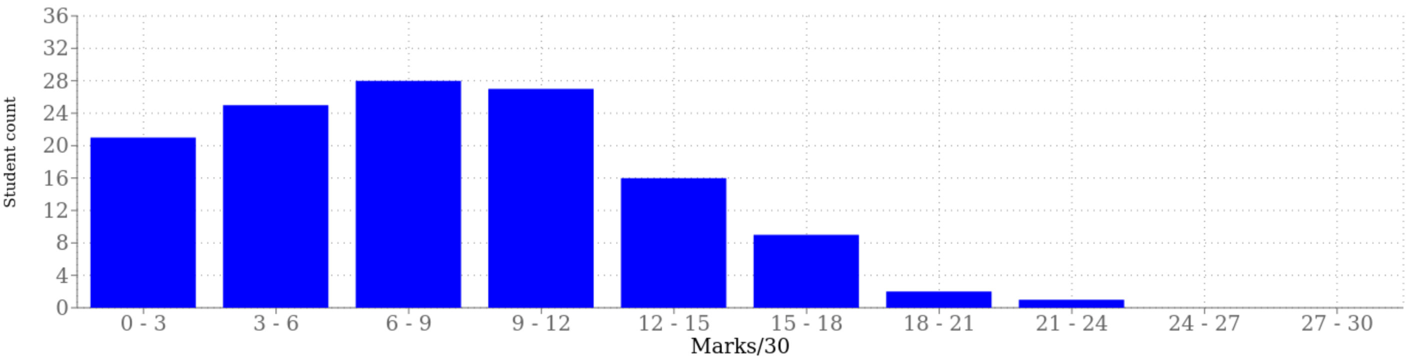
Mid-sem marks - PAD's part



Mid-sem solution - PAD's part

Q04: Given,

$$V = \{(x, y) : x, y \in \mathbb{R}\} \quad -(1)$$

$$\otimes = (a_1, b_1) + (a_2, b_2)$$
$$= (3b_1 + 3b_2, -a_1 - a_2) \quad -(2)$$

$$\odot = k(a_1, b_1) = (3kb_1, -ka_1) \quad -(3)$$
$\forall (a_i, b_i) \in V$, $k \in \mathbb{R}$

(a) Verification of $\otimes$ as a binary operation on V:

Since $(a_i, b_i) \in V$, $a_i, b_i \in \mathbb{R}$, from Eq^n (2),
$3b_1 + 3b_2 \in \mathbb{R}$ and
$-a_1 - a_2 \in \mathbb{R}$

Let $3b_1 + 3b_2 = c_1$ and $-a_1 - a_2 = c_2$

$\Rightarrow (a_1, b_1) + (a_2, b_2) = (c_1, c_2)$, $c_1, c_2 \in \mathbb{R}$

$\Rightarrow \otimes : V \times V \longrightarrow V$

$\therefore \otimes$ is a binary operation on V

$\boxed{2 \text{ marks}}$

Verification of $\odot$ as a binary operation on V.

If R is a set of real numbers the $k \in R$. Hence, $3kb_1 \in \mathbb{R}$ and $-ka_1 \in \mathbb{R}$.

$\Rightarrow k(a_1, b_1) \in V$

According to Eq^n (3),

$\odot : R \times V \longrightarrow V$

However, for a binary operation, $\odot : V \times V \longrightarrow V$.

Hence, $\odot$ is not a binary operation on V

$\boxed{\text{2 marks}}$

(b) Verification of distributivity of $\otimes$ over $\odot$.

Let $k \in R$ and $(a_1, b_1), (a_2, b_2) \in V$. If $\otimes$ is distributive over $\odot$ then

$$k \otimes [(a_1, b_1) \odot (a_2, b_2)] = [k \otimes (a_1, b_1)] \odot [k \otimes (a_2, b_2)]$$

and $[(a_1, b_1) \odot (a_2, b_2)] \otimes k = [(a_1, b_1) \otimes k] \odot [(a_2, b_2) \otimes k]$

(left and right distributivity)

$(a_1, b_1) \odot (a_2, b_2) \Leftarrow$ this operation is not defined.

Hence, $\otimes$ cannot be distributive over $\odot$

$\boxed{\text{3 marks}}$

(c) Verification of distributivity of $\odot$ over $\otimes$.

If $\odot$ is left distributive over $\otimes$ then

$$k \odot [(a_1, b_1) \otimes (a_2, b_2)] = [k \odot (a_1, b_1)] \otimes [k \odot (a_2, b_2)] \quad -(4)$$

From Eq^n (2),

$(a_1, b_1) \otimes (a_2, b_2) = (3b_1 + 3b_2, -a_1 - a_2)$

$\Rightarrow k \odot [(a_1, b_1) \otimes (a_2, b_2)]$

$= (-3ka_1 - 3ka_2, -3kb_1 - 3kb_2)$ — (5)

$k \odot (a_1, b_1) = (3kb_1, -ka_1)$

$k \odot (a_2, b_2) = (3kb_2, -ka_2)$

$\Rightarrow [k \odot (a_1, b_1)] \otimes [k \odot (a_2, b_2)]$

$= (-3ka_1 - 3ka_2, -3kb_1 - 3kb_2)$ — (6)

From Eq^n (5) and (6),

$k \odot [(a_1, b_1) \otimes (a_2, b_2)]$

$= [k \odot (a_1, b_1)] \otimes [k \odot (a_2, b_2)]$ — (7)

Hence, $\odot$ is left-distributive over $\otimes$.

Similarly, it can be shown that $\odot$ is right-distributiveover $\otimes$.

$\therefore$ $\odot$ is distributive over $\otimes$.

$\boxed{\text{3 marks}}$

Q05: System 1

$$2x_1 + 6x_2 + 9x_3 + 7x_4 = p_1 \quad -(1)$$

$$x_1 + 3x_2 + 3x_3 + 2x_4 = p_2 \quad -(2)$$

$$-x_1 - 3x_2 + 3x_3 + 4x_4 = p_3 \quad -(3)$$

The above system of equations can be written as a matrix equation of the form $\underline{\underline{A}}\,\underline{x} = \underline{p}$ as follows:

$$\begin{bmatrix} 2 & 6 & 9 & 7 \\ 1 & 3 & 3 & 2 \\ -1 & -3 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \quad -(4)$$

$$\Rightarrow [\underline{\underline{A}} | \underline{p}] = \left[\begin{array}{cccc|c} 2 & 6 & 9 & 7 & p_1 \\ 1 & 3 & 3 & 2 & p_2 \\ -1 & -3 & 3 & 4 & p_3 \end{array}\right]$$

$$\begin{array}{l} R_2 \longrightarrow 2R_2 - R_1 \\ R_3 \longrightarrow 2R_3 + R_1 \end{array} \quad \left[\begin{array}{cccc|c} 2 & 6 & 9 & 7 & p_1 \\ 0 & 0 & -3 & -3 & 2p_2 - p_1 \\ 0 & 0 & 15 & 15 & 2p_3 + p_1 \end{array}\right]$$

$$R_3 \longrightarrow R_3 + 5R_2 \quad \left[\begin{array}{cccc|c} 2 & 6 & 9 & 7 & p_1 \\ 0 & 0 & -3 & -3 & 2p_2 - p_1 \\ 0 & 0 & 0 & 0 & 2p_3 + 10p_2 - 4p_1 \end{array}\right]$$

The condition for solvability dictates that

$$\rho(\underline{\underline{A}} | \underline{p}) = \rho(\underline{\underline{A}})$$

$$\Rightarrow 2p_3 + 10p_2 - 4p_1 = 0 \quad -(5)$$

1 mark till this point

Degree of freedom analysis:

# of variables = 3

# of equations = 1

$\Rightarrow$ DOF = 2

Let $p_2 = \alpha$ and $p_3 = \beta$

$$\Rightarrow \quad 2\beta + 10\alpha - 4p_1 = 0$$

$$\Rightarrow \quad p_1 = \frac{1}{2}\beta + \frac{5}{2}\alpha$$

$$\Rightarrow \quad \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{2}\alpha + \frac{1}{2}\beta \\ \alpha \\ \beta \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \alpha' \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix} + \beta' \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad - (6)$$

$\therefore$ The dimension of the range space of system 1 is 2. Two vectors in the range space basis are $[5 \; 2 \; 0]^T$ and $[1 \; 0 \; 2]^T$.

3 marks till this point

System 2 :

$$3y_1 + y_2 + 10y_3 + 14y_4 = q_1 \qquad -(7)$$

$$y_1 + 2y_3 + 3y_4 = q_2 \qquad -(8)$$

$$y_2 + 4y_3 + 5y_4 = q_3 \qquad -(9)$$

Writing the above set of equations as $\underline{\underline{B}}\,\underline{y} = \underline{q}$ we get

$$\begin{bmatrix} 3 & 1 & 10 & 14 \\ 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \qquad -(10)$$

$$\Rightarrow [\underline{\underline{B}} \,|\, \underline{q}] = \left[\begin{array}{cccc|c} 3 & 1 & 10 & 14 & q_1 \\ 1 & 0 & 2 & 3 & q_2 \\ 0 & 1 & 4 & 5 & q_3 \end{array}\right]$$

$$R_2 \rightarrow 3R_2 - R_1 \quad \left[\begin{array}{cccc|c} 3 & 1 & 10 & 14 & q_1 \\ 0 & -1 & -4 & -5 & 3q_2 - q_1 \\ 0 & 1 & 4 & 5 & q_3 \end{array}\right]$$

$$R_3 \rightarrow R_3 + R_2 \quad \left[\begin{array}{cccc|c} 3 & 1 & 10 & 14 & q_1 \\ 0 & -1 & -4 & -5 & 3q_2 - q_1 \\ 0 & 0 & 0 & 0 & q_3 + 3q_2 - q_1 \end{array}\right] \qquad -(11)$$

$$\Rightarrow q_3 + 3q_2 - q_1 = 0 \qquad -(12)$$

$\boxed{\text{4 marks till here}}$

As before,

\# of variables = 3

\# of equations = 1

$\Rightarrow$ DoF = 2

Let $q_2 = \alpha''$ and $q_3 = \beta''$

$\Rightarrow q_1 = 3\alpha'' + \beta''$

$$\Rightarrow \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 3\alpha'' + \beta'' \\ \alpha'' \\ \beta'' \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \alpha'' \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \beta'' \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad - (13)$$

Hence, the dimension of the range space of system 2 is 2. Two vectors in the basis of the range space of system 2 are $[3 \; 1 \; 0]^T$ and $[1 \; 0 \; 1]^T$.

6 marks till here

Both the systems have the same dimension. If the field is assumed to be $\mathbb{R}$ for both the systems then the range spaces would be identical if the basis of one system can be expressed as a linear combination of the basis of the other system i.e.

$$\begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad -(14)$$

and

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = d_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad -(15)$$

Eq^n (14) can be written as

$$\begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$$

$$\underline{\underline{D}} \; \underline{c} = \underline{\gamma}$$

$$\Rightarrow [\underline{\underline{D}} \,|\, \underline{\gamma}] = \left[\begin{array}{cc|c} 3 & 1 & 5 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{array}\right]$$

$$R_2 \longrightarrow 3R_2 - R_1 \qquad \left[\begin{array}{cc|c} 3 & 1 & 5 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{array}\right]$$

$R_3 \longrightarrow R_3 + R_2$ $\left[\begin{array}{cc|c} 3 & 1 & 5 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{array}\right]$

$\rho(\underline{\underline{D}}) = 2$ while $\rho(\underline{\underline{D}}|\underline{x}) = 3$. Hence, the system is not solvable. Therefore, the basis of system 1 cannot be written as a linear combination of the basis of system 2. Hence, the range spaces of the two systems are <u>not</u> identical.

$\boxed{10 \text{ marks}}$

<u>Q06</u>: Given the system of equations:

$$\frac{dy}{dx} - 2y = 0 \qquad -(1)$$

$$\frac{dy}{dx} - \frac{y}{x} = 0 \qquad -(2)$$

From Eq^n (1),

$$\frac{dy}{y} = 2x$$

$$\Rightarrow \quad y = c_1 e^{2x} \qquad -(3)$$

From Eq^n (2),

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\Rightarrow y = c_2 x \qquad -(4)$$

$\boxed{\text{4 marks till here}}$

The simultaneous equations have the solutions which satisfy both Eq^n (3) and Eq^n (4). We need to test whether the family of Eq^n (3) & (4) are linearly independent.

$$|W| = \begin{vmatrix} c_1 e^{2x} & c_2 x \\ 2c_1 e^{2x} & c_2 \end{vmatrix}$$

$$\Rightarrow |W| = c_1 c_2 e^{2x}(1 - 2x) \qquad -(5)$$

The Wronskian given by Eq^n (5) is not necessarily zero everywhere. Hence, the solutions are linearly independent.

At the solution,

$$c_1 e^{2x} = c_2 x$$

$$\Rightarrow x = \left(\frac{c_1}{c_2}\right) e^{2x}$$

$$\Rightarrow x = \alpha e^{2x} \qquad -(6)$$

Any x satisfying the transcendental eq^n (6) will be in the basis of the linear vector solution space. I <u>do not</u> expect you to solve Eq^n (6) but to just mention the above condition.

$\boxed{\text{10 marks}}$