$\int_{1-\delta}^{1} \cos(xy)(1-y^2)^{-1/2} dy \leq \frac{\varepsilon}{2}.$$

With this choice of δ, since $(1-y^2)^{-1/2}$ is a smooth function on $[0, 1-\delta]$, we can apply Lemma 3.7 to the first integral to conclude that there is a positive number M such that

$$\left| \int_{0}^{1-\delta} \cos(xy)(1-y^2)^{-1/2} dy \right| \leq \frac{\varepsilon}{2} \quad \text{for all } x > M.$$

Consequently $\left| \int_0^{\pi/2} \cos(x \sin\theta) d\theta \right| \leq \varepsilon$ for all $x > M$.

5.33 (a) $\langle 1, J_0(\mu_k x) \rangle_x = \int_0^b J_0(\mu_k x) x dx = \dfrac{b}{\mu_k} J_1(\mu_k b)$, $\|J_0(\mu_k x)\|_x^2 = \dfrac{b^2}{2} J_1^2(\mu_k b)$.

Therefore

$$1 = \frac{2}{b} \sum_{k=1}^{\infty} \frac{1}{\mu_k J_1(\mu_k b)} J_0(\mu_k x).$$

(b) $\langle x, J_0(\mu_k x) \rangle_x = \int_0^b J_0(\mu_k x) x^2 dx = \dfrac{b^2}{\mu_k} J_1(\mu_k b) - \dfrac{1}{\mu_k^2} \int_0^b J_0(\mu_k x) dx.$

The integral $\int_0^b J_0(\mu_k x) dx$ cannot be expressed more simply, so that

$$x = \frac{2}{b^2} \sum_{k=1}^{\infty} \frac{\mu_k b^2 J_1(\mu_k b) - \int_0^b J_0(\mu_k x) dx}{\mu_k^2 J_1(\mu_k b)} J_0(\mu_k x).$$

(c) $\left\langle x^2, J_0(\mu_k x) \right\rangle_x = \left(\dfrac{b^3}{\mu_k} - \dfrac{4b}{\mu_k^3} \right) J_1(\mu_k b)$. Hence

$$x^2 = \frac{2}{b} \sum_{k=1}^{\infty} \frac{\mu_k^2 b^2 - 4}{\mu_k^3 J_1(\mu_k b)} J_0(\mu_k x).$$

(d) From (a) and (c) we have

$$b^2 - x^2 = \frac{8}{b} \sum_{k=1}^{\infty} \frac{1}{\mu_k^3 J_1(\mu_k b)} J_0(\mu_k x).$$

(e) $\langle f, J_0(\mu_k x) \rangle_x = \displaystyle\int_0^{b/2} J_0(\mu_k x) x dx = \frac{b}{2\mu_k} J_1(\mu_k b/2)$. Hence

$$f(x) = \frac{1}{b} \sum_{k=1}^{\infty} \frac{J_1(\mu_k b/2)}{\mu_k J_1^2(\mu_k b)} J_0(\mu_k x).$$

5.34 Since $J_0'(x) = -J_1(x)$, it follows that $J_1(\mu_k) = 0$ for all k. The first (nonnegative) eigenvalue is $\mu_0 = 0$, corresponding to the eigenfunction $J_0(0) = 1$. The remaining eigenvalues μ_k are the positive zeros of J_1, corresponding to the eigenfunctions $J_0(\mu_k x)$. Since

$$\begin{aligned}
\langle 1, J_0(0)\rangle_x &= \int_0^1 x\ dx = \frac{1}{2}, \\
\langle 1, J_0(\mu_k x)\rangle_x &= \frac{1}{\mu_k^2}\ xJ_1(x)\big|_0^{\mu_k} = 0 \quad \text{for all } k \in \mathbb{N}, \\
\|J_0(0)\|_x^2 &= \frac{1}{2},
\end{aligned}$$

we see that the Bessel series expansion of 1 reduces to the single term 1.

5.35 From Exercises 5.13 and 5.14(a) we have $\langle x, J_1(\mu_k x)\rangle_x = \int_0^1 J_1(\mu_k x)x^2 dx = -J_0(\mu_k)/\mu_k = J_2(\mu_k)/\mu_k$, and, from Equation (5.34), $\|J_1(\mu_k x)\|_x^2 = \frac{1}{2}J_2^2(\mu_k)$. Therefore

$$x = 2\sum_{k=1}^{\infty} \frac{1}{\mu_k J_2(\mu_k)} J_1(\mu_k x), \quad 0 < x < 1.$$

5.36 Using the identity $[x^n J_n(x)]' = x^n J_{n-1}(x)$,

$$\begin{aligned}
\langle x^n, J_n(\mu_k x)\rangle_x &= \int_0^1 x^{n+1} J_n(\mu_k x)dx \\
&= \frac{1}{\mu_k^{n+2}}\ t^{n+1} J_{n+1}(t)\big|_0^{\mu_k} \\
&= \frac{1}{\mu_k} J_{n+1}(\mu_k).
\end{aligned}$$

From equation (5.36) we obtain $\|J_n(\mu_k x)\|_x^2 = \frac{1}{2\mu_k^2}\left(\mu_k^2 - n^2\right) J_n^2(\mu_k)$. Therefore

$$x^n = 2\sum_{n=1}^{\infty} \frac{\mu_k J_{n+1}(\mu_k)}{(\mu_k^2 - n^2)\, J_n^2(\mu_k)} J_n(\mu_k x).$$

5.37 Using the relation $x^2 J_1(x) = \left[x^2 J_2(x)\right]'$ we have

$$\langle f, J_1(\mu_k x)\rangle_x = \int_0^1 x^2 J_1(\mu_k x)dx = \frac{1}{\mu_k} J_2(\mu_k).$$

Equation (5.36) also implies

$$\|J_1(\mu_k x)\|_x^2 = \frac{4\mu_k^2 - 1}{2\mu_k^2} J_1^2(2\mu_k).$$

Consequently,

$$f(x) = 2\sum_{k=1}^{\infty} \frac{\mu_k J_2(\mu_k)}{(4\mu_k^2 - 1)J_1^2(2\mu_k)} J_1(\mu_k x), \quad 0 < x < 2.$$

This representation is not pointwise. At $x = 1$, $f(1) = 1$ whereas the right-hand side is $\frac{1}{2}[f(1^+) + f(1^-)] = \frac{1}{2}$.

5.38 With $\mu = 0$ Equation (5.26) takes the form

$$x^2 u'' + xu' - \nu^2 u = 0,$$

which is of the Cauchy-Euler type. Its general solution is $c_1 x^\nu + c_2 x^{-\nu}$. The eigenfunction associated with the eigenvalue 0 is therefore x^ν, the solution which is bounded at $x = 0$ when $\nu > 0$. Applying the boundary condition (5.28) to this eigenfunction leads to the relation $\beta_1 b + \nu\beta_2 = 0$.

5.39 Assuming $u(r,t) = v(r)w(t)$ leads to

$$\frac{w'}{kw} = \frac{1}{v}\left(v'' + \frac{1}{r}v'\right) = -\mu^2.$$

The solutions of the resulting pair of equations are $w(t) = e^{-\mu^2 kt}$ and $v(r) = cJ_0(\mu r) + dY_0(\mu r)$. Discarding the unbounded solution Y_0, and setting $v(1) = cJ_0(\mu) = 0$, yields the sequence of solutions

$$u_n(r,t) = c_n e^{-\mu^2 kt} J_0(\mu_n r),$$

where μ_n are the positive zeros of J_0. Note that $\mu = 0$ yields the trivial solution. The general solution is then the formal sum $\sum u_n(r,t)$.

5.40

$$u(r,0) = \sum_{n=0}^{\infty} u_n(r,0) = \sum_{n=0}^{\infty} c_n J_0(\mu_n r) = f(r).$$

Assuming the function f belongs to $\mathcal{L}_x^2(0,1)$, the coefficients in the Fourier-Bessel series expansion of f are

$$c_n = \frac{\langle f(r), J_n(\mu_n r)\rangle_x}{\|J_0(\mu_k r)\|_x^2} = \frac{2}{J_1^2(\mu_n)} \int_0^1 f(r) J_0(\mu_n r) r \; dr.$$

5.41 Use separation of variables to conclude that

$$u(r,t) = \sum_{k=1}^{\infty} J_0(\mu_k r)[a_k \cos \mu_k ct + b_k \sin \mu_k ct],$$

$$a_k = \frac{2}{R^2 J_1^2(\mu_k R)} \int_0^R f(r) J_0(\mu_k r) r dr,$$

$$b_k = \frac{2}{c\mu_k R^2 J_1^2(\mu_k R)} \int_0^R g(r) J_0(\mu_k r) r dr.$$

Chapter 6

6.1 (a) $\hat{f}(\xi) = \int_{-1}^{1}(1-|x|)e^{-i\xi x}dx = 2\int_0^1 (1-x)\cos\xi x \; dx = \frac{2}{\xi^2}(1-\cos\xi)$.

(b) $\hat{f}(\xi) = \int_{-\pi}^{\pi} \cos x \; e^{-i\xi x}dx = 2\int_0^{\pi} \cos x \;\; \cos\xi x \; dx = 2\frac{\xi \sin \pi\xi}{1-\xi^2}$.

(c) $\hat{f}(\xi) = \frac{1}{i\xi}\left(1-e^{-i\xi}\right)$.

6.2 (a) Suppose $I = [a, b]$. By the CBS inequality,

$$\int_a^b |f(x)|\, dx \leq \left[\int_a^b |f(x)|^2\, dx\right]^{1/2} \left[\int_a^b dx\right]^{1/2} = \sqrt{b-a}\,\|f\| < \infty,$$

hence $f \in \mathcal{L}^1(a, b)$.

(b) Let $|f(x)| \leq M$ for all $x \in I$, where M is a positive number. Then

$$\|f\| = \left[\int_I |f(x)|^2\, dx\right]^{1/2} \leq M \left[\int_I |f(x)|\, dx\right]^{1/2} < \infty.$$

6.3 For any fixed point $\xi \in J$, let ξ_n be a sequence in J which converges to ξ. Since

$$|F(\xi_n) - F(\xi)| \leq \int_I |\varphi(x, \xi_n) - \varphi(x, \xi)|\, dx,$$

and $|\varphi(x, \xi_n) - \varphi(x, \xi)| \leq 2g(x) \in \mathcal{L}^1(I)$, we can apply Theorem 6.4 to the sequence of functions $\varphi_n(x) = \varphi(x, \xi_n) - \varphi(x, \xi)$ to conclude that

$$\begin{aligned}\lim_{n\to\infty} |F(\xi_n) - F(\xi)| &\leq \lim_{n\to\infty} \int_I |\varphi(x, \xi_n) - \varphi(x, \xi)|\, dx \\ &= \int_I \lim_{n\to\infty} |\varphi(x, \xi_n) - \varphi(x, \xi)|\, dx = 0.\end{aligned}$$

6.4 From Exercise 6.3 we know that F is continuous where $\varphi(x, \cdot)$ is continuous. Suppose $\varphi(x, \cdot)$ has a jump discontinuity at $\xi \in J$. It suffices to show that F also has a jump discontinuity at ξ. Let ξ_n be a sequence in J which tends to ξ from the left. Applying Theorem 6.4 to the sequence $\varphi_n(x) = \varphi(x, \xi_n)$, which is dominated by $g(x) \in \mathcal{L}^1(I)$, we obtain

$$\begin{aligned}F(\xi^-) &= \lim_{n\to\infty} F(\xi_n) \\ &= \lim_{n\to\infty} \int_I \varphi(x, \xi_n) dx \\ &= \int_I \lim_{n\to\infty} \varphi(x, \xi_n) dx.\end{aligned}$$

Hence the integral $F(\xi^-) = \int_I \varphi(x, \xi^-)dx$ exists. Similarly, by taking a sequence in J which approaches ξ from the right, we conclude that $F(\xi^+)$ also exists. Thus F has no worse than a jump discontinuity at ξ.

6.5 For any $\xi \in J$, let ξ_n be a sequence in J such that $\xi_n \neq \xi$ for all n and $\xi_n \to \xi$. We have

$$\frac{F(\xi_n) - F(\xi)}{\xi_n - \xi} = \int_I \frac{\varphi(x, \xi_n) - \varphi(x, \xi)}{\xi_n - \xi} dx.$$

Define

$$\psi_n(x, \xi) = \frac{\varphi(x, \xi_n) - \varphi(x, \xi)}{\xi_n - \xi},$$

then $\psi_n(x, \xi) \to \varphi_\xi(x, \xi)$ pointwise. ψ_n is integrable on I and, by the mean value theorem, $\psi_n(x, \xi) = \varphi_\xi(x, \eta_n)$ for some η_n between ξ_n and ξ. Therefore, for all n and ξ, $|\psi_n(x, \xi)| \leq |h(x)|$ on I. Now use the dominated convergence theorem to conclude that $\int_I \psi_n(x, \xi)dx \to \int_I \varphi_\xi(x, \xi)dx$ as $n \to \infty$. This proves that

$$F'(\xi) = \lim_{n\to\infty} \frac{F(\xi_n) - F(\xi)}{\xi_n - \xi} = \int_I \varphi_\xi(x, \xi)dx.$$

The continuity of F' follows from Exercise 6.3.

6.6 The formula $\int_0^\infty x^n e^{-\xi x}dx = n!/\xi^{n+1}$ for all $\xi \geq a > 0$ may be proved by induction on n, using integration by parts. But we shall follow an approach which utilizes the result of Exercise 6.5 with $I = [0, \infty)$ and $J = [a, \infty)$. The function $e^{-\xi x}$ is differentiable with respect to ξ on $[a, \infty)$ any number of times, and its derivatives are all continuous on J for any $x \in I$. Furthermore, $e^{-\xi x} \leq e^{-ax}$ for all $\xi \geq a$ and e^{-ax} is integrable on $[0, \infty)$. Therefore

$$\frac{d}{d\xi}\int_0^\infty e^{-\xi x}dx = \int_0^\infty \frac{\partial}{\partial \xi} e^{-\xi x}dx = -\int_0^\infty xe^{-\xi x}dx,$$

which implies

$$\int_0^\infty xe^{-\xi x}dx = -\frac{d}{d\xi}\left(\frac{1}{\xi}\right) = \frac{1}{\xi^2}.$$

Now the function $xe^{-\xi x}$ is dominated by $a^{-1}e^{-ax}$, which belongs to $\mathcal{L}^1(a, \infty)$, hence

$$\frac{d}{d\xi}\int_0^\infty xe^{-\xi x}dx = \int_0^\infty x\frac{\partial}{\partial \xi} e^{-\xi x}dx = -\int_0^\infty x^2 e^{-\xi x}dx$$

and

$$\int_0^\infty x^2 e^{-\xi x}dx = -\frac{d}{d\xi}\left(\frac{1}{\xi^2}\right) = \frac{2!}{\xi^3}.$$

In the n-th step, since $x^{n-1}e^{-\xi x} \leq (n-1)a^{-1}e^{-ax} \in \mathcal{L}^1(a,\infty)$, we can write

$$\frac{d}{d\xi}\int_0^\infty x^{n-1}e^{-\xi x}dx = -\int_0^\infty x^n e^{-\xi x}dx,$$

and we conclude that

$$\int_0^\infty x^n e^{-\xi x}dx = -\frac{d}{d\xi}\left(\frac{(n-1)!}{\xi^n}\right) = \frac{n!}{\xi^{n+1}}, \quad \xi \geq a.$$

6.7 For every $x \geq 0$ the integrand $e^{-x}x^{\xi-1}$ is a C^∞ function in the variable ξ on $[a,b]$. It is dominated by the $\mathcal{L}^1(0,\infty)$ function $e^{-x}x^{b-1}$, hence $\Gamma(x)$ is a continuous function on $[a,b]$. Since this is true for every $a > 0$ and $b > a$, $\Gamma(x)$ is continuous on $(0,\infty)$. Its n-th derivative is dominated by the $\mathcal{L}^1(0,\infty)$ function $e^{-x}x^{b-1}(\log x)^n$, so $\Gamma^{(n)}(x)$ is also continuous on $(0,\infty)$ for any $n \in \mathbb{N}$.

6.8 (a) 1, (b) 1/2, (c) 0.

6.9 Express the integral over (a,b) as a sum of integrals over the subintervals $(a,x_1),\cdots,(x_n,b)$. Since both f and g are smooth over each sub-interval, the formula for integration by parts applies to each integral in the sum.

6.10 (a) f is even, hence $B(\xi) = 0$, $A(\xi) = 2\int_0^\pi \sin x \cos \xi x \; dx = 2\dfrac{1+\cos \pi\xi}{1-\xi^2}$, and

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{1+\cos \pi\xi}{1-\xi^2}\cos x\xi \; d\xi.$$

(b) $\hat{f}(\xi) = i(e^{-i\xi}-1)/\xi$. Therefore

$$f(x) = \frac{i}{2\pi}\lim_{L\to\infty}\int_{-L}^{L}\frac{e^{i\xi(x-1)}-e^{i\xi x}}{\xi}d\xi = \frac{1}{\pi}\int_0^\infty \frac{\sin(1-x)\xi + \sin\xi}{\xi}d\xi.$$

(c) f is odd, $B(\xi) = 2(\xi - \sin\xi)/\xi^2$, and

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\xi - \sin\xi}{\xi^2}\sin x\xi \; d\xi.$$

(d) f is odd, $B(\xi) = 2\int_0^{\pi/2}\cos x \sin \xi x \; dx = 2\dfrac{\xi - \sin(\pi\xi/2)}{\xi^2-1}$, and

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\xi - \sin(\pi\xi/2)}{1-\xi^2}\sin x\xi \; d\xi.$$

(e) $A(\xi) = -\dfrac{\xi \sin \pi\xi}{\xi^2 - 1}$, $B(\xi) = \dfrac{\xi + \xi \cos \pi\xi}{\xi^2 - 1}$, and

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{\xi}{\xi^2 - 1}[\sin(x - \pi)\xi + \sin x\xi]d\xi.$$

6.11 At $x = 0$ the Fourier integral representation of f in Exercise 6.10(e) gives

$$\frac{f(0^+) + f(0^-)}{2} = \frac{1}{2} = \frac{1}{\pi}\int_0^\infty \frac{\xi}{1 - \xi^2} \sin \pi\xi \; d\xi,$$

from which the desired formula follows.

6.12 Let $f(x)$ be the odd extension of $e^{-\alpha x}$ from $(0, \infty)$ to $\mathbb{R}$, that is,

$$f(x) = \begin{cases} e^{-\alpha x}, & x > 0 \\ -e^{\alpha x}, & x < 0. \end{cases}$$

Its Fourier sine transform is

$$B(\xi) = 2\int_0^\infty e^{-\alpha x} \sin \xi x \; dx = \frac{2\xi}{\xi^2 + \alpha^2}.$$

Therefore, by the Fourier integral formula (6.28),

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\xi}{\xi^2 + \alpha^2} \sin x\xi \; d\xi,$$

which gives the desired formula when $x > 0$.

6.13 Define f as the odd extension of $e^{-x} \cos x$ from $(0, \infty)$ to $\mathbb{R}$,

$$f(x) = \begin{cases} e^{-x} \cos x, & x > 0 \\ -e^{x} \cos x, & x < 0. \end{cases}$$

Its Fourier sine transform is

$$B(\xi) = 2\int_0^\infty e^{-x} \cos x \sin \xi x \; dx = \frac{2\xi^3}{\xi^4 + 4}.$$

Now $f(x)$ may be represented on $\mathbb{R}$ by the inversion formula (6.28), which yields

$$e^{-x} \cos x = \frac{2}{\pi}\int_0^\infty \frac{\xi^3}{\xi^4 + 4} \sin x\xi \; d\xi$$

when $x > 0$. Since f is not continuous at $x = 0$, this integral is not uniformly convergent.

6.14 The even extension of $e^{-x}\cos x$ from $[0,\infty)$ to $\mathbb{R}$ is $e^{-|x|}\cos x$, whose cosine transform is

$$A(\xi) = 2\int_0^\infty e^{-x}\cos x\cos\xi x\ dx = \frac{1}{1+(1+\xi)^2} + \frac{1}{1+(1-\xi)^2}.$$

Hence

$$e^{-x}\cos x = \frac{1}{\pi}\int_0^\infty \left[\frac{1}{1+(1+\xi)^2} + \frac{1}{1+(1-\xi)^2}\right]\cos x\xi\ d\xi, \quad \text{for all } x \geq 0.$$

This representation is pointwise because $e^{-|x|}\cos x$ is continuous on $\mathbb{R}$.

6.15 Extend

$$f(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$$

as an odd function to $\mathbb{R}$ and show that its sine transform is $B(\xi) = 2(1 - \cos\pi\xi)/\xi$. Equation (6.28) then gives

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos\pi\xi}{\xi}\sin x\xi\ d\xi = \begin{cases} 1, & 0 < x < \pi \\ 1/2, & x = \pi \\ 0, & x > \pi. \end{cases}$$

6.16 Being even, f has a cosine transform which is given by

$$A(\xi) = 2\int_0^\infty \frac{\cos\xi x}{1+x^2}dx.$$

From Example 6.3 we know that the cosine transform of $e^{-|x|}$ is

$$2\int_0^\infty \frac{\cos\xi x}{1+x^2}dx = \frac{2}{1+\xi^2}.$$

Therefore, by the inversion formula,

$$e^{-|x|} = \frac{2}{\pi}\int_0^\infty \frac{\cos\xi x}{1+\xi^2}d\xi, \quad x \in \mathbb{R},$$

and hence

$$A(\xi) = \pi e^{-|\xi|}.$$

6.17 The cosine transform of the even function f is

$$A(\xi) = 2\int_0^1 (1-x)\cos\xi x\ dx = 2\frac{1-\cos\xi}{\xi^2} = \frac{\sin^2(\xi/2)}{(\xi/2)^2}.$$

Hence

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{\sin^2(\xi/2)}{(\xi/2)^2}\cos x\xi\ d\xi.$$

At $x = 0$, which is a point of continuity of f, this yields

$$1 = \frac{1}{\pi}\int_0^\infty \frac{\sin^2(\xi/2)}{(\xi/2)^2}d\xi = \frac{1}{\pi}\int_{-\infty}^\infty \frac{\sin^2\xi}{\xi^2}d\xi.$$

6.18 $2\pi \left\| e^{-|x|} \right\|^2 = 4\pi \int_0^\infty e^{-2x} dx = 2\pi$. From Example 6.3, $\hat{f}(\xi) = 2(1+\xi^2)^{-1}$, hence $\left\| \hat{f} \right\|^2 = 4 \int_0^\infty (1+\xi^2)^{-1} dx = 2\pi$.

6.19 From Equations (6.21) and (6.31) it follows that $\left\| \hat{f} \right\|^2 = \|A\|^2 + \|B\|^2 = 2\pi \|f\|^2$.

6.20 $\mathcal{F}[f(x-a)](\xi) = \int_{-\infty}^\infty f(x-a)e^{-i\xi x}dx = \int_{-\infty}^\infty f(x)e^{-i\xi(x+a)}dx = e^{-ia\xi}\hat{f}(\xi)$.

$\mathcal{F}[e^{ia\xi}f(x)](\xi) = \int_{-\infty}^\infty f(x)e^{-i\xi(x+a)}dx = \int_{-\infty}^\infty f(x-a)e^{-i\xi x}dx = \mathcal{F}[f(x-a)](\xi)$.

6.21 $\psi_n(x)$ decays exponentially as $|x| \to \infty$, so it belongs to $\mathcal{L}^1(\mathbb{R})$ and $\hat{\psi}$ therefore exists. From Example 6.17 we have $\hat{\psi}_0(\xi) = \sqrt{2\pi}\psi_0(\xi)$. Assuming $\hat{\psi}_n(\xi) = (-i)^n\sqrt{2\pi}\psi_n(\xi)$, we have

$$\begin{aligned}
\hat{\psi}_{n+1}(\xi) &= \mathcal{F}\left(e^{-x^2/2}H_{n+1}(x)\right)(\xi) \\
&= \mathcal{F}\left[e^{-x^2/2}(2xH_n(x) - H_n'(x))\right](\xi) \\
&= \mathcal{F}\left[x\psi_n(x) - \psi_n'(x)\right](\xi) \\
&= i\hat{\psi}_n'(\xi) - i\xi\hat{\psi}_n(\xi) \\
&= (-i)^{n+1}\sqrt{2\pi}[-\psi_n'(\xi) + \xi\psi_n(\xi)] \\
&= (-i)^{n+1}\sqrt{2\pi}\psi_{n+1}(x),
\end{aligned}$$

where we used the identity $H_{n+1}(x) = 2xH_n(x) - H_n'(x)$ and Theorem 6.15. Thus, by induction, $\hat{\psi}_n(\xi) = (-i)^n\sqrt{2\pi}\psi_n(\xi)$ is true for all $n \in \mathbb{N}_0$.

6.22 Since $\hat{u}(\xi) = 2\int_0^\infty u(x)\cos\xi x \; dx$,

$$\hat{u}(\xi) = \begin{cases} 2, & 0 < \xi < \pi \\ 0, & x > \pi. \end{cases}$$

Consequently,

$$u(x) = \frac{1}{\pi}\int_0^\pi 2\cos x\xi \; d\xi = \frac{2}{\pi x}\sin \pi x, \quad x > 0.$$

6.23 Let $I(z) = \int_0^\infty e^{-b\xi^2}\cos z\xi \; d\xi$. Since the integrand is dominated by the $\mathcal{L}^1$ function $e^{-b\xi^2}$ and differentiable with respect to z, we have $I'(z) = -\int_0^\infty e^{-b\xi^2}\xi\sin z\xi \; d\xi$. Integrating by parts,

$$I'(z) = -\frac{z}{2b}I(z).$$

The solution of this equation is $I(z) = ce^{-z^2/4b}$, where the integration constant is $c = I(0) = \int_0^\infty e^{-b\xi^2}d\xi = \frac{1}{2}\sqrt{\pi/b}$.

6.24 From Exercise 6.18 the temperature u is given by

$$
\begin{aligned}
u(x,t) &= \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} f(y)e^{-(y-x)^2/4kt}dy \\
&= \frac{T_0}{\sqrt{\pi}}\int_{-a}^{a}\frac{1}{2\sqrt{kt}}e^{-(y-x)^2/4kt}dy \\
&= \frac{T_0}{\sqrt{\pi}}\int_{-(a+x)/2\sqrt{kt}}^{(a-x)/2\sqrt{kt}} e^{-p^2}dp \\
&= \frac{T_0}{2}\left[\frac{2}{\sqrt{\pi}}\int_{-(a+x)/2\sqrt{kt}}^{0} e^{-p^2}dp + \frac{2}{\sqrt{\pi}}\int_{0}^{(a-x)/2\sqrt{kt}} e^{-p^2}dp\right] \\
&= \frac{T_0}{2}\left[\operatorname{erf}\left(\frac{x+a}{2\sqrt{\pi kt}}\right) + \operatorname{erf}\left(\frac{a-x}{2\sqrt{\pi kt}}\right)\right].
\end{aligned}
$$

If we set $T_0 = 1$ this solution coincides with that of Example 6.19. That is because the temperature $u(x,0)$ in the example, though initially defined on $0 < x < \infty$, is then extended as an *even* function to $\mathbb{R}$, so it coincides with $f(x)$ as defined in this exercise.

6.25 The boundary condition at $x = 0$ implies $A(\lambda) = 0$ in the representation of $u(x,t)$ given by (6.39), so that u is now an odd function of x. By extending $f(x)$ as an odd function from $(0,\infty)$ to $(-\infty,\infty)$ we can see that $B(\lambda)$ is the sine transform of f and the same procedure followed in Example 6.18 leads to

$$
\begin{aligned}
u(x,t) &= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(y)\sin\lambda y\sin\lambda x\; e^{-k\lambda^2 t}dyd\lambda \\
&= \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(y)\sin\lambda y\sin\lambda x\; e^{-k\lambda^2 t}dyd\lambda \\
&= \frac{1}{\pi}\int_0^{\infty}\int_0^{\infty} f(y)\left[\cos(y-x)\lambda - \cos(y+x)\lambda\right]e^{-k\lambda^2 t}dyd\lambda \\
&= \frac{1}{2\sqrt{\pi kt}}\int_0^{\infty} f(y)\left[e^{-(y-x)^2/4kt} - e^{-(y+x)^2/4kt}\right]dy.
\end{aligned}
$$

6.26 Substituting into the formula in Exercise 6.25,

$$
\begin{aligned}
u(x,t) &= \frac{1}{2\sqrt{\pi kt}}\int_0^{1}\left[e^{-(y-x)^2/4kt} - e^{-(y+x)^2/4kt}\right]dy \\
&= \frac{1}{\sqrt{\pi}}\int_{-x/2\sqrt{kt}}^{(1-x)/2\sqrt{kt}} e^{-p^2}dp - \frac{1}{\sqrt{\pi}}\int_{x/2\sqrt{kt}}^{(1+x)/2\sqrt{kt}} e^{-p^2}dp \\
&= \operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right) - \frac{1}{2}\left[\operatorname{erf}\left(\frac{x-1}{2\sqrt{kt}}\right) + \operatorname{erf}\left(\frac{x+1}{2\sqrt{kt}}\right)\right], \quad x > 0,\ t > 0.
\end{aligned}
$$

6.27 The transformed wave equation is $\hat{u}_{tt}(\xi,t) = -c^2\xi^2\hat{u}(\xi,t)$, under the initial conditions $\hat{u}(\xi,0) = \hat{f}(\xi)$, $\hat{u}_t(\xi,0) = 0$, is solved by $\hat{u}(\xi,t) = \hat{f}(\xi)\cos c\xi t$. Taking the inverse Fourier transform yields

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\cos c\xi t \; e^{ix\xi} d\xi.$$

Using the relation $\cos c\xi t \; e^{ix\xi} = \frac{1}{2}\left[e^{i\xi(x+ct)} + e^{i\xi(x-ct)}\right]$, we obtain

$$u(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\left[e^{i\xi(x+ct)} + e^{i\xi(x-ct)}\right] d\xi = \frac{1}{2}[f(x+ct) + f(x-ct)].$$

6.28 The condition $u(0,t) = 0$ allows us to extend both f and u as odd functions of x, so that

$$B(\xi) = 2\int_0^{\infty} f(x)\sin\xi x \; dx,$$

and

$$u(x,t) = \frac{1}{\pi}\int_0^{\infty} B(\xi)\cos c\xi t\sin x\xi \; d\xi.$$

6.29 Assuming f is continuous,

$$\begin{aligned}
y(x) &= \frac{1}{2}\int_{-\infty}^{\infty} f(t)e^{-|x-t|}dt \\
&= \frac{1}{2}\int_{-\infty}^{x} f(t)e^{-(x-t)}dt + \frac{1}{2}\int_{x}^{\infty} f(t)e^{-(t-x)}dt \\
y'(x) &= \frac{1}{2}\left[f(x) - \int_{-\infty}^{x} f(t)e^{-(x-t)}dt\right] + \frac{1}{2}\left[-f(x) + \int_{x}^{\infty} f(t)e^{-(t-x)}dt\right] \\
&= -\frac{1}{2}\int_{-\infty}^{x} f(t)e^{-(x-t)}dt + \frac{1}{2}\int_{x}^{\infty} f(t)e^{-(t-x)}dt
\end{aligned}$$

$$\begin{aligned}
y''(x) &= -\frac{1}{2}f(x) + \frac{1}{2}\int_{-\infty}^{x} f(t)e^{-(x-t)}dt - \frac{1}{2}f(x) + \frac{1}{2}\int_{x}^{\infty} f(t)e^{-(t-x)}dt \\
&= -f(x) + \frac{1}{2}\int_{-\infty}^{\infty} f(t)e^{-|x-t|}dt.
\end{aligned}$$

Consequently, $y - y'' = f$.

Chapter 7

7.1 (a) $\dfrac{2a^2}{s^3}+\dfrac{2ab}{s^2}+\dfrac{b^2}{s}$.

(b) $\dfrac{s}{s^2-1}, \quad s>1$.

(c) $\dfrac{2}{s(s^2+4)}$

(d) $\dfrac{1}{s^2+4}$.

(e) $\dfrac{c}{s}\left(1-e^{-as}\right)$.

(f) $\dfrac{a}{s}-\dfrac{a}{bs^2}(1-e^{-bs})$.

(g) $\dfrac{2s}{(s^2-1)^2}, \quad s>1$.

(h) $\dfrac{2}{(s-1)^3}, \quad s>1$.

(i) $\sqrt{\pi/s}$.

7.2 (a) ae^{-bx}.

(b) $2\cosh 3x-\dfrac{5}{3}\sinh 3x$.

(c) $1-e^{-x}$.

(d) $\dfrac{1}{2}\left(1-e^{-2x}\right)$.

(e) $3\left(\cosh\sqrt{6}x-\dfrac{1}{\sqrt{6}}\sinh\sqrt{6}x\right)$.

(f) $2\sqrt{x/\pi}$.

(g) $\dfrac{1}{2}(5e^{-x}+3e^{-2x}+6e^{-3x})$.

7.3

$$\begin{aligned}\mathcal{L}(f(ax))(s) &= \int_0^\infty f(ax)e^{-sx}dx \\ &= \frac{1}{a}\int_0^\infty f(x)e^{-sx/a}dx \\ &= \frac{1}{a}F(s/a), \quad a>0.\end{aligned}$$

7.4 (a) e^{-s}/s^2.

(b) $2e^{-s}/s^3$.

(c) $e^{-s}(s^{-1}+2s^{-2}+2s^{-3}) - e^{-3s}(9s^{-1}+6s^{-2}+2s^{-3})$.

(d) $-e^{-\pi s/2}/(s^2+1)$.

(e) $(1-e^{-s})s^{-1} - (1-e^{-(s+1)})(s+1)^{-1}$.

7.5 $f(x) = x[H(x) - H(x-1)] + e^{1-x}H(x-1)$.

$\mathcal{L}(f)(\xi) = \dfrac{1}{s^2}\left(1-e^{-s}\right) - \dfrac{1}{s}e^{-s} + \dfrac{1}{s+1}e^{-s}$.

7.6 (a) $\dfrac{1}{2}H(x-6)(x-6)^2$.

(b) $H(x-1)e^{-(x-1)}\sin(x-1)$.

(c) $H(x-3) + H(x-1)$.

(d) $H(x-3)e^{x-3} + H(x-1)e^{x-1}$.

(e) $\dfrac{1}{3}[\sin 3x + H(x-\pi)\sin 3(x-\pi)] = \dfrac{1}{3}[\sin 3x - H(x-\pi)\sin 3x]$.

7.7 If f has jump discontinuities at the points $x_1, \cdots, x_n$ then the sum $f(x_1^-) - f(x_1^+) + \cdots + f(x_n^-) - f(x_n^+)$ has to be added to the right-hand side of (7.6).

7.8 (a) The transformed equation is gives

$$Y(s) = \frac{s+5}{s^2+4s+5} = \frac{s+2}{(s+2)^2+1} + 3\frac{1}{(s+2)^2+1},$$

hence

$$y(x) = e^{-2x}(\cos x + 3\sin x).$$

(b) $y(x) = 3e^{x/3}$.

(c) $y(x) = e^{-x}(\sin 2x + \sin x)$.

(d) $y(x) = 15 + 36x + 24x^2 + 32x^3$.

(e) $y(x) = (e^{2x} - e^x)H(x) + \left[\dfrac{1}{2}e^{2(x-1)} - e^{x-1} + \dfrac{1}{2}\right]H(x-1)$.

(f) $y(x) = \dfrac{1}{4}\left(e^{-2x} + 2x - 1\right)H(x) + \dfrac{1}{4}(e^{-2(x-1)} - 2x + 1)H(x-1)]$.

(g) $y(x) = \dfrac{1}{2}(e^{-x} - \cos x + \sin x) + \dfrac{3}{2}[e^{-(x-\pi)} + \cos x - \sin x]H(x-1)$.

7.9 The first three answers may be obtained by using Equations (7.9), (7.10) or Theorem 7.14.

(a) $\dfrac{1}{6}x\sin 3x$.

(b) $\dfrac{1}{x}(e^x - 1)$.

(c) $\dfrac{1}{x}\left(e^{-bx} - e^{-ax}\right)$.

(d) Differentiate $F(s) = \operatorname{arccot}(s+1)$, use Equation (7.14), and then invert to arrive at $\mathcal{L}^{-1}(F)(x) = \dfrac{1}{x}e^{-x}\sin x$.

7.10 Using Equation (7.9),

$$\begin{aligned}\mathcal{L}\left(\frac{\sin x}{x}\right)(s) &= \int_s^\infty \frac{1}{z^2+1}dz \\ &= \frac{\pi}{2} - \arctan s \\ &= \arctan\left(\frac{1}{s}\right).\end{aligned}$$

Now Theorem 7.6 implies

$$\mathcal{L}[\mathrm{Si}(x)](s) = \frac{1}{s}\arctan\left(\frac{1}{s}\right).$$

7.11 (a) Write

$$\begin{aligned}\mathcal{L}(f)(s) &= \int_0^\infty f(x)e^{-sx}dx \\ &= \sum_{n=0}^\infty \int_{np}^{(n+1)p} f(x)e^{-sx}dx \\ &= \sum_{n=0}^\infty \int_0^p f(x+np)e^{-s(x+np)}dx,\end{aligned}$$

then use the equations $f(x+np) = f(x)$ for all n and $\sum_{n=0}^\infty e^{-nps} = (1-e^{-ps})^{-1}$, with $p > 0$ and $s > 0$, to arrive at the answer.

(b) $\mathcal{L}(f)(s) = \dfrac{1}{1-e^{-s}}\left[\dfrac{1}{s^2}(1-e^{-s}) - \dfrac{e^{-s}}{s}\right]$.

7.12 For any function f in $\mathcal{E}$ there are positive constants M, α, and b such that

$$|f(x)| \leq Me^{\alpha x} \quad \text{for all } x \geq b.$$

Therefore

$$\begin{aligned}
|\mathcal{L}(f)(s)| &\le \int_0^\infty |f(x)|\, e^{-x\,\mathrm{Re}\,s} dx \\
&\le \int_0^b |f(x)|\, e^{-x\,\mathrm{Re}\,s} dx + M \int_b^\infty e^{-x\,\mathrm{Re}\,s} dx \\
&\le \int_0^b |f(x)|\, e^{-x\,\mathrm{Re}\,s} dx + \frac{M}{\mathrm{Re}\,s} e^{-b\,\mathrm{Re}\,s}.
\end{aligned}$$

Since f is locally integrable the right hand-side tends to 0 as $\mathrm{Re}\,s \to \infty$.

7.13 The left-hand side is the convolution of x^3 and $y(x)$. Applying Theorem 7.14 to the integral equation gives $3!Y(s)/s^4 = F(s)$, from which $Y(s) = s^4F(s)/6$. From Corollary 7.7 we conclude that

$$y(x) = \frac{1}{6} f^{(4)}(x) + \frac{1}{6}\mathcal{L}^{-1}[f(0^+)s^3 + f'(0^+)s^2 + f''(0^+)s + f'''(0^+)].$$

The integral expression for $f(x)$ implies that $f^{(n)}(0^+) = 0$ for $n = 0, 1, 2, 3$ (we also know from Exercise 7.12 that s^n cannot be the Laplace transform of a function in $\mathcal{E}$ for any $n \in \mathbb{N}_0$). Assuming that f is differentiable to fourth order (or that y is continuous), the solution is $y(x) = f^{(4)}(x)/6$.

7.14

$$f * g(x) = \int_0^x f(x-t)g(t)dt = \int_0^x e^{x-t}\frac{1}{\sqrt{\pi t}}dt = e^x \frac{1}{\sqrt{\pi}}\int_0^x e^{-t}\frac{1}{\sqrt{t}}dt.$$

Under the change of variable $t = p^2$, we obtain

$$f * g(x) = e^x \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} e^{-p^2} dp = e^x \operatorname{erf}(\sqrt{x}).$$

Therefore

$$\begin{aligned}
\mathcal{L}[e^x \operatorname{erf}(\sqrt{x})](s) &= \mathcal{L}(f)\mathcal{L}(g)(s) \\
&= \frac{\Gamma(1/2)}{\sqrt{\pi}} \frac{1}{s-1} \frac{1}{\sqrt{s}} \\
&= \frac{1}{s-1}\frac{1}{\sqrt{s}}, \quad s > 1,
\end{aligned}$$

from which

$$\begin{aligned}
\mathcal{L}[\operatorname{erf}(\sqrt{x})](s) &= \mathcal{L}[e^{-x}e^x \operatorname{erf}(\sqrt{x})](s) \\
&= \frac{1}{s\sqrt{s+1}}.
\end{aligned}$$

7.15

$$\begin{aligned}\mathcal{L}([x])(s) &= \int_0^\infty [x]e^{-sx}dx \\ &= \sum_{n=0}^\infty n\int_n^{n+1} e^{-sx}dx \\ &= \frac{1}{s}\sum_{n=0}^\infty n(e^{-ns}-e^{-(n+1)s}) \\ &= \frac{1}{s}\sum_{n=0}^\infty ne^{-ns}(1-e^{-s}) \\ &= \frac{1-e^{-s}}{s}\left(-\frac{d}{ds}\sum_{n=0}^\infty e^{-ns}\right) \\ &= -\frac{1-e^{-s}}{s}\frac{d}{ds}\frac{1}{1-e^{-s}} \\ &= \frac{e^{-s}}{s(1-e^{-s})}.\end{aligned}$$

7.16 (a) Suppose, without loss of generality, that f is continuous.

$$\begin{aligned}|f*g(x+\delta)-f*g(x)| &\le \int_0^{x+\delta} |f(x+\delta-t)g(t)-f(x-t)g(t)|\,dt \\ &= \int_0^x |f(x+\delta-t)-f(x-t)|\,|g(t)|\,dt \\ &+ \int_x^{x+\delta} |f(x+\delta-t)-f(x-t)|\,|g(t)|\,dt.\end{aligned}$$

Since f is continuous and g is locally integrable, the right-hand side tends to 0 as $\delta \to 0$.

(b) Suppose f is smooth and g is continuous. Then

$$(f*g)'(x) = f(0)g(x) + \int_0^x f'(x-t)g(t)dt,$$

and, based on the result of (a), the right-hand side is continuous.

(c) If f is piecewise continuous, then the convolution integral over $(0,x)$ can be expressed as a finite sum of integrals over subintervals (x_i, x_{i+1}), $0 \le i \le n$, in each of which f is continuous, and the conclusion of (a) follows for each subinterval. Furthermore, the one-sided limits at x_i clearly exist, hence $f*g$ is piecewise continuous. A similar argument can be used to prove that $f*g$ is piecewise smooth if f is piecewise smooth and g is piecewise continuous. In the latter case the interval $(0,x)$ is partitioned by the points of discontinuity of f, f', and g for the argument to go through.

7.17 The transformed wave equation is $U_{xx} = s^2 U/c^2$, and its solution

$$U(x,s) = a(s)e^{-sx/c} + b(s)e^{sx/c}$$

becomes unbounded as $\operatorname{Re} s \to \infty$ unless $b(s) = 0$. From the boundary condition at $x = 0$,

$$U(0,s) = a(s) = \mathcal{L}(\cos^2 t),$$

and therefore

$$\begin{aligned} u(x,s) &= \mathcal{L}^{-1}\left[e^{-sx/c}\mathcal{L}(\cos^2 t)\right] \\ &= H(t - x/c)\cos^2(t - x/c). \end{aligned}$$

7.18 The (bounded) solution of the transformed equation is

$$U(x,s) = a(s)e^{-\sqrt{(s-a)/k}x}.$$

The boundary condition at $x = 0$ implies $a(s) = \hat{f}(s)$, hence

$$\begin{aligned} u(x,t) &= \mathcal{L}^{-1}\left[\hat{f}(s)e^{-\sqrt{(s-a)/k}x}\right] \\ &= \int_0^t f(t-\tau)\mathcal{L}^{-1}\left(e^{-\sqrt{(\tau-a)/k}x}\right)d\tau. \end{aligned}$$

The inverse transform of $e^{-\sqrt{\tau/k}x}$ was obtained in Exercise 7.21, and that of $e^{-\sqrt{(\tau-a)/k}x}$ follows from applying Theorem 7.10.

7.19 The transformed equation is

$$U_{xx} = \frac{1}{c^2}(s+1)^2 U,$$

hence

$$U(x,s) = F(s)e^{-(s+1)x/c},$$

where $F(s) = U(0,s) = \mathcal{L}(\sin t)$. The solution is therefore

$$\begin{aligned} u(x,t) &= \mathcal{L}^{-1}\left[e^{-(s+1)x/c}\mathcal{L}(\sin t)\right] \\ &= e^{-x/c}H(t - x/c)\sin(t - x/c). \end{aligned}$$

7.20 By changing the variable of integration from τ to $p = x/2\sqrt{k\tau}$, Equation (7.20) becomes

$$u(x,t) = \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{kt}}^{\infty} f\left(t - x^2/4kp^2\right) e^{-p^2} dp$$
$$\lim_{x\to 0} u(x,t) = \lim_{x\to 0} \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{kt}}^{\infty} f\left(t - x^2/4kp^2\right) e^{-p^2} dp.$$

For any function in $\mathcal{E}$, the integrand $f\left(t - x^2/4kp^2\right) e^{-p^2}$ is clearly dominated by an $\mathcal{L}^1$ function for large values of p, therefore we can take the limit inside the integral to obtain

$$\lim_{x\to 0} u(x,t) = \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{kt}}^{\infty} f(t)\, e^{-p^2} dp = f(t).$$

7.21 $F(s) = e^{-a\sqrt{s}}/\sqrt{s}$ is analytic in the complex plane cut along the negative axis $(-\infty, 0]$. Using Cauchy's theorem, the integral along the vertical line $(\beta - i\infty, \beta + i\infty)$ can be reduced to two integrals, one along the bottom edge of the cut from left to right, and the other along the top edge from right to left. This yields

$$\begin{aligned}
\mathcal{L}^{-1}(F)(x) &= \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} F(s) e^{xs} ds \\
&= \frac{1}{2\pi} \int_0^{\infty} \frac{e^{ia\sqrt{s}} + e^{-ia\sqrt{s}}}{\sqrt{s}} e^{-xs} ds \\
&= \frac{1}{\pi} \int_0^{\infty} \frac{\cos\left(a\sqrt{s}\right)}{\sqrt{s}} e^{-sx} ds \\
&= \frac{2}{\pi} \int_0^{\infty} e^{-xt^2} \cos at\ dt.
\end{aligned}$$

Noting that the last integral is the Fourier transform of e^{-xt^2}, and using the result of Example 6.17, we obtain the desired expression for $\mathcal{L}^{-1}(F)(x)$.

To evaluate $\mathcal{L}^{-1}\left(e^{-a\sqrt{s}}\right)$ one is tempted to use Theorem 7.14, whereby

$$\begin{aligned}
\mathcal{L}^{-1}\left(e^{-a\sqrt{s}}\right)(x) &= \mathcal{L}^{-1}\left(\sqrt{s} \cdot e^{-a\sqrt{s}}/\sqrt{s}\right) \\
&= \mathcal{L}^{-1}\left(\sqrt{s}\right) * \mathcal{L}^{-1}\left(e^{-a\sqrt{s}}/\sqrt{s}\right),
\end{aligned}$$

and attempt to compute the resulting convolution. But the the problem here is that $\mathcal{L}^{-1}\left(\sqrt{s}\right)$ does not exist, at least not in $\mathcal{E}$ (see Example 7.4 or

Exercise 7.12). Another approach is to try to remove $\sqrt{s}$ from the denominator of $F(s)$ by integration (this is actually suggested by the formula we seek to prove).

$$\int_s^\infty \frac{e^{-a\sqrt{z}}}{\sqrt{z}} dz = \frac{1}{2} \int_{\sqrt{s}}^\infty e^{-az} dz = \frac{2}{a} e^{-a\sqrt{s}}.$$

Using the formula (7.10), we therefore conclude that

$$\begin{aligned}
\mathcal{L}^{-1}\left(e^{-a\sqrt{s}}\right)(x) &= \frac{a}{2}\mathcal{L}^{-1}\left[\int_s^\infty \left(e^{-a\sqrt{z}}/\sqrt{z}\right) dz\right] \\
&= \frac{a}{2x}\mathcal{L}^{-1}\left(e^{-a\sqrt{s}}/\sqrt{s}\right)(x) \\
&= \frac{a}{2\sqrt{\pi x^3}} e^{-a^2/4x}.
\end{aligned}$$

http://www.springer.com/978-1-84628-971-2

Sturm-Liouville Theory and its Applications
Al-Gwaiz, M.
2008, X, 264 p., Softcover
ISBN: 978-1-84628-971-2