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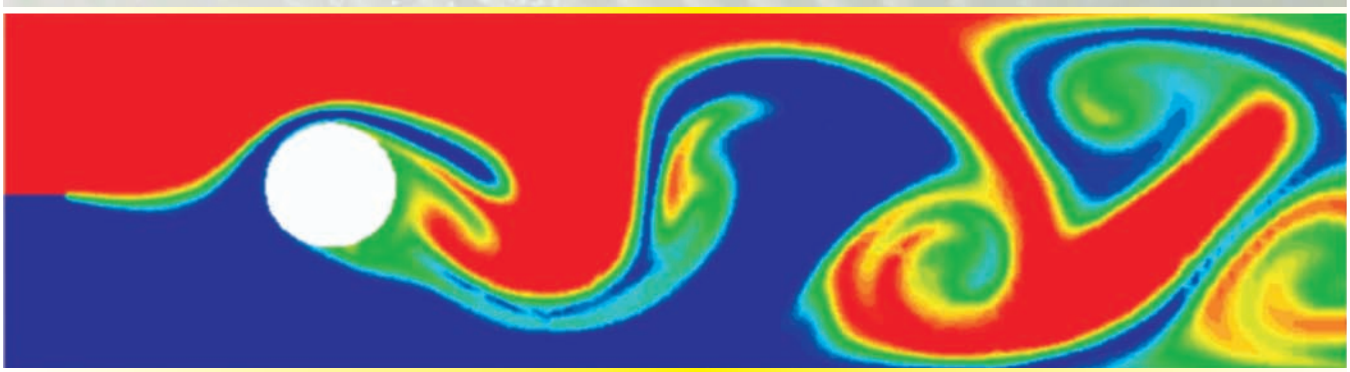
INTRODUCTION TO HEAT TRANSFER



S.K. SOM

Introduction to Heat Transfer

INTRODUCTION TO HEAT TRANSFER

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INTRODUCTION TO HEAT TRANSFER
S.K. Som

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In memory of my beloved father *A.N. Som*
and

To my wife *Banani*
for her loving cooperation, understanding and patience

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Preface

This text has been written primarily as an introductory course in heat transfer. It is an outcome of my teaching experience at the Indian Institute of Technology Kharagpur for a period of over thirty years.

The objective of teaching is to transmit knowledge and to lay a strong foundation based on which future knowledge can readily be acquired and utilized for a useful purpose. In the wake of technological growth, a need has been felt to update the engineering curriculum at the undergraduate engineering level. Heat transfer is one of the basic courses in some engineering disciplines such as mechanical, metallurgical, aerospace, and chemical. In a basic core level course, a host of topics are usually covered. It is the job of the instructor to redistribute the emphasis on topics so that the student can focus on those which need to be studied in more detail. A pertinent question that often arises in relation to teaching of such basic subjects as the first-level course, is whether the course should be heavy with intensive analytical treatments without much of practical demonstrations, or should it involve mostly the description of theory as information along with the practical applications of different formulae with limited analytical derivations and justifications. In one situation of an analytically-biased course, the student may fail to develop interest in the subject from an engineering point of view, while in another situation of limited analytical treatment, the student may not get the benefit of acquiring in-depth knowledge of the subject required for innovative work and advanced studies. The merit of a basic course therefore lies in its well-balanced coverage of mathematical operations, physical concepts and practical demonstrations. Moreover, an orderly presentation of materials in a lucid language is an essential feature of an introductory course for an effective interaction between the teaching and the learning process. The purpose of writing this book is to strive towards these goals and thus provide a useful foundation on heat transfer to all undergraduate engineering students pursuing courses in the above disciplines.

The book contains twelve chapters. The topics of the chapters are so chosen that it would require a one-semester (15 weeks) course with three 1-hour lecture classes and one 1-hour tutorial class per week. The final Chapter 12 presents the basic principles of mass transfer. This chapter has been included in recognition of the fact that many practical problems are associated with coupled phenomena of heat and mass transfer, and also strong analogies exist between heat transfer and mass transfer by diffusion. The text contains many worked-out examples. These are selected in a way so that the nuances of the principles are better explained. The exercise problems are aimed at enhancing the creative capability of the students. It is expected that on completion of this course, the students will be able to apply the basic principles of heat and mass transfer to engineering design in an appropriate manner.

I am indebted to many of my colleagues for their valuable suggestions towards achieving the objectives stated here. I gratefully acknowledge the persistent encouragement of my wife Banani throughout the work and her untiring efforts in typing the original manuscript. Also I like to express a special sense of appreciation to my loving daughter Trina who benevolently forwent her share of attention during preparation of the manuscript.

I will be happy to receive constructive criticisms from my colleagues, teachers and students on the content or pedagogy with to view to improving the text in its next edition.

S.K. Som

1

Fundamental Concepts

The definition of 'heat' is provided by classical thermodynamics. It is defined as an energy that flows due to a difference in temperature. It flows in a direction from higher to lower temperature. This energy can neither be observed nor be measured directly. However, the effects produced by the transfer of this energy are amenable to observations and measurements. The science of thermodynamics provides the relationship of this energy transfer with the change in the internal energy of a system through which the energy transfer takes place. Therefore, thermodynamic studies prescribe how much heat has to be supplied to or rejected from a system between specified equilibrium end states, but fail to predict the rate of energy transfer. This is because of the fact that thermodynamics deals with changes between equilibrium states and does not include time as a variable. The rate of energy transfer, on the other hand, is a time dependent non-equilibrium phenomenon and deals with systems that lack thermal equilibrium.

The science that deals with the mechanism and the rate of energy transfer due to a difference in temperature is known as heat transfer. The subject 'heat transfer' has a wide scope and is of prime importance in almost all fields of engineering. From an engineering point of view, the studies on heat transfer provide the fundamental information needed to estimate the size and hence the cost of an equipment necessary to transfer a specified amount of heat in a given time.

Learning objectives

The reading of this chapter will enable the students

- to know the various modes of heat transfer,
- to understand the physical mechanisms of different modes of heat transfer and the basic laws that govern the process of heat transfer in different modes,
- to get acquainted with the transport and thermodynamic properties of a system involved in the process of heat transfer, and
- to formulate and analyze simple problems related to different modes of heat transfer.

1.1 MODES OF HEAT TRANSFER

The process of heat transfer takes place by three distinct modes: conduction, convection and radiation.

1.1.1 Conduction

The mechanism of heat transfer due to a temperature gradient in a stationary medium is called conduction. The medium may be a solid or a fluid. A very popular example of conduction heat transfer is that when one end of a metallic spoon is dipped into a cup of hot tea, the other end becomes gradually hot. The law which describes the rate of heat transfer in conduction is known as Fourier's law. According to this law, the rate of heat transfer per unit area normal to the direction of heat flow is directly proportional to the temperature gradient along that direction. Let us consider a temperature distribution $T(x)$, as illustrated in Figure 1.1, in a medium through which heat is being conducted.

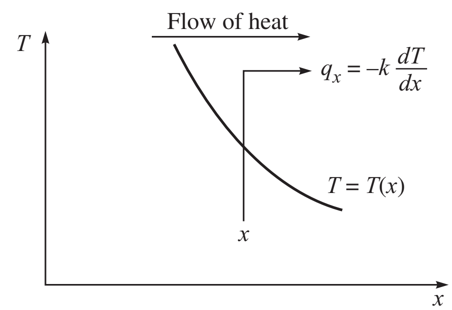


Figure 1.1 Quantitative description of Fourier's law.

Here, the temperature is a function of x only. Therefore, heat flows along the x-direction in which the temperature gradient dT/dx exists. According to Fourier's law,

$$q_x = -k\frac{dT}{dx} \tag{1.1}$$

where q_x is the rate of heat flow per unit area normal to the direction of heat flow. The minus sign in Eq. (1.1) indicates that heat flows in the direction of decreasing temperature. In the present case (Figure 1.1), dT/dx is negative and hence heat flows in the positive direction of the x-axis. The proportionality constant k is known as *thermal conductivity* and is a transport property of the medium through which heat is conducted.

Heat conduction takes place in any direction along which a temperature gradient does exist. When the temperature becomes a function of three space coordinates, say, x, y, z in a rectangular Cartesian frame, heat flows along the three coordinate directions. Equation (1.1), under the situation, is written in vector form as

$$\mathbf{q} = -k\nabla T \tag{1.2}$$

where

$$\mathbf{q} = \mathbf{i}\,q_x + \mathbf{j}\,q_y + \mathbf{k}\,q_z$$

and

$$\nabla T = \mathbf{i}\frac{\partial T}{\partial x} + \mathbf{j}\frac{\partial T}{\partial y} + \mathbf{k}\frac{\partial T}{\partial z}$$

with **i**, **j**, **k** as the unit vectors along the coordinate axes x, y, and z respectively.

For an isotropic medium, the thermal conductivity k is a scalar quantity which depends upon temperature only. Equation (1.2) is known as the generalized law of Fourier's heat conduction. In SI units, q is expressed in W/m^2 and ∇T in K/m. Therefore the unit of k, as found from Eq. (1.2), is W/(m K).

Let us consider a plane solid wall whose thickness L is small compared to other dimensions as shown in Figure 1.2(a). The two faces are kept at constant temperatures T_1 and T_2 ($T_1 > T_2$). Heat flows by conduction from the left face (at T_1) to the right one (at T_2). This is an example of one-dimensional heat transfer problem where heat flows in the x-direction only and the rate of heat flow is given by Eq. (1.1).

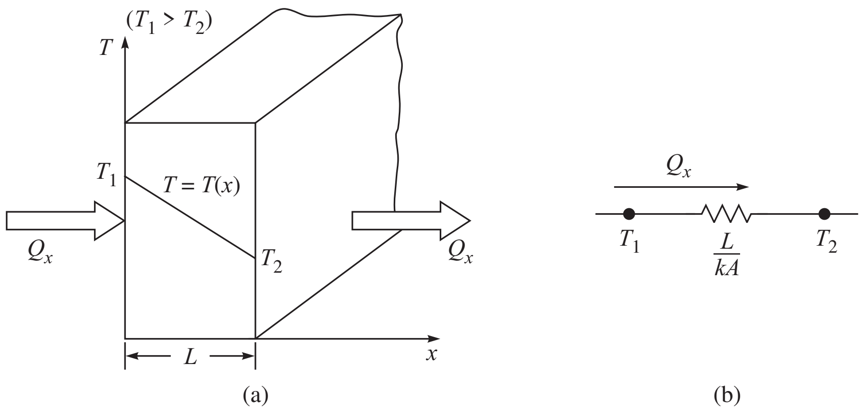


Figure 1.2 (a) Heat conduction through a plane wall. (b) The analogous electrical circuit of Figure 1.2(a).

It will be shown later (in Chapter 2) that under steady-state conditions (when the temperature at any point is independent of time), the temperature distribution $T(x)$ becomes linear. At steady-state, the amount of heat which flows into the wall from its left face will flow out of the wall from its right face. This means that the rate of heat flux at any section of the wall will be the same. Accordingly, from Eq (1.1), the value of dT/dx will be constant, i.e. the temperature distribution becomes linear. Hence, we can write

$$\frac{dT}{dx} = \frac{T_2 - T_1}{L}$$

Therefore, Eq. (1.1) becomes

$$q_x = k\frac{T_1 - T_2}{L} \tag{1.3}$$

The rate of heat flow Q_x through the wall is then given by

$$Q_x = q_x A = \frac{kA(T_1 - T_2)}{L}$$

or

$$Q_x = \frac{T_1 - T_2}{L/kA} \tag{1.4}$$

where A is the cross-sectional area of the wall normal to the direction of heat flow. Equation (1.4) is similar to the current–voltage relation in a dc electric circuit (Ohm's law). The term L/kA represents the thermal resistance of the wall through which heat flows at a rate of Q_x under a temperature difference of ($T_1 - T_2$). The analogous electrical circuit is shown in Figure 1.2(b).

The mechanism of conduction is, in fact, a little complicated. In liquids and gases, conduction is due to the collisions of molecules in course of their random motions. Temperature is a measure of the kinetic energy of the molecules or atoms of a substance. The kinetic energy of a molecule is attributed to its translational, vibrational, and rotational motions. A molecule having a higher kinetic energy is at a higher temperature than another having a lower kinetic energy. When two such molecules collide with each other, a part of the kinetic energy of the more energetic (higher temperature) molecule is transferred to the less energetic (lower temperature) molecule. The process is being conceived as a transfer of heat by conduction from a higher temperature to a lower one. In solids, the conduction of heat is attributed to two effects: (i) the flow of free electrons and (ii) the lattice vibrational waves caused by the vibrational motions of the molecules at relatively fixed positions called a lattice.

EXAMPLE 1.1 Determine the steady-state heat transfer rate per unit area through an 80 mm thick homogeneous slab with its two faces maintained at uniform temperatures of 40°C and 20°C respectively. The thermal conductivity of the material is 0.20 W/(m K).

Solution: The physical problem refers to the situation shown in Figure 1.2(a), with $T_1 = 40°C$, $T_2 = 20°C$, $L = 0.08$ m and $k = 0.20$ W/(m K). Therefore, using Eq. (1.3), we have

$$\frac{Q}{A} = q = k\frac{T_1 - T_2}{L}$$

$$= 0.20\frac{(40 - 20)}{0.08}$$

$$= 50 \text{ W/m}^2$$

EXAMPLE 1.2 What is the thickness required of a masonry wall having the thermal conductivity of 0.8 W/(m K), if the steady rate of heat flow per unit surface area through it is to be 80 per cent of heat flow through a composite structural wall having a thermal conductivity of 0.2 W/(m K) and a thickness of 100 mm? Both the walls are of the same surface area and subjected to the same surface temperature difference.

Solution: Using Eq. (1.3), for the composite wall,

$$q_c = \frac{k_c (T_1 - T_2)}{L_c}$$

For the masonary wall,

$$q_m = \frac{k_m (T_1 - T_2)}{L_m}$$

Therefore,

$$\frac{q_m}{q_c} = \frac{k_m}{k_c}\frac{L_c}{L_m}$$

or

$$0.8 = \frac{0.8}{0.2}\frac{0.1}{L_m}$$

which gives

$$L_m = 0.5 \text{ m} = 500 \text{ mm}$$

EXAMPLE 1.3 For many materials, the thermal conductivity can be approximated as a linear function of temperature over a limited temperature range as

$$k(T) = k_0(1 + \beta T)$$

where β is an empirical constant and k_0 is the value of thermal conductivity at a reference temperature. Show that in such cases, the rate of heat flow at steady-state through a plane wall, as shown in Figure 1.2(a), can be expressed as

$$Q = \frac{k_{av}A}{L}(T_1 - T_2)$$

where k_{av} is the value of k at the average temperature $(T_1 + T_2)/2$.

Solution: From Eq. (1.1), we can write

$$q_x = \frac{Q_x}{A} = -k\frac{dT}{dx}$$

Integrating both sides,

$$\int_1^2 \frac{Q_x}{A}dx = -\int_1^2 k_0(1 + \beta T)dT$$

$$\frac{Q_x}{A}L = k_0\left[(T_1 - T_2) + \frac{\beta}{2}(T_1^2 - T_2^2)\right]$$

$$= k_0\left[1 + \frac{\beta(T_1 + T_2)}{2}\right](T_1 - T_2)$$

or

$$Q_x = \frac{k_{av}A}{L}(T_1 - T_2)$$

1.1.2 Convection

The mode by which heat is transferred between a solid surface and the adjacent fluid in motion when there is a temperature difference between the two is known as convection heat transfer. The temperature of the fluid stream refers either to its bulk or free stream temperature. Let us consider the flow of a fluid over a heated surface as shown in Figure 1.3. If the temperature T_∞ with which the fluid approaches the surface is less than the temperature T_s of the surface, then heat is transferred from the surface to the flowing fluid, and as a result, the surface is cooled. It is our common experience that an increase in fluid motion cools the surface faster which means that the rate of heat transfer increases with an increase in the fluid motion. This implies a sense that the heat is being convected away from the surface by the stream of fluid and hence the mode of heat transfer is known as convective heat transfer.

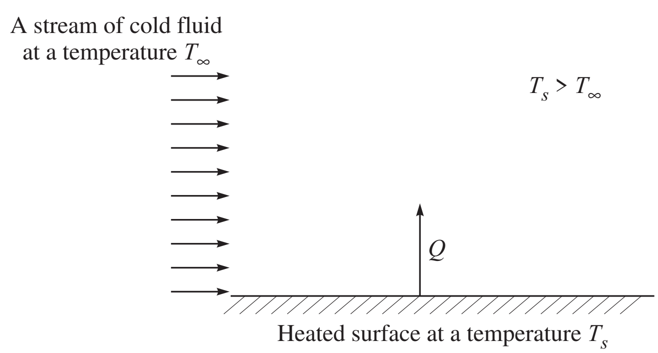


Figure 1.3 Convective heat transfer from a solid surface to a moving fluid.

The mode of convective heat transfer, in fact, comprises two mechanisms: (i) conduction at the solid surface and (ii) advection by the bulk or macroscopic motion of the fluid a little away from the solid surface. This can be explained through the velocity and temperature distributions in the fluid flowing past the solid surface. A consequence of the fluid–surface interaction in case of momentum transfer due to flow is the development of a region in the fluid flow near the surface within which the flow velocity varies from zero at the surface to a finite value U_∞ at the outer flow as shown in Figure 1.4. The velocity U_∞ is termed the *free stream velocity*. This region of fluid flow is known as the hydrodynamic boundary layer. In a similar fashion, there is a region of fluid flow where the temperature varies from T_s at the surface to T_∞ in the outer flow as shown in Figure 1.4. This region is known as the *thermal boundary layer*. The temperature T_∞ is known as the *free stream temperature* and equals to the uniform temperature with which the fluid approaches the surface.

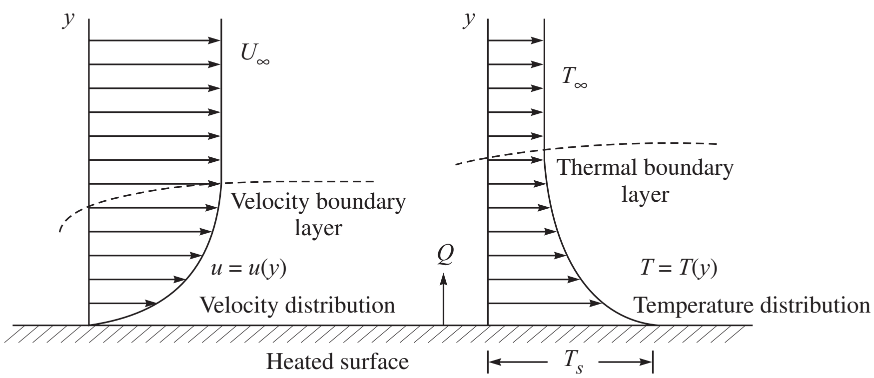


Figure 1.4 Velocity and temperature distributions in boundary layers in convective heat transfer mode.

At the interface between the surface and the fluid ($y = 0$, Figure 1.4), the fluid velocity is zero. Therefore, heat is transferred from the surface to the fluid by conduction, i.e. by the transfer of kinetic energy between the molecules due to their random motions. The contribution of advection due to bulk fluid motion arises with the growth of the boundary layer in the direction of flow. The heat that is conducted into this layer is swept downstream and is eventually transferred to the fluid outside the boundary layer. Therefore a knowledge of the boundary layer is essential to understand the mechanism of convective heat transfer clearly. This will be discussed in detail in Chapters 5, 6, 7 and 8.

The convection is of two types: *forced convection* and *free convection*. In forced convection, the fluid is forced to flow over a solid surface by external means such as fan, pump or atmospheric wind. When the fluid motion is caused by the buoyancy forces that are induced by density differences due to the variation in temperature or species concentration (in case of multicomponent systems) in the fluid, the convection is called *natural* (or *free*) convection. Figure 1.5 shows an example of forced convection from a horizontal surface where the fluid motion past the surface is caused by a fan, whereas Figure 1.6 shows a situation of free convection from a heated vertical surface in air in the absence of any forced flow.

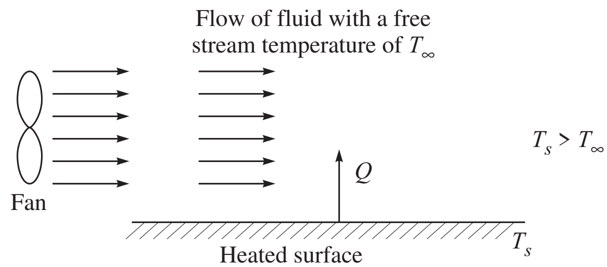


Figure 1.5 Forced convective heat transfer from a horizontal surface.

In the case of free convection, the buoyancy induced air flow takes place near the surface. The flow of air, under the situation, can be well explained by comparing the hydrostatic pressure at a point A adjacent to the surface with that at a point B on the same horizontal level but in the region of cold air away from the surface. The density of air near the surface is lower because of high temperature compared to that of cold air away from the plate. Therefore p_B, the pressure at B, is higher than p_A, the pressure at A. This difference in pressure $(p_B - p_A)$ induces the flow past the surface as shown in Figure 1.6.

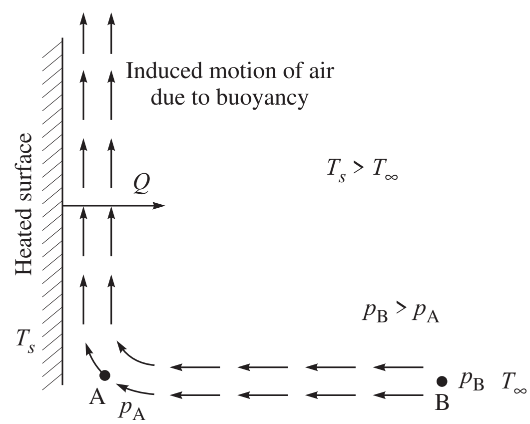


Figure 1.6 Free convective heat transfer from a heated vertical surface.

Irrespective of the details of the mechanism, the rate of heat transfer by convection (both forced and free) between a solid surface and a fluid is calculated from the relation

$$Q = \bar{h} A \Delta T \tag{1.5}$$

where

Q is the rate of heat transfer by convection
A is the heat transfer area
$\Delta T = (T_s - T_f)$, is the difference between the surface temperature T_s and the temperature of the fluid T_f at some reference location. In the case of external flows, T_f is the free stream temperature T_∞ (Figure 1.6), and in the case of internal flows, it is the bulk temperature of the fluid.
$\bar{h}$ is the average convective heat transfer coefficient over the area A.

The relation expressed by Eq. (1.5) was originally proposed by Sir Isaac Newton and is known as the Newton's law of cooling. In fact, Eq. (1.5) defines the average convective heat transfer coefficient over an area A. Therefore, it can also be written as

$$\bar{h} = \frac{Q}{A\Delta T} \tag{1.6}$$

Thus the average convective heat transfer coefficient over an area at a constant temperature is defined to be the rate of heat transfer per unit area per unit temperature difference (the difference between the surface temperature and the reference temperature of the fluid). In SI units, Q is expressed in W, A in m^2 and ΔT in K or °C. Therefore, $\bar{h}$ is expressed in W/(m^2 K) or W/(m^2 °C). Since the mode of heat transfer at the surface is conduction, as explained earlier, the convective heat transfer coefficient $\bar{h}$ is related to the thermal conductivity of the fluid and the temperature gradient at the surface. This will be discussed later in Chapters 5 and 7.

Equation (1.5) can be written in terms of heat flow, thermal potential and thermal resistance for convective heat transfer (similar to Eq. (1.4) for conduction) as

$$Q = \frac{\Delta T}{1/(\bar{h}A)} \tag{1.7}$$

For the convection problems, described by Figures 1.3 and 1.4, Eq. (1.7) takes the form

$$Q = \frac{T_s - T_\infty}{1/(\bar{h}A)} \tag{1.8}$$

The analogous electrical circuit is shown in Figure 1.7.

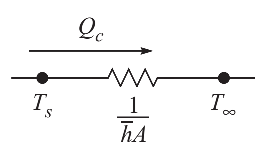


Figure 1.7 The analogous electrical circuit of convective heat transfer problems shown in Figures 1.3 and 1.4.

EXAMPLE 1.4 The average forced convective heat transfer coefficient for a hot fluid flowing over a cold surface is 200 W/(m^2 °C). The fluid temperature upstream of the cold surface is 100°C, and the surface is held at 20°C. Determine the heat transfer rate per unit surface area from the fluid to the surface.

Solution: The rate of heat transfer per unit area, q, is to be determined by making use of Eq. (1.5) as

$$\begin{aligned} q &= \frac{Q}{A} = \bar{h}(T_\infty - T_s) \\ &= 200(100 - 20) \\ &= 16{,}000 \text{ W/m}^2 \\ &= 16 \text{ kW/m}^2 \end{aligned}$$

EXAMPLE 1.5 Forced air flows over a convective heat exchanger in a room heater resulting in an average convective heat transfer coefficient, $\bar{h}$ = 800 W/(m^2 °C). The surface temperature of the heat exchanger may be considered constant at 75°C, the air being at 25°C. Determine the heat exchanger surface area required for 20 MJ/h of heating.

Solution: From Eq. (1.5),

$$A = \frac{Q}{\bar{h}\,\Delta T}$$

$$= \frac{20 \times 10^6}{3600 \times 800 \times (75 - 25)}$$

$$= 0.139 \text{ m}^2$$

EXAMPLE 1.6 The average free convective heat transfer coefficient on a thin hot vertical plate suspended in still air can be determined from observations of the change in plate temperature with time as it cools. Assuming that the plate is of uniform temperature at any instant of time and the radiation exchange between the plate and surroundings is negligible, evaluate the convective heat transfer coefficient at the instant when the plate temperature is 225°C and the change in plate temperature with time, $dT/dt = -0.02$ K/s. The ambient temperature is 25°C, the plate has a total surface area of 0.1 m^2 and a mass of 4 kg. The specific heat of the plate material is 2.8 kJ/(kg K).

Solution: The physical situation is similar to that shown in Figure 1.6.
We use Eq. (1.8), i.e.

$$Q = \bar{h}A(T_s - T_\infty)$$

Here, $T_\infty = 25°\text{C}, \quad T_s = 225°\text{C}, \quad A = 0.1 \text{ m}^2$

The rate of heat transfer from the plate Q can also be written, from an energy balance, as

$$Q = mc_p\left|\frac{dT}{dt}\right| = 4 \times 2.8 \times 10^3 \times 0.02$$

where m and c_p are respectively the mass and specific heat of the plate.

Therefore, $4 \times 2.8 \times 10^3 \times 0.02 = \bar{h} \times 0.1 \times (225 - 25)$

or $\bar{h} = 11.2 \text{ W/(m}^2\text{ °C)}$

1.1.3 Radiation

Any substance at a finite temperature emits energy in the form of electromagnetic waves in all directions and at all wavelengths (from a very low one to a very high one). The energy emitted within a specific band of wavelength (0.1–100 μm) is termed *thermal radiation*. The exchange of such radiant energy between two bodies at different temperatures is defined as heat transfer between the bodies by radiation. We have seen earlier that the heat transfer by conduction or convection requires the presence of a medium. But the radiation heat transfer does not necessarily require a medium, rather it occurs most efficiently in a vacuum.

An ideal radiator is definsed to be a body that emits and absorbs, at any temperature, the maximum possible amount of radiation at any given wavelength. An ideal radiator is termed a *blackbody*. The amount of thermal radiation emitted by a surface depends upon the absolute temperature and the nature of the surface. The rate of radiant energy emitted from the surface of a blackbody is given by

$$E_b = \sigma A\, T^4 \tag{1.9}$$

where

A is the surface area
T is the absolute temperature of the surface from which the energy is being radiated
σ is a constant.

A blackbody radiates uniformly in all directions.

Equation (1.9) is known as the Stefan–Boltzmann law after two Austrian scientists, J. Stefan who established the equation experimentally in 1879 and L. Boltzmann who derived it theoretically from quantum mechanics in1884. The constant σ is independent of surface, medium and temperature, and is known as Stefan–Boltzmann constant. In SI units, E_b is expressed in W, A in m^2 and T in K. Therefore, the unit of σ becomes W/(m^2 K^4), and its value is 5.6697×10^{-8} W/(m^2 K^4).

Real bodies do not behave like an ideal radiator (or blackbody). They emit radiation at a lower rate than that of a blackbody. The bodies (or surfaces) which emit, at any temperature, a constant fraction of blackbody emission at that temperature at each wavelength, are called gray bodies (or gray surfaces). Therefore, the rate of emission of radiant energy from the surface of a gray body is written as

$$E = A\varepsilon\sigma\, T^4 \tag{1.10}$$

where ε is defined as emissivity which is equal to the ratio of the rate of emission from a gray body (or surface) to that from a blackbody (or surface) at the same temperature T.

A net transfer of radiant heat requires a difference in the surface temperature of any two bodies between which the exchange takes place. At the same time, all the radiant energy emitted by a body may not be received by another body since the electromagnetic radiation is emitted in all directions and travels in straight lines. The net radiant heat transfer between two black surfaces of area A_1 and A_2 and at temperatures T_1 and T_2 can be written as

$$Q_r = A_1 F_{1-2}\sigma(T_1^4 - T_2^4) = A_2 F_{2-1}\sigma(T_1^4 - T_2^4) \tag{1.11}$$

Here, F_{1-2} and F_{2-1} are the dimensionless parameters which take care of the geometric relationship of the surfaces in determining the fraction of the energy emitted by one being received by the other. They are termed *view factor* or *shape factor*. F_{1-2} represents the fraction of the energy leaving surface 1 that reaches surface 2 and F_{2-1} represents the fraction of the energy leaving surface 2 that reaches surface 1. The equality of A_1F_{1-2} and A_2F_{2-1} is known as the reciprocity theorem and will be proved in Chapter 11.

EXAMPLE 1.7 After sunset, radiant energy can be sensed by a person standing near a brick wall. Such walls frequently have a surface temperature around 50°C, and the typical brick emissivity value is approximately 0.9. What would be the radiant heat flux per square metre from a brick wall at this temperature?

Solution: Applying Eq. (1.10), we have

$$\frac{E}{A} = \varepsilon\sigma T^4 = 0.9 \times 5.6697 \times 10^{-8} \times (50 + 273.15)^4$$
$$= 556.44 \text{ W/m}^2$$

EXAMPLE 1.8 Asphalt pavements on hot summer days exhibit a surface temperature of approximately 50°C. Consider such a surface to emit as a blackbody and calculate the emitted radiant energy per unit surface area.

Solution: We use Eq. (1.9) which gives

$$\frac{E_b}{A} = \sigma T^4$$
$$= 5.6697 \times 10^{-8} \times (50 + 273.15)^4$$
$$= 618.27 \text{ W/m}^2$$

1.2 COMBINED MODES OF HEAT TRANSFER

1.2.1 Combined Conduction and Convection

The combined conduction and convection heat transfer usually arises in a situation where heat is transferred from a fluid at a higher temperature to a fluid at a lower temperature via a solid partition. The two modes, under the situation, act in series. A practical example is the roof heating of a house in winter (Figure 1.8(a)), where the heat from the hot air outside the roof is transferred to the colder ambient environment inside the room. Another example is the heat transfer from hot tea to the surrounding cold air through the side wall of a cup containing the tea. (Figure 1.8(b)).

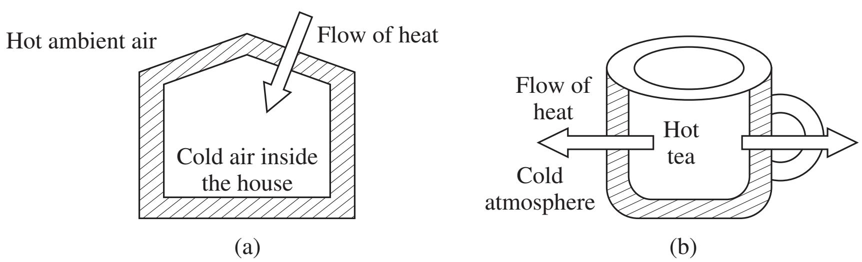


Figure 1.8 (a) Roof heating of a house. (b) Heat transfer from hot tea in a cup to the surrounding cold atmosphere.

Let us analyze the situation by considering a plane wall (Figure 1.9) one side of which is exposed to a hot fluid of temperature T_A, while the other side is in contact with a cold fluid of temperature T_B. The surface temperatures of the wall are T_1 and T_2 as shown in Figure 1.9.

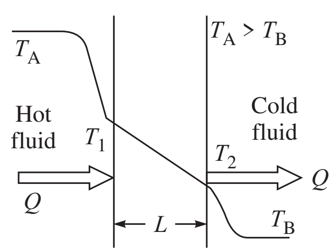


Figure 1.9 Heat transfer from a hot fluid to a cold fluid via a plane wall.

The transfer of heat will take place from the hot fluid temperature T_A to the surface temperature

T_1 by convection. At steady-state, the same heat will flow through the wall by conduction from its surface at temperature T_1 to that at temperature T_2 and will finally be transferred out by convection from the surface at temperature T_2 to the cold fluid at T_B. Therefore, we can write

$$Q = \bar{h}_1 A(T_A - T_1) = \frac{kA}{L}(T_1 - T_2) = \bar{h}_2 A(T_2 - T_B) \tag{1.12}$$

where

$\bar{h}_1$, $\bar{h}_2$ are the average convective heat transfer coefficients over the surfaces at T_1 and T_2 respectively

k, A, L are the thermal conductivity, surface area, and thickness of the wall respectively.

From Eq. (1.12), we can write

$$Q = \frac{T_A - T_B}{\dfrac{1}{\bar{h}_1 A} + \dfrac{L}{kA} + \dfrac{1}{\bar{h}_2 A}} \tag{1.13}$$

Equation (1.13) can be represented by a resistance network system as shown in Figure 1.10.

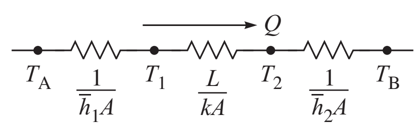


Figure 1.10 Analogous electrical network of a combined conduction–convection system as shown in Figure 1.9.

The difference in temperature $(T_A - T_B)$ is the overall thermal potential for heat to flow through three thermal resistances in series. The first and last ones ($1/\bar{h}_1 A$ and $1/\bar{h}_2 A$) are the convective resistances, while the middle one L/kA is the conductive resistance.

An overall heat transfer coefficient, under the situation, is defined as

$$\begin{aligned} Q &= UA(\Delta T)_{\text{overall}} \\ &= UA(T_A - T_B) \end{aligned} \tag{1.14}$$

By comparing Eq. (1.14) with Eq. (1.13), we have

$$UA = \frac{1}{\dfrac{1}{\bar{h}_1 A} + \dfrac{L}{kA} + \dfrac{1}{\bar{h}_2 A}} \tag{1.15}$$

or

$$\frac{1}{UA} = \frac{1}{\bar{h}_1 A} + \frac{L}{kA} + \frac{1}{\bar{h}_2 A} \tag{1.16}$$

EXAMPLE 1.9 A 150 mm thick concrete wall having thermal conductivity $k = 0.8$ W/(m °C) is exposed to air at 60°C on one side and to air at 20°C on the opposite side. The average convective heat transfer coefficients are 40 W/(m^2 °C) on the 60°C side and 10 W/(m^2 °C) on the 20°C side. Determine the heat transfer rate per unit surface area of the wall and the surface temperatures of the wall on both the sides. Draw the equivalent electrical network system.

Solution: To determine the rate of heat transfer Q, we use Eq. (1.13), i.e.

$$\frac{Q}{A} = \frac{60 - 20}{\dfrac{1}{40} + \dfrac{0.15}{0.8} + \dfrac{1}{10}}$$
$$= 128 \text{ W/m}^2$$

Let T_1 and T_2 be the surface temperatures on the 60°C-side and 20°C-side respectively. Then we can write from Eq. (1.12),

$$T_1 = 60 - \frac{128}{40} = 56.8°\text{C}$$

and
$$T_2 = 20 + \frac{128}{10} = 32.8°\text{C}$$

Figure 1.11 shows the equivalent electrical network system.

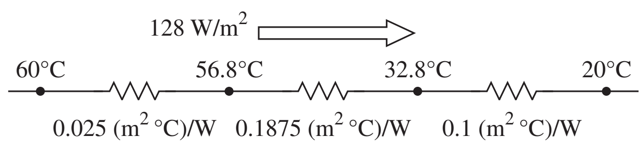


Figure 1.11 The equivalent electrical network (Example 1.9).

EXAMPLE 1.10 A flat panel used on a spacecraft is fabricated from a single layer of SiC composite 0.01 m thick. The spacecraft has the inner air temperature $T_i = 298$ K and it is in orbit with the panel exposed only to deep space, where $T_0 = 0$ K. The material has a thermal conductivity $k = 5.0$ W/(m K) and an emissivity of $\varepsilon = 0.8$. The inner surface of the panel is exposed to airflow resulting in an average heat transfer coefficient of $\bar{h}_i = 70$ W/(m^2 K). Determine the outer surface temperature T_2 and the rate of heat transfer Q/A from it.

Solution: The electrical network is shown in Figure 1.12. T_1 and T_2 are respectively the inner and the outer surface temperatures of the panel and b is its thickness.

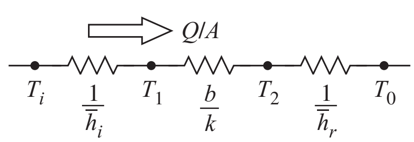


Figure 1.12 The equivalent electrical network (Example 1.10).

Heat transfer from the outer surface takes place only by radiation, and we can write

$$\frac{Q}{A} = \varepsilon\sigma\,(T_2^4 - T_0^4) = 0.8(5.67 \times 10^{-8})T_2^4 \tag{1.17}$$

We define here a radiation heat transfer coefficient h_r at the outer surface as

$$\frac{Q}{A} = h_r(T_2 - 0) = 0.8(5.67 \times 10^{-8})(T_2^4 - 0^4)$$

which gives $$h_r = 4.536 \times 10^{-8}\, T_2^3$$

Following the network shown in Figure 1.12, we can write

$$\frac{Q}{A} = \frac{298 - 0}{\dfrac{1}{70} + \dfrac{0.01}{5} + \dfrac{1}{4.536 \times 10^{-8}\, T_2^3}} \tag{1.18}$$

Equating Eqs. (1.17) and (1.18), we have

$$\frac{298}{[0.01428 + 0.002 + (1/4.536 \times 10^{-8}\, T_2^3)]} = 4.536 \times 10^{-8}\, T_2^4$$

The temperature T_2 is solved from the above equation by trial and error. The results of the solution in consideration of the fact $T_2 < 298$ K, are shown below:

Assumed T_2 (K)	*Q/A from Eq.* (*1.18*) (W/m^2)	*Q/A from Eq.* (*1.17*) (W/m^2)
285	307.6	299.3
287	314.1	307.7
293	333.8	334.3
292	330.5	329.8
292.5	332.1	332.03

Therefore the satisfactory solution (where the values of Q/A in columns 2 and 3 match) is

$$T_2 = 292.5 \text{ K and } Q/A = 332 \text{ W/m}^2$$

1.2.2 Combined Convection and Radiation

In many practical situations, a surface loses or receives heat by convection and radiation simultaneously. Here the two modes act in parallel to determine the total heat transfer rate. One example is the heat loss from the hot wall of a furnace or a combustion chamber to the surrounding air. Another similar example is the heat transfer from the roof of a house heated from the interior to the relatively cold ambient air. In both the cases, convection and radiation heat transfers take place simultaneously under the same temperature difference as a potential of heat transfer. Figure 1.13(a) represents such a physical system of simultaneous convective and radiative heat transfer from a surface to its surrounding air.

The total rate of heat transfer Q is given by the sum of the rates of heat flow by convection Q_c and by radiation Q_r as

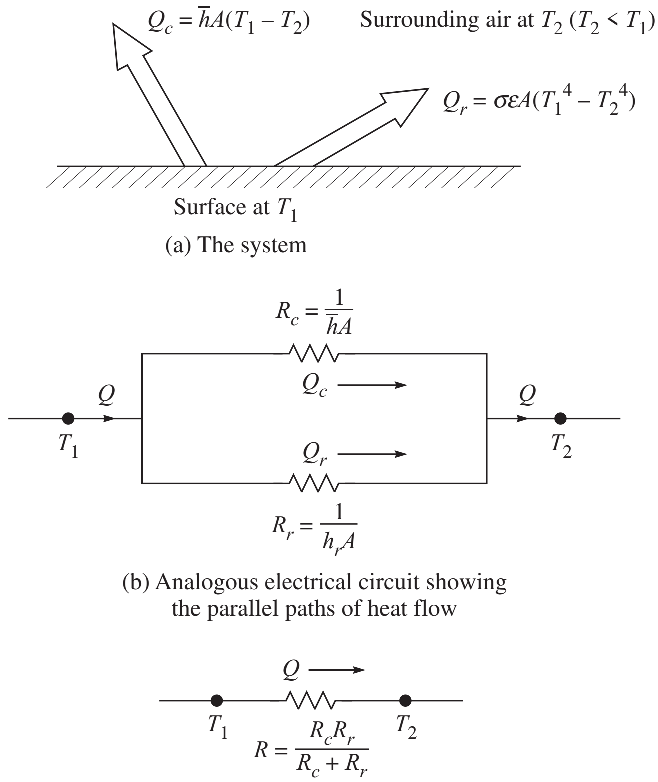
$$\begin{aligned} Q &= Q_c + Q_r \\ &= \bar{h} A(T_1 - T_2) + \sigma \varepsilon A(T_1^4 - T_2^4) \\ &= \bar{h} A(T_1 - T_2) + h_r A(T_1 - T_2) \end{aligned} \tag{1.19}$$

where

A is the area of the surface whose emissivity is ε

$\bar{h}$ is the average convective heat transfer coefficient of the surface

h_r is the radiation heat transfer coefficient.



(a) The system

(b) Analogous electrical circuit showing the parallel paths of heat flow

(c) Electrical circuit with a single equivalent resistance.

Figure 1.13 Combined convection and radiation heat transfer from a hot surface to the surrounding air.

The radiation heat transfer coefficient h_r is given by

$$h_r = \frac{\varepsilon\sigma(T_1^4 - T_2^4)}{T_1 - T_2}$$

$$= \varepsilon\sigma\,(T_1^2 + T_2^2)(T_1 + T_2) \tag{1.20}$$

Equation (1.19) can be written as

$$Q = \frac{T_1 - T_2}{\dfrac{1}{\bar{h}A} + \dfrac{1}{h_r A}} \tag{1.21}$$

The electrical analogous circuit represented by Eq. (1.21) is shown in Figures 1.13(b) and 1.13(c).

EXAMPLE 1.11 A horizontal steel pipe having an outer diameter of 80 mm is maintained at a temperature of 60°C in a large room where the air and wall temperature are at 20°C. The average free convection heat transfer coefficient between the outer surface of the pipe and the surrounding air is 6.5 W/(m^2 K), and the surface emissivity of steel is 0.8. Calculate the total heat loss by the pipe per unit length.

Solution: Here, we use Eq. (1.19).
If L be the length of the pipe in metre, then the surface area of the pipe

$$A = (\pi \times 0.08 \times L) \text{ m}^2$$

Therefore, we can write

$$\begin{aligned}\frac{Q}{L} &= 65(\pi \times 0.08)(333.15 - 293.15) + 5.67 \times 10^{-8} (0.8)(\pi \times 0.08)(333.15^4 - 293.15^4)\\ &= 65.345 + 56.242\\ &= 121.591 \text{ W/m}\end{aligned}$$

In this example, we observe that the rate of heat transfer by convection and the rate of heat transfer by radiation are almost the same. Therefore, we cannot neglect any one of them.

1.3 THERMAL CONTACT RESISTANCE

When two different conducting surfaces are placed in contact (Figure 1.14(a)), a thermal resistance to the flow of heat is always present at the interface which is known as *thermal contact resistance*. Because of this resistance, a sharp fall in temperature at the interface is always observed in the direction of heat flow as shown in Figure 1.14(b).

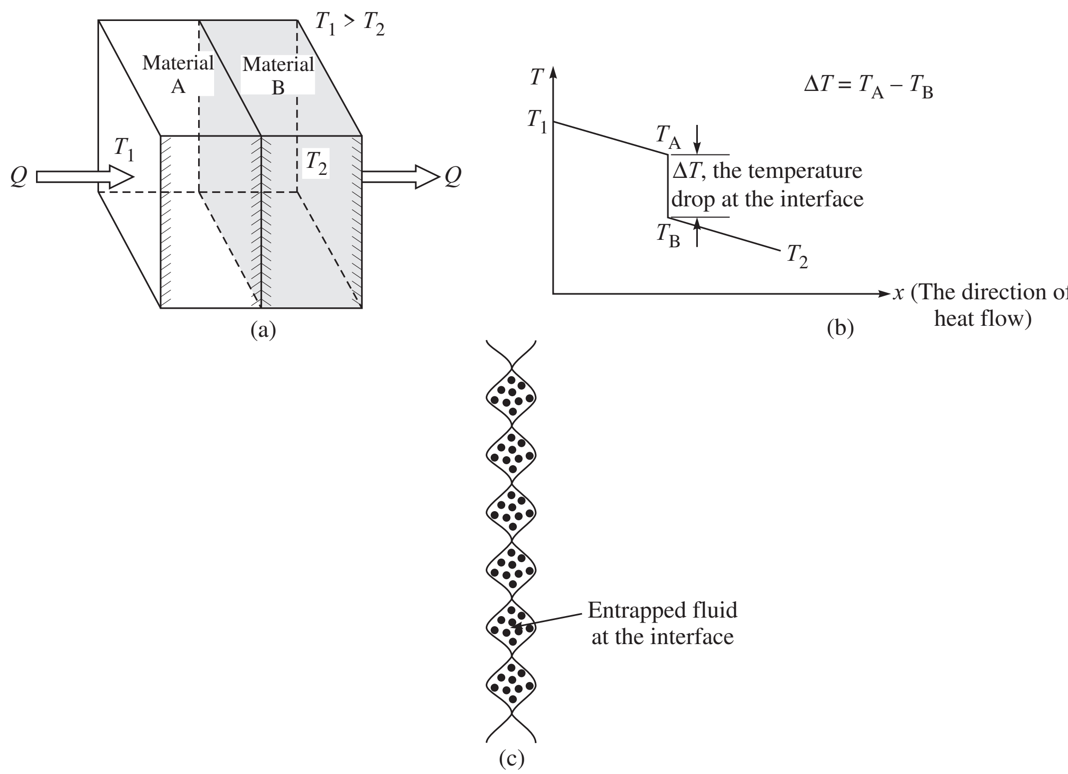


Figure 1.14 (a) Two conducting materials in physical contact. (b) Temperature distribution in the materials through the interface. (c) Enlarged view of the interface.

The physical explanation for the existence of contact resistance can be offered as follows: No real surface is perfectly smooth. When two surfaces are kept in contact, they do not fit tightly together because of the surface roughness, rather a thin fluid (usually air) is always trapped between them. The resistance to heat flow through this fluid between the contact surfaces plays the central role in determining the contact resistance. The physical mechanism may be better understood by examining a joint in greater detail from an enlarged view (Figure 1.14(c)) of the contact between the surfaces. It is observed that the two solids touch only at the peaks in the surfaces and the valleys in the mating surfaces are either filled by a fluid (usually air) or vacuum. Heat transfer at the joint, therefore, takes place through two parallel paths:

(a) The solid-to-solid conduction at the spots of contact.
(b) The conduction through the fluid entrapped in between the mating surfaces.

The second path offers the major resistance to heat flow since the thermal conductivity of the entrapped fluid (which is usually air or a gas) is quite small in comparison to that of the solids.

In consideration of a steady one-dimensional heat flow through two solid surfaces as shown in Figure 1.14(a), the thermal contact resistance R_{th} is defined as

$$R_{th} = \frac{T_A - T_B}{Q/A} \tag{1.22}$$

where Q is the rate of heat flow and A is the cross-sectional area (area normal to heat flow) at the interface.

The thermal contact resistance primarily depends on the surface roughness, the pressure holding the two surfaces in contact, the type of interface fluid, and the interface temperature. No satisfactory theory is available on contact resistance. Despite numerous measurements of the contact resistance between various solids, a unique correlation, applicable for all types of interfaces, is yet to be available.

Table 1.1 shows the influence of contact pressure on thermal contact resistance between metal surfaces under vacuum conditions. It is observed that an increase in contact pressure decreases the contact resistance. This is due to an increase in the area of the contact spots. Table 1.2 shows the effect of interfacial fluid on the thermal contact resistance between the two aluminium surfaces. Any interstitial substance that fills the void space between the contacting surfaces will decrease the contact resistance provided its thermal conductivity is more than that of air or the gas initially occupying the void space. Efforts are made to reduce the contact resistance by placing a soft metallic foil, a grease, or a viscous liquid at the interface between the contacting materials.

Table 1.1 The influence of contact pressure on thermal contact resistance for different interface materials

Interface material	*Resistance, R_{th}* [(m^2 K)/W × 10^4)]	
	Contact pressure (100 kN/m^2)	*Contact pressure* (10,000 kN/m^2)
Stainless steel	6–25	0.7–4.0
Copper	1–10	0.1–0.5
Magnesium	1.5–3.5	0.2–0.4
Aluminium	1.5–5.0	0.2–0.4

Table 1.2 The effect of interfacial fluid on thermal contact resistance between aluminum surfaces

Interfacial fluid	*Resistance,* R_{th} [(m^2 K)/(W × 10^4)]
Air	2.75
Helium	1.05
Hydrogen	0.720
Silicon oil	0.525
Glycerin	0.265

If the two mating surfaces are perfectly smooth (which is never realized in practice), there is no void space in the interface because of complete solid-to-solid contact. In this situation, there is no discontinuity in the value of the temperature at the interface. In the case of steady one-dimensional heat flow, the interface has a single fixed temperature as shown in Figure 1.15. At steady-state the rate of heat conduction through both the solids at the interface must be equal. Hence, we can write

$$k_A \left(\frac{dT}{dx}\right)_{\substack{\text{for solid A}\\ \text{at the interface}}} = k_B \left(\frac{dT}{dx}\right)_{\substack{\text{for solid B}\\ \text{at the interface}}} \tag{1.23}$$

Therefore we see that though the temperature continuity exists at the interface, but the temperature gradients are different on the two sides of the interface and are inversely proportional to the thermal conductivities of the solids.

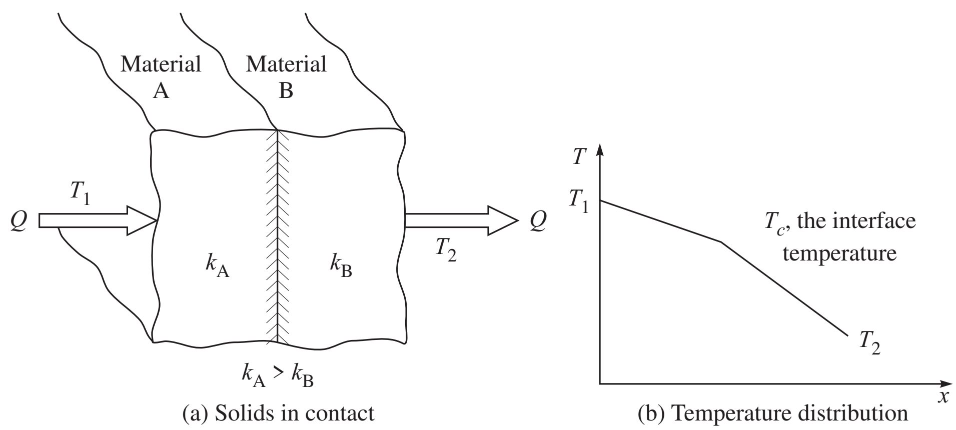


Figure 1.15 Heat conduction through solids with ideal interface due to perfectly smooth mating surfaces.

1.4 THERMAL PROPERTIES OF MATTER

Several physical properties of matter are required in the analysis of heat transfer problems. They are usually referred to as thermal properties or in a broader sense *thermophysical properties* of matter. These properties can be classified into two distinct categories: transport properties and thermodynamic properties. The transport properties are those which are related to the transport of mass, momentum and energy through a system composed of matter. They are defined by equations that relate the rate of transport of the quantity concerned with the potential difference that causes the process of transport. The examples of such properties are the mass diffusivity D (for diffusion of mass as defined by Fick's law of diffusion) the viscosity μ (for momentum transfer as defined by Newton's law of viscosity) and the thermal conductivity (for conduction of heat as defined by Fourier's law of heat conduction). The thermodynamic properties, on the other hand, pertain to the equilibrium state of a system. The density ρ and the specific heat c_p are two such properties which are required in the analysis of heat transfer problems. Several thermodynamic properties are defined in the science of thermodynamics. In fact, any state variable (a point function) is a thermodynamic property. However, a discussion on it is beyond the scope of this book. Interested readers can consult any standard book on classical thermodynamics.

1.4.1 Thermal Conductivity

The thermal conductivity has been defined earlier by the Fourier's law of heat conduction (Eq. (1.1)) as $k = -q_x/(\partial T/\partial x)$. The physical mechanism of conduction in solids, liquids and gases has also been discussed earlier. The thermal conductivities of materials vary over a wide range as shown in Figure 1.16. Usually, the thermal conductivity of a solid is greater than that of a liquid which is again greater than that of a gas. Table 1.3 shows the thermal conductivities of some materials at room temperature.

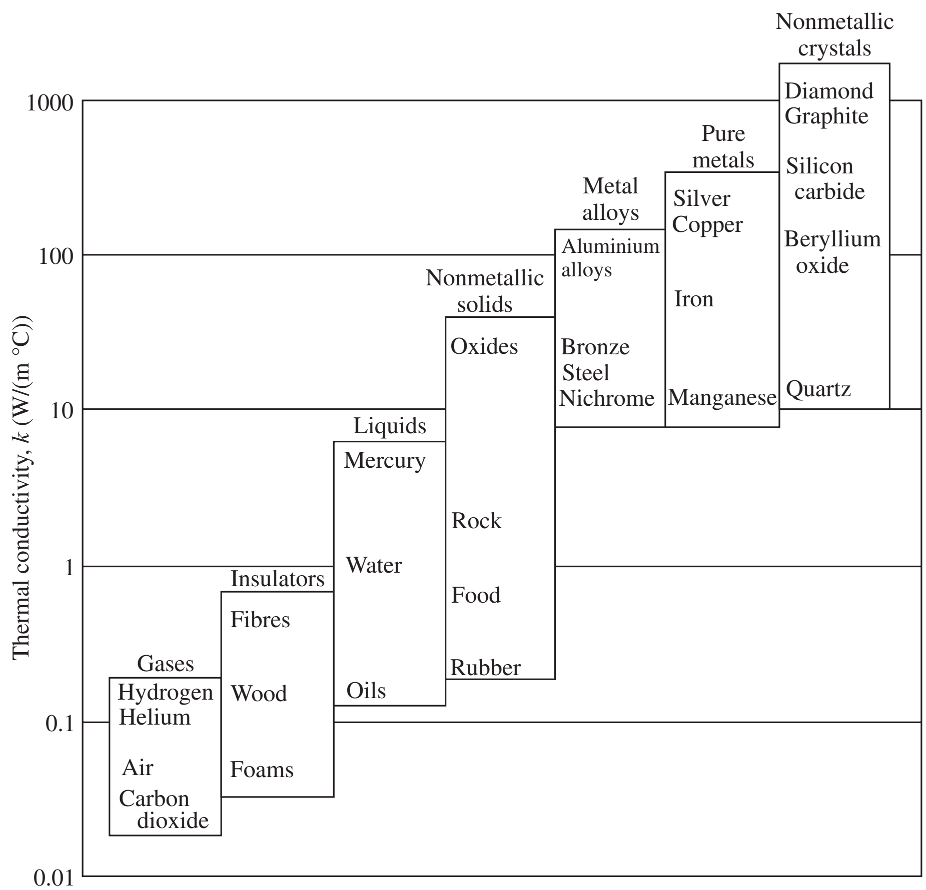


Figure 1.16 The range of thermal conductivity of various materials at room temperature (300 K).

Table 1.3 Thermal conductivities of some materials at room temperature (300 K)

Material	*k* (W/(m °C))
Diamond	2300
Silver	429
Copper	401
Gold	317
Aluminium	237
Iron	80.2
Mercury (l)	8.54
Glass	0.78
Brick	0.72
Water (l)	0.613
Human skin	0.37
Wood (oak)	0.17
Helium (g)	0.152
Soft rubber	0.13
Refrigerant-12	0.072
Glass fibre	0.043
Air (g)	0.026
Urethane, rigid foam	0.026

Solids

In solids, heat conduction is due to two effects—flow of free electrons and propagation of lattice vibrational waves. The thermal conductivity is therefore determined by the addition of these two components. In a pure metal, the electronic component is more prominent than the component of lattice vibration and gives rise to a very high value of thermal conductivity. Because of this, the metals are good conductors of both heat and electricity. For nonmetals, on the other hand, the lattice vibrational component is dominant over the electronic component. The lattice component of thermal conductivity strongly depends on the way the molecules are arranged. Highly ordered crystalline non-metallic solids like diamond, silicon, quartz exhibit very high thermal conductivities (more than that of pure metals) due to lattice vibration only, but are poor conductors of electricity. Therefore, such materials have widespread use in electronic industry for cooling of sensitive electronic components. Another interesting fact is that the thermal conductivity of an alloy of two metals is usually much lower than that of either metals. Table 1.4 shows such examples. The variations of thermal conductivity with temperature for some metals and alloys are shown in Figure 1.17 (Frank Kreith and Mark S. Bohan, *Principles of Heat Transfer*, 5th ed., PWS Publishing Company).

Table 1.4 The comparison of thermal conductivities of metallic alloys with those of constituting pure metals

Pure metal or alloy	k_2 ((W/(m °C))
Copper	401
Nickel	91
Constantan (55% Cu, 45% Ni)	23
Copper	401
Aluminium	237
Commercial bronze (90% Cu, 10% Al)	52

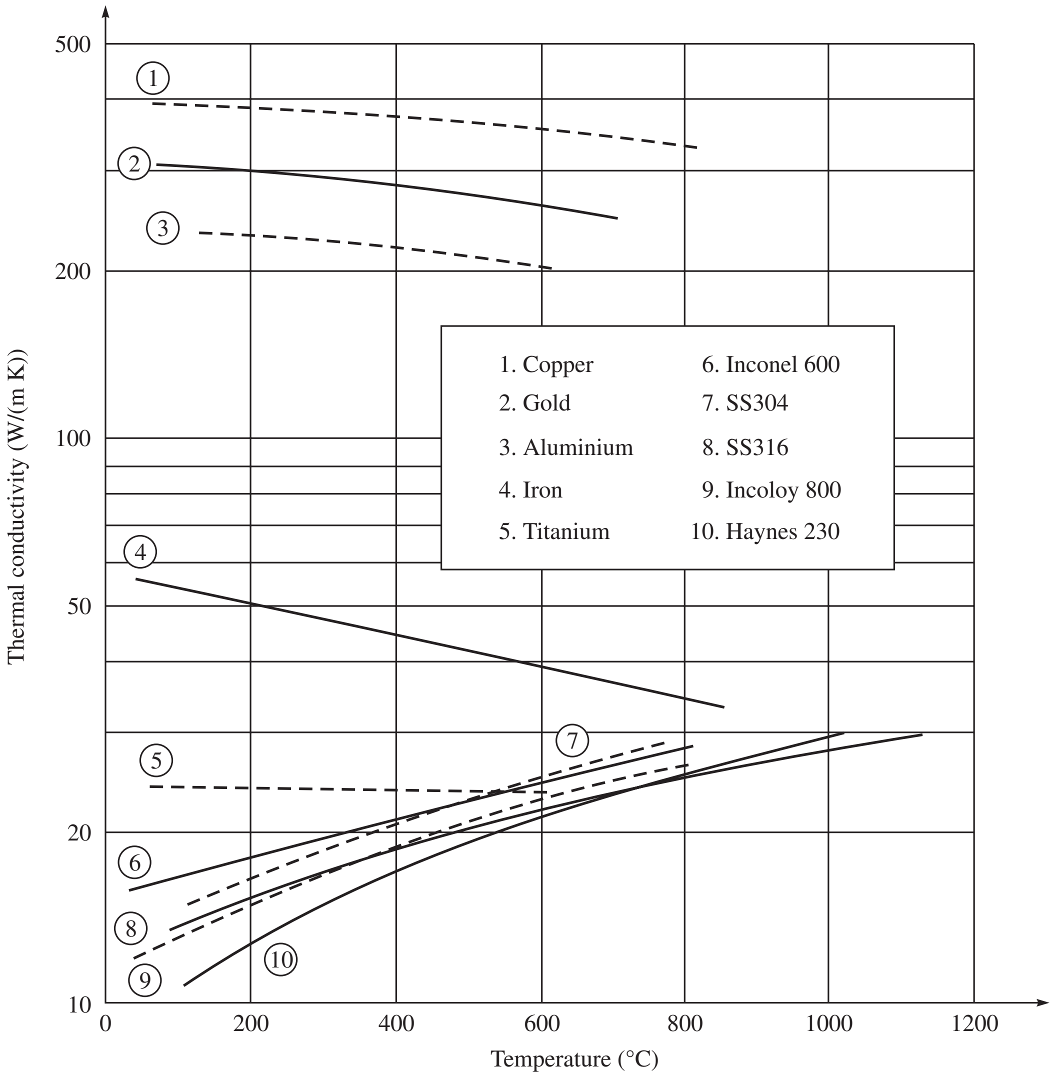


Figure 1.17 The variation of thermal conductivity with temperature for typical metals and their alloys.

Liquids sand gases

The thermal conductivity for liquids and gases is attributed to the transfer of kinetic energy between the randomly moving molecules due to their collisions. Because of large intermolecular spaces and hence a smaller number of molecular collisions, the thermal conductivities exhibited by gases are lower than those of the liquids. Figures. 1.18(a) and 1.18(b) show the variations of thermal conductivity with temperature for some liquids and gases (Frank Kreith and Mark S. Bohan, *Principles of Heat Transfer*, 5th ed., PWS Publishing Company).

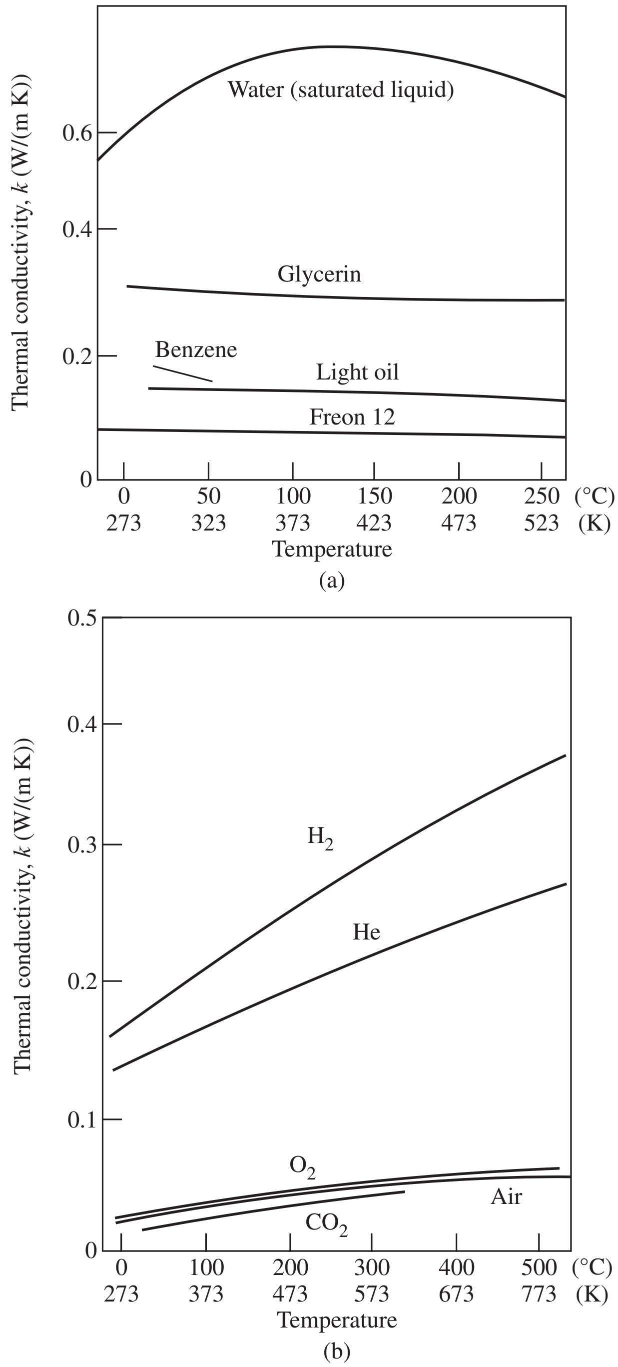


Figure 1.18 The variation of thermal conductivity with temperature for (a) liquids and (b) gases.

Experimental measurement of thermal conductivity of a material

The thermal conductivity of a material is usually measured by placing an electric thermofoil heater in a sandwiched form between the two identical samples of the material as shown in Figure 1.19. The thickness of the resistance heater including its cover is usually less than 0.5 mm. The heater cover is made of thin silicon rubber. The lateral surfaces of the samples are properly insulated so that heat flows only in one direction. The exposed ends of the sample are kept cooled at a constant temperature with the help of a circulating fluid such as tap water. Two thermocouples are embedded into each sample, some distance L apart, to measure the temperature drop ΔT across the distance L in each sample. At steady state, the sum total of the rate of heat transfer through both the samples becomes equal to the electrical power drawn by the heater. Therefore, we can write

$$VI = Q = 2KA\frac{\Delta T}{L} \tag{1.24}$$

where

- V is the voltage across the heater
- I is the electrical current flowing through the heater
- Q is the total rate of heat flow through the two samples
- A is the cross-sectional area (area normal to the direction of heat flow) of the samples.

The thermal conductivity of the sample material is determined from Eq. (1.24) with the help of the measured values of all other parameters.

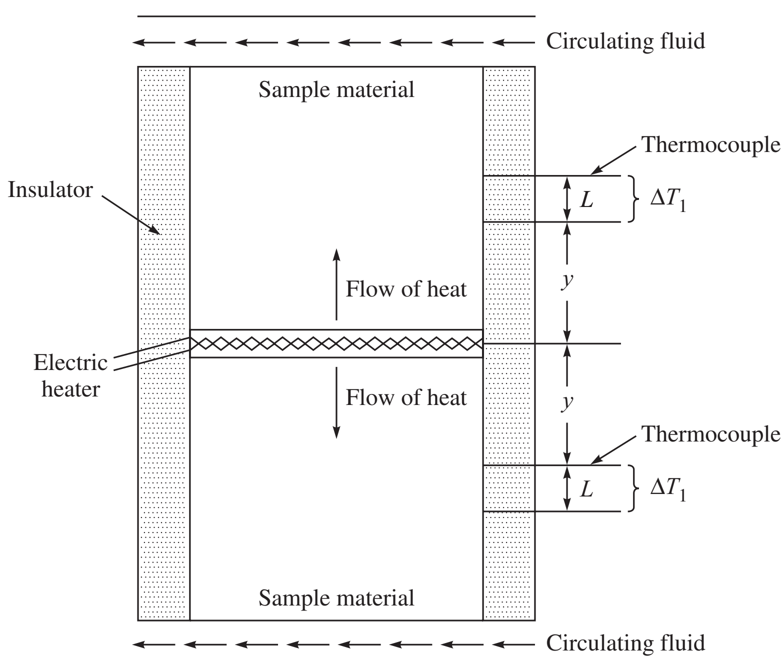


Figure 1.19 Experimental measurement of thermal conductivity.

Thermal diffusivity

In the analysis of heat transfer problems, the ratio of thermal conductivity to the heat capacity appears to be an important property and is termed *thermal diffusivity* α. Therefore,

$$\alpha = \frac{k}{\rho c} \tag{1.25}$$

The unit of α, as given by Eq. (1.25), is m^2/s.

The thermal diffusivity of a material is the measure of its ability to conduct thermal energy relative to its ability to store thermal energy. Materials having large values of α will respond quickly to a change in the thermal environment in establishing a steady-state temperature field within the material in transporting heat, while materials having small values of α will do it sluggishly.

SUMMARY

- The energy in transit due to a difference in temperature is known as heat. Heat flows in a direction of decreasing temperature. The transfer of heat takes place in three distinct modes, namely conduction, convection and radiation.
- The conduction heat transfer takes place in a stationary medium and is described by Fourier's law. The law states that the rate of heat transfer per unit area normal to the direction of heat flow is directly proportional to the temperature gradient. The constant of proportionality is known as *thermal conductivity* which is a property of the medium through which heat is conducted.
- Convection is the mode by which heat is transferred between a solid surface and the adjacent fluid in motion when there is a temperature difference between the two. If the fluid is forced to flow over a solid surface by external means such as fan or pump, the convection is called *forced convection*, while in case of fluid motion caused only by buoyancy forces that are induced by density difference due to variation of temperature in the fluid, the convection is termed *free* (or *natural*) *convection*. The average convective heat transfer coefficient over an area at a constant temperature is defined to be the rate of heat transfer per unit area per unit temperature difference (the difference between the surface temperature and the reference temperature of the fluid). The reference temperature of the fluid is the free stream temperature in the case of external flows and bulk mean temperature in the case of internal flows.
- The thermal radiation is the energy emitted by any substance, at a finite temperature, in the form of electromagnetic waves in all directions but within a specific band of wavelength (0.1–100 μm). The exchange of such radiant energy between the bodies at different temperatures is defined as radiation heat transfer. The radiation heat transfer does not necessarily require a medium.
- An ideal radiator is defined to be a body that emits and absorbs, at any temperature, the maximum possible amount of radiation at any given wavelength. An ideal radiator is termed a *blackbody*. An ideal radiator (or a blackbody) emits uniformly in all directions.
- The emission of radiant energy from a blackbody is described by the Stefan–Boltzmann law which states that the rate of emission from a blackbody per unit surface area is proportional to the fourth power of its absolute temperature. The constant of

proportionality is independent of surface, and of medium and temperature and is known as the Stefan–Boltzmann constant whose value is 5.6697×10^{-8} W/(m^2 K^4).

- Real bodies emit radiation at a lower rate than that of a blackbody. The bodies which emit, at any temperature, a constant fraction of blackbody emission at that temperature at each wavelength, are called gray bodies. The ratio of the rate of emission from a gray body to that from a blackbody at the same temperature is defined as *emissivity*. All the radiant energy emitted by a body may not be received by another. The fraction of energy leaving the surface of one body that reaches the other body is defined as *view factor*.
- A combined conduction and convection heat transfer usually arises in a situation where heat is transferred from a fluid at a higher temperature to a fluid at a lower temperature via a solid partition. The two modes, under the situation, act in series. A combined mode of convection and radiation heat transfer takes place in a situation when a solid surface either loses heat to or receives heat from the surrounding ambient. The two modes act in parallel in determining the total heat transfer rate.
- There exists a thermal resistance to the flow of heat at the interface of two mating solid surfaces. This is termed contact resistance which results in a sharp fall in the temperature at the interface in the direction of heat flow. The existence of contact resistance is attributed to the poor thermal conductivity of the entrapped fluid (usually air or gas) at the interface. The contact resistance decreases with an increase in contact pressure.
- The thermophysical properties are classified into two groups: transport properties and thermodynamics properties. Thermal conductivity is an important transport property in determining the rate of heat flow by conduction and is defined by Fourier's law of heat conduction.
- The thermal conductivity of a solid is usually greater than that of a liquid which is again greater than that of a gas. The thermal conductivity of solids is attributed to the flow of free electrons and lattice vibrational waves, while in the case of fluids, it is attributed to the transfer of kinetic energy between the randomly moving and colliding molecules. The thermal diffusivity of a material is defined to be the ratio of thermal conductivity to the heat capacity. It is an important property used in analyzing the unsteady-state heat conduction through the material.

REVIEW QUESTIONS

1. (a) What is the driving potential for heat transfer?
(b) Why is it that the science of thermodynamics cannot predict the rate of heat transfer?

2. (a) Is it true that conduction necessarily requires the existence of matter and depends upon the molecular activity within the matter?
(b) Why is it that metals are good conductors of both heat and electricity, while some non-metallic crystalline solids are very good conductors of heat but very poor conductors of electricity? Explain.
(c) Is a material medium necessary for heat transfer by radiation?

3. A person who sits in front of a fireplace feels warm. Name the mode of heat transfer through which he receives heat.

4. (a) Can all three modes of heat transfer (conduction, convection and radiation) occur simultaneously (in parallel)in a medium? Give examples if affirmative.
   (b) Can a medium involve (i) conduction and convection (in both series and parallel) or (ii) convection and radiation (in both series and parallel)? Give examples if affirmative.
5. Identify the different modes of heat transfer in the following systems/operations.
   (a) Steam raising in a steam boiler
   (b) Air/water cooling of an IC engine cylinder
   (c) Condensation of steam in a condenser
   (d) Heat loss from a vacuum flask
   (e) Heating of water in a bucket with an immersion heater.
6. Does conduction play any role in the process of convective heat transfer? Explain.

PROBLEMS

1.1 The heat flux through a wooden slab of 50 mm thickness, whose inner and outer surface temperatures are 40°C and 20°C, respectively, is determined to be 40 W/m^2. Calculate the thermal conductivity of the wood?

[**Ans.** 0·1 W/(m K)]

1.2 A fibre glass insulating board of thermal conductivity 0.038 W/(m °C) is used to limit the heat losses to 80 W/m^2 for a temperature difference of 160°C across the board. Determine the thickness of the insulating board.

[**Ans.** 76 mm]

1.3 An aluminium pan whose thermal conductivity is 237 W/(m °C) has a flat bottom with diameter 100 mm and thickness 6 mm. Heat is transferred steadily to boiling water in the pan through its bottom at a rate of 500 W. If the inner surface of the bottom of the pan is at 150°C, determine the temperature of the outer surface of the bottom of the pan.

[**Ans.** 151·61°C]

1.4 An ice chest whose outer dimensions are 300 mm × 400 mm × 400 mm is made of 30 mm thick Styrofoam (k = 0.033 W/(m °C)). Initially the chest is filled with 40 kg of ice at 0°C, and the inner surface temperature of the ice chest can be taken to be 0°C at all times. The heat of fusion of ice at 0°C is 333.7 kJ/kg, and the surrounding ambient air is at 30°C. Neglecting any heat transfer from the 400 mm × 400 mm base of the ice chest, determine how long will it take for the ice in the chest to melt completely if the outer surfaces of the ice chest are at 8°C.

[**Ans.** 658 h]

1.5 A 200 mm diameter sphere at 120°C is suspended in air at 20°C. If the natural convective heat transfer coefficient between the sphere and the air is 15 W/ (m^2 °C), determine the rate of heat loss from the sphere.

[**Ans.** 188.50 W]

1.6 Heat is supplied to a plate from its back surface at a rate of 50 W/m^2 and is removed from its front surface by air flow at 30°C. If the heat transfer coefficient between the air and the plate surface is 50 W/(m^2 °C), what is the temperature of the front surface of the plate at steady-state heat transfer?

[**Ans.** 31°C]

1.7 For heat transfer purposes, a standing man can be modelled as a vertical cylinder 1.8 m in height and 0.3 m in diameter with both top and bottom surfaces insulated and with the side surface at an average temperature of 30°C. For a convection heat transfer coefficient of 15 W/(m^2 K), determine the rate of heat loss from this man by convection in an environment at 20°C.

[**Ans.** 254 W]

1.8 One surface of a vertical plate having a uniform temperature T_w is exposed to a free convection environment of surrounding air at a temperature of T_∞ ($T_\infty < T_w$). The free convective heat transfer coefficient between the plate and the air is 5 W/(m^2 K). If the rate of heat transfer from the plate to the air is simulated by conduction through a stagnant layer of air adhering to the plate, determine the thickness of the layer. Take the thermal conductivity of air as k = 0.025 W/(m K).

[**Ans.** 5 mm]

1.9 A sphere 100 mm in diameter is suspended inside a large evacuated chamber whose walls are kept at 300 K. If the surface of the sphere has an emissivity of 0.8 and is maintained at 500 K, determine the rate of heat loss from the sphere to the walls of the chamber.

[**Ans.** 2·47 kW]

1.10 An aluminium plate 4 mm thick is mounted in a horizontal position, and its bottom surface is well insulated. A special, thin coating is applied to the top surface such that it absorbs 80 per cent of any incident solar radiation. The density and specific heat of aluminium are known to be 2700 kg/m^3 and 900 J/(kg K), respectively. Disregard the radiation heat loss from the plate.

(a) Consider a situation when the plate is at a temperature of 25°C and its top surface is suddenly exposed to ambient air at T_∞ = 20°C and a solar radiation that provides an incident flux of 900 W/m^2. The convective heat transfer coefficient between the surface and the air is 20 W/(m^2 K). What is the initial rate of change of the plate temperature?

[**Ans.** 0·064 K/s]

(b) What will be the temperature of the plate when the steady-state conditions are reached?

[**Ans.** 56°C]

1.11 The hot combustion gases of a furnace are separated from the surrounding ambient air at 25°C by a brick wall having a thermal conductivity of 0.2 W/(m K) and a surface emissivity of 0.8. At steady-state conditions, the inner surface temperature is found to be 450°C. Determine the thickness of the brick wall to maintain the outer surface temperature below a maximum value of 150°C. Neglect radiation between the combustion gases and the inner surface. The convective heat transfer coefficients at the inner and the outer surfaces are 50 W/(m^2 K) and 20 W/(m^2 K) respectively.

[**Ans.** 16.69 mm]

1.12 The temperature of a surface is maintained at 300°C by separating it from an air flow by a layer of insulation 20 mm thick for which the thermal conductivity is 0.01 W/(m K). If the air temperature is 30°C and the convective heat transfer coefficient between the air and the outer surface of the insulation is 400 W/(m^2 K), determine the temperature of this outer surface.

[**Ans.** 30·34°C]

1.13 A refrigerator whose dimensions are 1.8 m × 1.2 m × 0.8 m has a wall and door thickness of 30 mm, consumes 600 W and has a COP of 3.5. It is observed that the motor of the refrigerator remains ON for 5 minutes and then remains OFF for 15 minutes periodically. If the temperatures at the inner and outer surfaces of the refrigerator walls are 6°C and 17°C respectively, determine the thermal conductivity of the refrigerator walls and the annual cost of operating the refrigerator if the unit cost of electricity is Rs 3.00/kWh. (The COP of a refrigerator is defined to be the ratio of the rate of heat removal to the power consumed by the refrigerator.) Neglect the heat transfer from the top and the bottom faces of the refrigerator.

[**Ans.** 0.795 W/(m °C), Rs 3942]

1.14 Consider a steady-state heat transfer between two large parallel plates of emissivity 0.8 and at constant temperatures of T_1 = 300 K and T_2 = 200 K, respectively. The plates are separated by a distance of 20 mm. Determine the rate of heat transfer between the plates per unit surface area assuming the gap between the plates to be (a) evacuated, (b) filled with air (k = 0.025 W/(m K), (c) filled with glass wool (k = 0.045 W/(m K) and (d) filled with superinsulation (k = 0.00002 W/(m K)).

[**Ans.** (a) 295 W, (b) 420 W, (c) 295 W, (d) 295 W]

2

One-Dimensional Steady-State Heat Conduction

We have seen in Chapter 1 that a temperature difference is responsible for conduction of heat through a medium, and the flow of heat takes place in the direction of decreasing temperature. The gradient of temperature distribution determines the rate of heat transfer by conduction (Fourier's law of heat conduction, Eq. (1.1) refers). In a steady-state heat conduction, the temperature at any point in the medium remains invariant with time. This happens when the rate of heat conducted into the medium equals the rate of heat conducted out of the medium. In an unsteady heat conduction problem, the inflow and outflow of heat due to conduction through a medium are not the same. This causes the temperature at any point in the medium to change with time.

The conduction of heat takes place in any direction along which a temperature gradien does exist. When it takes place in all the three coordinate directions (in a frame of reference), it is termed *three-dimensional heat conduction*. In an unsteady three-dimensional heat conduction through a medium, we can express temperature distribution (in a Cartesian frame of coordinate axes) as

$$T = T(x, y, z, t) \tag{2.1}$$

In the case of steady-state heat conduction, the temperature ceases to be a function of time. It becomes a function of space coordinates only. Hence, we can express steady-state conduction as

$$T = T(x, y, z) \tag{2.2}$$

Under many practical situations, the conduction of heat is significant in only one direction of the coordinate axes, and is negligible in the other two directions. The heat conduction is then termed *one-dimensional conduction* and the temperature becomes a function of time and one space coordinate. In the case of a steady one-dimensional heat conduction, the temperature is a function of one space coordinate only.

A relationship of the temperature with space coordinates and time in the form of a differential equation can be obtained by making use of the principle of conservation of energy along with the Fourier's law of heat conduction. The resulting equation is known as *heat conduction equation*. Once we know the temperature distribution for a heat conduction problem, we can find out the rate of heat transfer from Eq. (1.1) (Fourier's law of heat conduction). In this chapter we will first derive the three-dimensional unsteady heat conduction equation, and then discuss some of the one-dimensional steady-state heat conduction problems.

Learning objectives

The reading of this chapter will enable the students

- to understand the difference between steady- and unsteady-state heat transfer,
- to identify the different types of one-dimensional heat conduction problems,
- to analyze mathematically the different types of one-dimensional heat conduction problems with and without convection at boundary surfaces, and
- to understand how the combined conduction and convection heat transfer plays the role of enhancing the rate of heat transfer by the incorporation of extended surfaces.

2.1 DERIVATION OF HEAT CONDUCTION EQUATION

The heat conduction equation is derived by considering the energy balance of a small element of the heat conducting medium as a control volume appropriate to a coordinate frame of reference. A rectangular parallelepiped is considered (Figure 2.1) as the control volume in a rectangular Cartesian frame of coordinate axes. Let us also consider that thermal energy is being generated along with the conduction of heat within the control volume.

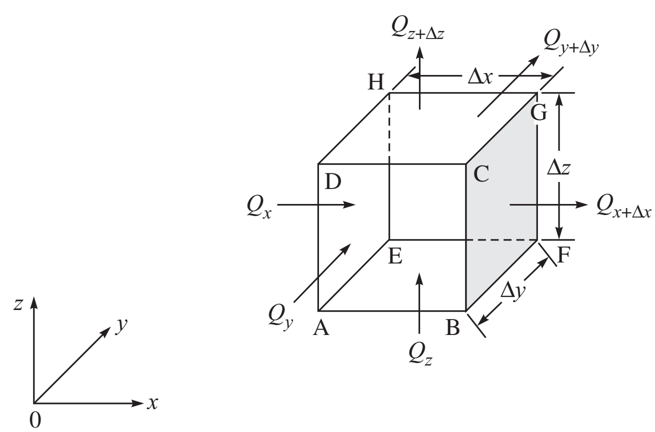


Figure 2.1 A control volume appropriate to a rectangular Cartesian coordinate system.

The principle of conservation of energy for the control volume can be described as

The rate of heat conducted into the control volume + the rate of generation of thermal energy within the control volume	=	The rate of heat conducted out of the control volume + the rate of increase in the internal energy of the control volume

or

The rate of increase in internal energy of the control volume	=	The net rate of heat conducted into the volume + the rate of generation of thermal energy within the control volume

(2.3)

Now we use the Fourier's law (Eq. (1.1)) to determine the conduction of heat through the different faces of the control volume (Figure 2.1).

Let Q_x be the rate of heat conducted in, through the face AEHD. Therefore, we can write

$$Q_x = -k\Delta y\Delta z\frac{\partial T}{\partial x}$$

Heat conducted out, through the face BFGC, can be written as

$$Q_{x+\Delta x} = -k\Delta y\Delta z\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(-k\Delta y\Delta z\frac{\partial T}{\partial x}\right)\Delta x$$

[Neglecting the higher-order terms in the expansion of $\partial T/\partial x$ by Taylor series]

Hence, the net rate of inflow of heat to the control volume due to conduction in the x-direction is

$$Q_x - Q_{x+\Delta x} = \frac{\partial}{\partial x}\left(k\,\Delta y\,\Delta z\frac{\partial T}{\partial x}\right)\Delta x$$

$$= \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)\Delta x\,\Delta y\,\Delta z$$

$$= \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)d\mathbb{V}$$

where $d\mathbb{V}$ (= Δx Δy Δz) is the elemental volume of the parallelepiped. In a similar fashion, the net rate of inflow of heat to the control volume due to conduction in the y-direction (through the faces ABCD and EFGH) can be written as

$$Q_y - Q_{y+\Delta y} = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)d\mathbb{V}$$

Also, the net rate of inflow of heat to the control volume due to conduction in the z-direction (through the faces ABFE and DCGH) = $\dfrac{\partial}{\partial z}\left(k\dfrac{\partial T}{\partial z}\right)d\mathbb{V}$

Therefore, the net rate of conduction of heat into the control volume $= \left\{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right\}d\mathbb{V}$

The rate of increase in internal energy of the control volume can be written as

$$\frac{\partial}{\partial t}(\rho c T d\mathbb{V}) = \frac{\partial}{\partial t}(\rho c T)d\mathbb{V}$$

since by the definition of control volume, $d\mathbb{V}$ is invariant with time. The term c is the specific heat of the conducting medium.

If q_G is the rate of generation of thermal energy per unit volume within the elemental control volume, then we can write with the help of Eq. (2.3)

$$\frac{\partial}{\partial t}(\rho c T)d\mathbb{V} = \left[\left\{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right\}+q_G\right]d\mathbb{V} \tag{2.4}$$

This equation is valid for any elemental volume $d\mathbb{V}$. Therefore, we can write

$$\frac{\partial}{\partial t}(\rho c T) = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + q_G \tag{2.4a}$$

Equation (2.4a) is the three-dimensional unsteady heat conduction equation in a rectangular Cartesian coordinate system. In the case of constant properties (thermal conductivity, specific heat, and density), Eq. (2.4a) can be written as

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{q_G}{\rho c} \tag{2.4b}$$

where $\alpha(= k/\rho c)$ is the thermal diffusivity of the medium.

For a steady-state heat conduction, the term $\partial T/\partial t = 0$. Moreover, if there is no generation of thermal energy within the medium, we have $q_G = 0$. Under this situation, Eq. (2.4b) becomes

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2} = 0 \tag{2.4c}$$

The three-dimensional heat conduction equations in the rectangular Cartesian coordinate system for constant properties of the medium under different situations are summarized below:

1. Unsteady with thermal energy generation:
$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q_G}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$
2. Unsteady without thermal energy generation (the **diffusion equation**):
$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$
3. Steady state with thermal energy generation (the **Poisson equation**):
$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q_G}{k} = 0$$
4. Steady-state without thermal energy generation (the **Laplace equation**):
$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2} = 0$$

Equation (2.4a) can be written in vector form as

$$\frac{\partial}{\partial t}(\rho c T) = \nabla \cdot (k \nabla T) + q_G \tag{2.5a}$$

Similarly, Eqs. (2.4b) and (2.4c) can be written in vector forms as

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{q_G}{\rho c} \tag{2.5b}$$

and

$$\nabla^2 T = 0 \tag{2.5c}$$

respectively.

2.1.1 Heat Conduction Equation in Cylindrical and Spherical Coordinate Systems

The heat conduction equation in any coordinate system can be derived in two ways: (a) either by expanding the vector form of the first term in the right-hand side of Eq. (2.5a) with respect to the particular coordinate system, (b) or by considering an elemental control volume appropriate to the reference frame of coordinates and then applying the fundamental principle of conservation of energy as given by Eq. (2.3).

We will first derive the heat conduction equation in the cylindrical coordinate system following the way (a) described above.

Cylindrical coordinate system

The term $\nabla \cdot (k\ \nabla T)$ in Eq. (2.5a) can be expanded in the cylindrical coordinate system r, θ, z (Figure 2.2(a)) as

$$\begin{aligned}\nabla \cdot (k \nabla T) &= \nabla \cdot \left\{ \mathbf{i}_r \left(k \frac{\partial T}{\partial r} \right) + \mathbf{i}_\theta \left(\frac{k}{r} \frac{\partial T}{\partial \theta} \right) + \mathbf{i}_z \left(k \frac{\partial T}{\partial z} \right) \right\} \\ &= \frac{1}{r} \frac{\partial}{\partial r}\left(r k \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta}\left(k \frac{\partial T}{\partial \theta} \right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z} \right)\end{aligned} \tag{2.6}$$

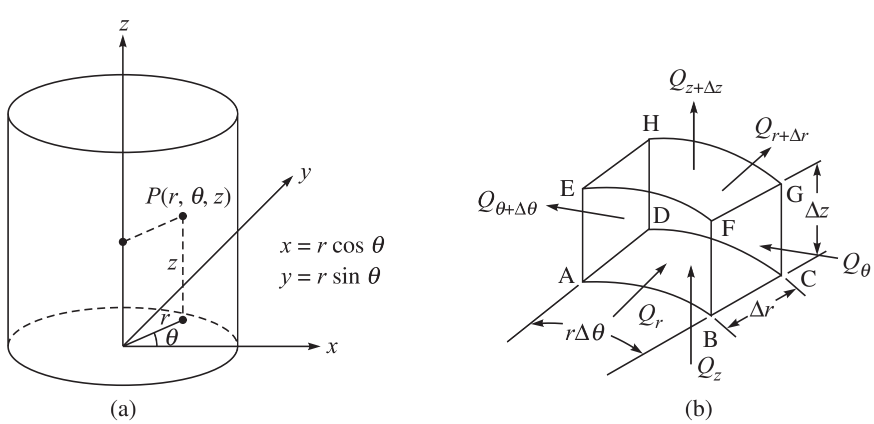


Figure 2.2 (a) Cylindrical coordinate system. (b) An elemental control volume in the cylindrical coordinate system.

Therefore, in the cylindrical coordinate system, the equation of unsteady three-dimensional heat conduction with generation of thermal energy becomes

$$\frac{\partial}{\partial t}(\rho c T) = \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + q_G \tag{2.7a}$$

For constant properties,

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{q_G}{\rho c} \tag{2.7b}$$

For a steady state with constant properties and without generation of thermal energy,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{2.7c}$$

Equation (2.7a) will now be derived by considering the energy fluxes through the control volume as shown in Figure 2.2(b).

The net rate of heat inflow to the control volume due to conduction in the r-direction (through faces ABFE and DCGH), as shown in the figure

$$= Q_r - Q_{r+\Delta r}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)d\mathbb{V} \quad \left[\because Q_r = -k(r\Delta\theta\Delta z)\frac{\partial T}{\partial r} \text{ and } Q_{r+\Delta r} = Q_r + \frac{\partial}{\partial r}(Q_r)\Delta r\right]$$

where $d\mathbb{V}$ (the elemental volume) $= r\Delta\theta\Delta r\Delta z$.

In a similar fashion, the net rate of heat inflow due to conduction in the θ-direction (through the faces BCGF and ADHE), as shown in the figure becomes

$$= Q_\theta - Q_{\theta+\Delta\theta}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right)d\mathbb{V} \quad \left[\because Q_\theta = -k(\Delta r\,\Delta z)\frac{1}{r}\frac{\partial T}{\partial \theta} \text{ and } Q_{\theta+\Delta\theta} = Q_\theta + \frac{\partial}{\partial \theta}(Q_\theta)\Delta\theta\right]$$

The net rate of heat inflow due to conduction in the z-direction (through the faces ABCD and EFGH) is

$$= Q_z - Q_{z+\Delta z}$$

$$= \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)d\mathbb{V} \quad \left[\because Q_z = -k(r\,\Delta\theta\,\Delta r)\frac{\partial T}{\partial z} \text{ and } Q_{z+\Delta z} = Q_z + \frac{\partial}{\partial z}(Q_z)\Delta z\right]$$

The rate of increase in internal energy of the control volume

$$= \frac{\partial}{\partial t}(\rho\Delta r\, r\Delta\theta\Delta z c T)$$

$$= \frac{\partial}{\partial t}(\rho c T)d\mathbb{V}$$

If q_G is the volumetric rate of thermal energy generation within the elemental control volume, following Eq. (2.3), we can write the heat conduction equation in the cylindrical coordinate system as

$$\frac{\partial}{\partial t}(\rho c T) = \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q}_G \tag{2.8}$$

Spherical coordinate system

The terms ∇T and $\nabla\cdot(k\nabla T)$ can be expanded in the spherical coordinate system R, ϕ, θ (Figure 2.3) as

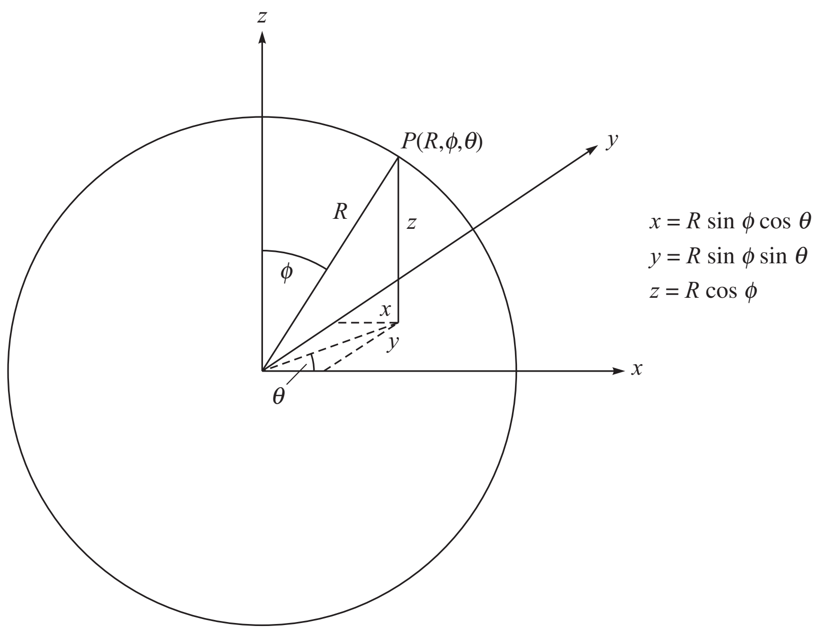


Figure 2.3 Spherical polar coordinate system.

$$\nabla T = \mathbf{i}_R\frac{\partial T}{\partial R} + \mathbf{i}_\phi\frac{1}{R}\frac{\partial T}{\partial \phi} + \mathbf{i}_\theta\frac{1}{R\sin\phi}\frac{\partial T}{\partial \theta}$$

and $$\nabla\cdot(k\nabla T) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2k\frac{\partial T}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\sin\phi\right) + \frac{1}{R^2\sin^2\phi}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right)$$

Substituting the expression for $\nabla\cdot(k\nabla T)$ in Eq. (2.5a), we can write the heat conduction equation in the spherical coordinate system as

$$\frac{\partial}{\partial t}(\rho c T) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2k\frac{\partial T}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\sin\phi\right) + \frac{1}{R^2\sin^2\phi}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right) + q_G \tag{2.9a}$$

where q_G is the volumetric rate of thermal energy generation within the medium.

For constant properties,

$$\frac{\partial T}{\partial t} = \alpha\left[\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial T}{\partial R}\right)+\frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial T}{\partial\phi}\right)+\frac{1}{R^2\sin^2\phi}\frac{\partial^2 T}{\partial\theta^2}\right]+\frac{q_G}{pc} \tag{2.9b}$$

For a steady state with constant properties and without generation of thermal energy,

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial T}{\partial R}\right)+\frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial T}{\partial\phi}\right)+\frac{1}{R^2\sin^2\phi}\frac{\partial^2 T}{\partial\theta^2} = 0 \tag{2.9c}$$

The derivation of Eq. (2.9a) by considering an elemental control volume appropriate to the spherical coordinate system is left as an exercise to the student.

The one-dimensional heat conduction equations in different coordinate systems are summarized below: They are obtained from Eqs. (2.4a), (2.7a) and (2.9a).

Rectangular Cartesian coordinate system (Figure 2.1): **(Heat flow in the *x*-direction)**

Variable thermal conductivity	$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+q_G=\rho c\frac{\partial T}{\partial t}$
Constant thermal conductivity	$\frac{\partial^2 T}{\partial x^2}+\frac{q_G}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$

Cylindrical coordinate system (Figure 2.2(a)): **(Heat flow in the *r*-direction)**

Variable thermal conductivity	$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)+q_G=\rho c\frac{\partial T}{\partial t}$
Constant thermal conductivity	$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{q_G}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$

Spherical coordinate system (Figure 2.3): **(Heat flow in the *R*-direction)**

Variable thermal conductivity	$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2k\frac{\partial T}{\partial R}\right)+q_G=\rho c\frac{\partial T}{\partial t}$
Constant thermal conductivity	$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial T}{\partial R}\right)+\frac{q_G}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$

2.1.2 Derivation of Heat Conduction Equation by Vector Approach Using a Finite Control Volume of Arbitrary Shape

Let us consider a finite control volume $\mathbb{V}$, as shown in Figure 2.4, of an arbitrary shape bounded by the control surface S.

The net rate of efflux of heat due to conduction through the control volume

$$= \iint_S \mathbf{q}\cdot(\mathbf{n}\,dA)$$

$$= \iint_S (-k\nabla T)\cdot(\mathbf{n}\,dA)$$

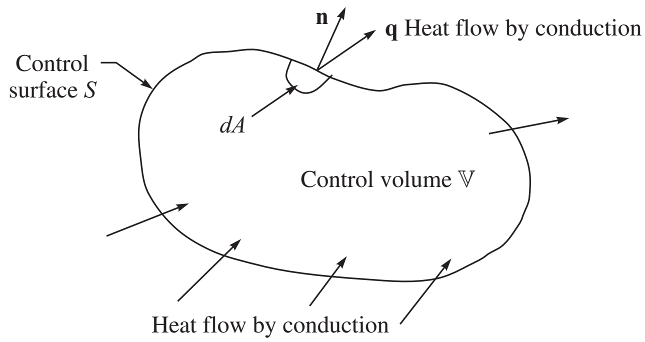


Figure 2.4 A finite control volume of arbitrary shape used in the derivation.

where **q** is the heat flux vector at a point enclosed by an elemental surface area dA (Figure 2.4) and **n** is a unit vector along the outward direction of the normal to the elemental surface of area dA.

The rate of increase in internal energy of the control volume $= \dfrac{\partial}{\partial t}\iiint_{\mathbb{V}}(\rho c T)d\mathbb{V}$

The rate of generation of thermal energy within the control volume $= \iiint_{\mathbb{V}} q_G \, d\mathbb{V}$

where q_G is the rate of thermal energy generation per unit volume. Therefore, following Eq. (2.3), we can write

$$\frac{\partial}{\partial t}\iiint_{\mathbb{V}}(\rho c T)d\mathbb{V} - \iint_S (k\nabla T)\cdot(\mathbf{n}dA) - \iiint_{\mathbb{V}} q_G \, d\mathbb{V} = 0 \tag{2.10}$$

The second term of Eq. (2.10) can be converted into a volume integral by the use of Gauss divergence theorem as

$$\iint_S (k\nabla T).(\mathbf{n}dA) = \iiint_{\mathbb{V}} \nabla.(k\nabla T)d\mathbb{V}$$

Since the volume $\mathbb{V}$ does not change with time, the sequence of differentiation and integration in the first term of Eq. (2.10) can be interchanged.

Hence Eq. (2.10) can be written as

$$\iiint_{\mathbb{V}}\left[\frac{\partial}{\partial t}(\rho c T) - \nabla\cdot(k\nabla T) - q_G\right]d\mathbb{V} = 0 \tag{2.11}$$

Equation (2.11) is valid for any arbitrary control volume irrespective of its shape and size. So, we can write

$$\frac{\partial}{\partial T}(\rho c T) - \nabla\cdot(k\nabla T) - q_G = 0$$

or

$$\frac{\partial}{\partial T}(\rho c T) = \nabla\cdot(k\nabla T) + q_G$$

This is the required heat conduction equation.

2.2 THE STEADY-STATE ONE-DIMENSIONAL HEAT CONDUCTION IN SIMPLE GEOMETRIES

2.2.1 The Plane Wall without Generation of Thermal Energy

The conduction of heat through a solid plane wall whose thickness is small compared to other dimensions is a typical one-dimensional heat conduction problem where heat flows in the direction of x only (Figure 1.2(a)).The problem has already been addressed in Section 1.2.1. We will discuss the problem here in more detail. The rate of heat transfer through the wall can be written according to Fourier's law of heat conduction (Eq. (1.2)) as

$$q_x = -k\frac{dT}{dx}$$

or

$$Q_x = q_x A = -kA\frac{dT}{dx} \tag{2.12}$$

Upon integration of Eq. (2.12) in consideration of the fact that Q_x and A are both independent of x, we have

$$Q_x\int_1^2 dx = -kA\int_1^2 \frac{dT}{dx}\,dx$$

The thermal conductivity k is assumed to be constant. The subscripts 1 and 2 correspond to the two faces of the wall which are kept at temperatures T_1 and T_2 respectively (Figure 1.2(a)).

Therefore,

$$Q_x = \frac{T_1 - T_2}{\dfrac{L}{kA}} \tag{2.13}$$

Alternative method

Another way of obtaining Eq. (2.13) is to first solve for the temperature distribution from the heat conduction equation and then proceed to calculate the rate of heat transfer from Eq. (2.12).

In case of one-dimensional heat conduction, T is a function of x. Hence $\dfrac{\partial T}{\partial y} = \dfrac{\partial T}{\partial z} = 0$.

Therefore, we can write from Eq. (2.4c) as

$$\frac{d^2T}{dx^2} = 0 \tag{2.14}$$

The solution of Eq. (2.14) is

$$T = c_1 x + c_2$$

The constants c_1 and c_2 are determined from boundary conditions (Figure 1.2(a)):

$$\text{At } x = 0 \qquad T = T_1$$
$$\text{At } x = L \qquad T = T_2$$

Hence, we get

$$T = T_1 - \frac{x}{L}(T_1 - T_2)$$

This gives
$$\frac{dT}{dx} = \frac{T_2 - T_1}{L}$$

Therefore, we can write from Eq. (2.12)

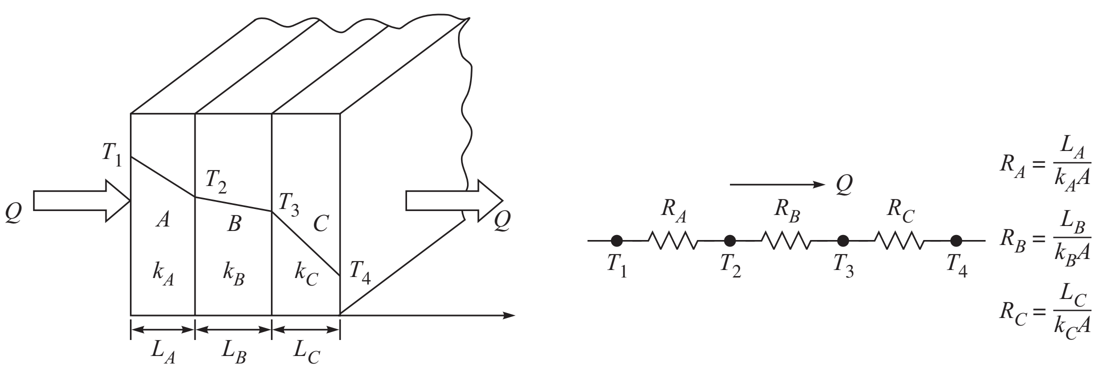
$$Q_x = \frac{T_1 - T_2}{\dfrac{L}{kA}}$$

The electrical analogous circuit has been shown in Figure 1.2(b).

If the temperatures of the surrounding fluids of the two surfaces are specified instead of surface temperatures, the rate of heat transfer through the wall has to be determined in consideration of the convective heat transfer at the surfaces. This is a case of combined conduction and convection heat transfer and has already been discussed in Section 1.3.1.

Wall with multiple layers of different materials

If more than one material is present in a multilayered wall as shown in Figure 2.5(a), heat flows in series through the different layers. The temperature gradients in three materials will be different depending upon their thermal conductivities. The contact between any two layers is considered to be ideal without having any contact resistance.



(a) The multilayered wall (b) The electrical analogous circuit

Figure 2.5 Steady-state one-dimensional heat conduction.

Under steady state, we can write

$$Q = \frac{T_1 - T_2}{\dfrac{L_A}{k_A A}} = \frac{T_2 - T_3}{\dfrac{L_B}{k_B A}} = \frac{T_3 - T_4}{\dfrac{L_C}{k_C A}} \tag{2.15}$$

Equation (2.15) can also be written in a different fashion in terms of an overall temperature difference ($T_1 - T_4$) as

$$Q = \frac{T_1 - T_4}{\dfrac{L_A}{k_A A} + \dfrac{L_B}{k_B A} + \dfrac{L_C}{k_C A}} \tag{2.16}$$

Here, T_1 is the surface temperature of material A and T_4 is the surface temperature of material C. The temperatures T_2 and T_3 are the interface temperatures between materials A and B, and between B and C, respectively. The area A is normal to heat flow. The materials A, B, and C, have thermal conductivities k_A, k_B, k_C and thicknesses L_A, L_B, L_C respectively. The electrical analogous circuit is shown in Figure 2.5(b).

Single-layer wall with materials having different thermal conductivities

In this situation (Figure 2.6(a)), heat conduction takes place in two parallel paths. The analogous electrical circuit is shown in Figure 2.6(b).

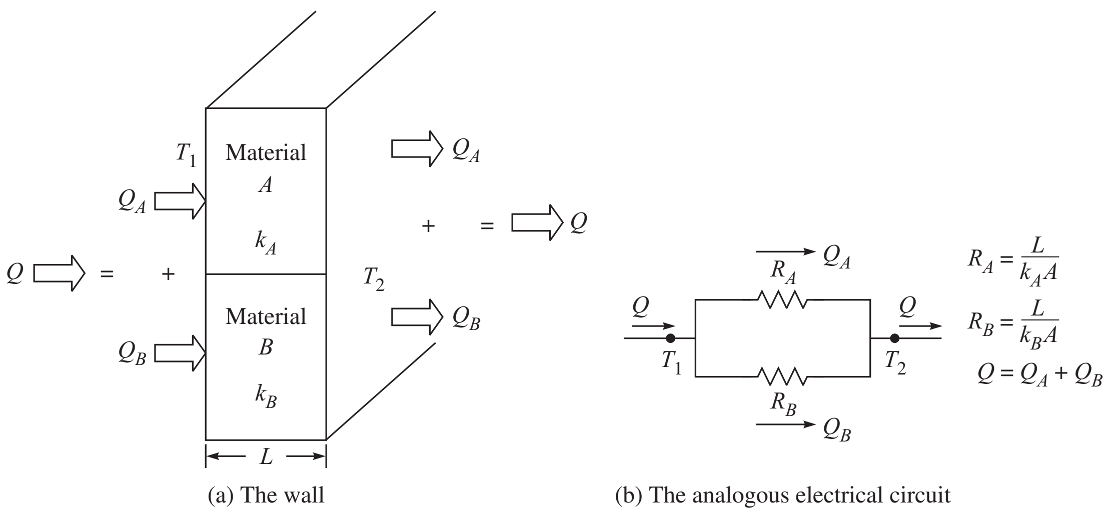


Figure 2.6 Heat conduction through a single-layer wall with materials of different thermal conductivities.

The rate of heat transfer Q is given by

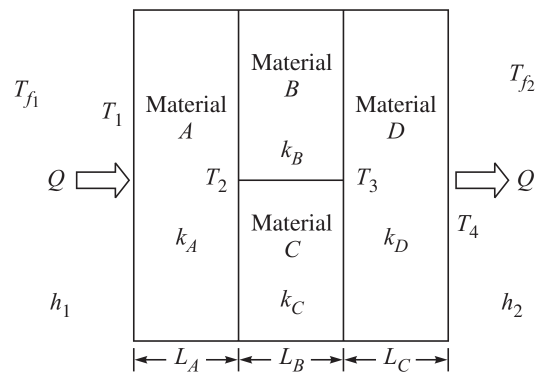
$$Q = \frac{T_1 - T_2}{R}$$

where

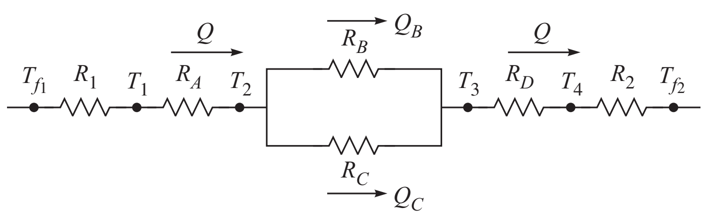
$$\frac{1}{R} = \frac{1}{R_A} + \frac{1}{R_B}$$

Multilayered wall with convection at the boundary surfaces

Figure 2.7(a) shows a multilayered wall comprising materials of different thermal conductivities. The left face of the wall is exposed to a surrounding fluid at a temperature of T_{f_1} while the right face is exposed to a surrounding fluid at a temperature of T_{f_2} ($T_{f_2} < T_{f_1}$). The thickness of the wall ($L_A + L_B + L_C$) is small compared to its dimensions in other directions. This is a problem of one-dimensional combined series and parallel heat transfer by conduction and convection. The thermal network is shown in Figure 2.7(b). The contacts between the layers are considered to be ideal.



(a) The multilayered wall



(b) The analogous electrical circuit

Figure 2.7 Heat conduction in combined series and parallel paths through a multilayered wall with materials of different thermal conductivities.

The rate of heat transfer through the wall Q can be written as

$$Q = \frac{T_{f1} - T_{f2}}{R_1 + R_A + \dfrac{R_B R_C}{R_B + R_C} + R_D + R_2}$$

where

$$R_1 \text{ (convective resistance at the left surface)} = \frac{1}{h_1 A}$$

$$R_A \text{ (conduction resistance of material } A\text{)} = \frac{L_A}{k_A A}$$

$$R_B \text{ (conduction resistance of material } B\text{)} = \frac{L_B}{k_B A}$$

$$R_C \text{ (conduction resistance of material } C\text{)} = \frac{L_C}{k_C A}$$

R_D (conduction resistance of material D) $= \dfrac{L_D}{k_D A}$

R_2 (convective resistance at the right surface) $= \dfrac{1}{h_2 A}$

Here h_1, h_2 are the convective heat transfer coefficients at the left and right faces respectively; A is the cross-sectional area of the wall.

EXAMPLE 2.1 A plane wall of a refrigerated van is made of 1.5 mm steel sheet (k_S = 25 W/(m K)) at the outer surface, 10 mm plywood (k_P = 0.05 W/(m K)) at the inner surface and 20 mm glass wool (k_G = 0.01 W/(m K)) in between the outer and inner surfaces. The temperature of the cold environment inside the van is –15°C, while the outside surface is exposed to a surrounding ambient at 24°C. The average values of convective heat transfer coefficients at the inner and outer surfaces of the wall are 12 W/(m K) and 20 W/(m K) respectively. The surface area of the wall is 0.75 m^2. Determine:

(a) The individual components of the thermal resistance to heat flow.
(b) The rate of heat flow through the wall.
(c) The temperatures at (i) the outer surface of the wall, (ii) the interface between steel sheet and glass wool, (iii) the interface between glass wool and plywood, and (iv) the inner surface of the wall

Solution: The composite wall and the electrical analogous thermal circuit are shown in Figure 2.8.

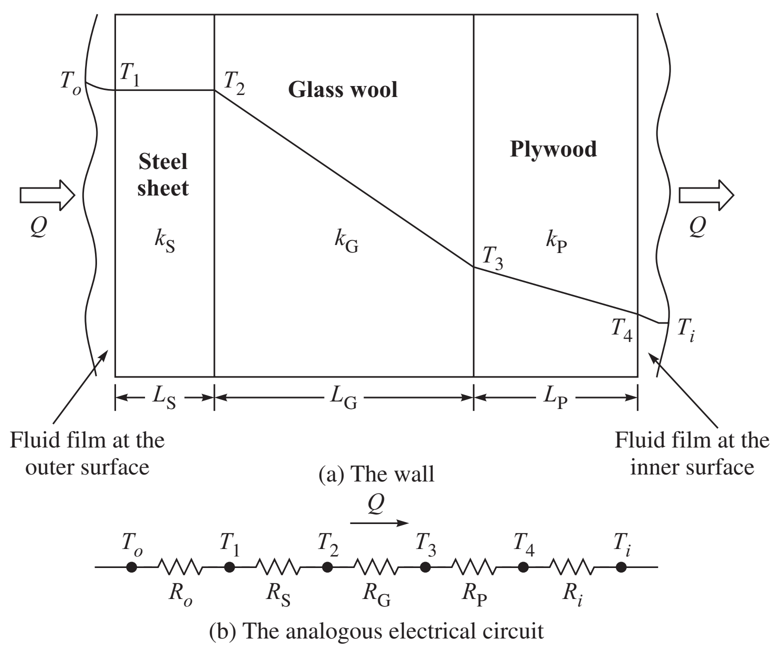


Figure 2.8 The composite wall and the analogous electrical circuit (Example 2.1).

For the problem,

$$\begin{aligned} h_i &= 12 \text{ W/(m K)} & L_S &= 1.5 \text{ mm} \\ k_S &= 25 \text{ W/(m K)} & L_G &= 20 \text{ mm} \\ k_G &= 0.01 \text{ W/(m K)} & L_P &= 10 \text{ mm} \\ k_P &= 0.05 \text{ W/(m K)} & T_o &= 24°\text{C} \\ h_o &= 20 \text{ W/(m K)} & T_i &= 15°\text{C} \end{aligned}$$

(a) R_o (convective resistance at the outer surface) $= \dfrac{1}{20 \times 0.75} = 0.067 \text{ K W}^{-1}$

R_S (conduction resistance of steel sheet) $= \dfrac{1.5 \times 10^{-3}}{25 \times 0.75} = 8 \times 10^{-5} \text{ K W}^{-1}$

R_G (conduction resistance of glass wool) $= \dfrac{20 \times 10^{-3}}{0.01 \times 0.75} = 2.67 \text{ K W}^{-1}$

R_P (conduction resistance of plywood) $= \dfrac{10 \times 10^{-3}}{0.05 \times 0.75} = 0.267 \text{ K W}^{-1}$

R_i (convective resistance at the inner surface) $= \dfrac{1}{12 \times 0.75} = 0.111 \text{ K W}^{-1}$

(b) The rate of heat flow, $Q = \dfrac{24 - (-15)}{0.067 + 8 \times 10^{-5} + 2.67 + 0.267 + 0.111}$

$= 12.52$ W

(c) The temperatures at the interfaces are marked in Figure 2.8.

$$\begin{aligned} T_1 &= 24 - (12.52 \times 0.067) = 23.16°\text{C} \\ T_2 &= 23.16 - (12.52 \times 8 \times 10^{-5}) \approx 23.16°\text{C} \\ T_3 &= 23.16 - (12.52 \times 2.67) = -10.27°\text{C} \\ T_4 &= -10.27 - (12.52 \times 0.267) = -13.61°\text{C} \end{aligned}$$

Check for T_i

$T_i = -13.61 - (12.52 \times 0.111) = -15°\text{C}$ (the value given in the problem)

EXAMPLE 2.2 A laboratory furnace wall is constructed of 0.2 m thick fire clay bricks (k = 1.0 W/(m K)). The wall is covered on the outer surface with a thick layer of insulating material (k = 0.07 W/(m K)). The furnace inner brick surface is at 1250 K and the outer surface of the insulating material is at 310 K. If the maximum allowable heat transfer rate through the wall of the furnace is 900 W/m^2, how thick must the insulating layer be?

Solution: Let L be the thickness of the insulating material in metre. Here, we will use Eq. (2.16). Therefore, we can write

$$900 = \frac{1250 - 310}{\dfrac{0.2}{1.0} + \dfrac{L}{0.07}}$$

which gives

$$L = 0.07\left(\frac{940}{900} - 0.2\right)$$
$$= 0.059 \text{ m}$$

EXAMPLE 2.3 Obtain an analytical expression for the steady-state temperature distribution $T(x)$ in the plane wall of Figure 2.9 having uniform surface temperatures T_1 and T_2 at x_1 and x_2 respectively and a thermal conductivity which varies linearly with temperature as $k = k_0(1 + bT)$, where k_0 and b are constants.

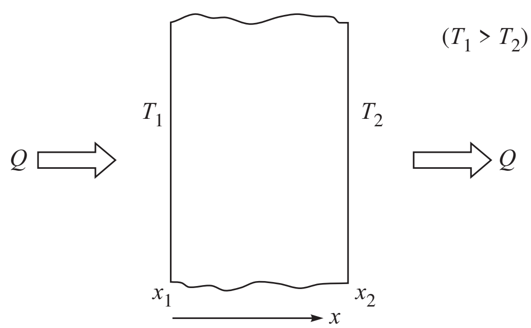


Figure 2.9 The plane wall (Example 2.3).

Solution: The heat conduction equation Eq. (2.4a), under the situation, becomes

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$$

This gives upon integration

$$k\frac{dT}{dx} = C \text{ (constant)}$$

or
$$k_0(1 + bT)\,dT = Cdx$$

or
$$(1 + bT)dT = C_1 dx \qquad (\text{where } C_1 = C/k_0)$$

Integrating again the above equation, we get

$$T + \frac{bT^2}{2} = C_1 x + C_2 \qquad (\text{where } C_2 \text{ is a constant}) \tag{2.17}$$

Equation (2.17) can be written as

$$\left(T + \frac{1}{b}\right)^2 = \frac{2}{b}(C_1 x + C_2) + \frac{1}{b^2} \tag{2.18}$$

From the boundary conditions, we get

$$\left(T_1 + \frac{1}{b}\right)^2 = \frac{2}{b}(C_1 x_1 + C_2) + \frac{1}{b^2}$$

$$\left(T_2 + \frac{1}{b}\right)^2 = \frac{2}{b}(C_1 x_2 + C_2) + \frac{1}{b^2}$$

The preceding two equations finally give

$$\frac{2}{b}C_1 = \frac{1}{x_1 - x_2}\left[\left(T_1 + \frac{1}{b}\right)^2 - \left(T_2 + \frac{1}{b}\right)^2\right] - \frac{1}{b^2}$$

and
$$\frac{2}{b}C_2 = \frac{1}{x_2 - x_1}\left[x_2\left(T_1 + \frac{1}{b}\right)^2 - x_1\left(T_2 + \frac{1}{b}\right)^2\right] - \frac{1}{b^2}$$

Substituting the values of C_1 and C_2 in Eq (2.18), we can write after some algebraic rearrangements

$$T = -\frac{1}{b} + \left[\left(T_1 + \frac{1}{b}\right)^2 - 2\left(T_m + \frac{1}{b}\right)\left(\frac{T_1 - T_2}{x_2 - x_1}\right)(x - x_1)\right]^{1/2}$$

where
$$T_m = \frac{T_1 + T_2}{2}$$

EXAMPLE 2.4 In a manufacturing process, a transparent film is bonded to a substrate as shown in Figure 2.10. To cure the bond at a temperature T_0, a radiant source is used to provide a heat flux q_0 (W/m^2), all of which is absorbed at the bonded surface. The back of the substrate is maintained at a temperature T_1 while the free surface of the film is exposed to air at a temperature T_∞ and a convective heat transfer coefficient h.

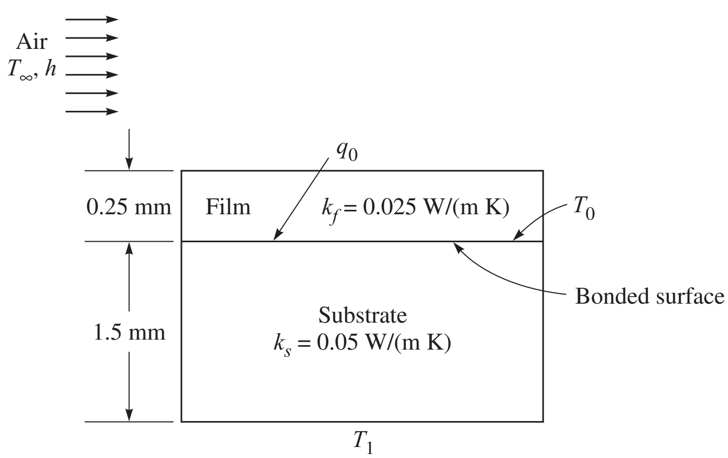


Figure 2.10 The film and substrate (Example 2.4).

(a) Calculate the heat flux q_0 that is required to maintain the bonded surface at T_0 = 60°C, when T_∞ = 20°C, h = 25 W/(m^2 K) and T_1 = 30°C.

(b) If the film is not transparent and all of the radiant heat flux is absorbed at its upper surface, determine the heat flux required to achieve bonding.

Solution: (a) $q_0 = q_f + q_s$

where q_f and q_s are the rates of heat transfer per unit surface area through the film and the substrate respectively.

$$q_f = \frac{60 - 20}{\dfrac{1}{25} + \dfrac{0.25 \times 10^{-3}}{0.025}}$$

$$= 800 \text{ W/m}^2$$

$$q_s = \frac{60 - 30}{\dfrac{1.5 \times 10^{-3}}{0.05}}$$

$$= 1000 \text{ W/m}^2$$

Therefore, $q_0 = 800 + 1000 = 1800 \text{ W/m}^2$

(b) In this situation, we can write

$$q_0 = q_1 + q_2$$

where q_1 is the rate of heat conduction through the film and the substrate, and q_2 is the rate of convective heat transfer from the upper surface of the film to air.

$$q_1 = \frac{60 - 30}{\dfrac{1.5 \times 10^{-3}}{0.05}}$$

$$= 1000 \text{ W/m}^2$$

To determine q_2 we have to find the temperature of the top surface of the film. Let this temperature be T_2. Then

$$1000 = \frac{T_2 - 60}{\dfrac{0.25 \times 10^{-3}}{0.025}}$$

which gives $T_2 = 70°\text{C}$

Therefore, $q_2 = 25 \times (70 - 20)$

$$= 1250 \text{ W/m}^2$$

Hence, $q_0 = 1000 + 1250$

$$= 2250 \text{ W/m}^2$$

EXAMPLE 2.5 Steady-state one-dimensional conduction occurs in a rod (Figure 2.11) of constant thermal conductivity k and of variable cross-sectional area $A_x = A_0 e^{ax}$, where A_0 and a are constants. The length of the rod is L. The lateral surface of the rod is well insulated. The surfaces at $x = 0$ and $x = L$ are kept at constant temperatures of T_1 and T_2 respectively. Find the expressions for (a) the temperature distribution in the rod and (b) the rate of heat transfer through the rod.

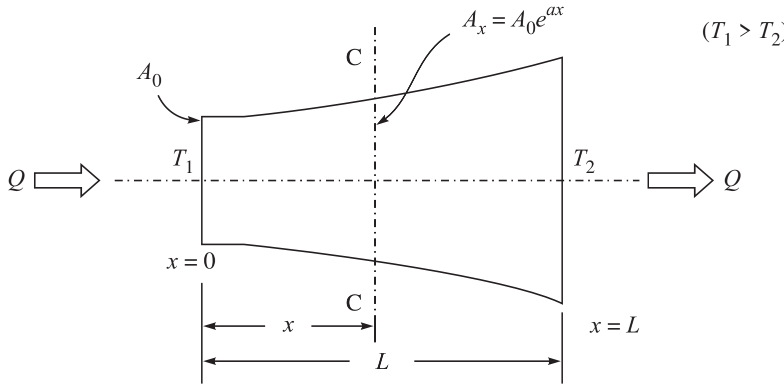


Figure 2.11 The rod (Example 2.5).

Solution: Late Q be the rate of heat flow through the rod at steady state. Therefore, we can write at any section C–C (Figure 2.11)

$$Q = -kA_0 e^{ax}\frac{dT}{dx}$$

or
$$dT = -\frac{Q}{kA_0}e^{-ax}dx$$

Integrating the above equation, we get

$$\int_{T_1}^{T} dT = \int_0^x \frac{-Q}{kA_0}e^{-ax}dx$$

or
$$T - T_1 = \frac{Q}{kaA_0}(e^{-ax} - 1) \tag{2.19}$$

where T is the temperature of the rod at the section C–C

Again, we can write

$$\int_{T_1}^{T_2} dT = \int_0^L -\frac{Q}{kA_0}e^{-ax}dx$$

or
$$T_2 - T_1 = \frac{Q}{kaA_0}(\mathrm{e}^{-aL} - 1) \tag{2.20}$$

From Eqs. (2.19) and (2.20), we have

$$\frac{T_1 - T}{T_1 - T_2} = \frac{1 - e^{-ax}}{1 - e^{-aL}} \tag{2.21}$$

This is the required temperature distribution.

The rate of heat transfer Q can be written from Eq. (2.20) as

$$Q = \frac{kaA_0}{1 - e^{-aL}}(T_1 - T_2)$$

2.2.2 The Plane Wall with Generation of Thermal Energy

Let us consider a plane wall (Figure 2.12) whose thickness $2L$ is small compared to other dimensions and has a constant thermal conductivity k. Thermal energy is generated uniformly within the wall while its two faces are kept at a constant temperature T_w. Let q_G be the rate of generation of thermal energy per unit volume. We shall determine the steady-state temperature distribution within the wall and shall show that the rate of heat loss from the two faces of the wall equals the rate of generation of thermal energy within the wall (a requirement of steady state). Because of symmetry, the mid-plane of the wall has been chosen as the axis $x = 0$ as shown in Figure 2.12. We can write from Eq. (2.4b)

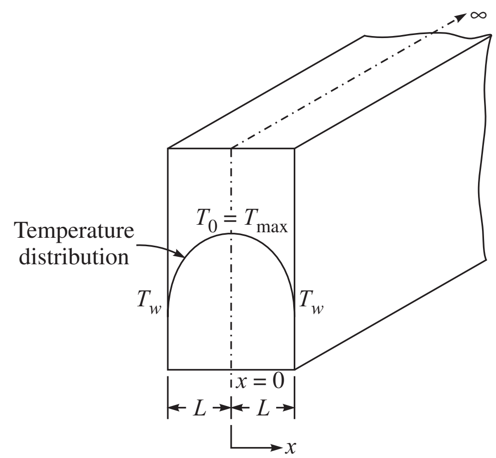


Figure 2.12 One-dimensional steady-state heat conduction in a plane wall with uniform generation of thermal energy.

$$\frac{d^2T}{dx^2} + \frac{q_G}{k} = 0$$

Integrating the above equation twice, we get

$$T = -\frac{q_G}{2k}x^2 + c_1 x + c_2 \tag{2.22}$$

where c_1 and c_2 are constants which have to be determined from the boundary conditions of the problem as shown below.

At $\quad x = 0, \quad \dfrac{dT}{dx} = 0 \quad$ (symmetric condition about $x = 0$)

At $\quad x = \pm L, \quad T = T_w$

The above two conditions give

$$c_1 = 0$$

and

$$c_2 = T_w + \frac{q_G}{2k}L^2$$

Hence Eq. (2.22) becomes

$$T - T_w = \frac{q_G}{2k}(L^2 - x^2) \tag{2.23}$$

Equation (2.23) shows that the temperature distribution is parabolic (Figure 2.12). The temperature at $x = 0$ is maximum, and let it be denoted by T_0. Then it becomes

$$T_0 = T_w + \frac{q_G}{2k}L^2 \tag{2.24}$$

We can eliminate q_G from Eqs. (2.23) and (2.24) to express the temperature distribution by any one of the following two forms.

$$\frac{T - T_w}{T_0 - T_w} = 1 - \frac{x^2}{L^2} \tag{2.24a}$$

or

$$\frac{T - T_0}{T_w - T_0} = \left(\frac{x}{L}\right)^2 \tag{2.24b}$$

The rate of heat loss from the two faces can be written as

$$Q = \left| -kA\left(\frac{dT}{dx}\right)_{x=L} \right| + \left| -kA\left(\frac{dT}{dx}\right)_{x=-L} \right|$$

$$= \frac{4kA(T_0 - T_w)}{L} \tag{2.25}$$

where A is the surface area of the wall.

Substituting for $(T_0 - T_w)$ from Eq. (2.24) in Eq. (2.25), we get

$$Q = 2LAq_G$$

Therefore the heat lost from the faces equals the generation of thermal energy within the wall which is the requirement for steady state.

EXAMPLE 2.6 Consider a 20 mm thick plate with uniform heat generation of 80 MW/m^3. The left and right faces are kept at constant temperatures of 160°C and 120°C respectively. The plate has a constant thermal conductivity of 200 W/(m K). Determine (a) the expression for temperature distribution in the plate, (b) the location and the value of maximum temperature, and (c) the rate of heat transfer (i) at the left face (ii) at the right face and (iii) at the plate centre.

Solution: The plate is schematically shown in Figure 2.13.

(a) We start with the equation

$$\frac{d^2T}{dx^2} + \frac{q_G}{k} = 0$$

or

$$T = -\frac{q_G}{2k}x^2 + c_1x + c_2$$

With $q_G = 80 \times 10^6$ W/m^3 and $k = 200$ W/(m K)

the above equation becomes

$$T = -2 \times 10^5 x^2 + c_1x + c_2$$

The boundary conditions (Figure 2.13) are

At $x = 0$, $T = 160°C$

At $x = 0.02$ m, $T = 120°C$

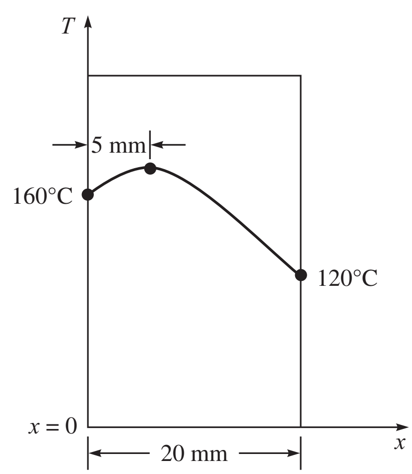


Figure 2.13 The plate (Example 2.6).

Therefore, $c_2 = 160°\text{C}, \quad c_1 = 2000°\text{C m}^{-1}$

Hence $T = 160 + 2 \times 10^3(x - 100x^2)$

(b) For maximum temperature,

$$\frac{dT}{dx} = 2 \times 10^3 - 4 \times 10^5 x = 0$$

which gives $x = 0.005 \text{ m} = 5 \text{ mm}$

Therefore, the maximum temperature occurs 5 mm away from the left surface towards the right

Therefore, $T_{\max} = 160 + 2 \times 10^3 \times (0.005) - 2 \times 10^5 \times (0.005)^2$
$= 165°\text{C}$

(c) (i) $$\left(\frac{q}{A}\right)_{x=0} = -k\left(\frac{dT}{dx}\right)_{x=0}$$

Now, $$\left(\frac{dT}{dx}\right)_{x=0} = 2 \times 10^3$$

Therefore, $$\left(\frac{q}{A}\right)_{x=0} = -200 \times 2 \times 10^3 \text{ W/m}^2$$
$$= -0.4 \text{ MW/m}^2$$

The minus sign indicates that heat flows in the negative x-direction.

(c) (ii) $$\left(\frac{q}{A}\right)_{x=0.02} = -k\left(\frac{dT}{dx}\right)_{x=0.02}$$

Now, $$\left(\frac{dT}{dx}\right)_{x=0.02} = 2 \times 10^3 - 4 \times 10^5 \times (0.02)$$
$$= -6 \times 10^3$$

Therefore, $$\left(\frac{q}{A}\right)_{x=0.02} = 200 \times 6 \times 10^3 \text{ W/m}^2$$
$$= 1.2 \text{ MW/m}^2$$

The minus sign for $\left(\frac{q}{A}\right)_{x=0}$ and the plus sign for $\left(\frac{q}{A}\right)_{x=0.02}$ indicate that heat is lost from both the surfaces to the surroundings.

(c) (iii) $$\left(\frac{q}{A}\right)_{x=0.01} = -k\left(\frac{dT}{dx}\right)_{x=0.01}$$

Here, $$\left(\frac{dT}{dx}\right)_{x=0.01} = 2 \times 10^3 - 4 \times 10^5 \times (0.01)$$
$$= -2 \times 10^3$$

Therefore, $$\left(\frac{q}{A}\right)_{x=0.01} = 200 \times 2 \times 10^3 \text{ W/m}^2$$

$$= 0.4 \text{ MW/m}^2$$

A check for the above results can be made from an energy balance of the plate as

$$\left|\frac{q}{A}\right|_{x=0} + \left|\frac{q}{A}\right|_{x=0.02} = q_G \times 0.02$$

EXAMPLE 2.7 A pressure vessel for a nuclear reactor is approximated as a large flat plate of thickness L. The inside surface of the plate at $x = 0$ is insulated, the outside surface at $x = L$ is maintained at a uniform temperature T_w. The gamma-ray heating of the plate can be represented as a generation of thermal energy within the plate in the form

$$q_G = q_0 e^{-\beta x} \text{ W/m}^3$$

where q_0 and β are constants and x is measured from the insulated inner surface.

Develop expressions for:

(a) The temperature distribution in the plate.
(b) The temperature at the insulated surface ($x = 0$) of the plate.
(c) The heat flux at the outer surface ($x = L$).

Solution: The present situation pertains to a steady one-dimensional heat conduction problem with generation of thermal energy.

Therefore, $$\frac{d^2T}{dx^2} + \frac{q_G}{k} = 0$$

or $$\frac{d^2T}{dx^2} = -\frac{q_0}{k} e^{-\beta x}$$

Integrating the above equation, we have

$$\frac{dT}{dx} = \frac{q_0}{\beta k} e^{-\beta x} + c_1$$

and $$T = -\frac{q_0}{\beta^2 k} e^{-\beta x} + c_1 x + c_2 \tag{2.26}$$

The constants c_1 and c_2 are determined from the boundary conditions:

At $x = 0$, $\frac{dT}{dx} = 0$ (the insulated inner surface)

At $x = L$, $T = T_w$ (isothermal outer surface)

Therefore, we have

$$c_1 = -\frac{q_0}{\beta k}$$

and $$c_2 = T_w + \frac{q_0}{\beta^2 k} e^{-\beta L} + \frac{q_0}{\beta k} L$$

(a) Substituting the values of c_1 and c_2 in Eq. (2.26), we can write for the expression of temperature distribution as

$$T = T_w + \frac{q_0}{\beta^2 k}\left[e^{-\beta L}\left\{1 - e^{\beta L\left(1-\frac{x}{L}\right)}\right\} + \beta L\left(1 - \frac{x}{L}\right)\right]$$

(b) $(T)_{x=0} = T_w + \dfrac{q_0}{\beta^2 k}\,[e^{-\beta L}(1 - e^{\beta L}) + \beta L]$

(c) $(q)_{x=L} = -k\left(\dfrac{dT}{dx}\right)_{x=L}$

$$= -k\left[-\frac{q_0}{\beta k}(1 - e^{-\beta L})\right]$$

$$= \frac{q_0}{\beta}(1 - e^{-\beta L}) \text{ W/m}^2$$

2.2.3 Cylinders and Spheres without Generation of Thermal Energy

In this section we shall discuss the steady-state heat conduction through solids of cylindrical and spherical geometries without generation of thermal energy.

Cylindrical geometry

Let us consider a long hollow cylinder having inner and outer radii of r_1 and r_2 respectively (Figure 2.14). The inner and outer surfaces are kept at fixed temperatures T_1 and T_2 respectively ($T_1 > T_2$). Therefore heat will be transferred by conduction through the wall of the cylinder in the radial direction from inner to outer surface. This describes a model for steady one-dimensional heat conduction in a cylindrical coordinate system. One of the practical examples is the heat loss from hot water flowing through a pipe. Usually the length of the pipe is much larger compared to its wall thickness and therefore no significant heat transfer takes place in other directions except in the radial direction. Therefore, the temperature will depend on the radial direction only, i.e. $T = T(r)$. We further assume that thermal conductivity of the pipe wall is constant and there is no generation of thermal energy within the material. Let L be the length of the pipe.

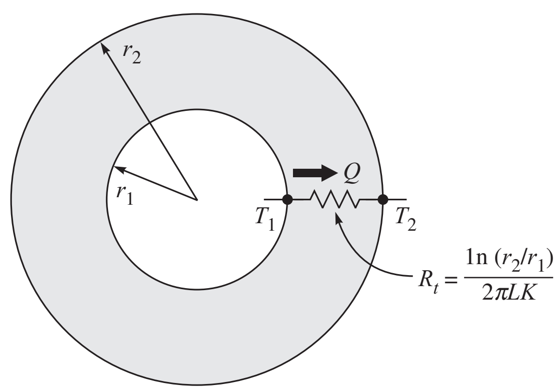


Figure 2.14 Heat conduction through a cylindrical wall.

The Fourier's law of heat conduction can be written, under the situation, as

$$Q = -kA\frac{dT}{dr}$$

where A (= $2\pi rL$) is the heat transfer area at a location r.

Therefore,
$$Q = -k(2\pi rL)\frac{dT}{dr} \tag{2.27}$$

It is important to note in this context that the heat transfer area A is a function of r, the direction of heat transfer, while in the case of a plane wall the heat transfer area is constant in the direction of heat flow. Separating the variables in Eq. (2.27) and integrating from $r = r_1$ where $T(r_1) = T_1$ to $r = r_2$, where $T(r_2) = T_2$, we get

$$\int_{r_1}^{r_2} \frac{Q}{2\pi rL}dr = -\int_{T_1}^{T_2} kdT \tag{2.28}$$

At steady state, the heat flow Q at any radial location r will be the same. Therefore, Eq. (2.28) gives

$$Q = 2\pi Lk\frac{T_1 - T_2}{\ln(r_2/r_1)} \tag{2.29}$$

This equation can be rearranged as

$$Q = \frac{T_1 - T_2}{\dfrac{\ln(r_2/r_1)}{2\pi Lk}} \tag{2.30}$$

The denominator of Eq. (2.30) represents the thermal resistance R_1 as shown in Figure 2.14. Therefore, we can write

$$Q = \frac{T_1 - T_2}{R_t}$$

where
$$R_t = \frac{\ln(r_2/r_1)}{2\pi Lk}$$

Equation (2.30) can also be derived by determining first the expression for temperature as a function of radial coordinate r from the solution of heat conduction equation in cylindrical coordinates and then by using Eq. (2.27).

The heat conduction equation Eq. (2.7a) under the situation becomes

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0$$

or
$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

This gives

$$T = c_1 \ln r + c_2 \tag{2.31}$$

Applying the boundary conditions

$$T = T_1 \quad \text{at} \quad r = r_1$$
$$T = T_2 \quad \text{at} \quad r = r_2$$

we obtain

$$c_1 = \frac{T_2 - T_1}{\ln(r_2/r_1)}$$

and

$$c_2 = T_1 - \frac{T_2 - T_1}{\ln(r_2/r_1)} \ln r_1$$

Then Eq. (2.31) becomes

$$\frac{T_1 - T}{T_1 - T_2} = \frac{\ln(r/r_1)}{\ln(r_2/r_1)} \tag{2.32}$$

Equation (2.32) is the expression for temperature distribution in the cylindrical wall. The rate of heat transfer Q at any radial location r can be written, following Eq. (2.27), and making use of Eq. (2.32) as

$$Q = -k(2\pi rL)\frac{dT}{dr}$$
$$= \frac{T_1 - T_2}{\dfrac{\ln(r_2/r_1)}{2\pi Lk}}$$

Convection boundary conditions

Let us now consider that the cylindrical wall is exposed to convection on both sides to fluids at temperatures T_{f_i} and T_{f_o} ($T_{f_i} > T_{f_o}$) with convective heat transfer coefficients h_1 and h_2 respectively as shown in Figure 2.15. The thermal resistance network in this case consists of one conduction and two convection resistances in series. The rate of heat transfer under steady conditions can be expressed as

$$Q = \frac{T_{f_i} - T_{f_o}}{R_{\text{total}}} \tag{2.33}$$

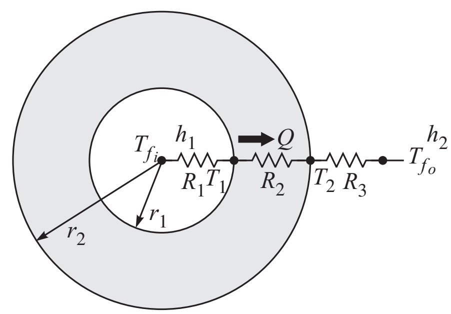


Figure 2.15 A cylindrical wall subjected to convection from both the inner and the outer surfaces.

The total resistance R_{total} is given by

$$R_{\text{total}} = R_1 + R_2 + R_3 \tag{2.34}$$

where

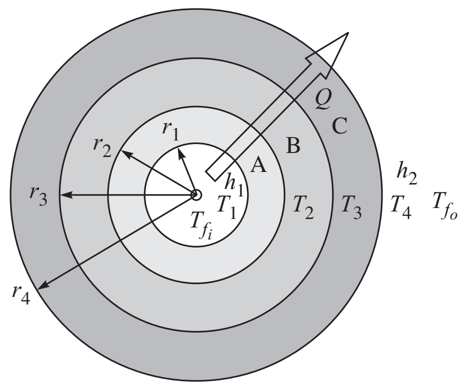
$$R_1 \text{ (convective resistance at the inner surface)} = \frac{1}{(2\pi r_1 L)h_1} \tag{2.35a}$$

$$R_2 \text{ (conduction resistance of solid wall)} = \frac{\ln(r_2/r_1)}{2\pi Lk} \tag{2.35b}$$

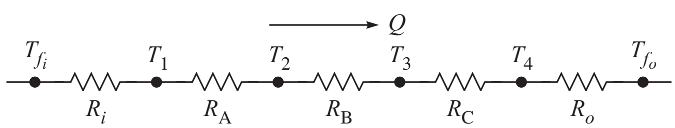
$$R_3 = \text{(convective resistance at the outer surface)} = \frac{1}{(2\pi r_2 L)h_2} \tag{2.35c}$$

Multilayered cylindrical wall

Steady-state heat transfer through a cylindrical wall of multiple layers of different materials can be analyzed by the network principle of thermal resistances in series as adopted in case of a plane wall. Let us consider a cylindrical wall of three layers which is exposed to convection on both sides to fluids at temperatures T_{f_i} and T_{f_o} ($T_{f_i} > T_{f_o}$) with convective heat transfer coefficients h_1 and h_2 respectively as shown in Figure 2.16(a). The thermal network is shown in Figure 2.16(b).



(a) The physical system



(b) The thermal resistance network

Figure 2.16 One-dimensional heat flow through a three-layered cylindrical wall.

The rate of heat flow is given by

$$Q = \frac{T_{f_i} - T_{f_o}}{R_{\text{total}}} \tag{2.36}$$

The total thermal resistance in this case is given by

$$R_{\text{total}} = R_i + R_{\text{A}} + R_{\text{B}} + R_{\text{C}} + R_o \tag{2.37}$$

where

$$R_i \text{ (convective resistance at the inner surface)} = \frac{1}{(2\pi r_1 L) h_1} \tag{2.38a}$$

$$R_{\text{A}} \text{ (conduction resistance of layer A)} = \frac{\ln (r_2/r_1)}{2\pi L k_{\text{A}}} \tag{2.38b}$$

$$R_{\text{B}} \text{ (conduction resistance of layer B)} = \frac{\ln (r_3/r_2)}{2\pi L k_{\text{B}}} \tag{2.38c}$$

R_C (conduction resistance of layer C) $= \dfrac{\ln(r_4/r_3)}{2\pi L k_C}$ (2.38d)

R_o (convective resistance at the outer surface) $= \dfrac{1}{(2\pi r_4 L) h_2}$ (2.38e)

Here L is the length of the cylinder.

It is evident from Eqs. (2.36), (2.37) and (2.38) that the rate of heat flow can be determined if the inner and outer ambient temperatures and the geometrical dimensions of the cylindrical wall are known.

Once Q is known, any intermediate temperature T_k can be determined by applying the relation $Q = (T_j - T_k)/\sum_j^k R$ across any layer or layers such that T_j is a known temperature at location j and $\sum_j^k R$ is the total thermal resistance between locations j to k. For example, the interface temperature T_2 (Figure 2.16) can be determined from

$$Q = \frac{T_{f_i} - T_2}{R_i + R_A} \tag{2.39a}$$

The temperature T_2 can also be calculated from

$$Q = \frac{T_2 - T_{f_o}}{R_B + R_C + R_o} \tag{2.39b}$$

Critical thickness of Insulation

In the case of combined convection and conduction heat transfer through a plane wall, an addition of insulating material (materials with low thermal conductivity) to the wall always decreases the rate of heat transfer. This is so because the addition of any material increases the total thickness of the wall keeping the heat transfer area constant. This results in an increase in the conduction resistance of the wall without affecting the convection resistance.

In the case of a cylindrical wall, the situation is a little different. The addition of insulation on the cylindrical surface increases the conduction resistance but at the same time decreases the convection resistance of the surface by increasing the outer surface area. Therefore, the heat transfer from the wall may increase or decrease depending upon the fact as to which of these counterweighing effects become dominant over the other.

Let us consider a cylindrical pipe of outer radius r_1 with a constant outer surface temperature T_1 (Figure 2.17). The pipe is now insulated with a material of thermal conductivity k so that the outer radius becomes r_2. Let us consider that the surrounding medium is at a temperature T_∞ ($T_\infty < T_1$) with a convective heat transfer coefficient h.

The rate of heat transfer from the pipe to the surrounding can be written as

$$Q = \frac{T_1 - T_\infty}{\dfrac{\ln(r_2/r_1)}{2\pi L k} + \dfrac{1}{h(2\pi r_2 L)}} \tag{2.40}$$

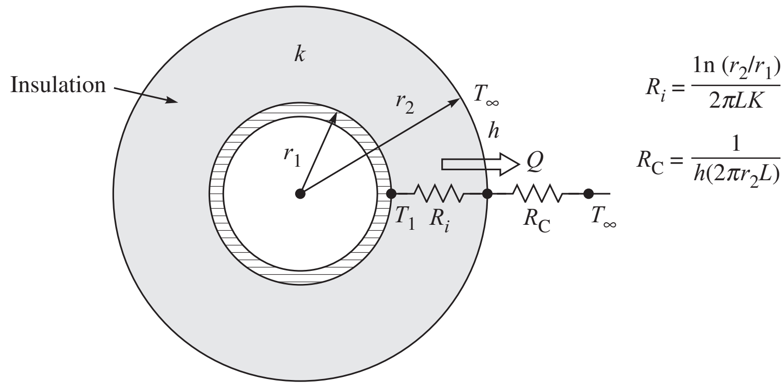


Figure 2.17 Heat transfer from an insulated cylindrical pipe exposed to convection from the outer surface.

where L is the length of the pipe.

It is observed from Eq. (2.40) that with an increase in the value of r_2, the first term of the denominator increases while the second term decreases. Therefore Q is expected to have either a maximum or a minimum in relation to its variation with r_2. This can be checked mathematically from the values of dQ/dr_2 and d^2Q/dr_2^2.

From Eq. (2.40),

$$\frac{dQ}{dr_2} = \frac{-2\pi L(T_1 - T_\infty)\left(\dfrac{1}{kr_2} - \dfrac{1}{hr_2^2}\right)}{\left[\dfrac{\ln(r_2/r_1)}{k} + \dfrac{1}{hr_2}\right]^2} \tag{2.41}$$

By putting $dQ/dr_2 = 0$ as the condition for either maximum or minimum, we have

$$\frac{1}{kr_2} - \frac{1}{hr_2^2} = 0$$

which gives

$$r_2 = \frac{k}{h}$$

It can be shown from Eq. (2.41) that the second derivative d^2Q/dr_2^2 is always negative. Therefore, Q has a maximum with r_2 and this is attained when $r_2 = k/h$. This is shown in Figure 2.18. This value of r_2 is termed the critical radius of insulation r_c and we can write accordingly

$$r_c = \frac{k}{h} \tag{2.42}$$

If the outer radius is less than the value given by Eq. (2.42), the heat transfer will be increased by adding more insulation. For outer radius greater than the critical radius r_c, an increase in insulation thickness will result in a decrease in heat transfer. Therefore for the purpose of insulating a hot pipe, the outer radius of insulation must be greater than the critical radius r_c.

The critical insulation radius r_c, as given by Eq. (2.42), is directly proportional to the thermal conductivity k of the insulating material and inversely proportional to the convective heat transfer

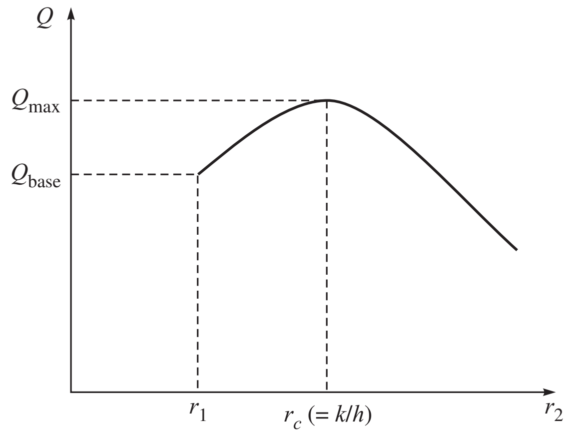


Figure 2.18 Variation in the rate of heat transfer with the outer radius of insulation of an insulated pipe.

coefficient h at the outer surface. The lowest value of h encountered in practice is about 5 W/(m^2 K) in the case of natural convection of gases, and the thermal conductivity of common insulating materials is about 0.05 W/(m^2 K). Therefore, the largest value of r_c likely to be encountered in practice becomes

$$r_{c,\text{largest}} = \frac{0.05}{5} = 0.01 \text{ m}$$

The critical radius would be much less in forced convection because of the much larger value of h. The value of $r_{c,\text{largest}}$ as given above serves a useful purpose in practice. We do not need to always calculate the critical insulation radius for its application in different situations provided we ensure that the outer radius of insulation is anything more than 0.01 m.

The radius of electric wires is usually smaller than the critical radius. Therefore, the plastic electrical insulation actually enhances the rate of heat transfer from electrical wires and keeps the steady operating temperature at a lower and safer value.

EXAMPLE 2.8 Consider a tube wall of inner and outer radii r_i and r_o whose temperatures are maintained at T_i and T_o, respectively. The thermal conductivity of the cylinder is temperature dependent and may be represented by an expression of the form $k = k_o(1 + at)$ where k_o and a are constants. Obtain an expression for the heat transfer per unit length of the tube.

Solution: We start with Eq. (2.28), i.e.

$$\int_{r_1}^{r_2} \frac{Q}{2\pi rL} dr = -\int_{T_1}^{T_2} kdT$$

Here

$$k = k_o(1 + aT)$$

Therefore,

$$\int_{r_1}^{r_2} \frac{Q}{2\pi rL} dr = -k_o \int_{T_1}^{T_2} (1 + aT)dT$$

or

$$\frac{Q}{2\pi L} \ln\left(\frac{r_2}{r_1}\right) = k_o(T_1 - T_2)\left[1 + a\left(\frac{T_1 + T_2}{2}\right)\right]$$

or

$$Q = 2\pi L k_m \frac{T_1 - T_2}{\ln(r_2/r_1)}$$

where $k_m = k\left(1 + a\dfrac{T_1 + T_2}{2}\right)$ is the thermal conductivity at mean temperature of $(T_1 + T_2)/2$.

EXAMPLE 2.9 A thin-walled copper tube of outside metal radius $r = 0.01$ m carries steam at 400 K. It is inside a room where the surrounding air temperature is 300 K. The tube is insulated with magnesia insulation of an approximate thermal conductivity of 0.07 W/(m K).

(a) What is the critical thickness of insulation for an external convective coefficient $h = 4.0$ W/(m^2 K)? (Assume negligible conduction resistance due to the wall of the copper tube.)

(b) Under these conditions, determine the rate of heat transfer per metre of tube length for

(i) a 0.002 m thick layer of insulation
(ii) the critical thickness of insulation
(iii) a 0.05 m thick layer of insulation.

Solution: (a) $r_c = \dfrac{k}{h} = \dfrac{0.07}{4.0} = 0.0175$ m = 17.5 mm

(b) For the solution of this part, we have to use Eq. (2.40), i.e.

$$Q = \frac{T_1 - T_\infty}{\dfrac{\ln(r_2/r_1)}{2\pi Lk} + \dfrac{1}{h(2\pi r_2 L)}}$$

Here, $T_1 = 400$ K, $T_\infty = 300$ K, $L = 1$ m

(i) $r_1 = 0.01$ m, $r_2 = 0.01 + 0.002 = 0.012$ m

Therefore,

$$Q = \frac{400 - 300}{\dfrac{\ln(0.012/0.01)}{2\pi \times 0.07} + \dfrac{1}{2\pi \times 4 \times 0.012}}$$

$$= 26.80 \text{ W/m}$$

(ii) $r_1 = 0.01$ m, $r_2 = 0.0175$ m

Therefore,

$$Q = \frac{400 - 300}{\dfrac{\ln(0.0175/0.01)}{2\pi \times 0.07} + \dfrac{1}{2\pi \times 4 \times 0.0175}}$$

$$= 28.20 \text{ W/m}$$

(iii) $r_1 = 0.01$ m, $r_2 = 0.01 + 0.05 = 0.06$ m

Therefore,

$$Q = \frac{400 - 300}{\dfrac{\ln(0.06/0.01)}{2\pi \times 0.07} + \dfrac{1}{2\pi \times 4 \times 0.06}}$$

$$= 21.11 \text{ W/m}$$

It is important to note that Q increases by 5·2% when the insulation thickness increases from 0·002 m to the critical value of 0·0175 m. Addition of insulation beyond the critical thickness decreases the value of Q (the heat loss).

EXAMPLE 2.10 A copper pipe having 35 mm OD and 30 mm ID carries liquid oxygen to the storage site of a space shuttle at –182°C and 0.06 m^3/min. The ambient air is at 20°C and has a dew point of 10°C. How much insulation with a thermal conductivity of 0.02 W/(m K) is needed to prevent condensation on the exterior of the insulation if the convective heat transfer coefficient h = 17 W/(m^2 K) on the outside? Take thermal conductivity of copper k_{Cu} = 400 W/(m K).

Solution: The physical system along with the thermal resistance network is shown in Figure 2.19.

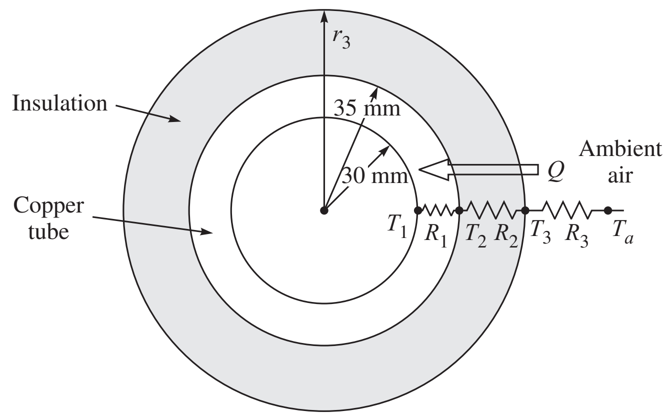


Figure 2.19 The pipe and its thermal resistance network (Example 2.10).

We can write

$$Q = \frac{T_a - T_1}{R_1 + R_2 + R_3} = \frac{T_a - T_3}{R_3}$$

or

$$\frac{R_1 + R_2 + R_3}{R_3} = \frac{T_a - T_1}{T_a - T_3} \tag{2.43}$$

Here, T_a = 20°C, T_1 = –182°C, T_3 = 10°C

$$R_1 \text{ (conduction resistance of copper pipe)} = \frac{\ln(0.035/0.03)}{2\pi L \times 400}$$

$$= \frac{3.85 \times 10^{-4}}{2\pi L} \text{ K/W}$$

$$R_2 \text{ (conduction resistance of insulating material)} = \frac{\ln(r_3/0.035)}{2\pi L \times 0.02}$$

$$= \frac{1}{2\pi L}[50\ln(r_3/0.035)] \text{ K/W}$$

$$R_3 \text{ (convective resistance at the outer surface)} = \frac{1}{2\pi L \times 17 \times r_3}$$

$$= \frac{1}{2\pi L}\left(\frac{0.06}{r_3}\right) \text{ K/W}$$

where r_3 is the outer radius of insulation in metre (Figure 2.19).

Substituting the values in Eq. (2.43), we have

$$1 + \frac{50\ln(r_3/0.035) + 3.85\times10^{-4}}{\dfrac{0.06}{r_3}} = \frac{20-(-182)}{20-10}$$

A rearrangement of the above equation gives

$$r_3 \ln r_3 + 3.35r_3 = 0.023$$

The equation is solved for r_3 by trial-and-error method which finally gives

$$r_3 = 0.054 \text{ m}$$

Therefore the thickness of insulation = 0.054 − 0.035
= 0.019 m = 19 mm

Cylinders with generation of thermal energy

Let us consider a long cylinder of radius r_o (Figure 2.20) and having a uniform thermal conductivity k. Thermal energy is generated uniformly within the cylinder whose surface is kept at a temperature T_w. Let q_G be the rate of generation of thermal energy per unit volume. We will determine the expression for the temperature distribution within the cylinder and show that the rate of heat loss from the surface equals the generation of thermal energy within the cylinder at steady state. This is a problem of one-dimensional heat conduction in cylindrical coordinates with generation of thermal energy.

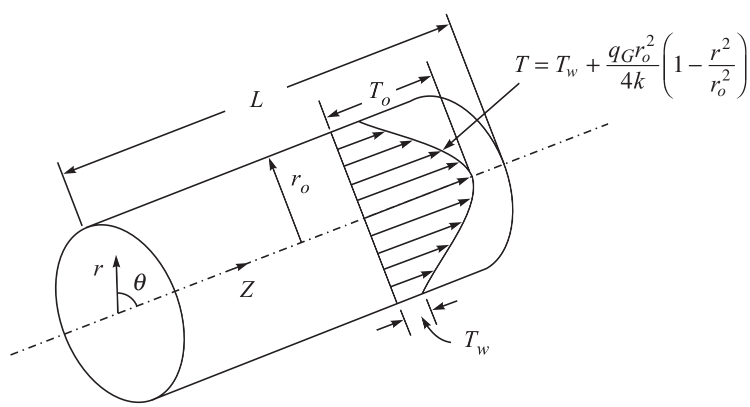


Figure 2.20 A long cylinder with internal generation of thermal energy.

The heat conduction equation in this case becomes

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + \frac{q_G}{k} = 0$$

or
$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{q_G}{k}r$$

Upon integration, we get

$$\frac{dT}{dr} = -\frac{q_G r}{2k} + \frac{C_1}{r}$$

or
$$T = \frac{q_G r^2}{4k} + C_1 \ln r + C_2$$

The constant C_1 has to be zero, otherwise the temperature field would have a singularity at the centre of the pipe, which is not physically possible. Moreover, at $r = 0$, $dT/dr = 0$ (condition of symmetry). The constant C_2 can be determined from the boundary condition, i.e.

$$\text{at } r = r_o \qquad T = T_w$$

which gives
$$C_2 = T_w + \frac{q_G r_o^2}{4k}$$

Therefore, the temperature distribution becomes

$$T = T_w + \frac{q_G r_o^2}{4k}\left(1 - \frac{r^2}{r_o^2}\right) \tag{2.44}$$

It is observed from Eq. (2.44) that the maximum temperature occurs at the centre of the cylinder ($r = r_o$). If the temperature at the centre be denoted by T_o, then we can write from Eq. (2.44)

$$T_o = T_w + \frac{q_G r_o^2}{4k} \tag{2.45}$$

With the help of Eqs. (2.44) and (2.45), the temperature distribution can also be written in another form as

$$\frac{T - T_w}{T_o - T_w} = \left(1 - \frac{r^2}{r_o^2}\right) \tag{2.46}$$

The temperature distribution, given by Eq. (2.44) is shown in Figure 2.20.

Let us calculate the heat loss from the surface of the cylinder.

The rate of heat transfer Q_r at any radial location can be written as

$$Q_r = -k(2\pi rL)\frac{dT}{dr}$$

Evaluating dT/dr from Eq. (2.44) and then substituting it in the above equation, we get

$$Q_r = \pi r^2 L q_G$$

Hence,
$$(Q_r)_{\text{at}\, r=r_o} = \pi r_o^2 L q_G \tag{2.47}$$

Equation (2.47) indicates that the rate at which the thermal energy is generated within the cylinder is the rate at which it is transferred out as heat from the surface of the cylinder—a requirement of steady state.

EXAMPLE 2.11 An electrical resistance wire 2·5 mm in diameter and 0.5 m long has a measured voltage drop of 25 V for a current flow of 40 A. The thermal conductivity of the wire material is 24 W/(m K). Determine (a) the rate of generation of thermal energy per unit volume within the wire and (b) the maximum temperature in the wire if the surface temperature is kept at 650 K.

Solution: (a) The rate of generation of thermal energy per unit volume is given by

$$q_G = \frac{25 \times 40 \times 4}{\pi (2.5)^2 \times 10^{-6} \times 0.5}$$

$$= 4.07 \times 10^8 \text{ W/m}^3$$

$$= 407 \text{ MW/m}^3$$

(b) Here we will use Eq. (2.45) for the temperature T_o at the centre of the wire as

$$T_o = T_w + \frac{q_G r_o^2}{4k}$$

$$= 650 + \frac{4.07 \times 10^8 \times (2.5)^2 \times 10^{-6}}{4 \times 24}$$

$$= 676.5 \text{ K}$$

Conduction in spherical geometries

Consider a spherical cell (Figure 2.21) of inner radius r_1 and outer radius r_2 and of a constant thermal conductivity k. The inner and outer surfaces of the cell are kept at constant temperatures of T_1 and T_2 ($T_1 > T_2$). Heat is conducted at a steady rate from the inner surface to the outer surface in the radial direction only. The Fourier's law of heat conduction can be written, under the situation, as

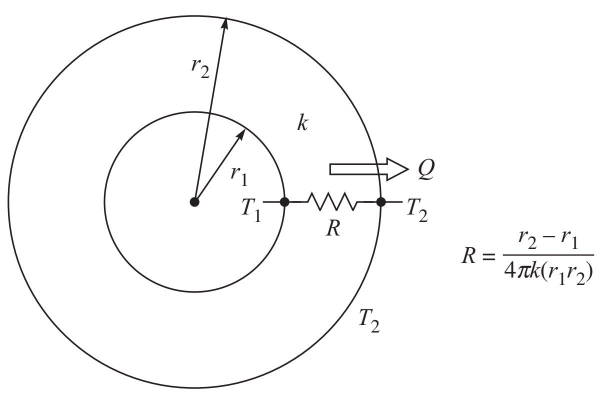


Figure 2.21 Heat conduction through a spherical cell.

$$Q = -kA\frac{dT}{dr}$$

$$Q = -k(4\pi r^2)\frac{dT}{dr}$$

or

$$\int_{r_1}^{r_2}\frac{Q}{4\pi r^2}\,dr = -\int_{T_1}^{T_2} kdT$$

At steady state, Q is constant, and hence it becomes

$$\frac{Q}{4\pi}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) = k(T_1 - T_2)$$

The equation can be rearranged as

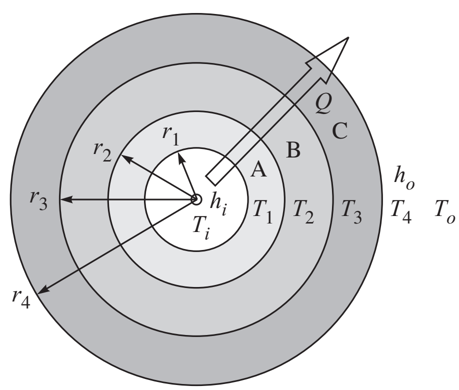
$$Q = \frac{T_1 - T_2}{\{(r_2 - r_1)/[4\pi k(r_1 r_2)]\}} \tag{2.48}$$

The denominator in Eq. (2.48) represents the conduction resistance R of the spherical cell (Figure 2.21).

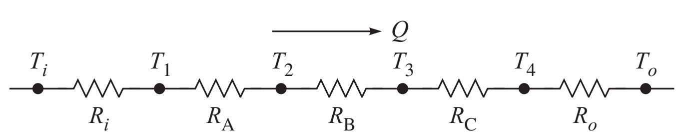
Multilayered spherical cell

For a multilayered spherical cell in consideration of convective heat transfer at inner and outer surfaces (Figure 2.22), we can write

$$Q = \frac{T_i - T_o}{R_{\text{total}}} \tag{2.49a}$$



(a) The physical system



(b) The thermal resistance network

Figure 2.22 One-dimensional heat flow through a three-layered spherical cell.

The total thermal resistance R_{total} is given by

$$R_{\text{total}} = R_i + R_A + R_B + R_C + R_o$$

where R_i, R_A, R_B, R_C, R_o are the convective resistance at the inner surface, conduction resistance of layer A, conduction resistance of layer B, conduction resistance of layer C, convective resistance at the outer surface respectively and are given by

$$R_i = \frac{1}{(4\pi r_1^2)h_i},\; R_A = \frac{r_2 - r_1}{4\pi k (r_1 r_2)},\; R_B = \frac{r_3 - r_2}{4\pi k (r_2 r_3)},\; R_C = \frac{r_4 - r_3}{4\pi k (r_3 r_4)},\; R_o = \frac{1}{(4\pi r_4^2)h_o} \quad (2.49b)$$

where h_i and h_o are the convective heat transfer coefficients at the inner and outer surfaces respectively.

Critical radius of insulation for a spherical wall

In a similar manner as done in the case of a cylindrical pipe, the critical radius of insulation of a spherical wall can be determined as

$$r_{c,\text{spherical wall}} = \frac{2k}{h}$$

where k is the thermal conductivity of the wall and h is the convective heat transfer coefficient of the surrounding medium. The proof of the above expression is left as an exercise to the student.

Spheres with generation of thermal energy

Let us consider a solid sphere of radius r_o and of thermal conductivity k with internal generation of thermal energy at the rate of q_G per unit volume. The outer surface is kept at a fixed temperature T_s. We have to find out the temperature distribution within the sphere. The heat conduction equation under the situation becomes

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} + \frac{q_G}{k} = 0$$

or

$$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = -\frac{q_G r^2}{k}$$

Integrating the above equation, we get

$$\frac{dT}{dr} = -\frac{q_G r}{3k} + \frac{C_1}{r^2}$$

or

$$T = -\frac{q_G r^2}{6k} - \frac{C_1}{r} + C_2$$

The boundary conditions are:

At $\quad r = 0, \quad \dfrac{dT}{dr} = 0$

At $\quad r = r_o, \quad T = T_s$

These give

$$T = T_s + \frac{q_G r_o^2}{6k}\left(1 - \frac{r^2}{r_o^2}\right) \tag{2.50}$$

It is observed that the maximum temperature occurs at the centre of the sphere ($r = r_o$). The rate of heat transfer at any radius r is given by

$$Q = -k(4\pi r^2)\frac{dT}{dr}$$

$$= \frac{4}{3}\pi r^3 q_G$$

Therefore, the rate of heat loss from the outer surface of the sphere becomes

$$Q_{\text{at}\, r=r_o} = \frac{4}{3}\pi r_o^3 q_G = \text{rate of thermal energy generation within the sphere}$$

EXAMPLE 2.12 A spherical tank of 5 m internal diameter, made of 25 mm thick stainless steel (k = 15 W/(m K)), is used to store iced water at 0°C. The tank is located in a room whose temperature is 20°C. The outer surface of the tank is black (emissivity $\varepsilon = 1$) and heat transfer between the outer surface of the tank and the surroundings is by natural convection and radiation. The convection heat transfer coefficients at the inner and the outer surfaces of the tank are 80 W/(m^2 K) and 10 W/(m^2 K) respectively. Determine (a) the rate of heat transfer to the iced water in the tank and (b) the amount of ice at 0°C that melts during a 24-hour period. The heat of fusion of ice at atmospheric pressure is Δh_f = 334 kJ/kg. The Stefan–Boltzmann constant $\sigma = 5.67 \times 10^{-8}$ W/m^2.

Solution: The thermal resistance network is shown in Figure 2.23.

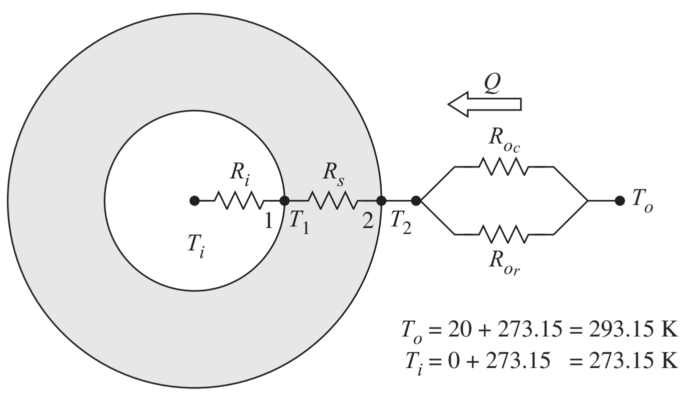


Figure 2.23 The spherical tank and its thermal resistance network (Example 2.12).

(a) The inner and outer surface areas of the tank are

$$A_1 = \pi(5)^2 = 78.54 \text{ m}^2$$
$$A_2 = \pi(5.05)^2 = 80.12 \text{ m}^2$$

The individual thermal resistances can be determined as

R_i (convective resistance at the inner surface) $= \dfrac{1}{80 \times 78.54} = 0.000159$ K/W

R_s (conduction resistance of the tank) $= \dfrac{5.05 - 5}{2 \times 15 \times \pi \times 5.05 \times 5} = 0.000021$ K/W

R_{o_c} (convective resistance at the outer surface) $= \dfrac{1}{10 \times 80.12} = 0.00125$ K/W

R_{o_r} (radiative resistance at the outer sufrace) $= \dfrac{1}{80.12 \times h_r}$

The radiative heat transfer coefficient h_r is determined following Eq. (1.20) with the value of emissivity $\varepsilon = 1$ as

$$h_r = 5.67 \times 10^{-8}(T_2^2 + 293.15^2)(T_2 + 293.15) \tag{2.51}$$

But we do not know the outer surface temperature T_2 of the tank and hence we cannot determine the value of h_r.

Therefore, we adopt an iterative procedure as follows:

- Assume a value of T_2 and calculate the rate of heat transfer Q.
- Calculate T_2 from the value of Q obtained.
- Compare the new value of T_2 with the earlier one (assumed at the first step).
- Repeat the calculation if necessary using the updated value of T_2.

The value of T_2 must be closer to 0°C since the heat transfer coefficient inside the tank is much larger. We assume T_2 = 4°C = 277.15 K. Putting the value in Eq (2.51), we have

$$h_r = 5.26 \text{ W/(m}^2\text{ K)}$$

Therefore, $$R_{o_r} = \frac{1}{80.12 \times 5.26} = 0.00237 \text{ K/W}$$

The two parallel resistances R_{o_c} and R_{o_r} (Figure 2.23) can be replaced by an equivalent resistance R_o as

$$\frac{1}{R_o} = \frac{1}{R_{o_c}} + \frac{1}{R_{o_r}} = 10 \times 80.12 + 5.26 \times 80.12$$

which gives $$R_o = 0.000818 \text{ K/W}$$

Now the resistances R_i, R_s and R_o are in series, and hence the total resistance becomes

$$\begin{aligned} R_{\text{total}} &= R_i + R_s + R_o \\ &= 0.000159 + 0.000021 + 0.000818 \\ &= 0.001 \text{ K/W} \end{aligned}$$

The rate of heat transfer, $Q = \dfrac{T_o - T_i}{R_{\text{total}}}$

$$= \frac{20 - 0}{0.001}$$

$$= 20 \times 10^3 \text{ W}$$

The outer surface temperature T_2 is now calculated as

$$T_2 = T_o - QR_o$$

$$= 20 - 20 \times 10^3 \times 0.000818$$

$$= 3.64°\text{C}$$

which is sufficiently close to the assumed value of 4°C. Therefore, there is no need for any further iteration.

(b) The total amount of heat transfer during a 24-hour period is $= 20 \times 10^3 \times 24 \times 3600$

$$= 1.73 \times 10^9 \text{ J}$$

$$= 1.73 \times 10^6 \text{ kJ}$$

Therefore, the amount of ice which will melt during a 24-hour period is

$$m_{\text{ice}} = \frac{1.73 \times 10^6}{334} = 5180 \text{ kg}$$

2.3 HEAT TRANSFER FROM EXTENDED SURFACE

An extended surface is employed in practice to cause a combined conduction and convection heat transfer from the surface to the surroundings. While conduction takes place within the solid material, the convection heat transfer takes place between its boundary surfaces and the surroundings. The system enhances the total heat transfer rate between the solid surface and the surrounding fluid. Such an extended surface is usually termed *fin*. To understand the physical phenomenon, let us consider a plane wall with a built-in extended surface or fin as shown in Figure 2.24. Heat is transferred by conduction through the fin along its length, i.e. in the direction

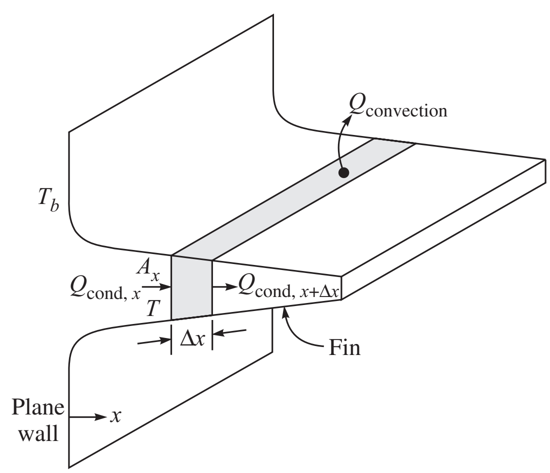


Figure 2.24 A plane wall with a built-in extended surface.

of x, and at the same time, heat is transferred by convection from the surface of the fin to the adjacent fluid. Therefore, we see that the fin provides an increased area to convective heat transfer so that the rate of heat transfer from the wall is increased. A variety of fin configurations is available in market, suitable for different purposes. Some of them are shown in Figure 2.25.

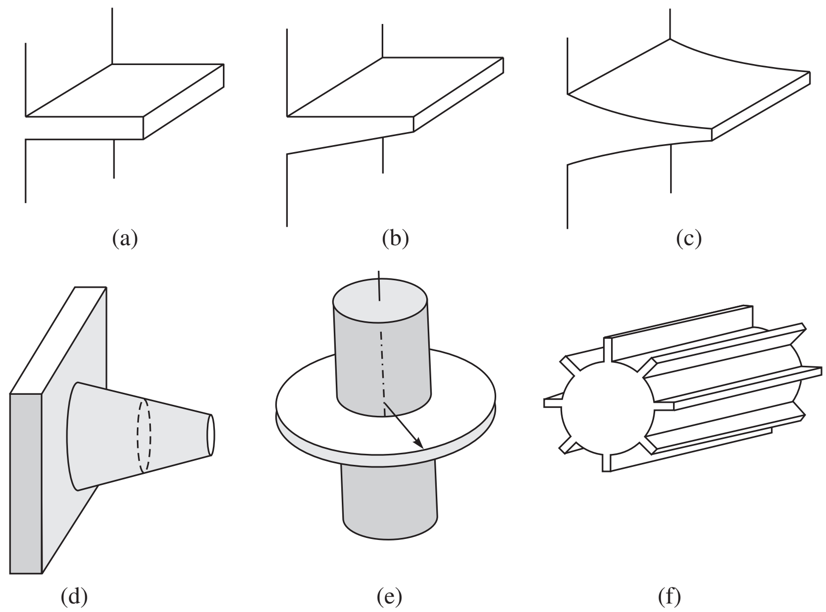


Figure 2.25 Fin configurations: (a) straight fin of uniform cross-section, (b) straight fin of trapezoidal profile with non-uniform rectangular cross-section, (c) fin of parabolic profile with non-uniform cross-section, (d) fin with non-uniform circular cross-section (pin fin), (e) annular fin, and (f) longitudinal fins in cylindrical tube.

The next task is to determine the temperature distribution along the fin which is strongly dependent on thermal conductivity of the fin material and also on the convective heat transfer coefficient of the adjacent fluid. The temperature distribution in the fin influences the degree to which the heat transfer rate is enhanced. Ideally, the fin material should have a large thermal conductivity to minimize temperature variations from its base to its tip. In the limit of infinite thermal conductivity, the entire fin would be at the temperature of the base surface (the plane wall in Figure 2.24) thereby providing the maximum possible enhancement in heat transfer.

Governing equation for temperature distribution in a fin

The analysis is made for a one-dimensional steady-state heat transfer through a fin (Figure 2.24) of constant thermal conductivity and without generation of thermal energy within the fin. Consider a volume element (Figure 2.24) having a length of Δx, cross-sectional area of A_x and a perimeter of P. The energy balance on the volume element can be expressed as

$$\begin{bmatrix}\text{Rate of heat conduction}\\ \text{to the element at } x\end{bmatrix} = \begin{bmatrix}\text{Rate of heat conduction}\\ \text{from the element at } x+\Delta x\end{bmatrix} + \begin{bmatrix}\text{Rate of heat convection}\\ \text{from the element}\end{bmatrix}$$

or
$$Q_{\text{cond},x} = Q_{\text{cond},x+\Delta x} + Q_{\text{conv}} \tag{2.52}$$

We can write according to Fourier's law of heat conduction (Eq. 1.1),

$$Q_{\text{cond},x} = -kA_x \frac{dT}{dx} \tag{2.53}$$

Therefore,
$$Q_{\text{cond},x+\Delta x} = Q_x + \frac{d}{dx}(Q_{\text{cond},x})\Delta \text{x}$$

$$= -kA_x \frac{dT}{dx} - k\frac{d}{dx}\left(A_x \frac{dT}{dx}\right)\Delta x \tag{2.54}$$

The rate of heat transfer by convection can be written as

$$Q_{\text{conv}} = hP\Delta x\,(T - T_\infty) \tag{2.55}$$

where T is the temperature of the volume element at location x, T_∞ is the temperature of the surrounding fluid and h is the average convective heat transfer coefficient over the surface of the fin.

Substituting Eqs. (2.53), (2.54) and (2.55) in Eq. (2.52), we can write, after some rearrangements

$$\frac{d^2T}{dx^2} + \left(\frac{1}{A_x}\frac{dA_x}{dx}\right)\frac{dT}{dx} - \frac{hP}{kA_x}(T - T_\infty) = 0 \tag{2.56}$$

Equation (2.56) is the governing differential equation for temperature distribution in the fin. In a special case of constant cross-sectional area, Eq. (2.56) becomes

$$\frac{d^2T}{dx^2} - \frac{hP}{kA}(T - T_\infty) = 0 \tag{2.57}$$

[A_x is replaced by A, the constant cross-sectional area]

Substituting $\theta = (T - T_\infty)$ and $m = \sqrt{hP/kA}$ in Eq. (2.57), we have

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \tag{2.58}$$

Equation (2.58) is a linear, homogeneous, second order, ordinary differential equation. The general solution of the equation can be written as

$$\theta = C_1 e^{mx} + C_2 e^{-mx} \tag{2.59}$$

The constants C_1 and C_2 are found out from the boundary conditions which depend upon the physical situation.

Boundary condition at fin base ($x = 0$). The temperature at the fin base T_b is usually specified as a known parameter. Therefore, we can write

$$\theta_{\text{at}\,x=0} = \theta_b = T_b - T_\infty \tag{2.60}$$

Boundary condition at fin tip. At fin tip, there are several possibilities as follows:

(a) Infinitely long fin: if the fin is infinitely long, we can assume $T_{\text{fin tip}} = T_\infty$

Therefore, $$\theta = 0 \text{ as } x \rightarrow \infty \tag{2.61}$$

(b) Fin of finite length with insulated tip:

At $x = L$ (length of the fin), $$\frac{d\theta}{dx} = 0 \tag{2.62}$$

(c) Fin of finite length with convective heat loss at the tip:

At $x = L$, $$-k\frac{d\theta}{dx} = h\theta \tag{2.63}$$

(d) Fixed temperature at the fin tip:

At $x = L$ (length of the fin), $$\theta = \theta_L \tag{2.64}$$

where, $\theta_L = T_L - T_\infty$, and T_L is the prescribed temperature at fin tip

Therefore, we see that the explicit expression for the temperature distribution in the fin depends upon the physical situation at the fin tip as stated above. This is discussed below in detail.

Infinitely long fin

The boundary conditions given by Eqs. (2.60) and (2.61) determine C_1 and C_2 from Eq. (2.59) as

$$C_1 = 0,\ C_2 = \theta_b$$

Therefore, we can write from Eq. (2.59)

$$\frac{\theta}{\theta_b} = e^{-mx}$$

or

$$\frac{T - T_\infty}{T_b - T_\infty} = e^{-mx} \tag{2.65}$$

The fin temperature under the situation, decreases exponentially along the length of the fin. This is shown in Figure 2.26(a).

The heat transfer rate Q is given by

$$\begin{aligned} Q &= -kA\left(\frac{dT}{dx}\right)_{x=0} \\ &= kAm\theta_b \\ &= \sqrt{hPkA}\,(T_b - T_\infty) \qquad (\because m = \sqrt{hP/kA}) \end{aligned} \tag{2.66}$$

Fin with insulated tip

The boundary conditions given by Eqs. (2.60) and (2.62) as applied to Eq. (2.59) give

$$C_1 = \frac{e^{-mL}}{e^{mL} + e^{-mL}}\theta_b$$

$$C_2 = \frac{e^{mL}}{e^{mL} + e^{-mL}}\theta_b$$

With the values of C_1 and C_2, Eq. (2.59) can be written as

$$\frac{\theta}{\theta_b} = \frac{\cosh[m(L-x)]}{\cosh(mL)}$$

or

$$\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh[m(L-x)]}{\cosh(mL)} \tag{2.67}$$

The temperature distribution is shown in Figure 2.26(b).

The rate of heat transfer can be determined as

$$\begin{aligned} Q &= -kA\left(\frac{dT}{dx}\right)_{x=0} \\ &= kAm(T_b - T_\infty)\tanh(mL) \\ &= \sqrt{hPkA}\,\theta_b \tanh(mL) \end{aligned} \tag{2.68}$$

Convective heat transfer from the fin tip

By making use of the boundary conditions given by Eqs. (2.60) and (2.63), we get from Eq. (2.59)

$$\theta_b = C_1 + C_2$$

$$h(C_1 e^{mL} + C_2 e^{-mL}) = km(C_2 e^{-mL} - C_1 e^{mL})$$

Solving for C_1 and C_2 from the above two equations, it can be shown from Eq. (2.59), after some algebraic manipulations, that

$$\frac{\theta}{\theta_b} = \frac{\cosh\left[m(L-x)\right] + (h/mk)\sinh\left[m(L-x)\right]}{\cosh(mL) + (h/mk)\sinh(mL)} \tag{2.69}$$

The temperature distribution given by Eq. (2.69) is shown in Figure 2.26(c). The magnitude of the temperature gradient decreases along the length x of the fin. This trend is a consequence of the reduction in the conduction heat transfer with increasing x due to continuous convection losses from the fin surface.

The rate of heat loss from the fin can be written as

$$Q = -kA\left(\frac{dT}{dx}\right)_{x=0} = -kA\left(\frac{d\theta}{dx}\right)_{x=0} \tag{2.70}$$

With the help of Eq. (2.69), Eq. (2.70) can be written as

$$Q = \sqrt{hPkA}\,\theta_b \frac{\sinh(mL) + (h/mk)\cosh(mL)}{\cosh(mL) + (h/mk)\sinh(mL)} \tag{2.71}$$

Fixed temperature at the fin tip

The situation arises when the two ends of a fin are attached to two surfaces kept at fixed temperatures. We can write from Eq. (2.59), with the help of Eqs. (2.60) and (2.64) prescribing the boundary conditions, as

$$\theta_b = C_1 + C_2$$
$$\theta_L = C_1 e^{mL} + C_2 e^{-mL}$$

Solving for C_1 and C_2 from the above two equations and after some algebraic manipulations, it can be shown that

$$\frac{\theta}{\theta_b} = \frac{(\theta_L/\theta_b)\sinh(mx) + \sinh[m(L-x)]}{\sinh(mL)} \tag{2.72}$$

The temperature distribution given by Eq. (2.72) is shown in Figure 2.26(d).

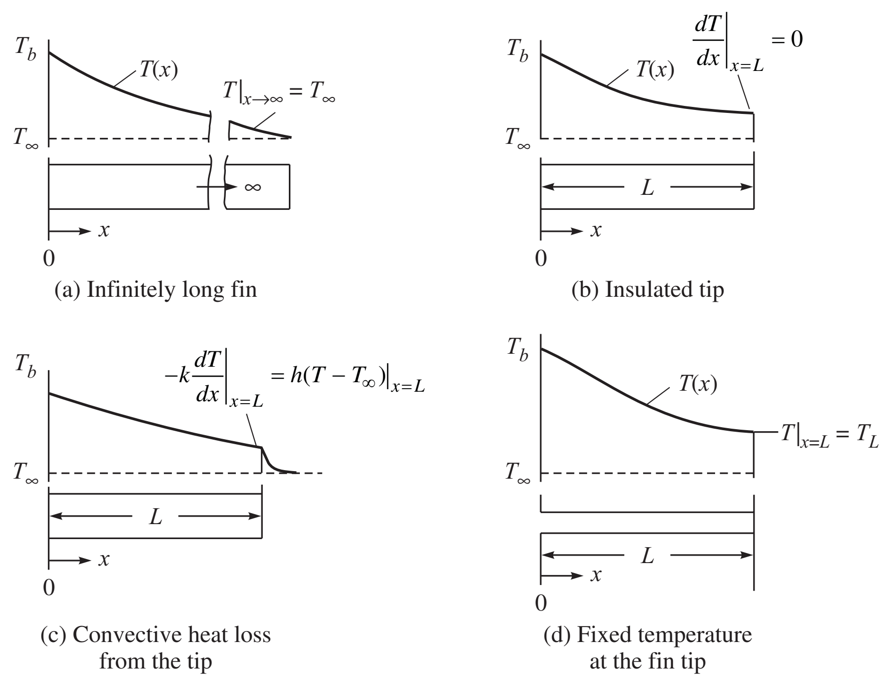


Figure 2.26 Temperature distribution along the length of a fin for four boundary conditions at the fin tip.

The rate of heat loss from the fin can be determined by making use of Eq. (2.70) with the help of Eq. (2.72) as

$$Q = \sqrt{hPkA}\ \theta_b \frac{\cosh(mL) - (\theta_L/\theta_b)}{\sinh(mL)} \tag{2.73}$$

The heat transfer characteristics of fins of constant cross-sectional area but with different tip conditions, as described above, are summarized in Table 2.1.

Table 2.1 Expressions of temperature distribution and heat loss for fins of uniform cross-section

Case	*Tip condition* ($x = L$)	*Temperature distribution* (θ/θ_b)	*Rate of heat loss*
(a)	Infinitely long fin ($L \to \infty$)	e^{-mx}	$\sqrt{hPkA}\,\theta_b$
(b)	Adiabatic $(d\theta/dx)_{x=L} = 0$	$\dfrac{\cosh[(m(L-x)]}{\cosh(mL)}$	$\sqrt{hPkA}\,\theta_b \tanh(ml)$
(c)	Convective heat loss $h\theta_L = -k(d\theta/dx)_{x=L}$	$\dfrac{\cosh[m(L-x)] + (h/mk)\sinh[m(L-x)]}{\cosh(mL) + (h/mk)\sinh(mL)}$	$\sqrt{hPkA}\,\theta_b \dfrac{\sinh(mL)+(h/mk)\cosh(mL)}{\cosh(mL)+(h/mk)\sinh(mL)}$
(d)	Prescribed temperature $\theta_{x=L} = \theta_L$	$\dfrac{(\theta_L/\theta_b)\sinh(mx) + \sinh[m(L-x)]}{\sinh(mL)}$	$\sqrt{hPkA}\,\theta_b \dfrac{\cosh(mL) - (\theta_L/\theta_b)}{\sinh(mL)}$

Annular fin of uniform thickness

Consider the annular fin shown in Figure 2.27. If there is no circumferential variation in temperature and if the thickness t is small compared to the dimension ($r_1 - r_2$), the temperature can be assumed to be a function of r only.

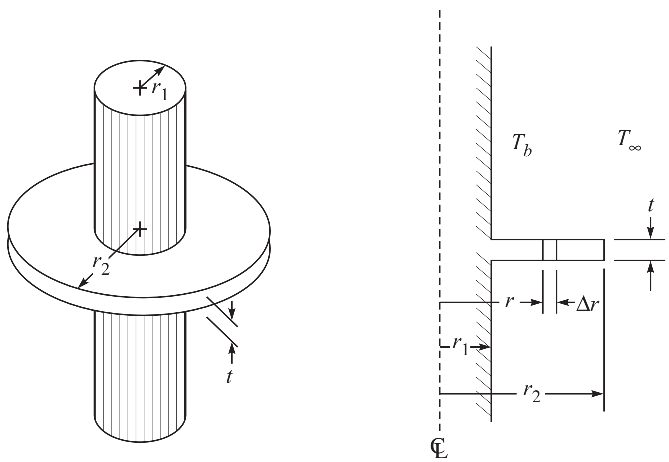


Figure 2.27 An annular fin

Here, the cross-sectional area $A = 2\pi rt$; the perimeter, $P = 2(2\pi r) = 4\pi r$. Therefore, we can write, following Eq. (2.56),

$$\frac{d^2\theta}{dr^2} + \frac{1}{2\pi rt}\frac{d}{dr}(2\pi rt)\frac{d\theta}{dr} - \frac{h(4\pi r)}{k2\pi rt}\theta = 0$$

or

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - \frac{2h}{kt}\theta = 0 \tag{2.74}$$

This is a form of Bessel's differential equation of zero order which has the general solution

$$\theta = C_1 I_0(mr) + C_2 k_0(mr) \tag{2.75}$$

where

$m = \sqrt{2h/kt}$

I_0 = modified Bessel function of the first kind

k_0 = modified Bessel function of the second kind

The constants C_1 and C_2 are determined from the boundary conditions, which are

$$\theta(r_1) = T_b - T_\infty = \theta_b$$

$$\left.\frac{d\theta}{dr}\right|_{r=r_2} = 0$$

where T_b is the temperature of the cylindrical surface to which the fin is attached. The second boundary condition implies no heat loss from the end of the fin.

With C_1 and C_2 evaluated, Eq. (2.75) becomes

$$\frac{\theta}{\theta_b} = \frac{I_0(mr)\,k_1(mr_2) + k_0(mr)\,I_1(mr_2)}{I_0(mr_1)\,k_1(mr_2) + k_0(mr_1)\,I_1(mr_2)}$$

The heat loss from the fin is determined as

$$Q = -k(2\pi r_1 t)\left.\frac{d\theta}{dr}\right|_{r=r_1}$$

which gives

$$Q = 2\pi k\theta_b(mr_1)\frac{k_1(mr_1)\,I_1(mr_2) - I_1(mr_1)\,k_1(mr_2)}{I_0(mr_1)\,k_1(mr_2) + k_0(mr_1)\,I_1(mr_2)}$$

Efficiency of a fin: fin selection and design

Before characterizing the performanc of a fin, we should have a clear physical understanding of how does a fin enhance the heat transfer rate from a base surface to which the fin is attached. When a fin is attached to a surface (Figure 2.24), it is obvious that it provides a conduction resistance over that portion of the base surface to which it is attached. But at the same time, it provides an additional surface area for convective heat transfer to the surrounding fluid. Even though the temperatures at different locations along the length of this connective surface are lower than the base temperature ($T < T_b$) due to conduction through the fin, the total convective heat transfer from the surface counterweighs the adverse effect of conduction resistance and finally enhances the rate of heat transfer from the base surface.

The fin efficiency η_f is defined as

$$\eta_f = \frac{\text{actual heat transfer rate from the base surface through the fin}}{\text{heat transfer rate that could have been obtained if the entire fin were at the base temperature}}$$

In case of a fin with infinitely high thermal conductivity, the entire fin will be at the base temperature and the fin efficiency η_f will be equal to its maximum value of unity.

For a rectangular straight fin of uniform cross-sectional area (Figure 2.25(a)) and with insulated tip, the actual heat transfer rate is given by Eq. (2.68). Therefore,

$$\eta_f = \frac{\sqrt{hPkA}}{hPL\theta_b}\theta_b \tanh(mL)$$

$$= \frac{\tanh(mL)}{mL}$$

$$= \frac{\tanh\sqrt{hPL^2/kA}}{\sqrt{hPL^2/kA}} \tag{2.76}$$

If a rectangular fin is long, wide and thin, $P/A \approx 2/t$, the heat loss from the end can be taken into account approximately by increasing L by $t/2$ and assuming that the end is insulated. This approximation keeps the surface area from which heat is lost the same as in the real case and the fin efficiency then becomes

$$\eta_f = \frac{\tanh\sqrt{2hL_c^2/kt}}{\sqrt{2hL_c^2/kt}}$$

where $$L_c = (L + t/2)$$

The error that results from this approximation will be less than 8% when

$$\left(\frac{ht}{2k}\right)^{1/2} \leq \frac{1}{2}$$

Figure 2.28 shows a comparison of the efficiencies of triangular and rectangular fins. The profile area is represented by A_m. In case of a rectangular fin, $A_m = Lt$. while for triangular fin, $A_m = Lt/2$.

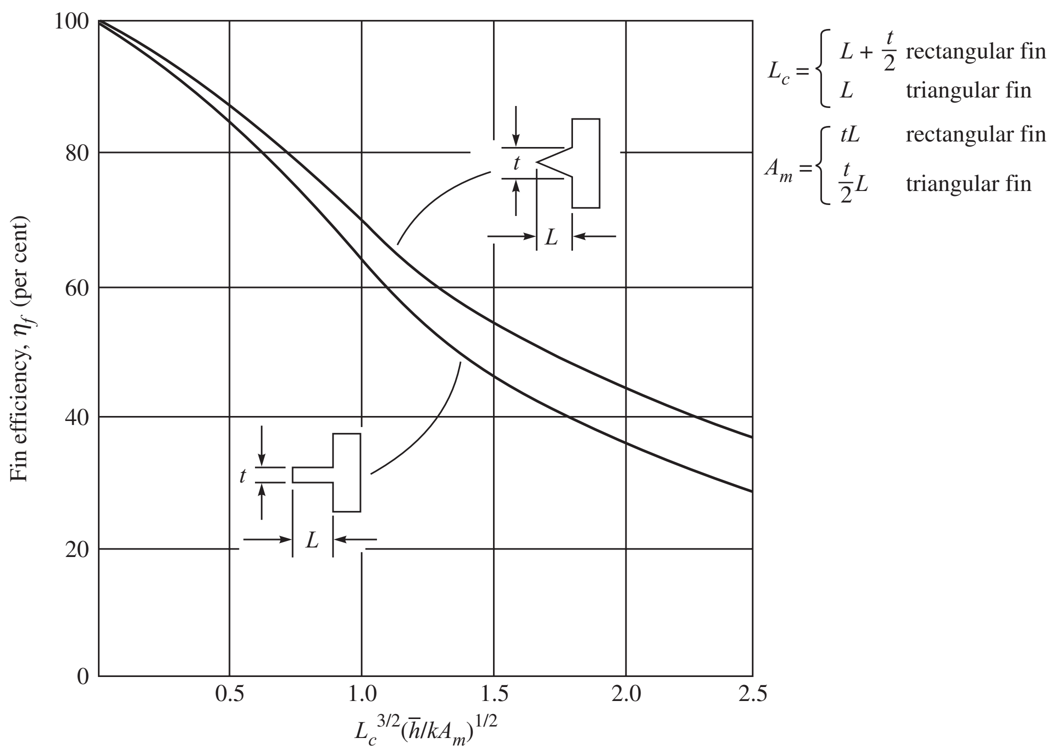


Figure 2.28 Efficiencies of triangular and rectangular fins.

The efficiency η_f, as discussed above, characterizes the performance of a single fin. The efficiency of an array of fins attached to a base surface as shown in Figure 2.29 is characterized by a parameter η_0 defined as

$$\eta_0 = \frac{q_t \text{ (the total heat transfer rate from the base surface)}}{q_{t(\max)} \text{ (the total heat transfer rate from the base surface if the entire length of all the fins is at the base temperature)}}$$

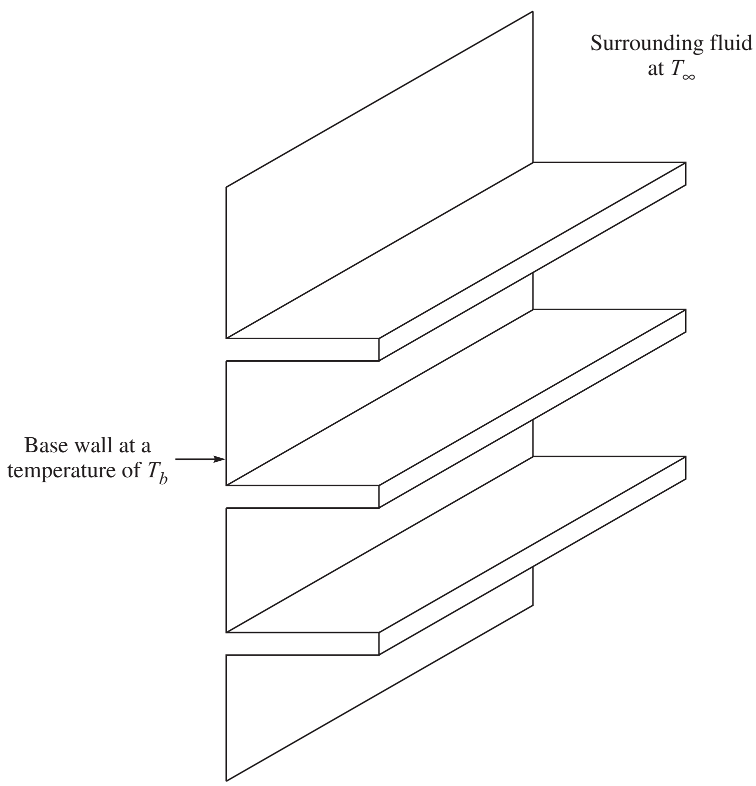


Figure 2.29 A representative array of fins.

The total heat transfer rate q_t from the base surface accounts for both the heat transfer rate through the fins and that from the unfinned portion of the base surface. We can thus write

$$q_t = \eta_f h A_f \theta_b + h(A_0 - A_f)\theta_b$$

and

$$q_{t(\max)} = hA_0\theta_b$$

where

η_f = efficiency of the individual fins
h = convective heat transfer coefficient
A_f = heat transfer area of all the fins
A_0 = total heat transfer area (A_f + area of the unfinned portion of the base surface)
θ_b = $T_b - T_0$ (T_b and T_0 are the temperatures of the base surface and the surrounding fluid respectively).

Therefore, it becomes

$$\eta_0 = \frac{\eta_f hA_f \theta_b + h(A_0 - A_f)\theta_b}{hA_0\theta_b}$$

$$= 1 - \frac{A_f}{A_0}(1 - \eta_f) \tag{2.77}$$

When $\eta_f = 1,\ \eta_0 = 1$

The various shapes and forms of fins usually fabricated in practice are shown in Figure 2.25. The geometry of a fin is determined mainly from the criterion of a high fin efficiency. However, the determination of the most desirable shape and size of a fin, in practice, pertains to a problem of optimization among the heat transfer characteristic, pressure drop of the fluid flowing through the channels of fins, the weight, the cost and the available space.

Effectiveness of a fin in justifying its use

It has already been made clear that the incorporation of a fin on a base surface enhances the rate of heat transfer by providing an additional surface area for convective heat transfer. But at the same time it puts an additional conduction resistance to the portion of the base surface to which the fin is attached. Now the question arises that, is it always economically justified to incorporate fins? The answer to this question lies in the fact that the gain in the rate of heat transfer by incorporating fins decreases with an increase in the convective heat transfer coefficient of the surrounding fluid. When the surrounding medium is a liquid in forced convection, the value of h is high and the enhancement in heat transfer by fins is quite low. At still further higher values of h in the case of boiling liquids or condensing vapour as the medium, fins produce a marginal or almost negligible increase in the rate of heat transfer from the base surface. Therefore, the incorporation of fins is not recommended under these situations from an economic point of view. A valid method of evaluating the performance of a fin in this regard is to compare the heat transfer obtained with the fin to that which would be obtained without the fin. The ratio of these two quantities is defined as the effectiveness $\eta_{\text{effectiveness}}$ of a fin, i.e.

$$\eta_{\text{effectveness}} = \frac{Q_{\text{with fin}}}{Q_{\text{without fin}}} = \frac{\eta_f A_f h\theta_0}{hA_b\theta_0} = \eta_f \frac{A_f}{A_b} \tag{2.78}$$

where A_f is the total surface area of the fin and A_b is the base area.

EXAMPLE 2.13 A very long, 10 mm diameter copper rod (k = 370 W/(m K)) is exposed to an environment at 20°C. The base temperature of the rod is maintained at 120°C. The heat transfer coefficient between the rod and the surrounding air is 10 W/(m^2 K). (a) Determine the heat transfer rate for finite lengths, 0.02, 0.04, 0.08, 0.2, 0.4, 0.8, 1 and 10 metres assuming heat loss at the end, and (b) compare the results with that of an infinitely long fin whose tip temperature equals the environment temperature of 20°C.

Solution: (a) The rate of heat transfer for all finite lengths will be given by Eq. (2.71). Here

$$\frac{P}{A} = \frac{4\pi(0.01)}{\pi(0.01)^2} = 400\ \text{m}^{-1}$$

$$m = \sqrt{\frac{10 \times 400}{370}} = 3.288 \text{ m}^{-1}$$

$$\frac{h}{mk} = \frac{10}{3.288 \times 370} = 0.0082$$

$$\sqrt{hPkA} = \sqrt{\frac{10 \times \pi (0.01) \times 370 \times \pi \times (0.01)^2}{4}}$$

$$= 0.0955 \text{ W/K}$$

$$\theta_b = 120 - 20 = 100°\text{C}$$

Equation (2.71) can be written in a different form as

$$Q = \sqrt{hPkA}\ \theta_b \frac{(h/mk) + \tanh(mL)}{1 + (h/mk)\tanh(mL)}$$

For $L = 0.02$ m

$$Q = 0.0955 \times 100 \frac{0.0082 + \tanh(3.288 \times 0.02)}{1 + 0.0082 \times \tanh(3.288 \times 0.02)}$$

$$= 0.704 \text{ W}$$

Repeating the calculations for lengths of 0.04, 0.08, 0.2, 0.4, 0.8, 1 and 10 m, we obtain the results which are shown below:

x(m)	Q(W)
0·02	0·704
0·04	1·336
0·08	2·530
0·20	5·561
0·40	8·286
0·80	9·450
1·00	9·524
10·00	9·550

(b) For an infinitely long rod, we use Eq. (2.66)

Therefore,

$$Q_{L \to \infty} = \sqrt{hPkA}\,\theta_b$$

$$= 0.0955 \times 100$$

$$= 9.55$$

We observe that since k is large, there are significant differences between the finite length and the infinite length cases. However, when the length of the rod approaches 1 m, the result becomes almost the same with that of the infinite length.

EXAMPLE 2.14 Consider two very long, slender rods of the same diameter but of different materials. One end of each rod is attached to a base surface maintained at 100°C, while the surfaces of the rods are exposed to ambient air at 20°C. By traversing the length of each rod with a thermocouple, it was observed that the temperatures of the rods were equal at the positions $x_A = 0.15$ m and $x_B = 0.075$ m, where x is measured from the base surface. If the thermal conductivity of rod A is known to be $k_A = 72$ W/(m K), determine the value of k_B for the rod B.

Solution: In the case of a very long slender rod we use the tip boundary condition $\theta_L = 0$ as $L \to \infty$ and accordingly we use Eq. (2.65) for the temperature distribution.

Therefore, we can write for the locations where the temperatures of both the fins are equal as

$$\theta_b e^{-m_A x_A} = \theta_b e^{-m_B x_B}$$

or

$$\frac{x_A}{x_B} = \frac{m_B}{m_A} \tag{2.79}$$

Again,

$$\frac{m_B}{m_A} = \frac{\sqrt{hP/k_B A}}{\sqrt{hP/k_A A}} = \sqrt{\frac{k_A}{k_B}} \tag{2.80}$$

From Eqs. (2.79) and (2.80), we can write

$$k_B = k_A \left(\frac{x_B}{x_A}\right)^2$$

$$= 72\left(\frac{0.075}{0.15}\right)^2$$

$$= 18 \text{ W/(m K)}$$

EXAMPLE 2.15 In a specific application, a stack (Figure 2.30) that is 300 mm wide and 100 mm deep contains 60 fins each of length L = 12 mm. The entire stack is made of aluminium which is everywhere 1.0 mm thick. The temperature limitations associated with electrical components joined to opposite plates dictate the maximum allowable plate temperature of T_b = 400 K and T_L = 350 K. Determine the rate of heat loss from the plate at 400 K, given h = 150 W/(m^2 K) and T_∞ = 300 K. Take $k_{\text{aluminium}}$ = 230 W/(m K).

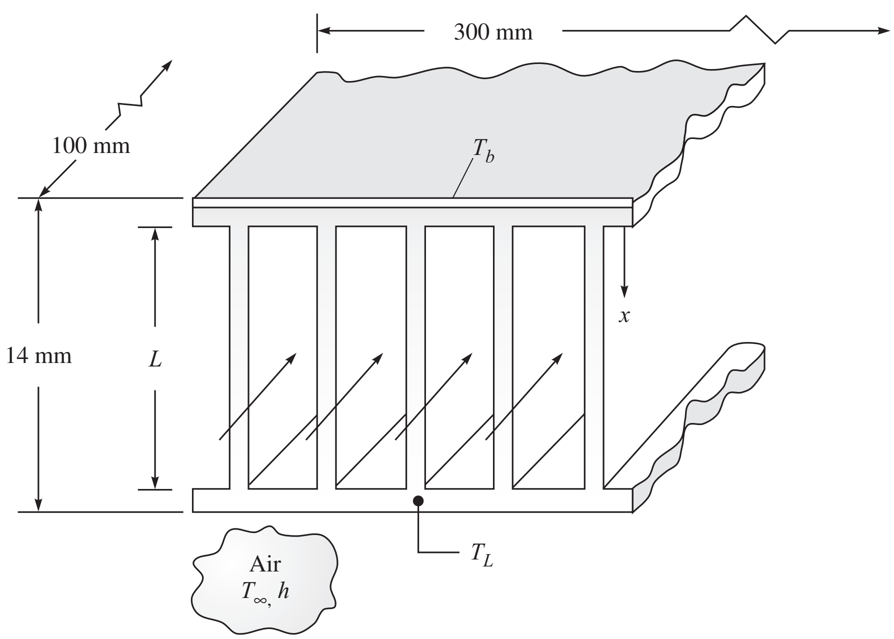


Figure 2.30 A stack containing fins as explained in Example 12.15.

Solution: Here, both the ends of the fins are at fixed temperatures. Therefore, we use Eq. (2.73) to find out the heat loss from the plates through the fins.

Let
$$M = \sqrt{hPkA}$$

and
$$m = \sqrt{\frac{hP}{kA}}$$

$$\theta_b = T_b - T_\infty$$

$$\theta_L = T_L - T_\infty$$

From the given data,

$$P \text{ (perimeter of each fin)} = 2(0.1 + 0.001) = 0{\cdot}2 \text{ m}$$

$$A \text{ (cross-sectional area of each fin)} = 0.1 \times 0.001 = 1 \times 10^{-4} \text{ m}^2$$

Therefore,

$$M = \sqrt{150 \times 0.2 \times 230 \times 10^{-4}} = 0.83$$

$$m = \sqrt{\frac{150 \times 0.2}{230 \times 10^{-4}}} = 36.12$$

$$\theta_b = 400 - 300 = 100 \text{ K}$$

$$\theta_L = 350 - 300 = 50 \text{ K}$$

Now, we get from Eq. (2.73)

$$Q_b \text{ (heat loss from the plate at 400 K)} = \underbrace{60 \times 0.83 \frac{\{100 \times \cosh(36.12 \times 0.012) - 50\}}{\sinh(36.12 \times 0.012)}}_{\text{heat loss from the fins}}$$

$$+ \underbrace{\{(0.1) \times (0.3) - 50 \times 10^{-4}\} \times 100 \times 150}_{\text{heat loss from the unfinned portion of the plate}} + \underbrace{(0.1)(0.3) \times 150 \times 100}_{\text{heat loss from the outer surface of the plate}}$$

$$= 6631 + 375 + 450$$

$$= 7456 \text{ W}$$

$$= 7.456 \text{ kW}$$

We observe that though the fins occupy a relatively less portion of the base surface (surface at 400 K), the heat loss from this portion through the fins is relatively much larger because of a large amount of convective heat loss from the exposed surfaces of the fin.

SUMMARY

- Heat conduction in a medium is said to be steady when the temperature at any location does not vary with time. Heat conduction in a medium is said to be one-dimensional when conduction is significant in only one direction of the coordinate axes and negligible in other two directions. When conduction is significant in all three directions, it is termed *three-dimensional conduction*.

- In heat conduction, the functional relationship of temperature with time and space coordinates in the form of a differential equation is known as *heat conduction equation*. The equation is derived by the application of the principle of conservation of energy along with the Fourier's conduction equation to the conducting medium.
- The rate of steady one-dimensional heat conduction through a plane, cylindrical or spherical wall of constant thermal conductivity can be, respectively, expressed as

$$Q_{\text{plane wall}} = kA\frac{T_1 - T_2}{L}$$

$$Q_{\text{cylinder}} = 2\pi kL\frac{T_1 - T_2}{\ln(r_2/r_1)}$$

$$Q_{\text{sphere}} = 4\pi k r_1 r_2\frac{T_1 - T_2}{r_2 - r_1}$$

  where T_1 and T_2 ($T_1 > T_2$) are the temperatures of the two boundary surfaces.
- The rate of steady one-dimensional heat transfer through composite walls exposed to convection from both sides to the surrounding mediums at temperatures T_i and T_o ($T_i > T_o$) can be expressed as

$$Q = \frac{T_i - T_o}{R_{\text{total}}}$$

  where R_{total} is the total thermal resistance to the flow of heat and is given by

$$R_{\text{total}} = R_{\text{conv},i} + R_{\text{wall}} + R_{\text{conv},o}$$

$$R_{\text{conv},i} = \frac{1}{h_i A_i}$$

$$R_{\text{conv},o} = \frac{1}{h_o A_o}$$

$$R_{\text{wall}}\ (\text{plane wall}) = \frac{L}{kA}$$

$$R_{\text{wall}}\ (\text{cylindrical wall}) = \frac{\ln(r_2/r_1)}{2\pi Lk}$$

$$R_{\text{wall}}\ (\text{spherical wall}) = \frac{r_2 - r_1}{4\pi r_1 r_2 k}$$

- The addition of insulation to a cylindrical pipe or a spherical shell increases the rate of heat transfer until the outer radius of insulation reaches a value, known as the critical radius of insulation. The rate of heat transfer then decreases with an increase in the outer radius. The critical radius of insulation is given by

$$r_{c,\text{cylinder}} = \frac{k}{h}$$

$$r_{c,\text{sphere}} = \frac{2k}{h}$$

- In case of a steady, one-dimensional heat conduction with generation of thermal energy in a medium, the maximum temperature occurs at the mid-section of the medium, provided the end surfaces are kept at the same and constant temperatures. The difference between this maximum temperature and the constant surface temperature is given by

$$\Delta T_{\text{plane wall}} = \frac{q_G L^2}{2k}$$

$$\Delta T_{\text{cylindrical wall}} = \frac{q_G r_o^2}{4k}$$

$$\Delta T_{\text{spherical wall}} = \frac{q_G r_o^2}{6k}$$

- Fins are commonly used in practice to enhance the rate of heat transfer from a surface by providing a large surface area for convection. The temperature distribution along the length of the fin and the rate of heat transfer from the base surface through the fin depends upon the thermal conductivity of the fin material, the convective heat transfer coefficient of the surrounding medium and the boundary conditions of the two ends of the fin.
- The temperature deceases from that of the base surface along the length of a fin due to conduction through the fin. The maximum heat transfer can be realized in an ideal situation if the entire fin were at the base temperature. This concept is used in defining the fin efficiency η_f as

$$\eta_f = \frac{\text{actual heat transfer rate through a fin}}{\text{heat transfer rate that could have been realized if the entire fin were at the base temperature}}$$

- The efficiency of an array of fins η is defined as

$$\eta = 1 - \frac{A_f}{A_0}(1 - \eta_f)$$

where A_0 is the total heat transfer area comprising that of fins and the unfinned base surface and A_f is the heat transfer area of the fins.
- The effectiveness of a fin in enhancing the rate of heat transfer is low when the convective heat transfer coefficient of a surrounding medium is high. Therefore, the incorporation of fins is not recommended under this situation from an economic point of view. Such a situation arises when the surrounding medium is a liquid in forced convection or a boiling liquid or condensing vapours.

REVIEW QUESTIONS

1. What is meant by steady-state heat conduction through a medium? Explain the situation physically in the case of medium (a) without generation of thermal energy and (b) with the generation of thermal energy.
2. What are the conditions for which the temperature distribution in a medium conducting heat is linear?

3. Why are the convection and the radiation resistances at a surface in parallel instead of being in series?
4. A pipe is insulated to reduce the heat loss from it. However, measurements indicate that the rate of heat loss has increased instead of having decreased. Can the measurement be right?
5. Why, in reducing the heat loss from a plane wall, is it that the concept of critical thickness of insulation does not arise?
6. Consider an insulated pipe exposed to the atmosphere. Will the critical radius of insulation be greater on calm days or windy days?
7. In the case of insulating an electrical wire, should the outer radius of insulation be more or less than the critical radius and why?
8. Hot water is to be cooled as it flows through the tubes exposed to atmospheric air. Fins can be attached in order to enhance the heat transfer. Would you recommend attaching the fins inside or outside the tubes? Explain why?
9. What are the influences of (a) fin length and (b) fin thickness on the efficiency of a fin?

PROBLEMS

2.1 Consider a 1.2 m high and 2 m wide glass window whose thickness is 6 mm and thermal conductivity is k = 0.78 W/(m° C). Determine the steady rate of heat transfer through this glass window and the temperature of its inner surface for a day during which the room is maintained at 24°C while the temperature of the outdoors is –5°C. Take the convection heat transfer coefficients on the inner and outer surfaces of the window to be h_1 = 10 W/(m^2 °C) and h_2 = 25 W/(m^2 °C), and disregard any heat transfer by radiation.

[**Ans.** 471.25 W, 4.36°]

2.2 A certain material 200 mm thick, with a cross-sectional area of 0.1 m^2, has one side maintained at 35°C and the other at 95°C. The temperature at the centre plane of the material is 62°C, and the heat flow through the material is 10 kW. Obtain an expression for the thermal conductivity of the material as a linear function of temperature.

[**Ans.** $K = a + bT$, where a = 478.24 W/(m° C), and b = –1.12 W/(m^2 °C)]

2.3 A truncated cone 0.3 m high is constructed of aluminium. The diameter at the top is 75 mm, and the diameter at the bottom is 125 mm. The lower surface is maintained at 93°C; the upper surface at 540°C. The other surface is insulated. Assuming one-dimensional heat flow, determine the rate of the heat transfer in watts. Take thermal conductivity of aluminium as 230 W/(m K).

[**Ans.** 1.31 kW]

2.4 The composite wall of an oven (Figure 2.31) consists of three materials, two of which are of known thermal conductivity, k_A = 20 W/(m K) and k_C = 50 W/(m K), and known thickness, L_A = 0.30 m and L_C = 0.15 m. The third material B, which is sandwiched between the materials A and C, is of known thickness L_B = 0.15 m, but of unknown thermal conductivity k_B.

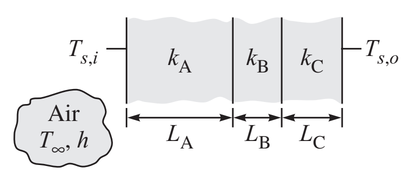


Figure 2.31 The composite wall (Problem 2.4).

Under steady-state operating conditions, measurements reveal an outer surface temperature of $T_{s,o}$ = 20°C, an inner surface temperature of $T_{s,i}$ = 600°C, and an oven air temperature of T_∞ = 800°C. The inside convection coefficient h is known to be 25 W/(m^2 K). Determine the value of k_B.

[**Ans.** 1.53 W/(m K)]

2.5 Consider a section of a brick wall as shown in Figure 2.32. The thermal conductivities of various materials used, in W/(m° C), are $k_A = k_F = 2$, $k_B = 8$, $k_C = 20$, $k_D = 15$, $k_E = 35$. The left and right surfaces of the wall are maintained at uniform temperatures of 600°C and 200°C, respectively. Assuming heat transfer through the wall to be one-dimensional, determine (a) the rate of heat transfer through the wall, and (b) the temperature drop across the material F. Disregard any contact resistance at the interfaces.

[**Ans.** (a) 12.74 kW, (b) 79.49°C]

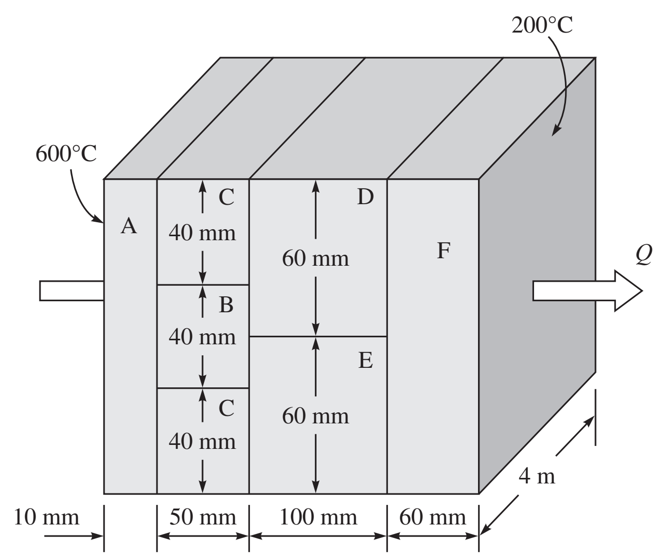


Figure 2.32 The multilayered wall (Problem 2.5).

2.6 Electric heater wires are installed in a solid wall having a thickness of 80 mm and k = 2.5 W/(m K). The right face is exposed to an environment with h = 50 W/(m^2 K) and T_∞ = 30°C, while the left face is exposed to h = 75 W/(m^2 K) and T_∞ = 50°C. What is the maximum allowable heat generation rate such that the maximum temperature in the solid does not exceed 450°C?

[**Ans.** 0.42 MW/m^3]

2.7 A plate having a thickness of 4.0 mm has an internal heat generation of 200 MW/m^3 and a thermal conductivity of 25 W/(m K). One side of the plate is insulated and the other side is maintained at 100°C. Calculate the maximum temperature in the plate.

[**Ans.** 164°C]

2.8 The rear window of an automobile is defogged by attaching a thin, transparent, film-type heating element to its inner surface. By electrically heating this element, a uniform heat flux may be established at the inner surface. For a 6 mm thick window glass (thermal conductivity k = 0·75 W/(m K)), determine the electrical power required per unit window area to maintain an inner surface temperature of 15°C when the interior air temperature and convection coefficient are $T_{\infty,i}$ = 25°C and h_i = 15 W/(m^2 K), while the exterior (ambient) air temperature and convection coefficients are $T_{\infty,o}$ = –15°C and h_o = 70 W/(m^2 K).

[**Ans.** 1196 W]

2.9 Consider a pipe of inside radius r_1 = 30 mm, outside radius r_2 = 60 mm, and thermal conductivity k_1 = 10 W/(m °C). The inside surface is maintained at a uniform temperature T_1 = 350°C, and the outside surface is to be insulated with an insulation material of thermal conductivity k_2 = 0.1 W/(m °C). The outside surface of the insulation material is exposed to an environment at T_2 = 20°C with a heat transfer coefficient h_2 = 10 W/(m^2 °C). Determine the thickness of the insulation material needed to reduce the heat loss by 25 per cent of that of the uninsulated pipe exposed to the same environmental conditions.

[**Ans.** 4.3 mm]

2.10 A 2 mm long steel tube (k = 15 W/(m °C)) with an outside diameter of 760 mm and a thickness of 130 mm is covered with an insulation (k = 0.2 W/(m °C)) of 20 mm thickness. A hot gas with a heat transfer coefficient of 400 W/(m^2 °C) flows inside the tube, and the outer surface of the insulation is exposed to cooler air with a heat transfer coefficient of 60 W/(m^2 °C). Calculate the total thermal resistance of the system for heat flow from hot gas to cooler air.

[**Ans.** 0.02675 °C/W]

2.11 Hot water at an average temperature of 80°C is flowing through a 10 m section of a cast iron pipe (k = 50 W/(m °C)) whose inner and outer diameters are 50 mm and 60 mm, respectively. The outer surface of the pipe whose emissivity is 0.7, is exposed to the cold air at 10°C in the basement with a heat transfer coefficient of 15 W/(m^2 °C). The heat transfer coefficient at the inner surface of the pipe is 120 W/(m^2 °C). Taking the walls of the basement to be at 10°C as well, determine the rate of heat loss from the hot water. Also, determine the average velocity of the water in the pipe if the temperature of the water drops by 2°C as it passes through the basement.

[**Ans.** 2.78 kW, 0.17 m/s]

2.12 A 250 mm OD steam pipe maintained at T_1 = 150°C is exposed to an ambient at T_1 = 25°C with a convection heat transfer coefficient h = 50 W/(m^2 °C). Calculate the thickness of the asbestos insulation (k = 0.1 W/(m °C)) required to reduce the heat loss from the pipe by 50 per cent.

[**Ans.** 2 mm]

2.13 A 2 mm diameter electrical wire at 50°C is covered by 0.5 mm thick plastic insulation (k = 0.15 W/(m K)). The wire is exposed to a medium at 10°C, with a combined convection and radiation heat transfer coefficient of 15 W/(m^2 K). Determine if the plastic insulation on the wire will increase or decrease heat transfer from the wire.

[**Ans.** increase]

2.14 A 1.0 mm diameter wire is maintained at a temperature of 400°C and exposed to a convection environment at 40°C with h = 120 W/(m^2 K). Calculate the thermal conductivity which will just cause an insulation thickness of 0.2 mm to produce a 'critical radius.' What should be the insulation thickness to reduce the heat transfer by 75 per cent from that which would be experienced by the bare wire?.

[**Ans.** 0.084 W/(m K), 134.50 mm]

2.15 Thermal energy is generated at a constant rate of $q_0 = 2 \times 10^8$ W/m^3 in a copper rod of radius $r = 5$ mm and thermal conductivity $k = 386$ W/(m °C). The rod is cooled by convection from its cylindrical surface into an ambient at 25°C with a heat transfer coefficient $h = 1000$ W/(m^2 °C). Determine the surface temperature of the rod.

[**Ans.** 52.5°C]

2.16 In a cylindrical fuel element for a gas-cooled nuclear reactor, the generation rate of thermal energy within the fuel element due to fission can be approximated by the relation

$$q(r) = q_0\left[1-\left(\frac{r}{a}\right)^2\right] \text{ W/m}^3$$

where a is the radius of the fuel element and q_0 is constant.The boundary surface at $r = a$ is maintained at a uniform temperature T_0.

(a) Assuming one-dimensional, steady-state heat flow, develop a relation for the temperature drop from the centreline to the surface of the fuel element.

(b) For a radius of $a = 30$ mm, the thermal conductivity $k = 10$ W/(m °C) and $q_0 = 2 \times 10^7$ W/m^3, calculate the temperature drop from the centreline to the surface.

[**Ans.** 337.5°C]

2.17 A hollow aluminium sphere ($k = 200$ W/(m K)), with an electrical heater in the centre, is used in tests to determine the thermal conductivity of insulating materials. The inner and outer radii of the sphere are 150 mm and 200 mm respectively, and testing is done under steady-state conditions with the inner surface of the aluminium maintained at 300°C. In a particular test, a spherical shell of insulation is cast on the outer surface of the sphere to a thickness of 120 mm. The system is in a room for which the air temperature is 20°C and the convection coefficient at the outer surface of the insulation is 25 W/(m^2 K). If a power of 80 W is dissipated by the heater under steady-state conditions, what is the thermal conductivity of the insulation?

[**Ans.** 0.043 (W/m K)]

2.18 A straight rectangular fin 20 mm thick and 150 mm long is constructed of steel and placed on the outside of a wall maintained at 200°C. The environment temperature is 15°C, and the heat transfer coefficient for convection is 20 W/(m^2 K). Calculate the heat lost from the fin per unit depth. Take thermal conductivity of steel as 60 W/(m K).

[**Ans.** 945.80 W/m]

2.19 A straight rectangular fin has a length of 20 mm and a thickness of 1.5 mm. The thermal conductivity is 60 W/(m K), and it is exposed to a convection environment at 20°C and $h = 500$ W/(m^2 K). Calculate the maximum possible heat loss per metre of fin depth for a base temperature of 200°C. What is the actual heat loss?

[**Ans.** 3.6 kW/m, 1.67 kW/m]

2.20 Consider a stainless steel spoon ($k = 20$ W/(m K)) partially immersed in boiling water at 100°C in a kitchen at 30°C. The handle of the spoon has a cross-section of 2 mm × 12 mm, and extends 180 mm in the air from the free surface of the water. If the heat transfer coefficient at the exposed surfaces of the spoon handle is 20 W/(m^2 K), determine the temperature difference across the exposed surface of the spoon handle. State any assumptions required.

[**Ans.** 69.71°C]

2.21 A triangular fin of stainless steel (k = 20 W/(m K)) is attached to a plane wall maintained at 500°C. The fin thickness is 15 mm, and the length is 250 mm. The environment is at 30°C and the convection heat transfer coefficient is 15 W/(m^2 K). Calculate the heat loss from the fin per unit depth. (Use Figure 2.28.)

[**Ans.** 653 W/m]

2.22 A 25 mm diameter glass rod (k = 0.8 W/(m K)) protrudes from a wall whose temperature is 150°C and is exposed to air at 30°C and having a convective heat triangular coefficient of 10 W/(m^2 °C). Determine (i) the temperature at the tip of the rod and (ii) the heat loss by the rod.

[**Ans.** (i) 32.14°C, (ii) 8.4 W]

2.23 One end of a 0.3 m long steel rod (k = 20 W/(m K)) is connected to a wall at 240°C (Figure 2.33), while the other end is connected to a wall that is maintained at 90°C. Air is blown across the rod so that a heat transfer coefficient of 17 W/(m^2 K) is maintained over the entire surface. If the diameter of the rod is 50 mm and the temperature of the air is 40°C, what is the net rate of heat loss to the air?

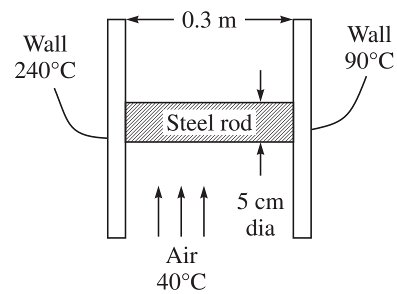


Figure 2.33 A rod connected between two walls (Problem 2.23).

2.24 Compare the tip temperatures in a straight fin of rectangular profile having a thickness of 20 mm, a length of 100 mm, and a width of 200 mm, exposed to a convection environment with h = 25 W/(m^2 K), for the fin materials: copper (k = 400 W/(m K)), stainless steel (k = 20 W/(m K)), and glass (k = 0.8 W/(m K). Also compare the heat transfer and fin efficiencies.

Ans.

Material	T_L(°C)	Q(W)	n(%)
Copper	222	234	97.46
Stainless steel	134	161	67.00
Glass	30.76	37.56	15.65

3

Multidimensional Steady-State Heat Conduction

In Chapter 2 we discussed the steady-state one-dimensional heat conduction problems where temperature is a function of only one space coordinate, and accordingly, the flow of heat takes place only in the direction in which the temperature gradient does exist. Though many practical problems fall into this category, there are situations where temperature becomes a function of more than one space coordinate. These situations arise when the boundaries of a system are irregular or when the temperature along a boundary is not constant. Some typical examples in practice are heat conduction through the walls of a short hollow cylinder, or the heat loss from a buried pipe.

The primary objective of analyzing a heat conduction problem is to determine the temperature distribution in a medium with prescribed temperatures at the boundaries, and then to determine the rate of heat flow in different directions. The temperature distribution is found out by solving the heat conduction equation (Eq. (2.5a)), while the heat flow vector is determined by making use of Fourier's equation (Eq. (1.2)). There are usually three methods used in the analysis of multidimensional heat conduction problems: (a) analytical method, (b) graphical method, and (c) numerical method. Though the analytical methods predict the most accurate results, they involve complicated mathematical functions and are applicable only for a restricted set of simple geometries and boundary conditions. In contrast to the analytical methods, the graphical and numerical methods are relatively simple and can accommodate complex geometries and boundary conditions. In this chapter, we will discuss only the two-dimensional steady-state heat conduction.

Learning objectives

The reading of this chapter will enable the students

- to recognize the physical situations for multidimensional heat conduction,
- to understand the different methods of solution of multidimensional heat conduction problems,

- to develop the analytical solutions of simple steady-state two-dimensional heat conduction problems,
- to appreciate the graphical method for the solution of two-dimensional heat conduction problems,
- to understand the basis of numerical solution and its advantage over the analytical solution in multidimensional heat conduction problems, and
- to develop numerical solutions for two-dimensional steady-state heat conduction problems.

3.1 ANALYTICAL METHOD

Let us consider a steady-state two-dimensional heat conduction problem as shown in Figure 3.1. The three sides of a rectangular plate ABCD are maintained at a constant temperature T_1, while the fourth side is maintained at a constant temperature T_2 ($T_2 \neq T_1$). We have to find out the temperature distribution in the plate. A Cartesian coordinate system x–y has been taken for the purpose as shown in Figure 3.1. The general heat conduction equation (Eq. (2.5a)) can be written, under the situation, as

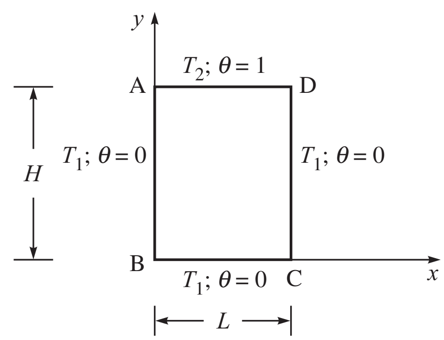


Figure 3.1 Two-dimensional heat conduction in a rectangular plate.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{3.1}$$

Equation (3.1) is a linear, second-order partial differential equation with two independent variables and is known as Laplace equation. One of the powerful methods for the solution of this equation is the method of separation of variables. To simplify the solution, the temperature T is transferred to a non-dimensional temperature as

$$\theta = \frac{T - T_1}{T_2 - T_1} \tag{3.2}$$

Substituting Eq. (3.2) into Eq. (3.1), the transformed differential equation becomes

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 \tag{3.3}$$

The boundary conditions are

$$\theta(0, y) = 0 \quad \text{(the side AB)} \tag{3.4a}$$
$$\theta(x, 0) = 0 \quad \text{(the side BC)} \tag{3.4b}$$
$$\theta(L, y) = 0 \quad \text{(the side CD)} \tag{3.4c}$$
$$\theta(x, H) = 1 \quad \text{(the side DA)} \tag{3.4d}$$

We now apply the separation of variables method. In this method, we seek a solution of θ in the form of the product of two functions, one of which depends only on x and the other depends only on y. Hence we can write

$$\theta(x, y) = X(x)\, Y(y) \tag{3.5}$$

Equation (3.3) can now be written with the help of Eq. (3.5) as

$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} \tag{3.6}$$

Since X is a function of x, the left-hand side of Eq. (3.6) is either a function of x or a constant depending upon the type of relationship of X with x. Similarly, the right-hand side of Eq. (3.6) is either a function of y or a constant. Therefore, the only possibility for equality of both the sides, as shown by Eq. (3.6), is that both of them be equal to a same constant value, say λ^2. We then have

$$\frac{d^2X}{dx^2} + \lambda^2 x = 0 \tag{3.7a}$$

$$\frac{d^2Y}{dy^2} - \lambda^2 y = 0 \tag{3.7b}$$

Therefore, we see that the partial differential equation (Eq. (3.3)) has been reduced to two ordinary differential equations ((3.7a), (3.7b)). The general solutions of Eqs. (3.7a) and (3.7b) are

$$X = A\cos\lambda x + B\sin\lambda x$$

$$Y = Ce^{\lambda y} + De^{-\lambda y}$$

Hence, $$\theta = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y}) \tag{3.8}$$

The important point to be noted in this context is that we have deliberately chosen the sign of the constant λ^2, in forming Eqs. (3.7a) and (3.7b) from Eq (3.6), so that we have a trigonometric function of x and an exponential function of y in the solution of θ. This is because of the typical boundary conditions given by Eqs. (3.4a) to (3.4d) where we have a non-homogeneous condition (the condition for which $\theta \neq 0$) along the edge DA. This will be well understood while applying the boundary conditions to determine the constants A, B, C, and D of Eq. (3.8).

The boundary condition (3.4a) gives $A = 0$. For the boundary condition (3.4b), we can write

$$B(C + D)\sin\lambda x = 0$$

This may be satisfied if $B = 0$ or $C = -D$. However the requirement $B = 0$ does not yield to any solution of θ and hence it is not accepted. Therefore, we have $C = -D$. Using the boundary condition (3.4c), we get

$$BC(\sin\lambda L)\,(e^{\lambda y} - e^{-\lambda y}) = 0 \tag{3.9}$$

The only way to satisfy the condition for a solution of θ is

$$\sin\lambda L = 0$$

or $$\lambda L = n\pi$$

or $$\lambda = \frac{n\pi}{L} \qquad n = 1, 2, 3, \ldots$$

Now, we have

$$\theta = BC\left(\sin\frac{n\pi x}{L}\right)(e^{n\pi y/L} - e^{-n\pi y/L}) \tag{3.10}$$

Combining the constants B and C into a single constant and recognizing its dependence on n, we can write

$$\theta(x, y) = b_n\left(\sin\frac{n\pi x}{L}\right)\left(\sinh\frac{n\pi y}{L}\right) \tag{3.11}$$

where $b_n = 2BC$ and $(e^{n\pi y/L} - e^{-n\pi y/L}) = 2\sinh(n\pi y/L)$.

Equation (3.11) shows that we have infinite number of solutions (depending upon the value of n) for the differential Eq. (3.3). Since Eq. (3.3) is linear, we can write the general solution as a superimposition of all the solutions.

Hence,
$$\theta(x, y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right) \tag{3.12}$$

We make use of the last boundary condition, Eq. (3.4d), for determination of b_n as

$$1 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi H}{L}\right) \tag{3.13}$$

To determine b_n from Eq. (3.13), we have to express 1 in terms of an orthogonal function $f_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ as

$$1 = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \tag{3.14}$$

For the determination of A_n in Eq (3.14), we have to recall the property of orthogonal function. An infinite set of functions $f_1(x), f_2(x),..., f_n(x)$ is said to be orthogonal in the domain $a \le x \le b$ if

$$\int_a^b f_m(x)\, f_n(x)\; dx = 0 \qquad m \ne n \tag{3.15}$$

Let a function $g(x)$ be expressed as

$$g(x) = \sum_{n=1}^{\infty} A_n\, f_n(x) \tag{3.16}$$

where $f_n(x)$ is an orthogonal function in the domain $a \le x \le b$.

Multiplying both sides of Eq. (3.16) by $f_n(x)$ and integrating between the limits a and b, we have

$$\int_a^b g(x)\, f_n(x)dx = \int_a^b f_n(x)\sum_{n=1}^{\infty} A_n f_n(x)dx \tag{3.17}$$

However, from Eq (3.17), it is evident that all but one of the terms in the right-hand side are zero, and thus we get

$$\int_a^b g(x)\, f_n(x)dx = A_n \int_a^b f_n^2(x)\,dx$$

or

$$A_n = \frac{\int_a^b g(x)\, f_n(x)\, dx}{\int_a^b f_n^2(x)\, dx} \tag{3.18}$$

To determine A_n in Eq. (3.14), we have to use Eq. (3.18) with $g(x) = 1$ and $f_n(x) = \sin\left(\frac{n\pi x}{L}\right)$. Hence, we get

$$A_n = \frac{\int_0^L \sin\left(\frac{n\pi x}{L}\right) dx}{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx} = \frac{2}{\pi}\left\{\frac{(-1)^{n+1}+1}{n}\right\}$$

Therefore, Eq. (3.14) may be written as

$$1 = \sum_{n=1}^{\infty} \frac{2}{\pi}\left\{\frac{(-1)^{n+1}+1}{n}\right\} \sin\left(\frac{n\pi x}{L}\right) \tag{3.19}$$

This is simply the expansion of unity in a Fourier series. Comparing Eqs. (3.13) and (3.19), we get

$$b_n = \frac{2\{(-1)^{n+1}+1\}}{n\pi \sinh\left(\frac{n\pi H}{L}\right)} \qquad n = 1, 2, 3, \ldots \tag{3.20}$$

Substituting Eq. (3.20) into Eq. (3.12), we obtain the final solution for the temperature distribution $\theta(x, y)$ as

$$\theta(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}+1}{n} \sin\left(\frac{n\pi x}{L}\right) \frac{\sinh(n\pi y/L)}{\sinh(n\pi H/L)} \tag{3.21}$$

Equation (3.21) is a convergent series and hence the finite values of θ can be computed from the equation for given values of x and y. The representative values for the problem of two-dimensional rectangular plate (Figure 3.1) are shown in the form of isotherms in Figure 3.2.

Let us consider a case, where, instead of a constant temperature, a sinusoidal temperature distribution of $\theta = \theta_0 \sin\left(\frac{\pi x}{L}\right)$ is imposed on the edge AD of the plate in Figure 3.1. In this situation, we can write Eq. (3.12) as

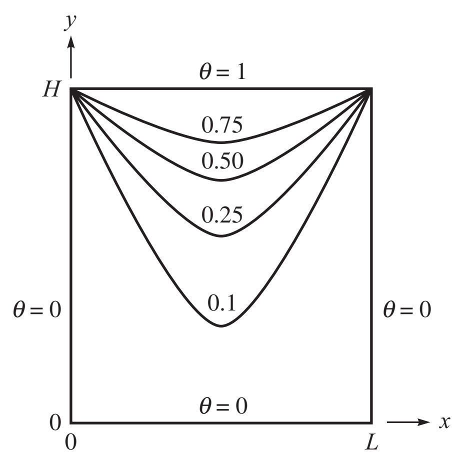


Figure 3.2 Isothermal lines for two-dimensional heat conduction in a rectangular plate.

$$\theta_0 \sin\left(\frac{\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi H}{L}\right) \tag{3.22}$$

This condition holds only if $b_2 = b_3 = b_4 = \cdots = 0$

and
$$b_1 = \frac{\theta_0}{\sinh\left(\dfrac{\pi H}{L}\right)}$$

Therefore,
$$\theta = \frac{\theta_0 \sinh(\pi y/L)}{\sinh(\pi H/L)} \sin\left(\frac{\pi x}{L}\right) \tag{3.23}$$

If the boundary conditions of the plate in Figure 3.1 are changed so that the edges DA, AB and BC are at a temperature T_1, while the edge CD is at T_2, then the solution of Eq. (3.3) becomes

$$\theta(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}+1}{n} \frac{\sinh(n\pi x/H)}{\sinh(n\pi L/H)} \sin\left(\frac{n\pi y}{H}\right) \tag{3.24}$$

To deduce Eq. (3.24), we have to change the sign of the arbitrary constant λ^2 in Eqs. (3.7a) and (3.7b) to induce trigonometric functions of y and exponential functions of x as the solution of θ.

Heat flux boundary condition

More useful boundary conditions are such where one of the sides exchanges heat by convection to a surrounding fluid, two of the other sides are insulated and the fourth side is at a fixed known temperature. This is illustrated in Figure 3.3, where $\theta = T - T_\infty$, the excess temperature over the ambient temperature T_∞.

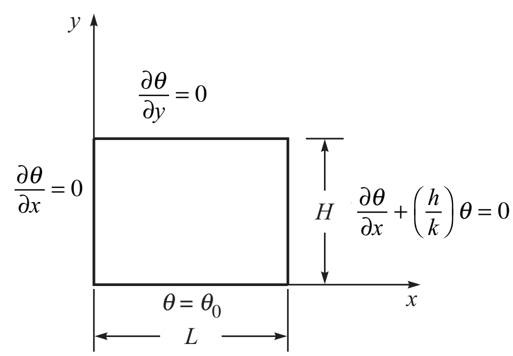


Figure 3.3 A rectangular plate with insulated and non-insulated sides.

The solution of Eq. (3.1) under the situation is

$$\theta = \frac{2\theta_0 h}{k} \sum_{n=1}^{\infty} \frac{\cos(p_n x)\cosh[p_n(H-y)]}{\left[\left\{\left(\dfrac{h}{k}\right)^2 + p_n^2\right\} L + \left(\dfrac{h}{k}\right)\right] \cos(p_n L)\cosh(p_n H)} \tag{3.25}$$

where p_n are the positive roots of $p_n \tan p_n L = \dfrac{h}{k}$.

EXAMPLE 3.1 Determine the temperature at the centre of the plate shown in Figure 3.4.

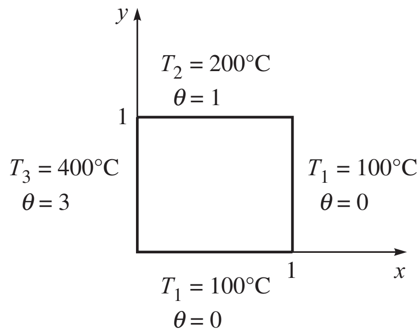


Figure 3.4 The plate (Example 3.1).

Solution: Let us define a non-dimensional temperature as

$$\theta = \frac{T - T_1}{T_2 - T_1}$$
$$= \frac{T - 100}{100}$$

The values of θ at different edges are also shown in Figure 3.4. A problem having more than one non-homogeneous boundary condition (the boundary condition for which θ is not zero) can be resolved into a set of simpler problems, each with the physical geometry of the original problem and having only one non-homogeneous boundary condition. The solutions to the simpler problems can be superimposed (at the geometric point being considered) to yield the solution to the original problem. This is possible because the governing equation (Eq. (3.3)) is linear in θ. Therefore, the problem shown in Figure 3.4 is separated into two modular problems as shown in Figure 3.5.

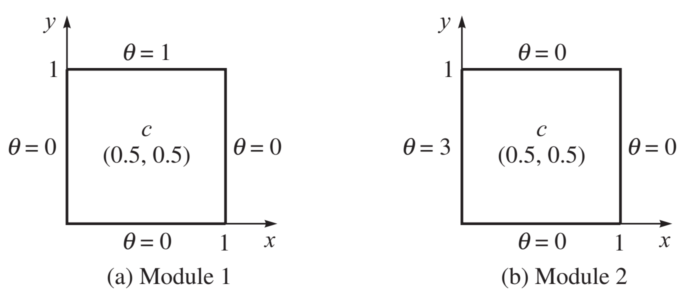


Figure 3.5 Modules of the original problem shown in Figure 3.4.

The solution to module 1 (Figure 3.5(a)) is given by Eq. (3.21). Hence we can write

$$\theta(0.5,\ 0.5) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\left(\frac{n\pi}{2}\right)\frac{\sinh\left(\frac{n\pi}{2}\right)}{\sinh(n\pi)} \tag{3.26}$$

Since the series is a rapidly converging one, we can determine the value of $\theta(0.5, 0.5)$ by taking the first three terms. Hence,

$$\theta(0.5,\ 0.5) = \frac{2}{\pi}\sum_{n=1}^{3}\frac{(-1)^{n+1}+1}{n}\sin\left(\frac{n\pi}{2}\right)\frac{\sinh\left(\dfrac{n\pi}{2}\right)}{\sinh(n\pi)}$$

$$= 0.25$$

The solution to module 2 (Figure 3.5(b)) can be written following Eq. (3.24) as

$$\theta(x,\ y) = 3\left(\frac{2}{\pi}\right)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\frac{\sinh n\pi(1-x)}{\sinh(n\pi)}\ \sin(n\pi y)$$

Hence,

$$\theta(0.5,\ 0.5) = \frac{6}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\frac{\sinh(n\pi/2)}{\sinh(n\pi)}\sin\left(\frac{n\pi}{2}\right)$$

We take the first three terms of the series, because of its rapid convergence, in determining the value of $\theta(0.5, 0.5)$. This gives

$$\theta(0.5,\ 0.5) = 0.75$$

By the principle of superimposition, the temperature at the centre of the plate of the original problem as shown in Figure 3.4 becomes

$$\theta(0.5,\ 0.5) = 0.25 + 0.75 = 1$$

Therefore,

$$T(0.5,\ 0.5) = 100 + 100 \times 1 = 200°\text{C}$$

EXAMPLE 3.2 A thin square plate as shown in Figure 3.6 has the following temperature distribution on its boundary:

At edge AB, $T = 300$ K

At edge BC, $T = 300 + 200\sin\left(\dfrac{\pi x}{a}\right)$ K

At edge CD, $T = 300 + 100\sin\left(\dfrac{\pi y}{a}\right)$ K

At edge DA, $T = 300$ K

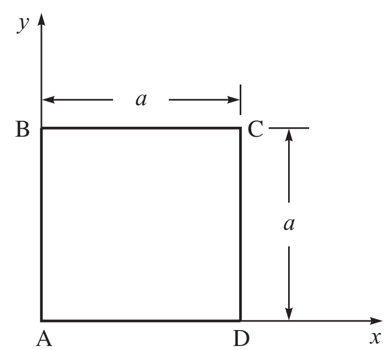


Figure 3.6 The plate (Example 3.2).

Determine (a) the steady-state temperature distribution and (b) the centre temperature.

Solution: Let us define a non-dimensional temperature $\theta = \dfrac{T-300}{100}$ and split the problem into two modules as shown in Figure 3.7 so that the results of each module can be superimposed to get the result of the problem shown in Figure 3.6.

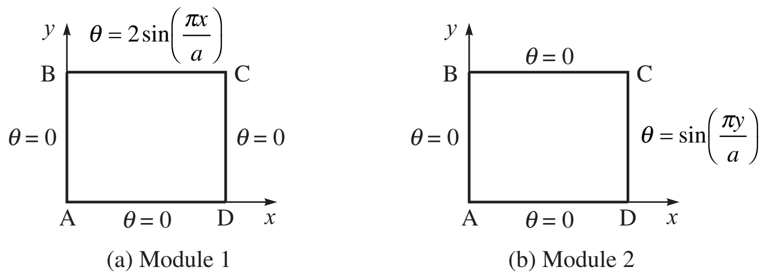


Figure 3.7 Modules of the problem shown in Figure 3.6.

The temperature distribution of module 1 can be written following Eq. (3.23) as

$$\theta(x,\ y)_1 = \frac{2\sinh(\pi y/a)}{\sinh(\pi)} \sin\left(\frac{\pi x}{a}\right)$$

The temperature distribution of module 2 can be written following Eq. (3.24) as

$$\theta(x,\ y)_2 = \frac{\sinh(\pi x/a)}{\sinh(\pi)} \sin\left(\frac{\pi y}{a}\right)$$

(a) Therefore the temperature distribution of the plate becomes

$$\begin{aligned}\theta(x,\ y) &= \theta(x,\ y)_1 + \theta(x,\ y)_2 \\ &= 2\frac{\sinh(\pi y/a)}{\sinh(\pi)} \sin\left(\frac{\pi x}{a}\right) + \frac{\sinh(\pi x/a)}{\sinh(\pi)} \sin\left(\frac{\pi y}{a}\right)\end{aligned}$$

(b) At the centre, $x = \dfrac{a}{2}, \quad y = \dfrac{a}{2}$

Hence,

$$\begin{aligned}\theta\left(\frac{a}{2}, \frac{a}{2}\right) &= \frac{2\sinh(\pi/2)}{\sinh(\pi)} + \frac{\sinh(\pi/2)}{\sinh(\pi)} \\ &= 3\frac{\sinh(\pi/2)}{\sinh(\pi)} \\ &= 0.6\end{aligned}$$

Therefore,

$$\begin{aligned}T\left(\frac{a}{2}, \frac{a}{2}\right) &= 0.6 \times 100 + 300 \\ &= 360\ \text{K}\end{aligned}$$

3.2 GRAPHICAL METHOD

The framework of the graphical method is based on the fact that the lines of constant temperature are perpendicular to the lines indicating the direction of heat flow by conduction. The lines of constant temperatures are known as *isotherms*, while lines indicating the directions of heat flow are known as *heat flux lines* and they represent the lines of constant heat flow rate. The objective of the method is to construct a network consisting of isotherms and heat flux lines. The two sets of lines intersect at right angles and form a network of curvilinear squares. This is analogous to a flow net in the solution of ideal flow, where a family of streamlines (lines of constant stream function) intersect a family of equipotential lines (lines of constant velocity potential function). The isotherms are analogous to equipotential lines, while the heat flux lines are analogous to streamlines. However, the application of the graphical method is limited to two-dimensional systems with isothermal and insulated boundaries. Despite its limitations, the method yields a reasonable and fair estimation of the temperature distribution and heat flow in a system.

Let us explain the method by considering the two-dimensional system shown in Figure 3.8. The inside surface is maintained at some constant temperature T_1 and the outer surface is maintained at T_2. We have to determine the temperature distribution in the system and the rate of heat transfer. The methodology is described below.

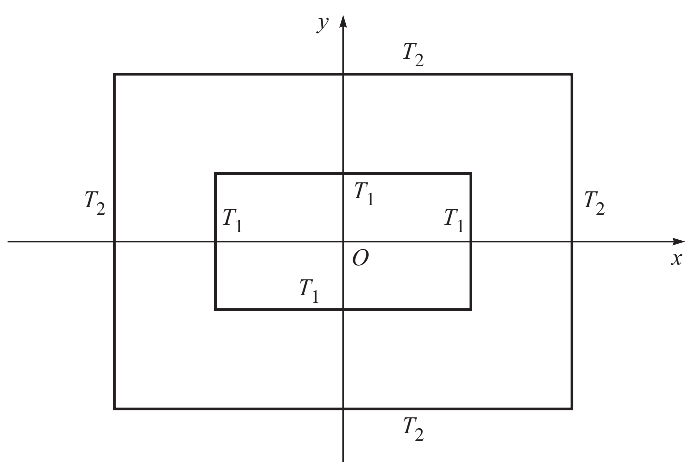


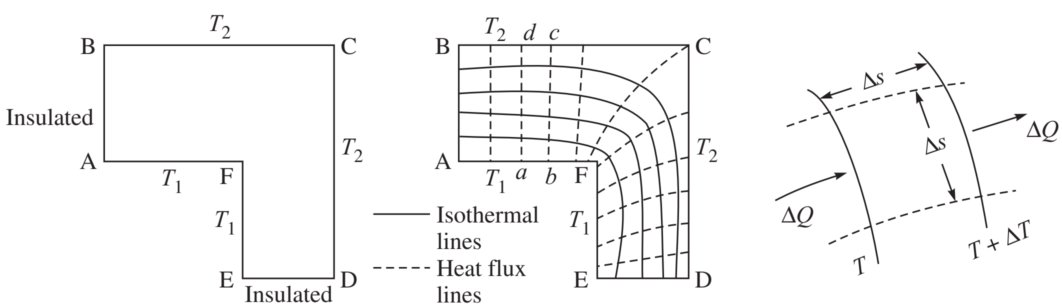
Figure 3.8 A two-dimensional system with isothermal boundaries.

- The first step is to identify all relevant lines of symmetry. For the system shown in Figure 3.8, such lines are the designated horizontal and vertical lines Ox and Oy. Therefore it is possible to consider only one-fourth of the system configuration ABCDEFA as shown in Figure 3.9(a).
- The second step is to identify the isothermal and insulated faces as the boundary lines of isotherms and adiabates (constant heat flow lines). Here BC and CD are the isothermal lines having a temperature T_2, while AF and FE are the similar lines having a temperature T_1 (Figure 3.9(a)). The faces AB and ED appear from the symmetry lines and therefore they represent the insulated surfaces or the boundary heat flux lines.
- The third step is to draw the isothermal and heat flux lines freehand, by trial and error, until they form a network of curvilinear squares. This is the most important step of the method. Such a curvilinear network is shown in Figure 3.9(b). It should be noted that the heat flux lines emanating from isothermal boundaries are perpendicular to the boundary, except when they come from a corner. The heat flux lines coming from or

leading to a corner of an isothermal boundary (like C and F in Figures 3.9(a) and 3.9(b)) bisect the angle between the surfaces forming the corner.

- The fourth step is to determine heat flux. Consider a curvilinear square mesh of Figure 3.9(b) as one element shown separately in Figure 3.9(c). Let the two bounded isotherms of the element be of temperatures T and $T + \Delta T$ and the length of the sides of the square be Δs. Then the rate of heat flow ΔQ through the element can be written according to Fourier's law as

$$\Delta Q = -k(\Delta s \times 1)\frac{\Delta T}{\Delta s} = -k\Delta T \tag{3.27}$$



(a) One-fourth part of the system shown in Figure 3.6

(b) Network consisting of isotherms and heat flow lines

(c) An element bounded by isotherms and adiabates

Figure 3.9 Representative diagrams for the graphical method of heat conduction analysis of the system of Figure 3.8.

Here we have considered a unit depth of the corner section (Figure 3.9(a)) in a direction perpendicular to the plane of the figure.

A heat flow lane is defined as the passage formed by any two adjoining heat flux lines. For example, *abcda* is a heat flow lane as shown in Figure 3.9(b). Therefore, the rate of heat flow will remain the same across any square within any one heat flow lane from the boundary T_1 to the boundary T_2. If the number of temperature increments between the boundaries T_1 and T_2 is N and the increment is same for all the adjoining isotherms, then we can write for the temperature difference ΔT across any one element in a heat flow lane as

$$\Delta T = \frac{T_1 - T_2}{N} \tag{3.28}$$

It is observed from Eqs. (3.27) and (3.28) that the heat flow will be the same for all lanes. Therefore, the total rate of heat transfer can be written with the help of Eqs. (3.27) and (3.28) as

$$Q = \frac{M}{N}k(T_2 - T_1) \tag{3.29}$$

where M is the number of heat flow lanes. The ratio M/N depends on the shape of the system and hence it is defined as conduction shape factor S_F. Therefore,

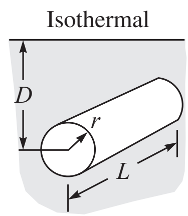
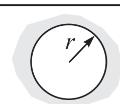
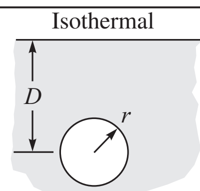
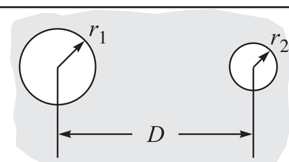
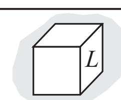
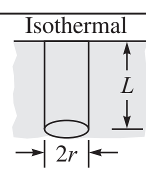
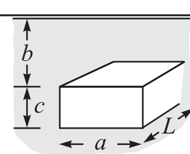
$$S_F = \frac{M}{N}$$

Equation (3.29) can now be written as

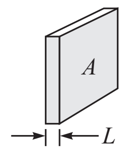
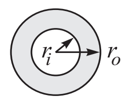
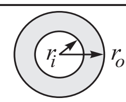
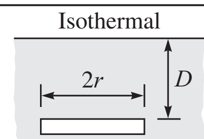
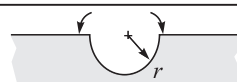
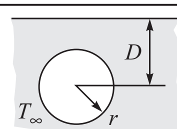
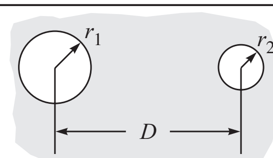
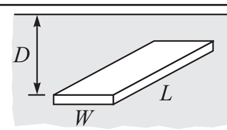
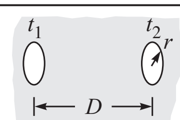
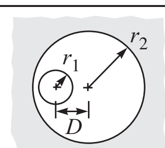
$$Q = S_F k(T_2 - T_1) \tag{3.30}$$

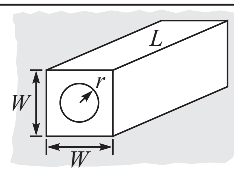
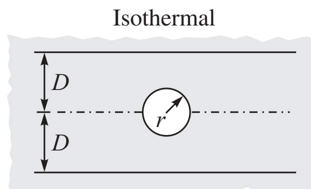
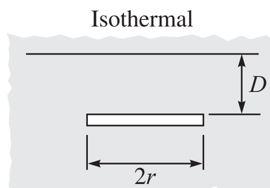
The values of S_F for several two-dimensional systems of practical significance are summarized in Table 3.1.

Table 3.1 Conduction shape factors for some common physical systems after Holman†

Physical system	*Schematic*	*Shape factor*	*Restriction*
Isothermal cylinder of radius r buried in semi-infinite medium having isothermal surface		$\dfrac{2\pi L}{\cosh^{-1}(D/r)}$ $\dfrac{2\pi L}{\ln(D/r)}$	$L \gg r$ $L \gg r$, $D > 3r$
Isothermal sphere of radius r buried in infinite medium		$4\pi r$	
Isothermal sphere of radius r buried in semi-infinite medium having isothermal surface $\Delta T = T_{surf} - T_{far\ field}$		$\dfrac{4\pi r}{1-(r/2D)}$	
Conduction between two isothermal cylinders of length L buried in infinite medium		$\dfrac{2\pi L}{\cosh^{-1}\left(\dfrac{D^2 - r_1^2 - r_2^2}{2r_1 r_2}\right)}$	$L \gg r$ $L \gg D$
Buried cube in infinite medium, L on a side		$8.24L$	
Isothermal cylinder of radius r placed in semi-infinite medium as shown		$\dfrac{2\pi L}{\ln(2L/r)}$	$L \gg 2r$
Isothermal rectangular parallelepiped buried in semi-infinite medium having isothermal surface		$1.685L\left[\log\left(1+\dfrac{b}{a}\right)\right]^{-0.59} \times \left(\dfrac{b}{c}\right)^{-0.078}$	

†Holman, J.P., *Heat Transfer*, 7th ed., Tata McGraw-Hill, New Delhi, 1992.

Physical system	*Schematic*	*Shape factor*	*Restriction*
Plane wall		$\dfrac{A}{L}$	One-dimensional heat flow
Hollow cylinder, length L		$\dfrac{2\pi L}{\ln(r_o/r_i)}$	$L \gg r$
Hollow sphere		$\dfrac{4\pi r_o r_i}{r_o - r_i}$	
Thin horizontal disk buried in semi-infinite medium with isothermal surface		$4r$ $8r$ $\dfrac{4\pi r}{\pi/2 - \tan^{-1}(r/2D)}$	$D = 0$ $D \gg 2r$ $D/2r > 1$ $\tan^{-1}(r/2D)$ in radians
Hemisphere buried in semi-infinite medium $\Delta T = T_{\text{sphere}} - T_{\text{far field}}$		$2\pi r$	
Isothermal sphere buried in semi-infinite medium with insulated surface		$\dfrac{4\pi r}{1 + r/2D}$	
Two isothermal spheres buried in infinite medium		$\dfrac{4\pi r_2}{\dfrac{r_2}{r_1}\left[1 - \dfrac{(r_1/D)^4}{1-(r_2/D)^2}\right] - \dfrac{2r_2}{D}}$	$D > 5r_{\max}$
Thin rectangular plate of length L, buried in semi-infinite medium having isothermal surface		$\dfrac{\pi W}{\ln(4W/L)}$ $\dfrac{2\pi W}{\ln(4W/L)}$ $\dfrac{2\pi W}{\ln(2\pi D/L)}$	$D = 0$ $W > L$ $D \gg W$ $W > L$ $W \gg L$ $D > 2W$
Parallel disks buried in infinite medium		$\dfrac{4\pi r}{\left[\dfrac{\pi}{2} - \tan^{-1}(r/D)\right]}$	$D > 5r$ $\tan^{-1}(r/D)$ in radians
Eccentric cylinders of length L		$\dfrac{2\pi L}{\cosh^{-1}\left(\dfrac{r_1^2 + r_2^2 - D^2}{2r_1 r_2}\right)}$	$L \gg r_2$

Physical system	*Schematic*	*Shape factor*	*Restriction*
Cylinder centred in a square of length L		$\dfrac{2\pi L}{\ln(0.54W/r)}$	$L \gg W$
Horizontal cylinder of length L centred in infinite plate		$\dfrac{2\pi L}{\ln(4D/r)}$	
This horizontal disk buried in semi-infinite medium with adiabatic surface $\Delta T = T_{\text{disk}} - T_{\text{far field}}$		$\dfrac{4\pi r}{(\pi/2) + \tan^{-1}(r/2D)}$	$D/2r > 1$ $\tan^{-1}(r/2D)$ in radians

3.3 NUMERICAL METHOD

It is already recognized that the analytical solution of the two-dimensional heat conduction equation involves complex mathematical functions. Moreover, the solution becomes extremely complicated if the geometry is irregular and the prescribed boundary conditions are not simple. On the other hand, a numerical method is much simple and offers flexibility in its solution with respect to system geometry and the boundary conditions. While the analytical solutions of partial differential equations involve closed-form expressions which give the continuous variation of the dependent variables throughout the domain, the numerical solutions predict the values of the dependent variables at only discrete points in the domain, called the grid points. For example, consider a section of a discrete grid network in the x–y plane as shown in Figure 3.10. For convenience, let us assume that the spacings of the grid points in the x and y directions are uniform and are given by Δx and Δy respectively. The grid points are identified by an index i which runs in the x-direction and an index j which runs in the y-direction. Let a point P in Figure 3.10 be indexed (i,j), then the neighbouring grid points are indexed accordingly as shown in the figure.

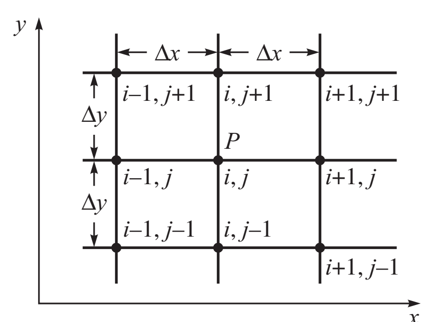


Figure 3.10 A section of a grid network in x–y plane.

The numerical method to be discussed here is based on a finite difference technique in which a partial derivative is replaced by a suitable algebraic difference quotient. Most common finite-difference representations of derivatives are based on the Taylor's series expansion.

Let temperature T be a function of x and y, i.e. $T = T(x, y)$, and $T_{i,j}$ denote the value of T at the point $P(i,j)$. The temperature $T_{i+1,j}$ at the point $(i+1, j)$ can be expressed in terms of a Taylor series expanded about the point (i,j) as follows:

$$T_{i+1,j} = T_{i,j} + \left(\frac{\partial T}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2!} + \left(\frac{\partial^3 T}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^3}{3!} + \ldots \tag{3.31}$$

Solving Eq. (3.31) for $\left(\frac{\partial T}{\partial x}\right)_{i,j}$, we can write

$$\left(\frac{\partial T}{\partial x}\right)_{i,j} = \underbrace{\frac{T_{i+1,j} - T_{i,j}}{\Delta x}}_{\text{Finite difference representation}} \underbrace{- \left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j} \frac{\Delta x}{2!} - \left(\frac{\partial^3 T}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^2}{3!}}_{\text{Truncation error}} \tag{3.32}$$

In Eq. (3.32), the first term on the right side, namely, $(T_{i+1,j} - T_{i,j})/\Delta x$, is a finite difference representation of the partial derivative $(\partial T/\partial x)_{i,j}$. The remaining terms on the right-hand side constitute the truncation error. If Δx is very small , we can approximate the partial derivative with the algebraic finite difference quotient as

$$\left(\frac{\partial T}{\partial x}\right)_{i,j} \approx \frac{T_{i+1,j} - T_{i,j}}{\Delta x} \tag{3.33}$$

We find from Eq. (3.32), that the lowest-order term in the truncation error involves Δx to the first power, and hence the finite difference expression in Eq. (3.33) is called first-order accurate. Equation (3.32) is usually written as

$$\left(\frac{\partial T}{\partial x}\right)_{i,j} = \frac{T_{i+1,j} - T_{i,j}}{\Delta x} + 0(\Delta x) \tag{3.34}$$

In Eq. (3.34), the symbol $0(\Delta x)$ represents formally the terms of order Δx. With reference to Figure 3.10, it is observed that the finite difference expression in Eq. (3.33) uses information to the right of grid point (i,j), i.e. $T_{i+1,j}$. No information to the left of (i, j) is used. Hence the finite difference expression in Eq. (3.33) is called a *forward difference*.

A finite difference expression of $\left(\frac{\partial T}{\partial x}\right)_{i,j}$ which uses information to the left of grid point (i, j) is known as the *backward difference*. This can be developed by expanding $T_{i-1,j}$ about $T_{i,j}$ as follows:

$$T_{i-1,j} = T_{i,j} + \left(\frac{\partial T}{\partial x}\right)_{i,j} (-\Delta x) + \left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j} \frac{(-\Delta x)^2}{2!} + \left(\frac{\partial^3 T}{\partial x^3}\right)_{i,j} \frac{(-\Delta x)^3}{3!} + \ldots \tag{3.35}$$

Solving for $(\partial T/\partial x)_{i,j}$, we obtain

$$\left(\frac{\partial T}{\partial x}\right)_{i,j} = \frac{T_{i,j} - T_{i-1,j}}{\Delta x} + 0(\Delta x) \tag{3.36}$$

or
$$\left(\frac{\partial T}{\partial x}\right)_{i,j} \approx \frac{T_{i,j} - T_{i-1,j}}{\Delta x} \tag{3.37}$$

Equation (3.37) represents the backward difference of $\left(\frac{\partial T}{\partial x}\right)_{i,j}$ which uses information to the left of grid point (i,j).

In most of the applications of numerical methods, the first-order accuracy is not sufficient. A finite difference quotient of $\left(\frac{\partial T}{\partial x}\right)_{i,j}$ of second-order accuracy is obtained by subtracting Eq. (3.35) from Eq. (3.31) as

$$T_{i+1,j} - T_{i-1,j} = 2\left(\frac{\partial T}{\partial x}\right)_{i,j} \Delta x + 2\left(\frac{\partial^3 T}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^3}{3!} + \ldots$$

Solving for $\left(\frac{\partial T}{\partial x}\right)_{i,j}$, it becomes

$$\left(\frac{\partial T}{\partial x}\right)_{i,j} = \frac{T_{i+1,j} - T_{i-1,j}}{2\Delta x} + 0(\Delta x)^2 \tag{3.38}$$

The finite difference quotient in Eq. (3.38) uses information from both the sides of the grid point (i,j). Moreover, the order of the truncation terms is the second power of Δx. Therefore the quotient in Eq. (3.38) is known as second-order central difference.

The analogous difference expressions for the y derivatives are obtained in exactly the same fashion. The expressions are

$$\left(\frac{\partial T}{\partial y}\right)_{i,j} = \begin{cases} \dfrac{T_{i,j+1} - T_{i,j}}{\Delta y} + 0(\Delta y) & \text{First-order forward difference} \\ \dfrac{T_{i,j} - T_{i,j-1}}{\Delta y} + 0(\Delta y) & \text{First-order backward difference} \\ \dfrac{T_{i,j+1} - T_{i,j-1}}{2\Delta y} + 0(\Delta y)^2 & \text{Second-order central difference} \end{cases}$$

Since the heat conduction equation involves the second-order derivatives, there is a need for developing the finite difference expressions of these derivatives. This is done as follows:
By adding Eqs. (3.31) and (3.35), we have

$$T_{i+1,j} + T_{i-j} = 2T_{i,j} + 2\left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2!} + 2\left(\frac{\partial^4 T}{\partial x^4}\right)_{i,j} \frac{(\Delta x)^4}{4!} + \cdots$$

Solving for $\left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j}$, we get

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + 0(\Delta x)^2 \tag{3.39}$$

In Eq. (3.39), the first term on the right-hand side is a central finite difference for the second derivative with respect to x at the grid point (i,j). The truncation terms are of order $(\Delta x)^2$. Therefore, the central difference is of second-order accuracy. The analogous expression of Eq. (3.39) for the derivative $\left(\frac{\partial^2 T}{\partial y^2}\right)_{i,j}$ can be written as

$$\left(\frac{\partial^2 T}{\partial y^2}\right)_{i,j} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} + 0(\Delta y)^2 \tag{3.40}$$

Table 3.2 summarizes the finite difference forms of first-order and second-order derivatives of T with respect to x and y.

Table 3.2 Finite difference forms of first-order and second-order derivatives of T with respect to x and y

T = T (x, y): T is a continuous function of x, y	
1. First-order forward difference of first-order derivative with respect to x	$\left(\frac{\partial T}{\partial x}\right)_{i,j} = \frac{T_{i+1,j} - T_{i,j}}{\Delta x}$
2. First-order backward difference of first-order derivative with respect to x	$\left(\frac{\partial T}{\partial x}\right)_{i,j} = \frac{T_{i,j} - T_{i-1,j}}{\Delta x}$
3. Second-order central difference of first-order derivative with respect to x	$\left(\frac{\partial T}{\partial x}\right)_{i,j} = \frac{T_{i+1,j} - T_{i-1,j}}{2\Delta x}$
4. Second-order central difference of second-order derivative with respect to x	$\left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2}$
5. First-order forward difference of first-order derivative with respect to y	$\left(\frac{\partial T}{\partial y}\right)_{i,j} = \frac{T_{i,j+1} - T_{i,j}}{\Delta y}$
6. First-order backward difference of first-order derivative with respect to y	$\left(\frac{\partial T}{\partial y}\right)_{i,j} = \frac{T_{i,j} - T_{i,j-1}}{\Delta y}$
7. Second-order central difference of first-order derivative with respect to y	$\left(\frac{\partial T}{\partial y}\right)_{i,j} = \frac{T_{i,j+1} - T_{i,j-1}}{2\Delta y}$
8. Second-order central difference of second-order derivative with respect to y	$\left(\frac{\partial^2 T}{\partial y^2}\right)_{i,j} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}$

Finite difference form of the heat conduction equation

Let us first write the steady-state two-dimensional heat conduction equation with generation of thermal energy in a Cartesian coordinate system as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\dot{q}_G}{k} = 0 \tag{3.41}$$

If we replace the partial derivatives of Eq. (3.41) by their approximate finite difference counterparts from Table 3.2, then Eq. (3.41) becomes

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} + \frac{\dot{q}_G}{k} = 0$$

if we assume $\Delta x = \Delta y$ in the grid network, then it becomes

$$T_{i,j+1} + T_{i,j-1} + T_{i+1,j} + T_{i-1,j} - 4T_{i,j} + \frac{\dot{q}_G (\Delta x)^2}{k} = 0 \tag{3.42}$$

If there is no generation of thermal energy, Eq. (3.42) becomes

$$T_{i,j+1} + T_{i,j-1} + T_{i+1,j} + T_{i-1,j} - 4T_{i,j} = 0 \tag{3.43}$$

Therefore we see that the differential equation for heat conduction at a point is transferred to an algebraic equation for that point referred to as a node in the grid network . Equation (3.42) or (3.43) is known as nodal equation for the node (i, j) and is valid for any interior node.

Derivation of nodal equation from the method of energy balance

The algebraic nodal equation may also be obtained from the principle of conservation of energy applied to a region about the node as a control volume as shown in Figure 3.11. The direction of heat flow is not known before hand. We assume that the heat fluxes are directed towards the node as shown in Figure 3.11, and write the rate equation accordingly. The depth of the control volume in a direction perpendicular to the plane of the figure is considered to be unity.

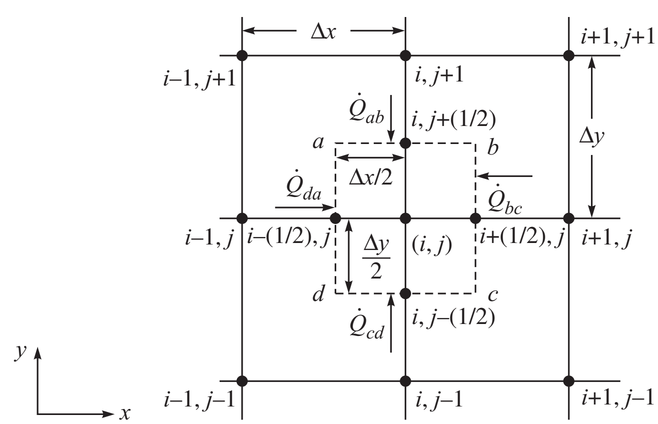


Figure 3.11 A control volume around a node (i, j) in a grid network.

Now, we can write

$$\dot{Q}_{ab} = k\cdot(\Delta x\cdot 1)\cdot\left(\frac{\partial T}{\partial y}\right)_{i,j+(1/2)}$$

$$= k\cdot(\Delta x)\frac{T_{i,j+1} - T_{i,j}}{\Delta y} \tag{3.44a}$$

$$\dot{Q}_{bc} = k\cdot(\Delta y\cdot 1)\left(\frac{\partial T}{\partial x}\right)_{i+(1/2),j}$$

$$= k\cdot\Delta y\frac{T_{i+1,j} - T_{i,j}}{\Delta x} \tag{3.44b}$$

$$\dot{Q}_{cd} = -k\cdot(\Delta x\cdot 1)\left(\frac{\partial T}{\partial y}\right)_{i,j-(1/2)}$$

$$= k\cdot(\Delta x)\frac{T_{i,j-1} - T_{i,j}}{\Delta y} \tag{3.44c}$$

$$\dot{Q}_{da} = -k\cdot(\Delta y\cdot 1)\left(\frac{\partial T}{\partial x}\right)_{i-(1/2),j}$$

$$= k\cdot\Delta y\frac{T_{i-1,j} - T_{i,j}}{\Delta x} \tag{3.44d}$$

The rate of generation of thermal energy within the control volume can be written as

$$\dot{Q}_G = \dot{q}_g\cdot\Delta x\cdot\Delta y\cdot 1 \tag{3.44e}$$

where $\dot{q}_g$ is the rate of generation of thermal energy per unit volume.

According to the principle of conservation of energy, we can write

$$\dot{Q}_{ab} + \dot{Q}_{bc} + \dot{Q}_{cd} + \dot{Q}_{da} = \dot{Q}_G \tag{3.45}$$

With the help of Eqs. (3.44a) to (3.44e) and in consideration of $\Delta x = \Delta y$, we obtain from Eq. (3.45)

$$T_{i,j+1} + T_{i,j-1} + T_{i+1,j} + T_{i-1,j} - 4T_{i,j} + \frac{\dot{q}_G(\Delta x)^2}{k} = 0 \tag{3.46}$$

In absence of the generation of any thermal energy ($\dot{q}_G = 0$), we have

$$T_{i,j+1} + T_{i,j-1} + T_{i+1,j} + T_{i-1,j} - 4T_{i,j} = 0$$

Therefore, we find that the nodal equation for heat conduction can be developed either by writing directly the differential equation of heat conduction in finite difference form or by the application of the fundamental principle of conservation of energy to a shell surrounding the node and by expressing the temperature gradients determining the heat fluxes (Fourier's law) in finite difference forms.

Treatment of nodal points at the boundary

The algebraic nodal equation (Eq. (3.42) or Eq. (3.43)) developed so far is valid for an *interior* node (i,j) whose temperature is unknown. For any heat transfer problem pertaining to a physical system, the boundaries are always specified by some given conditions. For example, the boundaries may be specified by fixed temperatures. In this situation, no nodal equation for the nodes at the boundary is needed, since the temperatures at all these points are already known. Sometimes the temperatures at the boundaries are not specified, rather the heat flux is specified or a condition of convective heat transfer in a surrounding fluid medium of given temperature is specified. Under these situations, the algebraic equation for the boundary nodes has to be developed. This equation is known as *exterior* or *boundary* nodal equation and is discussed below:

(a) *Node on a plane boundary surface with a uniform heat flux*

Let us consider the node A on a plane boundary surface which receives a uniform heat flux $\dot{q}'$ per unit surface area from the surrounding. The nodal equation for A is derived by considering the energy balance of the control volume shown by the dotted boundaries (Figure 3.12), as follows:

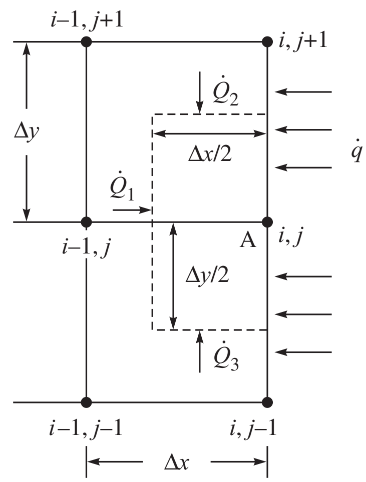


Figure 3.12 Grid network around a nodal point on a plane boundary surface with a prescribed heat flux.

$$\dot{Q}_1 = k\Delta y \cdot \frac{T_{i-1,j} - T_{i,j}}{\Delta x} \tag{3.47a}$$

$$\dot{Q}_2 = k\frac{\Delta x}{2} \cdot \frac{T_{i,j+1} - T_{i,j}}{\Delta y} \tag{3.47b}$$

$$\dot{Q}_3 = k\frac{\Delta x}{2} \cdot \frac{T_{i,j-1} - T_{i,j}}{\Delta y} \tag{3.47c}$$

We can write for the energy balance of the control volume

$$\dot{Q}_1 + \dot{Q}_2 + \dot{Q}_3 + \dot{q}'\Delta y = 0$$

Substituting the values of $\dot{Q}_1$, $\dot{Q}_2$, and $\dot{Q}_3$ from Eqs. (3.47a) to (3.47c) and in consideration of $\Delta x = \Delta y$, we finally have

$$2T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} + \frac{2\dot{q}'\Delta x}{k} = 0 \tag{3.48}$$

(b) *Node on a plane boundary surface with convection to the surrounding medium*

Let us consider the nodal point A on a plane boundary surface from which heat is being transferred by convection to the surrounding ambient at a temperature of T_∞. We consider in the similar way, as done before, the principle of conservation of energy for the control volume shown by the dotted boundaries (Figure 3.13).

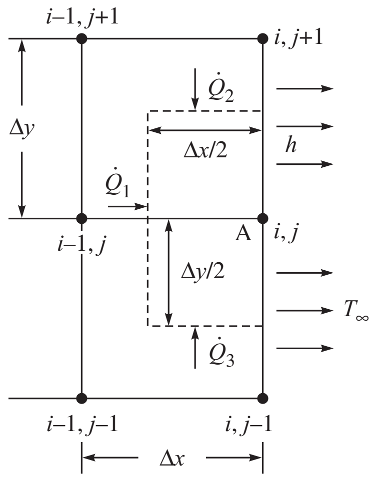


Figure 3.13 Grid network around a nodal point on a plane boundary surface with convection.

The expressions of $\dot{Q}_1$, $\dot{Q}_2$, and $\dot{Q}_3$ remain the same as given by Eqs. (3.47a) to (3.47c). The heat loss from the boundary surface of the control volume = $h\Delta y(T_{i,j} - T_\infty)$.

Therefore, we can write

$$k\Delta y\ \frac{(T_{i-1,j} - T_{i,j})}{\Delta x} + k\frac{\Delta x}{2}\frac{(T_{i,j+1} - T_{i,j})}{\Delta y} + k\frac{\Delta x}{2}\frac{(T_{i,j-1} - T_{i,j})}{\Delta y} - h\Delta y(T_{i,j} - T_\infty) = 0$$

After a simplification and in consideration of $\Delta x = \Delta y$, it becomes

$$(2T_{i-1,j} + T_{i,j+1} + T_{i,j-1}) + \frac{2h\Delta x}{k}\ T_\infty - 2\left(\frac{h\Delta x}{k} + 2\right)T_{i,j} = 0 \qquad (3.49)$$

It can be mentioned in this context that the nodal equation for an adiabatic boundary surface can be obtained by putting either $\dot{q}' = 0$ in Eq. (3.48) or $h = 0$ in Eq. (3.49). This gives

$$2T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0 \qquad (3.50)$$

(c) *Node at a corner on the boundary surface with convection*

Let us consider the corner A (Figure 3.14) on the boundary surface which is subject to a heat loss due to convection to the surrounding at a temperature of T_∞. Here $\dot{Q}_1$ and $\dot{Q}_2$ are the heat fluxes coming into the control volume across its interior surfaces (Figure 3.14) while $\dot{Q}_3$ and $\dot{Q}_4$ are the rates of heat loss by convection from the boundary surfaces.

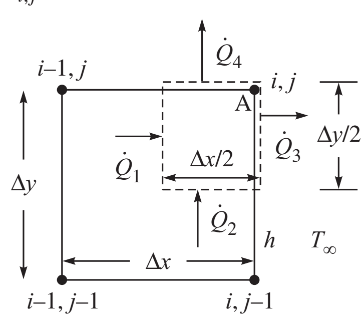


Figure 3.14 Grid network around a nodal point at the corner of a plane boundary surface with convection.

We can thus write

$$\dot{Q}_1 = k\frac{\Delta y}{2}\left(\frac{T_{i-1,j} - T_{i,j}}{\Delta x}\right) \tag{3.51a}$$

$$\dot{Q}_2 = k\frac{\Delta x}{2}\left(\frac{T_{i,j-1} - T_{i,j}}{\Delta y}\right) \tag{3.51b}$$

$$\dot{Q}_3 = h\frac{\Delta y}{2}(T_{i,j} - T_\infty) \tag{3.51c}$$

$$\dot{Q}_4 = h\frac{\Delta x}{2}(T_{i,j} - T_\infty) \tag{3.51d}$$

From the energy balance of the control volume, we can write

$$\dot{Q}_1 + \dot{Q}_2 - \dot{Q}_3 - \dot{Q}_4 = 0 \tag{3.52}$$

Substituting the expressions for $\dot{Q}_1, \dot{Q}_2, \dot{Q}_3,$ and $\dot{Q}_4$ from Eqs. (3.51a) to (3.51d) in Eq. (3.52) and in consideration of $\Delta x = \Delta y$, we finally have

$$(T_{i,j-1} + T_{i-1,j}) + \frac{2h\Delta x}{k}T_\infty - 2\left(\frac{h\Delta x}{k} + 1\right)T_{i,j} = 0 \tag{3.53}$$

(d) *Node at an interior corner on a boundary surface with convection*

Let us consider a nodal point A at the interior corner of a boundary and the control volume surrounding the point A as shown in Figure 3.15. There is a loss of heat from the boundary surfaces by convection to the surrounding at a temperature of T_∞. We can write under the situation

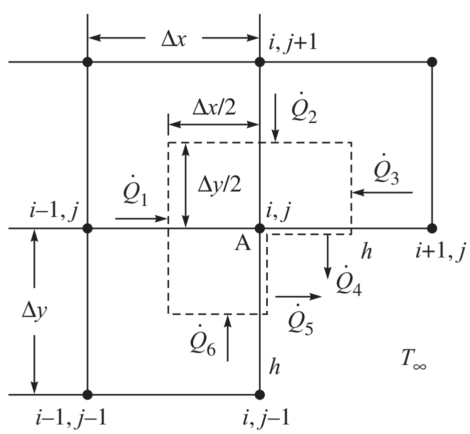


Figure 3.15 Grid network around a nodal point at the interior corner of a plane boundary surface with convection.

$$\dot{Q}_1 = k\Delta y\ \frac{T_{i-1,j} - T_{i,j}}{\Delta x} \tag{3.54a}$$

$$\dot{Q}_2 = k\Delta x \frac{T_{i,j+1} - T_{i,j}}{\Delta y} \tag{3.54b}$$

$$\dot{Q}_3 = k\frac{\Delta y}{2}\frac{T_{i+1,j} - T_{i,j}}{\Delta x} \tag{3.54c}$$

$$\dot{Q}_4 = h\frac{\Delta x}{2}(T_{i,j} - T_\infty) \tag{3.54d}$$

$$\dot{Q}_5 = h\frac{\Delta y}{2}(T_{i,j} - T_\infty) \tag{3.54e}$$

$$\dot{Q}_6 = k\frac{\Delta x}{2}\frac{T_{i,j-1} - T_{i,j}}{\Delta y} \tag{3.54f}$$

From the energy balance of the control volume

$$\dot{Q}_1 + \dot{Q}_2 + \dot{Q}_3 - \dot{Q}_4 - \dot{Q}_5 + \dot{Q}_6 = 0 \tag{3.55}$$

Substituting the values of $\dot{Q}_1, \dot{Q}_2, \dot{Q}_3, \dot{Q}_4, \dot{Q}_5$ and $\dot{Q}_6$ from Eqs. (3.54a) to (3.54f) into Eq. (3.55) and considering $\Delta x = \Delta y$, we finally get

$$2(T_{i-1,j} + T_{i,j+1}) + (T_{i+1,j} + T_{i,j-1}) + \frac{2h\Delta x}{k}T_\infty - 2\left(3 + \frac{h\Delta x}{k}\right)T_{i,j} = 0 \tag{3.56}$$

Solution of nodal equations

The algebraic nodal equation is written for any node whose temperature is unknown. While Eq. (3.43) is used for an interior node, any one of the Eqs. (3.48), (3.49), (3.50), (3.53), and (3.56) may be used for an exterior node depending upon the situation. Therefore, the number of nodal equations always equals the number of unknown temperatures. In other words, we have a system of linear algebraic equations with the number of unknowns being equal to the number of equations. A system of N such algebraic equations corresponding to N unknown temperatures is given below:

$$\begin{aligned}
a_{11}T_1 + a_{12}T_2 + a_{13}T_3 + \cdots + a_{1N}T_n &= c_1 \\
a_{21}T_1 + a_{22}T_2 + a_{23}T_3 + \cdots + a_{2N}T_n &= c_2 \\
a_{31}T_1 + a_{32}T_2 + a_{33}T_3 + \cdots + a_{3N}T_n &= c_3 \\
\vdots \qquad\qquad\qquad\qquad & \vdots \\
a_{N1}T_1 + a_{N2}T_2 + a_{N3}T_3 + \cdots + a_{NN}T_n &= c_N
\end{aligned}$$

where $T_1, T_2, T_3, \ldots, T_n$ are the unknown nodal temperatures which are specified by a single subscript rather than by the double subscript (i, j). $a_{11}, a_{12}, \ldots, a_{NN}$ and $c_1, c_2, \ldots, c_N$ are constants. The system of equations can be written in matrix notation as

$$[A]\ [T] = [C] \tag{3.57}$$

where

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \ldots a_{1N} \\ a_{21} & a_{22} & a_{23} \ldots a_{2N} \\ a_{31} & a_{32} & a_{33} \ldots a_{3N} \\ \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & a_{N3} \ldots a_{NN} \end{bmatrix}$$

$$[T] = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_N \end{bmatrix}$$

$$[C] = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}$$

There are a number of methods like (i) Matrix inversion, (ii) Gauss-seidel iteration, (iii) Cramer's rule, etc. for the solution of Eq. (3.57). However, the discussion on any of these methods is beyond the scope of this book. The readers may consult any standard textbook on mathematics or numerical analysis for the purpose.

EXAMPLE 3.3 Three sides of a square plate of 1 m by 1 m are kept at a temperature of 600 K (Figure 3.16) while the one side is exposed to a stream of air at 400 K with a convective heat transfer coefficient h = 10 W/(m^2 K). With the help of a numerical method, determine the steady-state temperature distribution within the plate. Take the thermal conductivity of the plate k = 2.5 W/(m K).

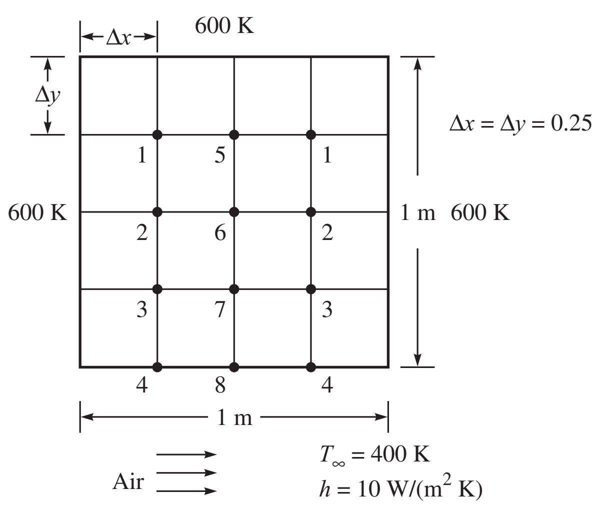


Figure 3.16 The square plate and the grid network (Example 3.3).

Solution: The grid network, as shown in Figure 3.16, consists of 12 nodal points whose temperatures are unknown. However, the number of unknowns is reduced to 8 through symmetry, in which case the temperatures of the nodal points to the left of symmetry line must be equal to the temperatures of those to the right. Nodes 1, 2, 3, 5, 6, 7 are the interior points for which Eq. (3.43) is applicable. Hence we can write for the nodal equations as

node 1: $$1200 + T_5 + T_2 - 4T_1 = 0 \tag{3.58a}$$

node 2: $$600 + T_1 + T_6 + T_3 - 4T_2 = 0 \tag{3.58b}$$

node 3: $$600 + T_2 + T_7 + T_4 - 4T_3 = 0 \tag{3.58c}$$

node 5: $$600 + 2T_1 + T_6 - 4T_5 = 0 \tag{3.58d}$$

node 6: $$2T_2 + T_5 + T_7 - 4T_6 = 0 \tag{3.58e}$$

node 7: $$2T_3 + T_6 + T_8 - 4T_7 = 0 \tag{3.58f}$$

Nodes 4 and 8 are the exterior points on the boundary with convection to surrounding air for which Eq. (3.49) is applicable. Here,

$$\frac{h\Delta x}{k} = \frac{10 \times 0.25}{2.5} = 1$$

Therefore, we can write for

node 4: $$2T_3 + T_8 + 600 + 800 - 6T_4 = 0 \tag{3.58g}$$

node 8: $$2T_4 + 2T_7 + 800 - 6T_8 = 0 \tag{3.58h}$$

Equations (3.58a) to (3.58h) can be written for the eight unknown temperatures T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8 as follows:

$$\begin{aligned}
-4T_1 + T_2 + 0 + 0 + T_5 + 0 + 0 + 0 &= -1200 \\
T_1 - 4T_2 + T_3 + 0 + 0 + T_6 + 0 + 0 &= -600 \\
0 + T_2 - 4T_3 + T_4 + 0 + 0 + T_7 + 0 &= -600 \\
2T_1 + 0 + 0 + 0 - 4T_5 + T_6 + 0 + 0 &= -600 \\
0 + 2T_2 + 0 + 0 + T_5 - 4T_6 + T_7 + 0 &= 0 \\
0 + 0 + 2T_3 + 0 + 0 + T_6 - 4T_7 + T_8 &= 0 \\
0 + 0 + 2T_3 - 6T_4 + 0 + 0 + 0 + T_8 &= -1400 \\
0 + 0 + 0 + 2T_4 + 0 + 0 + 2T_7 - 6T_8 &= -800
\end{aligned}$$

The above set of eight equations can be written in matrix notation as

$$[A]\ [T] = [B] \tag{3.59}$$

where

$$[A] = \begin{bmatrix} -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & -4 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & -6 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 & -6 \end{bmatrix}$$

$$[T] = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{bmatrix}$$

and

$$[B] = \begin{bmatrix} -1200 \\ -600 \\ -600 \\ -600 \\ 0 \\ 0 \\ -1400 \\ -800 \end{bmatrix}$$

Equation (3.59) is solved by the method of matrix inversion as

$$[T] = [A]^{-1} \; [B]$$

$$= \begin{bmatrix} 592.2186 \\ 579.7197 \\ 554.4782 \\ 498.0590 \\ 589.1548 \\ 572.1821 \\ 540.1340 \\ 479.3976 \end{bmatrix}$$

Therefore,

T_1 = 592.2186 K, T_2 = 579.7197 K, T_3 = 554.4782 K, T_4 = 498.0590 K,
T_5 = 589.1548 K, T_6 = 572.1821 K, T_7 = 540.1340 K, T_8 = 479.3976 K

EXAMPLE 3.4 Consider one-dimensional heat conduction through a fin subject to convection at boundary surfaces with insulated tip as shown in Figure 3.17. Develop a nodal equation for (a) the interior nodes along the length of the fin and (b) the boundary node at the insulated tip.

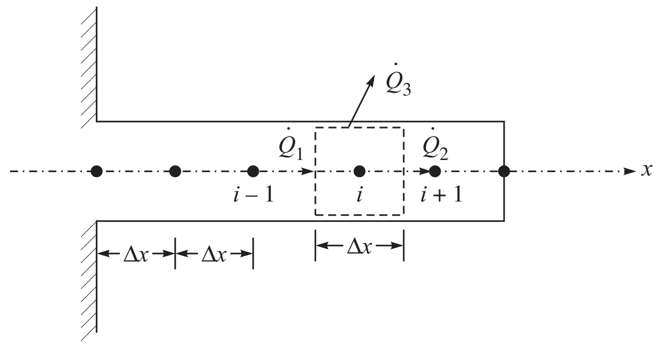


Figure 3.17 One-dimensional heat conduction through a fin (Example 3.4(a)).

Solution: (a) Since this is a one-dimensional conduction problem, the nodes are spaced only along the x-direction by a uniform distance of Δx. Consider a node (i) and the control volume surrounding the node as shown by the dotted line boundaries.

$\dot{Q}_1$ (the rate of heat conduction to the control volume from its left face) = $\dfrac{kA(T_{i-1} - T_i)}{\Delta x}$

$\dot{Q}_2$ (the rate of heat conduction out of the control volume from its right face) = $\dfrac{kA(T_i - T_{i+1})}{\Delta x}$

$\dot{Q}_3$ (the rate of convective heat loss from the control surface) = $hP\ \Delta x(T_i - T_\infty)$

where A, P and k are the cross-sectional area, the perimeter and the thermal conductivity of the fin respectively. T_∞ is the temperature of the surrounding fluid and h is the convective heat transfer coefficient.

The energy balance of the control volume at steady state gives

$$\dot{Q}_1 - \dot{Q}_2 - \dot{Q}_3 = 0$$

or

$$\frac{kA}{\Delta x}(T_{i-1} - T_i) - \frac{kA(T_i - T_{i+1})}{\Delta x} - hP\ \Delta x(T_i - T_\infty) = 0$$

After some rearrangements, we can write

$$\left\{\frac{hP(\Delta x)^2}{kA} + 2\right\}T_i - (T_{i-1} + T_{i+1}) - \frac{hP\,(\Delta x)^2}{kA}T_\infty = 0 \tag{3.60}$$

This is the required nodal equation.

(b) Here we specify the boundary node by the index i and consider a control volume of length $\Delta x/2$ as shown by the dotted line boundaries in Figure 3.18.

$\dot{Q}_1$ (the rate of heat conduction to the control volume from its left face) = $\dfrac{kA\,(T_{i-1} - T_i)}{\Delta x}$.

$\dot{Q}_2$ (the rate of heat convection out from the surfaces) = $\dfrac{hP\,\Delta x}{2}(T_i - T_\infty)$

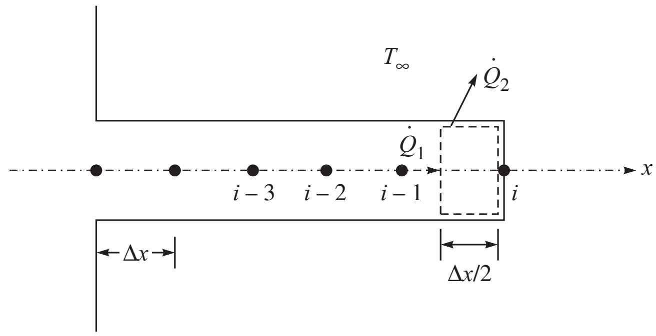


Figure 3.18 Control volume around a boundary node of a fin (Example 3.4(b)).

From the energy balance of the control volume, we can write

$$\dot{Q}_1 - \dot{Q}_2 = 0$$

or

$$\frac{kA(T_{i-1} - T_i)}{\Delta x} - \frac{hP\,\Delta x}{2}(T_i - T_\infty) = 0$$

After some rearrangements, it becomes

$$\left\{\frac{hP\,(\Delta x)^2}{2kA} + 1\right\}T_i - T_{i-1} - \frac{hP\,(\Delta x)^2}{2kA}T_\infty = 0 \tag{3.61}$$

This is the nodal equation for the boundary node.

EXAMPLE 3.5 An aluminium rod (k = 200 W/(m K)) of 20 mm diameter and 0.2 m long protrudes from a wall which is maintained at 400°C as shown in Figure 3.19. The end of the rod is insulated and the surface of the rod is exposed to air at 30°C. The air flowing around the rod gives a convection coefficient 40 W/(m^2 K). With the help of a numerical method, (a) calculate the temperature of the six nodes as shown in the figure, (b) determine the rate of heat loss from the base through the fin, and (c) compare the results with those obtained analytically using Eqs. (2.67) and (2.68) as described in Section 2.3.4.

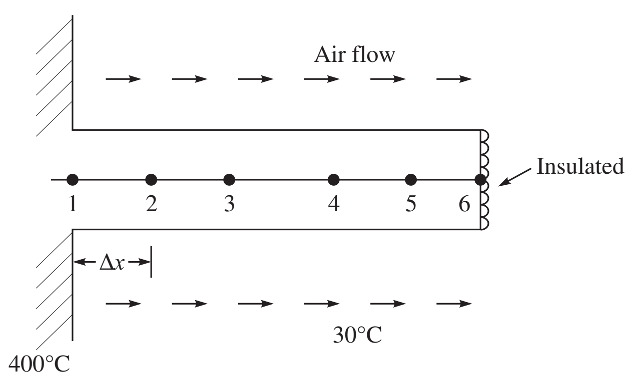


Figure 3.19 The aluminium rod (Example 3.5).

Solution: (a) We have to use the nodal equations developed in Example 3.4. Here we are given six nodes. Hence,

$$\Delta x = \frac{0.2}{6-1} = 0.04 \text{ m}$$

The node 1 is at the base temperature of 400°C. Therefore we have to find the temperature of the five nodes 2 to 6. The nodes 2, 3, 4, 5 are the interior nodes while the node 6 is a boundary node.

$$\frac{hP(\Delta x)^2}{kA} = \frac{40 \times \pi \times (0.02) \times (0.04)^2}{200 \times \pi \times (0.02)^2/4}$$
$$= 0.064$$

For nodal equations, we use Eq. (3.60) for an interior node and Eq. (3.61) for the boundary node. The nodal equations are as follows:

node 2: $2.064T_2 - (400 + T_3) - 0.064 \times 30 = 0$

or $$2.064T_2 - T_3 = 398.08 \tag{3.62}$$

node 3: $2.064T_3 - T_2 - T_4 - 0.064 \times 30 = 0$

or $$2.064T_3 - T_2 - T_4 = 1.92 \tag{3.63}$$

node 4: $$2.064T_4 - T_3 - T_5 = 1.92 \tag{3.64}$$

node 5: $$2.064T_5 - T_4 - T_6 = 1.92 \tag{3.65}$$

node 6: $$1.032T_6 - T_5 = 0.96 \tag{3.66}$$

Equations (3.62) to (3.66) are solved for the variables T_2, T_3, T_4, T_5, and T_6. We adopt the matrix inversion method for the solution. The equations are written in matrix form for the purpose as

$$\begin{bmatrix} 2.064 & -1 & 0 & 0 & 0 \\ -1 & 2.064 & -1 & 0 & 0 \\ 0 & -1 & 2.064 & -1 & 0 \\ 0 & 0 & -1 & 2.064 & -1 \\ 0 & 0 & 0 & -1 & 1.032 \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} 398.08 \\ 1.92 \\ 1.92 \\ 1.92 \\ 0.96 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} 2.064 & -1 & 0 & 0 & 0 \\ -1 & 2.064 & -1 & 0 & 0 \\ 0 & -1 & 2.064 & -1 & 0 \\ 0 & 0 & -1 & 2.064 & -1 \\ 0 & 0 & 0 & -1 & 1.032 \end{bmatrix}^{-1} \begin{bmatrix} 398.08 \\ 1.92 \\ 1.92 \\ 1.92 \\ 0.96 \end{bmatrix}$$

$$= \begin{bmatrix} 328.370 \\ 279.6820 \\ 246.9705 \\ 228.1452 \\ 222.01 \end{bmatrix} °C$$

(b) $Q = -kA\left(\dfrac{dT}{dx}\right)_{x=0}$

$$= kA\frac{T_1 - T_2}{\Delta x}$$

$$= \frac{200 \times \pi \times (0.02)^2 (400 - 328.370)}{4 \times 0.04}$$

$$= 112.516 \text{ W}$$

(c) Here we have to use Eq. (2.67) for the temperature distribution

$$m = \sqrt{\frac{40 \times \pi \times (0.02) \times 4}{200 \times \pi \times (0.02)^2}}$$

$$= 6.32 \text{ m}^{-1}$$

Following Eq. (2.67),

$$T = 30 + 370 \, \frac{\cosh\{6.32(L - x)\}}{\cosh(6.32 \times 0.2)}$$

$$= 30 + 193.61 \times \cosh\{6.32(L - x)\}$$

Therefore, we have

at node 2 (x = 0.04 m), T_2 = 331.32°C
at node 3 (x = 0.08 m), T_3 = 282.01°C
at node 4 (x = 0.12 m), T_4 = 248.89°C
at node 5 (x = 0.16 m), T_5 = 229.83°C
at node 6 (x = 0.2 m), T_6 = 223.61°C

From Eq. (2.68),

$$Q = \sqrt{40 \times \pi \times (0.02) \times 200 \times \frac{\pi}{4} \times (0.02)^2} \tanh(6.32 \times 0.2)$$

$$= 125.295 \text{ W}$$

Table 3.3 shows the comparison of the numerical and analytical results.

Table 3.3 Comparison of the numerical and analytical results of Example 3.5

Parameter	*Value obtained from the analytical formula*	*Value obtained from the numerical method*
T_2	331.32°C	328.37°C
T_3	282.01°C	279.68°C
T_4	248.89°C	246.97°C

(*Contd.*)

Table 3.3 Comparison of the numerical and analytical results of Example 3.5 (*Contd.*)

Parameter	*Value obtained from the analytical formula*	*Value obtained from the numerical method*
T_5	229.83°C	228.14°C
T_6	223.61°C	222.00°C
Q	125.29 W	112.52 W

EXAMPLE 3.6 The cross-section of a square chimney is shown in Figure 3.20. The chimney is made of concrete (k = 2.0 W/(m K)) and the flow section is 0.2 m × 0.2 m and the thickness of the wall is 0.2 m. The average temperature of the hot gas in the chimney is T_g = 400°C. The chimney is losing heat from its outer surface to the ambient air T_∞ = 20°C by convection. With the help of a numerical method and taking a grid size of $\Delta x = \Delta y$ = 0.1 m, determine the temperature at the nodal points of the cross-section. Take the average convective heat transfer coefficient both inside and outside the chimney as 20 W/(m^2 K).

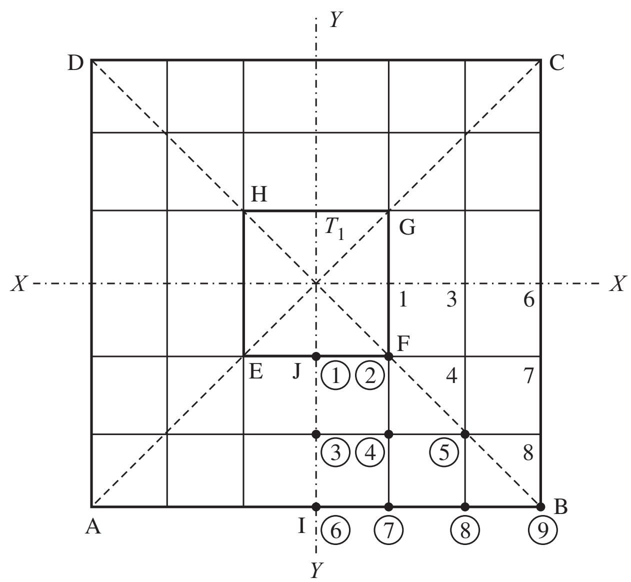


Figure 3.20 The cross-section of the chimney (Example 3.6).

Solution: In the cross-section of the chimney shown in Figure 3.20, ABCDA is the outer boundary, while EFGHE is the inner boundary and also represents the cross-section of the flow. The most pertinent aspect of the problem is the apparent symmetry about the horizontal and vertical lines (lines *XX* and *YY* respectively) passing through the midpoint of the chimney as well as the diagonal axes AC and DB.

Therefore, we need to consider only one-eighth of the geometry in the solution. Let this be the part JIBFJ as shown in the figure. The nodal network of this part consists of nine equally spaced nodes (1, 2, 3, 4, 5, 6, 7, 8 and 9). Therefore, we have to find out the temperatures of these nine nodal points. The temperatures of other nodal points of the entire cross-section of the chimney will be found by the advantage of the symmetry lines which will act as mirrors. The nodal points in the section JIBFJ considered for the solution, are shown by encircled numbers in Figure 3.20, while the nodal points as the mirror images about the symmetry lines FB are shown by the corresponding numbers without being encircled.

Here, $$\frac{h\,\Delta x}{k} = \frac{20\times 0.1}{2} = 1$$

Nodal equations:

Nodes 3, 4 and 5 are the interior nodes and therefore we use Eq. (3.43).

Node 3: $$2T_4 + T_1 + T_6 - 4T_3 = 0 \tag{3.67}$$

Node 4: $$T_3 + T_2 + T_5 + T_7 - 4T_4 = 0 \tag{3.68}$$

Node 5: $$2T_4 + 2T_8 - 4T_5 = 0 \tag{3.69}$$

Nodes 1, 6, 7, and 8 are the boundary nodes with convection and therefore we have to use Eq. (3.49) for these nodes.

Node 1: $$2T_3 + 2T_2 + 2\times 1\times 400 - 2(1 + 2)T_1 = 0$$

or $$-3T_1 + T_2 + T_3 = -400 \tag{3.70}$$

Node 6: $$2T_3 + 2T_7 + 2\times 1\times 20 - 2(1 + 2)T_6 = 0$$

or $$T_3 - 3T_6 + T_7 = -20 \tag{3.71}$$

Node 7: $$2T_4 + T_6 + T_8 + 2\times 1\times 20 - 2(1 + 2)T_7 = 0$$

or $$2T_4 + T_6 - 6T_7 + T_8 = -40 \tag{3.72}$$

Node 8: $$2T_5 + T_7 + T_9 + 2\times 1\times 20 - 2(1 + 2)T_8 = 0$$

or $$2T_5 + T_7 - 6T_8 + T_9 = -40 \tag{3.73}$$

Node 9 is an exterior corner on the boundary with convection and hence we use Eq. (3.53) as follows:

Node 9: $$2T_8 + 2\times 1\times 20 - 2(\,1 + 1)T_9 = 0$$

or $$T_8 - 2T_9 = -20 \tag{3.74}$$

Node 2 is an interior corner on the boundary with convection and hence Eq. (3.56) is used

Node 2: $$2(T_4 + T_4) + (T_1 + T_1) + 2\times 1\times 400 - 2(3 + 1)T_2 = 0$$

or $$T_1 - 4T_2 + 2T_4 = -400 \tag{3.75}$$

Now the nine simultaneous equations [Eqs. (3.67) to (3.75)] have to be solved for the nine unknown temperatures T_1 to T_9. We make use of the matrix inversion method for the solution. Equations (3.67) to (3.75) are written in matrix form for the purpose as

$$\begin{bmatrix} 1 & 0 & -4 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 0 & 0 & 2 & 0 \\ -3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & -6 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & -6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 1 & -4 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -400 \\ -20 \\ -40 \\ -40 \\ -20 \\ -400 \end{bmatrix} \tag{3.76}$$

Hence

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{bmatrix} = [A]^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -20 \\ -20 \\ -40 \\ -40 \\ -20 \\ -20 \end{bmatrix} = \begin{bmatrix} 267.442 \\ 240.224 \\ 162.101 \\ 146.727 \\ 104.158 \\ 87.509 \\ 80.426 \\ 61.589 \\ 40.795 \end{bmatrix}$$

where A is the coefficient matrix of Eq. (3.76).

The temperatures at all the nodal points on the entire cross-section of the chimney are shown in Figure 3.21.

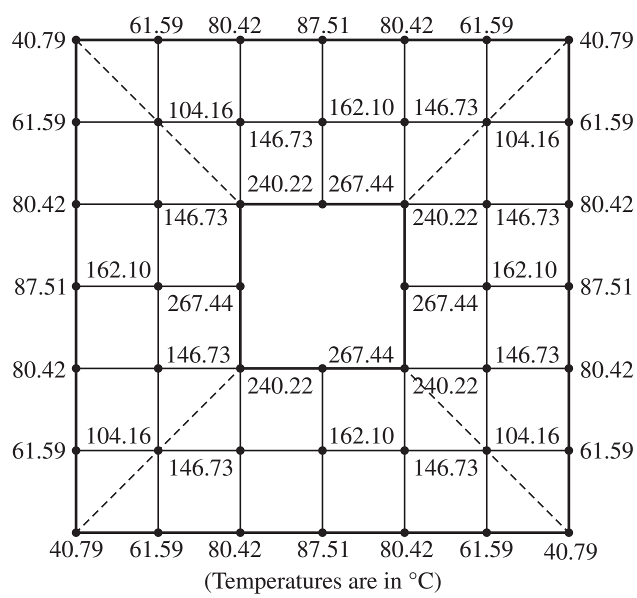


Figure 3.21 Temperatures at the nodal points of the entire cross-section of the chimney (Example 3.6).

SUMMARY

- In a multidimensional steady-state heat conduction, the temperature becomes a function of more than one space coordinate in the conducting system. There are usually three methods of analysis of multidimensional heat conduction problems, namely: (a) the analytical method, (b) the graphical method, and (c) the numerical method. The analytical methods are more accurate but complicated and restricted to simple geometries. The graphical and numerical methods, on the other hand, are less accurate, but relatively simple and also suitable for complex geometries.
- The framework of the graphical method is based on the fact that the lines of constant temperature (isotherms) are perpendicular to the lines of constant heat flux (adiabates). The objective of the method is to construct a network of curvilinear squares consisting of isotherms and adiabates. The isothermal and insulated faces are identified as the boundary lines of isotherms and adiabates respectively, while the symmetry lines always represent the adiabates. Heat flux is calculated by making use of Fourier's heat conduction equation to a square mesh. At steady state, the heat flux is same through a heat flow lane defined as the passage formed by any two adjoining heat flux lines. The heat flux through all heat flow lanes will be equal if the number of temperature increments is same for all the adjoining isotherms in a heat flow lane. The total heat flux then becomes equal to the heat flux through a curvilinear square mesh multiplied by the number of lanes. A conduction shape factor is defined as the ratio of the number of heat flow lanes to the number of temperature increments.
- In the numerical method, the temperatures are defined only at discrete grid points in a grid network in the conducting medium. The differential form of heat conduction equation is replaced by an algebraic equation by substituting the partial derivatives in the form of algebraic difference quotients with the help of Taylor's series expansion. The resulting algebraic equation for a node is known as nodal equation. If the temperatures at the boundaries are known, the nodal equations for the boundary nodes are not needed. Otherwise, the nodal equations for the boundary nodes are developed from the principle of energy conservation of a control volume surrounding the node. The number of nodal equations are always equal to the number of unknown temperatures at the grid points. Therefore, the temperatures are determined from the solution of simultaneous linear algebraic equations (the nodal equations).

REVIEW QUESTIONS

1. (a) Write down the governing differential equations for steady-state two-dimensional heat conduction in (i) a Cartesian coordinate system, (ii) a cylindrical coordinate system, and (iii) a spherical coordinate system.

(b) What is the main assumption in the separation of variables method for solving a two-dimensional heat conduction equation?

2. Why are numerical methods relatively less accurate but more advantageous over analytical methods?

3. Which one of the following two expressions of the first-order derivative of temperature in finite difference form is more accurate from the truncation error point of view?

$$\frac{\partial T}{\partial x} = \frac{T(i+1, j) - T(i-1, j)}{2\Delta x}$$

$$\frac{\partial T}{\partial x} = \frac{T(i+1, j) - T(i, j)}{\Delta x}$$

4. For the steady conduction of heat in a medium, suppose the nodal equation for the temperature T of an interior node is given by

$T_{\text{node}} = (T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}})/4,$

then answer the following questions:

(a) Is there any thermal energy generation in the medium?
(b) Is the nodal spacing constant or variable?
(c) Is the thermal conductivity of the medium constant or variable?

5. Derive the nodal temperature equation for an exterior corner node A with one adjacent side insulated and another adjacent side subjected to a convective heat transfer as shown in Figure 3.22.

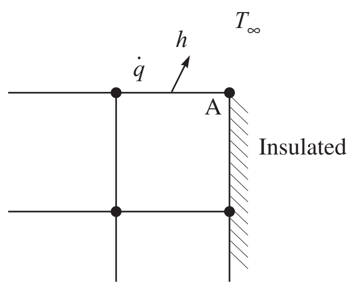


Figure 3.22 An exterior corner node (Review Question 5).

6. Derive the nodal temperature equation for the case of an exterior corner node A as shown in Figure 3.23 when both the adjacent surfaces are insulated.

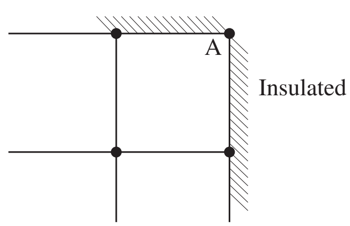


Figure 3.23 An exterior corner node (Review Question 6).

PROBLEMS

3.1 Find analytically the steady-state temperature at the centre of the rectangular plate as shown in Figure 3.24.

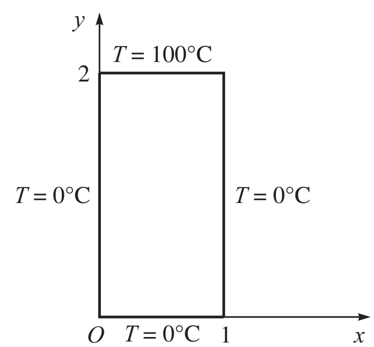


Figure 3.24 The rectangular plate (Problem 3.1).

[**Ans.** 5.5°C]

3.2 For a square plate as shown in Figure 3.25, determine analytically the steady-state temperature at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

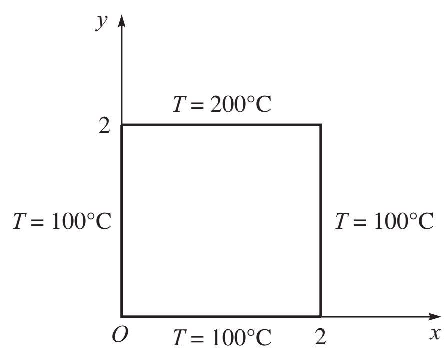


Figure 3.25 The square plate (Problem 3.2).

[**Ans.** 106.8°C]

3.3 Consider a triangular fin as shown in Figure 3.26(a).

(a) Applying the principle of conservation of energy to a control volume surrounding a node i as shown in Figure 3.26(b), show that the nodal equation can be written as

$$\left[1-\left(i-\frac{1}{2}\right)\frac{\Delta x}{L}\right](T_{i-1} - T_i) + \left[1-\left(i+\frac{1}{2}\right)\frac{\Delta x}{L}\right](T_{i+1} - T_i) + \frac{h\,(\Delta x)^2}{kL\sin\theta}(T_\infty - T_i) = 0$$

(b) With the help of the following data, determine (i) the temperatures at the nodes as shown in Figure 3.26(a), and (ii) the rate of heat transfer from the fin, and (iii) the fin efficiency.

$W = 1$ m; $L = 50$ mm; $b = 10$ mm; $T_0 = 300$°C; $T_\infty = 30$°C;
$h = 20$ W/(m^2 K); $k = 200$ W/(m K).

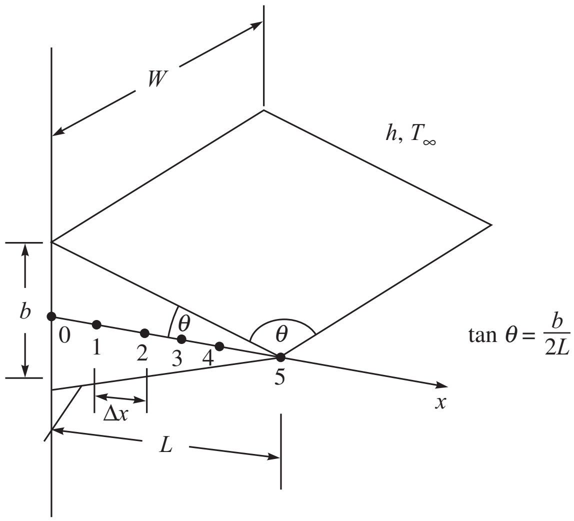


Figure 3.26(a) The triangular fin (Problem 3.3).

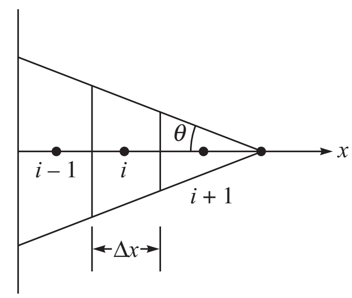


Figure 3.26(b) Control volume (Problem 3.3).

[**Ans.** (i) T_1 = 297.66°C, T_2 = 295.41°C, T_3 = 293.33°C, T_4 = 291.61°C, T_5 = 289.03°C; (ii) 531.65 W; (iii) 98%]

3.4 Consider a steady two-dimensional heat transfer in a long solid body. The cross-section of the body is shown in Figure 3.27 with the measured values of temperatures at selected points at boundary surfaces. The thermal conductivity of the body is k = 50 W/(m K) and there is no heat generation. With the help of the numerical method and in consideration of a mesh size $\Delta x = \Delta y$ = 30 mm, determine the temperatures at the indicated points on the cross-section of the body.

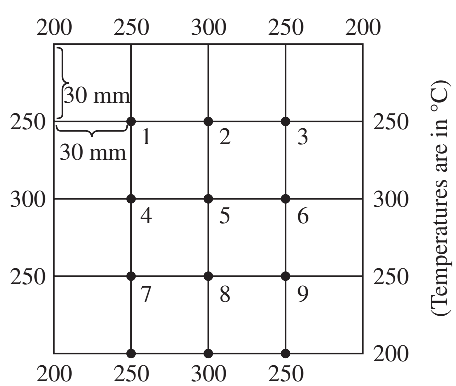


Figure 3.27 The cross-section of a long solid body (Problem 3.4).

[**Ans.** $T_1 = T_3 = T_7 = T_9$ = 262.50°C, $T_2 = T_8$ = 275°C, $T_4 = T_6$ = 275°C, T_5 = 275°C]

3.5 Consider a fin as shown in Figure 3.28. The base temperature is maintained at 400°C and the fin is exposed to a convective environment at 20°C. The heat transfer coefficient h = 50 W/(m^2 K). Determine the steady-state temperatures of the nodes shown in the figure. Also calculate the rate of heat transfer from the base through the fin. Take the thermal conductivity of the fin as $k = 10$ W/(m K).

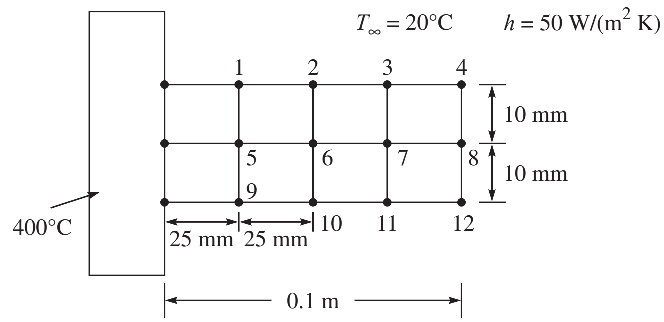


Figure 3.28 The fin (Problem 3.5).

[**Ans.** $T_1 = T_9 = 264.95$°C, $T_2 = T_{10} = 182.57$°C, $T_3 = T_{11} = 133.96$°C, $T_4 = T_{12} = 108.75$°C, $T_5 = 268.77$°C, $T_6 = 185.30$°C, $T_7 = 135.91$°C, $T_8 = 110.92$°C; 1.89 kW]

3.6 Consider a steady two-dimensional heat transfer through a long solid bar whose cross-section is shown in Figure 3.29. One of the boundary surfaces is insulated and the temperatures at the selected points on other boundary surfaces are shown in Figure 3.29. With the help of the numerical method and using a mesh size of $\Delta x = \Delta y = 10$ mm, determine the temperatures at the nodal points in the medium as shown in the figure. Take the thermal conductivity of the medium $k = 40$ W/(m K).

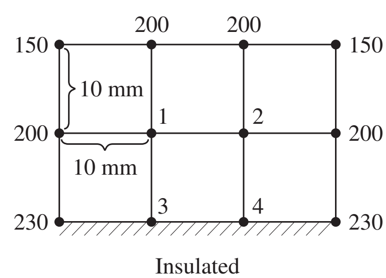


Figure 3.29 The cross-section of a long solid bar (Problem 3.6).

[**Ans.** $T_1 = 204.29$°C, $T_2 = 204.29$°C, $T_3 = 212.86$°C, $T_4 = 212.86$°C]

3.7 A 3 m long pipe with 30 mm outer diameter carries hot water. The outer surface temperature is 50°C and the pipe is buried 0.3 m deep in damp soil. The soil upper surface temperature is 10°C. The damp soil conductivity $k = 0.8$ W/(m K). Using Table 3.1 for the expression of S_F (shape factor), determine the rate of heat transfer from the pipe to the soil.

[**Ans.** 0.2 kW]

3.8 The cross-section of a long solid bar, 10 m long, 0.5 m wide and 0.3 m high is shown in Figure 3.30. The two side surfaces are kept at a constant temperature of 10°C. The lower surface is insulated, while the upper surface is heated uniformly by a 20 kW resistance heater. With the help of the numerical method and taking a mesh size of $\Delta x = \Delta y = 0.1$ m, determine the temperatures at the nodal points shown. The thermal conductivity of the bar material is 20 W/(m K).

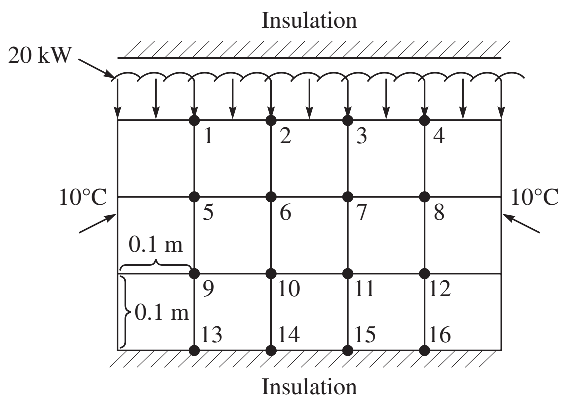


Figure 3.30 The cross-section of a long solid bar (Problem 3.8).

[**Ans.** $T_1 = 141.11°C$, $T_2 = 192.43°C$, $T_3 = 192.43°C$, $T_4 = 141.11°C$, $T_5 = 81°C$, $T_6 = 118.09°C$, $T_7 = 118.09°C$, $T_8 = 81°C$, $T_9 = 54.79°C$, $T_{10} = 80.85°C$, $T_{11} = 80.85°C$, $T_{12} = 54.79°C$, $T_{13} = 47.31°C$, $T_{14} = 69.67°C$, $T_{15} = 69.67°C$, $T_{16} = 47.31°C$]

4

Unsteady Conduction

In the preceding chapters we discussed steady-state heat conduction through solid bodies. However, in most of the practical applications, a solid body is subject to a change in its thermal environment. Therefore, some time must elapse before the body attains a steady state. During this time, the process of heating or cooling of the body takes place with temperatures at different locations within the body changing with time. The heat conduction during this transient process of heating or cooling of the system is usually referred to as transient or unsteady conduction. In this chapter, we shall discuss the different types of transient or unsteady heat conduction problems.

Learning objectives

The reading of the chapter will enable the students

- to recognize an unsteady or transient heat transfer problem,
- to develop the mathematical formulation of an unsteady heat conduction problem from physical understanding,
- to recognize the physical implications of a lumped-capacity system and to analyze the process of transient heat transfer for a lumped-capacity system,
- to obtain analytical solutions to the problems of transient heat flow through infinite and semi-infinite plates, infinite cylinders, and spheres,
- to obtain analytical solutions for unsteady heat conduction through a semi-infinite plate with a periodic change in temperature at the surface,
- to obtain analytical solutions for unsteady heat conduction in a solid medium due to a moving heat source,
- to realize the advantages of the numerical method over the analytical method in analyzing transient heat transfer problems,
- to get acquainted with explicit and implicit schemes of the numerical method and to appreciate the relative merits and demerits of the two schemes, and
- to formulate and solve the transient heat transfer problems by numerical methods.

4.1 LUMPED HEAT-CAPACITY SYSTEM

The system in which the temperature is uniform throughout, is defined as a lumped heat-capacity system. There will be no conduction of heat within the system because of a lack of spatial temperature gradient. It is found from Fourier heat conduction equation that the representative magnitude of the temperature gradient within a conducting medium, for a given heat flow, depends upon the geometrical dimension and the thermal conductivity of the medium. The heat conduction through a body of small size and of high thermal conductivity always results in a very small temperature difference within the body and thus makes the assumption of uniform temperature more realistic.

Let us consider that a hot solid ball of any arbitrary shape but of a uniform temperature T_i is suddenly dropped into a pool of liquid at a temperature of $T_f (T_f < T_i)$ as shown in Figure 4.1.

If the size of the ball is small and its thermal conductivity is high, we can consider it to be a lumped heat-capacity system so that, at any instant, the temperature is uniform throughout the ball. Since $T_i > T_f$, heat will be convected out from the surface of the ball into the pool of liquid. Because of heat flow, the ball loses its internal energy and hence its temperature is decreased. The process continues till the ball attains the temperature T_f of the surrounding fluid. The analysis for the transient cooling of the ball is made as follows.

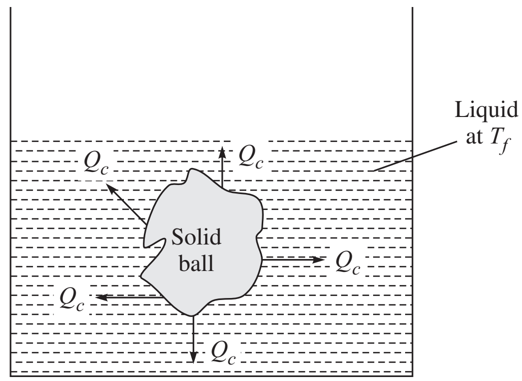


Figure 4.1 Cooling of a hot solid ball in a pool of liquid.

Let T be the temperature of the ball at any instant. The rate of heat convected out from the surface of the ball, is then given by

$$Q_c = hA_s(T - T_f)$$

where h is the convective heat transfer coefficient assumed uniform over the surface and A_s is the surface area of the ball.

Again, the rate of change of internal energy of the ball, $E = \rho \mathbb{V} c \dfrac{dT}{dt}$

where $\mathbb{V}$ and c are respectively the volume and specific heat of the solid ball.

From the principle of conservation of energy, we can write

$$E + Q_c = 0$$

or
$$\frac{dT}{dt} = -\frac{hA_s}{\rho c \mathbb{V}}(T - T_f) \tag{4.1}$$

Substituting $\theta = (T - T_f)$ in Eq. (4.1), we have

$$\frac{d\theta}{dt} = -\frac{hA_s}{\rho c \mathbb{V}}\theta \tag{4.1a}$$

Integrating Eq. (4.1a), it becomes

$$\ln \theta = -\frac{hA_s}{\rho c \mathbb{V}} t + A \tag{4.2}$$

The constant A is found out from the initial condition that at $t = 0$, $\theta = \theta_i$ (where $\theta_i = T_i - T_f$). This gives

$$A = \ln \theta_i$$

Finally, Eq. (4.2) becomes

$$\frac{\theta}{\theta_i} = \exp\left(-\frac{hA_s}{\rho c \mathbb{V}} t\right)$$

or

$$\frac{T - T_f}{T_i - T_f} = \exp\left(-\frac{t}{t_c}\right) \tag{4.3}$$

where $t_c = \rho c \mathbb{V}/hA_s$. The parameter t_c is identified as time constant.

Equation (4.3) is shown graphically in Figure 4.2.

It is evident from Eq. (4.3) and Figure 4.2 that the ball attains the surrounding fluid temperature T_f asymptotically, at an infinitely large time, i.e. as $t \to \infty$, $T \to T_f$. Moreover, for a lower value of t_c, the temperature response of the ball is faster which means that the ball attains any desired temperature in a shorter period time.

When $t = t_c$, we have from Eq. (4.3)

$$\frac{T - T_f}{T_i - T_f} = e^{-1} = 0.368$$

or

$$\frac{T_i - T}{T_i - T_f} = 0.632 \tag{4.4}$$

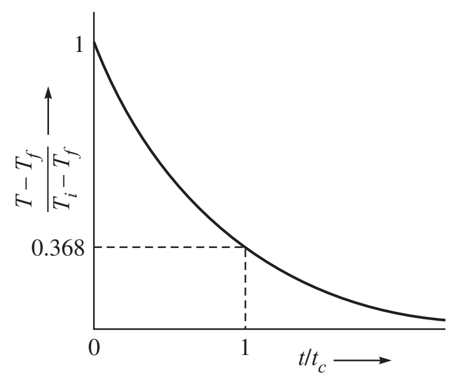


Figure 4.2 Transient temperature response of a lumped heat-capacity system.

Equation (4.4) gives a physical definition of time constant t_c as the time required to make the change in temperature to a value of 63.2 per cent of the final or maximum change in its value.

The analogous electrical circuit is shown in Figure 4.3.

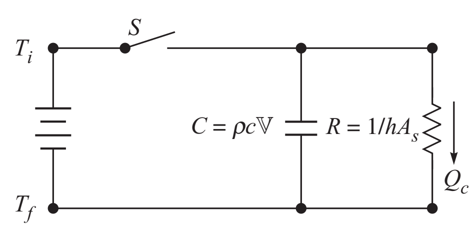


Figure 4.3 The analogous electrical circuit of a lumped heat-capacity system.

The capacitor is charged initially to a potential T_i by closing the switch S. The analogous process is the discharge of the capacitor through the resistance when the switch is opened.

Applicability of lumped heat-capacity system

We have mentioned so far that when the size of a system is small and its thermal conductivity is large, it can be approximated by a lumped heat-capacity system having a uniform temperature. This is because the surface convection resistance is large compared to the internal conduction resistance.

The surface convection resistance, $R_{\text{conv}} \sim \dfrac{1}{hA_s}$

The internal conduction resistance, $R_{\text{cond}} \sim \dfrac{L_c}{kA_s}$

where L_c is a *characteristic* dimension or length of the conducting system.

Hence,
$$\frac{R_{\text{cond}}}{R_{\text{conv}}} \sim \frac{hL_c}{k} \tag{4.5}$$

The term hL_c/k is defined as Biot number, Bi, which is the magnitude ratio of conduction resistance to convection resistance. Therefore, we can write

$$\text{Bi} = \frac{hL_c}{k} \tag{4.6}$$

If we express t_c in terms of Biot number Bi (= hL_c/k) in Eq. (4.3), we have

$$\frac{T - T_f}{T_i - T_f} = \exp\left[-\text{Bi}\left(\frac{\alpha t}{L_c^2}\right)\right]$$

where, α (= $k/\rho c$) is the thermal diffusivity of the solid material.

The term within the round brackets in the exponent of e is a dimensionless term and is defined as Fourier number Fo.

Therefore,
$$\text{Fo} = \frac{\alpha t}{L_c^2} \tag{4.7}$$

The physical significance of Fourier number will be discussed later on. The analysis of lumped heat-capacity system is valid provided Bi << 1. The characteristic dimension L_c is defined as $L_c = \mathbb{V}/A_s$, where, $\mathbb{V}$ and A_s are the volume and surface area respectively of the conducting system. The Biot number provides a measure of the temperature drop in the solid relative to the temperature difference between the surface and the bulk fluid. This is illustrated in Figure 4.4.

It is generally accepted that lumped heat-capacity system analysis is applicable when Bi < 0.1. The error associated with the analysis is very small. When the criterion Bi < 0.1 is satisfied, the variations in temperatures at different locations within the body at any instant of time remain within 5 per cent. However, one may still use the lumped heat-capacity analysis even when the Biot number is a little more than 0.1, if high accuracy is not a major concern.

EXAMPLE 4.1 An apple of 100 mm diameter is subject to a cold environment. The thermal conductivity of apple, k = 0.6 W/(m K) and the convective heat transfer coefficient of the surrounding environment, h = 10 W/(m^2 °C). Comment whether the lumped parameter analysis is suitable for predicting the temperature–time history of the apple.

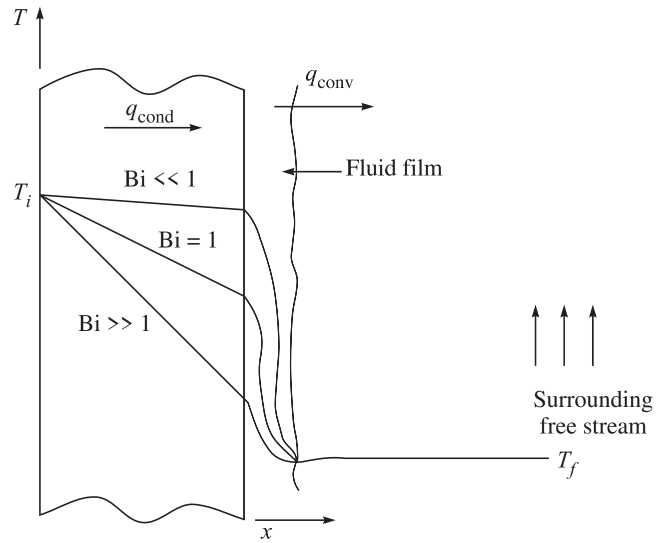


Figure 4.4 Relative magnitudes of temperature drops due to conduction and that due to surface convection at different values of Bi.

Solution: Here the characteristic dimension,

$$L_c = \frac{4/3\,\pi r^3}{4\pi r^2} = \frac{r}{3} = \frac{0.05}{3}\text{ m}$$

Hence the Biot number,

$$\text{Bi} = \frac{hL_c}{k} = \frac{10 \times 0.05}{0.6 \times 3} = 0.28$$

Since Bi > 0.1, the lumped parameter analysis is not very accurate, and is therefore not suitable.

EXAMPLE 4.2 The temperature of a gas stream is to be measured by a thermocouple whose junction can be approximated as a 1.5 mm diameter sphere. The properties of the junction are k = 40 W/(m K), ρ = 8000 kg/m^3, c = 300 J/(kg K) and the heat transfer coefficient between the junction and the gas is h = 75 W/(m^2 °C). (a) Determine the time constant of the thermocouple junction and (b) the time it takes to read a temperature which corresponds to a temperature difference (the difference in temperature between the gas stream and the thermocouple junction) of 99 per cent of its initial value.

Solution: (a) Time constant,

$$t_c = \frac{\rho c \mathbb{V}}{hA_s} = \frac{8000 \times (300) \times \frac{4}{3}\pi\,(0.00075)^3}{75 \times 4\pi\,(0.00075)^2} = 8\text{ s}$$

(b) We have to use Eq. (4.4). Thus, we have

$$\frac{T - T_i}{T_f - T_i} = 0.99\ \text{(given)}$$

Then, $$\frac{T_f - T}{T_f - T_i} = 1 - 0.99 = 0.01$$

Therefore, we can write

$$0.01 = e^{-t/8}$$

which gives $$t = 36.84 \text{ s}$$

EXAMPLE 4.3 What is the maximum edge dimension of a solid aluminum cube subject to a convective heat transfer with h = 30 W/(m^2 K) for a lumped parameter analysis to be fairly accurate? Take $k_{\text{aluminium}}$ = 250 W/(m K).

Solution: Let the maximum edge dimension be a which must correspond to Bi = 0·1.

The characteristic dimension, $$L_c = \frac{a^3}{6a^2} = \frac{a}{6}$$

Therefore, $$\text{Bi} = 0.1 = \frac{hL_c}{k} = \frac{30 \times a}{250 \times 6}$$

which gives $$a = 5 \text{ m}$$

EXAMPLE 4.4 A glass of diameter 50 mm contains some hot milk. The height of the milk in the glass is 100 mm. To cool the milk, the glass is placed into a large pan filled with cold water at 25°C. The initial temperature of the milk is 80°C. The milk is stirred slowly and continuously so that its temperature remains uniform at all times. The heat transfer coefficient between the water and the glass is 100 W/(m^2 °C). Can the milk in this case be treated as a lumped-parameter system? If so, determine (a) the time taken for the milk to cool from 80°C to 30°C and (b) the total amount of energy transferred from milk to water during the cooling process.

k_{milk} = 0.6 W/(m K), ρ_{milk} = 900 kg/m^3, $c_{p_{\text{milk}}}$ = 4.2 kJ/(kg K). Neglect the effect of stirring work.

Solution: Since the temperature of the milk is always maintained uniform by stirring, it can be considered as a lumped-parameter system.

(a) We have to use the equation,

time constant, $$t_c = \frac{\rho c \mathbb{V}}{hA_s} = \frac{900 \times 4200 \times (\pi/4) \times (0.05)^2 \times 0.1}{100 \times \pi \times (0.05) \times 0.1}$$

$$= 472.50 \text{ s} = 7.87 \text{ min}$$

Using Eq. (4.3), we can now write

$$\frac{30 - 25}{80 - 25} = e^{-t/7.87}$$

This gives $$t = 18.87 \text{ min}$$

(b) The energy transferred during the time of cooling,

$$E = \int_0^{18.87 \times 60} hA_s (T - T_f)\, dt$$

With the help of Eq. (4.3), we can write

$$
\begin{aligned}
E &= 100 \times \pi \times 0.05 \times 0.1 \int_0^{18.87 \times 60} (80 - 25)\, e^{-t/472.5}\, dt \\
&= 0.5 \times \pi \times 55 \times 472.5\, (1 - e^{-18.87 \times 60/472.5}) \\
&= 0.5 \times \pi \times 55 \times 472.5 \left(1 - \frac{5}{55}\right) \\
&= 0.5 \times \pi \times 472.5 \times 50 \text{ J} \\
&= 37.11 \text{ kJ}
\end{aligned}
$$

Composite system

The lumped heat-capacity method of analysis can also be applied to composite systems. One has to identify each system, defining the composite one for the analysis. Let us consider a thick container filled with a liquid at a temperature of T_0 which is exposed to an environment at a temperature of T_∞ ($T_\infty < T_0$) as shown in Figure 4.5. The liquid in the container is considered as system 1, while the thick container wall is considered as system 2. Let the convective heat transfer coefficient on the liquid side (inside the container) and that on the surrounding fluid side (outside the container) be h_1 and h_2 respectively.

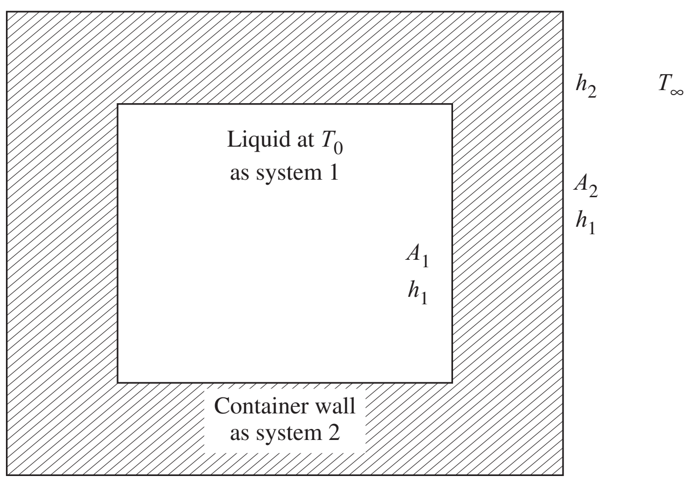


Figure 4.5 A composite lumped heat-capacity system.

First of all, one has to justify the lumped-capacity method of analysis by checking the criterion of Biot number (Bi), the magnitude ratio of conduction-to-convection resistance, of both the systems as follows.

$$\text{Bi}_{\text{sys1}}\ (= h_1 \mathbb{V}_1 / A_1 k_1) < 0.1$$

$$\text{Bi}_{\text{sys2}}\ (= h_2 \mathbb{V}_2 / A_2 k_2) < 0.1$$

where $\mathbb{V}_1$, A_1, k_1 and $\mathbb{V}_2$, A_2, k_2 are the volume, surface area, and thermal conductivity of system 1 and system 2 respectively.

If the above inequalities are satisfied, then we can go for the lumped-capacity method in which the temperature variations within the container wall and within the liquid are neglected. Let T_1 and T_2 be the instantaneous temperatures of the liquid and the container wall respectively.

Energy balance of the liquid gives

$$\rho_1 \mathbb{V}_1 c_1 \frac{dT_1}{dt} = -h_1 A_1 (T_1 - T_2) \tag{4.8}$$

and energy balance of the container wall gives

$$\rho_2 \mathbb{V}_2 c_2 \frac{dT_2}{dt} = h_1 A_1 (T_1 - T_2) - h_2 A_2 (T_2 - T_\infty) \tag{4.9}$$

where c_1 and c_2 are the specific heats of liquid and the container material respectively.

Thus we have two simultaneous linear ordinary differential equations which are to be solved for the temperatures T_1 and T_2 as functions of time, i.e. the temperature–time history of the liquid and the container. The initial conditions are as follows.

At

$$t = 0$$

$$T_1 = T_0 \tag{4.10a}$$

$$T_2 = T_0 \tag{4.10b}$$

Equations (4.8) and (4.9) can be rearranged as

$$\frac{d\theta_1}{dt} + b_1(\theta_1 - \theta_2) = 0 \tag{4.11a}$$

and

$$\frac{d\theta_2}{dt} - b_2(\theta_1 - \theta_2) + b_3\theta_2 = 0 \tag{4.11b}$$

where

$$\theta_1 = T_1 - T_\infty$$

$$\theta_2 = T_2 - T_\infty$$

$$b_1 = \frac{h_1 A_1}{\rho_1 \mathbb{V}_1 c_1}$$

$$b_2 = \frac{h_1 A_1}{\rho_2 \mathbb{V}_2 c_2}$$

$$b_3 = \frac{h_2 A_2}{\rho_2 \mathbb{V}_2 c_2}$$

Equation (4.11a) can be written as

$$\theta_2 = \frac{1}{b_1}\frac{d\theta_1}{dt} + \theta_1 \tag{4.11c}$$

or

$$\frac{d\theta_2}{dt} = \frac{1}{b_1}\frac{d^2\theta_1}{dt^2} + \frac{d\theta_1}{dt} \tag{4.12}$$

Again from Eq. (4.11b),

$$\frac{d\theta_2}{dt} = b_2\theta_1 - (b_2 + b_3)\theta_2 \tag{4.13}$$

Equating $\frac{d\theta_2}{dt}$ from Eqs. (4.12) and (4.13), and making use of Eq. (4.11c), we have

$$\frac{d^2\theta_1}{dt^2} + (b_1 + b_2 + b_3)\frac{d\theta_1}{dt} + b_1 b_3 \theta_1 = 0 \tag{4.14}$$

The solution of Eq. (4.14) is

$$\theta_1 = A\exp(m_1 t) + B\exp(m_2 t) \tag{4.15}$$

where $$m_1 = \frac{-(b_1 + b_2 + b_3) + \sqrt{(b_1 + b_2 + b_3)^2 - 4b_1 b_3}}{2}$$

and $$m_2 = \frac{-(b_1 + b_2 + b_3) - \sqrt{(b_1 + b_2 + b_3)^2 - 4b_1 b_3}}{2}$$

The constants A and B are found out from the initial conditions given by Eqs. (4.10a) and (4.10b). The two conditions can be written together as

At $t = 0$ $$\theta_1 = \theta_2 = \theta_0 \tag{4.16a}$$
(where $\theta_0 = T_0 - T_\infty$)

Using the relation given by Eq. (4.16a) in Eq. (4.11a), we have

at $t = 0$ $$\frac{d\theta_1}{dt} = 0 \tag{4.16b}$$

Now we use the two conditions given by Eqs. (4.16a) and (4.16b). This gives

$$A + B = \theta_0$$
$$m_1 A + m_2 B = 0$$

Finally, $$A = \frac{m_2\theta_0}{m_2 - m_1}$$

$$B = \frac{-m_1\theta_0}{m_2 - m_1}$$

Substituting the expressions of A and B in Eq. (4.15), we have

$$\frac{\theta_1}{\theta_0} = \frac{T_1 - T_\infty}{T_0 - T_\infty} = \frac{1}{m_2 - m_1}\{m_2\exp(m_1 t) - m_1\exp(m_2 t)\} \tag{4.17}$$

The expression of T_2 as a function of time t can be found out by substituting θ_1 from Eq. (4.17) in Eq. (4.11c). This is left as an exercise to the readers.

The electrical analogy for the composite lumped heat-capacity system is shown in Figure 4.6.

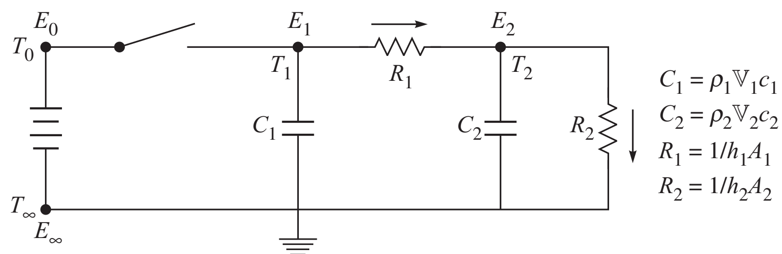


Figure 4.6 The analogous electrical circuit of a two-lumped-heat-capacity system.

Initially, the switch is closed and the two capacitors C_1 and C_2 are charged to the same potential difference ($E_0 - E_\infty$). This is similar to the initial temperature T_0 of liquid and container wall in the thermal system (Figure 4.5) while the environment is at T_∞. When the switch is opened the two capacitors discharge through two resistances R_1 and R_2 which is similar to the heat flow from the liquid and container through the respective convective resistances in the thermal system of Figure 4.5.

4.2 ONE-DIMENSIONAL UNSTEADY CONDUCTION

We shall now discuss a few simple cases of one-dimensional unsteady heat conduction where temperature is a function time and one space coordinate.

4.2.1 Transient Heat Conduction in Infinite Plates, Infinitely Long Cylinders and Spheres

Infinite plate

Let us consider a plate of thickness L such that the dimensions in the other two directions are very large compared to the thickness as shown in Figure 4.7. Such plates are usually referred to as infinite plates. Initially, the plate is at a uniform temperature T_i, and then the surfaces are suddenly lowered to a temperature T_1 ($T_1 < T_i$) and are kept at that temperature. We have to find out the temperature distribution within the plate at different instants of time, i.e. we have to determine mathematically the temperature T as a function of x and t.

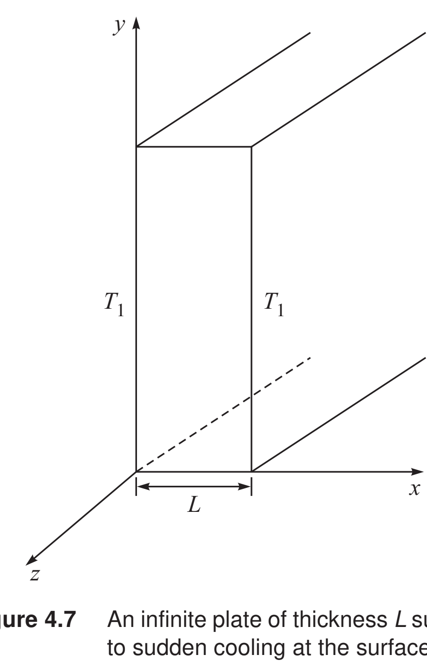


Figure 4.7 An infinite plate of thickness L subject to sudden cooling at the surfaces.

Since the dimensions of the plate in y and z directions are much larger compared to that in x-direction, the problem can be considered as one-dimensional where the temperature T is a function of only one space coordinate x. The governing equation in this case can be written as

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{4.18}$$

where α is the thermal diffusivity of the plate. By the introduction of the variable $\theta = T - T_1$, Eq. (4.18) becomes

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \tag{4.19}$$

The initial and boundary conditions are

$$\theta = \theta_i \qquad \text{at } t = 0 \qquad 0 \le x \le L \tag{4.20a}$$
$$\theta = 0 \qquad \text{at } x = 0 \qquad t > 0 \tag{4.20b}$$
$$\theta = 0 \qquad \text{at } x = L \qquad t > 0 \tag{4.20c}$$

where $\theta_i = T_i - T_1$.

We solve Eq. (4.19) by the method of separation of variables. Therefore, we assume a product solution as

$$\theta(x, t) = X(x)Y(t)$$

With the help of the above expression, Eq. (4.19) can be written as

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{\alpha Y}\frac{dY}{dt}$$

Since X is a function of x only and Y is a function of t only, we can write

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{\alpha Y}\frac{dY}{dt} = -\lambda^2$$

where λ^2 is a constant.

This results in two ordinary differential equations as follows:

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \tag{4.21a}$$

$$\frac{dY}{dt} + \alpha\lambda^2 Y = 0 \tag{4.21b}$$

In order to satisfy the boundary conditions, it is necessary that $\lambda^2 > 0$ so that the form of the solution becomes

$$\theta = (A \cos \lambda x + B \sin \lambda x)e^{-\lambda^2 \alpha t} \tag{4.22}$$

From the boundary condition (4.20b), $A = 0$. The boundary condition (4.20c) gives

$$\sin L\lambda = 0$$

or

$$\lambda = \frac{n\pi}{L} \qquad n = 1, 2, 3, \ldots$$

Hence we can write the solution of θ in the form of a series, recognizing the dependence of constant B on n as

$$\theta = \sum_{n=1}^{\infty} B_n e^{-(n\pi/L)^2 \alpha t} \sin \frac{n\pi x}{L} \tag{4.23}$$

The coefficient B_n can be found out from the boundary condition (4.20a) and by using the property of the orthogonal function $\sin \dfrac{n\pi x}{2L}$ as follows:

$$\int_0^L \theta_i \sin\left(\frac{n\pi x}{L}\right) dx = \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) dx$$

or

$$B_n = \theta_i \frac{\int_0^L \sin\left(\frac{n\pi x}{L}\right) dx}{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx}$$

$$= \frac{2\theta_i}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{4}{n\pi} \theta_i \qquad n = 1, 3, 5, \ldots$$

Therefore the final solution for the temperature becomes

$$\frac{\theta}{\theta_i} = \frac{T - T_1}{T_i - T_1} = \frac{4}{\pi} \sum \frac{1}{n} e^{-(n\pi)^2 \mathrm{Fo}} \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 3, 5, \ldots \tag{4.24}$$

where Fo (Fourier number) = $\dfrac{\alpha t}{L^2}$. (4.25)

Convection boundary condition—infinite plate

From a practical point of view, convective boundary conditions at two surfaces of an infinite plate are more appropriate than the constant temperature boundary conditions we discussed above. This happens when an infinite plate which is initially at a uniform temperature is suddenly exposed to a fluid at a different temperature so that the plate is either heated or cooled, depending upon the case, due to convective heat transfer from the surfaces.

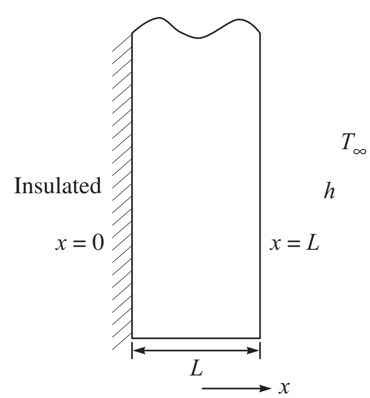


Figure 4.8 An infinite plate with one face insulated and the other face suddenly exposed to a surrounding fluid for convective heat transfer.

Let us consider that the plate as shown in Figure 4.8, is initially at a uniform temperature T_i and is suddenly exposed to a fluid at a temperature of T_∞ ($T_\infty < T_i$) with the surface at $x = 0$ being insulated. In this situation, the governing equation (Eq. 4.18) and the functional form of the solution given by Eq. (4.22) remain the same. Therefore, we can write

$$\theta = (A \cos \lambda x + B \sin \lambda x)e^{-\lambda^2 \alpha t} \tag{4.26}$$

Here, $\theta = T - T_\infty$

The boundary conditions are:

$$\text{At } t = 0, \quad \theta = \theta_i = T_i - T_\infty \tag{4.27a}$$

$$\text{At } x = 0, \quad \frac{\partial \theta}{\partial x} = 0 \tag{4.27b}$$

$$\text{At } x = L, \quad \frac{\partial \theta}{\partial x} = -\frac{h}{k}\theta \tag{4.27c}$$

where h is the average convective heat transfer coefficient over the surface at $x = L$.

The boundary condition (4.27b) gives $B = 0$.

Hence, we have

$$\theta = A \cos(\lambda x)\, e^{-\lambda^2 \alpha t} \tag{4.28}$$

Now we apply the boundary condition (4.27c) and obtain

$$\left(\frac{\partial \theta}{\partial x}\right)_{x=L} = -A\lambda \sin(\lambda L)\, e^{-\lambda^2 \alpha t}$$

Therefore,

$$A\lambda \sin(\lambda L) = \frac{h}{k} A \cos(\lambda L)$$

or

$$\cot(\lambda L) = \frac{\lambda L}{\text{Bi}} \tag{4.29}$$

where Bi (Biot number) $= \dfrac{hL}{k}$.

An infinite number of values of λ satisfy Eq. (4.29) and these values of λ are called the characteristic values or eigenvalues. The simplest way to determine the numerical values of λ is to plot $\cot(\lambda L)$ and $\lambda L/\text{Bi}$ against λL. This is shown in Figure 4.9. The values of λ at the points of intersection of these curves are the characteristic values which satisfy Eq. (4.29).

The value of $\lambda = 0$ is disregarded since it gives the trivial solution $\theta = A$ (Eq. (4.28)). A particular solution given by Eq. (4.28) corresponds to each value of λ which in turn yields different values of A. To identify the correspondence between A and λ, a subscript notation is adopted. The complete solution can now be written as

$$\theta = \sum_{n=1}^{\infty} A_n \cos(\lambda_n x)\, e^{-\alpha \lambda_n^2 t} \tag{4.30}$$

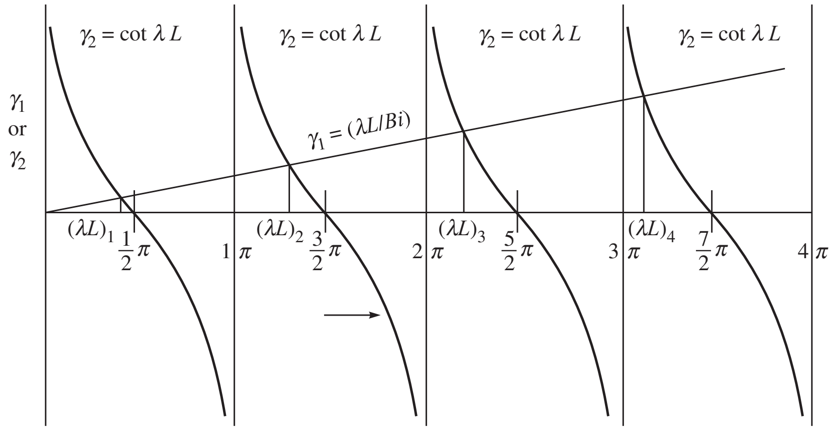


Figure 4.9 Graphical solution of Eq. (4.29).

The constant A_n is found out by making use of the boundary condition (4.27a), which gives

$$\theta_i = \sum_{n=1}^{\infty} A_n \cos(\lambda_n x) \tag{4.31}$$

It can be shown that the function $\cos(\lambda_n x)$ is orthogonal between $x = 0$ and $x = L$, provided $\cot(\lambda_m L)/\lambda_m L = \cot(\lambda_n L)/\lambda_n L$ which is a corollary of Eq. (4.29).

This means

$$\int_0^L \cos\lambda_m x \cos\lambda_n x \, dx \qquad \left[\begin{array}{l} = 0 \text{ if } m \neq n \\ \neq 0 \text{ if } m = n \end{array}\right.$$

Utilizing this property, we can write from Eq. (4.31),

$$\theta_i \int_0^L \cos(\lambda_n x)\, dx = A_n \int_0^L \cos(\lambda_n x) \sum_{n=1}^{\infty} A_n \cos(\lambda_n x)$$

$$= A_n \int_0^L \cos^2(\lambda_n x)\, dx$$

or

$$A_n = \theta_i \frac{\int_0^L \cos(\lambda_n x)\, dx}{\int_0^L \cos^2(\lambda_n x)\, dx}$$

$$= \theta_i \frac{\frac{1}{\lambda_n}\sin(\lambda_n L)}{\frac{L}{2} + \frac{1}{2\lambda_n}\sin(\lambda_n L)\cos(\lambda_n L)}$$

$$= \frac{2\theta_i \sin(\lambda_n L)}{L\lambda_n + \sin(\lambda_n L)\cos(\lambda_n L)}$$

Substituting the value of A_n in Eq. (4.30), we obtain the general solution of θ as

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = 2\sum_{n=1}^{n=\infty} \frac{\sin(\lambda_n L)\cos(\lambda_n x)\, e^{-\alpha\lambda_n^2 t}}{L\lambda_n + \sin(\lambda_n L)\cos(\lambda_n L)} \tag{4.32}$$

At any instant, the rate of heat transfer per unit area, from the surface (at $x = L$) of the plate to the fluid is given by

$$q = -k\left(\frac{\partial T}{\partial x}\right)_{x=L} \tag{4.33}$$

From Eq. (4.32),

$$\left(\frac{\partial T}{\partial x}\right)_{x=L} = (T_i - T_\infty)\, 2\sum_{n=1}^{\infty} \frac{-\lambda_n \sin^2(\lambda_n L)\, e^{-\alpha\lambda_n^2 t}}{L\lambda_n + \sin(\lambda_n L)\cos(\lambda_n L)}$$

Substituting the value of $(\partial T/\partial x)_{x=L}$ in Eq. (4.33), we have

$$q = 2k\,(T_i - T_\infty)\sum_{n=1}^{\infty} \frac{\lambda_n \sin^2(\lambda_n L)\, e^{-\alpha\lambda_n^2 t}}{L\lambda_n + \sin(\lambda_n L)\cos(\lambda_n L)} \tag{4.34}$$

The change in internal energy of the plate or the total amount of heat transferred per unit area during a time t is given by

$$Q = \int_0^t q\,dt = 2k(T_i - T_\infty)\int_0^t \sum_{n=1}^{\infty} \frac{\lambda_n \sin^2(\lambda_n L)\, e^{-\alpha\lambda_n^2 t}}{L\lambda_n + \sin(\lambda_n L)\cos(\lambda_n L)}\,dt$$

$$= \frac{2k(T_i - T_\infty)}{\alpha}\sum_{n=1}^{\infty} \frac{\sin^2(\lambda_n L)\,(1 - e^{-\alpha\lambda_n^2 t})}{[L\lambda_n + \sin(\lambda_n L)\cos(\lambda_n L)]\,\lambda_n} \tag{4.35}$$

Dimensionless representation: The expression for temperature distribution (Eq. (4.32)) and that for heat transfer (Eq. (4.35)) can be written in dimensionless forms. For this purpose, we define a dimensionless parameter δ_n as

$$\delta_n = \lambda_n L$$

If we substitute λ_n in terms of δ_n in Eq. (4.32), we have

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = 2\sum_{n=1}^{\infty} \frac{\sin(\delta_n)\cos[\delta_n (x/L)]\, e^{-\mathrm{Fo}\,\delta_n^2}}{\delta_n + \sin(\delta_n)\cos(\delta_n)} \tag{4.36}$$

where Fo is the Fourier number defined by Eq. (4.25). Equation (4.36) is the required expression for temperature distribution in dimensionless form. In order to make Eq. (4.35) dimensionless, an initial energy Q_i of the plate per unit surface area is defined as

$$Q_i = \rho c L(T_i - T_\infty) \tag{4.37}$$

Dividing Q of Eq. (4.35) by Q_i of Eq. (4.37) and in consideration of $\lambda_n = \delta_n/L$, we have

$$\frac{Q}{Q_i} = 2\sum_{n=1}^{\infty} \frac{\sin^2\delta_n\,[1 - e^{-\mathrm{Fo}\,\delta_n^2}]}{\delta_n\,(\delta_n + \sin\delta_n\cos\delta_n)} \tag{4.38}$$

We can also write Eq. (4.29) in terms of δ_n as

$$\delta_n \tan \delta_n = \text{Bi} \tag{4.39}$$

The first four roots of δ_n from Eq. (4.39) for different values of Bi are shown in Table 4.1.

Table 4.1 The first four roots of the transcendental equation, $\delta_n \tan \delta_n = \text{Bi}$, for transient conduction in a plane wall

Bi = hL/k	δ_1	δ_2	δ_3	δ_4
0	0	3.1416	6.2832	9.4248
0.001	0.0316	3.1419	6.2833	9.4249
0.002	0.0447	3.1422	6.2835	9.4250
0.004	0.0632	3.1429	6.2838	9.4252
0.006	0.0774	3.1435	6.2841	9.4254
0.008	0.0893	3.1441	6.2845	9.4256
0.01	0.0998	3.1448	6.2848	9.4258
0.02	0.1410	3.1479	6.2864	9.4269
0.04	0.1987	3.1543	6.2895	9.4290
0.06	0.2425	3.1606	6.2927	9.4311
0.08	0.2791	3.1668	6.2959	9.4333
0.1	0.3111	3.1731	6.2991	9.4354
0.2	0.4328	3.2039	6.3148	9.4459
0.3	0.5218	3.2341	6.3305	9.4565
0.4	0.5932	3.2636	6.3461	9.4670
0.5	0.6533	3.2923	6.3616	9.4775
0.6	0.7051	3.3204	6.3770	9.4879
0.7	0.7506	3.3477	6.3923	9.4983
0.8	0.7910	3.3744	6.4074	9.5087
0.9	0.8274	3.4003	6.4224	9.5190
1.0	0.8603	3.4256	6.4373	9.5293
1.5	0.9882	3.5422	6.5097	9.5801
2.0	1.0769	3.6436	6.5783	9.6296
3.0	1.1925	3.8088	6.7040	9.7240
4.0	1.2646	3.9352	6.8140	9.8119
5.0	1.3138	4.0336	6.9096	9.8928
6.0	1.3496	4.1116	6.9924	9.9667
7.0	1.3766	4.1746	7.0640	10.0339
8.0	1.3978	4.2264	7.1263	10.0949
9.0	1.4149	4.2694	7.1806	10.1502
10.0	1.4289	4.3058	7.2281	10.2003
15.0	1.4729	4.4255	7.3959	10.3898
20.0	1.4961	4.4915	7.4954	10.5117
30.0	1.5202	4.5615	7.6057	10.6543
40.0	1.5325	4.5979	7.6647	10.7334
50.0	1.5400	4.6202	7.7012	10.7832
60.0	1.5451	4.6353	7.7259	10.8172
80.0	1.5514	4.6543	7.7573	10.8606
100.0	1.5552	4.6658	7.7764	10.8871
∞	1.5708	4.7124	7.8540	10.9956

From Eqs. (4.36), (4.38), and (4.39), we can write

$$\frac{T - T_\infty}{T_i - T_\infty} = \text{a function of (Bi, Fo, } x/L)$$

$$\frac{Q}{Q_i} = \text{a function of (Bi, Fo, } x/L)$$

Therefore we conclude that in an unsteady state heat conduction in a solid with convection at the surfaces, the pertinent dimensionless parameters which govern the temperature distribution and heat flow are Biot number (Bi) and Fourier number (Fo).

Infinitely long cylinder

An infinitely long cylinder is one whose diameter is very small compared to its length. Let the radius of such a cylinder be r_0. The cylinder is initially at a temperature T_i, and at $t = 0$, the cylinder is suddenly exposed to a fluid at a temperature T_∞, with a convection heat transfer coefficient h at the cylinder surface. In such a situation, temperature T within the cylinder is a function of radial coordinate r (measured from the central axis of the cylinder) and time t. The governing equation is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{4.40}$$

The solution of Eq. (4.40) can be obtained by the method of separation of variables, similar to that described for the infinite flat plate, as

$$\frac{T - T_\infty}{T_i - T_\infty} = \sum_{n=1}^{\infty} A_n J_0[\delta_n (r/r_0)] e^{-\text{Fo}\,\delta_n^2} \tag{4.41a}$$

where
$$A_n = \frac{2}{\delta_n} \frac{J_1(\delta_n)}{[J_0^2(\delta_n) + J_1^2(\delta_n)]} \tag{4.41b}$$

The values of δ_n are the positive roots of characteristic equation

$$\delta_n \frac{J_1(\delta_n)}{J_0(\delta_n)} = \text{Bi}$$

where

$$\text{Fo (Fourier number)} = \frac{\alpha t}{r_0^2} \tag{4.41c}$$

$$\text{Bi (Biot number)} = \frac{h r_0}{k} \tag{4.41d}$$

The quantities J_0 and J_1 are Bessel functions of the first kind of zeroth and first order respectively. Their values are given in Table 4.2. The expression for dimensionless total heat transfer during a time interval of t can be obtained in the similar way as done in case of a plane wall, and is written as

$$\frac{Q}{Q_i} = 4\sum_{n=1}^{\infty} \frac{1}{\delta_n^2} \frac{J_1^2(\delta_n)}{[J_0^2(\delta_n) + J_1^2(\delta_n)]} (1 - e^{\text{Fo}\,\delta_n^2}) \tag{4.42}$$

The deductions of Eqs. (4.41a) and (4.42) are left as an exercise to the readers.

Table 4.2 Bessel function of the first kind

x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$
0.0	1.0000	0.0000	1.3	0.6201	0.5220
0.1	0.9975	0.0499	1.4	0.5669	0.5419
0.2	0.9900	0.0995	1.5	0.5118	0.5579
0.3	0.9776	0.1483	1.6	0.4554	0.5699
0.4	0.9604	0.1960	1.7	0.3980	0.5778
0.5	0.9385	0.2423	1.8	0.3400	0.5815
0.6	0.9120	0.2867	1.9	0.2818	0.5812
0.7	0.8812	0.3290	2.0	0.2239	0.5767
0.8	0.8463	0.3688	2.1	0.1666	0.5683
0.9	0.8075	0.4059	2.2	0.1104	0.5560
1.0	0.7652	0.4400	2.3	0.0555	0.5399
1.1	0.7196	0.4709	2.4	0.0025	0.520
1.2	0.6711	0.4983			

Sphere

The governing differential equation for one-dimensional transient heat conduction through a sphere is

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{4.43}$$

For a sphere of radius r_0 and of an initial uniform temperature T_i which is suddenly exposed at $t = 0$ to a fluid at T_∞, the solution of Eq. (4.43) is

$$\frac{T - T_\infty}{T_i - T_\infty} = \sum_{n=1}^{a} \frac{A_n}{\delta_n (r/r_0)} \sin[\delta_n (r/r_0)] e^{-\mathrm{Fo}\,\delta_n^2} \tag{4.44}$$

where
$$A_n = \frac{4[\sin\delta_n - \delta_n \cos\delta_n]}{2\,\delta_n - \sin 2\delta_n} \tag{4.45}$$

The values of δ_n are the positive roots of the characteristic equation

$$1 - \delta_n \cot \delta_n = \mathrm{Bi} \tag{4.46}$$

The expression for dimensionless total heat transfer during a time interval of t can be obtained from Eq. (4.44) as

$$\frac{Q}{Q_i} = 3\sum_{n=1}^{\infty} \frac{A_n}{\delta_n^3} (\sin\delta_n - \delta_n \cos\delta_n)(1 - e^{-\mathrm{Fo}\,\delta_n^2}) \tag{4.47}$$

The deductions of Eqs. (4.44) and (4.47) are left as an exercise to the readers.

4.2.2 The Heisler Charts for Transient Heat Flow

The analytical expressions for temperature distribution and the heat flux in case of an infinite plate, a very long cylinder and a sphere having uniform initial temperature and suddenly exposed to an environment at a different temperature have already been presented in the form of an infinite series. Heisler [1] showed that for values of Fourier number Fo = ($\alpha t/L^2$ or $\alpha\, t/r_0^2$) > 0.2, the infinite series solution can be approximated with a fair accuracy (an error within 1%) by the first term of the series. The approximated expressions for temperature distributions and heat flux were then plotted graphically which are known as Heisler charts. The Heisfer charts are discussed below:

An infinite plate

For Fo > 0.2, Eq. (4.36) can be approximated by its first term as

$$\frac{\theta}{\theta_i} = Ae^{-\mathrm{Fo}\delta_1^2}\cos[\delta_1(x/L)] \tag{4.48a}$$

where

$$A = \frac{4\sin\delta_1}{2\delta_1 + \sin 2\delta_1} \tag{4.48b}$$

and

$$\delta_1 \tan\delta_1 = \mathrm{Bi} = \frac{hL}{k} \tag{4.48c}$$

Again, we can write

$$\frac{\theta_0}{\theta_i} = Ae^{-\mathrm{Fo}\delta_1^2} \tag{4.48d}$$

where θ_0 is the value of θ at $x = 0$

and

$$\frac{\theta}{\theta_0} = \cos\ \delta_1(x/L) \tag{4.48e}$$

Equation (4.38) for total heat transfer can accordingly be approximated by

$$\frac{Q}{Q_i} = A\frac{\sin\delta_1}{\delta_1}(1 - e^{-\mathrm{Fo}\delta_1^2}) \tag{4.48f}$$

Equations (4.48d), (4.48e) and (4.48f) are presented in graphical forms in Figures 4.10 to 4.13. These are known as Heisler charts for infinite plates. For problems of practical importance, 1/Bi lies between 0 and 100. Hence, the Heisler charts are available for 0 < (1/Bi) < 100. Moreover, θ_0 in Heisler charts is referred to as mid-plane temperature (Figures 4.10 and 4.11) of a plate of thickness $2L$ which is initially at a uniform temperature T_i but suddenly exposed to a surrounding fluid at a temperature of T_∞. This situation is same as that of a plate of thickness L insulated at one end ($x = 0$) and exposed to a fluid of temperature T_∞ at the other end ($x = L$).

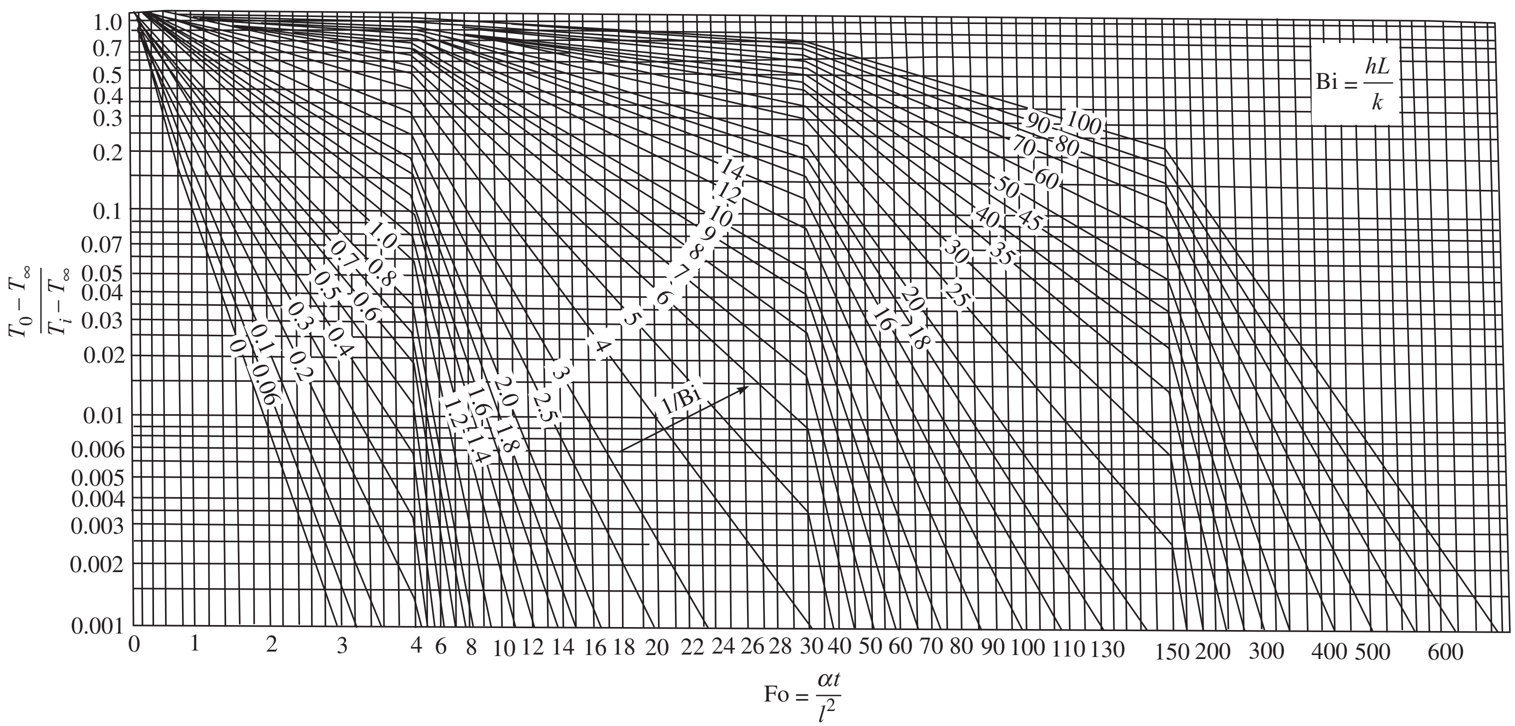


Figure 4.10 Mid-plane temperature for an infinite flat plate of thickness 2 *L*.

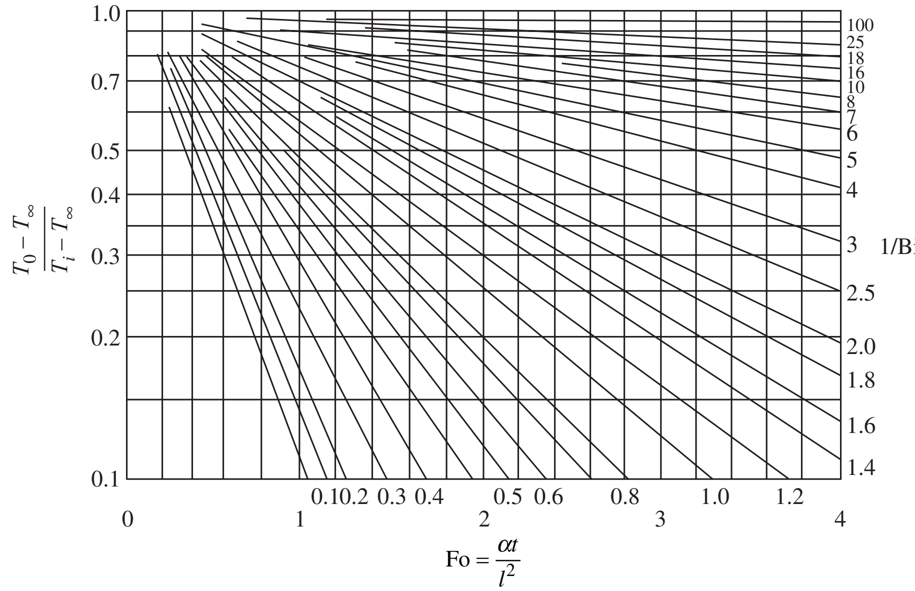


Figure 4.11 Mid-plane temperature for an infinite flat plate of thickness $2L$ in an expanded scale ($0 < Fo < 4$).

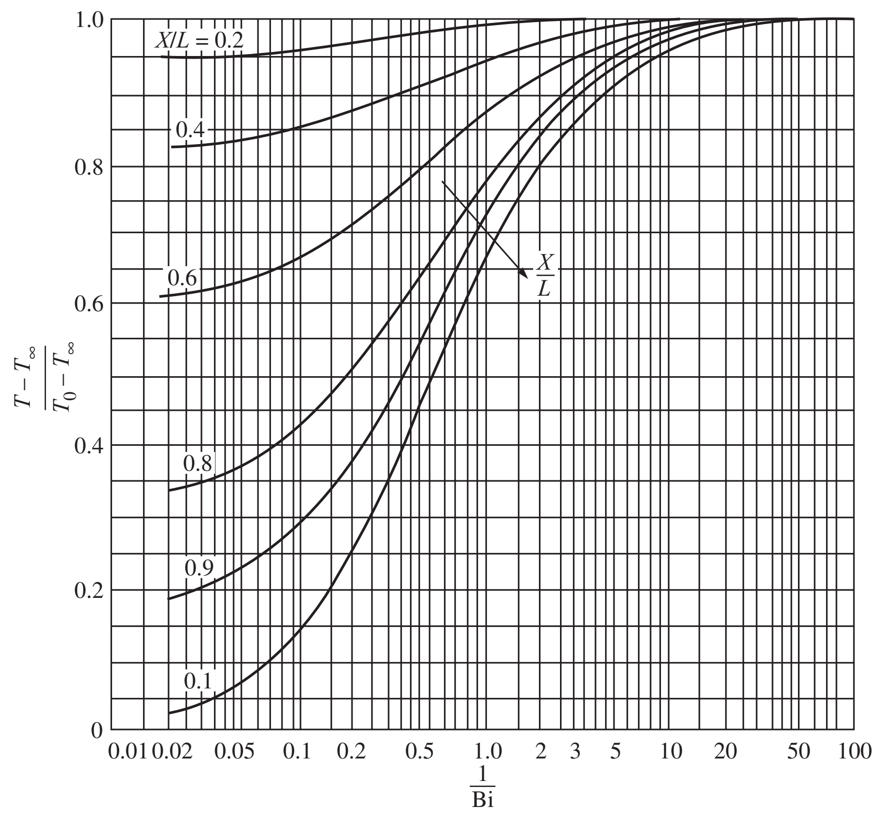


Figure 4.12 The ratio of temperature at any plane to that at the mid-plane in an infinite flat plate of thickness $2L$.

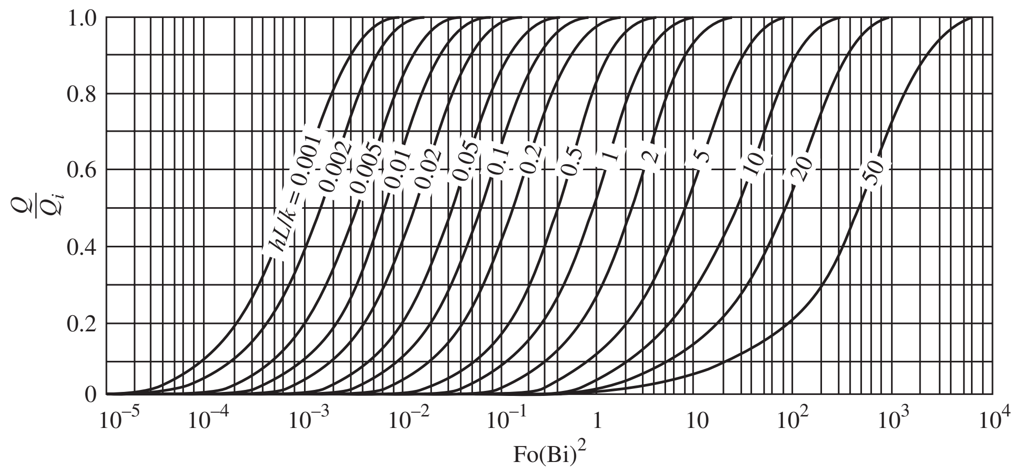


Figure 4.13 Dimensionless heat transfer Q/Q_i for an infinite flat plate of thickness $2L$.

Long cylinder

In a similar fashion, for Fo > 0.2, the approximated expressions θ/θ_i, θ/θ_0 and Q/Q_i for a long cylinder can be written, following Eqs. (4.41a) and (4.42), as

$$\frac{\theta}{\theta_i} = A_1 J_0[\delta_1(r/r_0)]e^{-\mathrm{Fo}\delta_1^2} \tag{4.49a}$$

where

$$A_1 = \frac{2}{\delta_1}\frac{J_1(\delta_1)}{[J_0^2(\delta_1) + J_1^2(\delta_1)]} \tag{4.49b}$$

and

$$\delta_1 \frac{J_1(\delta_1)}{J_0(\delta_1)} = \mathrm{Bi} \tag{4.49c}$$

Again,

$$\frac{\theta_0}{\theta_i} = A_1 e^{-\mathrm{Fo}\delta_1^2} \tag{4.49d}$$

Therefore,

$$\frac{\theta}{\theta_0} = J_0[\delta_1(r/r_0)] \quad \{\text{dividing (4.49a) by (4.49d)}\} \tag{4.49e}$$

where θ_0 is the value of θ at $r = r_0$.

$$\frac{Q}{Q_i} = \frac{2A_1}{\delta_1}(1 - e^{-\mathrm{Fo}\delta_1^2}) \tag{4.49f}$$

Figures 4.14 to 4.17 which are the graphical representations of Eqs. (4.49d), (4.49e), and (4.49f) are known as Heisler charts for long cylinders.

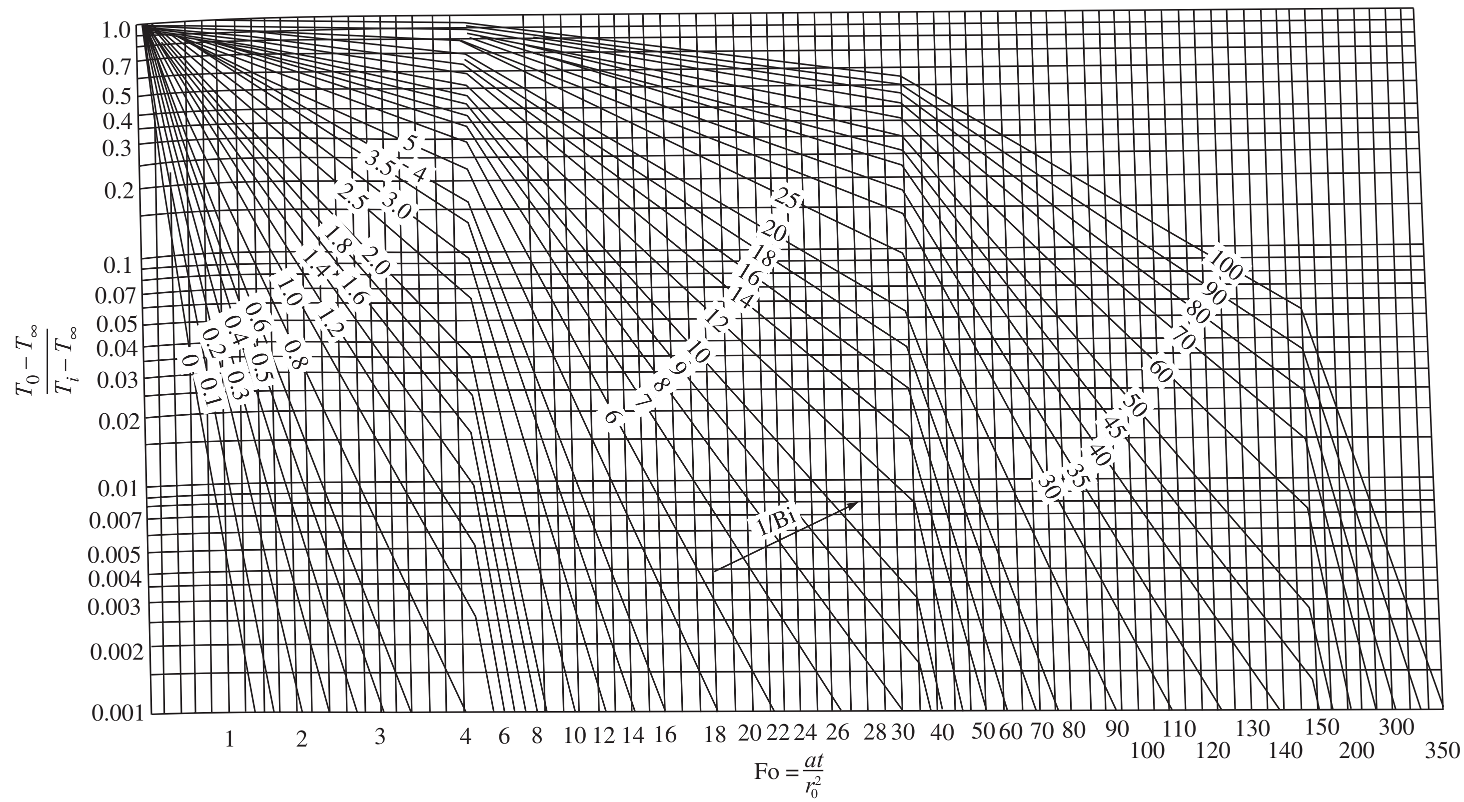


Figure 4.14 The temperature at the axis of an infinitely long cylinder of radius r_0.

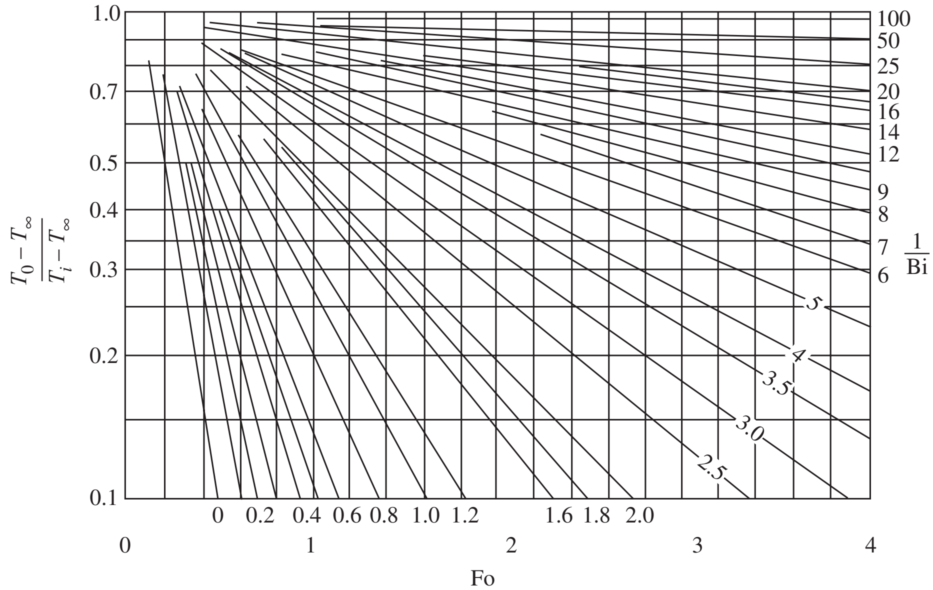


Figure 4.15 The temperature at the axis of an infinitely long cylinder of radius r_0 in an expanded scale (0 < Fo < 4).

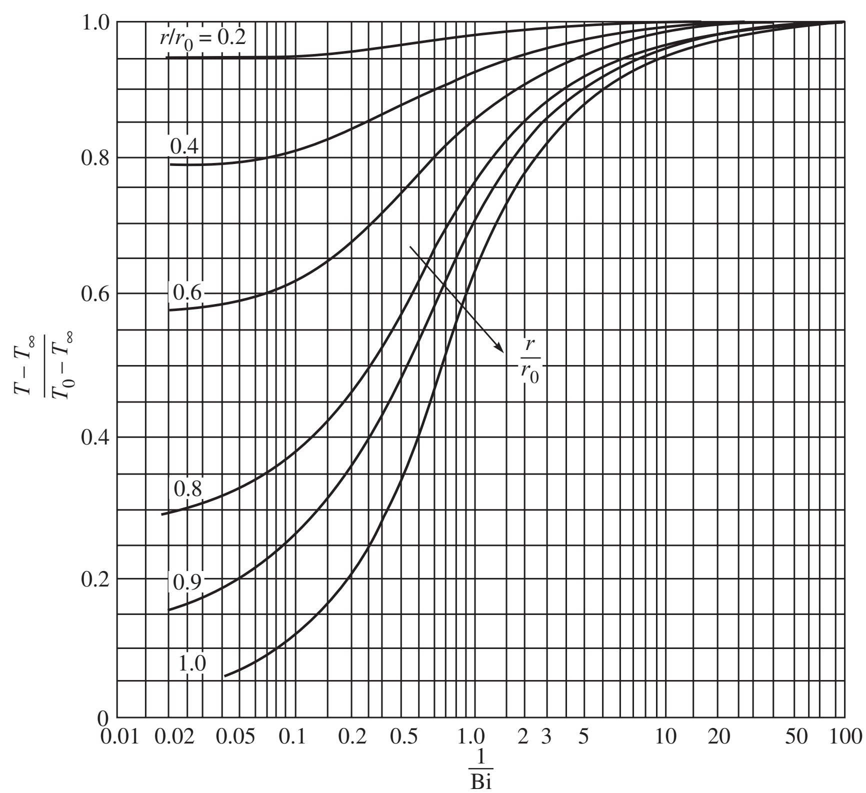


Figure 4.16 The ratio of temperature at any radial location to that at the axis of an infinitely long cylinder of radius r_0.

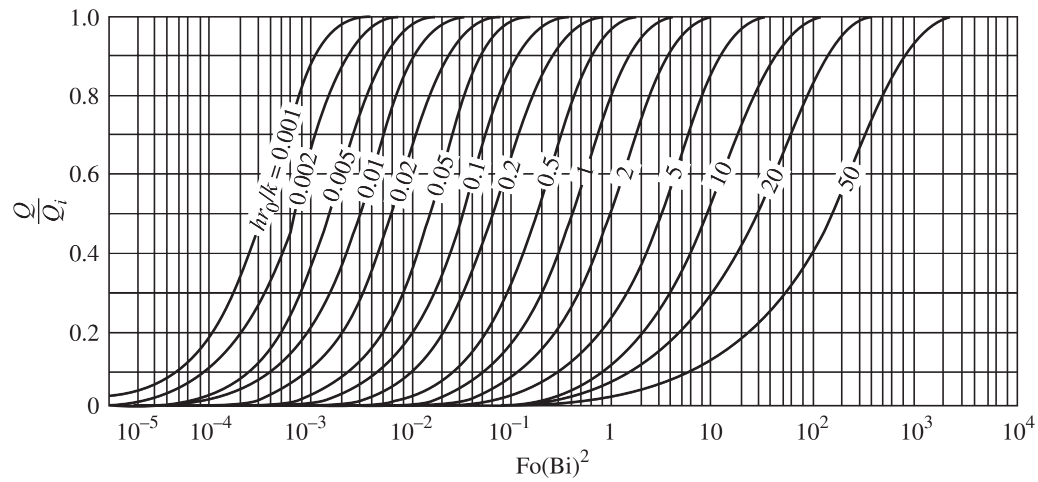


Figure 4.17 Dimensionless heat transfer Q/Q_i for an infinitely long cylinder of radius r_0.

Sphere

The approximate expressions of temperature distribution and heat flux in case of a sphere for Fo > 0.2 can be written, following Eqs. (4.44) and (4.47), as

$$\frac{\theta}{\theta_i} = A_1 \frac{\sin [\delta_1 (r/r_0)] e^{-\text{Fo}\delta_1^2}}{\delta_1 (r/r_0)} \tag{4.50a}$$

where
$$A_1 = \frac{4 \sin \delta_1 - \delta_1 \cos \delta_1}{2\delta_1 - \sin 2\delta_1} \tag{4.50b}$$

$$1 - \delta_1 \cot \delta_1 = \text{Bi} \tag{4.50c}$$

$$\frac{\theta_0}{\theta_i} = A_1 e^{-\text{Fo}\delta_1^2} \tag{4.50d}$$

$$\frac{\theta}{\theta_0} = \frac{\sin [\delta_1 (r/r_0)]}{\delta_1 (r/r_0)} \tag{4.50e}$$

where θ_0 is the value of θ at $r = r_0$.

$$\frac{Q}{Q_i} = 3\frac{A_1}{\delta_1^3}(\sin \delta_1 - \delta_1 \cos \delta_1)(1 - e^{-\text{Fo}\delta_1^2}) \tag{4.50f}$$

The graphical forms of Eqs. (4.50d), (4.50e), and (4.50f), as represented by Figures 4.18 to 4.21, are known as Heisler charts for a sphere. The coefficients of Heisler charts, i.e. the values of A_1 and δ_1 for infinite plates, long cylinders and spheres are shown in Table 4.3.

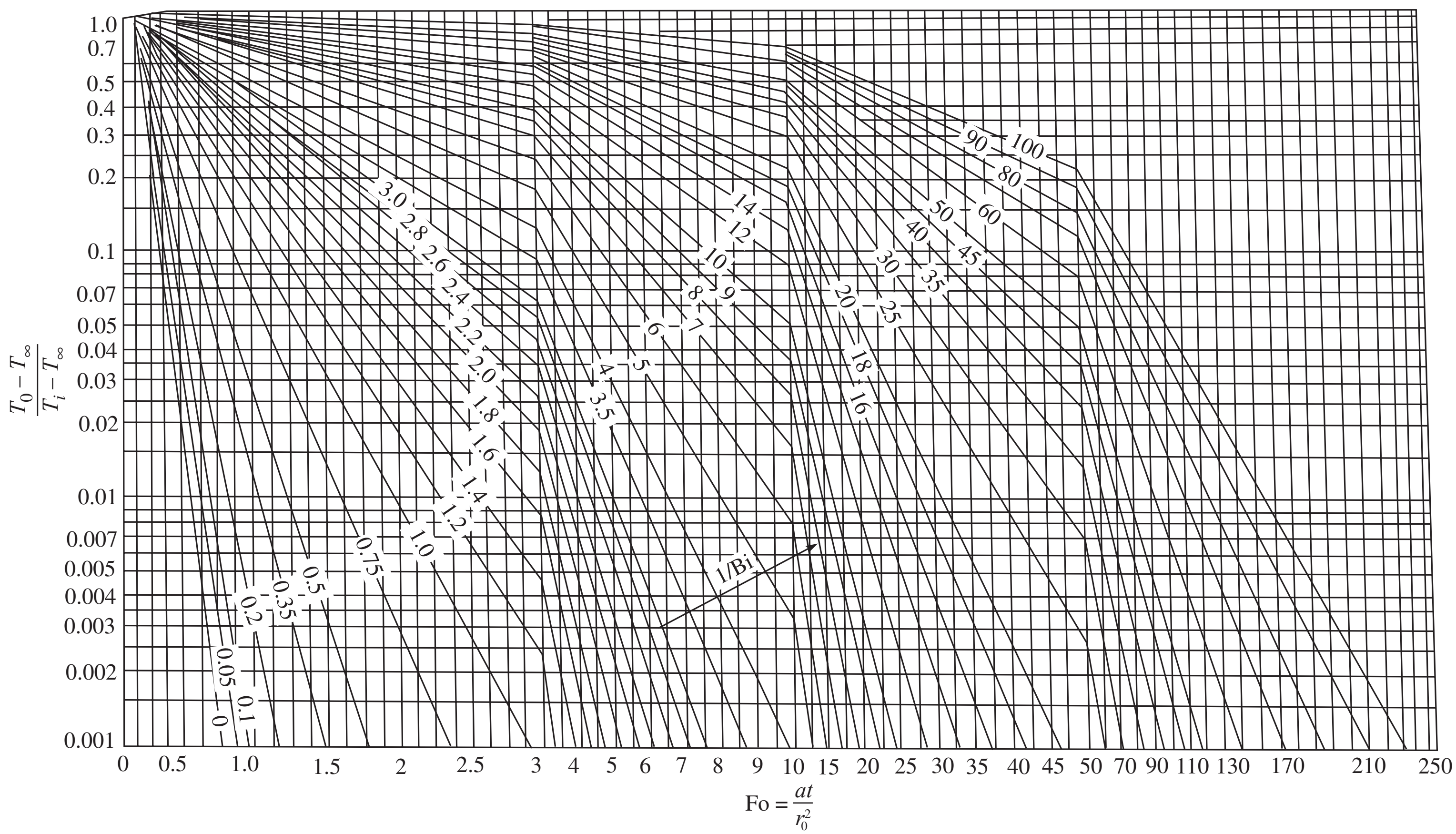


Figure 4.18 The centre temperature of a sphere of radius r_0.

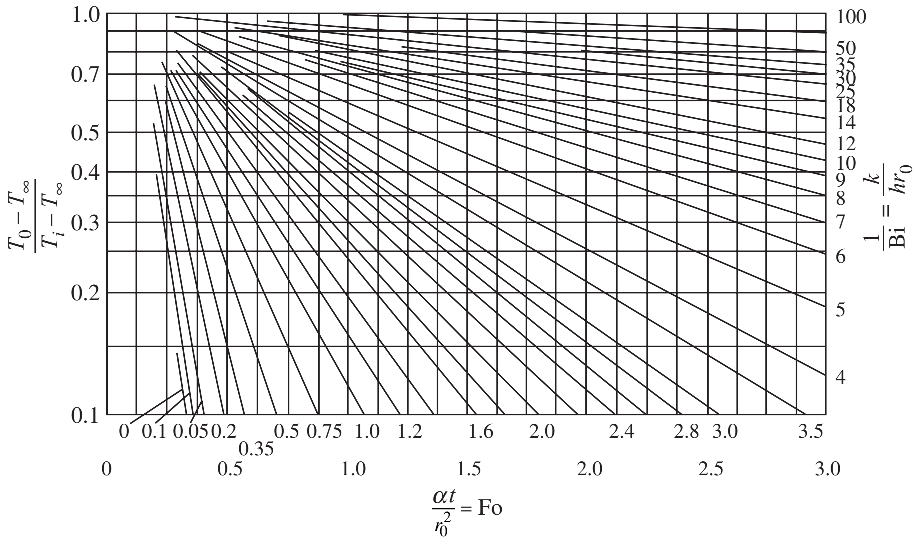


Figure 4.19 The centre temperature of a sphere of radius r_0 in an expanded scale ($0 < \mathrm{Fo} < 4$).

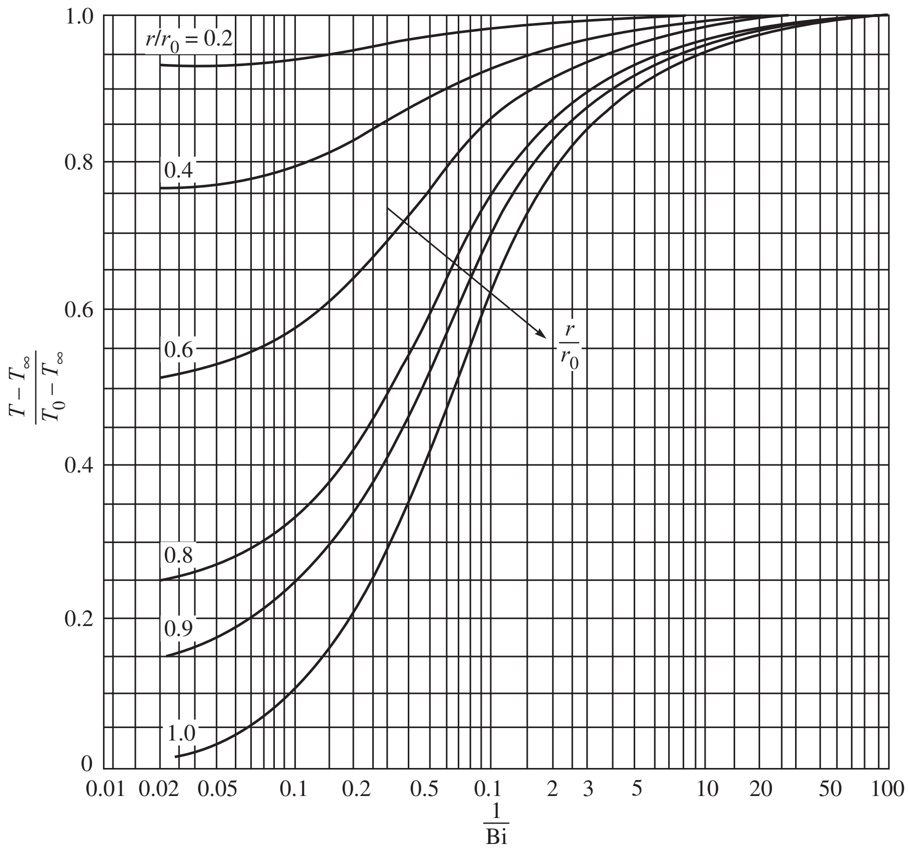


Figure 4.20 The ratio of temperature at any radial location to the temperature at the centre of a sphere of radius r_0.

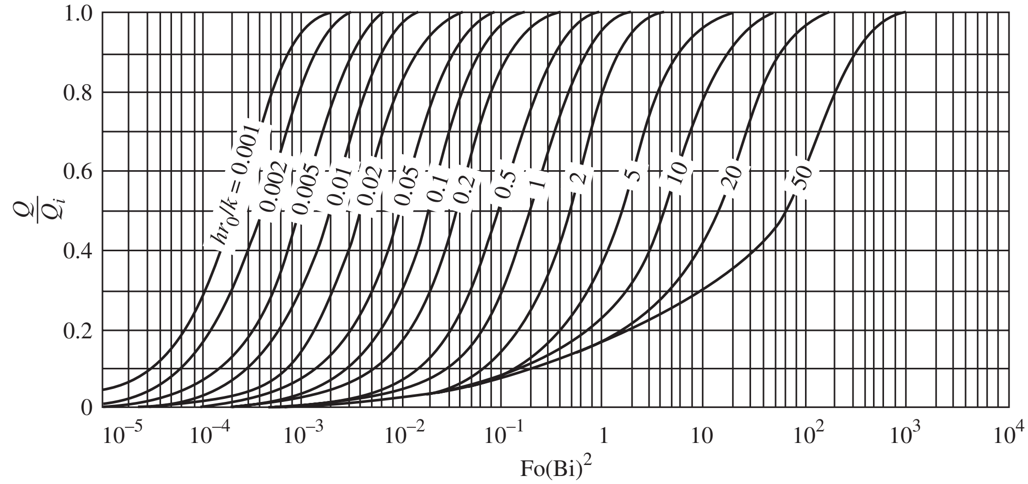


Figure 4.21 Dimensionless heat transfer Q/Q_i for a sphere of radius r_0.

Table 4.3 Coefficients for Heisler solution

hL/k or hr_0/k	Infinite plate δ_1	Infinite plate A_1	Long cylinder δ_1	Long cylinder A_1	Sphere δ_1	Sphere A_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.3	0.5218	1.0451	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0931	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0185	1.1345	1.2644	1.1713
0.7	0.7506	1.0919	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2403	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8674
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2881	1.5919	2.9857	1.9781
30.0	1.4202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990

EXAMPLE 4.5 A masonry brick wall (k = 0.60 W/(m K) and $\alpha = 5 \times 10^{-7}$ m^2/s) of 150 mm thickness and originally at 30°C is suddenly exposed on one side to hot gases at 780°C. The other side of the wall is already insulated. The convection heat transfer coefficient on the hot gas side is 20 W/(m^2 K). Determine, by making use of Heisler charts, (a) the time required for the insulated surface to attain a temperature of 480°C and (b) the heat transferred to the wall per unit area during that time.

Solution: The insulated surface corresponds to the centre line of a wall of thickness $2L$ = 2 × 150 = 300 mm which is exposed on both sides to hot environment. This has to be kept in mind for the use of Heisler charts. The dimensionless temperature ratio at the insulated surface is given by

$$\frac{T_0 - T_\infty}{T_i - T_\infty} = \frac{480 - 780}{30 - 780} = 0.4$$

$$\text{Bi} = \frac{hL}{k} = \frac{20 \times 0.15}{0.60} = 5$$

$$\frac{1}{\text{Bi}} = 0.2$$

(a) From Figure 4.11,

$$\text{Fo} = 0.6 \text{ at } \frac{1}{\text{Bi}} = 0.2 \quad \text{and} \quad \frac{T_0 - T_\infty}{T_i - T_\infty} = 0.4$$

Therefore,

$$\text{Fo} = \frac{\alpha t}{L^2} = 0.6$$

or

$$t = \frac{0.6 \times (0.15)^2}{5 \times 10^{-7}} = 2.70 \times 10^2 \text{ s} = 7.5 \text{ h}$$

(b)

$$\text{Fo (Bi)}^2 = 0.6 \times 25 = 15$$

From Figure 4.13,

$$\frac{Q}{Q_i} = 0.69 \qquad \text{at Bi} = 5 \text{ and Fo (Bi)}^2 = 15$$

$$Q = 0.69 \times Q_i = 0.69 \times \rho c 2L\, (T_\infty - T_i)$$

$$= 0.69 \times \left(\frac{0.60}{5 \times 10^{-7}} \right) \times 0.3 \times (780 - 30)$$

$$= 180.3 \times 10^6 \text{ J} = 186.3 \text{ MJ}$$

EXAMPLE 4.6 A large aluminium plate of thickness 200 mm originally at a temperature of 530°C is suddenly exposed to an environment at 30°C. The convective heat transfer coefficient between the plate and the environment is 500 W/(m^2 K). Determine with the help of Heisler charts, the temperature at a depth of 20 mm from one of the faces 225 seconds after the plate is exposed to the environment. Also calculate how much energy has been lost per unit area of the plate during this time? Take for aluminium, $\alpha = 8 \times 10^{-5}$ m^2/s and $k = 200$ W/(m K).

Solution: For the use of Heisler charts, we have

$$T_i = 530°\text{C}, \quad T_\infty = 30°\text{C}, \quad h = 500 \text{ W/(m}^2\text{ K)} \quad k = 200 \text{ W/(m K)}$$

$$t = 225 \text{ s} \quad 2L = 0.2 \text{ m} \quad x = (0.1 - 0.02) = 0.08 \text{ m} \quad \alpha = 8 \times 10^{-5} \text{ m}^2/\text{s}$$

$$\text{Bi} = \frac{hL}{k} = \frac{500 \times 0.1}{200} = 0.25$$

or

$$\frac{1}{\text{Bi}} = \frac{1}{0.25} = 4$$

$$\text{Fo} = \frac{\alpha t}{L^2} = \frac{8 \times 10^{-5} \times 225}{(0.1)^2} = 1.8$$

From Figure 4.11, $\dfrac{T_0 - T_\infty}{T_i - T_\infty} = 0.7$ at $\dfrac{1}{\text{Bi}} = 4$ and Fo = 1.8

$$\frac{X}{L} = \frac{0.08}{0.1} = 0.8$$

From Figure 4.12, $\dfrac{T - T_\infty}{T_0 - T_\infty} = 0.93$ at $\dfrac{1}{\text{Bi}} = 4$ and $\dfrac{X}{L} = 0.8$

Therefore, $T - T_\infty = 0.93(T_0 - T_\infty) = 0.93 \times 0.70(T_i - T_\infty) = 0.93 \times 0.70 \times 500 = 325.5°\text{C}$

or $T = 325.5 + 30 = 355.5°\text{C}$

Now, $\text{Fo(Bi)}^2 = 1.8 \times (0{\cdot}25)^2 = 0.1125$

From Figure 4.13, $\dfrac{Q}{Q_i} = 0.4$ at $\text{Fo(Bi)}^2 = 0.1125$ and Bi = 0.25

Therefore,

$$Q = 0.4 \times (\rho c) \times 2L\,(T_i - T_\infty)$$

$$= 0.4 \times \left(\frac{200}{8 \times 10^{-5}}\right) \times 0.2 \times (530 - 30)$$

$$= 100 \times 10^6 \text{ J} = 100 \text{ MJ}$$

EXAMPLE 4.7 A long cylinder of radius 150 mm and at an initial uniform temperature of 530°C is suddenly exposed to an environment at 30°C. The convection heat transfer coefficient between the surface of the cylinder and the environment is 380 W/(m^2 K). The thermal conductivity and thermal diffusivity of the cylinder material are 200 W/(m K) and 8.5×10^{-5} m^2/s respectively. Determine (a) the temperature at a radius of 120 mm and (b) the heat transferred per metre length of the cylinder 265 seconds after the cylinder is exposed to the environment (use Heisler charts).

Solution: (a) $\text{Fo} = \dfrac{\alpha t}{r_0^2} = \dfrac{8.5 \times 10^{-5} \times 265}{(0.15)^2} = 1$

$$\text{Bi} = \frac{hr_0}{k} = \frac{380 \times (0.15)}{200} = 0.285$$

Therefore,
$$\frac{1}{\text{Bi}} = \frac{1}{0.285} = 3.5$$

From Figure 4.15,

$$\frac{T_0 - T_\infty}{T_i - T_\infty} = 0.6 \qquad \text{at Fo} = 1 \text{ and } \frac{1}{\text{Bi}} = 3.5$$

From the given data,
$$\frac{r}{r_0} = \frac{120}{150} = 0.8$$

Again from Figure 4·16,

$$\frac{T - T_\infty}{T_0 - T_\infty} = 0.9 \qquad \text{at } \frac{1}{\text{Bi}} = 3.5 \text{ and } \frac{r}{r_0} = 0.8$$

Therefore,
$$\begin{aligned} T - T_\infty &= 0.9(T_0 - T_\infty) \\ &= 0.9 \times 0.6(T_i - T_\infty) \\ &= 0.9 \times 0.6 \times (530 - 30) \\ &= 270°\text{C} \end{aligned}$$

Hence,
$$T = 270 + 30 = 300°\text{C}$$

(b)
$$\text{Fo(Bi)}^2 = 1 \times (0.285)^2 = 0.0812$$

From Figure 4.17,

$$\frac{Q}{Q_i} = 0.4 \qquad \text{at} \qquad \text{Fo(Bi)}^2 = 0.0812 \qquad \text{and} \qquad \text{Bi} = 0.285$$

Here Q_i is the initial energy of the cylinder and is given by

$$\begin{aligned} Q_i &= \rho c \mathbb{V}(T_i - T_\infty) \\ &= \frac{200}{8.5 \times 10^{-5}} \times \pi(0.15)^2 \times L(530 - 30) \\ &= (83.16 \times L \times 10^6) \text{ J} \\ &= (83.16 \times \text{L}) \text{ MJ} \end{aligned}$$

where L is the length of the cylinder in metre.

Therefore,
$$\frac{Q}{L} = 0.4 \times 83.16 = 33.26 \text{ MJ/m}$$

EXAMPLE 4.8 A steel sphere of radius 50 mm is initially at a uniform temperature of 530°C and is suddenly immersed in an oil bath of 30°C with a convection heat transfer coefficient of 500 W/(m^2 K). How long will it take for the centre of the sphere to reach a temperature of 105°C? Take the thermal conductivity of steel as 50 W/(m K) and thermal diffusivity as 1.5×10^{-5} m^2/s (use Heisler charts).

Solution: We have to use Figure 4.19 for the present situation. From the given data, we find

$$\frac{T_0 - T_\infty}{T_i - T_\infty} = \frac{105 - 30}{530 - 30} = 0.15$$

$$\text{Bi} = \frac{hr_0}{k} = \frac{500 \times 0.05}{50} = 0.5$$

or
$$\frac{1}{\text{Bi}} = 2$$

At $\frac{T_0 - T_\infty}{T_i - T_\infty} = 0.15$ and $\frac{1}{\text{Bi}} = 2$, we have from Figure 4.19,

$$\text{Fo} = 1.5$$

Therefore,
$$\frac{\alpha t}{r_0^2} = 1.5$$

or
$$t = \frac{1.5 \times (0.05)^2}{1.5 \times 10^{-5}}$$
$$= 2505 \text{ s}$$
$$= 4.17 \text{ min}$$

4.2.3 Applications of the Solutions of One-dimensional Transient Heat Conduction to Multidimensional Systems

The Heisler charts discussed so far are useful to determine the temperature distribution in an infinite plate, in a long cylinder, or in a sphere. But in most of the practical situations we have to deal with the plates whose height and depth are not large compared to the thickness, or cylinders whose length is not large compared to the diameter. In such cases heat flow is not one-dimensional and hence the Heisler charts are no longer applicable. Additional space coordinates have to be considered to express the heat conduction equation and the method of its solution has to be sought for. Fortunately, it is possible to obtain the solution of multidimensional systems from a straightforward combination of one-dimensional systems. This is discussed below.

Consider a rectangular bar of infinite length but of finite width of $2L_1$ and finite height of $2L_2$ as shown in Figure 4.22. The bar is initially at a uniform temperature T_i. At time $t = 0$, the bar is exposed to an environment at T_∞, with a convective heat transfer coefficient of h. The heat conduction equation, under the situation, can be written in a Cartesian coordinate system (Figure 4.22) as

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \tag{4.51}$$

Equation (4.51) can be solved by the method of separation of variables, where we assume a solution in the product form as

$$T(x, y, t) = X(x)Y(y)\tau(t) \tag{4.52}$$

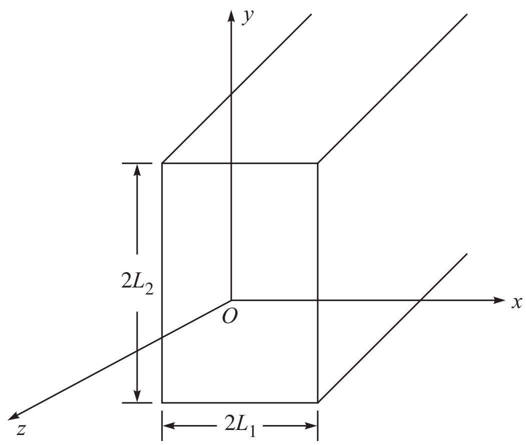


Figure 4.22 A long rectangular bar.

The rectangular bar shown in Figure 4.22 can be formed by two infinite plates of thickness $2L_1$ and $2L_2$ respectively as shown in Figure 4.23. For these two infinite plates, the heat conduction equations are as follows:

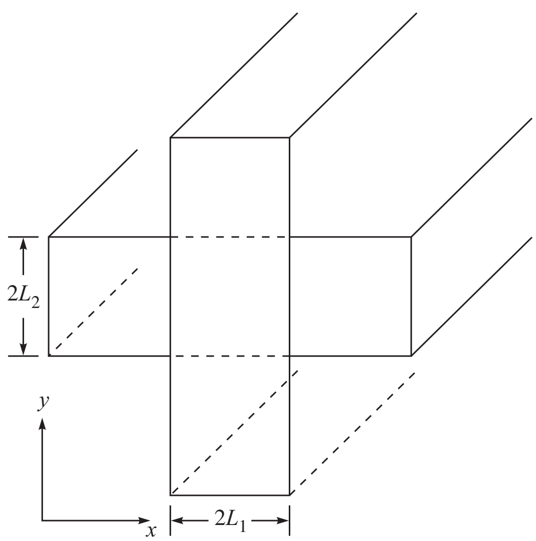


Figure 4.23 Interpenetration of two infinite plates to form a rectangular bar of Figure 4.22.

For plate 1 of thickness $2L_1$,

$$\frac{\partial T_1}{\partial t} = \alpha \frac{\partial^2 T_1}{\partial x^2} \tag{4.53}$$

For plate 2 of thickness $2L_2$,

$$\frac{\partial T_2}{\partial t} = \alpha \frac{\partial^2 T_2}{\partial y^2} \tag{4.54}$$

The suffices 1 and 2 are used to specify the plate of thickness $2L_1$ and that of thickness $2L_2$ respectively.

The solutions of Eqs. (4.53) and (4.54) are of the form

$$T_1 = T_1(x,\ t) \tag{4.55a}$$

$$T_2 = T_2(y,\ t) \tag{4.55b}$$

Now it will be shown that the product solution of the rectangular bar given by Eq. (4.52) can be obtained as a simple product of the solutions given by Eqs. (4.55a) and (4.55b). That is,

$$T(x,\ y,\ t) = T_1(x, t)\cdot T_2(y, t) \tag{4.56}$$

Let us first assume that Eq. (4.56) holds good.

From Eq. (4.56), we get

$$\frac{\partial^2 T}{\partial x^2} = T_2\frac{\partial^2 T_1}{\partial x^2} \tag{4.57a}$$

$$\frac{\partial^2 T}{\partial y^2} = T_1\frac{\partial^2 T_2}{\partial y^2} \tag{4.57b}$$

$$\frac{\partial T}{\partial t} = T_1\frac{\partial T_2}{\partial t} + T_2\frac{\partial T_1}{\partial t} \tag{4.57c}$$

With the help of Eqs. (4.53) and (4.54), Eq. (4.57c) can be written as

$$\frac{\partial T}{\partial t} = \alpha\left(T_1\frac{\partial^2 T_2}{\partial y^2} + T_2\frac{\partial^2 T_1}{\partial x^2}\right) \tag{4.58}$$

Substituting for $\dfrac{\partial^2 T}{\partial x^2}$, $\dfrac{\partial^2 T}{\partial y^2}$, and $\dfrac{\partial T}{\partial t}$ from Eqs. (4.57a), (4.57b), and (4.58) respectively into Eq (4.51), we have

$$\alpha\left(T_1\frac{\partial^2 T_2}{\partial y^2} + T_2\frac{\partial^2 T_1}{\partial x^2}\right) = \alpha\left(T_2\frac{\partial^2 T_1}{\partial x^2} + T_1\frac{\partial^2 T_2}{\partial y^2}\right) \tag{4.59}$$

Equation (4.59) shows that the assumed product solution given by Eq. (4.56) satisfies the differential heat conduction equation Eq. (4.51) for the rectangular bar shown in Figure 4.23. Therefore, we can write

$$\left(\frac{T - T_\infty}{T_i - T_\infty}\right)_{\text{bar}} = \left(\frac{T - T_\infty}{T_i - T_\infty}\right)_{\text{plate}\,1} \times \left(\frac{T - T_\infty}{T_i - T_\infty}\right)_{\text{plate}\,2} \tag{4.60}$$

In the same way, the solution for a three-dimensional block, as shown in Figure 4.24, can be expressed as a product of the solutions of three infinite plates—plate 1 of thickness $2L_1$, plate 2 of thickness $2L_2$, and plate 3 of thickness $2L_3$—as shown below.

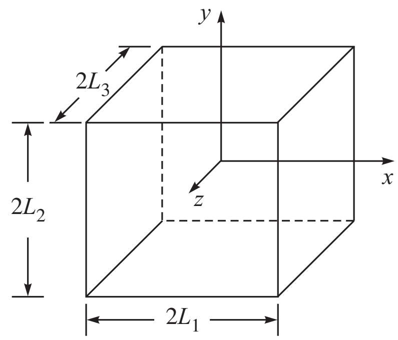


Figure 4.24 A three-dimensional block.

$$\left(\frac{T-T_\infty}{T_i-T_\infty}\right)_{\text{block}} = \left(\frac{T-T_\infty}{T_i-T_\infty}\right)_{\text{plate 1}} \times \left(\frac{T-T_\infty}{T_i-T_\infty}\right)_{\text{plate 2}} \times \left(\frac{T-T_\infty}{T_i-T_\infty}\right)_{\text{plate 3}} \tag{4.61}$$

In a similar fashion, the solution for a cylinder of finite length as shown in Figure 4.25 can be expressed as a product of the solutions of an infinitely long cylinder and of an infinite plate of thickness that equals the length of the cylinder.

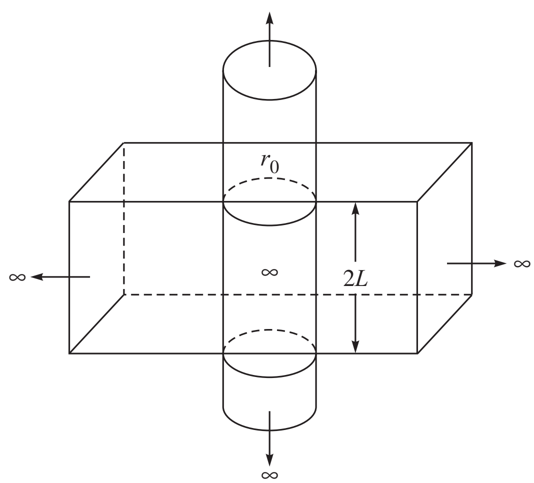


Figure 4.25 Interpenetration of an infinitely long cylinder and of an infinite plate of thickness equal to the length of the cylinder to form a cylinder of finite length.

Hence,

$$\left(\frac{T-T_\infty}{T_i-T_\infty}\right)_{\substack{\text{cylinder of length}\\ 2L \text{ and radius } r_0}} = \left(\frac{T-T_\infty}{T_i-T_\infty}\right)_{\substack{\text{infinte cylinder}\\ \text{of radius } r_0}} \times \left(\frac{T-T_\infty}{T_i-T_\infty}\right)_{\substack{\text{infinite plate of}\\ \text{thickness } 2L}} \tag{4.62}$$

The heat transfer for two- and three-dimensional bodies can be obtained by superimposing the heat transfer solutions from one-dimensional bodies as shown below:

For a two-dimensional body obtained as an intersection of two one-dimensional bodies 1 and 2,

$$\left(\frac{Q}{Q_i}\right)_{\text{2-D body}} = \left(\frac{Q}{Q_i}\right)_1 + \left(\frac{Q}{Q_i}\right)_2 \left[1 - \left(\frac{Q}{Q_i}\right)_1\right] \tag{4.63}$$

For a three-dimensional body obtained as an intersection of three one-dimensional bodies 1, 2 and 3,

$$\left(\frac{Q}{Q_i}\right)_{\text{3-D body}} = \left(\frac{Q}{Q_i}\right)_1 + \left(\frac{Q}{Q_i}\right)_2 \left[1 - \left(\frac{Q}{Q_i}\right)_1\right] + \left(\frac{Q}{Q_i}\right)_3 \left[1 - \left(\frac{Q}{Q_i}\right)_1\right]\left[1 - \left(\frac{Q}{Q_i}\right)_2\right] \tag{4.64}$$

EXAMPLE 4.9 A large aluminium bar (k = 198 W/(m K)) of cross-section 180 mm × 104 mm is initially at a uniform temperature of 730°C and is suddenly exposed to an environment at 30°C with a convection heat transfer coefficient of 1100 W/(m^2 K). Determine, by making use of Heisler charts, the temperature at the centre of the bar 100 s after it is exposed to the environment. (Thermal diffusivity of the bar, $\alpha = 8.1 \times 10^{-5}$ m^2/s.)

Solution: The bar can be considered as the intersection of an infinite plate 1 of thickness $2L_1$ = 180 mm and a second infinite plate 2 of thickness $2L_2$ = 104 mm.

For plate 1,

$$(\text{Fo})_1 = \frac{\alpha t}{L_1^2} = \frac{8.1 \times 10^{-5} \times 100}{(0.09)^2} = 1$$

$$(\text{Bi})_1 = \frac{hL_1}{k} = \frac{1100 \times 0.09}{198} = 0.5$$

or

$$\frac{1}{(\text{Bi})_1} = 2$$

At Fo = 1 and 1/Bi = 2, we have from Figure 4.11

$$\frac{T_0 - T_\infty}{T_i - T_\infty} = 0.7$$

For plate 2,

$$(\text{Fo})_2 = \frac{\alpha t}{L_2^2} = \frac{8.1 \times 10^{-5} \times 100}{(0.052)^2} = 3$$

$$(\text{Bi})_2 = \frac{hL_2}{k} = \frac{1100 \times (0.052)}{198} = 0.29$$

or

$$\frac{1}{(\text{Bi})_2} = 3.46$$

At Fo = 3 and 1/Bi = 3.46, we have from Figure 4.11

$$\frac{T_0 - T_\infty}{T_i - T_\infty} = 0.47$$

Therefore, $$\left(\frac{T_0 - T_\infty}{T_i - T_\infty}\right)_{\text{bar}} = \left(\frac{T_0 - T_\infty}{T_i - T_\infty}\right)_{\text{plate1}} \times \left(\frac{T_0 - T_\infty}{T_i - T_\infty}\right)_{\text{plate2}}$$

or $$\frac{T_0 - 30}{730 - 30} = 0.7 \times 0.47$$

or $$T_0 = 260{\cdot}3°\text{C}$$

EXAMPLE 4.10 An iron beam of rectangular cross-section of size 300 mm × 200 mm is used in the construction of a building. Initially, the beam is at a uniform temperature of 30°C. Due to an accidental fire, the beam is suddenly exposed to hot gases at 730°C, with a convection heat transfer coefficient of 100 W/(m^2 K). Determine the time required for the centre plane of the beam to reach a temperature of 310°C. (Take thermal conductivity of the beam k = 73 W/(m K) and thermal diffusivity of the beam $\alpha = 2.034 \times 10^{-5}$ m^2/s; use Heisler chart.)

Solution: The rectangular iron beam can be considered as an intersection of an infinite plate 1 having a thickness $2L_1$ = 300 mm and a second infinite plate 2 of thickness $2L_2$ = 200 mm

Here, $$\left(\frac{T_0 - T_\infty}{T_i - T_\infty}\right)_{\text{beam}} = \frac{310 - 730}{30 - 730} = 0.6$$

Therefore, we can write

$$0.6 = \left(\frac{T_0 - T_\infty}{T_i - T_\infty}\right)_{\text{plate1}} \times \left(\frac{T_0 - T_\infty}{T_i - T_\infty}\right)_{\text{plate2}}$$

A straightforward solution is not possible. We have to adopt an iterative method of solution. At first, a value of t (time) is assumed to determine the centre-line temperature of the beam. The value of t at which

$$\left(\frac{T_0 - T_\infty}{T_i - T_\infty}\right)_{\text{beam}} = 0.6$$

is satisfied, gives the required time t.
Let us first assume t = 900 s
For plate 1,

$$(\text{Bi})_1 = \frac{hL_1}{k} = \frac{100 \times 0.15}{73} = 0.2055$$

$$\frac{1}{(\text{Bi})_1} = 4.87$$

$$(\text{Fo})_1 = \frac{\alpha t}{L_1^2} = \frac{2.034 \times 10^{-5} \times 900}{(0.15)^2} = 0.814$$

At Fo = 0.814 and (1/Bi) = 4.87, we read from Figure 4.11

$$\left(\frac{T_0 - T_\infty}{T_i - T_\infty}\right)_{\text{plate1}} = 0.85$$

For plate 2,

$$(\text{Bi})_2 = \frac{hL_2}{k} = \frac{100 \times 0.1}{73} = 0.137$$

or
$$\frac{1}{(\text{Bi})_2} = 7.3$$

$$(\text{Fo})_2 = \frac{\alpha t}{L_2^2} = \frac{2.034 \times 10^{-5} \times 900}{(0.1)^2} = 1.83$$

At Fo = 1.83 and (1/Bi) = 7.3, we have from Figure 4.11

$$\left(\frac{T_0 - T_\infty}{T_i - T_\infty}\right)_{\text{plate2}} = 0.8$$

Therefore,
$$\left(\frac{T_0 - T_\infty}{T_i - T_\infty}\right)_{\text{beam}} = 0.85 \times 0.8 = 0.68$$

Since the calculated value of 0.68 is greater than the required value of 0.60 and $T_\infty > T_0 > T_i$, the assumed value of t is less. This means that the actual value of t will be greater than 900 s to make $(T_0 - T_\infty)/(T_i - T_\infty) = 0.6$.

Let us take t = 1200 s for the second iteration.

Then,
$$(\text{Fo})_1 = \frac{0.814 \times 1200}{900} = 1.08$$

$$(\text{Fo})_2 = \frac{1.83 \times 1200}{900} = 2.44$$

$(\text{Bi})_1$ and $(\text{Bi})_2$ will remain the same.

At Fo = 1.08 and (1/Bi) = 4.87, we read from Figure 4.11

$$\left(\frac{T_0 - T_\infty}{T_i - T_\infty}\right)_{\text{plate1}} = 0.83$$

At Fo = 2.44 and (1/Bi) = 7.3, we have from Figure 4.11

$$\left(\frac{T_0 - T_\infty}{T_i - T_\infty}\right)_{\text{plate2}} = 0.72$$

Therefore,
$$\left(\frac{T_0 - T_\infty}{T_i - T_\infty}\right)_{\text{beam}} = 0.83 \times 0.72 = 0.598$$

The calculated value 0.598 is very close to the required value of 0.6. Hence the time required for the centre of beam to reach 310°C is nearly 1200 s or 20 minutes.

4.2.4 Transient Heat Flow in a Semi-infinite Solid

A semi-infinite solid is an idealized body that has a single identifiable plane surface and extends to infinity in all directions as shown in Figure 4.26. Let us consider that the solid is at an initial temperature T_i and the temperature of the identifiable surface is suddenly lowered and maintained at a temperature of T_0. We have to find an expression for the temperature distribution in the solid as a function of time and subsequently the heat flux at any position, x, in the solid as a function of time. Here heat flows in a direction (x-direction) in which the body extends to infinity.

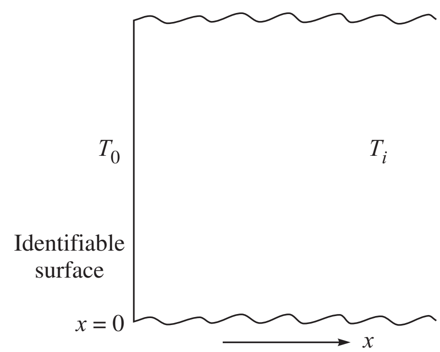


Figure 4.26 A semi-infinite solid.

This is a one-dimensional unsteady heat conduction problem. The governing differential equation can be written as

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{4.65}$$

The boundary and initial conditions are

$$T(x,\ 0) = T_i \tag{4.66a}$$

$$T(0,\ t) = T_0 \qquad \text{for } t > 0 \tag{4.66b}$$

There are several methods of solving Eq. (4.65). One of the most convenient methods of solution to accommodate the boundary conditions given by Eqs. (4.66a) and (4.66b) is the method of similarity transformation. This method recognizes the existence of a similarity parameter η which reduces the partial differential equation (Eq. (4.65)) to an ordinary differential equation. This requirement is satisfied when

$$\eta = \frac{x}{\sqrt{4\alpha t}} \tag{4.67}$$

so that $T = T(\eta)$.

(The proof of this is given in Appendix B.1.)

With the help of Eq. (4.67), we can write

$$\frac{\partial T}{\partial x} = \frac{dT}{d\eta}\cdot\frac{d\eta}{dx}\,\frac{1}{\sqrt{4\alpha t}}\,\frac{dT}{d\eta}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{d}{d\eta}\left(\frac{\partial T}{\partial x}\right)\cdot\frac{\partial \eta}{\partial x}$$

$$= \frac{d}{d\eta}\left(\frac{1}{\sqrt{4\alpha t}}\,\frac{dT}{d\eta}\right)\cdot\frac{1}{\sqrt{4\alpha t}}$$

$$= \frac{1}{4\alpha t} \frac{d^2T}{d\eta^2}$$

$$\frac{\partial T}{\partial t} = \frac{dT}{d\eta} \cdot \frac{\partial \eta}{\partial t}$$

$$= \frac{-x}{2t\sqrt{4\alpha t}} \frac{dT}{d\eta}$$

Substituting the values of $\frac{\partial^2 T}{\partial x^2}$ and $\frac{\partial T}{\partial t}$ in Eq. (4.65), we have

$$\frac{d^2T}{d\eta^2} = -2\eta \frac{dT}{d\eta} \tag{4.68}$$

The boundary conditions given by Eqs. (4.66a) and (4.66b) are now reduced to

$$T(\eta = 0) = T_0 \tag{4.69a}$$

$$T(\eta \to \infty) = T_i \tag{4.69b}$$

Here the variables x and t are clubbed into a single dimensionless variable η as described by Eq. (4.67). The variable η is named similarity variable, since T (the temperature) depends only on η. This means that whatever may be the values of x (the position) and t (time), the value of T remains the same so long η remains the same.

Equation (4.68) can be written as

$$\frac{d^2T}{d\eta^2} + 2\eta \frac{dT}{d\eta} = 0$$

Multiplying both sides by the integrating factor e^{η^2}, we have

$$e^{\eta^2} \frac{d^2T}{d\eta^2} + 2\eta e^{\eta^2} \frac{dT}{d\eta} = 0$$

or
$$\frac{d}{d\eta}\left(e^{\eta^2} \frac{dT}{d\eta}\right) = 0$$

or
$$\frac{dT}{d\eta} = c_1 e^{-\eta^2}$$

or
$$T = c_1 \int_0^{\eta} e^{-\eta^2}\, d\eta + c_2 \tag{4.70}$$

The constants c_1 and c_2 are found out from the boundary conditions. From the first boundary condition, Eq. (4.69a), we get

$$c_2 = T_0$$

From the second boundary condition, Eq. (4.69b), we get

$$c_1 = \frac{T_i - T_0}{\int_0^{\infty} e^{-\eta^2}\, d\eta}$$

$$= \frac{2}{\sqrt{\pi}}(T_i - T_0)$$

Therefore, Eq. (4.70) can be written as

$$\frac{T - T_0}{T_i - T_0} = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2}\, d\eta = \text{erf}\ \eta \tag{4.71}$$

The function erf η is known as Gaussian error function which is a standard mathematical function and is tabulated in Table 4.4. The temperature distribution as given by Eq. (4.71) is shown in Figure 4.27.

Table 4.4 Gaussian error function

η	erf η	η	erf η	η	erf η
0.00	0.00000	0.36	0.38933	1.04	0.85865
0.02	0.02256	0.38	0.40901	1.08	0.87333
0.04	0.04511	0.40	0.42839	1.12	0.88679
0.06	0.06762	0.44	0.46622	1.16	0.89910
0.08	0.09008	0.48	0.50275	1.20	0.91031
0.10	0.11246	0.52	0.53790	1.30	0.93401
0.12	0.13476	0.56	0.57162	1.40	0.95228
0.14	0.15695	0.60	0.60386	1.50	0.96611
0.16	0.17901	0.64	0.63459	1.60	0.97635
0.18	0.20094	0.68	0.66378	1.70	0.98379
0.20	0.22270	0.72	0.69143	1.80	0.98909
0.22	0.24430	0.76	0.71754	1.90	0.99279
0.24	0.26570	0.80	0.74210	2.00	0.99532
0.26	0.28690	0.84	0.76514	2.20	0.99814
0.28	0.30788	0.88	0.78669	2.40	0.99931
0.30	0.32863	0.92	0.80677	2.60	0.99976
0.32	0.34913	0.96	0.82542	2.80	0.99992
0.34	0.36936	1.00	0.84270	3.00	0.99998

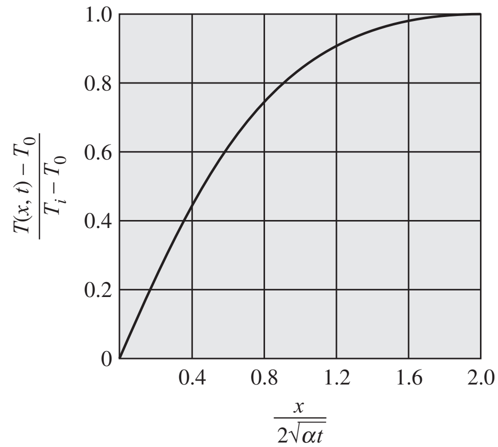


Figure 4.27 Temperature distribution in a semi-infinite solid

The heat flux from the surface ($x = 0$) may be obtained by making use of the Fourier's law of heat conduction as

$$(q)_{x=0} = -k\left(\frac{\partial T}{\partial x}\right)_{x=0}$$

$$= -k\left[\left(\frac{\partial T}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial x}\right)\right]_{\eta=0}$$

$$= \frac{-k\,\frac{2}{\sqrt{\pi}}\,(T_i - T_0)}{\sqrt{4\,\alpha t}}\,(e^{-\eta^2})_{\eta=0}$$

$$= \frac{k(T_0 - T_i)}{\sqrt{\pi\alpha t}}$$

Analytical solutions of Eq. (4.65) are also available for conditions of (a) constant heat flux at the surface at $x = 0$ and (b) convection from the surface at $x = 0$ to an environment at a temperature of T_∞. They are summarized below:

(a) Constant heat flux q_0 at the surface $x = 0$

$$T(x,\ t) - T_i = \frac{2\,q_0\,(\alpha t/\pi)^{1/2}}{k}\exp\left(\frac{-x^2}{4\,\alpha t}\right) - \frac{q_0\,x}{k}\,\text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \tag{4.72}$$

(b) surface convection $\left[-k\left(\frac{\partial T}{\partial x}\right)_{x=0} = h(T_\infty - T(0,t))\right]$

$$\frac{T(x,t) - T_i}{T_\infty - T_i} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \left[\exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right)\text{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{\sqrt{\alpha t}}{k}\right)\right] \tag{4.73}$$

The function erfc in the expressions (4.72) and (4.73) is known as complementary error function and is defined as erfc $\eta = 1 - \text{erf}\,\eta$.

EXAMPLE 4.11 A large slab of wrought–iron is at a uniform temperature of 550°C. The temperature of one surface is suddenly changed to 50°C. Calculate the time required for the temperature to reach 255°C at a depth of 80 mm (Take, for the slab, k = 60 W/(m K), $\alpha = 1.6 \times 10^{-5}$ m²/s.)

Solution:

$$\eta = \frac{x}{2\sqrt{\alpha t}} = \frac{0.08}{2\sqrt{1.6 \times 10^{-5}\, t}} = \frac{10}{\sqrt{t}}$$

$$\frac{T - T_\infty}{T_i - T_\infty} = \text{erf}\left(\frac{10}{\sqrt{t}}\right)$$

or

$$\frac{255 - 50}{550 - 50} = \text{erf}\left(\frac{10}{\sqrt{t}}\right)$$

or

$$0.41 = \text{erf}\left(\frac{10}{\sqrt{t}}\right)$$

We read, from Table 4.4, the value of η (= $10/\sqrt{t}$) = 0.38 corresponding to erf η = 0.41

Therefore,

$$\frac{10}{\sqrt{t}} = 0.38$$

or

$$t = 692.525 \text{ s} = 11.54 \text{ min}$$

EXAMPLE 4.12 A large block of nickel-steel (k = 20 W/(m K), $\alpha = 0.518 \times 10^{-5}$ m²/s) is at a uniform temperature of 30°C. One surface of the block is suddenly exposed to a constant surface heat flux of 6 MW/m². Determine the temperature at a depth of 100 mm after a time of 100 seconds

Solution:

$$\eta = \frac{x}{\sqrt{4\,\alpha t}} = \frac{0.1}{\sqrt{4 \times 0.518 \times 10^{-5} \times 100}} = 2.20$$

or

$$\frac{x^2}{4\,\alpha t} = (2.20)^2 = 4.84$$

From Table 4.4, we have erf (2.20) = 0.99814.
Therefore, we can write from Eq. (4.72)

$$T - 30 = \frac{2 \times 6 \times 10^6 \times \sqrt{0.518 \times 10^{-5} \times 100/\pi}}{20}\, e^{-4.84} - \frac{6 \times 10^6 \times 0.1}{20}\,(1 - 0.99814)$$

which gives $T = 135.12$°C

4.2.5 Semi-infinite Solid: Surface Temperature Periodic with Time

There are many practical situations of heat conduction where the temperature (or heat flux) boundary conditions change with time in a periodic manner. These are observed in cylinders of a reciprocating internal combustion engine, in cyclic regenerators, in earth's soil due to a cyclic change in daily or annual temperature of earth's surface, and in many other industrial processes where thermal cycling is desirable from a control point of view.

Let us consider a semi-infinite solid as shown in Figure 4.26, where the temperature of the surface at $x = 0$ is varying periodically with time over a mean temperature T_m given by

$$T = T_m + (T_0 - T_m)\cos\,\omega t \tag{4.74a}$$

The initial temperature of the semi-infinite solid is considered to be T_m.

If we define an excess temperature θ as $\theta = T - T_m$, we can write Eq. (4.74a) as

$$\theta = \theta_0 \cos\,\omega t \tag{4.74b}$$

where θ_0 $(= T_0 - T_m)$ represents the amplitude of the periodic variation of θ at $x = 0$.

We seek a solution for θ as a function of x and t. The governing differential equation in this situation is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{4.75}$$

Equation (4.75) can be written in terms of θ as

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2} \tag{4.76}$$

The initial and boundary conditions to be satisfied are

$$\text{At } x = 0 \qquad \theta = \theta_0 \cos\,\omega t \qquad (t \ge 0) \tag{4.77a}$$

$$\text{At } x \to \infty \qquad \theta = 0 \qquad (t \ge 0) \tag{4.77b}$$

We adopt the method of separation of variables and accordingly assume the temperature to be given by

$$\theta(x,\ t) = F(t)\cdot X(x) \tag{4.78}$$

Equation (4.76) can be written by making use of Eq. (4.78) as

$$\frac{F'(t)}{\alpha F(t)} = \frac{X''(x)}{X(x)} \tag{4.79}$$

where
$$F'(t) = \frac{dF(t)}{dt}$$

$$X''(x) = \frac{d^2 X(x)}{dx^2}$$

Since $F(t)$ is a function of t only and $X(x)$ is a function of x only, the equality of the left-hand side and the right-hand side of Eq. (4.79) holds good when both of them are equal to a constant. Let us take the constant to be $\pm i\lambda^2$ (where λ is real) to accommodate the boundary conditions given by Eqs. (4.77a) and (4.77b). Therefore, we can write

$$\frac{F'(t)}{\alpha F(t)} = \frac{X''(x)}{X(x)} = \pm i\lambda^2$$

In consideration of the positive sign with the term '$i\lambda^2$', we have the following solution.

$$\theta = \exp(i\lambda^2 \alpha t)\cdot[c_1 \exp(\sqrt{i}\,\lambda x) + c_2 \exp(-\sqrt{i}\,\lambda x)]$$

$$= c_1 \exp\left[\frac{\lambda x}{\sqrt{2}} + i\left(\lambda^2\alpha t + \frac{\lambda x}{\sqrt{2}}\right)\right] + c_2 \exp\left[\frac{-\lambda x}{\sqrt{2}} + i\left(\lambda^2\alpha t - \frac{\lambda x}{\sqrt{2}}\right)\right] \tag{4.80}$$

$$\left(\text{since, } (1+i)^2 = 1 + i^2 + 2i = 2i \quad \text{or} \quad \sqrt{i} = \frac{1+i}{\sqrt{2}}\right)$$

In a similar way we can write the solution of θ in consideration of the negative sign with $i\lambda^2$ as

$$\theta = c_3 \exp\left[\frac{\lambda x}{\sqrt{2}} - i\left(\lambda^2\alpha t + \frac{\lambda x}{\sqrt{2}}\right)\right] + c_4 \exp\left[\frac{-\lambda x}{\sqrt{2}} - i\left(\lambda^2\alpha t - \frac{\lambda x}{\sqrt{2}}\right)\right] \tag{4.81}$$

The final solution for θ will be the sum of the two solutions given by Eqs. (4.80) and (4.81). However, the first terms of both the equations (Eqs. (4.80) and (4.81)) cannot satisfy the boundary condition given by Eq. (4.77b). Therefore these two terms should be dropped out, or in other words, $c_1 = c_3 = 0$. Hence we have

$$\theta = c_2 \exp\left[-\frac{\lambda x}{\sqrt{2}} + i\left(\lambda^2\alpha t - \frac{\lambda x}{\sqrt{2}}\right)\right] + c_4 \exp\left[-\frac{\lambda x}{\sqrt{2}} - i\left(\lambda^2\alpha t - \frac{\lambda x}{\sqrt{2}}\right)\right]$$

$$= \exp\left(-\frac{\lambda x}{\sqrt{2}}\right)\left[A\cos\left(\lambda^2\alpha t - \frac{\lambda x}{\sqrt{2}}\right) + B\sin\left(\lambda^2\alpha t - \frac{\lambda x}{\sqrt{2}}\right)\right]$$

$$= c\exp\left(-\frac{\lambda x}{\sqrt{2}}\right)\cos\left(\lambda^2\alpha t - \frac{\lambda x}{\sqrt{2}} - \Phi\right) \tag{4.82}$$

where $$\Phi = \tan^{-1}\left(\frac{B}{A}\right)$$

$$c = \frac{1}{\sqrt{A^2 + B^2}} \qquad \text{(a constant)}$$

Applying the boundary condition (4.77a), we have

$$c = \theta_0,\ \lambda^2 = \omega/\alpha \text{ and } \phi = 0$$

Therefore, we can write Eq. (4.82) as

$$\theta = \theta_0 \exp(-\lambda x/\sqrt{2}) \cos \omega\{t - (x/\sqrt{2\alpha\omega})\} \tag{4.83}$$

This is the required solution for θ as a function of x and t. It depicts the following important features.

- The temperature at any depth x oscillates periodically with the same time period $2\pi/\omega$ as that of the temperature oscillations at the surface $x = 0$.
- The amplitude of oscillation is diminished exponentially with x by a factor of $\exp(-\lambda x/\sqrt{2})$ from the amplitude of temperature variation at the surface $x = 0$.
- The periodic oscillations of temperature with time at any depth x bear a phase lag of $x/\sqrt{2\alpha\omega}$ with respect to the oscillation at $x = 0$.

Figures 4.28 and 4.29 show these three features while Figure 4.30 shows the penetration of temperature oscillation into the solid.

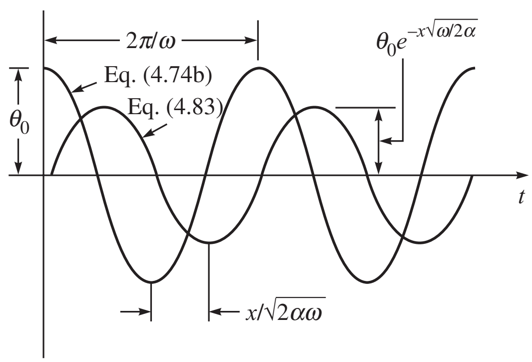


Figure 4.28 Periodic variation of temperature with time. (Graphical representations of Eqs. (4.74a) and (4.83).)

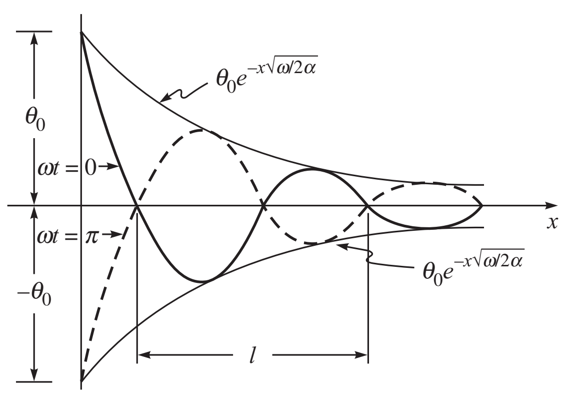


Figure 4.29 Periodic variation of temperature with depth x. (Graphical representation of Eq. (4.83) for $\omega t = 0$ and π.)

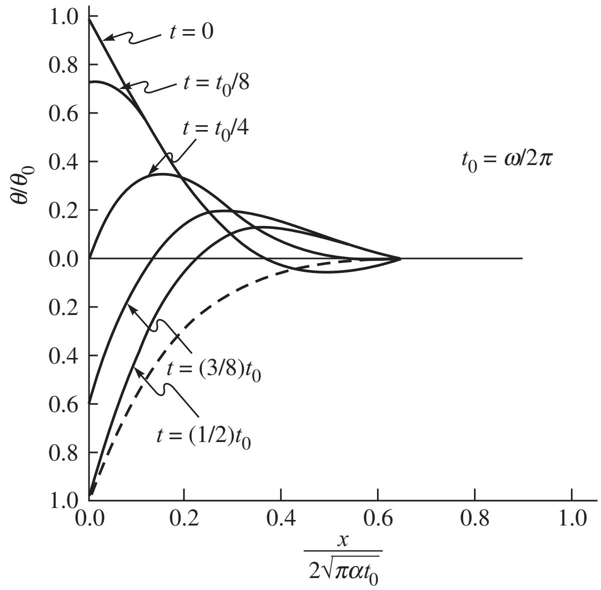


Figure 4.30 Penetration of a temperature oscillation.

4.2.6 Heat Conduction in a Body Resulting from a Moving Heat Source

Heat conduction in a body because of transfer of heat between the body and a moving heat source (or sink) has got wide industrial applications in the fields of arc welding, surface hardening, continuous casting or quenching, etc. The conduction of heat under these situations becomes unsteady since the temperature at any point in the body with reference to a fixed coordinate system changes with time. However if a heat source moves with a constant rectilinear velocity past a body of sufficiently large dimensions, the system appears to be steady with respect to a reference frame of coordinates which moves with the same velocity as that of the source. This means physically that the system will appear steady to an observer travelling with the heat source.

Let us consider a long thin rod of constant cross-sectional area as shown in Figure 4.31. The rod is being heated by a point heat source S moving with a uniform velocity U, and is simultaneously cooled by convection to a surrounding fluid medium at a temperature T_f. If the

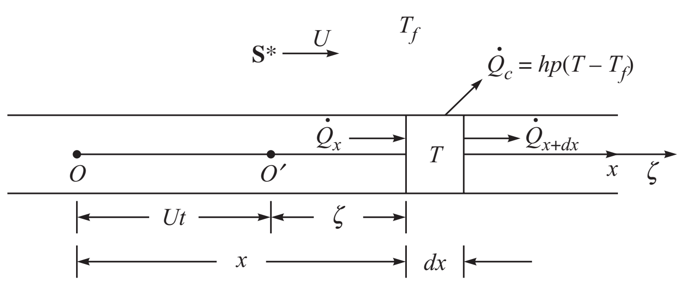


Figure 4.31 Heat conduction in a long thin rod due to heat flux from a moving heat source.

rod is thin and is having a high thermal conductivity, then the conduction resistance across a section will be very low compared to the convection resistance at the surface. Therefore the rod can be assumed to have a uniform temperature across a section and hence the heat conduction will take place only along the length of the rod. Let us take an axis *Ox* with origin *O* as a fixed point.

The governing equation for temperature distribution can be obtained from an energy balance of an element of length *dx* at a distance *x* from the fixed origin *O* (Figure 4.31) as shown below.

$$\rho cAdx\frac{\partial T}{\partial t} = \dot{Q}_x - \dot{Q}_{x+dx} - \dot{Q}_c$$

$$= -kA\frac{dT}{dx} - \left[kA\frac{dT}{dx} + \frac{\partial}{\partial x}\left(-kA\frac{\partial T}{dx}\right)dx\right] - hPdx\,(T - T_f) \tag{4.84}$$

where

ρ is the density of the rod material
c is the specific heat of the rod material
A is the cross-sectional area of the rod
P is the perimeter of the rod
h is the convective heat transfer coefficient of the fluid film at the rod surface (assumed to be constant over the entire surface).

Equation (4.84) can finally be reduced to

$$\frac{\partial \theta}{\partial t} = \alpha\left(\frac{\partial^2 \theta}{\partial x^2} - m^2\theta\right) \tag{4.85}$$

where the excess temperature $\theta = T - T_f$

and $m = \sqrt{\dfrac{hP}{kA}}$

If we use a moving coordinate ζ in a sense that it is measured from an origin O' which is moving with the velocity U of the source, then the system will appear to be quasi-steady with respect to the coordinate ζ. To transform Eq. (4.85) into ζ coordinate, we have to first find out the relationship between ζ and x as

$$\zeta = x - Ut$$

Therefore,

$$\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial \zeta}\cdot\frac{\partial \zeta}{\partial t} = -U\frac{\partial \theta}{\partial \zeta}$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial \zeta}\cdot\frac{\partial \zeta}{\partial x} = \frac{\partial \theta}{\partial \zeta}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial \zeta}\left(\frac{\partial \theta}{\partial x}\right)\cdot\frac{\partial \zeta}{\partial x} = \frac{\partial^2 \theta}{\partial \zeta^2}$$

Substituting for $\frac{\partial \theta}{\partial t}$ and $\frac{\partial^2 \theta}{\partial x^2}$ from the above relations in Eq. (4.85), we have

$$-U\frac{\partial \theta}{\partial \zeta} = \alpha\left(\frac{\partial^2 \theta}{\partial \zeta^2} - m^2\theta\right)$$

or

$$\frac{\partial^2 \theta}{\partial \zeta^2} + \frac{U}{\alpha}\frac{\partial \theta}{\partial \zeta} - m^2\theta = 0 \tag{4.86}$$

The solution of Eq. (4.86) is

$$\theta = A\exp\left[\left\{\sqrt{\left(\frac{U}{2\alpha}\right)^2 + \frac{hP}{kA}} - \frac{U}{2\alpha}\right\}\zeta\right] + B\exp\left[\left\{-\sqrt{\left(\frac{U}{2\alpha}\right)^2 + \frac{hP}{kA}} - \frac{U}{2\alpha}\right\}\zeta\right] \tag{4.87}$$

The appropriate boundary conditions are

$$\theta = 0, \quad \text{at} \quad \zeta \to \pm\infty$$

The solution takes two forms depending on whether ζ is greater or less than zero. Therefore,

$$\theta = A\exp\left[\left\{\sqrt{\left(\frac{U}{2\alpha}\right)^2 + \frac{hP}{kA}} - \frac{U}{2\alpha}\right\}\zeta\right] \qquad \text{for } \zeta < 0$$

$$= B\exp\left[\left\{-\sqrt{\left(\frac{U}{2\alpha}\right)^2 + \frac{hP}{kA}} - \frac{U}{2\alpha}\right\}\zeta\right] \qquad \text{for } \zeta > 0$$

The physical condition that the temperature at $\zeta = 0$ has a unique value gives

$$A = B = \theta_0$$

where θ_0 is the temperature at $\zeta = 0$ and is supposed to be the maximum one due to the presence of the heat source.

Hence, we can write

$$\theta = \theta_0\exp\left[\left\{\sqrt{\left(\frac{U}{2\alpha}\right)^2 + \frac{hP}{kA}} - \frac{U}{2\alpha}\right\}\zeta\right] \qquad \text{for } \zeta < 0 \tag{4.88a}$$

$$= \theta_0\exp\left[\left\{-\sqrt{\left(\frac{U}{2\alpha}\right)^2 + \frac{hP}{kA}} - \frac{U}{2\alpha}\right\}\zeta\right] \qquad \text{for } \zeta > 0 \tag{4.88b}$$

The heat flow in both the positive and negative directions of ζ can be found out by making use of the expression

$$Q = -kA\left(\frac{\partial \theta}{\partial \zeta}\right)_{\zeta=0}$$

With the help of Eqs. (4.88a) and (4.88b), we have

$$Q_1 = kA\theta_0\left[\sqrt{\left(\frac{U}{2\alpha}\right)^2 + \frac{hP}{kA}} - \frac{U}{2\alpha}\right] \tag{4.89a}$$

$$Q_2 = kA\theta_0\left[\sqrt{\left(\frac{U}{2\alpha}\right)^2 + \frac{hP}{kA}} + \frac{U}{2\alpha}\right] \tag{4.89b}$$

where Q_1 is the magnitude of the rate of heat conduction from $\zeta = 0$ to its negative direction, i.e. towards left of O', and Q_2 is the magnitude of the rate of heat conduction from $\zeta = 0$ to its positive direction, i.e. towards right of O'.

The temperature θ_0 should be a known parameter which can be evaluated if we know the heat flux from the heat source. Let $\dot{Q}_s$ be the rate of heat flow from the source. Then from an energy balance of a small element around the point O', we have

$$\dot{Q}_s = \dot{Q}_1 + \dot{Q}_2$$

Finally, we can write with the help of Eqs. (4.89a) and (4.89b)

$$\theta_0 = \frac{\dot{Q}_s}{2kA\sqrt{\left(\frac{U}{2\alpha}\right)^2 + \frac{hP}{kA}}} \tag{4.90}$$

4.3 NUMERICAL METHODS FOR TRANSIENT HEAT FLOW

The numerical methods to be described here are based on finite difference technique as discussed in Section 3.3 of Chapter 3. Let us first consider one-dimensional transient heat conduction. The governing differential equation for this case in a Cartesian coordinate system is given by

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2} \tag{4.91}$$

This partial differential equation is parabolic in nature and yields to a marching solution in terms of the marching variable t. There are, in general, two different approaches: an explicit approach and an implicit approach.

4.3.1 Explicit Approach

First we have to write Eq. (4.91) in a *difference* form. The temporal derivative of T is written in a forward difference scheme, whereas the second-order space derivative (right-hand side of Eq. (4.91)) is written in a central difference scheme at a time level where the values of T at all grid points are known. Then the difference equation becomes

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha\frac{T_{i+1}^n - 2\,T_i^n + T_{i-1}^n}{(\Delta x)^2} \tag{4.92a}$$

or
$$T_i^{n+1} = T_i^n + \frac{\alpha \Delta t}{\Delta x^2} (T_{i+1}^n - 2T_i^n + T_{i-1}^n) \tag{4.92b}$$

The time level is indicated by the superscript, while grid indexing in the x-direction is indicated by the suffix. Let us assume that the values of T are known at all grid points at time level n. Therefore in Eq. (4.92b), we have all known values on the right-hand side and hence the only unknown T_i^{n+1} cen be determined straightforward by this equation. We have a single equation with a single unknown. Time matching solution means that we calculate the values of T at all grid points at time level $n + 1$ from known values of T at level n with the help of Eq. (4.92b). Then the values of T at all grid points at time level n+2 are calculated from the known values at time level n+1 by making use of the same equation (Eq. (4.92b)). In this fashion, the solution is progressively obtained by marching in steps of time. This is illustrated in Figure 4.32. The solution starts from the time level $t = 0$, where we know the values of T at all grid points from the initial condition of the problem. The important feature in the solution is that one cannot find out the values of T_1 (T at $x = 0$, the left-hand boundary in Figure 4.32) and T_n (T at $x = L$, the right-hand boundary in Figure 4.32) at any time level from Eq. (4.92b). To calculate T_1^{n+1} from Eq. (4.92b), one needs the value of T_0^{n+1} and again to calculate T_N^{n+1}, the value of T_{N+1}^{n+1} is needed. But neither T_0^{n+1} nor T_{N+1}^{n+1} are known at any time level since they are outside the computational or physical domain of the problem.

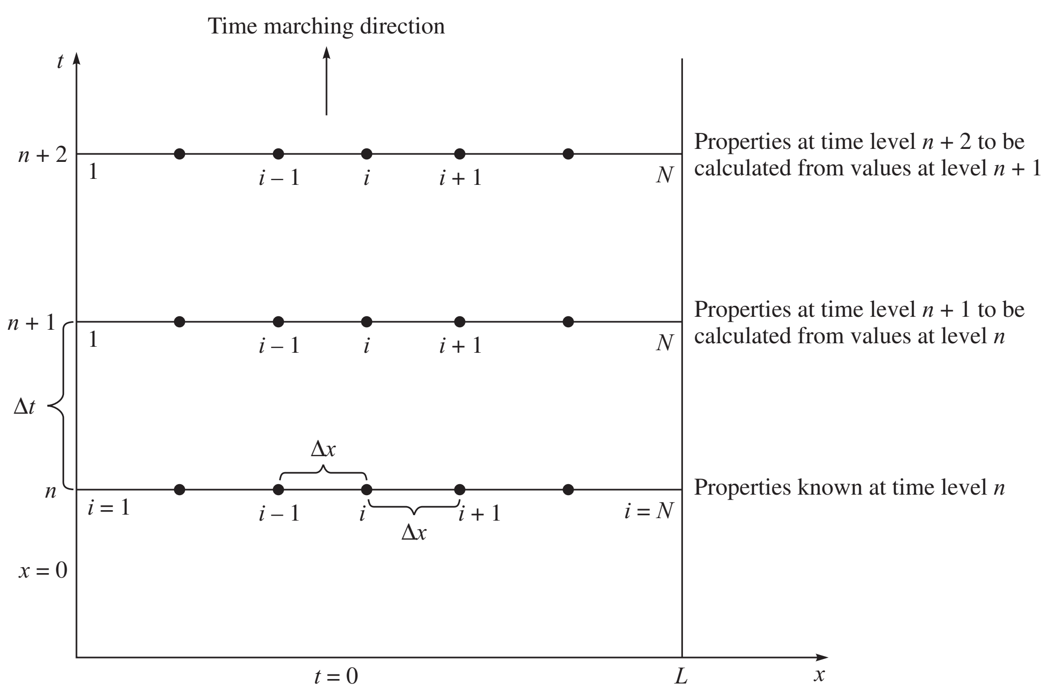


Figure 4.32 Illustration of time marching solution.

For any heat transfer problem pertaining to a physical system, the boundaries are always specified by some given conditions known as boundary conditions. For example, temperatures may be specified at the boundaries. In this situation, the temperatures at all boundary nodes at all time levels are known. This type of boundary condition is known as Dirichlet type boundary condition. The other types of boundary conditions encountered in a physical problem are as follows:

(a) Convective heat transfer from the boundary surface to the surrounding gas of a given temperature.
(b) The boundary surface is subjected to a constant heat flux.
(c) The boundary surface is insulated.

In all these cases, the boundary conditions are imposed through the temperature gradient at the boundary. These types of boundary conditions are known as **Neumann type condition**. Under these situations, the algebraic equation for the boundary nodes (the finite difference form of the heat conduction equation at boundary nodes) has to be developed and will be discussed later on.

Let us define a network Fourier number Fo_N as

$$Fo_N = \frac{\alpha \Delta t}{\Delta x^2} \tag{4.93}$$

Then Eq. (4.92b) can be written as

$$T_i^{n+1} = Fo_N(T_{i+1}^n + T_{i-1}^n) + (1 - 2Fo_N)T_i^n \tag{4.94}$$

Stability criterion and restriction on the value of Fo_N

The accuracy of finite difference solution depends upon the value of Δx and Δt. Though the accuracy increases with a decrease in Δx and Δt, the computational time also increases due to the increased number of nodal grid points and the time steps to reach a prescribed final time. However, an undesirable feature of the explicit method is that it is not unconditionally stable. By stability we mean that the solution for the nodal temperatures should continuously approach final (steady state) values with increasing time. However, the solution may be characterized by numerically induced oscillations which are physically impossible. If errors are already present at some stage of the solution from a real computer with finite accuracy, then the solution will be stable if the errors shrink, or at best remain the same as the solution progresses from step n to n+1. On the other hand, if the errors grow larger during the progression of the solution from step n to n+1, then the solution is unstable. The requirement for the stability of the solution of Eq. (4.94) in determining the values of T at all grid points at different time levels is that the coefficient of T_i^n should be greater than or equal to 0. This gives

$$Fo_N \le \frac{1}{2} \tag{4.95}$$

This restriction implies that for given values of α and Δx, the time step Δt cannot exceed the limit imposed on it by Eq. (4.95). This is called the stability criterion. A mathematical proof of this has been kept beyond the scope of this chapter. Interested readers can consult the reference Anderson [2]. Figure 4.33 shows, as an example, what happens if the stability criterion given by Eq. (4.95) is violated. The figure shows that numerical results obtained from Eq. (4.94) with a value of $Fo_N = 5/11$, satisfying the stability criterion, are in good agreement with the exact solution. But when the numerical solution is made with $Fo_N = 5/9$, associated with a larger time step and thus violating the stability criterion, the solution becomes unstable and oscillations grow with time, making the results diverge from the actual ones.

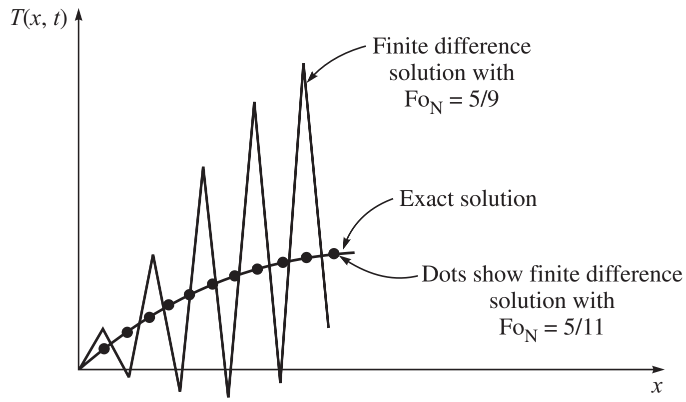


Figure 4.33 The influence of parameter Fo_N on stability of the solution of Eq. (4.94).

It can be demonstrated that violation of the restriction on Fo_N expressed by Eq. (4.95) is not physically possible since it violates otherwise the second law of thermodynamics as shown below.

Let us consider a situation for which

$$T_{i+1}^n = T_{i-1}^n$$

and

$$T_i^n < T_{i+1}^n$$

Then we can write from Eq. (4.94),

$$\begin{aligned} T_i^{n+1} &= 2Fo_N T_{i+1}^n + (1 - 2Fo_N)T_i^n \\ &= T_{i+1}^n + (2Fo_N - 1)(T_{i+1}^n - T_i^n) \end{aligned} \tag{4.96}$$

Since $T_{i+1}^n > T_i^n$, the second term on the right-hand side of Eq. (4.96) becomes positive if $Fo_N > \dfrac{1}{2}$, and results in

$$T_i^{n+1} > T_{i+1}^n$$

This is not physically possible since it violates the second law of thermodynamics.

4.3.2 Boundary or Exterior Nodal Equations

The boundary or exterior nodal equations, under different boundary conditions, are developed in a similar fashion as done in the case of steady-state conduction in Section 3.3 of Chapter 3. This is described below.

Convective heat loss from boundary surface

Let us consider a boundary node as shown in Figure 4.34. Applying the principle of conservation of energy to a control volume shown by dotted lines, we have

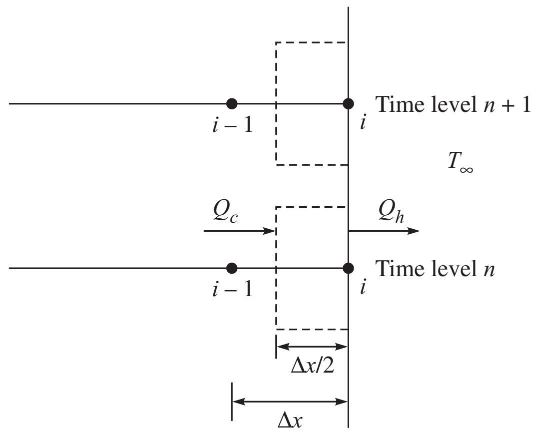


Figure 4.34 Energy balance for a boundary node.

$$Q_c - Q_h = \rho c A \,\frac{\Delta x}{2}\left(\frac{T_i^{n+1} - T_i^n}{\Delta t}\right) \tag{4.97a}$$

where

$$Q_c \text{ (heat conducted into the control volume) } = \frac{kA}{\Delta x}(T_{i-1}^n - T_i^n) \tag{4.97b}$$

$$Q_h \text{ (convective heat loss to the surrounding) } = hA(T_i^n - T_\infty) \tag{4.97c}$$

A = cross-sectional area of the control volume.

Therefore, Eq. (4.97a) becomes

$$\frac{kA\,(T_{i-1}^n - T_i^n)}{\Delta x} - hA(T_i^n - T_\infty) = \rho c A\,\frac{\Delta x}{2}\left(\frac{T_i^{n+1} - T_i^n}{\Delta t}\right)$$

or

$$T_i^{n+1} = \frac{2\,h\Delta t}{\rho c \Delta x}(T_\infty - T_i^n) + \frac{2\alpha\Delta t}{(\Delta x)^2}(T_{i-1}^n - T_i^n) + T_i^n \tag{4.98}$$

If we define a network Biot number (or cell Biot number) $\mathrm{Bi_N}$, in a similar way as done for $\mathrm{Fo_N}$ (Eq. (4.93)), as

$$\mathrm{Bi_N} = \frac{h(\Delta x)}{k}$$

we can write

$$\frac{2h\Delta t}{\rho c \Delta x} = 2\left(\frac{h(\Delta x)}{k}\right)\left(\frac{\alpha\,\Delta t}{(\Delta x)^2}\right)$$
$$= 2\,\mathrm{Bi_N}\,\mathrm{Fo_N}$$

Therefore, Eq. (4.98) can be written as

$$T_i^{n+1} = 2\text{Fo}_\text{N}(T_{i-1}^n + Bi_N T_\infty) + (1 - 2\text{Fo}_\text{N} - 2\text{Bi}_\text{N}\text{Fo}_\text{N})T_1^n \tag{4.99}$$

Recalling the procedure for determining the criterion of stability, we require that the coefficient of T_i^n must be greater than or equal to zero. Therefore,

$$1 - 2\text{Fo}_\text{N} - 2\text{Bi}_\text{N}\text{Fo}_\text{N} \geq 0$$

or

$$\text{Fo}_\text{N} \leq \frac{1}{2(1 + \text{Bi}_\text{N})} \tag{4.100}$$

Therefore we find that the stability criterion for the solution of the interior nodal equation (Eq. (4.94)) is given by Eq. (4.95), while that for the solution of exterior nodal equation (Eq. (4.99)) is given by Eq. (4.100). Now we have to determine which requirement is more stringent. Since $\text{Bi}_\text{N} > 0$, it is apparent that the limiting value of Fo_N determined from Eq. (4.100) will be less than that determined from Eq. (4.96). Therefore, Eq. (4.100) will serve as the criterion of stability to limit the maximum value of Fo_N and hence of Δt for a given grid spacing Δx and the given values of thermo-physical properties like α, h, and k.

4.3.3 Exterior Nodal Equations for Constant Heat Flux and Insulated Surfaces

If the boundary surface receives a constant heat flux q_0, then we modify Eq. (4.97a) as

$$Q_c + q_0 A = \rho c A \frac{\Delta x}{2}\left(\frac{T_1^{n+1} - T_1^n}{\Delta t}\right) \tag{4.101}$$

With the help of Eq. (4.97b), it becomes

$$\frac{kA\,(T_{i-1}^n - T_i^n)}{\Delta x} + q_0 A = \rho c A \frac{\Delta x}{2}\left(\frac{T_i^{n+1} - T_i^n}{\Delta t}\right)$$

or

$$T_i^{n+1} = 2\text{Fo}_\text{N}T_{i-1}^n + (1 - 2\text{Fo}_\text{N})T_i^n + \frac{2\,q_0}{\rho c\,\Delta x} \tag{4.102}$$

For an insulated surface, $q_0 = 0$ and hence we have

$$T_i^{n+1} = 2\text{Fo}_\text{N}T_{i-1}^n + (1 - 2\text{Fo}_\text{N})T_i^n \tag{4.103}$$

The stability criterion given by Eq. (4.100) satisfies the stability requirement for the solution of Eqs. (4.102) and (4.103).

EXAMPLE 4.13 A large iron plate ($\alpha = 2 \times 10^{-5}$ m^2/s) of 600 mm thickness is initially at 20°C. Suddenly one face of the wall is raised to 580°C, while the other face is insulated. Estimate the temperature distribution that exists in the plate 25 minutes after the surface temperature is raised to 580°C. (Use the explicit scheme of numerical method.)

Solution: This is a one-dimensional transient heat flow problem and we have to use Eq. (4.94).

Let us choose Δx = 100 mm and assign $Fo_N = \dfrac{1}{2}$ to satisfy the stability criterion.

Then, $$Fo_N = \frac{1}{2} = \frac{\alpha \Delta t}{(\Delta x)^2} = \frac{2 \times 10^{-5} \times \Delta t}{(0.1)^2}$$

which gives $$\Delta t = 250 \text{ s}$$

Therefore the number of time increments $= \dfrac{25 \times 60}{250} = 6$

Since one face of the wall is insulated, we can consider the insulated surface as the centre of a plate of thickness 1.2 m, both the surfaces of which are maintained at 580°C as shown in Figure 4.35.

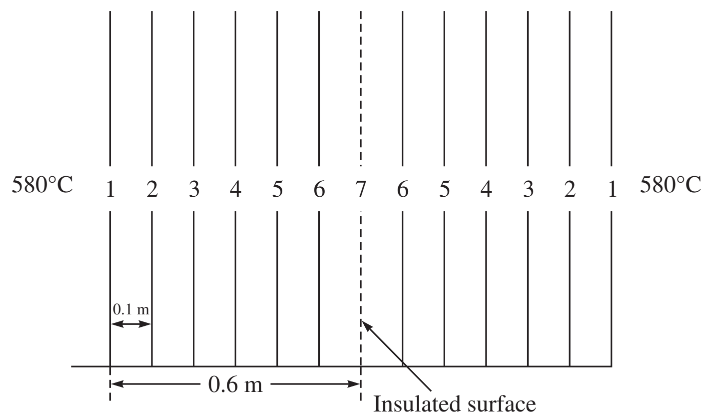


Figure 4.35 Grid spacing and nodal points in the plate (Example 4.13).

Substituting $Fo_N = \dfrac{1}{2}$ in Eq. (4.94), we have

$$T_i^{n+1} = \frac{T_{i+1}^n + T_{i-1}^n}{2} \tag{4.104}$$

At time $t = 0$, the surface (the node 1) is at 580°C while the rest of the plate (that is, all nodes from 2 to 7) is at 20°C. Equation (4.104) will be used to determine the temperatures at all nodes at different instants of time from $t = 0$ with a progressive increment of Δt = 250 s. The results are shown in Table 4.5.

Table 4.5 The results of Example 4.13

Time increments	*Temperature in °C at different nodes*							
	1	2	3	4	5	6	7	6
0	580	20	20	20	20	20	20	20
1	580	300	20	20	20	20	20	20
2	580	300	160	20	20	20	20	20

(*Contd.*)

Table 4.5 The results of Example 4.13 (*Contd.*)

Time increments	*Temperature in °C at different nodes*							
	1	2	3	4	5	6	7	6
3	580	370	160	90	20	20	20	20
4	580	370	230	90	55	20	20	20
5	580	405	230	142.50	55	37.50	20	37.50
6	580	405	273.75	142.50	90	37.50	37.50	37.50

Explicit scheme for multidimensional conduction

The differential equations for two- and three-dimensional unsteady conductions in a Cartesian coordinate system are written respectively as

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{4.105}$$

and

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{4.106}$$

With i, j, k as the grid indexing nomenclature for x, y, and z directions respectively, we can write the finite difference forms of the second-order space derivatives of Eq. (4.106) as

$$\frac{\partial^2 T}{\partial x^2} = \frac{T^n_{i+1,j,k} - 2T^n_{i,j,k} + T^n_{i-1,j,k}}{(\Delta x)^2} \tag{4.107a}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{T^n_{i,j+1,k} - 2T^n_{i,j,k} + T^n_{i,j-1,k}}{(\Delta y)^2} \tag{4.107b}$$

$$\frac{\partial^2 T}{\partial z^2} = \frac{T^n_{i,j,k+1} - 2T^n_{i,j,k} + T^n_{i,j,k-1}}{(\Delta z)^2} \tag{4.107c}$$

If we take $\Delta x = \Delta y = \Delta z$, we can write down the explicit expressions, similar to Eq. (4.94), for interior nodes in case of two- and three-dimensional systems as

$$T^{n+1}_{i,j} = \mathrm{Fo_N}\,(T^n_{i+1,j} + T^n_{i-1,j} + T^n_{i,j+1} + T^n_{i,j-1}) + (1 - 4\,\mathrm{Fo_N})T^n_{i,j} \quad \text{(for a two-dimensional system)} \tag{4.108}$$

and

$$T^{n+1}_{i,j,k} = \mathrm{Fo_N}\,(T^n_{i+1,j,k} + T^n_{i-1,j,k} + T^n_{i,j+1,k} + T^n_{i,j-1,k} + T^n_{i,j,k+1}\,T^n_{i,j,k-1}) + (1 - 6\,\mathrm{Fo_N})T^n_{i,j,k} \quad \text{(for a three-dimensional system)} \tag{4.109}$$

Therefore, the stability criterion dictates

$$\mathrm{Fo_N} \le \frac{1}{4} \quad \text{for two-dimensional systems}$$

$$\mathrm{Fo_N} \le \frac{1}{6} \quad \text{for three-dimensional systems}$$

Thus the values of temperatures at all internal grid points, i.e. for all values of i, j, k, are found out explicitly from Eq. (4.108) or Eq. (4.109), depending upon the case for a given value of $\mathrm{Fo_N}$ satisfying the stability criterion. The derivations of nodal equations for exterior nodes in case of two- and three-dimensional systems are left as an exercise to the readers. The nodal equations for one- and two-dimensional systems are summarized in Tables 4.6 and 4.7 respectively.

Table 4.6 The interior and exterior nodal equations in explicit method for one-dimensional unsteady heat conduction

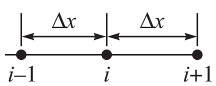
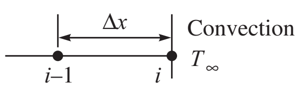
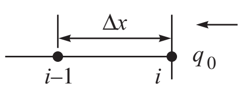
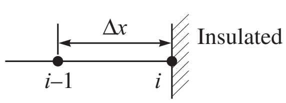
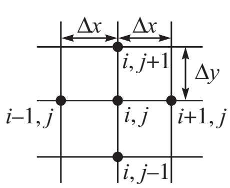
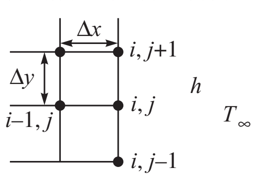
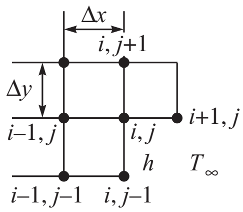
Physical situation	*Nodal equation*	*Stability criterion*
(a) Interior node 	$T_i^{n+1} = \mathrm{Fo_N}(T_{i+1}^n + T_{i-1}^n) + (1 - 2\mathrm{Fo_N})\, T_i^n$	$\mathrm{Fo_N} \le \frac{1}{2}$
(b) Exterior node with convection at the surface 	$T_i^{n+1} = 2\mathrm{Fo_N}(T_{i-1}^n + \mathrm{Bi_N} T_\infty) + (1 - 2\,\mathrm{Fo_N} - 2\,\mathrm{Fo_N}\,\mathrm{Bi_N})\; T_i^n$	$\mathrm{Fo_N} \le \dfrac{1}{2\,(1+\mathrm{Bi_N})}$
(c) Exterior node with a constant heat flux q_0 	$T_i^{n+1} = 2\mathrm{Fo_N}\; T_{i-1}^n + (1 - 2\mathrm{Fo_N})\; T_i^n + \dfrac{2q_0}{\rho c \Delta x}$	$\mathrm{Fo_N} \le \frac{1}{2}$
(d) Exterior node on an insulated surface 	$T_i^{n+1} = 2\,\mathrm{Fo_N} T_{i-1}^n + (1 - 2\,\mathrm{Fo_N}) T_i^n$	$\mathrm{Fo_N} \le \frac{1}{2}$

Table 4.7 The interior and exterior nodal equations in explicit method for two-dimensional unsteady heat conduction

Physical	*Nodal equation*	*Stability criterion*
(a) Interior node 	$T_{i,j}^{n+1} = \mathrm{Fo_N}\,(T_{i+1,j}^n - T_{i-1,j}^n + T_{i,j+1}^n + T_{i,j-1}^n) + (1 - 4\,\mathrm{Fo_N}) T_{i,j}^n$ $\Delta x = \Delta y$	$\mathrm{Fo_N} \le \frac{1}{4}$
(b) Exterior node at a plane surface with convection 	$T_{i,j}^{n+1} = \mathrm{Fo_N}(T_{i-1,j}^n + T_{i,j+1}^n + T_{i,j-1}^n + 2\,\mathrm{Bi_N} T_\infty) + (1 - 4\,\mathrm{Fo_N} - 2\,\mathrm{Bi_N}\,\mathrm{Fo_N}) T_{i,j}^n$	$\mathrm{Fo_N} \le \dfrac{1}{2\,(2+\mathrm{Bi_N})}$

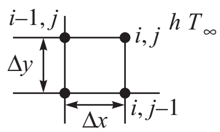
(c) Interior corner as an exterior node with convection

$$T_{i,j}^{n+1} = \frac{2}{3}\,\mathrm{Fo_N}\,(T_{i+1,j}^{n} + 2\,T_{i-1,j}^{n} + 2\,T_{i,j+1}^{n} + T_{i,j-1}^{n} + 2\,\mathrm{Bi_N}\,T_\infty) + \left(1 - 4\,\mathrm{Fo_N} - \frac{4}{3}\,\mathrm{Bi_N}\,\mathrm{Fo_N}\right) T_{i,j}^{n} \qquad \mathrm{Fo_N} \le \frac{3}{4\,(3 + \mathrm{Bi_N})}$$



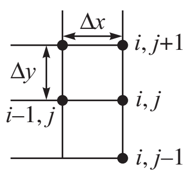
(d) Exterior corner as an exterior node

$$T_{i,j}^{n+1} = 2\,\mathrm{Fo_N}\,(T_{i-1,j}^{n} + T_{i,j-1}^{n} + 2\,\mathrm{Bi_N}\,T_\infty) + (1 - 4\,\mathrm{Fo_N} - 4\,\mathrm{Fo_N}\,\mathrm{Bi_N})\,T_{i,j}^{n} \qquad \mathrm{Fo_N} \le \frac{1}{4\,(1 + \mathrm{Bi_N})}$$



(e) Exterior node at an insulated boundary

$$T_{i,j}^{n+1} = \mathrm{Fo_N}\,(2\,T_{i-1,j}^{n} + T_{i,j+1}^{n} + T_{i,j-1}^{n}) + (1 - 4\,\mathrm{Fo_N})\,T_{i,j}^{n} \qquad \mathrm{Fo_N} \le \frac{1}{4}$$



4.3.4 Implicit Scheme

In the explicit method, the temperature at any nodal point (i, j, k) at a time $t+\Delta t$ is determined from the known values of temperatures at the same and neighbouring nodes at the preceding time t. This is done by expressing the space derivatives of temperature in finite difference form in terms of nodal temperature at time t, while the nodal temperature at time $t+\Delta t$ is induced only in the finite difference form of the temporal derivative of temperature. Though the method is very simple, it has an inherent limitation on the selection of time interval Δt based on the stability criterion. This dictates the use of a small value of Δt and thus requires a large number of time intervals to obtain a solution for a finite time. Therefore, the explicit method is time expensive. This disadvantage of the explicit method is overcome in the implicit method which is unconditionally stable and permits the use of a relatively large value of Δt. In the implicit method, there are two schemes (a) fully implicit scheme and (b) Crank Nicholson scheme.

Fully implicit scheme

Let us consider first a one-dimensional situation. There we have to deal with Eq. (4.91), the finite difference form of which in this scheme is given by

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha\,\frac{T_{i+1}^{n+1} - 2\,T_{i,}^{n+1} + T_{i-1}^{n+1}}{(\Delta x)^2} \tag{4.110}$$

A comparison of Eq. (4.110) with Eq. (4.92a) reveals that the left-hand sides of both the equations are same, while the right-hand side of Eq. (4.110) is evaluated at time step n+1. This way the unknown T_i^{n+1} is expressed in terms of other unknown quantities at time level n+1, namely T_{i+1}^{n+1} and T_{i-1}^{n+1}. In other words, Eq. (4.110) represents one equation with three unknown quantities, namely T_{i+1}^{n+1}, T_i^{n+1} and T_{i-1}^{n+1}. Equation (4.110) can be written in a rearranged form as

$$\mathrm{Fo_N}\, T_{i-1}^{n+1} - (1 + 2\mathrm{Fo_N})T_i^{n+1} + \mathrm{Fo_N}\, T_{i+1}^{n+1} = -\, T_i^n \tag{4.111}$$

where $\mathrm{Fo_N}$ (the network Fourier number) is defined by Eq. (4.93).

Crank–Nicholson scheme

This is a modified implicit scheme, where the left-hand side of Eq. (4.91) is written in the same fashion as done in Eq. (4.110), but the right-hand side is evaluated at the arithmetic average of explicit and fully implicit formulation, i.e. in terms of average temperatures between time level n and n+1. Therefore, the finite difference form of Eq. (4.91) becomes

$$\frac{T_i^{n+1} - T_{i,}^n}{\Delta t} = \alpha \frac{\frac{1}{2}(T_{i-1}^{n+1} + T_{i-1}^n) + \frac{1}{2}(-2\,T_{i,}^{n+1} - 2T_{i,}^n) + \frac{1}{2}(T_{i+1}^{n+1} + T_{i+1}^n)}{(\Delta x)^2} \tag{4.112}$$

Equation (4.112) can be rearranged in a simpler form as

$$\frac{\mathrm{Fo_N}}{2} T_{i-1}^{n+1} - (1 + \mathrm{Fo_N})T_i^{n+1} + \frac{\mathrm{Fo_N}}{2} T_{i+1}^{n+1} = -\, T_i^n - \frac{\mathrm{Fo_N}}{2}(T_{i+1}^n - 2T_i^n + T_{i-1}^n) \tag{4.113}$$

It is observed that in both Eqs. (4.111) and (4.113) there are three unknown quantities, namely T_{i+1}^{n+1}, T_i^{n+1} and T_{i-1}^{n+1}. However, the coefficients of the unknown quantities and the right-hand sides of the equations are all known quantities. Therefore, Eqs. (4.111) and (4.113) can be written in a generalized form as

$$AT_{i-1}^{n+1} + BT_i^{n+1} + AT_{i+1}^{n+1} = C_i \tag{4.114}$$

where the constants A, B, C_i are given as follows:

For the fully implicit scheme (Eq. (4.111)),

$$A = \mathrm{Fo_N} \tag{4.115a}$$

$$B = -(1 + 2\mathrm{Fo_N}) \tag{4.115b}$$

$$C_i = -\, T_i^n \tag{4.115c}$$

For the Crank–Nicholson scheme (Eq (4.113)),

$$A = \frac{\mathrm{Fo_N}}{2} \tag{4.116a}$$

$$B = -\,(1 + \mathrm{Fo_N}) \tag{4.116b}$$

$$C_i = -\, T_i^n - \frac{\mathrm{Fo_N}}{2}(T_{i+1}^n - 2T_i^n + T_{i-1}^n) \tag{4.116c}$$

Since Eq. (4.114) contains three unknown quantities, it does not standalone for a solution of T_i^{n+1}. Therefore, Eq. (4.114) is written at all interior grid points, resulting in a system of algebraic equations from which the unknown temperatures T_i^{n+1} for all i can be solved simultaneously. By definition, an implicit scheme is one where the unknowns are obtained by means of a simultaneous solution of the difference equations applied at all the points arrayed at a given time level. Let us consider the grid network as shown in Figure 4.32 where the temperatures at boundary nodes are known. Now, we write Eq. (4.114) for grid points from 2 to $N-1$ (i.e for $i = 2$ to $N-1$)

At grid point 2: $$AT_1 + BT_2 + AT_3 = C_2$$

or $$BT_2 + AT_3 = C'_2$$

where $C'_2 = C_2 - AT_1$; since T_1 is known, C'_2 is also a known quantity.

At grid point 3: $$AT_2 + BT_3 + AT_4 = C_3$$

At grid point 4: $$AT_3 + BT_4 + AT_5 = C_4$$

At grid point $N-1$: $$AT_{N-2} + BT_{N-1} + AT_N = C_{N-1}$$

or $$AT_{N-2} + BT_{N-1} = C'_{N-1}$$

where $C'_{N-1} = C_{N-1} - AT_N$; since T_N is known, C'_{N-1} is a known quantity.

Therefore, we get a system of $N-2$ linear algebraic equations which can be written in matrix form as

$$\begin{bmatrix} B & A & 0 & 0 & 0 & - & - & - & 0 \\ A & B & A & 0 & 0 & - & - & - & 0 \\ 0 & A & B & A & 0 & - & - & - & 0 \\ 0 & 0 & A & B & A & - & - & - & 0 \\ - & - & - & - & - & - & - & - & 0 \\ - & - & - & - & - & - & - & - & 0 \\ - & - & - & - & - & - & - & - & 0 \\ - & - & - & - & - & - & A & B & A \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & B \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ - \\ - \\ - \\ - \\ T_{N-1} \end{bmatrix} = \begin{bmatrix} C'_2 \\ C_3 \\ C_4 \\ C_5 \\ - \\ - \\ - \\ - \\ C'_{N-1} \end{bmatrix} \tag{4.117}$$

The coefficient matrix of Eq. (4.117) is a tri-diagonal matrix which has nonzero elements only along the three diagonals. The solution of the equation is usually done by the typical method of matrix inversion involving some manipulation of the tri-diagonal arrangements. The solution algorithm is known as 'Thomas algorithm'. A description of this algorithm is given in Appendix C.

Comparisons of implicit and explicit methods

The inherent advantage of the implicit method is that it is unconditionally stable. This means that any value of Δt, no matter how large, will result in a stable solution. Though on some occasions, oscillations are found, but the stability can be maintained over much larger values of Δt. However, the disadvantage of the method is that it is much more involved with algebraic equations and their solution compared to the explicit method which involves the simplest algebraic equation to determine the unknowns, straightforward, without any complicated solution algorithm. But the inherent disadvantage of the explicit method is the restriction on the value of Δt to ensure a stable solution. Once Δx is chosen, then Δt is not an independent parameter, rather it is restricted to be equal to or less than a certain value prescribed by the stability criterion. In many situations, the limiting value of Δt comes out to be so small to maintain the stability, that it results in a long computer running time to make calculations over a given interval of time. This is time expensive and also cost prohibitive. Therefore the choice of the method, whether explicit or implicit, depends mostly on the nature of the problem and is judged better by the implementer.

Implicit method for two-dimensional unsteady heat-conduction problem

The two-dimensional unsteady heat conduction equation is given by

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{4.118}$$

Let us consider the fully implicit scheme and write Eq. (4.118) accordingly in finite difference form as

$$\frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t} = \alpha\left[\frac{T_{i+1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i-1,j}^{n+1}}{(\Delta x)^2} + \frac{T_{i,j+1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j-1}^{n+1}}{(\Delta y)^2}\right] \tag{4.119}$$

In consideration of $\Delta x = \Delta y$, Eq. (4.119) can be written in a rearranged form as

$$AT_{i-1,j}^{n+1} + BT_{i,j}^{n+1} + AT_{i+1,j}^{n+1} + AT_{i,j-1}^{n+1} + AT_{i,j+1}^{n+1} = C_{i,j} \tag{4.120}$$

where

$A = \mathrm{Fo_N}$
$B = -(1 + 4\mathrm{Fo_N})$
$C_i = -T_{i,j}^{n}$

From a careful observation of Eq. (4.120) , we find that there are five unknowns in the left-hand side of the equation which destroys the tri-diagonal nature of the set of algebraic equations to be developed from Eq. (4.120) for different values of i and j. If there are N interior grid points in x-direction and M interior grid points in y-direction, then we have MN number of unknown temperatures to be found out. Equation (4.120) in this case results into a set of MN numbers of equations for different values of i from 1 to N for each value of j from 1 to M. The set of algebraic equations can be solved by any standard method of simultaneous solution of linear algebraic equations. However the solution requires much longer computer time compared to that required by Thomas algorithm for the solution of a set of equations of tri-diagonal in nature. Because of this, a scheme has been developed which allows Eq. (4.120) to be solved by means of tri-diagonal forms only. The scheme is known as **alternating-direction implicit scheme** (ADI) and is described below.

Let us consider that in the time marching solution of Eq. (4.120), we have to obtain the values of temperature at all grid points at a time $t + \Delta t$ from the known values of temperature at all grid points at a time t. The solution is achieved in a two-step process. In the first step over a time interval of $\Delta t/2$, the spatial derivatives in Eq. (4.118) are written in a central difference scheme, where only the x-derivative is treated implicitly. This gives

$$\frac{T_{i,j}^{n+1/2} - T_{i,j}^{n}}{\Delta t/2} = \alpha\left[\frac{T_{i+1,j}^{n+1/2} - 2T_{i,j}^{n+1/2} + T_{i-1,j}^{n+1/2}}{(\Delta x)^2} + \frac{T_{i,j+1}^{n} - 2T_{i,j}^{n} - T_{i,j-1}^{n}}{(\Delta y)^2}\right] \tag{4.121}$$

Equation (4.121) reduces to the tri-diagonal form as

$$AT_{i-1,j}^{n+1/2} + BT_{i,j}^{n+1/2} + AT_{i+1,j}^{n+1/2} = C_i, \tag{4.122}$$

where

$$A = \frac{\alpha \Delta t}{2(\Delta x)^2}$$

$$B = -\left(1 + \frac{\alpha \Delta t}{(\Delta x)^2}\right)$$

$$C_i = -T_{i,j}^{n} - \frac{\alpha \Delta t}{2(\Delta x)^2}(T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j-1}^{n})$$

For a solution using the Thomas algorithm, we proceed as follows: For a fixed value of j, we solve for $T_{i,j}^{n+1/2}$ for all values of i. This is termed a 'sweep' in the x-direction. If there are N grid points in the x-direction, we sweep from $i = 1$ to N. For a sweep like this, we can use the Thomas algorithm for the solution. This calculation is then repeated at the next row of grid points designated by $j + 1$, i.e. we have to replace j in Eq. (4.122) by $j + 1$ and solve for $T_{i,j+1}^{n+1/2}$ for all values of i from 1 to N. If there are M grid points in the y-direction, this process is repeated M times which means there will be M sweeps in the x-direction using Thomas algorithm M times. This is shown in Figure 4.36. At the end of the step, values of T at the intermediate time $t + \Delta t/2$ are known at all grid points, that is, $T_{i,j}^{n+1/2}$ is known at all (i,j).

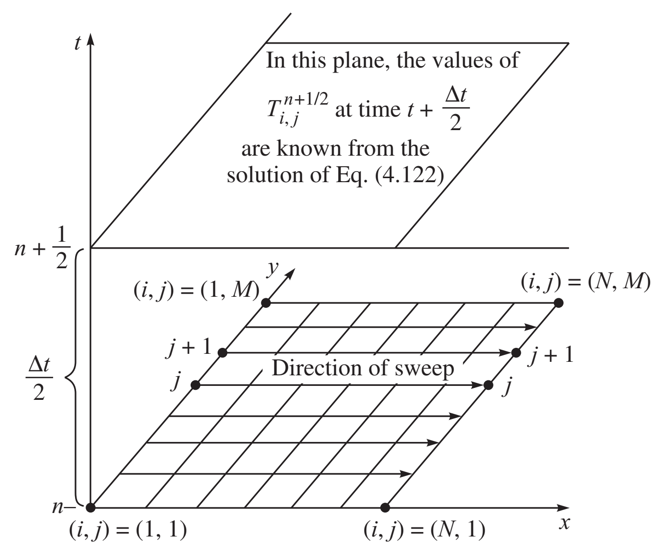


Figure 4.36 First step in the ADI process (sweeping in the x-direction to obtain T at time $(t + \Delta t/2)$.

In the second step of the ADI scheme, the solution is advanced from the time level $t + \Delta t/2$ to $t + \Delta t$, i.e. from the known values of $T_{i,j}^{n+1/2}$ to the values of $T_{i,j}^{n+1}$ for all (i, j). For this step, the spatial derivatives in Eq. (4.118) are written in a central difference scheme, where only the y-derivative is treated implicitly. This gives

$$\frac{T_{i,j}^{n+1} - T_{i,j}^{n+1/2}}{\Delta t/2} = \alpha \frac{(T_{i+1,j}^{n+1/2} - 2T_{i,j}^{n+1/2} + T_{i-1,j}^{n+1/2})}{(\Delta x)^2} + \alpha \frac{(T_{i,j+1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j-1}^{n+1})}{(\Delta y)^2} \tag{4.123}$$

Equation (4.123) is reduced in a tri-diagonal form as

$$DT_{i,j-1}^{n+1} + ET_{i,j}^{n+1} + DT_{i,j+1}^{n+1} = F_j \tag{4.124}$$

where

$$D = \frac{\alpha \Delta t}{2(\Delta y)^2}$$

$$E = \left(1 + \frac{\alpha \Delta t}{(\Delta y)^2}\right)$$

$$F_j = -T_{i,j}^{n+1/2} - \frac{\alpha \Delta t}{2(\Delta x)^2}(T_{i+1,j}^{n+1/2} - 2T_{i,j}^{n+1/2} - T_{i-1,j}^{n+1/2})$$

The value of $T_{i,j}^{n+1/2}$ is known at all grid points from the first step. Therefore, Eq. (4.124) is solved for $T_{i,j}^{n+1}$ in the second step. This is done in exactly the similar fashion as done in the first step. Here we sweep in the y-direction. In each sweep, the solution is made for $T_{i,j}^{n+1}$ for all values of j from 1 to M for a fixed value of i using the Thomas algorithm once. The process is repeated in another sweep where i is replaced by i + 1. Thus we require N sweeps to cover the x domain as shown in Figure 4.37. At the end of this step, the values of T at time $t + \Delta t$ are known at all grid points, i.e. $T_{i,j}^{n+1}$ is known at all (i,j).

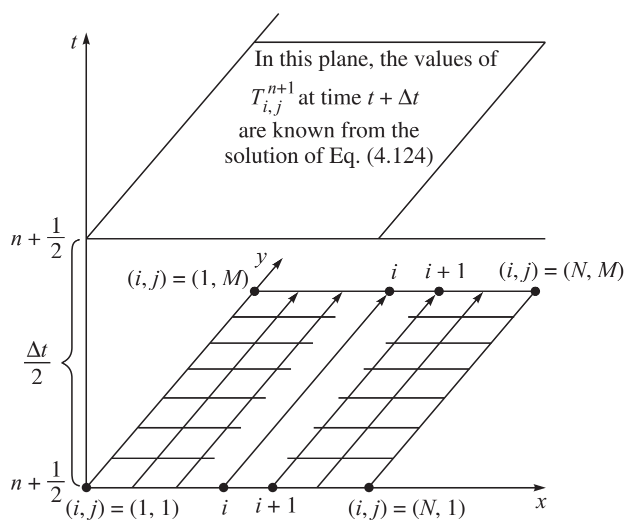


Figure 4.37 Second step in the ADI process (sweeping in the y-direction to obtain T at time $t + \Delta t$).

At the end of the two-step process, the dependent variable T has been marched a value Δt in the direction of t. Thus the solution of T is advanced in steps of Δt and each marching step of Δt comprises two sub-steps of $\Delta t/2$ so that in one of the sub-steps the equation is implicit in x, while in other, the equation is implicit in y with the repeated use of Thomas algorithm.

Exterior nodal equation (*finite difference form of heat conduction equation at nodes at the boundary surface*) *in implicit scheme*

The exterior nodal equations for different boundary conditions are developed in a similar way by considering the energy balance of a control volume enveloping the external node as done in the case of the explicit method. The starting equation is the same as Eq. (4.97a) (refer Figure 4.34). The only difference is that in the implicit scheme, Q_c and Q_h in Eq. (4.97a) are expressed in terms of the unknown nodal temperatures at the advanced time plane $n + 1$ instead of the known temperatures at the previous time plane n as done in the case of the explicit method. Therefore, the exterior nodal equations in implicit scheme contain two unknown quantities, namely T_i^{n+1}, T_{i-1}^{n+1} where i represents the boundary node. The exterior nodal equations are summarized in Tables 4.8, 4.9. The derivations are left as exercises to the reader.

EXAMPLE 4.14 Solve Example 4.13 by the fully implicit scheme.

Solution: Let us take $\Delta x = 0.1$ m (same as that taken in case of Example 4.13). We can take $\Delta t = 25$ min since there is no restriction on Δt due to stability, so that we can generate the results by one time step. However, the larger is the value of Δt, the smaller is the accuracy of the results. Therefore, we have to choose a reasonably lower value of Δt to expect a fairly accurate result. Let us choose $\Delta t = 500$ s.

Therefore the number of time increments or steps = $\dfrac{25 \times 60}{500} = 3$.

We have to use Eq. (4.114) for all interior grid points from $i = 2$ to 6.

Here,
$$A = \mathrm{Fo_N} = \frac{2 \times 10^{-5} \times 500}{(0.1)^2} = 1$$
$$B = -(1 + 2) = -3$$

Calculation for the first step.

From Eq. (4.114), we get

$$-3T_2 + T_3 = -600 \tag{4.125a}$$
$$T_2 - 3T_3 + T_4 = -20 \tag{4.125b}$$
$$T_3 - 3T_4 + T_5 = -20 \tag{4.125c}$$
$$T_4 - 3T_5 + T_6 = -20 \tag{4.125d}$$
$$T_5 - 3T_6 + T_7 = -20 \tag{4.125e}$$

For $i = 7$ (the node on the boundary), we use the exterior nodal equation for insulated boundary from Table 4.8. Thus, we have

$$2T_6 - 3T_7 = -20 \tag{4.125f}$$

Table 4.8 The exterior nodal equations in implicit scheme (one-dimensional conduction)

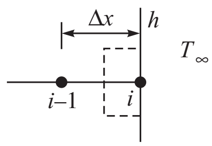
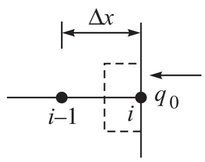
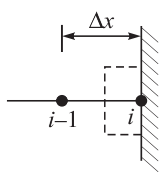
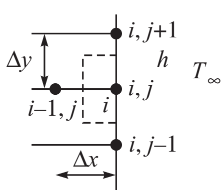
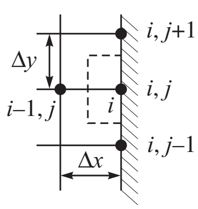
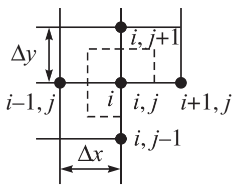
Physical situation	*Nodal equation*
(a) node on a plane surface with convection 	$2\text{Fo}_N\, T_{i-1}^{n+1} - (1 + 2\text{Fo}_N + 2\text{Fo}_N\, \text{Bi}_N)\, T^{\,n+1}$ $= -\, T_i^n - 2\text{Fo}_N\, \text{Bi}_N\, T_\infty$
(b) plane surface receiving a constant heat flux q_0 	$2\text{Fo}_N\, T_{i-1}^{n+1} - (1 + 2\text{Fo}_N) T_i^{n+1} = -\, T_i^n - \dfrac{2\Delta t}{\rho c \Delta x}\, q_0$
(c) Insulated plane surface 	$2\text{Fo}_N\, T_{i-1}^{n+1} - (1 + 2\text{Fo}_N) T_i^{n+1} = -\, T_i^n$

Table 4.8 The exterior nodal equations in implicit scheme (two-dimensional conduction)

Physical situation	*Nodal equation*
(a) Plane surface with convection 	$(1 + 2\text{Fo}_N\, (2 + \text{Bi}_N))\, T_{i,j}^{n+1} - \text{Fo}_N\, (2T_{i-1,j}^{n+1} + T_{i,j+1} + \,_{i,j-1}^{n+1})$ $= T_{i,j}^n + 2\text{Bi}_N\, \text{Fo}_N\, T_\infty$
(b) Plane Insulated surface 	$(1 + 4\text{Fo}_N)\, T_{i,j}^{n+1} - \text{Fo}_N(2T_{i-1,j}^{n+1} + T_{i,j+1}^{n+1} + T_{i,j-1}^{n+1}) = T_{i,j}^n$

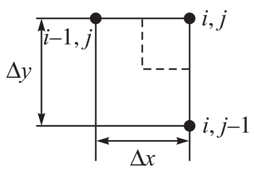
(*Contd.*)

(c) Interior corner with convection.



$$\left(1+4\mathrm{Fo_N}\left(1+\frac{1}{3}\mathrm{Bi_N}\right)\right)T_{i,j}^{n+1} - \frac{2}{3}\mathrm{Fo_N}(T_{i+1,j}^{n+1} + 2T_{i-1,j}^{n+1} + 2T_{i,j+1}^{n+1}$$
$$+ T_{i,j-1}^{n+1}) = T_{i,j}^{n} + \frac{4}{3}\mathrm{Bi_N}\ \mathrm{Fo_N}\ T_\infty$$

(d) Exterior corner with convection



$$(1 + 4\mathrm{Fo_N}(1 + \mathrm{Bi_N}))T_{i,j}^{n+1} - 2\mathrm{Fo_N}\,(T_{i-1,j}^{n+1} + T_{i,j-1}^{n+1}) = T_{i,j}^{n}$$
$$+ 4\,\mathrm{Bi_N}\,\mathrm{Fo_N}\,T_\infty$$

Equations (4.125a) to (4.125f) are written in matrix form as

$$\begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{bmatrix} = \begin{bmatrix} -600 \\ -20 \\ -20 \\ -20 \\ -20 \\ -20 \end{bmatrix} \tag{4.126}$$

By making use of Thomas algorithm, we get from Eq. (4.126)

$$\begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{bmatrix} = \begin{bmatrix} 233.913 \\ 101.739 \\ 51.304 \\ 32.174 \\ 25.217 \\ 23.478 \end{bmatrix}$$

The values of nodal temperatures as shown above are at $t = 500$ s. For the second step from 500 s to 1000 s, the set of equations similar to Eqs. (4.125a) to (4.125f) can be written, following Eq. (4.114), as

$$\begin{aligned} -3T_2 + T_3 &= -580 - 233.913 = -813.913 \\ T_2 - 3T_3 + T_4 &= -101.739 \\ T_3 - 3T_4 + T_5 &= -51.304 \\ T_4 - 3T_5 + T_6 &= -32.174 \\ T_5 - 3T_6 + T_7 &= -25.217 \\ 2T_6 - 3T_7 &= -23.478 \end{aligned}$$

Putting the above equations in matrix form, we have

$$\begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{bmatrix} = \begin{bmatrix} -813.913 \\ -101.739 \\ -51.304 \\ -32.174 \\ -25.217 \\ -23.478 \end{bmatrix}$$

which gives

$$\begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{bmatrix} = \begin{bmatrix} 329.630 \\ 174.977 \\ 93.562 \\ 54.404 \\ 37.478 \\ 32.811 \end{bmatrix}$$

For the third and final step of the problem, i.e. from t = 1000 s to 1500 s (25 min), we have the set of equations for the temperatures as follows:

$$\begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{bmatrix} = \begin{bmatrix} -909.630 \\ -174.977 \\ -93.562 \\ -54.404 \\ -37.478 \\ -32.811 \end{bmatrix}$$

which gives

$$\begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{bmatrix} = \begin{bmatrix} 379.631 \\ 229.892 \\ 135.069 \\ 81.753 \\ 55.786 \\ 48.128 \end{bmatrix}$$

Therefore, at the end of 25 minutes, the temperatures at nodal points (Figure 4.35) are:

$$T_1 = 580°\text{C},\ T_2 = 379.631°\text{C},\ T_3 = 229.892°\text{C},\ T_4 = 135.069°\text{C},\ T_5 = 81.753°\text{C},$$
$$T_6 = 55.786°\text{C, and } T_7 = 48.128°\text{C}$$

4.4 GRAPHICAL METHOD OF ANALYZING TRANSIENT HEAT CONDUCTION PROBLEM (THE SCHMIDT PLOT)

This method is based on the graphical representation of the explicit numerical scheme with a typical choice of $Fo_N = 1/2$. Let us explain the method in case of a one-dimensional transient conduction with constant surface temperature.

Consider a thick plane wall which is initially at a temperature of T_i. Let one of the surfaces of the wall be suddenly changed to a temperature T_s. We are now interested in determining the temperature distribution in the plate as a function of time. The heat conduction equation under the situation is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

The explicit finite difference form of the equation is written, following Eq. (4.94), as

$$T_i^{n+1} = \mathrm{Fo}_N(T_{i+1}^n + T_{i-1}^n) + (1 - 2\mathrm{Fo}_\mathrm{N})T_i^n$$

If we choose $\mathrm{Fo}_\mathrm{N} = \frac{1}{2}$, the above equation is reduced to

$$T_i^{n+1} = \frac{T_{i+1}^n + T_{i-1}^n}{2} \tag{4.127}$$

Equation (4.127) is the base of the graphical method known as Schmidt plot. The equation shows that the temperature for node i at a time level $(n + 1)$ is the arithmetic average of the temperature of two adjacent nodes $(i + 1)$ and $(i - 1)$ at time level n.

In this method the plane wall is divided into layers, each of thickness Δx, which are labelled with the index number of the nodes as shown in Figure 4.38. The initial temperature at each node is plotted. The time increment Δt is fixed by the choice of Δx according to the criterion

$$\mathrm{Fo}_\mathrm{N} = \frac{\alpha \Delta t}{(\Delta x)^2} = \frac{1}{2}$$

Now if we draw a straight line connecting the temperatures at nodes $(i - 1)$ and $(i + 1)$ at time level n, the point of intersection of this line with the mid-plane, denoting the node i, gives the temperatures at node i at the time level $(n + 1)$. The graphical construction is illustrated in Figure 4.38.

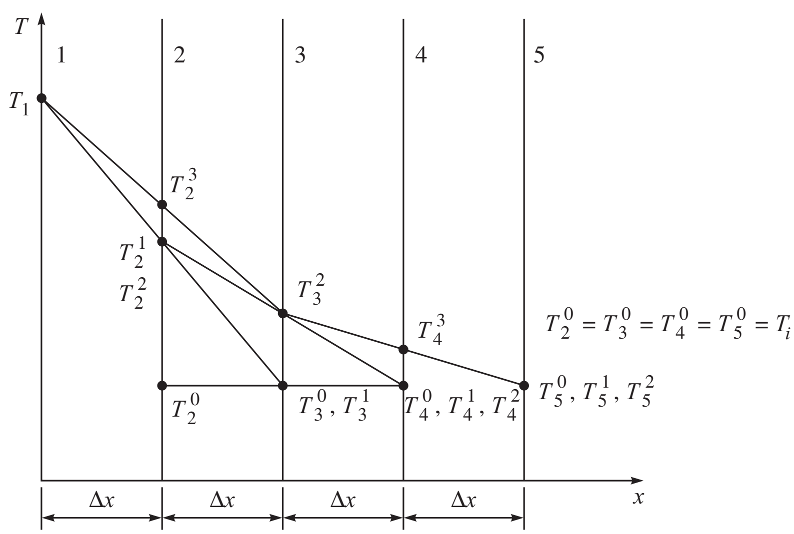


Figure 4.38 Schmidt plot for a one-dimensional unsteady conduction.

SUMMARY

- In an unsteady state heat conduction, the temperature becomes a function of both time and space coordinates. This happens in practice when a solid body is subjected to a change in its thermal environment.
- The system, in which the temperature is uniform throughout, is defined as a lumped heat-capacity system. This happens when the internal conduction resistance of the system is much smaller than the convection resistance of the fluid film at the surface. This is characterized by a low value of Biot number Bi ($=hL_c/k$). The lumped heat-capacity system analysis is applicable when Bi < 0.1. A lumped heat-capacity system may be described by an analogous capacitive–resistive electrical circuit.
- Analytical solutions are easily available for one-dimensional transient heat conduction problems pertaining to simple geometries like infinite plates, infinitely long cylinders and spheres subject to constant temperature or convection boundary condition at the surface. The temperature distribution is expressed as a product of an exponential function of time and an infinite series of trigonometric functions of space coordinate in case of flat plate and sphere or a Bessel's function of first kind of the space coordinate in case of cylinder. The graphical representations of the approximated analytical expressions of temperature distribution in case of infinite flat plates, infinitely long cylinders and spheres for Fo > 0.2 are known as Heisler charts.
- The solution for temperature distribution of a two-dimensional rectangular bar with dimension of $2L_1 \times 2L_2$ can be obtained from the product of the solutions of two infinite flat plates, namely plate 1 of thickness $2L_1$ and plate 2 of thickness $2L_2$. Similarly, the solution for a three-dimensional body with dimensions of $2L_1 \times 2L_2 \times 2L_3$ can be obtained from the product of the solution of three infinite flat plates, namely plate 1 of thickness $2L_1$, plate 2 of thickness $2L_2$ and plate 3 of thickness $2L_3$. The solution for a cylinder for a cylinder of radius r_0 and a finite length L can be expressed as a product of the solutions of an infinitely long cylinder of radius r_0 and an infinite flat plate having a thickness of L.
- Analytical solution for transient temperature distribution in a semi-infinite solid subject to a sudden change in thermal boundary condition (either a constant temperature or convection to a surrounding ambience at a given temperature) at one identifiable surface can be expressed in terms of the Gaussian error function of a similarity parameter η where $\eta = x/2\sqrt{\alpha t}$, x being the coordinate in the derection of heat flow
- Transient heat conduction problems are solved conveniently by numerical methods. In this method, temperatures are defined only at discrete grid points in a grid network in the conducting medium. The differential form of heat conduction equation is replaced by an algebraic equation by substituting the partial derivatives in the form of algebraic difference quotients. There are two methods: Explicit method and Implicit method.
- In an explicit method, the temperature at a nodal point at any time $t + \Delta t$ is determined straightforward from the known values of temperatures at the same and neighbouring nodes at the preceding time t. This is done by expressing the space derivatives of temperature in finite difference form in terms of nodal temperatures at time t, while the nodal temperature at $t + \Delta t$ is induced only in the temporal derivative of temperature. The

method is very simple but has an inherent limitation on time interval Δt based on the stability criterion. This results in the use of a small value of Δt and requires a large number of time intervals to obtain a solution for a finite time. The stability criteria under different situations are given by

$$\mathrm{Fo_N} \le \frac{1}{2} \qquad \text{for a one-dimensional system}$$

$$\mathrm{Fo_N} \le \frac{1}{4} \qquad \text{for a two-dimensional system}$$

$$\mathrm{Fo_N} \le \frac{1}{6} \qquad \text{for a three-dimensional system}$$

If the solid body is exposed to a convection environment, the stability criteria are given by

$$\mathrm{Fo_N} \le \frac{1}{2\,(1+\mathrm{Bi})} \qquad \text{for a one-dimensional system}$$

$$\mathrm{Fo_N} \le \frac{1}{2\,(2+\mathrm{Bi})} \qquad \text{for a two-dimensional system}$$

- In a fully implicit method, the second-order space derivatives of temperature in heat conduction equation are written in terms of the unknown nodal temperatures at the advanced time level $t + \Delta t$. In a modified implicit method (known as Crank–Nicholson method), the space derivatives of temperatures are written in terms of arithmetic average of unknown temperatures at time level $t + \Delta t$ and known temperatures at time level t. In both the methods, unknown temperatures are obtained by means of a simultaneous solution of the difference equations applied at all grid points arrayed at a given time level. The implicit method is unconditionally stable but involves manipulations of large matrices for the solution of difference equations.
- One-dimensional transient heat conduction problem can also be solved by the graphical method which is based on the explicit scheme of numerical method with the value of $\mathrm{Fo_N} = 1/2$. Under the situation, the unknown temperature at a node at time $t + \Delta t$ is the arithmetic average of the known temperatures of the two adjacent nodes at time t. The graphical plot in the method is known as Schmidt plot.

REVIEW QUESTIONS

1. (a) Tick the correct answer:
For a lumped-heat capacity system
(i) the conduction resistance within the body almost equals the convection resistance to heat flow at the surface.
(ii) the conduction resistance is much greater than the convection resistance.
(iii) the conduction resistance is much smaller than the convection resistance.

2. (a) Write the expression for Biot number and explain its physical significance.
(b) Is the Biot number more likely to be larger for highly conducting solid bodies or for poorly conducting ones?

(c) What is the limiting value of the Biot number for the lumped-heat-capacity analysis to be fairly accurate? Is it a maximum or minimum value?

3. Consider the two cases: In one case, a hot solid body is cooled in a pool of water in a large container. In another case, an identical hot solid body is cooled in air at the same temperature as that of water. In which case is the lumped-heat-capacity-system analysis more applicable and why?

4. Consider a hot boiled egg on a plate. The temperature of the egg is observed to drop by 5°C during the first minute. Will the temperature drop during the second minute be less than, equal to, or more than 5°C? Explain your answer with reasons.

5. If t_f is the time required for a 'lumped system' to reach the average temperature $(1/2)(T_i + T_\infty)$ where T_i is the initial temperature and T_∞ is the temperature of the environment, express t_f in terms of heat transfer coefficient h, thermal conductivity of the system k, and other thermo-physical properties.

6. What is meant by (a) infinite plate and (b) infinitely long cylinder in the analysis of transient heat conduction?

7. (a) Can you use Heisler chart for one-dimensional transient heat conduction problems at all Fourier numbers?
(b) Can you use Heisler chart for two- and three-dimensional systems undergoing transient heat conduction?

8. (a) Why do we need the numerical method when the analytical method can predict the exact result?
(b) Which one of the following two approaches you think is more accurate?
  (i) Prediction of the result from an analytical solution with simplifying assumptions
  (ii) Prediction of the result from the solution by numerical method of the realistic model without any simplifying assumptions.

9. Explain the relative merits and demerits of explicit and implicit schemes.

PROBLEMS

4.1 A cylindrical nickel-steel billet of 0.1m diameter and 0.5 m length, initially at 800°C, is suddenly dropped in a large vessel containing oil at 30°C. The convection heat transfer coefficient between the billet and the oil is 20 W/(m^2 K). Calculate the time required for the billet to reach a temperature of 250°C. Take for nickel-steel, k = 20 W/(m K), ρ = 8000 kg/m^3, c_p = 0.45kJ/(kg K).

[**Ans.** 1.57 h]

4.2 An aluminium sphere of 0.1 m diameter and at a uniform temperature of 500°C is suddenly exposed to an environment at 20°C, with convection heat transfer coefficient 30 W/(m^2 K). Calculate the temperature of the sphere (i) 100 s, (ii) 300 s, and (iii) 500 s after it is exposed to the environment. Justify any method you use for the analysis; take, for aluminium, k = 200 W/(m K), ρ = 2700 kg/m^3, c_p = 0.9 kJ/(kg K).

[**Ans.** (i) 466°C, (ii) 404°C, (iii) 350°C]

4.3 A chrome-nickel wire of 2 mm diameter, initially at 25°C, is suddenly exposed to hot gases at 725°C. If the convection heat transfer coefficient is 10 W/(m^2 K), calculate the time constant of the wire as a lumped-capacity system. Take k = 20 W/(m K), ρ = 7800 kg/m^3, c_p = 0.46 kJ/(kg K).

[**Ans.** 179 s]

4.4 Determine the time constant of a spherically shaped, copper-constantan thermocouple, exposed to a convective environment with an average heat transfer coefficient of 40 W/(m^2 K) for (i) bead diameter of 1.25 mm and (ii) bead diameter of 0.25 mm. The density and specific heat at constant pressure are almost same for both copper and constantan and are given as ρ = 8900 kg/m^3 and c_p = 0.4 kJ/(kg K); k_{cu} = 380 W/(m K), k_{con} = 20 W/(m K).

[**Ans.** (i) 18.54 s, (ii) 3.71 s]

4.5 A sphere made of copper is initially at a uniform temperature T_0 and is immersed in a fluid. Electric heaters are placed in the fluid and controlled so that the temperature of the fluid follows a periodic variation with time given by

$$T_\infty - T_m = M \sin \omega t$$

where

T_m is the time-average mean fluid temperature
M is the amplitude of temperature variation
ω is the frequency.

Derive an expression for the temperature of the sphere as a function of time. Assume that the temperatures of the sphere and fluid are uniform at any instant so that the lumped-capacity method may be used.

$$\left[\textbf{Ans.} \quad \left(T - T_0 = \frac{PM\omega}{P^2 + \omega^2}\left[1 + \frac{1}{\omega}(P \sin \omega t - \omega \cos \omega t)\right]\right) \text{ where } P = hA/\rho cV, \text{ and the nomenclature have their usual meanings as defined in the text.}\right]$$

4.6 Consider a 1 kW iron whose base plate is made of 30-mm thick aluminium alloy (ρ = 2800 kg/m^3, c_p = 0.87 kJ/(kg K), $\alpha = 7 \times 10^{-5}$ m^2/s). The base plate has a surface area of 0.05 m^2. Initially, the iron is in thermal equilibrium with the ambient air at 25°C. Taking the heat transfer coefficient at the surface of the base plate to be 10 W/(m^2 °C) and assuming that 80 per cent of the heat generated in the resistance wires is transferred to the plate, determine how long it will take for the plate temperature to reach 150°C? Is it realistic to assume the plate temperature to be uniform at all times?

[**Ans.** 9.9 min, yes]

4.7 An electronic device dissipating 50 W has a mass of 0.5 kg, a specific heat of 85 kJ/(kg K) , and a surface area of 600 mm^2. The device is on for 5 minutes and is then off for several hours, during which it cools to the ambient temperature of 25°C. Taking the heat transfer coefficient to be 10 W/(m^2 °C), determine the temperature of the device at the end of the 5-min operating period. (Use the lumped-capacity method.)

[**Ans.** 60.22°C]

4.8 A 2.0 kg aluminium household iron has a 1000 W heating element. The surface area is 0.06 m^2. The ambient temperature is 20°C, and the surface heat transfer coefficient is 20 W/(m^2 K). After the iron is switched on, how long will it take for its temperature to reach 110°C? (Justify the use of lumped-heat-capacity method ; take the thermophysical properties of aluminium as given in Problem 4.2.)

[**Ans.** 2.86 min]

4.9 A copper wire, 1.5 mm OD, 0.5 m long, is placed in an air stream whose temperature increases with time as given by $T_{air} = (20 + 8t)$°C, where t is the time in seconds. If the initial temperature of the wire is 20°C, determine its temperature after 2 s, 10 s, and 1 min. The heat transfer coefficient between the air and the wire is 40 W/(m^2 K). (Justify any assumption you make; take the properties of copper as given in Problem 4.4.)

[**Ans.** 35.37°C, 99.37°C, 499.37°C]

4.10 Spherical stainless steel vessel at 90°C contains 50 kg of water. The initial temperature of water is same as that of the vessel. If the entire system is suddenly immersed in ice water, determine (i) the time required for the water in the vessel to cool to 20°C, and (ii) the temperature of the walls of the vessel at that time. Assume that the heat transfer coefficient at the inner surface is 20 W/(m^2 K), the heat transfer coefficient at the outer surface is 30 W/(m^2 K), and the wall of the vessel is 30 mm thick. Assume that the lumped-capacity method is valid. Take ρ_{water} = 1000 kg/m^3, $\rho_{stainless\ steel}$ = 8000 kg/m^3, $c_{p_{water}}$ = 4000 J/(kg K).

[**Ans.** (i) 10.65 h, (ii) 8.1°C]

4.11 A large steel plate (1.0 per cent carbon) 50 mm thick is initially at a temperature of 425°C. It is suddenly exposed on both sides to a convective environment with an average heat transfer coefficient of 300 W/(m^2 °C), and a temperature of 60°C. Determine the centre-line temperature and the temperature inside the body 1.25 cm from the surface after 24 minutes. (Take, for the plate, k = 40 W/(m K), $\alpha = 1.12 \times 10^{-5}$ m^2/s.)

[**Ans.** 63.65°C, 63.43°C]

4.12 A large aluminium plate of 120 mm thickness is originally at 300°C. It is suddenly exposed to an environment at 20°C, with convection heat transfer coefficient of 600 W/(m^2 K). Calculate the temperature at a depth of 30 mm from one of the faces, 5 minutes after the plate is exposed to the environment. How much energy is transferred per unit area of the plate during this time? (Take the properties of aluminium as given in Problem 4.2.)

[**Ans.** 107°C, 37.56 MJ]

4.13 A large plate of aluminium of thickness 400 mm is originally at 500°C. It is suddenly exposed to a convection environment at 20°C, with convection heat transfer coefficient of 500 W/m^2 K. Calculate:

(i) the time required for the centre of the plate to reach a temperature of 200°C.
(ii) the surface temperature of the plate 8 minutes after it is exposed to the environment.
(iii) the temperature at a depth of 100 mm from one of the surfaces, 8 minutes after the plate is exposed to the environment. (Take the properties of aluminium as given in Problem 4.2.)

[**Ans.** (i) 20.25 min (ii) 356°C (iii) 339°C]

4.14 A long, 70 mm diameter solid cylinder of mild steel is initially at a uniform temperature T_i = 150°C. It is suddenly exposed to a convective environment at T_∞ = 50°C, and the surface convective heat transfer coefficient is $\bar{h}$ = 300 W/(m^2 K). Calculate the temperature (i) at the axis of the cylinder and (ii) at a 30 mm radial distance, after 5 minutes of exposure to the cooling environment. (iii) Determine the total energy transferred from the cylinder per metre of length during the first 5 minutes of cooling. Take k = 54 W/(m K), ρ = 7800 kg/m^3, and c_p = 460 J/(kg K).

[**Ans.** (i) 78°C, (ii) 73.24°C, (iii) 1 MJ]

4.15 A long 50-mm diameter cylindrical shaft made of stainless steel (k = 15 W/(m °C), ρ = 7900 kg/m^3, c_p = 470 J/(kg °C) comes out of an oven at a uniform temperature of 400°C. The shaft is then allowed to cool slowly in a chamber at 100°C with an average convection heat transfer coefficient of h = 60W/(m^2 °C). Determine the temperature at the centre of the shaft, 20 min after the start of the cooling process. Also, determine the heat transfer per metre length of the shaft during this time period.

[**Ans.** 172°C, 1.75 MJ]

4.16 A long cylindrical wood log (k = 0.17 W/(m K), ρ = 550 kg/m^3 c_p = 0.24 kJ/(kg K)) is 100 mm in diameter and is initially at a uniform temperature of 10°C. It is exposed to hot gases at 700°C in a fireplace with a heat transfer coefficient of 40 W/(m^2 °C). If the ignition temperature of the wood is 420°C, how long will it be before the log ignites?

[**Ans.** 6.5 min]

4.17 An 80-mm diameter orange originally at T_1 = 25°C is placed in a refrigerator where the air temperature T_∞ is 2°C and the average convective heat transfer coefficient over the surface of the orange is $\bar{h}$ = 60 W/(m^2 K). Estimate the time required for the centre temperature T_c of the orange to reach 4°C. (Use the properties of orange as k = 0.58 W/(m K), $\alpha = 1.4 \times 10^{-7}$ m^2/s.)

[**Ans.** 1.90 h]

4.18 For the purpose of heat transfer, an egg can be considered to be a 60-mm diameter sphere having the same properties as those given in Problem 4.17. An egg which is initially at 10°C is dropped into boiling water at 100°C. The heat transfer coefficient at the surface of the egg is estimated to be 500W/(m^2 °C). If the egg is considered cooked when its centre temperature reaches 60°C, how long the egg should be kept in the boiling water?

[**Ans.** 32 min]

4.19 A hailstone that is formed in a high-altitude cloud can be considered to be spherical in shape. Such a hailstone of 6-mm diameter and at –30°C begins to fall through warmer air at 5°C. How long will it take before the outer surface begins to melt? What is the temperature of the stone's centre at this point in time, and how much energy has been transferred to the stone? Assume a convection heat transfer coefficient of 300 W/(m^2 K), and take the properties of the hailstone as k = 2.0 W/(m K), ρ = 900 kg/m^3, c = 900 J/(kg K).

[**Ans.** 5.5 s, −1.3°C, 476 J]

4.20 A sphere 50 mm in diameter is initially at 1000 K. It is then quenched in a large bath having a constant temperature of 300 K with a convection heat transfer coefficient of 70 W/(m^2 K). The thermo-physical properties of the sphere material are: k = 1.7 W/(m K), ρ = 400 kg/m^3, c = 1600 J/(kg K).

(i) Calculate the time required for the surface of the sphere to reach 415 K.

(ii) Determine the heat flux (W/m^2) at the outer surface of the sphere at the time determined in part (i).

(iii) Determine the energy that has been lost by the sphere during the process of cooling to the surface temperature of 400 K.

[**Ans.** (i) 2.35 min, (ii) 8.05 kW/m^2, (iii) 24.63 kJ]

4.21 In a building, a long wooden beam of cross-section 0.4 m × 0.3 m is initially at a temperature of 30°C. Due to an accidental fire, the beam is suddenly exposed to hot gases at 800°C, with h = 50 W/(m^2 K). Estimate the time required for the wood to reach the ignition temperature of 450°C. (Take k = 70 W/(m K), $\alpha = 2 \times 10^{-5}$ m^2/s.)

[**Ans.** 1.28 h]

4.22 A cube of aluminium of 300 mm side length is initially at a temperature of 450°C. It is suddenly immersed in an oil at 30°C. The convection heat transfer coefficient is 600 W/(m^2 K). Calculate the temperature at the centre of the cube five minutes after it is immersed in the oil. For aluminium, k = 200 W/(m K), ρ = 2700 kg/m^3, and c = 900 J/(kg K).

[**Ans.** 174°C]

4.23 It is desired to anneal a glass cylinder (k = 0.8 W/(m K), $\alpha = 3.5 \times 10^{-7}$ m^2/s) of diameter 300 mm and length 600 mm to remove stresses before it is subjected to further processing. The glass cylinder is initially at 25°C. How long should the glass cylinder be kept in an oven so that every part of the glass attains at least a temperature of 400°C? The air in the oven is at 400°C, and h = 10 W/(m^2 K).

[**Ans.** 17.22 h]

4.24 A large slab of concrete (k = 1.37 W/(m K), ρ = 2000 kg/m^3 c = 0.890 kJ/(kg K) is initially at a temperature of 30°C. One surface of the slab is suddenly exposed to hot gases at 550°C, with convection heat transfer coefficient of 50 W/(m^2 K). Calculate the temperature at a depth of 100 mm in the slab after a time of 2 hours.

[**Ans.** 209°C]

4.25 A furnace wall is fabricated from fireclay bricks (k = 0.9 W/(m K), ρ = 1920 kg/m^3, c = 0.79 kJ/(kg K)). Its inner surface is maintained at 1200 K during furnace operation. The wall is designed according to the criterion that, for an initial temperature of 300 K, its midpoint temperature will not exceed 330 K after 4 hours of furnace operation. What is the minimum allowable wall thickness?

[**Ans.** 138 mm]

4.26 An 80-mm thick concrete slab having very large dimensions in the plane normal to the thickness is initially at a uniform temperature of 20°C. Both surfaces of the slab are suddenly raised to and held at 100°C. The material properties are k = 1·40 W/(m K) and $\alpha = 0.0694 \times 10^{-5}$ m^2/s. Using a nodal spacing of 10 mm, determine with the help of explicit method of numerical analysis, the temperature history in the slab during a period of 15 minutes.

4.27 Consider a large uranium plate of thickness L = 80 mm, thermal conductivity k = 30 W/(m °C), thermal diffusivity $\alpha = 12.5 \times 10^{-6}$ m^2/s and being initially at a uniform temperature of 100°C. Thermal energy is generated uniformly in the plate at a constant rate of 107 W/m^3. The left side of the plate is insulated while the other side is subjected to convection in an environment at T_∞ = 20°C with a heat transfer coefficient of h = 40 W/(m^2 °C). Using the explicit finite difference approach with a uniform nodal spacing of Δx = 20 mm, determine (i) the temperature distribution in the plate after 5 min and (ii) the time it will take for steady conditions to be reached in the plate.

4.28 Consider a two-dimensional transient heat transfer in an L-shaped long solid bar that is initially at a uniform temperature of 150°C and whose cross-section is given in Figure 4.39.

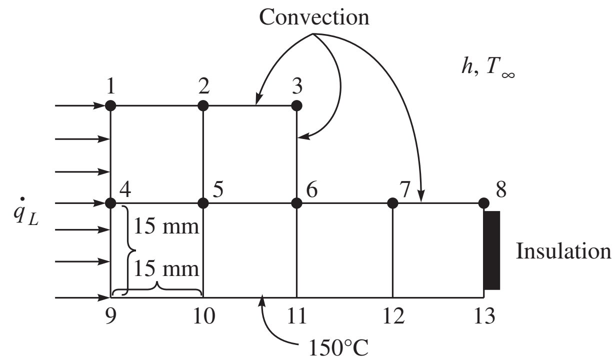


Figure 4.39 L-shaped solid bar (Problem 4.28).

The thermal conductivity and diffusivity of the body are k = 20 W/(m °C) and α = 3.2×10^{-6} m^2/s, respectively, and thermal energy is generated in the body at a rate of 2×10^7 W/m^3. The right surface of the body is insulated, and the bottom surface is maintained at a uniform temperature of 150°C at all times. At time t = 0, the entire top surface is subject to convection in ambient air at T_∞ = 25°C with a heat transfer coefficient of h = 80 W/(m^2 °C), and the left surface is subjected to uniform heat flux at a rate of $\dot{q}_L$ = 8000 W/m^2. The nodal network of the problem consists of 13 equally spaced nodes with $\Delta x = \Delta y$ = 15 mm as shown in the above figure. Using the explicit method, determine the temperature at all nodal points within the body after 2, 5, and 30 min.

4.29 Consider a long solid bar (k = 28 W/(m °C) and $\alpha = 12 \times 10^{-6}$ m^2/s) of square cross-section that is initially at a uniform temperature of 20°C. The cross-section of the bar is 200 mm × 200 mm in size, and thermal energy is generated in it uniformly at a rate of 2×10^{-5} W/m^3. All four sides of the bar are subject to convection to the ambient air at T_∞ = 20°C with a heat transfer coefficient of h = 50 W/(m^2 °C). Using the explicit finite difference method with a mesh size of $\Delta x = \Delta y$ = 10 mm, determine the centre-line temperature of the bar (i) after 10 min and (ii) after steady conditions are established.

4.30 A wall 0.12 m thick having a thermal diffusivity of 1.5×10^{-6} m^2/s is initially at a uniform temperature of 100°C. Suddenly one face is lowered to a temperature of 20°C, while the other face is perfectly insulated. Using the explicit finite difference technique with space and time increments of 30 mm and 300 s respectively, determine the temperature distribution at t = 40 min.

REFERENCES

[1] Heisler, M.P., ‘Temperature Charts for Induction and Constant Temperature Heating’, *Trans. ASME*, Vol. 69, pp. 227–236, 1947.

[2] Anderson, John D., Jr., *Computational Fluid Dynamics*, McGraw-Hill, 1995.

5

Convection

In this chapter we will discuss the basic mechanism and fundamental concepts of convective heat transfer.

Learning objectives

The reading of this chapter will enable the students

- to understand the mechanism of convective heat transfer,
- to understand the role of conduction in convective heat transfer,
- to understand the primary difference between natural and forced convection,
- to know the definition of heat transfer coefficient and its importance,
- to know the definition of Nusselt number and its physical significance,
- to identify the pertinent dimensionless numbers governing the phenomenon of convective heat transfer and understand their physical significances, and
- to solve problems through the application of the given working relations of heat transfer coefficient with the pertinent variables.

5.1 MECHANISM OF CONVECTIVE HEAT TRANSFER

We have already introduced the concept of convective heat transfer in Chapter 1 (Section 1.1.2). For the sake of clarity and continuity of the topic, we shall first repeat a part of Section 1.1.2 here.

Convection is the mode by which heat is transferred between a solid surface and the adjacent fluid in motion when there is a temperature difference between the two. The temperature of the fluid stream refers either to its bulk temperature (in case of internal flows) or to its free stream temperature (in case of external flows). Let us consider the flow of a fluid over a stationary heated surface as shown in Figure 5.1 (repeat of Figure 1.3). If the temperature T_∞ with which the fluid approaches the surface is less than the temperature T_s of the surface, heat is transferred from the surface to the flowing fluid, and as a result the surface is cooled. It is our common experience that an increase in fluid motion cools the surface faster which means that the rate of heat transfer from the surface increases with an increase in the fluid motion. This implies a sense that the heat is being convected away from the surface by the stream of fluid and hence this mode of heat transfer is known as convective heat transfer.

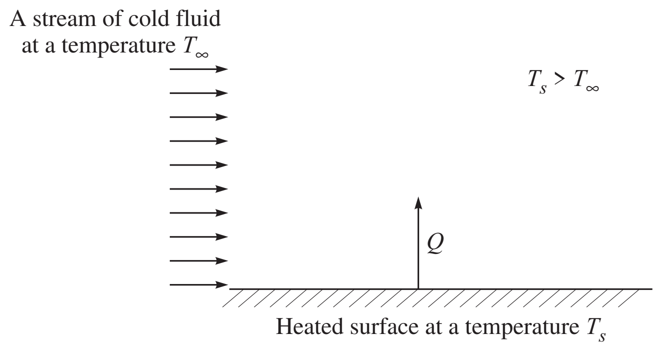


Figure 5.1 Convective heat transfer from a solid surface to a moving fluid

The mode of convective heat transfer, in fact, comprises two mechanisms: conduction at the solid surface and advection by the bulk or macroscopic motion of the fluid a little away from the solid surface. This can be explained through the velocity and temperature distributions in the fluid flowing past the solid surface. A consequence of the fluid–surface interaction in case of momentum transfer due to flow is the development of a region in the fluid flow near the surface within which the flow velocity varies from zero at the surface to a finite value U_∞ at the outer flow (Figure 5.2). The velocity U_∞ is termed *free stream velocity* (the uniform velocity with which the fluid approaches the surface). This region of fluid flow is known as the hydrodynamic boundary layer. In a similar fashion, there is a region of fluid flow where the temperature varies from T_s at the surface to T_∞ in the outer flow (Figure 5.2). This region is known as thermal boundary layer. The temperature T_∞ is known as *free stream temperature* and equals uniform temperature with which the fluid approaches the surface.

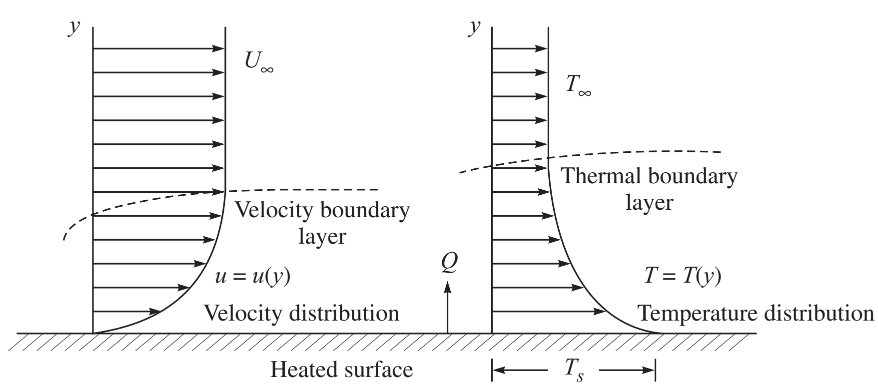


Figure 5.2 Velocity and temperature distributions in boundary layers in convective heat transfer mode.

At the interface between the surface and the fluid (y = 0, Figure 5.2), the fluid velocity is zero. This is known as “no-slip condition” of fluid flow which states that for the flow of a real fluid, obeying continuum, the velocity of fluid relative to the surface is zero at the surface. Therefore, heat is transferred from the surface to the fluid by conduction, i.e. by the transfer of kinetic energy between the molecules due to their random motions. The contribution of advection due to bulk fluid motion arises with the growth of boundary layer in the direction of flow. The heat that is conducted into this layer is swept downstream and is eventually transferred to the fluid outside the boundary layer. Therefore a knowledge of boundary layer is essential to understand the mechanism of convective heat transfer clearly.

The convection is of two types: *forced* convection and *free* convection. In forced convection, the fluid is forced to flow over a solid surface by external means such as fan, pump, or atmospheric wind. When the fluid motion is caused by buoyancy forces that are induced by density differences due to the variation of temperature or species concentration (in case of multicomponent systems) in the fluid, the convection is called natural (or free) convection. Figure 5.3 shows an example of forced convection from a horizontal surface where the fluid motion past the surface is caused by a fan, while Figure 5.4 shows a situation of free convection from a heated vertical surface in air in the absence of any forced flow.

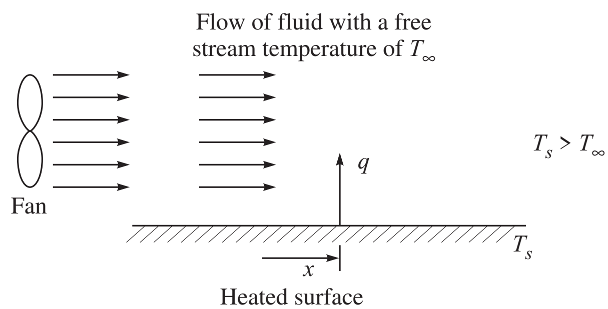


Figure 5.3 Forced convective heat transfer from a horizontal surface.

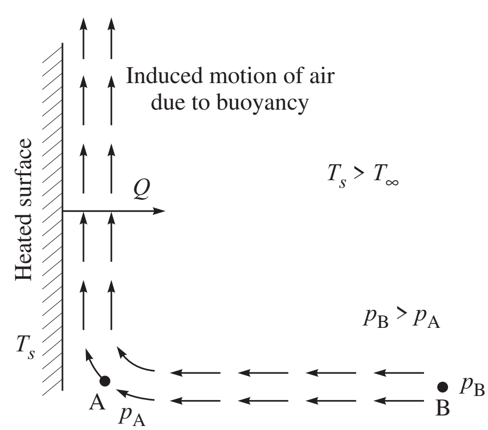


Figure 5.4 Free convective heat transfer from a heated vertical surface

In case of free convection, buoyancy induced flow takes place near the surface. The flow of air, under the present situation, can be well explained by comparing the hydrostatic pressure at a point A adjacent to the surface with that at a point B on the same horizontal plane but in the region of cold air away from the surface. The density of air near the surface is lower because of high temperature compared to that of cold air away from the plate. Therefore p_B, the pressure at B, is higher than p_A, the pressure at A. This difference in pressure ($p_B - p_A$) induces the flow past the surface as shown in Figure 5.4.

The rate of heat transfer by convection (both forced and free) between a solid surface and a fluid is usually calculated from the relation

$$Q = \bar{h} A \Delta T \tag{5.1}$$

where

Q is the rate of heat transfer by convection
A is the heat transfer area
$\Delta T = (T_s - T_f)$, the difference between the surface temperature T_s and the temperature of the fluid T_f at some reference location. In case of external flows, T_f is the free stream temperature T_∞ and in case of internal flows, it is the bulk temperature of the fluid.
$\bar{h}$ is the average convective heat transfer coefficient over the area A.

The relation expressed by Eq. (5.1) was originally proposed by Sir Isaac Newton and is known as Newton's law of cooling. In fact, Eq. (5.1) defines the average convective heat transfer coefficient over an area A. Therefore, it can be written as

$$\bar{h} = \frac{Q}{A \Delta T} \tag{5.2}$$

Thus the average convective heat transfer coefficient over an area at a constant temperature is defined to be the rate of heat transfer per unit area per unit temperature difference (the difference between the surface temperature and the reference temperature of the fluid). In SI units, Q is expressed in W, A in m^2 and ΔT in K or °C. Therefore, $\bar{h}$ is expressed in W/(m^2 K) or W/(m^2 °C). Since the mode of heat transfer at the surface is conduction, as explained earlier, the convective heat transfer coefficient $\bar{h}$ is expected to be related to thermal conductivity of the fluid and the temperature gradient at the surface.

The local heat transfer coefficient h is defined as

$$h = \frac{Q/A}{T_s - T_f} = \frac{q}{T_s - T_f} \tag{5.3}$$

The temperature at the solid surface T_s and the reference temperature of fluid stream T_f are the prescribed parameters. In the case of external flows $T_f = T_\infty$ (the free stream temperature), while in the case of internal flows, $T_f = T_b$ (the bulk mean temperature at a cross-section). The concept of bulk mean temperature will be discussed in Chapter 7. The temperature T_s may not be the same at different locations along the solid surface. Let us consider the case of heat transfer from a flat plate with a constant surface temperature T_s to a fluid stream having a free stream temperature of T_∞ ($T_\infty < T_s$). The local heat flux q at a location x (Figure 5.3) can be expressed in terms of local heat transfer coefficient h_x as

$$q = h_x (T_s - T_\infty) \tag{5.4}$$

Since the mode of heat transfer at the solid surface is conduction, we can write

$$q = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{5.5}$$

From Eqs. (5.4) and (5.5), we get

$$h_x = \frac{-k\left(\dfrac{\partial T}{\partial y}\right)_{y=0}}{T_s - T_\infty} \tag{5.6}$$

Therefore we see that the local heat transfer coefficient is directly related to the temperature gradient at the solid surface which is a local parameter even if T_s remains constant over the entire surface. The total heat transfer rate can be found out as

$$Q = \iint_A -k\left(\frac{\partial T}{\partial y}\right)_{y=0} dA = \iint_A h_x (T_s - T_\infty)\, dA \tag{5.7}$$

Again Q can be expressed in terms of the average heat transfer coefficient $\bar{h}$ defined by Eq. (5.1) as

$$Q = \bar{h} A(T_s - T_\infty) \tag{5.8}$$

From Eqs. (5.7) and (5.8), we have

$$\bar{h} = \frac{1}{A}\iint_A h_x\, dA \qquad \text{(since } T_s \text{ is constant over the entire surface)} \tag{5.9}$$

For a flat plate of length L, Eq. (5.9) can be written as

$$\bar{h}_L = \frac{1}{L}\int_0^L h_x\, dx \tag{5.10}$$

It is evident that the rate of heat transfer from a solid surface is governed by conduction and is controlled by the value of the temperature gradient in the direction of heat flow at the solid surface. However, the flow field has an effect on the temperature gradient since the heat conducted from the surface is being advected out by the flow of fluid a little away from the surface. The higher the flow velocity, the higher is the advection of heat and the lower is the temperature of the fluid near the wall and hence the steeper is the temperature gradient. Thus the phenomenon of convective heat transfer can be looked upon as conduction at the solid surface followed by partly conduction and partly advection within the thermal boundary layer and then by purely advection beyond the boundary layer.

The average or local convective heat transfer coefficient depends on flow velocity, flow geometry and physical properties such as thermal conductivity, specific heat, density, and viscosity of the fluid. Let us express this statement mathematically as

$$\bar{h} = f(V_{\text{ref}}, L_{\text{ref}}, k, c_p, \rho, \mu)$$

or

$$F(\bar{h}, V_{\text{ref}}, L_{\text{ref}}, k, c_p, \rho, \mu) = 0 \tag{5.11}$$

where

V_{ref} is a reference velocity
L_{ref} is a reference or characteristic geometrical dimension
k is the thermal conductivity of the fluid
c_p is the specific heat at constant pressure of the fluid
ρ is the density of the fluid
μ is the viscosity of the fluid.

The sole objective of the study of convective heat transfer is to determine the explicit functional relationship of h with the pertinent independent parameters as stated in Eq. (5.11) in different situations.

5.2 DIMENSIONLESS EXPRESSION OF HEAT TRANSFER COEFFICIENT

Equation (5.11) is a functional representation of a convective heat transfer problem in terms of the physical parameters involved. To reduce the number of dimensional variables, Eq. (5.11) can be expressed in terms of a lesser number of independent dimensionless variables known as π terms. This is done by the use of any standard method of dimensional analysis.

The number of variables in Eq. (5.11) is seven and the number of fundamental dimensions in which they can be expressed equals four. With the help of Buckingham's π-theorem and in consideration of ρ, μ, k and L_{ref} as the repeating variables, we have

$$\pi_1 = \frac{\bar{h}\, L_{\text{ref}}}{k}$$

$$\pi_2 = \frac{\rho V_{\text{ref}}\, L_{\text{ref}}}{\mu}$$

$$\pi_3 = \frac{\mu\, c_p}{k}$$

The first π term is defined as the average Nusselt number $\overline{\text{Nu}}$, the second π term is the well known Reynolds number Re, and the third π term is defined as the Prandtl number Pr.

Thus,
$$\overline{\text{Nu}} = \frac{\bar{h}\, L_{\text{ref}}}{k} \tag{5.12a}$$

$$\text{Re} = \frac{\rho V_{\text{ref}}\, L_{\text{ref}}}{\mu} \tag{5.12b}$$

$$\text{Pr} = \frac{\mu c_p}{k} \tag{5.12c}$$

Therefore, we can write

$$\overline{\text{Nu}} = \varphi(\text{Re, Pr}) \tag{5.13}$$

The explicit form of the functional relation can be found either by experiments or by a suitable theoretical analysis. The relationship depends upon the situation whether it is heat transfer from a flat surface, heat transfer from an aerofoil section, or heat transfer in internal flows etc. It also depends on whether the temperature at the solid surface is constant or varies in a certain fashion.

In case of external flows,

$V_{\text{ref}} = U_\infty$ (the free stream velocity)
$L_{\text{ref}} = L$ (the length of the surface in the direction of flow).

For internal flows,

$V_{\text{ref}} = V_{\text{av}}$ (average flow velocity defined to be the volumetric flow rate divided by the cross-sectional area)
$L_{\text{ref}} = D_h$ (the hydraulic diameter).

In case of free convection, there is no forced flow and hence there exists no reference velocity. The flow is caused by the difference in density of the fluid due to a change in temperature. This flow is termed buoyancy induced flow. The force of buoyancy F_B per unit volume causing the flow can be written as

$$F_B \sim g\Delta\rho \sim \rho g\beta\Delta T \tag{5.14}$$

where β is the volume expansivity and is defined as

$$\beta = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial t}\right)_p \tag{5.15}$$

It is customary in the field of heat transfer to replace $\Delta\rho$ in terms of β in expressing the buoyancy force F_B as done above. However in certain cases of combined heat and mass transfer, the density difference arises not only due to temperature gradient but also due to concentration gradient of species. In such cases F_B cannot be expressed in terms of β alone as done in Eq. (5.14); instead it is evaluated directly from the equation of state of the fluid.

Since the buoyancy force F_B is a pertinent parameter in determining the rate of heat transfer in a situation of free convection, we have to use '$\rho g\beta\Delta T$' as a single variable in expressing the functional relationship that describes a free convection problem. In fact, if we replace V_{ref} by '$\rho g\beta\Delta T$' in Eq. (5.11), we get the required functional form as

$$F(\bar{h}, \rho g\beta\Delta T, L_{\text{ref}}, k, c_p, \rho, \mu) = 0 \tag{5.16}$$

Following Buckingham's π-theorem with ρ, μ, k and L_{ref} as the repeating variables, we have

$$\pi_1 = \frac{\bar{h}\,L_{\text{ref}}}{k}$$

$$\pi_2 = \frac{g\beta\Delta T L_{\text{ref}}^3}{\nu^2}$$

$$\pi_3 = \frac{\mu c_p}{k}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid.

The second π term is known as Grashoff number Gr, and hence

$$\text{Gr} = \frac{g\beta \Delta T L_{\text{ref}}^3}{\nu^2} \tag{5.17}$$

Therefore, we can write for a free convection problem

$$\overline{\text{Nu}} = \varphi(\text{Gr, Pr}) \tag{5.18}$$

Physical implication of dimensionless numbers

Reynolds number: We know from the knowledge of fluid mechanics that Reynolds number Re is a ratio of representative magnitude of inertia force to that of viscous force. It is the characteristic parameter used as a criterion for dynamic similarity between flows governed by pressure force, viscous force, and inertia force.

Prandtl number: The Prandtl number Pr is given by $\mu c_p/k$. It can also be written in a form

$$\text{Pr} = \frac{\mu c_p}{k} = \frac{\nu}{\alpha}$$

We find that the Prandtl number is the ratio of kinematic diffusivity to thermal diffusivity. The kinematic diffusivity conveys information about the rate at which momentum diffuses through the fluid due to molecular motion. The thermal diffusivity, on the other hand, conveys information in regard to the diffusion of heat in the fluid. The extent of diffusion determines the thickness of boundary layer. Large values of ν and α mean that the viscous effect or temperature influence is felt farther out in the flow field, resulting in an increase in hydrodynamic boundary layer thickness and thermal boundary layer thickness. Therefore, the Prandtl number can be considered to be the representative magnitude of the ratio of thickness of hydrodynamic boundary layer to that of thermal boundary layer. Thus, the Prandtl number links the velocity field with the temperature field.

Nusselt number: The Nusselt number represents the dimensionless heat transfer coefficient. Let us write again the expression for the local heat transfer coefficient from Eq. (5.6).

$$h_x = \frac{-k\left(\dfrac{\partial T}{\partial y}\right)_{y=0}}{T_s - T_\infty}$$

Let L_{ref} be the characteristic geometrical dimension defining the local Nusselt number Nu_x, then we have

$$\text{Nu}_x = \frac{h_x L_{\text{ref}}}{k} = \frac{-\left(\dfrac{\partial T}{\partial y}\right)_{y=0} L_{\text{ref}}}{T_s - T_\infty}$$

If we define a dimensionless temperature θ as

$$\theta = \frac{T_s - T}{T_s - T_\infty} \tag{5.19a}$$

and the dimensionless y coordinate as

$$y' = \frac{y}{L_{\text{ref}}} \tag{5.19b}$$

we can write

$$\left(\frac{\partial \theta}{\partial y'}\right)_{y'=0} = \frac{-\left(\frac{\partial T}{\partial y}\right)_{y=0} L_{\text{ref}}}{T_s - T_\infty}$$

It finally becomes

$$\text{Nu}_x = \left(\frac{\partial \theta}{\partial y'}\right)_{y'=0} \tag{5.20}$$

Equation (5.20) implies that the local Nusselt number is equal to the dimensionless temperature gradient at that point on the surface provided the temperature is normalized by the reference temperature difference as shown in Eq. (5.19a), and the space coordinate defining the temperature gradient is normalized by the characteristic geometrical dimension used in defining the Nusselt number.

The physical significance of Grashoff number will be discussed in Chapter 8. However, it can be mentioned that the Grashoff number Gr is directly related to buoyancy force F_B. The higher is the value of F_B, the higher is the value of Gr.

The main objective in the analysis of convective heat transfer problem is to find out the explicit form of the functional relation of Nusselt number with the pertinent controlling parameters as expressed by Eqs. (5.13) and (5.18).

The Nusselt number always increases with an increase in the Reynolds number Re in case of forced convection and with an increase in the Grashoff number Gr in case of free convection. An increase in either Re or Gr for a fluid with given properties is associated with an increase in flow velocity past the solid surface. This enhances the advection of heat by fluid stream and reduces the fluid temperature near the surface. As a result, the thermal boundary layer thickness is reduced causing a steeper temperature gradient $\partial T/\partial y$ at the surface to justify for an enhanced rate of heat conduction there. It has to be kept in mind that the rate at which heat is being advected by the stream of fluid has to be conducted away by the fluid film adhering to the surface. Therefore, any effect in the advection of heat by a change in flow field is accompanied by a corresponding change in the conduction of heat by an adjustment of temperature gradient $\partial T/\partial y$ at the surface.

We find from the above discussion that a knowledge of velocity field along with the growth of hydrodynamic boundary layer and their interactions with temperature field and thermal boundary layer is essential to understand the theory of convective heat transfer and to develop the functional relationships expressed by Eqs. (5.13) and (5.18). The next two chapters discuss this aspect.

EXAMPLE 5.1 The local heat transfer coefficient for flow over a flat plate at constant temperature is found to be of the form

$$h_x = Ax^{-0.2}$$

where A is a coefficient (W/(m$^{1.8}$ K)) and x(m) is the distance from the leading edge of the plate. Show that $\bar{h}_L = 1.25h_L$ where $\bar{h}_L$ and h_L are the average heat transfer coefficient over a length L and the local heat transfer coefficient at $x = L$ respectively.

Solution: We have
$$h_x = Ax^{-0.2}$$
Therefore,
$$h_L = AL^{-0.2}$$
Now,
$$\begin{aligned}\bar{h}_L &= \frac{1}{L}\int_0^L h_x\,dx \\ &= \frac{1}{L}\int_0^L Ax^{0.2}\,dx \\ &= \frac{A}{0.8L}L^{0.8} \\ &= \frac{A}{0.8}L^{-0.2} \\ &= 1.25AL^{-0.2} = 1.25h_L\end{aligned}$$

EXAMPLE 5.2 Consider that a flat plate of length L is at a constant temperature T_s. Air with a free stream temperature of T_∞ ($T_\infty < T_s$) flows over the plate. The temperature distribution in air is given by

$$\frac{T - T_s}{T_\infty - T_s} = \frac{3}{2}\left(\frac{y}{A\sqrt{x}}\right) - \frac{1}{2}\left(\frac{y}{A\sqrt{x}}\right)^3$$

where x and y are the coordinate axes along and perpendicular to the plate. The coordinate x is measured from the leading edge of the plate, while y is measured from the plate surface. Here A is a constant with a dimension of m$^{1/2}$ and depends upon the properties of air and the nature of velocity distribution in the hydrodynamic boundary layer. Show that

$$\text{Nu}_x = \frac{3}{2}\frac{\sqrt{x}}{A}$$

$$\overline{\text{Nu}}_L = 2\text{Nu}_L$$

Solution: From Eq. (5.6), we have

$$h_x = \frac{-k\left(\dfrac{\partial T}{\partial y}\right)_{y=0}}{T_s - T_\infty}$$

Here,
$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = (T_\infty - T_s)\frac{3}{2A\sqrt{x}}$$

Therefore, $$h_x = \frac{3k}{2A\sqrt{x}}$$

$$\mathrm{Nu}_x = \frac{h_x\, x}{k} = \frac{3}{2}\frac{\sqrt{x}}{A}$$

$$\bar{h}_L = \frac{1}{L}\int_0^L h_x\, dx$$

$$= \frac{1}{L}\int_0^L \frac{3k}{2A\sqrt{x}}\, dx$$

$$= 2\left(\frac{3}{2}\frac{k}{A\sqrt{L}}\right) = 2h_L$$

Hence, $$\overline{\mathrm{Nu}}_L = \frac{\bar{h}_L\, L}{k} = \frac{2h_L\, L}{k} = 2\mathrm{Nu}_L$$

EXAMPLE 5.3 Air at 20°C and 1 atmospheric pressure flows over a flat plate with a free stream velocity of 1 m/s. The length of the plate is 1 m and it is heated over its entire length to a constant temperature of 100°C. Determine the rate of heat transfer from the plate per unit width to air. The following data are given.

For air at 60°C (the mean temperature of 100°C and 20°C)

$\mu = 1.9 \times 10^{-5}$ kg/(m s); $\rho = 1.05$ kg/m^3; $k = 0.03$ W/(m K); $c_p = 1.007$ kJ/(kg K); Pr = 0.7

For laminar flow over a flat plate, use the relation

$$\mathrm{Nu}_x = 0.332\ \mathrm{Re}_x^{1/2}\ \mathrm{Pr}^{1/3}$$

(The boundary layer flow over a flat plate will be laminar if Re_x ($= \rho U_\infty x/\mu$) $< 5 \times 10^5$)

Solution: First of all, we have to check whether the flow is laminar or not. Let us check at x = 1m (the length of the plate).

$$\mathrm{Re}_L = \frac{1.05 \times 1 \times 1}{1.9 \times 10^{-5}} = 0.5526 \times 10^5$$

Therefore the flow is entirely laminar and we can use the relationship of Nu_x with Re_x and Pr as given in the problem. Thus,

$$\mathrm{Re}_x = \frac{1.05 \times 1 \times x}{1.9 \times 10^{-5}} = 0.5526 \times 10^5 \times x$$

Therefore, we can write

$$\mathrm{Nu}_x = \frac{h_x\, x}{k} = (0.332)\,(0.5526 \times 10^5 \times x)^{1/2}\,(0.7)^{1/3}$$

or $$h_x = 0.332 \times 0.03 \times (0.5526 \times 10^5)^{1/2}\ (0.7)^{1/3}\ x^{-1/2}$$
$$= 2.08x^{-1/2}\ \text{W/(m}^2\ {}^\circ\text{C)}$$
$$\bar{h}_L = \frac{1}{L}\int_0^L h_x\, dx$$
$$= \int_0^1 2.08x^{-1/2}\, dx$$
$$= 4.16\ \text{W/m}^2$$

Hence, $$Q = 4.16 \times 1 \times (100 - 20)$$
$$= 332.8\ \text{W/m of width}$$

EXAMPLE 5.4 Air at atmospheric pressure is required to flow over a circuit board to cool the electronic elements mounted on it. Let us consider a chip of length 3 mm and width 3 mm located 0·1 m from the leading edge. The board surface is irregular and the Nusselt number correlation is given by $\text{Nu}_x. = 0.06\ \text{Re}_x^{0.85}\ \text{Pr}^{0.33}$. The chip has to dissipate 50 mW of energy while its surface temperature has to be kept below 45°C. What is the minimum flow velocity of air required for the purpose? Assume the free stream temperature of air to be 25°C. For air, $\rho = 1.2$ kg/m^3, $\mu = 1.8 \times 10^{-5}$ kg/(m s), $k = 0.03$ W/(m K), and $c_p = 1000$ J/(kg K).

Solution: Let the minimum flow velocity be U for which the temperature of the chip will be 45°C.

The local heat transfer coefficient h_x where the chip is mounted is determined as

$$h_x = \left(\frac{0.03}{0.1}\right) \times 0.06 \times \left(\frac{1.2 \times U \times 0.1}{1.8 \times 10^{-5}}\right)^{0.85} \left(\frac{1.8 \times 10^{-5} \times 1000}{0.03}\right)^{0.33}$$
$$= 27.063U^{0.85}$$

From an energy balance of the chip, we can write

$$27.063U^{0.85} \times 3 \times 3 \times 10^{-6}(45 - 25) = 50 \times 10^{-3}$$

which gives $U = 15{\cdot}48$ m/s.

EXAMPLE 5.5 A fluid flows through a pipe of diameter D with an average flow velocity of U_0. The tube is heated from an external source in such a way that the inner surface of the tube is kept at a constant temperature T_w over the entire length L of the tube. The bulk mean temperatures of the fluid at inlet to and outlet from the tube are T_{b_1} and T_{b_2} respectively. Show that the average Nusselt number $\overline{\text{Nu}}_L$ over the length of the tube L is given by

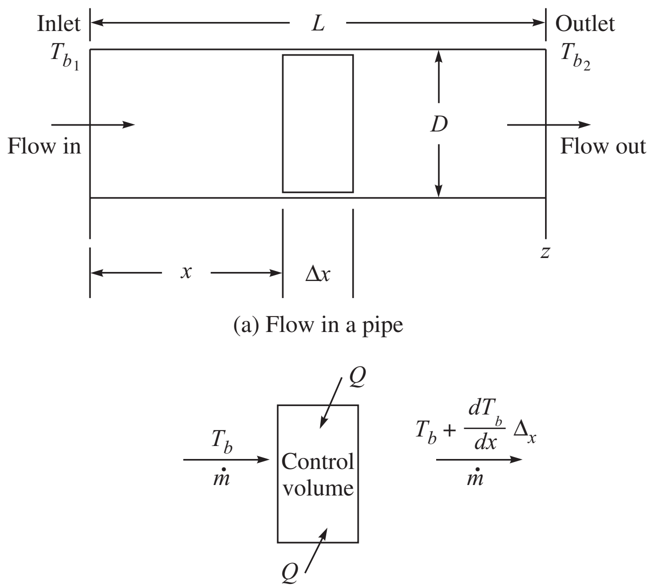
$$\overline{\text{Nu}}_L = \frac{1}{4}\frac{\text{Re Pr}}{(L/D)} \ln \frac{T_w - T_{b_1}}{T_w - T_{b_2}}$$

where the Reynolds number Re is defined as $\text{Re} = \rho U_0 D/\mu$ (consider the properties of fluid to be constant). Neglect axial conduction.

Solution: Let us consider a control volume of length Δx at a distance x from the inlet of the pipe as shown in Figure 5.5. The control volume is shown separately in Figure 5.5(b). From an energy balance of the control volume, we can write

$$h_x \pi D \Delta x (T_w - T_b) = \dot{m} c_p \left[\left(T_b + \frac{dT_b}{dx} \Delta x \right) - T_b \right]$$

where h_x is the local heat transfer coefficient at a distance x from the inlet.



(a) Flow in a pipe

(b) Control volume

Figure 5.5 Convective heat transfer in flow through a pipe (Example 5.5).

Mass flow rate $\dot{m}$ can be written as

$$\dot{m} = \rho \frac{\pi D^2}{4} U_0$$

Therefore, the energy balance relation becomes

$$h_x \pi D \Delta x (T_w - T_b) = \rho \frac{\pi D^2}{4} U_0 \, c_p \cdot \frac{dT_b}{dx} \Delta x$$

or

$$h_x dx = \rho \frac{D}{4} U_0 \, c_p \cdot \frac{dT_b}{T_w - T_b}$$

Integrating the above equation, we have

$$\int_0^L h_x \, dx = \rho \frac{D}{4} U_0 \, c_p \int_{T_{b1}}^{T_{b2}} \frac{dT_b}{T_w - T_b}$$

or
$$\frac{1}{L}\int_0^L h_x\,dx = \frac{\rho D}{4L} U_0\, c_p \ln \frac{T_w - T_{b_1}}{T_w - T_{b_2}}$$

or
$$\bar{h}_L = \frac{\rho D}{4L} U_0\, c_p \ln \frac{T_w - T_{b_1}}{T_w - T_{b_2}}$$

$$\overline{\text{Nu}}_L = \frac{\bar{h}_L D}{k} = \frac{1}{4(L/D)} \cdot \left(\frac{\rho D U_0}{\mu}\right)\left(\frac{\mu c_p}{k}\right) \ln \frac{T_w - T_{b_1}}{T_w - T_{b_2}}$$

$$= \frac{\text{Re Pr}}{4(L/D)} \ln \frac{T_w - T_{b_1}}{T_w - T_{b_2}}$$

EXAMPLE 5.6 Air at 1 atm pressure and 30°C enters a tube of 25 mm diameter with a velocity of 10 m/s. The tube is heated in a way that a constant-heat-flux of 2 kW/m^2 is maintained at the wall whose temperature is 20°C above the bulk mean air temperature throughout the length of the tube. Determine the exit bulk mean temperature of air and the overall Nusselt number if the length of the tube is 2 m. Take for air, ρ = 1·2 kg/m^3 and c_p = 1000 J/(kg K). Neglect axial conduction.

Solution: From an energy balance of a control volume of air as shown in Figure 5.6, we get

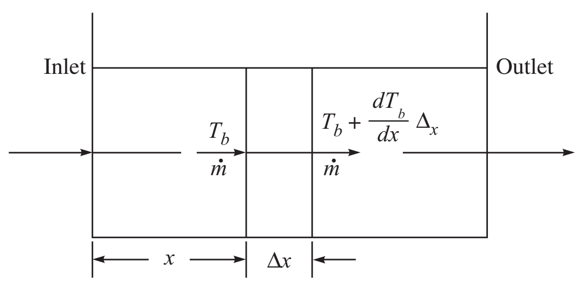


Figure 5.6 Air flow in a pipe (Example 5.6).

$$\dot{m}c_p\left(T_b + \frac{dT_b}{dx}\Delta x - T_b\right) = \dot{q}\,\pi D \cdot \Delta x$$

or
$$\frac{dT_b}{dx} = \frac{\dot{q}\pi D}{\dot{m}c_p}$$

$$= \frac{4\dot{q}\pi D}{\rho\pi D^2\, U \times c_p} = \frac{4\dot{q}}{\rho D U c_p}$$

$$= \frac{4 \times 2000}{1.2 \times 0.025 \times 10 \times 1000}$$

$$= 26{\cdot}7 \text{ °C/m}$$

Therefore,
$$T_{b_2} = 26.7 \times 2 + 30 = 83{\cdot}4\text{°C}$$

Again, we can write at any section x of the tube

$$h_x(T_w - T_b) = \dot{q}$$

or

$$h_x = \frac{\dot{q}}{T_w - T_b}$$

$$= \frac{2000}{20} = 100 \text{ W/(m}^2\ ^\circ\text{C)}$$

Since $(T_w - T_b)$ remains same, the heat transfer coefficients at all sections are the same.

$$\text{Nu}_L = \frac{\bar{h}LD}{k}$$

The thermal conductivity of air at a mean temperature of (30 + 83·4)/2 = 56·7°C is 0·0285 W/(m K).

Therefore,

$$\text{Nu}_L = \frac{100 \times 0.025}{0.0285} = 87{\cdot}72$$

EXAMPLE 5.7 Estimate the heat loss from a vertical wall exposed to nitrogen at one atmospheric pressure and 4°C. The wall is 2·0 m high and 2·5 m wide, and is maintained at 56°C. The average Nusselt number $\overline{\text{Nu}}_H$ over the height of the plate for natural convection is given by $\overline{\text{Nu}}_H = 0.13\,(\text{Gr Pr})^{1/3}$

The properties for nitrogen at a mean film temperature of (56 + 4)/2 = 30°C are given as

$\rho = 1.142$ kg/m^3; $k = 0.026$ W/(m K); $\nu = 15.630 \times 10^{-6}$ m^2/s; Pr = 0.713

Solution: We have to first determine the value of Grashoff number Gr. In consideration of nitrogen as an ideal gas, we can write

$$\beta \text{ (the volumetric coefficient of expansion)} = \frac{1}{T} = \frac{1}{303} = 3{\cdot}3 \times 10^{-3} \text{ K}^{-1}$$

Now,

$$\text{Gr} = \frac{g\beta(T_s - T_\infty)H^3}{\nu^2}$$

$$= \frac{9.8 \times 3.3 \times 10^{-3} \times (56 - 4) \times 2^3}{(15.630 \times 10^{-6})^2} = 5{\cdot}51 \times 10^{10}$$

and

$$\overline{\text{Nu}}_H = \frac{\bar{h}H}{k} = 0.13(\text{Gr Pr})^{1/3} = 0.13(5{\cdot}51 \times 10^{10} \times 0.713)^{1/3}$$

$$= 441{\cdot}934$$

Hence,

$$\bar{h} = \frac{441.934 \times 0.026}{2} = 5{\cdot}745 \text{ W/(m}^2\,^\circ\text{C)}$$

The heat loss from the plate is given by

$$Q = \bar{h}A(T_s - T_\infty)$$

$$= 5{\cdot}745 \times 2 \times 2{\cdot}5 \times (56 - 4)$$

$$= 1494 \text{ W}$$

EXAMPLE 5.8 Electric current passes through a 0.5 m long horizontal wire of 0.1 mm diameter. Determine the value of current to maintain the wire at 400 K in an atmosphere of quiescent air at 300 K. The resistance of the wire is 0.012 ohm per metre. Take the average Nusselt number over the length of the wire to be of 0.4. At mean film temperature of (400 + 300)/2 = 350 K, the thermal conductivity of air is 0·03 W/(m K).

Solution:

$$\overline{\mathrm{Nu}}_L = \frac{\bar{h}D}{k} = 0.4$$

Therefore,

$$\bar{h} = \frac{0.4 \times 0.03}{1 \times 10^{-4}} = 120 \text{ W/(m}^2 \text{ K)}$$

The heat loss from the wire,

$$\begin{aligned} Q &= \bar{h}\, \pi D L (T_{\text{wire}} - T_{\text{air}}) \\ &= 120 \times \pi \times 1 \times 10^{-4} \times 0.5 \times (400 - 300) \\ &= 1.88 \text{ W} \end{aligned}$$

At steady state the ohmic power loss in the wire equals the heat loss from its surface.

Therefore,

$$I^2 R = Q$$

or

$$I = \sqrt{\frac{1.88}{0.012 \times 0.5}} = 17{\cdot}70 \text{ A}$$

EXAMPLE 5.9 A water droplet undergoes a steady-state evaporation process in an ambient of hot and quiescent air. The free stream temperature of air is T_∞. The steady-state evaporation process means that the droplet temperature (which is uniform throughout its mass) remains constant during the process of evaporation. The average Nusselt number for the process of heat transfer from air to the droplet is given as $\overline{\mathrm{Nu}} = 2{\cdot}0$. Show that the square of the droplet diameter decreases linearly with time and also show that the time taken for the droplet diameter to be reduced to one-fourth of its initial value is given by

$$t = \frac{15 a_i^2\, \rho \Delta h_v}{128\, k\, (T_\infty - T_d)}$$

where

- T_d is the steady-state temperature of the droplet
- a_i is the initial droplet diameter
- ρ is the density of water at T_d
- Δh_v is the enthalpy of vaporization of water at T_d
- k is the thermal conductivity of air at film temperature $(T_\infty + T_d)/2$

Solution: Let m and a be the instantaneous mass and diameter of the droplet.
At steady state,

$$\left| \frac{dm}{dt} \right| \times \Delta h_v = \bar{h}\, (\pi a^2)(T_\infty - T_d)$$

In consideration of the fact that m is decreasing with time, we can write from the above equation

$$\frac{dm}{dt} = -\frac{\pi a^2 \bar{h}(T_\infty - T_d)}{\Delta h_v}$$

Again, $$m = \rho \frac{\pi a^3}{6}$$

Therefore, $$\frac{dm}{dt} = \frac{\rho(\pi a^2)}{2}\frac{da}{dt}$$

The overall heat transfer coefficient, $\bar{h} = \dfrac{\overline{\text{Nu}}k}{a} = \dfrac{2k}{a}$

Therefore, we can write

$$\frac{\rho(\pi a^2)}{2}\frac{da}{dt} = -\frac{2\pi a^2 k(T_\infty - T_d)}{a\,\Delta h_v}$$

or $$a\frac{da}{dt} = -\frac{4k(T_\infty - T_d)}{\rho\,\Delta h_v}$$

or $$\frac{d(a)^2}{dt} = -\frac{8k(T_\infty - T_d)}{\rho\,\Delta h_v}$$

At steady-state evaporation, T_d is constant and since T_∞ is constant, the properties like, k, ρ and Δh_v are all constants. Therefore the right-hand side of the above equation is constant and it shows that the square of the droplet diameter decreases linearly with time.

Integrating the above equation, and taking account of the condition at $t = 0$, $a = a_i$ (the initial diameter), we get

$$\frac{a^2}{a_i^2} = 1 - \frac{8k(T_\infty - T_d)}{a_i^2\,\rho\,\Delta h_v}t$$

For $a = a_i/4$, we get

$$t = \frac{15a_i^2\,\rho\Delta h_v}{128k(T_\infty - T_d)}$$

SUMMARY

- Convection is the mode by which heat is transferred between a solid surface and the adjacent fluid in motion when there is a temperature difference between the two. If the fluid is forced to flow over a solid surface by external means such as fan or pump, the convection is called forced convection. When the flow is caused only by buoyancy forces due to density variation in the fluid, the convection is termed free (or natural) convection.

- The mode of heat transfer at solid surface is always conduction and the heat conducted out from the solid surface is convected away by the stream of fluid.
- The average convective heat transfer coefficient over an area of a solid surface at a constant temperature is defined to be the rate of heat transfer per unit area per unit temperature difference (the difference between the surface temperature and the reference temperature of the fluid). The reference temperature is the free stream temperature in case of external flows and bulk mean temperature of fluid in case of internal flows. The local heat transfer coefficient h_x is related to temperature gradient at the solid surface as $h_x = -k\,(\partial T/\partial y)_{y=0}/(T_s - T_{\text{ref}})$, where y is the distance measured from the solid surface. The average heat transfer coefficient $\bar{h}$ over an area A is defined as $\bar{h} = \dfrac{1}{A}\displaystyle\int h_x\, dA$.
- In a region near the wall, the flow velocity varies from zero at the surface to a finite value U_∞, the free stream velocity, at the outer flow. This region of fluid flow is known as hydrodynamic boundary layer. In a similar fashion, the region of fluid flow near the surface, where the temperature varies from T_s (the surface temperature) to T_∞ (the free stream temperature), is known as thermal boundary layer.
- The pertinent dimensionless numbers which govern the phenomenon of forced convection are Nusselt number $\text{Nu}\left(= \dfrac{h\,L_{\text{ref}}}{k}\right)$, Reynolds number $\text{Re}\left(= \dfrac{\rho V_{\text{ref}}\, L_{\text{ref}}}{\mu}\right)$ and Prandtl number $\text{Pr}\,(= \mu c_p/k)$. When the Nusselt number is defined on the basis of average heat transfer coefficient $\bar{h}$ or the local heat transfer coefficient h_x, it is accordingly termed average Nusselt number $\overline{\text{Nu}}$ or local Nusselt number Nu_x. In case of external flows, L_{ref} is the distance from the leading edge along the solid surface and V_{ref} is the free stream velocity, while in case of internal flows, L_{ref} and V_{ref} are the hydraulic diameter and average flow velocity respectively.
- The pertinent dimensionless terms which govern the phenomenon of free convection are Nusselt number Nu, Grashoff number Gr $(= g\beta\,\Delta T\, L_{\text{ref}}/\nu^2)$ and Prandtl number Pr.

REVIEW QUESTIONS

1. In convection, what is the mode of heat transfer at the solid surface?

2. Define (a) hydrodynamic boundary layer and (b) thermal boundary layer.

3. Define local and average heat transfer coefficients. How is local heat transfer coefficient related to temperature gradient at the solid surface?

4. Tick the correct answer:

(a) An increase in flow velocity

(i) increases the thermal boundary layer thickness
(ii) decreases the thermal boundary layer thickness.

(b) A decrease in thermal boundary layer thickness, for prescribed surface temperature and free stream temperature
   (i) increases the rate of heat transfer
   (ii) decreases the rate of heat transfer.

5. What is the motive force that induces the flow of fluid past a solid surface in case of free convection?

6. Why is heat transfer coefficient in forced convection greater than that in free convection?

7. Write the pertinent dimensionless terms governing the phenomenon of (a) forced convection and (b) free convection.

8. Consider two identical flat plates one above another in quiescent air.
   (a) In one situation, the bottom plate is at 100°C and the top one is at 500°C.
   (b) In another situation, the bottom plate is at 500°C and the top one is at 100°C.

   State in which case the rate of heat transfer is expected be higher and why?

9. In a convective heat transfer problem, the Nusselt number Nu is proportional to Re^n where $n < 1$. Which of the following two will contribute to more effective enhancement in the rate of heat transfer?
   (a) A fluid with higher thermal conductivity
   (b) An increase in flow velocity of a given fluid.

PROBLEMS

5.1 Engine oil at 70°C flows with a velocity of 2 m/s over a 4 m long flat plate whose temperature is 20°C. Determine the rate of heat transfer per unit width of the plate. The properties of engine oil at mean film temperature of 45°C are given as
$\rho = 870\ \mathrm{kg/m^3}$; $\nu = 2{\cdot}4 \times 10^{-4}\ \mathrm{m^2/s}$; $k = 0.14$ W/(m K); $\mathrm{Pr} = 2{\cdot}8 \times 10^3$ (use the Nusselt number relation as given in Example 5.3).

[**Ans.** 5.98 kW/m]

5.2 Air flows with a velocity of U_∞ and temperature of T_∞ over a flat plate which is maintained at a constant temperature T_w ($T_w > T_\infty$). The temperature distribution in the thermal boundary layer is given by $(T_w - T)/(T_w - T_\infty) = 2(y/\delta_t) - \frac{1}{2}(y/\delta_t)^3$, where y is the distance normal to the surface and δ_t is the thickness of the thermal boundary layer. If δ is the hydrodynamic boundary layer, and it is given that $\delta_t/\delta = 1$ and $\delta/x = 5(\mathrm{Re}_x)^{-1/2}$, show that the local Nusselt number Nu_x is given by

$$\mathrm{Nu}_x = 0.4\sqrt{\mathrm{Re}_x}$$

5.3 For the same situation as described in Problem 5.2, the temperature distribution in the thermal boundary layer is given by

$$\frac{T_w - T}{T_w - T_\infty} = 1 - \exp\left[-\mathrm{Pr}^{0.3}\left(\frac{U_\infty\, y}{\nu}\right)\right]$$

If Pr = 0·7, T_w = 500 K, T_∞ = 300 K, U_∞ = 20 m/s and the properties of air at mean temperature of 400 K are $\nu = 26 \times 10^{-6}$ m^2/s, k = 0.033 W/(m K), determine the surface heat flux.

[**Ans.** 4.56 W/mm^2]

5.4 A long 150 mm diameter steam pipe whose external surface temperature is 120°C passes through some open area that is not protected against the cold winds. Determine the rate of heat loss per metre length of the pipe when the wind at 1 atm pressure and at 10°C is blowing across the pipe with a velocity of 10 m/s. Use the following correlation for Nusselt number

$$\text{Nu} = 0.3 + \frac{0.62\,\text{Re}^{1/2}\,\text{Pr}^{1/3}}{[1 + (0.4/\text{Pr})^{2/3}]^{1/4}}\left[1 + \left(\frac{\text{Re}}{28 \times 10^3}\right)^{5/8}\right]^{4/5}$$

Take the properties of air from Problem 5.3.

[**Ans.** 2.82 kW/m]

5.5 An electrical device that is 120 mm long, 56 mm wide and 40 mm high is to be cooled by attaching to its top surface a 120 mm × 56 mm wide polished aluminium heat sink which has seven fins 2 mm thick as shown in Figure 5.7. A fan blows air at 30°C parallel to the passage between the fins. The flow velocity of air is limited to 2 m/s by the capacity of the blower. The heat sink is to dissipate 30 W of heat and the base temperature of the heat sink is not to exceed 75°C. Assuming the fins and the base plate to be practically at the same temperature and the radiation heat transfer to be negligible, determine the minimum height of the fins needed to avoid overheating. Use the following relation for heat transfer coefficient in a parallel flow past an isothermal flat plate.

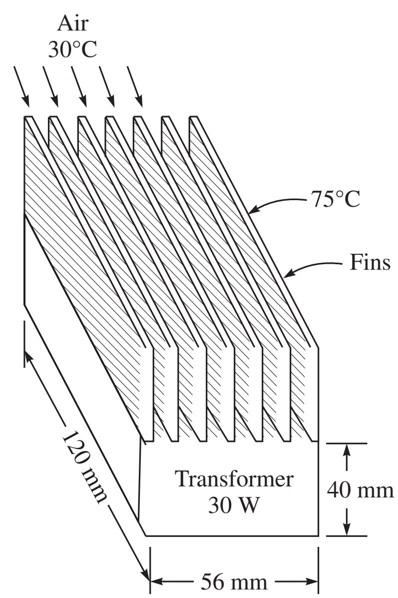


Figure 5.7 Electrical device and the cooling system (Problem 5.5).

$$\overline{\text{Nu}}_L = 0.664\,\text{Re}_L^{1/2}\,\text{Pr}^{1/3}$$

Use the same property values of air as given in Problem 5.3.

[**Ans.** 22.44 mm]

5.6 Water is to be heated from 20°C to 70°C as it flows through a 40 mm internal diameter and 600 mm long tube. The outer surface of the tube is wrapped with electrical wire which acts as an electrical resistance heater. The heater provides uniform heating throughout the surface of the tube. The outer surface of the heater is well insulated. At steady state if the system provides hot water at 15 litre/min, determine the power rating of the resistance heater and the inner surface temperature of the pipe at the exit. The expression for heat transfer coefficient in this case is given by

$$\text{Nu} = 0.023\ \text{Re}^{0.8}\ \text{Pr}^{0.4}$$

The Nusselt number is defined on the basis of tube diameter. The properties required for the use of the expression are to be evaluated at the mean bulk temperature and are given for water in the present case at a temperature of 45°C as

ρ = 990 kg/m^3; k = 0.631 W/(m K); c_p = 4180 J/(kg K); $\nu = 0.66 \times 10^{-6}$ m^2/s; Pr = 430

[**Ans.** 686 kW/m^2, 160.57°C]

5.7 Determine the rate of heat loss from a 60 W incandescent bulb at 130°C to 30°C airstreams moving at 0·5 m/s. Approximate the bulb as a 50 mm diameter sphere. What percentage of the power is lost by convection? Use the relation $\overline{\text{Nu}} = 0.37(\text{Re})^{0.6}$. The properties of air at a mean film temperature of 80°C are given as $\nu = 2.08 \times 10^{-5}$ m^2/s, k = 0.03 W/(m K).

[**Ans.** 12.28 W, 20.47 per cent]

5.8 A large fireplace in a ski lodge has a glass fire screen which covers a vertical opening in the fireplace. The opening is 1·5 m high and 30 m wide. Its surface temperature is 250°C and the ambient air temperature is 30°C. Determine the rate of convective heat transfer from the fireplace to the room. Use the relation for heat transfer coefficient as

$$\overline{\text{Nu}}_L = \left[0.825 + \frac{0.387\,(\text{Gr Pr})^{1/6}}{\{1 + (0.492/\text{Pr})^{9/16}\}^{8/27}}\right]^2$$

The properties of air at mean temperature of 140°C are $\nu = 26 \times 10^{-6}$ m^2/s, k = 0.033 W/(m K), Pr = 0.69.

[**Ans.** 65.93 kW]

5.9 A large vertical plate 2 m high is maintained at 150°C. The plate is exposed to air at 30°C. Determine the heat transfer coefficients at heights 0·5 m, 1·0 m, and 1·2 m. Also calculate the rate of heat loss per metre width of the plate to air. Use the relation

$$\overline{\text{Nu}}_x = 0.508\ \text{Pr}^{1/2}\ (0.952 + \text{Pr})^{-1}\ (\text{Gr}_x)^{1/4}$$

The properties of air at mean temperature of 90°C are given as

$$\nu = 21 \times 10^{-6}\ \text{m}^2/\text{s}, \qquad k = 0.034\ \text{W/(m K)}, \qquad \text{Pr} = 0.695$$

[**Ans.** 3.04 W/m^2, 2.56 W/m^2, 2.44 W/m^2, 689 W/m]

5.10 In Example 5.9, let the water droplets evaporate in an environment of air flowing with a free stream velocity of U so that the Nusselt number for heat transfer from air to the droplet is given by $\overline{\text{Nu}} = 2 + 0.66\text{Re}^{1/2}\text{Pr}^{1/3}$, where Re is defined on the basis of instantaneous droplet diameter 'a' as Re = $\rho U a/\mu$. Nomenclatures are same as in Example 5.9. Show that, during steady-state evaporation of the droplet, 3/2 power of the droplet diameter ($a^{3/2}$) decreases linearly with time. Find also a suitable expression for time taken for the droplet diameter to be reduced to one-fourth of its initial one.

6

Incompressible Viscous Flow: A Brief Review

It has been discussed in Chapter 5 that the rate of heat transfer in convection is greatly influenced by the velocity field near the solid surface. Therefore in this chapter, we will discuss in brief the basic principles of physics and the laws which govern the flow of a real fluid known as viscous flow.

Learning objectives

The reading of this chapter will enable the students

- to understand what is meant by incompressible viscous flow,
- to know how the principles of conservations of mass and momentum are applied to fluid flow to obtain the continuity equation and Navier–Stokes equations,
- to derive the velocity distributions from the exact solution of Navier–Stokes equation in laminar fully developed parallel flows,
- to have a concept of boundary layer flow,
- to derive the boundary layer equation and to have its solution for obtaining the boundary layer growth and velocity distribution within the boundary layer for external and internal flows,
- to know about (i) the fundamental concepts of turbulent flow, (ii) the characterization of turbulent flows and (iii) the ways by which it differs from laminar flow, and
- to know about turbulent stresses and the closure models in solving the Navier–Stokes equation in turbulent flow.

6.1 INTRODUCTORY CONCEPTS

The effect of viscosity is the most important factor in the analysis of flow of a real fluid. The viscosity is a fluid property the effect of which is understood when the fluid is in motion. In a flow field, when the fluid elements move with different velocities, each element feels some resistance to flow. Therefore, shear stresses are identified between the fluid elements with different velocities.

6.1.1 Definition of Viscosity

The relationship between the shear stress and the velocity field was given by Sir Issac Newton as

$$\tau = \mu \frac{\partial u}{\partial y} \tag{6.1}$$

where

τ is the shear stress
μ is the coefficient of viscosity
u is the velocity
y is the coordinate measured in a direction perpendicular to the direction of flow velocity u.

The law expressed by Eq. (6.1) is known as Newton's law of viscosity. Equation (6.1), in fact, defines the *coefficient of viscosity* μ. However, the law is valid for laminar and parallel flows. In a parallel flow, the velocity component exists only along one direction of the reference frame of coordinate axes. The concept of laminar flow will be discussed later on. The fluids which obey the Newton's law of viscosity given by Eq. (6.1) are known as newtonian fluids. In general, for newtonian fluids, the shear stress is proportional to the rate of shear strain where the proportionality constant is defined as the coefficient of viscosity or simply the viscosity μ. Such relations are known as constitutive equations. In a frame of Cartesian coordinate axes, we can write such relations as

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \tag{6.2a}$$

$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \tag{6.2b}$$

$$\tau_{xz} = \tau_{zx} = \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \tag{6.2c}$$

The terms within the brackets as a whole in the right-hand sides of Eqs. (6.2a), (6.2b), and (6.2c) are the rates of shear strain in x–y plane, y–z plane and x–z plane respectively.

6.1.2 Distinction between an Incompressible and a Compressible Flow

Compressibility of a substance is the measure of its change in volume (or density) under the action of external forces, namely the normal compressive forces. The normal compressive stress of any fluid element at rest is known as hydrostatic pressure p. The degree of compressibility of a substance is characterized by the bulk modulus of elasticity E defined as

$$\mathrm{E} = \lim_{\mathbb{V} \to 0} \frac{-\Delta p}{\Delta \mathbb{V}/\mathbb{V}} \tag{6.3}$$

where $\Delta\mathbb{V}$ and Δp are the changes in the volume and pressure respectively, and $\mathbb{V}$ is the initial volume. The values of E for liquids are very high compared to those of gases. Therefore, liquids are usually termed incompressible fluids.

For a flow to be compressible or incompressible we have to consider whether the change in pressure brought about by the fluid motion causes a large change in density or not. The flow velocity and the bulk modulus of elasticity of the fluid are the pertinent parameters in deciding whether a flow will be incompressible or compressible. A dimensionless number known as Mach number 'Ma', which is defined to be the ratio of velocity of flow to the acoustic velocity in the flowing medium at that condition, serves as the criterion for a flow to be incompressible or compressible. It can be shown that when the Mach number Ma is less than or equal to 0.33, the relative change in the density of fluid in a flow becomes less than or equal to 5 per cent. Therefore, flows with Ma $\leq$ 0.33 are considered to be incompressible flows, otherwise (Ma > 0.33) the flows are compressible ones. The speed of sound in a liquid is very high because of its high value of modulus of elasticity E. Therefore, a velocity corresponding to Ma > 0.33 for a liquid is so high that it is not usually encountered in practice. Hence flows of liquids, in practice, are always considered to be incompressible flows. However, this is not the case with a gas. In case of air, for example, at standard pressure and temperature, the acoustic velocity is about 335.28 m/s. Hence a Mach number of 0.33 corresponds to a velocity of about 110 m/s. Therefore, flow of air up to a velocity of 110 m/s under standard condition can be considered as incompressible flow.

6.1.3 Laminar and Turbulent Flow

There are two kinds of flow of a real fluid. In one kind of flow, the fluid particles move in layers or lamina gliding over one another in an orderly motion. At steady state of flow, the velocity components at any point in the flow field are absolutely invariant with time. Any small disturbance in velocity is dampened out by the action of viscosity. This kind of flow is known as *laminar flow*. On the other hand, the flows where various quantities like velocity components, pressure, density, etc. show an irregular and random variation with time and space, are known as *turbulent flows*. However, the irregularities or randomness of different quantities in turbulent flows are such that the statistical average of those quantities can be expressed quantitatively.

The important features of a turbulent flow are that the fluctuations are both diffusive and dissipative in nature. It is postulated that the fluctuations inherently come from disturbance (such as roughness of a solid surface) and may either get dampened out due to viscous damping or grow by taking energy from the free stream. At a Reynolds number less than a critical value, the kinetic energy of flow is not enough to sustain the random fluctuation against the viscous damping and in such cases the flow continuous to be laminar. At a somewhat higher Reynolds number than the critical one, the kinetic energy of flow supports the growth of fluctuations and therefore transition to turbulence takes place.

The fluctuations in velocity components and the transverse motion (motion in a direction perpendicular to the main flow direction) of fluid particles cause momentum transfer between the layers which is manifested in terms of an additional shear stress besides that caused by molecular momentum transport in the flow field. This is conceived as an enhancement in the fluid viscosity which is termed *eddy viscosity*. A relatively detailed discussion on turbulent flow will be made in Section 6.3.

6.2 CONSERVATION EQUATIONS

A fluid being a material body, must obey the law of conservation of mass, momentum and energy in course of its flow.

6.2.1 Conservation of Mass—the Continuity Equation

The law of conservation of mass states that mass can neither be created nor be destroyed. Conservation of mass is inherent to the definition of a closed system and can be written mathematically as

$$\frac{\Delta m}{\Delta t} = 0$$

where m is the mass of the system.

For a control volume (Figure 6.1), the principle of conservation of mass can be stated as

Rate at which mass enters the region = Rate at which mass leaves the region + Rate of accumulation of mass in the region

or

Rate of accumulation of mass in the control volume + Net rate of mass efflux from the control volume = 0 (6.4)

The above statement can be expressed analytically in terms of velocity and density field of a flow and the resulting expression is known as the *equation of continuity* or the *continuity equation*.

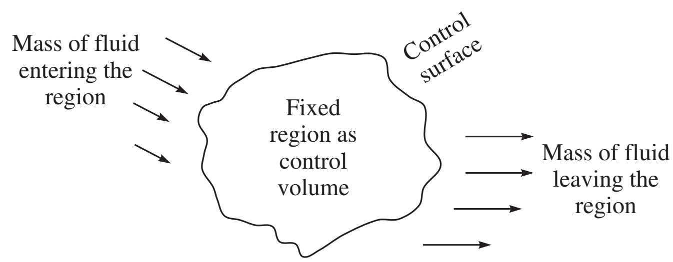


Figure 6.1 A control volume in a flow field.

Continuity equation: differential form

In order to derive the continuity equation at a point in a fluid, the point is enclosed by an elementary control volume appropriate to the coordinate frame of reference, and the influx and the efflux of mass across each surface as well as the rate of mass accumulation within the control volume are considered. A rectangular parallelepiped (Figure 6.2) is considered as the control volume in the rectangular Cartesian frame of coordinate axes. The net efflux of mass along the x-axis must be the excess outflow over inflow across faces normal to the x-axis. Let the fluid enter across one of such faces ABCD with a velocity u and density ρ. The velocity and density with which the fluid leaves the face EFGH will be $u + \frac{\partial u}{\partial x} dx$ and $\rho + \frac{\partial \rho}{\partial x} dx$ respectively (neglecting the higher-order terms in dx).

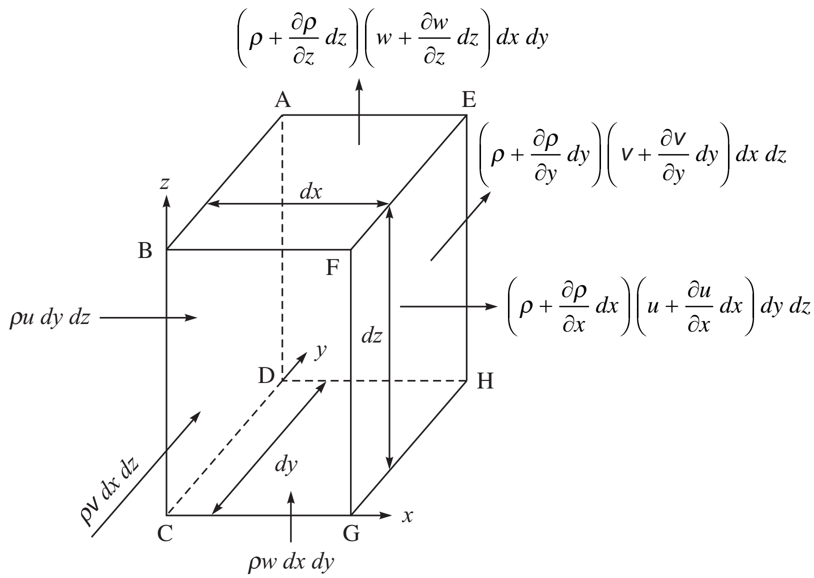


Figure 6.2 A control volume appropriate to the rectangular Cartesian frame of coordinate axes.

Therefore, the rate of mass entering the control volume through the face ABCD = $\rho\, u\, dy\, dz$, and the rate of mass leaving the control volume through the face EFGH

$$= \left(\rho + \frac{\partial \rho}{\partial x} dx\right)\left(u + \frac{\partial u}{\partial x} dx\right) dy\, dz$$

$$= \left[\rho u + \frac{\partial}{\partial x}(\rho u)\, dx\right] dy\, dz$$

(neglecting the higher-order terms in dx)

Hence, the net rate of mass efflux from the control volume in the x-direction

$$= \left[\rho u + \frac{\partial}{\partial x}(\rho u)\, dx\right] dy\, dz - \rho u\, dy\, dz$$

$$= \frac{\partial}{\partial x}(\rho u)\, dx\, dy\, dz$$

$$= \frac{\partial}{\partial x}(\rho u)\, d\mathbb{V}$$

where $d\mathbb{V}$ (= $dx\, dy\, dz$) is the elemental volume.

In a similar fashion, the net rate of mass efflux in the y-direction

$$= \left(\rho + \frac{\partial \rho}{\partial y} dy\right)\left(v + \frac{\partial v}{\partial y} dy\right) dx\, dz - \rho v\, dx\, dz$$

$$= \frac{\partial}{\partial y}(\rho v)\, d\mathbb{V}$$

and the net rate of mass efflux in the z-direction

$$= \left(\rho + \frac{\partial \rho}{\partial z} dz\right)\left(w + \frac{\partial w}{\partial z} dz\right) dx\, dy - \rho\, w dx\, dy$$

$$= \frac{\partial}{\partial z}(\rho w)\, d\mathbb{V}$$

The rate of accumulation of mass within the control volume is $\frac{\partial}{\partial t}(\rho d\mathbb{V}) = \frac{\partial \rho}{\partial t} d\mathbb{V}$ (by the definition of control volume, $d\mathbb{V}$ is invariant with time). Therefore, according to the statement of conservation of mass for a control volume (Eq. (6.4)), it can be written that

$$\left\{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right\} d\mathbb{V} = 0$$

Since this equation is valid irrespective of the size $d\mathbb{V}$ of the control volume, we can write

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{6.5}$$

This is the well known *equation of continuity of a compressible flow* in a rectangular Cartesian coordinate system. The equation can be written in vector form as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{6.6}$$

where $\mathbf{V}$ represents the velocity vector.

In case of a steady flow,

$$\frac{\partial \rho}{\partial t} = 0$$

Hence Eq. (6.6) becomes

$$\nabla \cdot (\rho \mathbf{V}) = 0 \tag{6.7}$$

or in a rectangular Cartesian coordinate system

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{6.8}$$

Equation (6.7) or (6.8) represents the continuity equation for a steady flow. In case of an incompressible flow,

$$\rho = \text{constant}$$

Hence $\partial \rho / \partial t = 0$ and moreover, $\nabla \cdot (\rho \mathbf{V}) = \rho \nabla \cdot (\mathbf{V})$.

Therefore, the continuity equation for an incompressible flow becomes

$$\nabla \cdot (\mathbf{V}) = 0 \tag{6.9}$$

or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{6.10}$$

Continuity equation in the cylindrical polar coordinate system

The continuity equation in any coordinate system can be derived in two ways: (i) either by expanding the vectorial form of Eq. (6.6) with respect to the particular coordinate system, (ii) or by considering an elemental control volume appropriate to the reference frame of coordinates and then by applying the fundamental principle of conservation of mass as given by Eq. (6.4). The term $\nabla \cdot (\rho \mathbf{V})$ in the cylindrical polar coordinate system (Figure 6.3(a)) can be written as

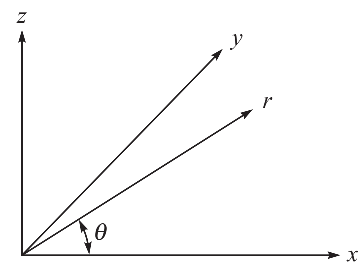


Figure 6.3(a) The cylindrical polar coordinate system.

$$\nabla \cdot (\rho \mathbf{V}) = \frac{\partial}{\partial r}(\rho v_r) + \frac{\rho v_r}{r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial}{\partial z}(\rho v_z) \tag{6.11}$$

where, v_r, v_θ and v_z are respectively the velocity components in r, θ and z directions in cylindrical polar coordinate system.

Therefore, the equation of continuity in a cylindrical polar coordinate system can be written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho v_r) + \frac{\rho v_r}{r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial}{\partial z}(\rho v_z) = 0 \tag{6.12}$$

The above equation can also be derived by considering the mass fluxes in the control volume shown in Figure 6.3(b).

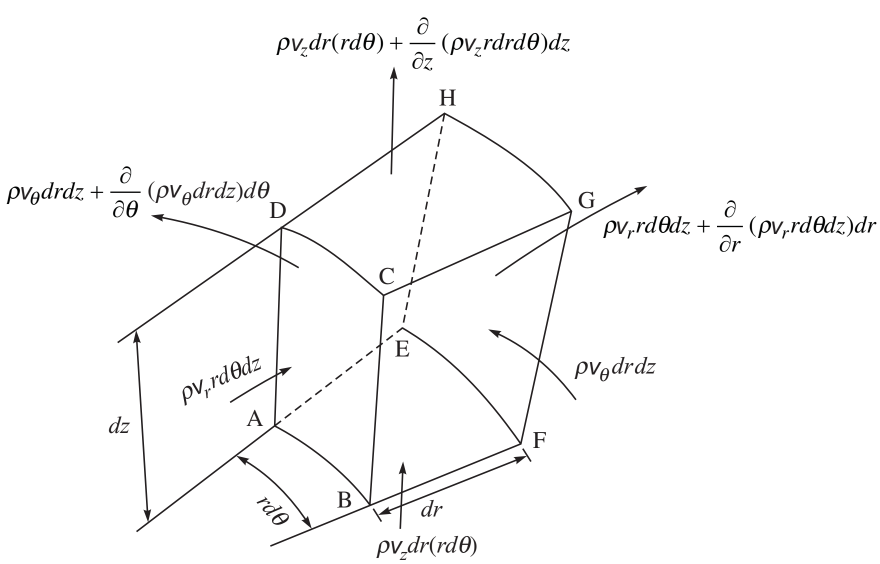


Figure 6.3(b) A control volume appropriate to the cylindrical polar coordinate system.

The rate of mass entering the control volume through the face ABCD

$$= \rho v_r \, r \, d\theta \, dz$$

The rate of mass leaving the control volume through the face EFGH

$$= \rho v_r \, r d\theta \, dz + \frac{\partial}{\partial r}(\rho v_r \, r d\theta \, dz) dr$$

Hence the net rate of mass efflux in the r-direction = $\dfrac{1}{r}\dfrac{\partial}{\partial r}(\rho v_r \, r) d\mathbb{V}$

where $d\mathbb{V} = r \, dr \, d\theta \, dz$ is the elemental volume.

The net rate of mass efflux from the control volume, in the θ-direction, is the difference of mass leaving through the face ADHE and the mass entering through the face BCGF and can be written as $\dfrac{1}{r}\dfrac{\partial}{\partial \theta}(\rho v_\theta) d\mathbb{V}$.

The net rate of mass efflux in the z-direction can be written in a similar fashion as

$$\frac{\partial}{\partial z}(\rho v_z) \, d\mathbb{V}$$

The rate of increase of mass within the control volume becomes

$$\frac{\partial}{\partial t}(\rho d\mathbb{V}) = \frac{\partial \rho}{\partial t}(d\mathbb{V})$$

Hence, following Eq. (6.4), the final form of continuity equation in a cylindrical polar coordinate system becomes

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho \, v_r \, r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho \, v_\theta) + \frac{\partial}{\partial z}(\rho \, v_z) = 0$$

or

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho \, v_r) + \frac{\rho v_r}{r} + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho \, v_\theta) + \frac{\partial}{\partial z}(\rho \, v_z) = 0$$

In case of an incompressible flow,

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \qquad (6.13)$$

Continuity equation in the spherical polar coordinate system

The equation of continuity in the spherical polar coordinate system (Figure 6.4) can be written by expanding the term $\nabla \cdot (\rho \mathbf{V})$ of Eq. (6.6) as

$$\frac{\partial \rho}{\partial t} + \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 \rho \, v_R) + \frac{1}{R \sin \varphi}\frac{\partial (\rho \, v_\theta)}{\partial \theta} + \frac{1}{R \sin \varphi} + \frac{\partial (\rho \, v_\varphi \sin \varphi)}{\partial \varphi} = 0 \qquad (6.14)$$

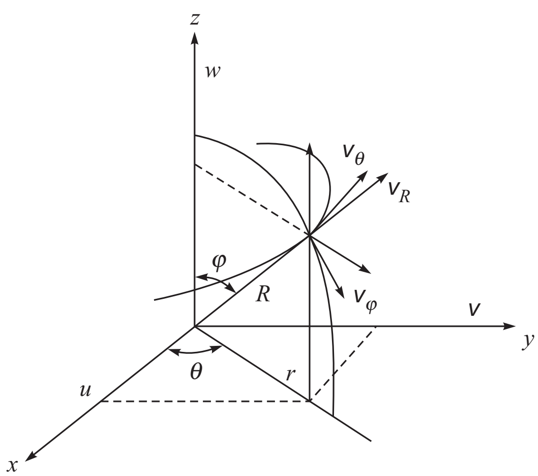


Figure 6.4 The spherical polar coordinate system.

For an incompressible flow,

$$\frac{1}{R}\frac{\partial}{\partial R}(R^2 v_R) + \frac{1}{\sin\varphi}\frac{\partial(v_\theta)}{\partial\theta} + \frac{1}{\sin\varphi} + \frac{\partial(v_\varphi \sin\varphi)}{\partial\varphi} = 0 \tag{6.15}$$

The derivation of Eq. (6.14) by considering an elemental control volume appropriate to the spherical polar coordinate system is left as an exercise to the student.

Continuity equation from a closed system approach

We know that the conservation of mass is inherent to the definition of a closed system as $Dm/Dt = 0$, m being the mass of the closed system. However, the general form of continuity, as expressed by Eq. (6.6), can also be derived from the basic equation of mass conservation of a closed system as follows:

Let us consider an elemental closed system of volume $\Delta\mathbb{V}$ and density ρ. Therefore, we can write

$$\frac{D}{Dt}(\rho\Delta\mathbb{V}) = 0$$

or

$$\frac{D\rho}{Dt} + \rho\frac{1}{\Delta\mathbb{V}}\frac{D}{Dt}(\Delta\mathbb{V}) = 0$$

The first term of the above equation is the material derivative of density with time which can be split up into its temporal and convective components, and hence we get

$$\frac{\partial\rho}{\partial t} + \mathbf{V}\cdot\nabla\rho + \rho\frac{1}{\Delta\mathbb{V}}\frac{D}{Dt}(\Delta\mathbb{V}) = 0$$

The term $\frac{1}{\Delta\mathbb{V}}\frac{D}{Dt}(\Delta\mathbb{V})$ is the rate of volumetric dilation per unit volume of the elemental system and equals the divergence of the velocity vector at the location enclosed by the system.

Therefore, we have

$$\frac{\partial \rho}{\partial t} + \mathbf{V}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{V} = 0$$

or

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{V}) = 0$$

6.2.2 Conservation of Momentum—Navier–Stokes Equation

Equations of motion of a real fluid were first derived in terms of flow parameters by the two scientists M. Navier and G.G. Stokes. Therefore these equations are named Navier–Stokes equations. However, they are derived from the Newton's second law which states that the product of mass and acceleration equals the sum of the external forces acting on a body.

The external forces acting on a fluid element are of two types: body forces and surface forces. The body forces act throughout the body of the fluid element and are distributed over the entire mass or volume of the element. These forces are generally caused by external agencies. The examples of these forces are gravitational force, electromagnetic force, etc. The surface forces, on the other hand, appear only at the surface of a fluid element and are originated by the action of the surrounding fluid from which the element has been isolated.

The first step in the derivation of Navier–Stokes equation is to express the net surface force acting on the fluid element in terms of the stress components. We consider, in the Cartesian coordinate system x–y–z, a differential fluid element in the form of a rectangular parallelepiped (Figure 6.5) so that the surfaces of the element are parallel to the coordinate planes.

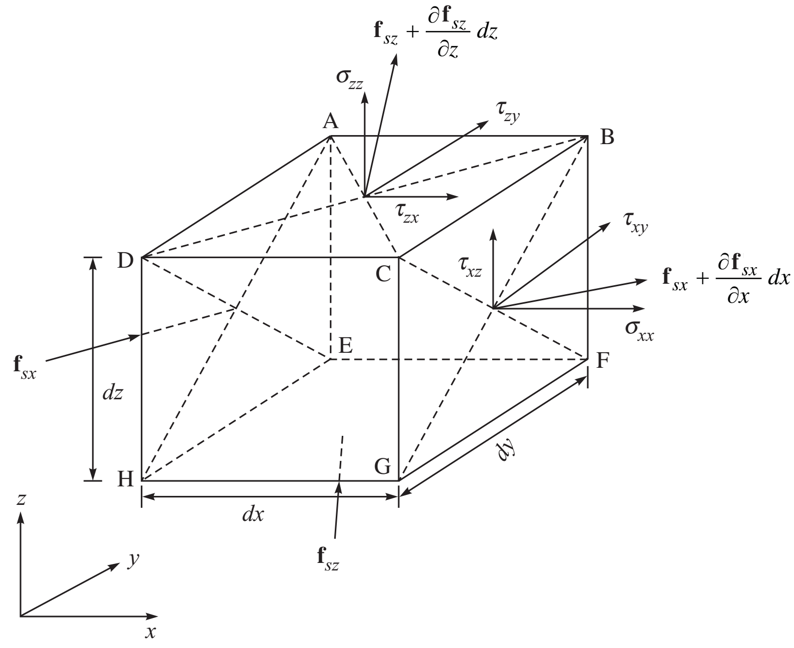


Figure 6.5 Components of stresses on the surfaces of a differential fluid element.

We now evaluate the net surface force acting on the element. Let $\mathbf{f}_{sx}$ be the surface force per unit area on the surface AEHD. Then the surface force per unit area on the surface BFGC becomes

$$\mathbf{f}_{sx} + \frac{\partial \mathbf{f}_{sx}}{\partial x}\, dx$$

Therefore, the net force due to imbalance of surface forces on the above two surfaces is

$$\frac{\partial \mathbf{f}_{sx}}{\partial x} dx\, dy\, dz \tag{6.16}$$

If $\mathbf{f}_{sy}$ and $\mathbf{f}_{sz}$ be the surface forces on surfaces HGCD and HGFE respectively per unit area, then it can be written that the resultant surface force on the fluid element for all six surfaces is

$$\left(\frac{\partial \mathbf{f}_{sx}}{\partial x} + \frac{\partial \mathbf{f}_{sy}}{\partial y} + \frac{\partial \mathbf{f}_{sz}}{\partial z}\right) dx\, dy\, dz \tag{6.17}$$

The resultant surface force per unit volume $\mathbf{F}_s$ can be written as

$$\mathbf{F}_s = \frac{\partial \mathbf{f}_{sx}}{\partial x} + \frac{\partial \mathbf{f}_{sy}}{\partial y} + \frac{\partial \mathbf{f}_{sz}}{\partial z} \tag{6.18}$$

The forces $\mathbf{f}_{sx}$, $\mathbf{f}_{sy}$ and $\mathbf{f}_{sz}$ can be resolved into components normal to and along the respective surfaces as shown in Figure 6.5. The normal components are known as normal stresses and are denoted by σ, while the components along a surface are known as shear stresses and are denoted by τ.

Hence we can write

$$\mathbf{f}_{sx} = \mathbf{i}\,\sigma_{xx} + \mathbf{j}\,\tau_{xy} + \mathbf{k}\,\tau_{xz} \tag{6.19}$$

$$\mathbf{f}_{sy} = \mathbf{i}\,\tau_{yx} + \mathbf{j}\,\sigma_{yy} + \mathbf{k}\,\tau_{yz} \tag{6.20}$$

$$\mathbf{f}_{sz} = \mathbf{i}\,\tau_{zx} + \mathbf{j}\,\tau_{zy} + \mathbf{k}\,\sigma_{zz} \tag{6.21}$$

The first subscript of stress components denotes the direction of the normal to the plane on which the stress acts, while the second subscript denotes the direction of the force component because of which the stress arises. The stress system contains nine scalar quantities which form a stress tensor. The set of nine components of the tensor is known as *stress matrix* and is described as

$$\pi = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \tag{6.22}$$

The above stress matrix is symmetric, which means that two shearing stresses with subscripts which differ only in their sequence are equal. This can be proved if we consider the equation of motion for instantaneous rotation of the fluid element. For rotation about the y-axis (Figure 6.5), we can write

$$\dot{\omega}_y \, dI_y = (\tau_{xz} \, dy \, dz) \, dx - (\tau_{zx} \, dx \, dy) \, dz$$

or

$$(\tau_{xz} - \tau_{zx}) = \frac{dI_y}{d\mathbb{V}} \dot{\omega}_y \tag{6.23}$$

where

$\dot{\omega}_y$ is the angular velocity about the y-axis

dI_y is the moment of inertia of the fluid element about the y-axis

$d\mathbb{V}$ (= $dx \, dy \, dz$) is the volume of the element. Since, dI_y is proportional to the fifth power of linear dimensions and $d\mathbb{V}$ is proportional to the third power of linear dimensions, the term ($dI_y/d\mathbb{V}$) on the right-hand side of Eq. (6.23) vanishes as the element contracts to a point. Hence, we get

$$\tau_{xz} = \tau_{zx}$$

In a similar way by considering the equation of motion for rotation about x and z axes, we have

$$\tau_{yz} = \tau_{zy}$$
$$\tau_{xy} = \tau_{yx}$$

By making use of the symmetric conditions, we can write the stress matrix as

$$\pi = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \tag{6.24}$$

Combining Eqs. (6.19), (6.20) and (6.21), we can write for the resultant surface force on the element per unit volume as

$$\mathbf{F}_s = \mathbf{i}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) + \mathbf{j}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right) + \mathbf{k}\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) \tag{6.25}$$

Let the body force per unit mass be

$$\mathbf{f}_b = \mathbf{i} f_{bx} + \mathbf{j} f_{by} + \mathbf{k} f_{bz} \tag{6.26}$$

If $\mathbf{V}$ is the velocity of the fluid element, following Newton's law of motion, we can write

$$\rho(dx \, dy \, dz) \frac{D\mathbf{V}}{Dt} = \rho \mathbf{f}_b (dx \, dy \, dz) + \mathbf{F}_s (dx \, dy \, dz)$$

or

$$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{f}_b + \mathbf{F}_s \tag{6.27}$$

The velocity vector **V** can be written in terms of its components as

$$\mathbf{V} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$$

Then

$$\frac{D\mathbf{V}}{Dt} = \mathbf{i}\frac{Du}{Dt} + \mathbf{j}\frac{Dv}{Dt} + \mathbf{k}\frac{Dw}{Dt} \tag{6.28}$$

Substituting Eqs. (6.25), (6.26), (6.28) in Eq. (6.27), we obtain

$$\rho\frac{Du}{Dt} = \rho f_{bx} + \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) \tag{6.29a}$$

$$\rho\frac{Dv}{Dt} = \rho f_{by} + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right) \tag{6.29b}$$

$$\rho\frac{Dw}{Dt} = \rho f_{bz} + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) \tag{6.29c}$$

The next step is to express the stress components appearing on the right-hand side of Eqs. (6.29a), (6.29b), (6.29c) in terms of the strain rates. These relations are known as constitutive equations of the fluid and are developed based on the following assumptions:

(a) The stress components may be expressed as a linear function of strain rate components. A class of fluids in practice obey such relations and are known as newtonian fluids.
(b) The relations between stress components and strain rate components must be invariant to coordinate transformation. This assumption implies that the constitutive equation is a physical law of the fluid.
(c) The stress components must reduce to the hydrostatic pressure when all velocities are zero.
(d) The fluids obey the Stoke's hypothesis which states that the second coefficient of viscosity is significant only when there is an exchange of molecular momentum between translational degrees of freedom and vibrational and rotational degrees of freedom. It has been verified that for an ideal gas the second viscosity coefficient is negligibly small. Fluids obeying this hypothesis are known as stokesian fluids.

Based on the above assumptions, the final expressions of the relations between stress components and strain rate components become

$$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{6.30a}$$

$$\sigma_{yy} = -p + 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{6.30b}$$

$$\sigma_{zz} = -p + 2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{6.30c}$$

$$\tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{6.30d}$$

$$\tau_{yz} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \tag{6.30e}$$

$$\tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \tag{6.30f}$$

The derivations of the above relations are outside the scope of this chapter.

From Eqs. (6.30a) to (6.30c), we get $p = -(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3$. This states that the scalar magnitude of thermodynamic pressure p is the arithmetic average of the normal stresses for all stokesian fluids.

Substituting the stress components from Eqs. (6.30a) to (6.30f) in Eqs. (6.29a) to (6.29c), we get

$$\rho\frac{Du}{Dt} = \rho f_{bx} - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla\cdot\mathbf{V}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] \tag{6.31a}$$

$$\rho\frac{Dv}{Dt} = \rho f_{by} - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\nabla\cdot\mathbf{V}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \tag{6.31b}$$

$$\rho\frac{Dw}{Dt} = \rho f_{bz} - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z} - \frac{2}{3}\nabla\cdot\mathbf{V}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] \tag{6.31c}$$

Equations (6.31a) to (6.31c) are the well known Navier–Stokes equations. The acceleration terms on the left-hand sides of the Eqs. (6.31a) to (6.31c) represent the total or substantial accelerations which can be split up into the respective temporal and convective components as

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \tag{6.32a}$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \tag{6.32b}$$

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \tag{6.32c}$$

By making use of the continuity equation (Eq. (6.5)) and Eqs. (6.32a) to (6.32c), the Navier–Stokes equations (Eqs. (6.31a) to (6.31c)) can be written in their popular form as

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho f_{bx} - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{\mu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{6.33a}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) = \rho f_{by}-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\frac{\mu}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \tag{6.33b}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) = \rho f_{bz}-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\frac{\mu}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \tag{6.33c}$$

For an incompressible flow,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0$$

Hence Eqs. (6.33a), (6.33b), and (6.33c) become

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right) = \rho f_{bx}-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \tag{6.34a}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) = \rho f_{by}-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \tag{6.34b}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) = \rho f_{bz}-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \tag{6.34c}$$

Equations (6.34a) to (6.34c) are the Navier–Stokes equations for an incompressible flow.

In short, the vector notation may be used to write the Navier–Stokes and continuity equation for incompressible flow as

$$\rho\frac{D\mathbf{V}}{Dt} = \rho\mathbf{f}_b-\nabla p+\nabla^2\mathbf{V} \tag{6.35}$$

$$\nabla\cdot\mathbf{V} = 0 \tag{6.36}$$

We have four unknown quantities, u, v, w and p, and four equations — equations of motion in three directions and the continuity equation. In principle, these equations are solvable, but to-date, a generalized solution is not available due to the complex nature of the set of these equations. The highest-order terms, which come from the viscous forces, are linear and of second order. The first-order convective terms are nonlinear and hence, the set is termed quasi-linear.

Navier–Stokes equations in cylindrical coordinates (Figure 6.6) are useful in solving many problems. If v_r, v_θ, and v_z denote the velocity components along the radial, azimuthal, and axial directions respectively, then for an incompressible flow, Eqs. (6.35) and (6.36), lead to the following system of equations:

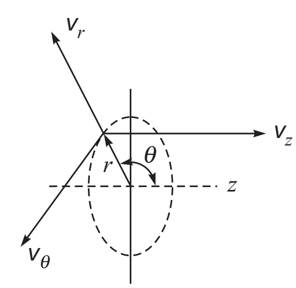


Figure 6.6 Cylindrical polar coordinate system and the velocity components.

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\cdot\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=\rho f_r-\frac{\partial p}{\partial r}$$

$$+\mu\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\cdot\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right) \tag{6.37a}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\cdot\frac{\partial v_\theta}{\partial \theta}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_r v_\theta}{r}\right)=\rho f_\theta-\frac{1}{r}\cdot\frac{\partial p}{\partial \theta}$$

$$+\mu\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r^2}+\frac{1}{r^2}\cdot\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\cdot\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right) \tag{6.37b}$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\cdot\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=\rho f_z-\frac{\partial p}{\partial z}$$

$$+\mu\left(\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial v_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right) \tag{6.37c}$$

$$\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{1}{r}\cdot\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}=0 \tag{6.38}$$

where, f_r, f_θ and f_z are the body forces per unit mass in r, θ and z directions respectively.

6.2.3 Exact Solutions of Navier–Stokes Equations

The basic difficulty in solving Navier–Stokes equations arises due to the presence of nonlinear (quadratic) inertia terms on the left-hand side. However, there are some nontrivial solutions of the Navier–Stokes equations in which the nonlinear inertia terms are identically zero. One such class of flows is termed *parallel flows* in which only one velocity term is nontrivial and all the fluid particles move in one direction only.

Let us consider a Cartesian coordinate system and choose x to be the direction along which does exist the velocity of flow, i.e. $u \neq 0$, $v = w = 0$. Invoking this in continuity equation, we get

$$\frac{\partial u}{\partial x}+\cancelto{0}{\frac{\partial v}{\partial y}}+\cancelto{0}{\frac{\partial w}{\partial z}}=0$$

or $$\frac{\partial u}{\partial x}=0, \qquad \text{which means } u=u(y, z, t)$$

Now, Navier–Stokes equations for incompressible flow become

$$\frac{\partial u}{\partial t}+u\cancelto{0}{\frac{\partial u}{\partial x}}+v\cancel{\frac{\partial u}{\partial y}}+w\cancel{\frac{\partial u}{\partial z}}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left[\cancelto{0}{\frac{\partial^2 u}{\partial x^2}}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]$$

$$\cancelto{0}{\frac{\partial v}{\partial t}} + u\,\cancelto{0}{\frac{\partial v}{\partial x}} + v\,\cancelto{0}{\frac{\partial v}{\partial y}} + w\,\cancelto{0}{\frac{\partial v}{\partial z}} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\cancelto{0}{\frac{\partial^2 v}{\partial x^2}} + \cancelto{0}{\frac{\partial^2 v}{\partial y^2}} + \cancelto{0}{\frac{\partial^2 v}{\partial z^2}}\right]$$

$$\cancelto{0}{\frac{\partial w}{\partial t}} + u\,\cancelto{0}{\frac{\partial w}{\partial x}} + v\,\cancelto{0}{\frac{\partial w}{\partial y}} + w\,\cancelto{0}{\frac{\partial w}{\partial z}} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\cancelto{0}{\frac{\partial^2 w}{\partial x^2}} + \cancelto{0}{\frac{\partial^2 w}{\partial y^2}} + \cancelto{0}{\frac{\partial^2 w}{\partial z^2}}\right]$$

So, we obtain

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0, \qquad \text{which means } p = p(x) \text{ alone,}$$

and

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\cdot\frac{dp}{dx} + \nu\left[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \tag{6.39}$$

Parallel flow in a straight channel (plane Poiseuille flow)

Consider a steady laminar and fully developed flow between two infinitely broad parallel plates as shown in Figure 6.7. Because of symmetry, the x-axis is taken along the centre line between the two plates. The flow is independent of any variation in the z-direction, hence, z dependence has been gotton rid of and Eq. (6.39) becomes

$$\frac{dp}{dx} = \mu\,\frac{d^2u}{dy^2} \tag{6.40}$$

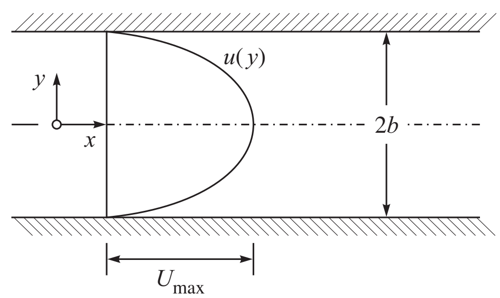


Figure 6.7 Parallel flow in a straight channel.

The boundary conditions are:

At $\quad y = b,\ u = 0;$

and at $\quad y = -b,\ u = 0.$

Integrating Eq. (6.40), we have

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx}\,y + C_1$$

and

$$u = \frac{1}{2\mu}\frac{dp}{dx}\,y^2 + C_1 y + C_2$$

Applying the boundary conditions, the constants are evaluated as

$$C_1 = 0 \qquad \text{and} \qquad C_2 = -\frac{1}{\mu}\frac{dp}{dx}\cdot\frac{b^2}{2}$$

So, the solution is

$$u = \frac{1}{2\mu}\left(-\frac{dp}{dx}\right)(b^2 - y^2) \tag{6.41}$$

which implies that the velocity profile is parabolic. This typical flow is usually termed *plane Poiseuille flow*.

The maximum and average velocities: From an inspection of Eq. (6.41), it is found that the maximum velocity occurs at $y = 0$. This gives

$$u_{\max} = \frac{b^2}{2\mu}\left(-\frac{dp}{dx}\right) \tag{6.42a}$$

The average velocity is given by

$$u_{av} = \frac{\text{flow rate}}{\text{flow area}} = \frac{1}{2b}\int_{-b}^{b} u\,dy$$

or

$$u_{av} = \frac{1}{2b}\int_0^b u\,dy = \frac{1}{b}\int_0^b \frac{1}{2\mu}\left(-\frac{dp}{dx}\right)(b^2 - y^2)\,dy$$

$$= \frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\cdot\frac{1}{b}\left\{[b^2 y]_0^b - \left[\frac{y^3}{3}\right]_0^b\right\}$$

Therefore,

$$u_{av} = \frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\cdot\frac{2}{3}b^2 \tag{6.42b}$$

So,

$$\frac{u_{av}}{u_{\max}} = \frac{2}{3} \quad \text{or} \quad u_{\max} = \frac{3}{2}u_{av} \tag{6.42c}$$

Wall shear stress: The shearing stress at the wall for the parallel flow in a channel can be determined from the velocity gradient as

$$\left|\tau_{yx}\right|_b = \mu\left(\frac{\partial u}{\partial y}\right)_b = b\left(\frac{dp}{dx}\right) = -2\mu\,\frac{u_{\max}}{b} \tag{6.43a}$$

Since the upper plate is a "minus y surface", a negative stress acts in the positive x direction, i.e. to the right.

The local friction coefficient C_f is defined by

$$C_f = \frac{\left|\tau_{yx}\right|_b}{\frac{1}{2}\rho u_{av}^2} = \frac{\frac{3\mu u_{av}}{b}}{\frac{1}{2}\rho u_{av}^2} = \frac{12}{\text{Re}} \tag{6.43b}$$

where $\text{Re} = u_{av}(2b)/\nu$ is the Reynolds number of flow based on average velocity and the channel height $2b$. Experiments show that Eq. (6.43b) is valid in the laminar regime of the channel flow. The maximum Reynolds number corresponding to fully developed laminar flow, for which a stable motion will persist, is 2300. In a reasonably careful experiment, laminar flow can be observed even up to Re = 10,000. But the value below which the flow will always remain laminar, is the critical value of Re which is 2300.

Hagen–Poiseuille flow

Consider a steady laminar and *fully developed* flow through a straight pipe of circular cross-section as shown in Figure 6.8. The axis of the pipe is considered to be the z-axis about which the flow is azimuthally symmetrical. All fluid particles travel along the z-axis, i.e.

$$v_z \neq 0, \; v_r = 0, \; v_\theta = 0$$

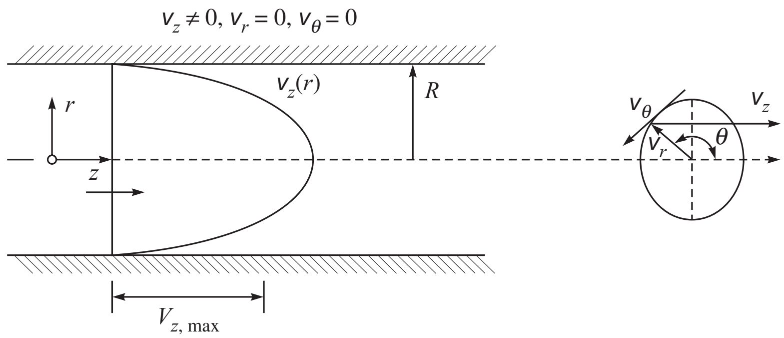


Figure 6.8 Hagen-Poiseuille flow.

From continuity equation, we obtain

$$\cancel{\frac{\partial v_r^0}{\partial r}} + \cancel{\frac{v_r^0}{r}} + \frac{\partial v_z}{\partial z} = 0 \qquad \left(\text{For azimuthal symmetry, } \frac{1}{r}\cdot\frac{\partial v_\theta}{\partial \theta} = 0\right)$$

$$\frac{\partial v_z}{\partial z} = 0 \qquad \text{which means } v_z = v_z(r)$$

Invoking $v_r = 0$, $v_\theta = 0$, $\dfrac{\partial v_z}{\partial z} = 0$, and $\dfrac{\partial}{\partial \theta}$ (any quantity) = 0 in the Navier–Stokes equations (Eqs. (6.37a) to (6.37c)), we finally obtain for a steady flow

$$\frac{d^2 v_z}{\partial r^2} + \frac{1}{r}\cdot\frac{d v_z}{\partial r} = \frac{1}{\mu}\frac{dp}{dz}$$

or
$$r\frac{d^2 v_z}{dr^2} + \frac{dv_z}{dr} = \frac{1}{\mu}\frac{dp}{dz}r$$

or
$$\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{1}{\mu}\frac{dp}{dz}r$$

or
$$r\frac{dv_z}{dr} = \frac{1}{2\mu}\frac{dp}{dz}r^2 + A$$

or
$$\frac{dv_z}{dr} = \frac{1}{2\mu}\frac{dp}{dz}r + \frac{A}{r}$$

or
$$v_z = \frac{1}{4\mu}\frac{dp}{dz}r^2 + A\ln r + B$$

The boundary conditions are:

At $r = 0$, v_z is finite and at $r = R$, $v_z = 0$

The condition at $r = 0$ implies that A should be equal to zero and the condition at $r = R$ yields

$$B = -\frac{1}{4\mu}\frac{dp}{dz}R^2$$

Therefore,
$$v_z = \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left(1-\frac{r^2}{R^2}\right) \tag{6.44}$$

This shows that the axial velocity profile in a fully developed laminar pipe flow is parabolic in nature. This typical flow is termed *Hagen–Poiseuille flow*.

The maximum and average velocities: The maximum velocity occurs at $r = 0$. This gives

$$v_{z,\max} = \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right) \tag{6.45}$$

The average velocity is given by

$$v_{z,\text{av}} = \frac{Q}{\pi R^2} = \frac{\int_0^R 2\pi r v_z r\,dr}{\pi R^2}$$

or
$$v_{z,\text{av}} = \frac{2\pi\frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left(\frac{R^2}{2}-\frac{R^4}{4R^2}\right)}{\pi R^2}$$

or
$$v_{z,\text{av}} = \frac{R^2}{8\mu}\left(-\frac{dp}{dz}\right) \tag{6.46a}$$

Therefore,
$$v_{z,\max} = 2v_{z,\text{av}} \tag{6.46b}$$

The discharge through the pipe is given by

$$Q = \pi R^2 v_{z,\text{av}} \tag{6.47}$$

or
$$Q = \pi R^2 \frac{R^2}{8\mu}\left(-\frac{dp}{dz}\right)$$

or
$$Q = -\frac{\pi D^4}{128\mu}\left(-\frac{dp}{dz}\right) \tag{6.48}$$

The shear stress at any point of the pipe flow is given by

$$\tau|_r = \mu\frac{dv_z}{dr}$$

From Eq. (6.44),
$$\frac{dv_z}{dr} = \frac{R^2}{4\mu}\left(\frac{dp}{dz}\right)\frac{2r}{R^2}$$

or
$$\frac{dv_z}{dr} = \frac{1}{2\mu}\left(\frac{dp}{dz}\right)r \tag{6.49a}$$

which means
$$\tau|_r = \frac{1}{2}\left(\frac{dp}{dz}\right)r \tag{6.49b}$$

This indicates that τ varies linearly with the radial distance from the axis. At the wall, τ assumes the maximum value.

At $r = R$,
$$\tau = \tau_{\max} = \frac{1}{2}\left(\frac{dp}{dz}\right)R \tag{6.50}$$

The skin friction coefficient for Hagen–Poiseuille flow can be expressed by

$$C_f = \frac{\left|\tau_{\text{at } r=R}\right|}{\frac{1}{2}\rho v_{z,\text{av}}^2}$$

With the help of Eqs. (6.50) and (6.46a), it can be written as

$$C_f = \frac{16}{\text{Re}} \tag{6.51}$$

where, $\text{Re} = \dfrac{\rho v_{z,\text{av}} D}{\mu}$ is the Reynolds number. The skin friction coefficient C_f is called the Fanning's friction factor.

In engineering applications (hydraulics), the Darcy friction coefficient of pipe flow is often used in the Darcy–Weisbach formula as

$$h_f = \frac{f L v_{z,\text{av}}^2}{2gD} \tag{6.52}$$

where h_f, the loss in pressure head due to friction over a finite length L, is given by

$$h_f = \frac{\Delta p}{\rho g}$$

The pressure drop Δp can be written from Eq. (6.50) as

$$\Delta p = \frac{4L}{D}\,\tau_{\text{at } r=R}$$

Using the above relation in Eq. (6.52), we have

$$f = \frac{4\tau_{\text{at } r=R}}{\frac{1}{2}\rho (v_{z,\text{av}})^2} \tag{6.53a}$$

With the help of Eqs. (6.50) and (6.46a), we can write

$$f = \frac{64}{\text{Re}} \tag{6.53b}$$

From a comparison of Eqs. (6.53b) and (6.51), we have

$$f = 4C_f = \frac{64}{\mathrm{Re}} \tag{6.53c}$$

EXAMPLE 6.1 Oil of specific gravity 0.90 and dynamic viscosity 0.1 Pa·s flows between two fixed plates kept 10 mm apart. If the average velocity of flow is 1.60 m/s, calculate (a) the maximum velocity, (b) the shear stress at the plate, (c) the shear stress at a distance of 2 mm from the plates and the pressure drop in a distance of 2 m.

Solution: (a) From Eq. (6.42c), we have

$$u_{\max} = \frac{3}{2}u_{\mathrm{av}} = \frac{3}{2} \times 1.60 = 2.4 \text{ m/s}$$

(b) From Eq. (6.43a),

$$|\tau|_{\text{at the plates}} = 2\mu\,\frac{u_{\max}}{b} = 2 \times 0.1 \times \frac{2.4}{0.005} = 96 \text{ N/m}^2$$

(c) $|\tau| = \mu \left|\dfrac{\partial u}{\partial y}\right|$

With the help of Eq. (6.41),

$$\tau = y\,\frac{dp}{dx}$$

Again from Eq. (6.42b),

$$\begin{aligned}\frac{dp}{dx} &= -\frac{3\mu}{b^2}\,u_{\mathrm{av}}\\ &= -\frac{3\times 0.1}{(0.005)^2} \times 1.6\\ &= -19.2 \times 10^3 \text{ N/m}^3\end{aligned}$$

At a distance of 0.002 m from the lower plate,

$$\begin{aligned}\tau &= -0.003 \times (19.2 \times 10^3)\\ &= -\ 576 \times 10^5 \text{ N/m}^2\end{aligned}$$

At a distance of 0.002 m from the upper plate

$$\begin{aligned}\tau &= 0.003 \times (19.2 \times 10^3)\\ &= 576 \times 10^3 \text{ N/m}^2\end{aligned}$$

The opposite signs in τ represent the opposite directions. The plus sign is in the direction of flow and the minus sign is in the direction opposite to the flow.

(d) The pressure drop over a distance of 2 m is

$$\Delta p = 19.2 \times 2 \times 10^3 = 38.4 \times 10^3 \text{ N/m}^2$$

EXAMPLE 6.2 A continuous belt (Figure 6.9) passing upwards at a velocity U_0' through a chemical bath picks up a liquid film of thickness h, density ρ and viscosity μ. Gravity tends to make the liquid drain down, but the movement of the belt keeps the fluid from running off completely. Assume that the flow is fully developed and the atmosphere produces no shear stress at the outer surface of the film. Obtain an expression for the velocity profile and the flow rate of the liquid per unit width of the belt.

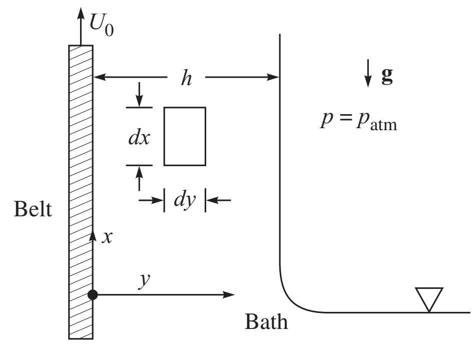


Figure 6.9 The belt and the liquid flow (Example 6.2).

Solution: The governing equation of the present problem can be written as

$$\mu \frac{d^2u}{dy^2} = \rho g$$

$$\mu \frac{du}{dy} = \rho g y + C_1$$

$$u = \frac{\rho g y^2}{2\mu} + \frac{C_1}{\mu} y + C_2$$

At $y = 0$ $\quad u = U_0,\quad$ which gives $C_2 = U_0$.

At $y = h,\quad \tau = 0,\quad$ so $\dfrac{du}{dy} = 0,\quad$ which gives $C_1 = -\rho g h$

Hence, the velocity distribution is given by

$$u = \frac{\rho g y^2}{2\mu} - \frac{\rho g h y}{\mu} + U_0$$

$$= U_0 + \frac{\rho g}{\mu}\left(\frac{y^2}{2} - hy\right)$$

Flow rate per unit width of the belt is given by

$$Q = \int_0^h u\,dy$$

$$= \int_0^h \left[U_0 + \frac{\rho g}{\mu}\left(\frac{y^2}{2} - hy\right)\right] dy$$

$$= U_0 h - \frac{\rho g h^3}{3\mu}$$

EXAMPLE 6.3 Oil of specific gravity 0.90 is discharged at a rate of 3 kg/s under a pressure difference of 10 kN/m^2 over a length of 5 m of a pipe having a diameter of 50 mm. Determine the viscosity of the oil.

Solution:

$$\frac{dp}{dz} = \frac{10 \times 10^3}{5} = 2000 \text{ N/m}^3$$

Flow rate,

$$Q = \frac{3}{0.9 \times 10^3} = 3.33 \times 10^{-3} \text{ m}^3/\text{s}$$

From Eq. (6.48),

$$\mu = \frac{\pi D^4}{128 Q}\frac{dp}{dz}$$

$$= \frac{\pi \times (0.05)^4 \times 2000}{128 \times 3.33 \times 10^{-3}}$$

$$= 0.092 \text{ kg/(m s)}$$

EXAMPLE 6.4 A viscous fluid moves in a steady flow parallel to the axis in the annular space between two coaxial cylinders of radii r_1 and r_2. Consider the flow to be fully developed. Show that the rate of volumetric flow is

$$Q = \frac{\pi r_1^4}{8\mu}\left(\frac{dp}{dz}\right)\left[(\eta^4 - 1) - \frac{(\eta^2 - 1)^2}{\ln(\eta)}\right]$$

where $\eta = r_2/r_1$.

Solution: The flow geometry is shown in Figure 6.10.

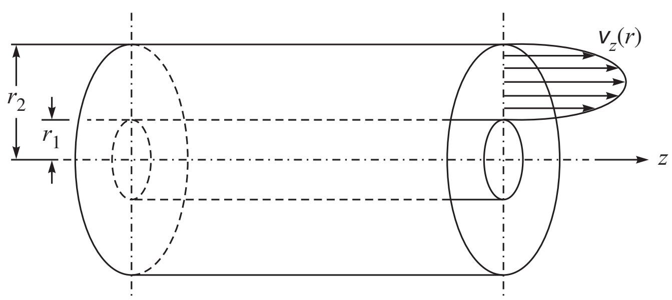


Figure 6.10 Parallel flow through the annulus of two concentric cylinders.

The governing equation of motion for the flow can be written as

$$\frac{d^2 v_z}{dr^2} + \frac{1}{r}\frac{dv_z}{dr} = \frac{dp}{dz}$$

or

$$\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = r\frac{dp}{dz}$$

Since dp/dz is constant, we get upon integration

$$r\frac{dv_z}{dr} = \frac{r}{2}\frac{dp}{dz} + \frac{A}{r}$$

Upon integration again,

$$v_z = \frac{r^2}{4\mu}\frac{dp}{dz} + A\ln r + B$$

$$= \frac{1}{4\mu}\frac{dp}{dz}(r^2 + A_1\ln r + B_1) \quad \text{(i)}$$

$$\text{At } r = r_1, \quad v_z = 0$$
$$\text{At } r = r_2, \quad v_z = 0$$

This gives

$$A_1 = \frac{(1-\eta^2)}{\ln(\eta)} r_1^2$$

$$B_1 = \frac{(\eta^2-1)}{\ln(\eta)} r_1^2 \ln r_1 - r_1^2$$

where $\eta = r_2/r_1$.

Substitution of A_1 and B_1 from the above equations into (i) gives

$$v_z = \frac{1}{4\mu}\left(\frac{dp}{dz}\right)\left[(r^2 - r_1^2) - \frac{(\eta^2-1)}{\ln(\eta)} r_1^2 \ln\frac{r}{r_1}\right]$$

The rate of volumetric flow in this case is

$$Q = 2\pi\int_{r_1}^{r_2} v_z\, r\, dr$$

$$= \frac{\pi}{4\mu}\frac{dp}{dz}\left[\frac{r_1^4}{2} - r_1^2 r^2 - \frac{(\eta^2-1)r_1^2}{\ln(\eta)}\left(\ln\frac{r}{r_1} - \frac{1}{2}\right)r^2\right]_{r_1}^{\eta r_1}$$

$$= \frac{\pi r_1^4}{8\mu}\left(-\frac{dp}{dz}\right)\left[(\eta^4-1) - \frac{(\eta^2-1)^2}{\ln(\eta)}\right]$$

The minus sign in the above expression may be discarded since Q is a scalar quantity

6.3 LAMINAR BOUNDARY LAYER

It has already been mentioned in Chapter 5 that the consequence of fluid surface interaction in momentum transfer in a flow past a solid surface is the development of a thin region close to the surface within which the fluid velocity (relative to the surface) changes from zero at the solid surface to the main stream velocity. The gradient of the velocity component in a direction normal to the surface is large compared to the gradient of this component in the stream-wise direction. The viscous stresses are very important within this region. This region is known as *boundary layer*. Therefore, close to the body is the boundary layer where shear stresses exert an increasingly larger effect on the fluid as one moves from free stream towards the solid boundary. However, outside the boundary layer, the effect of shear stresses on the flow is extremely small compared to values inside the boundary layer, and fluid particles experience no vorticity and hence the flow is similar to a potential flow. The boundary layer interface is, in fact, a fictitious one dividing the viscous boundary layer flow and the irrational free stream flow. In 1904, Ludwig Prandtl, the well known German scientist, introduced the concept of boundary layer. Prandtl's model of the boundary layer is shown in Figure 6.11

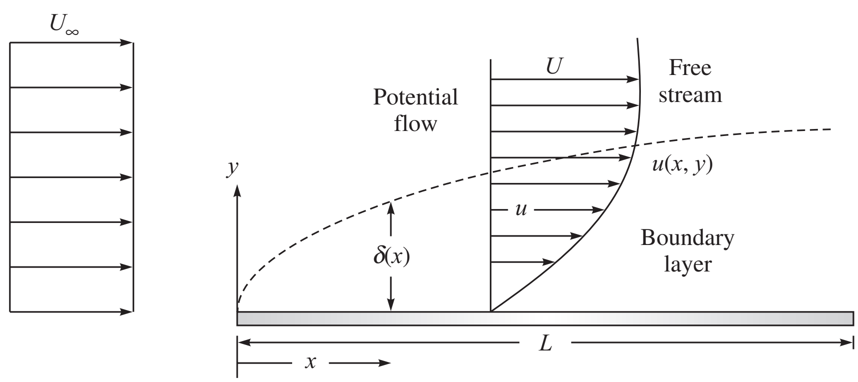


Figure 6.11 Prandtl's model of boundary layer on a flat plate.

According to boundary layer theory, the flow field is divided into two regions

(a) The region of thin boundary layer close to the surface where, due to fluid friction, the flow velocity is retarded from the free stream velocity to zero at the surface. In this region, viscous forces are of same order of magnitude as that of inertia forces.
(b) The region outside the boundary layer where vorticity is zero and viscous forces can be completely neglected in comparison with the inertia forces.

In case of flow with high Reynolds number Re, i.e. flow of fluids having low viscosity or high velocity, the thickness of boundary layer δ is very small compared to the characteristic length of the physical domain. This allows a simplification of Navier–Stokes equations as applied within the boundary layer by discarding a few terms of the equations. This is done by an order-of-magnitude-analysis. The simplified Navier–Stokes equations, i.e. the equations of motion within the boundary layer are known as boundary layer equations.

The flow within the boundary layer may also be of two types—laminar and turbulent, and accordingly the boundary layer is referred to as laminar boundary layer or turbulent boundary layer. We will discuss here the laminar boundary layer. A discussion on turbulent boundary layer will be made in Section 6.5

6.3.1 Derivation of Boundary Layer Equation

Consider a steady two-dimensional laminar incompressible flow. The Navier–Stokes equation together with the equation of continuity in the rectangular Cartesian coordinate system becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{6.54a}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{6.54b}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6.54c}$$

Let us express the above equations in non-dimensional forms. The reference quantities taken for nondimensionalizing the variables are (i) the reference length L and (ii) the reference velocity V_{ref} which is usually the free stream velocity. The dimensionless variables are defined by a star as the superscript. Thus

$$x^* = \frac{x}{L},\quad y^* = \frac{y}{L},\quad u^* = \frac{u}{V_{\text{ref}}},\quad v^* = \frac{v}{V_{\text{ref}}},\quad p^* = \frac{p}{\rho V_{\text{ref}}^2}$$

By making use of the above relations, Eqs. (6.54a) to (6.54c) are reduced to their respective nondimensional forms as

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\text{Re}}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right) \tag{6.55a}$$

$$0(1)\frac{0(1)}{0(1)} + 0(\delta^*)\frac{0(1)}{0(\delta^*)} = -\frac{0(1)}{0(1)} + \frac{1}{\text{Re}}\left(\frac{0(1)}{0(1)} + \frac{0(1)}{0(\delta^{*2})}\right)$$

$$u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{\text{Re}}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right) \tag{6.55b}$$

$$0(1)\frac{0(\delta^*)}{0(1)} + 0(\delta^*)\frac{0(\delta^*)}{0(\delta^*)} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{\text{Re}}\left(\frac{0(\delta^*)}{0(1)} + \frac{0(\delta^*)}{0(\delta^{*2})}\right) \tag{6.55c}$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$\frac{0(1)}{0(1)} + \frac{0(\delta^*)}{0(\delta^*)} = 0$$

We have to make an estimation of order-of-magnitude of each and every term of Eqs. (6.55a) to (6.55c). By order-of-magnitude, we mean the maximum value that a variable can attend. Therefore, we can write

$$0(\partial x^*) = 1,\quad 0(\partial x^{*2}) = 1,\quad 0(y^*) = \delta^*,\ 0(y^{*2}) = \delta^{*2},\quad 0(u^*) = 1,\quad (\partial u^*) = 1,\quad 0(p^*) = 1$$

where $\delta^* = \delta/L$.

From Eq. (6.55c), we find that if the two terms of the equation have to be of the same order, then $0(\partial v^*) = 0(\delta^*)$. Therefore, we have

$$0(\partial v^*) = 0(v^*) = 0(\delta^*)$$

The order-of-magnitude of the different terms of Eqs. (6.55a) to (6.55c) are shown below each term. It is observed that both the terms on left-hand side of Eq. (6.55a) are of order one. On right-hand side of Eq. (6.55a), the term $\partial^2 u^*/\partial x^{*2}$ within the bracket can be neglected in comparison with the term $\partial^2 u^*/\partial y^{*2}$, since $0(\delta^*) << 1$. If the order-of-magnitude of the viscous force has to be the same with that of the inertia force within the boundary layer, then

$$0\left(\frac{1}{\text{Re}}\right) \approx 0(\delta^{*2})$$

which gives

$$\delta^* \sim \frac{1}{\sqrt{\text{Re}}}$$

From Eq. (6.55b), we have

$$\frac{\partial p^*}{\partial y^*} \approx 0(\delta^*)$$

or

$$\Delta p^* \approx 0(\delta^{*2})$$

Since $(\delta^*) << 1$, we can set the above equation as

$$\frac{\partial p}{\partial y} = 0$$

This implies physically that the pressure gradient within the boundary layer in the direction normal to the flow is zero. This means that the pressure in the potential core outside the boundary layer is imprest on the boundary layer. Therefore, we can write $p^* = p^*(x^*)$ within the boundary layer. Now, reverting to the dimensional forms, we can write Eqs. (6.55a) to (6.55c) within the boundary layer as

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} \tag{6.56a}$$

$$\frac{\partial p}{\partial y} = 0 \tag{6.56b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6.56c}$$

Equation (6.56b) is taken care of by writing dp/dx instead of $\partial p/\partial x$ in Eq. (6.56a). Finally, we have

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{dp}{dx} + \mu\frac{\partial^2 u}{\partial y^2} \tag{6.57}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6.58}$$

Equation (6.57) is the two-dimensional boundary layer equation in the Cartesian coordinate system and is known as the Prandtl boundary layer equation.

6.3.2 Solution of Boundary Layer Equation for Flow Over a Flat Plate

The Blasius solution

Let us consider a steady two-dimensional incompressible flow over a flat plate at zero angle of incidence. The flow is extended to infinity in all directions as shown in Figure 6.11. It was H. Blasius who first solved the classical problem from an exact solution of boundary layer equation. Let the origin of the coordinates be at the leading edge of the plate with x-axis being the direction of the uniform stream, and the y-axis, normal to the plate (Figure 6.11).

Equations (6.57) and (6.58) can be written under the situation as

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = \mu\frac{\partial^2 u}{\partial y^2} \tag{6.59a}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \tag{6.59b}$$

The term dp/dx is zero in the present case. This is because we can write by the application of Euler's equation (equation of motion for an inviscid flow) in the potential core outside the boundary layer

$$\rho U_\infty \frac{dU_\infty}{dx} = -\frac{dp}{dx}$$

For the flow over a flat plate, dU_∞/dx and hence, $dp/dx = 0$. The boundary conditions to be satisfied for the solution of Eq. (6.59a) are

$$\text{At} \quad y = 0, \quad u = v = 0 \tag{6.60a}$$

$$\text{At} \quad y \to \infty, \quad u = U_\infty \tag{6.60b}$$

The characteristic parameters in this problem are U_∞, ν, x, y, which means that the flow velocity u depends on these four parameters and we can write

$$u = u(U_\infty, \nu, x, y)$$

By the principle of dimensional analysis, we can show that

$$\frac{u}{U_\infty} = F\left(\frac{y}{\sqrt{\nu x/U_\infty}}\right)$$

$$= F(\eta) \tag{6.61a}$$

where

$$\eta = \frac{y}{\sqrt{\nu x/U_\infty}} \tag{6.61b}$$

Equation (6.61a) implies that if u is scaled by U_∞ and y by the term $\sqrt{\nu x/U_\infty}$, the velocity profiles at all values of x become the same if they are expressed by the nondimensional variable η. In other words, the velocity profiles at all values of x will be concurrent, if they are plotted in coordinates u/U_∞ and η.

This is known as the principle of similarity in boundary layer. The stream function can now be obtained from Eq. (6.61a) as

$$\psi = \int u dy = \sqrt{U_\infty \nu x} \int F(\eta)\, d\eta = \sqrt{U_\infty \nu x}\, f(\eta) \tag{6.62a}$$

where $\int F(\eta)\, d\eta = f(\eta)$ and the constant of integration is zero if the stream function at the solid surface is set to zero.

Now, the velocity components and their derivatives are

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = U_\infty f'(\eta) \tag{6.62b}$$

$$v = -\frac{\partial \psi}{\partial x} = -\sqrt{U_\infty \nu}\left[\frac{1}{2} \cdot \frac{1}{\sqrt{x}} f(\eta) + \sqrt{x} f'(\eta)\left\{-\frac{1}{2} \frac{y}{\sqrt{\nu x/U_\infty}} \frac{1}{x}\right\}\right]$$

or

$$v = \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\, [\eta f'(\eta) - f(\eta)] \tag{6.62c}$$

$$\frac{\partial u}{\partial x} = U_\infty f''(\eta) \frac{\partial \eta}{\partial x} = U_\infty f''(\eta) \cdot \left[-\frac{1}{2} \frac{y}{\sqrt{\nu x/U_\infty}} \frac{1}{x}\right]$$

or

$$\frac{\partial u}{\partial x} = -\frac{U_\infty}{2} \cdot \frac{\eta}{x} f''(\eta) \tag{6.62d}$$

$$\frac{\partial u}{\partial y} = U_\infty f''(\eta) \cdot \frac{\partial \eta}{\partial y} = U_\infty f''(\eta) \cdot \left[\frac{1}{\sqrt{\nu x/U_\infty}}\right]$$

or

$$\frac{\partial u}{\partial y} = U_\infty \sqrt{\frac{U_\infty}{\nu x}}\, f''(\eta) \tag{6.62e}$$

$$\frac{\partial^2 u}{\partial y^2} = U_\infty \sqrt{\frac{U_\infty}{\nu x}}\, f'''(\eta)\left\{\frac{1}{\sqrt{\nu x/U_\infty}}\right\}$$

or

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{\nu x} f'''(\eta) \tag{6.62f}$$

Substituting Eqs. (6.62b) to (6.62f) in Eq. (6.59a), we have

$$-\frac{U_\infty^2}{2} \cdot \frac{\eta}{x} \cdot f'(\eta)\, f''(\eta) + \frac{U_\infty^2}{2x}\, [\eta f'(\eta) - f(\eta)]\, f''(\eta) = \frac{U_\infty^2}{x} f'''(\eta)$$

or

$$-\frac{U_\infty^2}{2x} \cdot f(\eta) f''(\eta) = \frac{U_\infty^2}{2x} f'''(\eta)$$

or

$$2f'''(\eta) + f(\eta) f''(\eta) = 0 \tag{6.63}$$

Equation (6.63) is known as Blasius equation. The boundary conditions for its solution are:

$$\text{At} \quad \eta = 0, \quad f(\eta) = 0, \quad f'(\eta) = 0 \tag{6.64a}$$

$$\text{At} \quad \eta = \infty, \quad f'(\eta) = 1 \tag{6.64b}$$

Blasius obtained the solution of Eq. (6.63) in the form of series expansion through analytical techniques which is beyond the scope of the text. However, a numerical solution of Eq. (6.63) is relatively simple and the interested readers can consult the text by Som and Biswas.[1] The values of f, f' and f'' for some values of η are shown in Table 6.1 and Figure 6.12.

Table 6.1 Values of f, f' and f'' after Schlichting[2]

η	f	f'	f''
0	0	0	0.33206
0.2	0.00664	0.06641	0.33199
0.4	0.02656	0.13277	0.33147
0.8	0.10611	0.26471	0.32739
1.2	0.23795	0.39378	0.31659
1.6	0.42032	0.51676	0.29667
2.0	0.65003	0.62977	0.26675
2.4	0.92230	0.72899	0.22809
2.8	1.23099	0.81152	0.18401
3.2	1.56911	0.87609	0.13913
3.6	1.92954	0.92333	0.09809
4.0	2.30576	0.95552	0.06424
4.4	2.69238	0.97587	0. 03897
4.8	3.08534	0.98779	0.02187
5.0	3.28329	0.99155	0.01591
8.8	7.07923	1.00000	0.00000

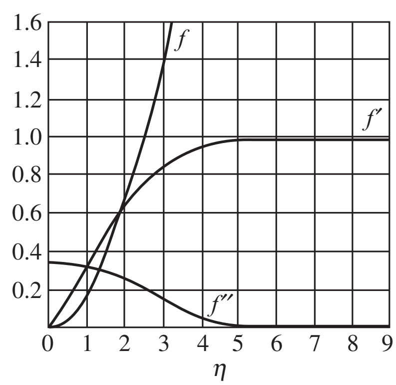


Figure 6.12 The variations of f, f' and f'' with η.

[1] Som, S.K., and G. Biswas, *Introduction to Fluid Mechanics and Fluid Machines*, 2nd ed, Tata McGraw-Hill, New Delhi, 2004

[2] Schlichting, H., *Boundary Layer Theory*, 7th ed., McGraw-Hill, New York, 1987.

Wall shear stress and boundary layer thickness

The shear stress at the plate surface can be calculated from the results of the Blasius solution, i.e.

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$= \mu U_\infty \frac{\partial}{\partial \eta}[f'(\eta)]\left(\frac{\partial \eta}{\partial y}\right)_{\eta=0}$$

$$= \mu U_\infty \times 0.33206 \times \frac{1}{\sqrt{\nu x/U_\infty}} \qquad (\because f''(0) = 0.33206 \text{ from Table 6.1})$$

Therefore,

$$\tau_w = \frac{0.332\rho U_\infty^2}{\sqrt{\mathrm{Re}_x}}$$

The local skin friction coefficient becomes

$$C_{f_x} = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} = \frac{0.664}{\sqrt{\mathrm{Re}_x}} \tag{6.65}$$

The total frictional force per unit width for one side of the plate of length L is given by

$$F\int_0^L \tau_w\, dx = 0.664\, \rho U_\infty^2 \sqrt{\frac{\nu L}{U_\infty}}$$

The average skin friction coefficient over the length L is given by

$$\overline{C}_{f_L} = \frac{F}{\frac{1}{2}\rho U_\infty^2 L} = \frac{1.328}{\sqrt{\mathrm{Re}_L}} \tag{6.66}$$

According to boundary condition (Eq. (6.60b) or (6.64b)), the velocity in the boundary layer does not reach the value of free-stream velocity until $y \to \infty$. Therefore, it is not possible, from this theory, to predict an exact boundary layer thickness. However, after a certain finite value of η, the velocity in the boundary layer asymptotically blends into the free-stream velocity. If an arbitrary limit of $u/U_\infty = 0.99$ is taken to define the boundary layer thickness, we can then write using Table 6.1

$$\frac{\delta}{\sqrt{\nu x / U_\infty}} = 5.0$$

or

$$\frac{\delta}{x} = \frac{5.0}{\sqrt{\mathrm{Re}_x}} \tag{6.67}$$

Since the above stated definition of boundary layer thickness is somewhat arbitrary, a more physically meaningful measure of boundary layer thickness is expressed by displacement thickness δ^* as

$$U_\infty \delta^* = \int_0^\infty (U_\infty - u)dy$$

or

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right)dy \tag{6.68}$$

The above mathematical definition of δ^* implies physically that the displacement thickness is the distance by which the external potential flow is displaced outwards due to the decrease in velocity in the boundary layer.

Substituting the values of u/U_∞ and η from Eqs. (6.62b) and (6.61b) respectively in Eq. (6.68), we have

$$\delta^* = \sqrt{\frac{\nu x}{U_\infty}} \int_0^\infty (1 - f')\,d\eta = \sqrt{\frac{\nu x}{U_\infty}} \lim_{n\to\infty} [\eta - f(n)]$$

$$= 1.7208 \sqrt{\frac{\nu x}{U_\infty}}$$

or

$$\frac{\delta^*}{x} = \frac{1.7208}{\sqrt{\mathrm{Re}_x}} \tag{6.69}$$

Following the analogy of displacement thickness, a momentum thickness δ^{**} may be defined in consideration of the loss of momentum in the boundary layer compared with that of potential flow. Thus

$$\rho U_\infty^2 \delta^{**} = \int_0^\infty \rho u (U_\infty - u) dy$$

or

$$\delta^{**} = \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy \tag{6.70}$$

Substituting the values of u/U_∞ and η from Eqs. (6.62b) and (6.61b) respectively in Eq. (6.70), we have

$$\delta^{**} = \sqrt{\frac{\nu x}{U_\infty}} \int_0^\infty f'(1 - f')\,d\eta$$

$$= 0.664 \sqrt{\frac{\nu x}{U_\infty}}$$

or

$$\frac{\delta^{**}}{x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}} \tag{6.71}$$

Momentum integral equation in boundary layer

An approximate but relatively simpler method of solution of boundary layer equation in engineering applications was proposed by Karman and Pohlhausen. The first step of this method is to derive an integral equation of the boundary layer. For this purpose, we define a finite thickness of the boundary layer δ at which u becomes U_∞. Then each and every term of the boundary layer equation (Eq. (6.57)) is integrated with respect to y from $y = 0$ to $y = \delta(x)$. Thus, we have

$$\int_0^\delta u \frac{\partial u}{\partial x} dy + \int_0^\delta v \frac{\partial u}{\partial y} dy = -\frac{1}{\rho}\int_0^\delta \frac{dp}{dx} dy + \frac{\mu}{\rho}\int_0^\delta \frac{\partial^2 u}{\partial y^2} dy \tag{6.72}$$

The second term on the left-hand side of Eq. (6.72) can be written as

$$\int_0^\delta v \frac{\partial u}{\partial y} dy = \int_0^\delta \frac{\partial}{\partial y}(uv)dy - \int_0^\delta u \frac{\partial v}{\partial y} dy$$

$$= [uv]_0^\delta - \int_0^\delta u \frac{\partial v}{\partial y} dy$$

or
$$\int_0^\delta v\frac{\partial u}{\partial y}dy = U_\infty(v)_\delta - \int_0^\delta u\frac{\partial v}{\partial y}dy \tag{6.73}$$

From continuity,
$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

or
$$\int_0^\delta \frac{\partial v}{\partial y}dy = -\int_0^\delta \frac{\partial u}{\partial x}dy$$

or
$$(v)_\delta = -\int_0^\delta \frac{\partial u}{\partial x}dy$$

With the above relations, Eq. (6.73) becomes
$$\int_0^\delta v\frac{\partial u}{\partial y}dy = -U_\infty\int_0^\delta \frac{\partial u}{\partial x}dy + \int_0^\delta u\frac{\partial u}{\partial x}dy$$

Therefore, Eq. (6.72) becomes
$$\int_0^\delta 2u\frac{\partial u}{\partial x}dy - U_\infty\int_0^\delta \frac{\partial u}{\partial x}dy = -\frac{1}{\rho}\frac{dp}{dx}\delta - \frac{\mu}{\rho}\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

or
$$\int_0^\delta \frac{\partial}{\partial x}(u^2)dy - U_\infty\int_0^\delta \frac{\partial u}{\partial x}dy = -\frac{1}{\rho}\frac{dp}{dx}\delta - \frac{\tau_w}{\rho} \tag{6.74}$$

Since δ is a function of x, the first and second terms of Eq. (6.74) can be written as
$$\int_0^\delta \frac{\partial}{\partial x}(u^2)dy = \frac{d}{dx}\int_0^\delta (u^2)dy - U_\infty^2\frac{d\delta}{dx}$$
$$U_\infty\int_0^\delta \frac{\partial u}{\partial x}dy = U_\infty\frac{d}{dx}\int_0^\delta u dy - U_\infty^2\frac{d\delta}{dx}$$

Hence Eq. (6.74) reduces to
$$\frac{d}{dx}\int_0^\delta u^2 dy - U_\infty\frac{d}{dx}\int_0^\delta u dy = -\frac{1}{\rho}\frac{dp}{dx}\delta - \frac{\tau_w}{\rho} \tag{6.75}$$

Equation (6.75) is called the momentum integral equation of the boundary layer and is also known as Karman integral equation.

The momentum integral equation can be written in terms of the displacement and momentum thickness. Equation (6.75) is rearranged as
$$\frac{d}{dx}\int_0^\delta u^2\,dy - U_\infty\frac{d}{dx}\int_0^\delta u dy = U_\infty\frac{dU_\infty}{dx}\delta - \frac{\tau_w}{\rho}$$
$$\left(\text{By the application of Euler's equation in free stream flow, } U_\infty\frac{dU_\infty}{dx} = -\frac{1}{\rho}\frac{dp}{dx}\right)$$

or
$$\frac{d}{dx}\int_0^\delta u^2\,dy - \frac{d}{dx}\int_0^\delta (U_\infty u)dy + \frac{dU_\infty}{dx}\int_0^\delta u dy - \frac{dU_\infty}{dx}\int_0^\delta U_\infty\,dy = -\frac{\tau_w}{\rho}$$

or
$$\frac{d}{dx}\int_0^{\delta} u(U_\infty - u)\,dy + \frac{dU_\infty}{dx}\int_0^{\delta}(U_\infty - u)\,dy = \frac{\tau_w}{\rho} \tag{6.76}$$

Since the integrals vanish outside the boundary layer, we can put the upper limit of integration as ∞. Thus, we can write Eq. (6.76) as

$$\frac{d}{dx}\int_0^{\infty} u(U_\infty - u)\,dy + \frac{dU_\infty}{dx}\,\delta\int_0^{\infty}(U_\infty - u)\,dy = \frac{\tau_w}{\rho} \tag{6.77}$$

If the displacement and momentum thickness are introduced from Eqs. (6.68) and (6.70) into Eq. (6.77), we obtain

$$\frac{d}{dx}(U_\infty^2\delta^{**}) + U_\infty\frac{dU_\infty}{dx}\delta^* = \frac{\tau_w}{\rho}$$

Solution of momentum integral equation—Karman Pohlhausen method

We will discuss the method for flow over a flat plate. In this case $\frac{dp}{dx} = \frac{dU_\infty}{dx} = 0$, and hence Eq. (6.77) becomes

$$\frac{d}{dx}\int_0^{\delta} u(U_\infty - u)\,dy = \frac{\mu}{\rho}\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{6.78}$$

The first step of the method is to assume a velocity profile within the boundary layer which satisfies all the requisite boundary conditions. Polhausen introduced a polynomial of third degree for the velocity function in terms of the nondimensional coordinate η (= y/δ) as

$$\frac{u}{U_\infty} = f(\eta) = a + b\eta + c\eta^2 + d\eta^3$$

The boundary conditions to be satisfied are:

$$\eta = 0, \qquad u = 0, \qquad \frac{\partial^2 u}{\partial y^2} = 0$$

$$\eta = 1, \qquad u = U_\infty, \qquad \frac{\partial u}{\partial y} = 0$$

From the above boundary conditions, we get

$$a = c = 0$$

$$b = \frac{3}{2}, \qquad d = -\frac{1}{2}$$

Therefore the velocity profile becomes

$$\frac{u}{U_\infty} = \frac{3}{2}\eta - \frac{1}{2}\eta^3 \tag{6.79}$$

Now we write Eq. (6.78) as

$$U_\infty^2\frac{d}{dx}\int_0^{\delta}\frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)dy = \frac{\mu}{\rho}\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{6.80}$$

Substituting the values of u/U_∞ from Eq. (6.79) and using $dy = \delta d\eta$, we have from Eq. (6.80)

$$U_\infty^2 \frac{d}{dx}\int_0^1 \left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)\left(1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3\right)\delta d\eta = \frac{\mu}{\rho}\left(\frac{3}{2\delta}\right)U_\infty$$

Performing the integration and after same rearrangements, we have

$$\frac{39\delta}{280}\frac{d\delta}{dx} = \frac{3\mu U_\infty}{2\rho U_\infty^2} \tag{6.81}$$

or
$$\int \delta d\delta = \frac{140}{13}\frac{\mu}{\rho U_\infty}\int dx + C_1$$

or
$$\frac{\delta^2}{2} = \frac{140}{13}\frac{\nu x}{U_\infty} + C_1$$

where C_1 is an arbitrary unknown constant. From the condition at $x = 0$ (the leading edge), $\delta = 0$, we have $C_1 = 0$. Therefore,

$$\delta^2 = \frac{280}{13}\frac{\nu x}{U_\infty} \tag{6.82}$$

or
$$\frac{\delta}{x} = \frac{4.64}{\sqrt{\mathrm{Re}_x}} \tag{6.83}$$

Though this method is an approximate one, the result is reasonably accurate. The value is slightly lower than that found from the exact solution given by Eq. (6.67).

6.4 BOUNDARY LAYER SEPARATION

Under certain conditions, the flow in the vicinity of the wall reverses its direction from that of the main stream. This flow reversal is known as *boundary layer separation*. We know that flow of a fluid takes place by virtue of the gradient of mechanical energy. In a flow with an adverse pressure gradient, i.e. when the pressure increases in the direction of flow ($dp/dx > 0$), the fluid particles in the main stream surmount the pressure hill due to their high kinetic energy which creates a favourable energy gradient for the flow in forward direction. But the fluid particles near the wall, because of their excessive loss in momentum due to fluid friction, become unable to move against the higher downstream pressure and follow the reverse path. Under the situation, the total mechanical energy of fluid particles in the vicinity of the wall is higher at the downstream compared to that at upstream.

Therefore for a boundary layer separation to take place, there has to be an adverse pressure gradient ($dp/dx > 0$) in the flow. In case of a parallel flow over a flat plate, $dp/dx = 0$, and hence, there is no chance of boundary layer separation. For an external flow, the pressure gradient in the direction of flow is caused by the geometrical shape of the body.

Let us consider the flow past a circular cylinder so that the direction of flow is perpendicular to the axis of the cylinder (Figure 6.13). Up to $\theta = 90°$, the flow area is like a converging passage, and hence the flow behaviour is similar to that in a nozzle. Beyond $\theta = 90°$, the flow area is diverged and the flow behaviour is much similar to that in a diffuser. Therefore, if we consider the flow of an ideal fluid, we will get a pressure distribution on the cylinder as shown by the firm line in Figure 6.13. Here p_∞ and U_∞ are the pressure and velocity in the free stream while p is the local pressure on the cylinder. In case of viscous flow, the influence of adverse pressure gradient beyond $\theta = 90°$ causes a flow separation as discussed in the earlier paragraph. Depending upon the magnitude of adverse pressure gradient, somewhere around $\theta = 90°$, the fluid particles, in the boundary layer are separated from the wall and driven in the upstream direction. However, the far field external stream pushes back these separated layers together with it and develops a broad pulsating wave behind the cylinder. This creates fluid eddies with rotational motion. The kinetic energy of rotational fluid eddies cannot be convected back to fluid pressure, rather it is dissipated into intermolecular energy by fluid friction. The actual pressure distribution is shown by the dotted line in Figure 6.13.

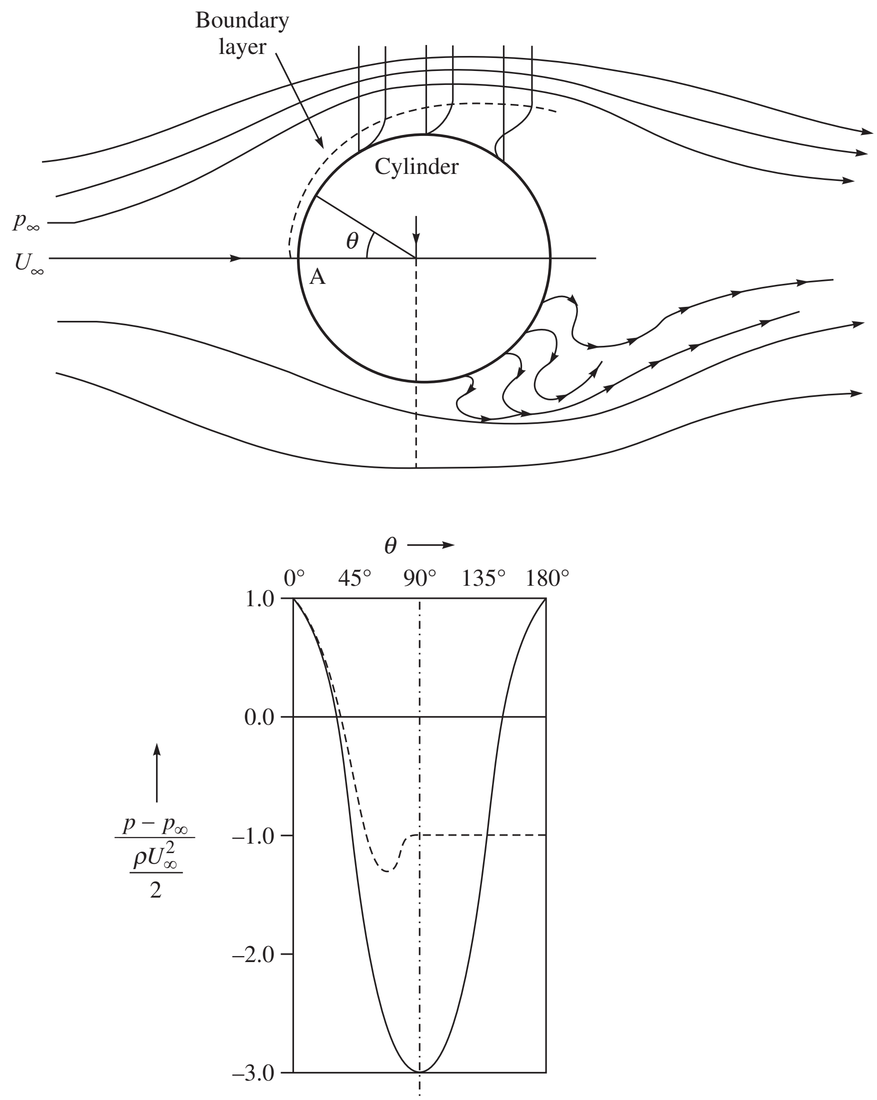


Figure 6.13 The pressure distribution in flow past a circular cylinder.

Since the wake zone pressure is less than that of the forward stagnation point (point A in Figure 6.13), the cylinder experiences a drag force in the direction of flow which is solely attributed to the pressure difference. This drag force is known as *form drag* whereas the shear stress at the wall gives rise to *skin friction drag*. These two drag forces together constitute the total drag on a body.

The point of separation may be defined as the limit between the forward and the reverse flow in the layer very close to the wall. This can be expressed mathematically that, at the point of separation,

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$$

This also indicates that the wall shear stress at the point of separation is zero. From a description of velocity profile with the aid of boundary layer equation, it can be shown that for separation to take place, there has to be a point of inflection in the velocity profile. If we write the boundary layer equation (Eq. (6.57)) at the wall, we have

$$\left(\frac{\partial^2 u}{\partial^2 y}\right)_{y=0} = \frac{1}{\mu}\frac{dp}{dx} \tag{6.84}$$

The velocity within the boundary layer increases from zero at the wall and reaches the value U_∞ asymptotically. This requires that $\partial u/\partial y$ should reach zero asymptotically from a positive value while $\partial^2 u/\partial y^2$ should reach zero asymptotically from a negative value. In case of flow without a separation, $dp/dx < 0$. Therefore from Eq. (6.84), $(\partial^2 u/\partial y^2)_{y=0} < 0$. Again $(\partial u/\partial y)_{y=0} > 0$ and the velocity u increases in a way that $\partial u/\partial y$ decreases at a continuously lesser rate in the y-direction. This implies that $\partial^2 u/\partial y^2$ always remains less than zero and hence the above stated condition is satisfied without a point of inflection as shown in Figure 6.14(a). In case of separation, there is an adverse pressure gradient, i.e. $dp/dx > 0$, we then have from Eq. (6.84), $(\partial^2 u/\partial y^2)_{y=0} > 0$. To satisfy the above stated condition near the boundary layer interface, $\partial^2 u/\partial y^2$ has to be zero at some location which implies a point of inflection (PI) in the velocity profile. This is shown in Figure 6.14(b).

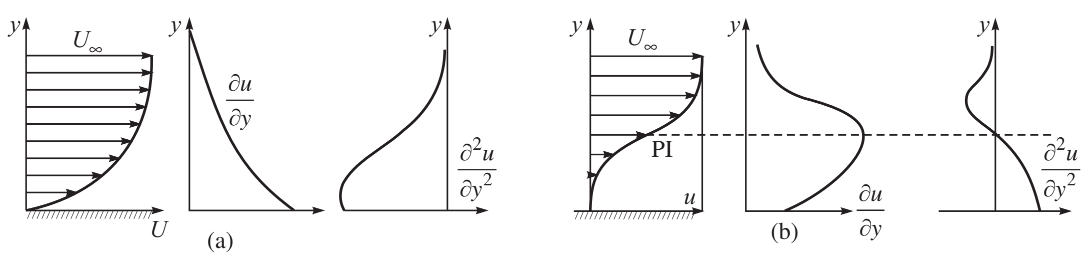


Figure 6.14 Velocity distribution within a boundary layer: (a) Favourable pressure gradient $dp/dx < 0$ and (b) adverse pressure gradient $dp/dx > 0$.

6.4.1 Concept of Entrance Region and Fully-developed Flow in a Duct

Let us consider a flow entering a pipe with uniform velocity (Figure 6.15). The growth of boundary layer on the wall takes place at the entrance of the tube. The boundary layer thickness goes on increasing in the direction of the flow and finally the boundary layers from the wall meet the axis of the pipe. The velocity profile is nearly rectangular at the entrance and it gradually changes to a parabolic one where the boundary layer meets the axis. The region in which the boundary layer develops, and its thickness is less than the pipe radius, is known as *entrance region* and the flow in this region is termed *developing flow*. The region, after the boundary layers from the wall, that meets the pipe axis and where the velocity profile becomes parabolic is known as *fully developed region* and the flow in this region is termed *fully developed flow*.

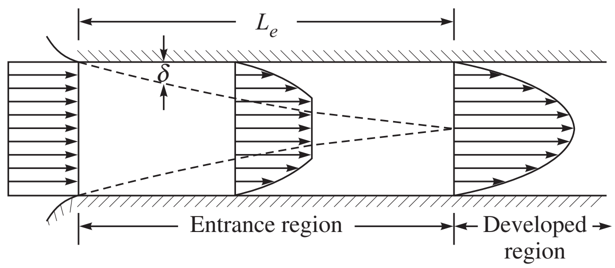


Figure 6.15 Growth of boundary layer in the entrance region of a duct.

In the entrance region, there prevails a potential core where the flow is uninfluenced by viscosity. The boundary layer becomes thicker in the direction of flow and hence the region of potential core is reduced and ultimately vanishes when the flow becomes fully developed. Since the fluid is retarded within the boundary layer which grows on in the direction of flow, the velocity in the diminishing potential core has to increase to keep the volume flow rate the same at every section (a consequence of continuity). Thus, the entrance region refers to a boundary layer flow with an accelerating potential core. In the entrance region, the velocity gradient is steeper at the wall causing a higher value of wall shear stress and consequently a higher pressure drop over a given length compared to that in the fully developed flow. In the fully developed flow, the entire flow region at any section is influenced by the viscosity. The velocity profile becomes parabolic and does not change along the direction of flow and for this, we can write $\partial u/\partial x = 0$ for a fully developed flow. The length from the entrance to the pipe at which the flow becomes fully developed is known as *entrance length* L_e. It can be shown that for laminar incompressible flows,

$$\frac{L_e}{D} = 0.05\text{Re} \qquad \text{where Re} = \frac{\rho U_{\text{av}} D}{\mu}$$

Here, D is the pipe diameter and U_{av} is the average velocity of flow which is defined as the ratio of volumetric flow rate to the cross-sectional area of the pipe.

EXAMPLE 6.5 Verify whether the following velocity profile satisfies all the boundary conditions in a laminar boundary layer flow over a flat plate.

$$\frac{u}{U_\infty} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

Solution:

$$\frac{\partial u}{\partial y} = \frac{U_\infty \pi}{2\delta}\cos\left(\frac{\pi y}{2\delta}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty \pi^2}{4\delta^2}\sin\left(\frac{\pi y}{2\delta}\right)$$

At $y = 0$,

$$u = U_\infty \sin\left(\frac{\pi}{2\delta}\times 0\right) = 0$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty \pi^2}{4\delta^2}\sin(0) = 0$$

At $y = \delta$,

$$u = U_\infty \sin\left(\frac{\pi}{2}\right) = U_\infty$$

$$\frac{\partial u}{\partial y} = \frac{U_\infty \pi}{2\delta}\cos\left(\frac{\pi}{2}\right) = 0$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty \pi^2}{4\delta^2}\sin\left(\frac{\pi}{2}\right) = \frac{U_\infty \pi^2}{4\delta^2}$$

Therefore, it is found that all boundary conditions, except the curvature condition at the boundary layer interface, i.e. at $y = \delta$, $\dfrac{\partial^2 u}{\partial y^2} = 0$, are satisfied by the given velocity distribution.

EXAMPLE 6.6 An approximate expression for the velocity profile in a steady, two-dimensional incompressible boundary layer is

$$\frac{u}{U_\infty} = 1 - e^{-\eta} + k\left(1 - e^{-\eta} - \sin\frac{\pi\eta}{6}\right) \qquad \text{for } 0 \le \eta \le 3$$

$$= 1 - e^{-\eta} - ke^{-\eta} \qquad \text{for } \eta \ge 3$$

where $\eta = y/\delta(x)$. Show that the profile satisfies the following boundary conditions:

At $y = 0$, $u = 0$,

At $y \to \infty$, $u = U_\infty$, $\dfrac{\partial u}{\partial y} = \dfrac{\partial^2 u}{\partial y^2} = 0$

Also find out k from an appropriate boundary condition.

Solution: At $y = 0$, $\eta = 0$,

and $u = U_\infty[1 - e^0 + k(1 - e^0 - \sin 0)]$

$= U_\infty \times 0 = 0$

When $y \to \infty$, $\eta \to \infty$

then, $u = U_\infty(1 - e^{-\infty} - ke^{-\infty})$

$$= U_\infty(1 - 0 - 0)$$
$$= U_\infty$$

For $\eta \geq 3$

$$\frac{\partial u}{\partial y} = U_\infty(e^{-\eta} + ke^{-\eta})\cdot\frac{1}{\delta}$$

$$\frac{\partial^2 u}{\partial y^2} = U_\infty(-e^{-\eta} - ke^{-\eta})\cdot\frac{1}{\delta^2}$$

For $$y \to \infty$$

$$\frac{\partial u}{\partial y} = 0$$

and $$\frac{\partial^2 u}{\partial y^2} = 0$$

To find the value of k, we use the boundary condition:

$$\text{At} \quad y = 0, \quad \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{dp}{dx}$$

From Euler's equation in the potential core, we have

$$\frac{dp}{dx} = -\rho U_\infty \frac{dU_\infty}{dx}$$

Hence we can write the boundary condition as:

At $$y = 0, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{1}{\nu}\cdot U_\infty \frac{dU_\infty}{dx}$$

For the given profile,

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = -\frac{U_\infty}{\delta^2}(1+k)$$

Hence,

$$-\frac{U_\infty}{\delta^2}(1+k) = \frac{1}{\nu}\cdot U_\infty \frac{dU_\infty}{dx}$$

which gives

$$k = \frac{\delta^2}{\nu}\cdot\frac{dU_\infty}{dx} - 1$$

EXAMPLE 6.7 A flat plate 1.2 m wide and of length L is kept parallel to a uniform stream of air of velocity 3.0 m/s in a wind tunnel. If it is desired to have a laminar boundary layer only on the plate, what is the maximum length L (in metre) of the plate? For the maximum length of the plate, determine the drag force on one side of the plate. (Assume that the laminar flow exists up to a Reynolds number of $\mathrm{Re}_x = 5 \times 10^5$. Take $\rho_{\mathrm{air}} = 1.2$ kg/m^3 and $\nu_{\mathrm{air}} = 1.5 \times 10^{-5}$ m^2/s.)

Solution: For maximum length of the plate,

$$\mathrm{Re}_L = \frac{U_\infty L}{\nu} = 5 \times 10^5$$

which gives

$$L = \frac{(5 \times 10^5)(1.5 \times 10^{-5})}{3.0} = 2.5 \text{ m}$$

From Eq. (6.66), we have

$$\overline{C}_{f_L} = \frac{1.328}{(\mathrm{Re}_L)^{1/2}} = \frac{1.328}{(5 \times 10^5)^{1/2}} = 1.878 \times 10^{-3}$$

Drag force on one side of the plate,

$$F_D = \overline{C}_{f_L} \left(\frac{1}{2}\rho U_\infty^2\right) \times B \times L$$

$$= 1.878 \times 10^{-3} \left(\frac{1}{2} \times 1.2 \times 9\right) \times 1.2 \times 2.5$$

$$= 0.03 \text{ N}$$

EXAMPLE 6.8 A smooth flat rectangular plate is placed edgewise in a stream of fluid. At what fraction of the length from the leading edge would the drag force on the front portion be equal to half of the total drag force? Assume the boundary layer to be laminar.

Solution: Let L be the length of the plate and x be the distance from the leading edge such that the drag force over the distance x is half of the total drag force over the entire length L of the plate, i.e.

$$F_{D_x} = \frac{1}{2} F_{D_L}$$

Again,

$$F_{D_x} = \overline{C}_{f_x} (B \cdot x) \frac{\rho U_\infty^2}{2}$$

$$F_{D_L} = \overline{C}_{f_L} (B \cdot L) \frac{\rho U_\infty^2}{2}$$

where L and B are respectively the length and breadth of the plate, and U_∞ is the free stream velocity. Now,

$$\frac{F_{D_x}}{F_{D_L}} = \frac{\overline{C}_{f_x}}{\overline{C}_{f_L}} \cdot \frac{x}{L} = \frac{1}{2}$$

We know from Eq. (6.66),

$$\overline{C}_{f_x} = \frac{1.328}{\sqrt{U_\infty x/\nu}}$$

$$\overline{C}_{f_L} = \frac{1.328}{\sqrt{U_\infty L/\nu}}$$

Hence
$$\frac{\overline{C}_{f_x}}{\overline{C}_{f_L}} = \sqrt{\frac{L}{x}}$$

Therefore, we can write
$$\left(\frac{L}{x}\right)^{1/2} \frac{x}{L} = \frac{1}{2}$$

or
$$\left(\frac{x}{L}\right)^{1/2} = \frac{1}{2}$$

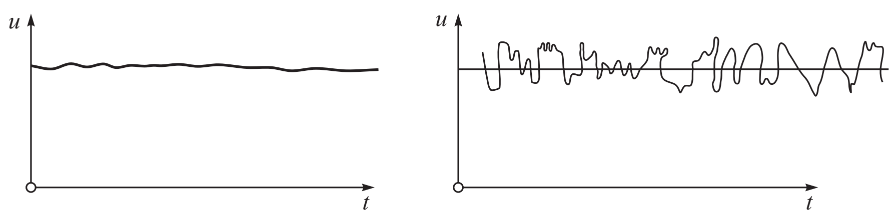
or
$$x = \frac{1}{4} L$$

6.5 A BRIEF INTRODUCTION TO TURBULENT FLOWS

The definition of turbulent flow and its main difference from laminar flow has been stated earlier in Section 6.1.3.

6.5.1 Characteristics of Turbulent Flow

The most important characteristic of a turbulent flow is that the velocity and pressure at a point fluctuate with time in a random manner. This is not so in case of a laminar flow. This is shown in Figure 6.16.



(a) Laminar flow (b) Turbulent flow

Figure 6.16 Variation of a velocity component u at a point with time for a steady laminar and turbulent flow.

It was Osborn Reynolds who first conducted experiments in 1883 with pipe flow by feeding into a stream a thin thread of liquid dye. For low Reynolds number, the dye traced a straight line and did not disperse. With increasing velocity, the dye thread got mixed in all directions and the flowing fluid appeared to be uniformly coloured in the downstream. It was thought that on the main motion in the direction of the axis, there existed a superimposed motion all along the main motion at right angles to it. This leads to the concept of decomposition of the velocity components in a turbulent flow into a time-average velocity component and a fluctuating velocity component. In the rectangular Cartesian coordinate system, we can write

$$u = \overline{u} + u'$$
$$v = \overline{v} + v'$$
$$w = \overline{w} + w'$$

where the quantities with bar represent the time average values over a finite time interval which is larger than the time scale of turbulence and smaller than any slow variation in the flow field which is not chaotic or turbulent in nature. The following rules apply to the fluctuating velocity components u', v', and w' in a turbulent flow:

$$\overline{u}' = \overline{v}' = \overline{w}' = 0$$

$$\overline{u'^2} \neq 0, \qquad \overline{v'^2} \neq 0, \qquad \overline{w'^2} \neq 0$$

$$\overline{u'v'} \neq 0, \qquad \overline{v'w'} \neq 0, \qquad \overline{u'w'} \neq 0$$

For the time averages of the spatial gradients of fluctuating components,

$$\overline{\frac{\partial u'}{\partial s}} = \overline{\frac{\partial^2 u'}{\partial s^2}} = 0$$

$$\overline{\frac{\partial v'}{\partial s}} = \overline{\frac{\partial^2 v'}{\partial s^2}} = 0$$

$$\overline{\frac{\partial w'}{\partial s}} = \overline{\frac{\partial^2 w'}{\partial s^2}} = 0$$

$$\overline{\frac{\partial (u'v')}{\partial s}} \neq 0, \qquad \overline{\frac{\partial (v'w')}{\partial s}} \neq 0, \qquad \overline{\frac{\partial (u'w')}{\partial s}} \neq 0$$

The square root of the average value of the square of the velocity fluctuations, $\sqrt{\overline{u'^2}}$, $\sqrt{\overline{v'^2}}$, $\sqrt{\overline{w'^2}}$, are positive and are measurable quantities. These values allow for the estimation of one of the main parameters of turbulence, namely the intensity. Its value is given by

$$\text{Turbulence intensity} = \sqrt{\frac{1}{3}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)}$$

Turbulence is called *isotropic* if the velocity fluctuations are independent of the axis of reference, i.e. invariant to translation, rotation and reflection of the axis. Hence, for an isotropic turbulence,

$$\sqrt{\overline{u'^2}} = \sqrt{\overline{v'^2}} = \sqrt{\overline{w'^2}}$$

and the turbulence intensity is $\sqrt{\overline{u'^2}}$.

Often the turbulence intensity is expressed as a fraction (in per cent) of the mean flow velocity and is called the *relative turbulence intensity*, $\sqrt{\overline{u'^2}}/\overline{u}$. If the turbulence has the same structure quantitatively in the entire flow field, then the turbulence is called *homogeneous turbulence*. Under the situation, $\sqrt{\overline{u'^2}}$, $\sqrt{\overline{v'^2}}$, and $\sqrt{\overline{w'^2}}$ are each constant over the entire flow field. By definition, an isotropic turbulence is always homogeneous.

Statistical correlations

Another way to describe the structure of turbulent flow is the one based on the statistical time and space correlations between the several velocity fluctuations of any elementary mass of the fluid stream. If u'_0 is the fluctuating velocity component of an elementary mass of the main flow at any instant of time and u'_1 be the same after a time t, we can then write a correlation coefficient between the two fluctuating velocity components in consideration of the movement of the point of reference with the stream as

$$R_t = \frac{\overline{u'_0 u'_1}}{\sqrt{\overline{u'^2_0}}\,\sqrt{\overline{u'^2_1}}}$$

when the bars again refer to the time average values. The quantity R_t is known as the *autocorrelation coefficient* and varies from 1 at $t = 0$ to zero at $t \to \infty$ as shown in Figure 6.17. When $R = 1$, there is complete correlation between u'_0 and u'_1, and when $R = 0$, there exists no correlation between u'_0 and u'_1.

The area under the curve (Figure 6.17), expressed by $\int_0^\infty R_t\,dt$, is a graphical representation of a temporal characteristic of the system. One characteristic quantity of the turbulence structure is the so-called *turbulence scale*, expressed in Lagrange's representation by the relation

$$l_1 = u' \int_0^\infty R_t\,dt \tag{6.85}$$

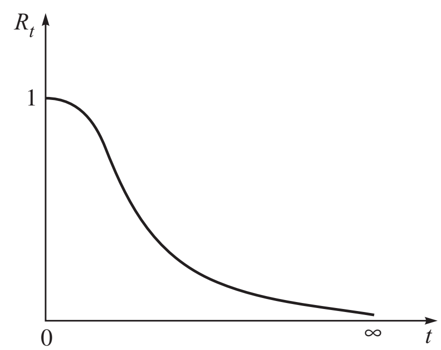


Figure 6.17 Variation of R_t with t.

If instead of considering the fluctuating velocities at two different instants of time, we consider them at the same instant of time but at two different points separated by a distance s in the main flow, a correlation coefficient can be written as

$$R_s = \frac{\overline{u'_0 u'_1}}{\sqrt{\overline{u'^2_0}}\,\sqrt{\overline{u'^2_1}}}$$

and thus the turbulence characteristic quantity called the Euler scale of turbulence will be

$$l_2 = \int_0^\infty R_t\,dt \tag{6.86}$$

The correlation studies reveal that the turbulent motion is composed of eddies which are convected by the mean motion. The turbulence scale is a function of the average size of fluid eddies. The size of the eddies varies over a wide range. The size of the largest eddies is comparable to the dimensions of flow passage. On the other hand the smallest eddies are much larger than the molecular mean free path so that the turbulent flow follows the continuum mechanics. The large eddies are always broken into smaller eddies which are dissipated due to friction in intermolecular energy near the wall. The generation of eddies is being sustained by the motion of the main stream on one hand, and their continuous breaking up into smaller eddies, and finally dissipation into intermolecular energy by fluid friction takes place on the other hand. This comprises the physical picture of a turbulent flow.

6.5.2 Reynolds Modification of the Navier–Stokes Equations for Turbulent Flow

In turbulent flow we are usually interested in time-averaged quantities. Therefore, we make the time average of each and every term of continuity equation and Navier–Stokes equations. We consider a steady incompressible flow in the Cartesian coordinate system.

Continuity equation:
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{6.87}$$

The Navier–Stokes equations (Eqs. (6.34a) to (6.34c)) are written in a little rearranged form with the help of equation of continuity as

$$\rho\left[\frac{\partial}{\partial x}(u^2)+\frac{\partial}{\partial y}(uv)+\frac{\partial}{\partial z}(uw)\right]=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \tag{6.88a}$$

$$\rho\left[\frac{\partial}{\partial x}(uv)+\frac{\partial}{\partial y}(v^2)+\frac{\partial}{\partial z}(vw)\right]=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \tag{6.88b}$$

$$\rho\left[\frac{\partial}{\partial x}(uw)+\frac{\partial}{\partial y}(vw)+\frac{\partial}{\partial z}(w^2)\right]=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \tag{6.88c}$$

The above form of the Navier–Stokes equations is known as the conservative form.

If we substitute the velocity components in terms of their mean and fluctuating components in continuity equation (Eq.(6.87)), it becomes

$$\frac{\partial}{\partial x}(\bar{u}+u')+\frac{\partial}{\partial y}(\bar{v}+v')+\frac{\partial}{\partial z}(\bar{w}+w')=0$$

or
$$\left(\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}\right)+\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)=0 \tag{6.89}$$

Again if we take the time-average of each term of Eq. (6.87), then we have

$$\overline{\frac{\partial u}{\partial x}}+\overline{\frac{\partial v}{\partial y}}+\overline{\frac{\partial w}{\partial z}}=0$$

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=0 \tag{6.90}$$

Comparing Eqs. (6.90) and (6.89), we get

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0$$

Therefore we find that the time-averaged velocity components and the fluctuating velocity components, each satisfies the continuity equation for incompressible flow. Let us take the time-average of each and every term of Navier–Stokes equations (Eqs. (6.88a) to (6.88c)), and accordingly we have

$$\rho\left[\overline{\frac{\partial}{\partial x}(u^2)}+\overline{\frac{\partial}{\partial y}(uv)}+\overline{\frac{\partial}{\partial z}(uw)}\right]=-\overline{\frac{\partial p}{\partial x}}+\mu\left(\overline{\frac{\partial^2 u}{\partial x^2}}+\overline{\frac{\partial^2 u}{\partial y^2}}+\overline{\frac{\partial^2 u}{\partial z^2}}\right) \tag{6.91a}$$

$$\rho\left[\overline{\frac{\partial}{\partial x}(uv)}+\overline{\frac{\partial}{\partial y}(v^2)}+\overline{\frac{\partial}{\partial z}(vw)}\right]=-\frac{\overline{\partial p}}{\partial y}+\mu\left(\overline{\frac{\partial^2 v}{\partial x^2}}+\overline{\frac{\partial^2 v}{\partial y^2}}+\overline{\frac{\partial^2 v}{\partial z^2}}\right) \tag{6.91b}$$

$$\rho\left[\overline{\frac{\partial}{\partial x}(uw)}+\overline{\frac{\partial}{\partial y}(vw)}+\overline{\frac{\partial}{\partial z}(w^2)}\right]=-\frac{\overline{\partial p}}{\partial z}+\mu\left(\overline{\frac{\partial^2 w}{\partial x^2}}+\overline{\frac{\partial^2 w}{\partial y^2}}+\overline{\frac{\partial^2 w}{\partial z^2}}\right) \tag{6.91c}$$

Before proceeding for the simplification of the different terms of Eqs. (6.91a) to (6.91c), we must look into some general rules of time-average quantities. If f and g are two quantities such that

$$f=\bar{f}+f'$$

$$g=\bar{g}+g'$$

and

$$\overline{f'}=\overline{g'}=0$$

then,

$$\overline{\frac{\partial f}{\partial s}}=\frac{\partial \bar{f}}{\partial s};\qquad \overline{\frac{\partial g}{\partial s}}=\frac{\partial \bar{g}}{\partial s}$$

$$\overline{\frac{\partial^2 f}{\partial s^2}}=\frac{\partial^2 \bar{f}}{\partial s^2};\qquad \overline{\frac{\partial^2 g}{\partial s^2}}=\frac{\partial^2 \bar{g}}{\partial s^2}$$

$$\overline{f^2}=\overline{(\bar{f}+f')^2}=\overline{(\bar{f}^2+2\bar{f}f'+f'^2)}$$

$$=\bar{f}^2+\overline{f'^2}$$

Similarly,

$$\overline{g^2}=\bar{g}^2+\overline{g'^2}$$

$$\overline{fg}=\overline{(\bar{f}+f')(\bar{g}+g')}$$

$$=(\overline{\bar{f}\bar{g}}+\overline{\bar{f}g'}+\overline{\bar{g}f'}+\overline{f'g'})$$

$$=(\bar{f}\bar{g}+\bar{f}\overline{g'}+\bar{g}\overline{f'}+\overline{f'g'})$$

$$=(\bar{f}\bar{g}+\overline{f'g'})\quad(\text{since } \overline{f'}=\overline{g'}=0)$$

By making use of these rules, Eqs. (6.91a) to (6.91c) can be written after certain rearrangements as

$$\rho\left[\frac{\partial}{\partial x}(\bar{u}^2)+\frac{\partial}{\partial y}(\bar{u}\bar{v})+\frac{\partial}{\partial z}(\bar{u}\bar{w})\right]=\frac{\partial \bar{p}}{\partial x}+\mu\left[\frac{\partial^2 \bar{u}}{\partial x^2}+\frac{\partial^2 \bar{u}}{\partial y^2}+\frac{\partial^2 \bar{u}}{\partial z^2}\right]$$
$$-\rho\left[\frac{\partial}{\partial x}(\overline{u'^2})+\frac{\partial}{\partial y}(\overline{u'v'})+\frac{\partial}{\partial z}(\overline{u'w'})\right] \tag{6.92a}$$

$$\rho\left[\frac{\partial}{\partial x}(\bar{u}\bar{v})+\frac{\partial}{\partial y}(\bar{v}^2)+\frac{\partial}{\partial z}(\bar{u}\bar{w})\right]=\frac{\partial \bar{p}}{\partial y}+\mu\left[\frac{\partial^2 \bar{v}}{\partial x^2}+\frac{\partial^2 \bar{v}}{\partial y^2}+\frac{\partial^2 \bar{v}}{\partial z^2}\right]$$
$$-\rho\left[\frac{\partial}{\partial x}(\overline{u'v'})+\frac{\partial}{\partial y}(\overline{v'^2})+\frac{\partial}{\partial z}(\overline{v'w'})\right] \tag{6.92b}$$

$$\rho\left[\frac{\partial}{\partial x}(\bar{u}\bar{w})+\frac{\partial}{\partial y}(\bar{v}\bar{w})+\frac{\partial}{\partial z}(\bar{w}^2)\right] = \frac{\partial \bar{p}}{\partial z}+\mu\left[\frac{\partial^2\bar{w}}{\partial x^2}+\frac{\partial^2\bar{w}}{\partial y^2}+\frac{\partial^2\bar{w}}{\partial z^2}\right]$$
$$-\rho\left[\frac{\partial}{\partial x}(\overline{u'w'})+\frac{\partial}{\partial y}(\overline{v'w'})+\frac{\partial}{\partial z}(\overline{w'^2})\right] \tag{6.92c}$$

The left-hand sides of Eqs. (6.92a) to (6.92c) are essentially similar to those of Navier–Stokes equation for steady state laminar flow, if the velocity components u, v, w are replaced by $\bar{u}$, $\bar{v}$, and $\bar{w}$. The same argument holds good for the first two terms on the right-hand side of Eqs. (6.92a) to (6.92c). However, the equations contain some additional terms on the right-hand side. These additional terms can be interpreted as components of a stress tensor so that we can write Eqs. (6.92a) to (6.92c) as

$$\rho\left[\frac{\partial}{\partial x}(\bar{u}^2)+\frac{\partial}{\partial y}(\bar{u}\bar{v})+\frac{\partial}{\partial z}(\bar{u}\bar{w})\right] = \frac{\partial \bar{p}}{\partial x}+\mu\nabla^2\bar{u}+\left[\frac{\partial}{\partial x}(\sigma'_{xx})+\frac{\partial}{\partial y}(\tau'_{xy})+\frac{\partial}{\partial z}(\tau'_{xz})\right] \tag{6.93a}$$

$$\rho\left[\frac{\partial}{\partial x}(\bar{u}\bar{v})+\frac{\partial}{\partial y}(\bar{v}^2)+\frac{\partial}{\partial z}(\bar{u}\bar{w})\right] = \frac{\partial \bar{p}}{\partial y}+\mu\nabla^2\bar{v}+\left[\frac{\partial}{\partial x}(\tau'_{xy})+\frac{\partial}{\partial y}(\sigma'_{yy})+\frac{\partial}{\partial x}(\tau'_{yz})\right] \tag{6.93b}$$

$$\rho\left[\frac{\partial}{\partial x}(\bar{u}\bar{w})+\frac{\partial}{\partial y}(\bar{v}\bar{w})+\frac{\partial}{\partial z}(\bar{w}^2)\right] = \frac{\partial \bar{p}}{\partial z}+\mu\nabla^2\bar{w}+\left[\frac{\partial}{\partial x}(\tau'_{xz})+\frac{\partial}{\partial y}(\tau'_{yz})+\frac{\partial}{\partial z}(\sigma'_{zz})\right] \tag{6.93c}$$

Comparing Eqs. (6.92) and (6.93), we can write

$$\begin{bmatrix}\sigma'_{xx} & \tau'_{xy} & \tau'_{xz}\\ \tau'_{xy} & \sigma'_{yy} & \tau'_{yz}\\ \tau'_{xz} & \tau'_{yz} & \sigma'_{zz}\end{bmatrix} = -\rho\begin{bmatrix}\overline{u'^2} & \overline{u'v'} & \overline{u'w'}\\ \overline{u'v'} & \overline{v'^2} & \overline{v'w'}\\ \overline{u'w'} & \overline{v'w'} & \overline{w'^2}\end{bmatrix} \tag{6.94}$$

These additional stresses are known as turbulent stresses or Reynolds stresses and are added with the laminar stresses to determine the total stresses in a turbulent flow. This is well depicted if we write the first two terms of the right-hand side of Eqs. (6.93a) to (6.93c) in terms of the laminar stress components following Eqs. (6.29a) to (6.29b). Then, we have

$$\rho\left[\frac{\partial}{\partial x}(\bar{u}^2)+\frac{\partial}{\partial y}(\bar{u}\bar{v})+\frac{\partial}{\partial z}(\bar{u}\bar{w})\right] = \frac{\partial}{\partial x}(\sigma_{xx}+\sigma'_{xx})+\frac{\partial}{\partial y}(\tau_{xy}+\tau'_{xy})+\frac{\partial}{\partial z}(\tau_{xz}+\tau'_{xz}) \tag{6.95a}$$

$$\rho\left[\frac{\partial}{\partial x}(\bar{u}\bar{v})+\frac{\partial}{\partial y}(\bar{v}^2)+\frac{\partial}{\partial z}(\bar{u}\bar{w})\right] = \frac{\partial}{\partial x}(\tau_{xy}+\tau'_{xy})+\frac{\partial}{\partial y}(\sigma_{yy}+\sigma'_{yy})+\frac{\partial}{\partial z}(\tau_{yz}+\tau'_{yz}) \tag{6.95b}$$

$$\rho\left[\frac{\partial}{\partial x}(\bar{u}\bar{w})+\frac{\partial}{\partial y}(\bar{v}\bar{w})+\frac{\partial}{\partial z}(\bar{w}^2)\right] = \frac{\partial}{\partial x}(\tau_{xz}+\tau'_{xz})+\frac{\partial}{\partial y}(\tau_{yz}+\tau'_{yz})+\frac{\partial}{\partial z}(\sigma_{zz}+\sigma'_{zz}) \tag{6.95c}$$

The concept of eddy viscosity

We have already observed that in a turbulent flow the additional stresses over the laminar ones arise due to the fluctuating velocity components. Each and every term of the matrix on the right hand side of Eq. (6.94) represents momentum fluxes across different surfaces parallel to the coordinate planes by the fluid eddies. For example, the term $\rho\overline{u'v'}$ represents the y-momentum flux across a surface perpendicular to the x-direction and the x-momentum flux across a surface perpendicular to the y-direction, and hence results in an additional shear stress τ'_{xy}. The additional sresses in a turbulent flow are known as *turbulent stresses*. Hence in a turbulent flow, the total stress is the sum of laminar or viscous stress and the turbulent stress. In analogy with the coefficient of viscosity in a laminar flow, J. Boussinesq introduced a turbulent viscosity to express the turbulent stresses in terms of the gradient of mean velocity as

$$\tau'_{xy} = \rho\overline{u'v'} = \mu_t \frac{d\bar{u}}{dy} \tag{6.96a}$$

where the term μ_t is defined as 'eddy viscosity'. It is apparent that eddy viscosity is attributed to random fluctuations in the velocity components and hence it is not a property of the fluid like laminar viscosity μ. However, the central concept of eddy viscosity modelling is to evaluate μ_t in terms of flow parameters in relating the turbulent stress with the gradient of mean velocity for a possible integration of the modified Navier–Stokes equations. Equation (6.96a) can be written as

$$\frac{\tau'_{xy}}{\rho} = \frac{\tau_t}{\rho} = \nu_t \frac{d\bar{u}}{dy}$$

where ν_t is known as eddy momentum diffusivity.

Hence for a turbulent flow,

$$\frac{\tau}{\rho} = \frac{\tau_1 + \tau_t}{\rho} = (\nu + \nu_t)\frac{d\bar{u}}{dy} \tag{6.96b}$$

Prandtl mixing length model

In analogy to the coefficient of viscosity in laminar flow and to the concept of mean free path of molecular collision in kinetic theory of matter, Prandtl introduced the concept of a mixing length in turbulent flow. In a turbulent flow, fluid elements move randomly in transverse directions to that of main flow and collide with one another. This creates a phenomenon of turbulent momentum transport. The mixing length ℓ is defined according to Prandtl, as the average transverse distance travelled by a fluid element before colliding with another one to change its original momentum. Prandtl showed with some physical arguments that the turbulent shear stress in case of a parallel flow can be expressed as

$$\tau = \rho\ell^2 \left|\frac{d\bar{u}}{dy}\right| \frac{d\bar{u}}{dy} \tag{6.97a}$$

The sign of τ is determined by the sign of $d\bar{u}/dy$ as shown in Eq. (6.97a). Prandtl argued that the mixing length ℓ must be proportional to the distance y from a wall and can be expressed, for a homogeneous and isotropic turbulence, as

$$\ell = ky \tag{6.97b}$$

where k is a proportionality constant which is determined from experiments.

Fully developed turbulent flow in a pipe

The parabolic velocity profile for a fully developed laminar flow in a pipe (Hagen–Poiseuille flow) has been derived from an exact solution of Navier–Stokes equations in Section 6.23. The theoretically determined wall frictional coefficient is in excellent agreement with the experimental data. However, this is not possible in case of turbulent flow. It has been found from experiments that a fully developed profile can be observed after a length of 25–40 diameters. Most of the information comes from the experimental data. Even in case of numerical computation with the aid of a turbulence closure model, the model constants are determined from experimental results only. Let us consider a fully developed turbulent flow in a long straight pipe of circular cross-section. From the force balance of a fluid element (Figure 6.18), we have

$$\tau_w = \frac{D}{4}\left(-\frac{dp}{dz}\right)$$

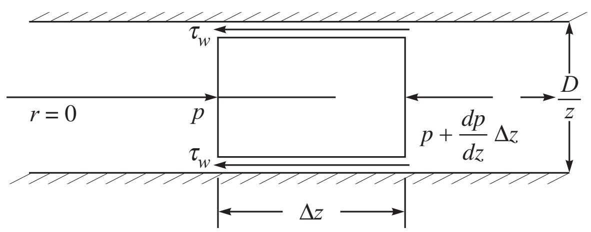


Figure 6.18 Force balance of a fluid element in a turbulent flow through a pipe.

Since the pressure gradient along the pipe can be measured, the wall shear stress is experimentally determined.

From Eq. (6.53a), we can write

$$\tau_w = \frac{f}{8}\rho\bar{u}_{\text{av}}^2 \tag{6.98}$$

where f is Darcy's friction coefficient for pipe flow. After exhaustive experimental investigation, Blasius formulated the following expression for the coefficient of friction f in turbulent flow as

$$f = \frac{0.316}{(\text{Re})^{1/4}} \tag{6.99}$$

However, the relation is valid in the range of Reynolds number $4 \times 10^3 \le \text{Re} \le 10^5$.

The shearing stress at the wall can be obtained by substituting f from Eq. (6.99) into Eq. (6.98). It becomes

$$\tau_w = 0.0395\rho(\bar{u}_{\text{av}})^{7/4}(\nu)^{1/4}(D)^{-1/4}$$

or

$$\tau_w = 0.03325\rho(\bar{u}_{\text{av}})^{7/4}(\nu)^{1/4}(R)^{-1/4} \tag{6.100}$$

where $R = D/2$, the pipe radius.

If we define a friction velocity $u_\tau = \sqrt{\tau_w/\rho}$, we can then write Eq. (6.98) as

$$\rho u_\tau^2 = 0.03325\rho(\bar{u}_{av})^{7/4}(\nu)^{1/4}(R)^{-1/4}$$

or
$$\rho u_\tau^{7/4} u_\tau^{1/4} = 0.03325\ \rho(\bar{u}_{av})^{7/4}(\nu)^{1/4}(R)^{-1/4} \tag{6.101}$$

Rearranging Eq. (6.101), we obtain

$$\frac{\bar{u}_{av}}{u_\tau} = 6.99\left(\frac{u_\tau R}{\nu}\right)^{1/7} \tag{6.102}$$

According to the experimental results of Nikuradse for turbulent pipe flow in the range of Reynolds number Re < 10^5, the ratio of $\bar{u}_{av}/\bar{u}_{max}$ is approximately 0.8. Then, we can write Eq. (6.102) as

$$\frac{\bar{u}_{max}}{u_\tau} = 8.74\left(\frac{u_\tau R}{\nu}\right)^{1/7} \tag{6.103}$$

It is reasonable to assume that Eq. (6.103) is valid not only for the centre line ($r = 0$) but also for any distance $r'(r' = R - r)$ from the wall. Hence it becomes

$$\frac{\bar{u}}{u_\tau} = 8.74\left(\frac{u_\tau(R-r)}{\nu}\right)^{1/7} = 8.74\left(\frac{u_\tau r'}{\nu}\right)^{1/7} \tag{6.104}$$

From Eqs. (6.103) and (6.104), we have

$$\frac{\bar{u}}{u_{max}} = \left(\frac{r'}{R}\right)^{1/7} \tag{6.105}$$

Equation (6.105) is known as Blasius 1/7th power law for the velocity distribution in turbulent flow through a pipe

Universal velocity distribution in a pipe flow

From a force balance on a cylindrical fluid element of radius r in a fully developed flow through a straight pipe of circular cross-section, we have

$$\tau = \frac{r}{2}\left(-\frac{dp}{dz}\right)$$

Again,

$$\tau_w = \frac{R}{2}\left(-\frac{dp}{dz}\right)$$

Therefore,

$$\frac{\tau}{\tau_w} = \frac{r}{R} \tag{6.106}$$

Here the shear stress τ at a radial location r comprises both the viscous stress and the turbulent stress.

If the turbulent stress from Eq. (6.97a) and the mixing length from Eq. (6.97b) are substituted in Eq. (6.106), we get

$$\frac{r}{R}\tau_w = \rho k^2 (R - r)^2 \left|\frac{du}{dr}\right| \frac{du}{dr} \tag{6.107}$$

While using Eqs. (6.105) and (6.106), we replace y by $(R - r)$. We have to remember in this context that Eq. (6.107) is not valid very near to the wall. However, Eq. (6.107) can be written as

$$\frac{\tau_w}{\rho}\frac{r}{R} = k^2 (R - r)^2 \left(\frac{du}{dr}\right)^2$$

or

$$u_\tau^2 \frac{r}{R} = k^2 (R - r)^2 \left(\frac{du}{dr}\right)^2$$

or

$$\frac{du}{dr} = \frac{1}{k}\frac{u_\tau \sqrt{r}}{\sqrt{R}\,(R - r)}$$

Upon integration,

$$\frac{u}{u_\tau} = -\frac{1}{k}\left[\ln\left(\frac{1+\sqrt{r/R}}{1-\sqrt{r/R}}\right) - 2\sqrt{\frac{r}{R}}\right] + C_1 \tag{6.108}$$

The constant C_1 can be found from the condition $u = u_{\max}$ at $r = 0$. Therefore,

$$C_1 = \frac{u_{\max}}{u_\tau}$$

and hence Eq. (6.108) becomes

$$\frac{u_{\max} - u}{u_\tau} = \frac{1}{k}\left[\ln\left(\frac{1+\sqrt{r/R}}{1-\sqrt{r/R}}\right) - 2\sqrt{\frac{r}{R}}\right] \tag{6.109}$$

This is the universal velocity distribution in turbulent pipe flow due to Prandtl. It holds good for all Reynolds numbers and all wall roughnesses in turbulent flow.

Universal law of resistance for pipe flow

Smooth pipe: A universal law of resistance has been derived by Prandtl from the average value of the universal logarithmic velocity distribution as

$$\frac{1}{\sqrt{f}} = A \log_{10} \mathrm{Re}\sqrt{f} + B$$

The constants A and B have been determined from experiments as $A = 2.0$ and $B = -0.8$. Finally, the above relation becomes

$$\frac{1}{\sqrt{f}} = 2 \log_{10} \mathrm{Re}\sqrt{f} - 0.8 \tag{6.110}$$

This is known as Prandtl universal resistance law for a smooth pipe and is valid for all Reynolds numbers. It is interesting to know that the universal resistance law agrees fairly well with the Blasius resistance formula, Eq. (6.97), up to $\mathrm{Re} = 10^5$.

Rough pipe: In rough pipes, the friction factor depends not only on the Reynolds number of flow but also on the roughness at the pipe wall. Roughness in commercial pipes is due to the protrusions at the surface which are random both in size and spacing. However, the commercial pipes are specified by the average roughness which is the measure of some average height of the protrusions. Nikuradse made extensive experimental studies to compare the flow rate and pressure drop in a commercial pipe with those of a pipe with artificial roughness created by gluing grains of sand of uniform size to the wall. A parameter known as *roughness Reynolds number* which equals $u_\tau \varepsilon/\nu$, where ε is a measure of the average roughness, was recognized as the criterion to characterize the pipe roughness. It was found that if $u_\tau \varepsilon/\nu < 4$, the surface may be considered hydraulically smooth. In this situation the mean height of the roughness element is smaller than the laminar sub-layer, and hence, the friction factor becomes the same as that of a smooth pipe. A comprehensive documentation of the experimental and theoretical investigations on the laws of friction for smooth and rough pipes in laminar and turbulent flows has been made in the form of a diagram by L.F. Mody. The diagram is known as Mody's diagram and is shown in Figure 6.19, and is employed till today as the best means for determining the values of *f*.

Universal velocity profile near a wall

We have discussed above the universal velocity profile in a turbulent flow through a straight pipe. The universal velocity profile near any solid surface can be described in a more generalized form as follows:

The turbulent flow along a wall should be divided, as suggested by Karman and Nikuradse, into three different zones. The first zone is a thin layer in immediate vicinity of the wall in which the viscous stress predominates over the insignificant turbulent stress. This region is called the laminar sublayer. The zone immediately above the laminar sub-layer is called the buffer zone in which the turbulent stresses and the viscous stresses are of the same order of magnitude. Still farther from the wall, we have the turbulent zone in which the viscous stresses may be completely neglected in comparison to the turbulent stresses. For convenience, a flow is usually divided into two zones, the laminar sublayer or viscous layer and the turbulent layer. For flows with zero pressure gradient in the stream-wise direction and a sufficiently large Reynolds number, the laminar sublayer becomes very thin and the shear stress τ can be considered to be constant in the flow field. Under the situation, we can write from the Prandtl's mixing length theory

$$\tau = \tau_w = \rho k^2 y^2 \left(\frac{d\bar{u}}{dy}\right)^2$$

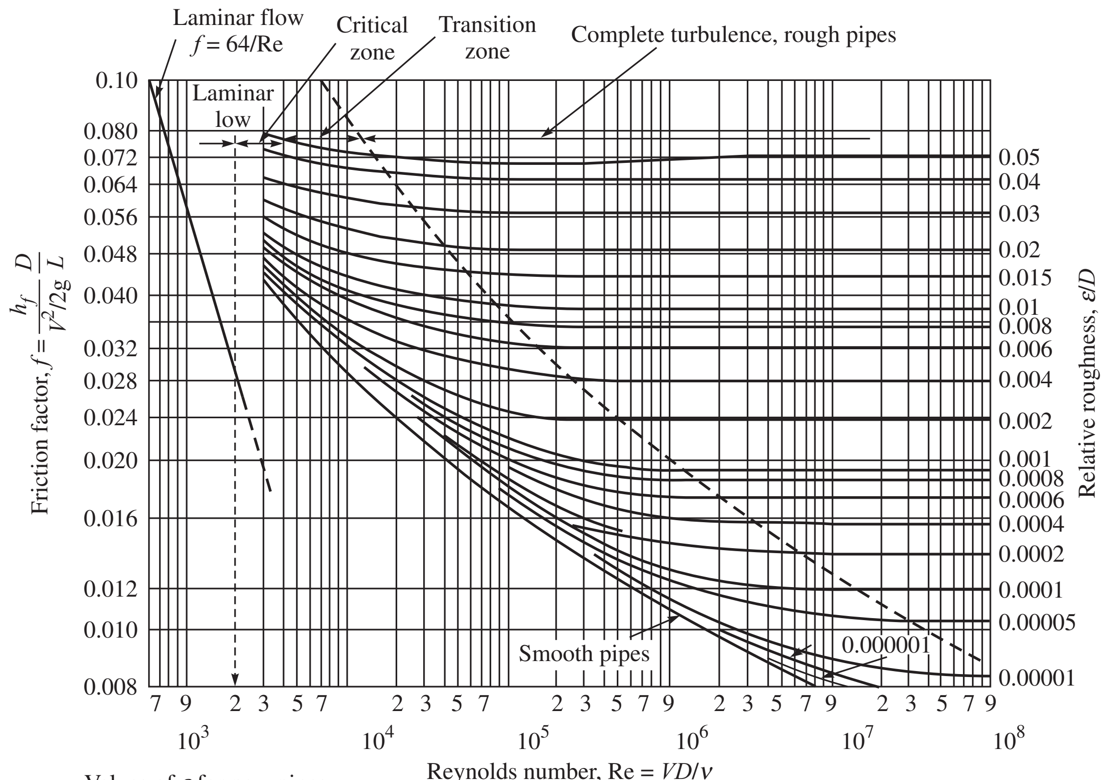
or

$$u_\tau = ky\frac{d\bar{u}}{dy}$$

On integration, this gives

$$\frac{u}{u_\tau} = \frac{1}{k}(\ln y + A) \tag{6.111}$$

The constant of integration A can be determined from an appropriate boundary condition. The no-slip condition at the wall cannot be satisfied with a finite constant of integration. This is



Values of ε for new pipes

Riveted steel	1–10 mm
Concrete	0.3–3 mm
Cast iron	0.25 mm
Galvanised iron	0.15 mm
Mild steel	0.045 mm
Drawn tubing, brass, lead, glass	0.0015 mm

Figure 6.19 Friction factors for pipes (Mody's diagram) (Adapted from Trans. ASME,66,672,1944)

expected since Eq. (6.111) is valid only in the turbulent zone. The appropriate condition will be that $\bar{u} = 0$ at a certain distance y from the wall. Hence Eq. (6.111) becomes

$$\frac{\bar{u}}{u_\tau} = \frac{1}{k}(\ln y - \ln y_0) \tag{6.112}$$

The distance y_0 is of the order of the thickness of viscous layer and its value can be determined by equating the turbulent and viscous stresses at y_0. It can be shown that

$$y_0 = \beta \frac{\nu}{u_\tau}$$

where β is a dimensionless constant. Substituting y_0 from the above equation into Eq. (6.112), we can write

$$\frac{\bar{u}}{u_\tau} = \frac{1}{k}\left(\ln \frac{u_\tau}{\nu} y - \ln \beta\right) \tag{6.113}$$

Equation (6.113) is the universal velocity profile near a wall. The values of k and β are to be determined from experiments. Nikuradse and Reichatdt's suggested, from their extensive experimental results, that $k = 0.40$ and $\beta = 0.111$. With these values of k and β, Eq. (6.113) becomes

$$\frac{\bar{u}}{u_\tau} = 2.5 \ln \frac{u_\tau}{\nu} y + 5.5 \tag{6.114}$$

The velocity profile given by Eq. (6.114) is not valid in the laminar sublayer (or viscous layer) where the viscous shear is predominant. In this layer a linear velocity distribution is obtained as follows:

Based on the fact that the shear stress remains constant in the flow field, it can be written

$$\tau_w = \rho u_\tau^2 = \mu \frac{d\bar{u}}{dy}$$

or

$$\frac{\bar{u}}{u_\tau} = \frac{u_\tau}{\nu} \cdot y \tag{6.115}$$

Von Kerman suggested a logarithmic equation for the buffer zone, $5 < u_\tau y/\nu < 70$ as

$$\frac{\bar{u}}{u_\tau} = 5.0 \ln \frac{u_\tau}{\nu} y - 3.05 \tag{6.116}$$

The velocity profiles given by Eqs. (6.114), (6.115) and (6.116) in different zones in a turbulent flow near a wall are shown in Figure 6.20 which are in good argument with the measured values.

Turbulent boundary layer over a flat plate

For a flow over a flat plate, the boundary layer grows from the leading edge and the flow is always laminar in its initial part and subsequently turns into transition flow and very shortly thereafter turns into a turbulent one. The turbulent boundary layer continues to grow in thickness. However, there exists always a laminar sublayer in the near vicinity of the plate (Figure 6.21).

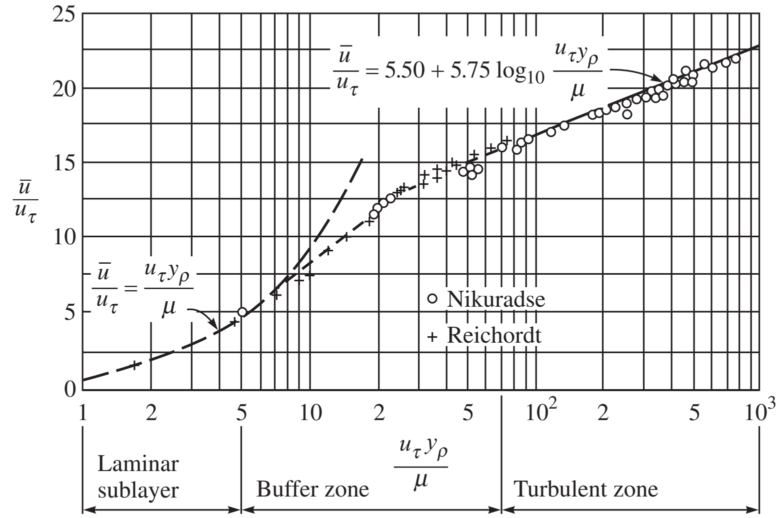


Figure 6.20 The velocity profile for a turbulent flow near a wall.

The transition from a laminar boundary layer to a turbulent one takes place at an axial location on the plate where the growth rate of the amplitude of the inherent disturbances becomes maximum. The source of the disturbances is attributed to free stream turbulence, wall roughness and acoustic signals. It has been found from experiments that the boundary layer remains laminar if the Reynolds number Re = $\rho U_\infty x/\mu$ is less than 5×10^5, where x is the distance along the plate from the leading edge.

In the present section we will show the application of Pohlausen method in solving the momentum integral equation to determine the growth of boundary layer and skin friction coefficient for a turbulent flow over a flat plate. We will assume here that the boundary layer is turbulent from the leading edge (x = 0) of the plate with respect to the Cartesian coordinate system as shown in Figure 6.21. The momentum integral equation can be written as

$$\frac{d}{dx}\int_0^\delta u(U_\infty - u)\,dy = \frac{\tau_w}{\rho} \tag{6.117}$$

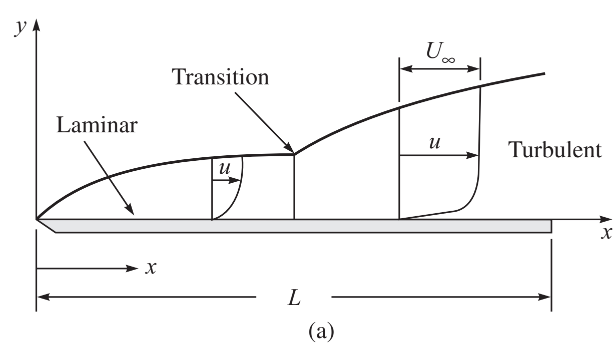


Figure 6.21 The growth of boundary layer from laminar to turbulent over a flat plate.

The shearing stress at the wall, τ_w in Eq. (6.117) depends upon the thickness of laminar sublayer and the velocity distribution within it. However, we will use an empirical relation for τ_w. Prandtl suggested that the velocity distribution within a turbulent boundary layer over a flat plate is essentially the same as that in a large diameter pipe and hence we can use the Blasius 1/7th power law velocity distribution given by Eq. (6.105) for a flat plate by replacing $u_{\max}$ by U_∞ and R by δ. It becomes

$$\frac{u}{U_\infty} = \left(\frac{y}{\delta}\right)^{1/7} \tag{6.118}$$

where $\delta = \delta(x)$ is the thickness of the boundary layer and U_∞ is the free stream velocity. The Blasius 1/7th power law for the velocity distribution given by Eq. (6.118) is valid for Reynolds number Re ($= U_\infty L/\nu$) < 10^6, where L is the length of the plate. For wall shear stress τ_w, the empirical formula for pipe flow as given by Eq. (6.100) can be used if we substitute $u_{av} = 0.817u_{\max}$ and then replace $u_{\max}$ by U_∞ and R by δ. Thus, the shearing stress τ_w for a flat plate becomes

$$\tau_w = 0.0233\rho U_\infty^2 \left(\frac{U_\infty \delta}{\nu}\right)^{-1/4} \tag{6.119}$$

Substitutions of Eqs. (6.119) and (6.118) into Eq. (6.117) gives

$$U_\infty^2 \frac{d}{dx}\left[\delta \int_0^1 \left(\frac{y}{\delta}\right)^{1/7} \left\{1 - \int_0^1 \left(\frac{y}{\delta}\right)^{1/7}\right\} d\left(\frac{y}{\delta}\right)\right] = 0.0233 U_\infty^2 \left(\frac{U_\infty \delta}{\nu}\right)^{-1/4}$$

After some rearrangement, we obtain

$$\frac{d\delta}{dx} = 0.239\left(\frac{\nu}{U_\infty \delta}\right)^{-1/4}$$

Integrating the above equation from 0 to x, we have

$$\frac{4}{5}\delta^{5/4} = 0.239\left(\frac{\nu}{U_\infty}\right)^{-1/4} x$$

or

$$\frac{\delta}{x} = \frac{0.379}{(\mathrm{Re}_x)^{1/5}} \tag{6.120}$$

A comparison of Eq. (6.120) with Eq. (6.83) indicates that the turbulent boundary layer grows at a more rapid rate along the length of the plate than the corresponding laminar boundary layer. Introducing the expression for δ from Eq. (6.120) into Eq. (6.119), the turbulent shear stress takes the form

$$\tau_w = 0.0296\rho U_\infty^2 (\mathrm{Re}_x)^{-1/5}$$

$$c_{f_x} = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} = \frac{0.0592}{(\mathrm{Re}_x)^{-1/5}} \tag{6.121}$$

The average skin friction coefficient $\bar{c}_{f_L}$ over a length L is given by

$$\bar{c}_{f_L} = \frac{1}{L}\int_0^L c_{f_x}\,dx$$
$$= 0.074(\mathrm{Re}_L)^{-1/5} \tag{6.122}$$

Equation (6.122) is derived assuming the boundary layer over the entire plate to be turbulent. In reality, a portion of the boundary layer from the leading edge up to some downstream location is laminar. In this case the coefficient of friction is reduced. In consideration of the transition Reynolds number to be 5×10^5, a fairly accurate empirical relation which is employed in practice is

$$\bar{c}_{f_L} = \frac{0.074}{(\mathrm{Re}_L)^{1/5}} - \frac{1742}{\mathrm{Re}_L} \tag{6.123}$$

Equation (6.123) is valid for the range $5 \times 10^5 < \mathrm{Re} < 10^7$.

EXAMPLE 6.9 Von Karman suggested that the mixing length can be expressed as

$$\ell = k\left|\frac{d\bar{u}/dy}{d^2\bar{u}/dy^2}\right|$$

where k is an empirical constant. With the help of this, show that the universal velocity distribution in a pipe flow can be written as

$$\frac{u_{\max} - u}{u_\tau} = -\frac{1}{k}\left[\ln\left(1 - \sqrt{\frac{r}{R}}\right) + \sqrt{\frac{r}{R}}\right]$$

where R is the radius of the pipe.

Solution: For a pipe flow,

$$\tau = \tau_w \frac{r}{R}$$

Therefore, we can write

$$\tau_w \frac{r}{R} = \rho u_\tau^2 \frac{r}{R} = \rho k^2 \frac{(d\bar{u}/dr)^4}{(d^2\bar{u}/dr^2)^2}$$

Taking the square root, we obtain

$$\frac{d^2\bar{u}/dr^2}{(d\bar{u}/dr)^2} = \pm\frac{k\sqrt{R}}{u_\tau}\cdot\frac{1}{\sqrt{r}}$$

Integrating,

$$-\left(\frac{d\bar{u}}{dr}\right)^{-1} = \pm\frac{2k}{u_\tau}\sqrt{\mathrm{R}}\sqrt{r} + C \tag{6.124}$$

The constant of integration is obtained from the condition when $r \to R$, $(du/dr)^{-1} \to 0$. (Since $r \to R$, the shear stress assumes a finite value while the mixing length approaches zero.)

Therefore, $$C = \pm \frac{2k}{u_\tau} R$$

With this value of C, Eq. (6.124) becomes

$$\frac{du}{dr} = -\frac{u_\tau}{2k\sqrt{R}(\sqrt{R} - \sqrt{r})} \tag{6.125}$$

The negative sign is used since du/dr is negative. The integration of Eq. (6.125) yields

$$u = -\frac{u_\tau}{k}\left[\left(1 - \sqrt{\frac{r}{R}}\right) - \ln(\sqrt{R} - \sqrt{r})\right] + C_1 \tag{6.126}$$

The constant C_1 is evaluated from the boundary condition at the centre of the pipe which is

$$\text{At } r = 0, \qquad u = u_{\max}$$

Then, $$C_1 = \frac{u_\tau}{k}(1 - \ln\sqrt{R}) - u_{\max}$$

Substituting the value of C_1 in Eq. (6.126), we obtain the final expression for universal velocity distribution as

$$u = u_{\max} + \frac{u_\tau}{k}\left[\ln\left(1 - \sqrt{\frac{r}{R}}\right) + \sqrt{\frac{r}{R}}\right]$$

or $$\frac{u_{\max} - u}{u_\tau} = -\frac{1}{k}\left[\ln\left(1 - \sqrt{\frac{r}{R}}\right) + \sqrt{\frac{r}{R}}\right]$$

EXAMPLE 6.10 Wind at a speed of 36 km/h blows over a flat plate of 6 m length. If the density and kinematic viscosity of air are 1.2 kg/m^3 and 1.5×10^{-5} m^2/s respectively, calculate the force at one side of the plate per metre width of the plate. Also estimate the thickness of the boundary layer at the trailing edge.

Solution: Wind velocity (free stream velocity), $U_\infty = \frac{36 \times 1000}{3600} = 10$ m/s

$$\text{Re}_L = \frac{10 \times 6}{1.5 \times 10^{-5}} = 4 \times 10^6$$

We consider that the transition of boundary layer from laminar to turbulent takes place at $\text{Re}_L = 5 \times 10^5$. Therefore, the corresponding friction coefficient is given by Eq. (6.123) as

$$\overline{C}_{fL} = \frac{0.074}{(\text{Re}_L)^{1/5}} - \frac{1742}{\text{Re}_L}$$

$$= \frac{0.074}{(4 \times 10^6)^{1/5}} - \frac{1742}{4 \times 10^6}$$

$$= 0.0031$$

Drag force on one side of the plate per unit metre width is

$$F_D = \overline{C}_{f_L} \times \frac{\rho U_\infty^2}{2} \times L$$

$$= 0.0031 \times \frac{1.2 \times 10^2}{2} \times 6$$

$$= 1.12 \text{ N}$$

The turbulent boundary layer thickness at the trailing edge is given by

$$\delta = L \times \frac{0.379}{(\mathrm{Re}_L)^{1/5}}$$

$$= 6 \times \frac{0.379}{(4 \times 10^6)^{0.2}}$$

$$= 0.108 \text{ m}$$

$$= 108 \text{ mm}$$

SUMMARY

- Viscosity is a property of the fluid which offers resistance to flow. It is defined to be the ratio of shear stress to the rate of shear strain.
- Continuity equation is the equation of conservation of mass in a fluid flow. The general form of the continuity equation for an unsteady compressible flow is given by

  $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

  where **V** is the velocity vector.
- Navier–Stokes equations are the equations of motion for the flow of a viscous fluid. In short form, the vector notation as follows may be used to express these equations.

  $$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{V} + \frac{\mu}{3} \nabla (\nabla \cdot \mathbf{V})$$
- The Navier–Stokes equations are not amenable to an analytical solution due to the presence of nonlinear inertia terms in them. However, there are certain special situations of flow where the nonlinear terms are reduced to zero and the exact solution of Navier–Stokes equations is possible. Some examples of such flows are plane Poiseuille flow and Hagen-Poiseuille flow which correspond to laminar, steady, fully developed flow through parallel fixed plates and circular pipe respectively. The knowledge of velocity field obtained through analytical methods permits the calculation of shear stress, pressure drop, and flow rate.
- A thin layer of fluid adjacent to a solid surface is known as boundary layer. The effect of viscosity is very prominent within the layer, and the inertia force and viscous force have the same order of magnitude within this layer. The main stream velocity undergoes a change from zero at the solid surface to the free stream velocity through the boundary layer. The thickness of the boundary layer is inversely proportional to the Reynolds number of flow and goes on increasing in the direction of flow.

- For high Reynolds number flow, the boundary layer thickness is very small compared to a physical dimension of the system. This allows us to simplify the Navier–Stokes equations discarding certain terms from an order-of-magnitude analysis as applicable within the boundary layer. The simplified Navier–Stokes equations for boundary layer flow are known as boundary layer equations.
- The pressure at any section is impressed on the boundary layer by the outer inviscid flow which can be calculated using Bernoulli's equation.
- The boundary layer equation is a second-order nonlinear partial differential equation. The exact solution of this equation is made by a similarity transformation of the independent variables governing the flow velocities. The similarity solution for flow over a flat plate is often referred to as the Blasius solution.
- Momentum integral equation is obtained by integrating the boundary layer equation within the boundary layer having a finite thickness. The growth of boundary layer and wall shear stress are obtained from the solution of momentum integral equation with the aid of a polynomial function for velocity distribution. This is referred to as approximate solution of boundary layer equation.
- The boundary layer separation takes place in a flow with an adverse pressure gradient. The fluid particles near the wall because of their excessive loss in momentum due to fluid friction become unable to move against the higher downstream pressure and therefore follow the reverse path.
- The flows where various quantities like velocity components, pressure, density at any point show an irregular and random variation with time are known as turbulent flows. The important features of a turbulent flow are that the fluctuations have statistical average for their quantitative representations and the fluctuations are both diffusive and dissipative in nature.
- In a turbulent flow, laminar stresses are increased by additional stresses arising out of the fluctuating velocity components. These additional stresses are known as apparent stresses of turbulent flow or Reynolds stresses.
- In analogy with the laminar shear stress, the turbulent shear stress can be expressed in terms of the mean velocity gradient and a mixing coefficient known as 'eddy viscosity'.

  The eddy momentum diffusivity ν_t can be expressed as $\nu_t = \ell^2 \left|\frac{d\bar{u}}{dy}\right|$, where ℓ is known as Prandtl mixing length. For a homogeneous, isotropic turbulence, ℓ is given by $\ell = ky$ where y is the distance from the wall and k is known as von Karman constant.
- The velocity distribution for a fully developed turbulent flow in a smooth pipe can be written for Re < 10^5, as

$$\frac{\bar{u}}{u_\tau} = \left(\frac{u_\tau (R - r)}{\nu}\right)^{1/7}$$

  where u_τ is the friction velocity. The friction factor in this regime is given by

$$= \frac{0.316}{(\text{Re})^{1/4}}$$

- The universal velocity distribution for a fully developed turbulent flow in a smooth pipe can be expressed

$$\frac{u_{\max} - u}{u_\tau} = \frac{1}{k}\left[\ln\left(\frac{1+\sqrt{r/R}}{1-\sqrt{r/R}}\right) - 2\sqrt{\frac{r}{R}}\right]$$

The universal law of resistance can be written as

$$\frac{1}{\sqrt{f}} = 2\log_{10} \mathrm{Re}\sqrt{f} - 0.8$$

- For a turbulent flow along a solid wall, there are three distinct zones. The zone in immediate vicinity of the wall is one in which the viscous stresses predominate over the turbulent stresses. This zone is known as laminar or viscous sublayer. The zone immediately above the laminar sublayer is called the buffer layer in which the turbulent stresses and viscous stresses are of the same order of magnitude. Still farther from the wall is the turbulent zone where the viscous stresses are negligible compared to turbulent stresses.

REVIEW QUESTIONS

Tick the correct answers (Questions 1–12).

1. The assumptions made in the derivation of Navier–Stokes equations are:
(a) continuum, incompressible flow, newtonian fluid and μ = constant
(b) steady, incompressible and irrotational flow
(c) continuum, non-newtonian fluid and incompressible flow
(d) continuum, newtonian fluid, isotropy, Stoke’s hypothesis and continuity to hydrostatics.

2. For a steady, laminar, parallel and fully developed flow, the pressure gradient in the direction of flow
(a) is zero
(b) is constant
(c) increases linearly in the direction of flow
(d) decreases linearly in the direction flow.

3. The pressure difference across the boundary layer over a solid surface is of the order of
(a) δ^2
(b) δ
(c) $1/\delta$
(d) 1
where δ is the boundary layer thickness.

4. A laminar boundary layer has a velocity distribution given by $u/U_\infty = y/\delta$. The displacement thickness δ^* for the boundary layer is

(a) $\delta/2$
(b) $\delta/4$
(c) δ
(d) $\delta/6$

5. The velocity distribution in a laminar boundary layer over a flat plate is expressed as $u/U_\infty = \sin(A\pi y/\delta)$ where δ is the thickness of the boundary layer. The appropriate value of A is
(a) 1.0
(b) –1/2
(c) 2
(d) 1/2

6. The boundary layer thickness in a laminar boundary layer varies with the longitudinal distance x as
(a) $x^{-1/2}$
(b) $x^{-1/5}$
(c) $x^{1/2}$
(d) x

7. Consider a laminar boundary layer over a flat plate. Let A and B be two points on the plate. The point A is at the middle of the plate while B is at the trailing edge. If τ_A and τ_B are the respective shear stresses at the points A and B, then
(a) $\tau_A < \tau_B$
(b) $\tau_A > \tau_B$
(c) $\tau_A = \tau_B$

8. The laminar sublayer exists
(a) only in laminar boundary layers
(b) in all turbulent boundary layers
(c) only in smooth turbulent boundary layers
(d) only in rough fully developed turbulent boundary layers.

9. The pressure drop in a fully developed laminar parallel flow through a 40 mm diameter pipe is 60 kPa over a length of 15 m. Then the wall shear stress τ_w in kPa is
(a) 0.02
(b) 0.04
(c) 0.01
(d) 0.05

10. The Darcy's friction factor in a laminar pipe flow is found to be 0.08. The Reynolds number of the flow is
(a) 2000
(b) 1000
(c) 800
(d) 1600

11. The intensity of turbulence refers to
(a) correlation of u' and v'
(b) average kinetic energy of turbulence per unit mass

(c) root mean square value of turbulent velocity fluctuations
(d) the Reynolds stress.

12. The turbulent shear stress in the *xy*-plane is given by

(a) $\overline{\rho u'}^2$
(b) $\rho u' v'$
(c) $-\overline{\rho u' v'}$
(d) $\overline{u' v'/\rho}$

PROBLEMS

6.1 A film of liquid moves down a plane inclined at 60° to the horizontal. The thickness of the liquid film is 3 mm and the velocity at the free surface is found to be 30 mm/s. If the dynamic viscosity of the liquid is 1.5 kg/(m s), determine (a) the specific gravity of the liquid, (b) the shear stress at the plate, and (c) the rate of discharge per unit width of the film. [**Ans.** (a) 1.18, (b) 30 N/m^2, (c) 60 cm^2/s]

6.2 Consider a laminar fully developed flow between two parallel fixed plates. At what distance from the centre-line does the local velocity equal the average velocity flow?

[**Ans.** At a distance of $1/2\sqrt{3}$ times of the distance between the plates on either side from the centre-line of the channel]

6.3 Consider a laminar, fully developed parallel flow between two flat plates where one is moving at a velocity U_∞ relative to other. The flow is known as coutte flow. Show that the velocity across the channel is given by

$$\frac{u}{U_\infty} = \frac{y}{h} + \alpha \frac{y}{h}\left(1 - \frac{y}{h}\right)$$

where
$$\alpha = \frac{h^2}{2uU_\infty}\left(-\frac{dp}{dx}\right)$$

with h as the distance between the two plates and dp/dx as the imposed pressure gradient. Also show that (a) for $\alpha = 3$, the volumetric flow across a section will be zero, (b) for $\alpha = -1$, the flow reversal starts at the fixed plate, (c) the pressure gradient at the mid-plane of the channel is independent of the pressure gradient dp/dx.

6.4 The velocity along the centre-line of the Hagen–Poiseuille flow in a 0.1m diameter pipe is 2 m/s. If the viscosity of the fluid is 0.07 kg/(m s) and the specific gravity is 0.92, calculate (a) the volumetric flow rate, (b) the shear stress at the pipe wall, and (c) the Dancy's friction coefficient. [**Ans.** (a) 7.85 l/s, (b) 5.6 N/m^2, (c) 0.049]

6.5 An oil of dynamic viscosity 0.15 kg/(m s) and specific gravity 0.9 flows through a 30 mm vertical pipe. Two pressure gauges are fixed 20 m apart. The gauge A fixed at the top records 200 kPa and the gauge B fixed at the bottom records 500 kPa. Find the direction of flow and the rate of flow. [**Ans.** Vertically upwards]

6.6 Calculate the power required to pump sulphuric acid (viscosity 0.04 kg/(m s) and specific gravity 1.83) at 30 litres per second from a supply tank into a storage tank through a 20 mm diameter pipe of length 4 m. The liquid level in the storage tank is 6 m above that in the supply tank. If the flow is turbulent and Re < 10^7, use the relation for the friction factor as $f = 0.0014(1 + 100\text{Re}^{-1/3})$. [**Ans.** 983 W]

6.7 Consider a flow between two concentric cylinders where the inner cylinder rotates with a constant angular velocity ω and the outer one is at rest. The radii of the inner and outer cylinders are r_1 and r_2 respectively. The length of the cylinder is L. Show that the torque transmitted by the inner cylinder to the outer one is given by

$$T = \frac{4\pi\mu L r_1^2 r_2^2 \omega}{r_2^2 - r_1^2}$$

6.8 Consider a laminar boundary flow over a flat plate. Find the ratio of skin friction drag on the front-half of the plate to that on the rear-half. [**Ans.** 2.414]

6.9 Water flows over a smooth flat plate along its length. The free stream velocity is 2 m/s. Find (a) the extent of laminar boundary layer on the plate, (b) the boundary layer thickness at the edge of laminar boundary layer and at the trailing edge, and (c) the shear stress at the trailing edge. Use the exact solution for the laminar boundary layer and the solution by the integral method for the turbulent boundary layer with 1/7th power law for the velocity profile. The length of the plate is 2 m and its width is 1.5 m (ρ = 1000 kg/m^3, $\nu = 1 \times 10^{-6}$ m^2/s).

[**Ans.** (a) up to a length of 0.25 m, (b) 1.77 mm, (c) 5.66 N/m^2]

6.10 In course of a flow over a flat plate, the laminar boundary layer undergoes a transition to turbulent boundary layer as the flow proceeds downstream. It is observed that a parabolic laminar profile is finally changed into 1/7th power law velocity profile in the turbulent regime. Find the ratio of the turbulent and laminar boundary layer thicknesses, if the momentum flux within the boundary layer remains constant. [**Ans.** 72/105]

6.11 If the velocity distribution in the turbulent boundary layer over a flat plate is given as

$$\frac{u}{U_\infty} = \left(\frac{y}{\delta}\right)^{1/\eta}$$

determine the ratio δ^*/δ and δ^{**}/δ in terms of η.

$$\left[\textbf{Ans.}\quad \frac{\delta^*}{\delta} = \frac{1}{(1+\eta)}, \frac{\delta^{**}}{\delta} = \frac{\eta}{(1+\eta)(2+\eta)}\right]$$

6.12 Determine the frictional drag force on one side of a plate 3 m long and 10 m wide which is placed in a wind tunnel at a velocity of 50 m/s under the following assumptions:

(a) The boundary layer is to remain laminar over the entire length of the plate.

(b) The transition occurs at $\text{Re}_x = 5 \times 10^5$.

The density and kinematic viscosity of air are 1.2 kg/m^3 and 1.5×10^{-5} m^2/s respectively. [**Ans.** (a) 8.78 N, (b) 48.11 N]

7

Principles of Forced Convection

This chapter will discuss the different theories and their mathematical background related to forced convection heat transfer in external and internal flows.

Learning objectives

The reading of this chapter will enable the students

- to understand the mathematical and physical implications of governing equation, namely the energy equation in forced convection,
- to grasp the concept of thermal boundary layer,
- to understand the implication of thermal boundary layer equation and its solution in different situations of external and internal flows,
- to understand the physical mechanism of forced convection heat transfer in different situations,
- to formulate forced convection heat transfer problems analytically, and
- to get acquainted with standard and popular empirical formulae for different situations of convective heat transfer.

7.1 DERIVATION OF ENERGY EQUATION

It has already been appreciated in Chapter 5 that to determine the rate of heat transfer from or to a solid body in a stream of fluid, one has to know the value of temperature gradient (in the direction of heat flow) at the solid surface, since conduction is the only mode of heat transfer at the surface. This requires the information about the temperature field in the fluid stream surrounding the solid body. The temperature distribution near a solid body in a flow is greatly influenced by the flow field, i.e. the velocity distribution around the body. The information about the velocity field is obtained from the solutions of equations of conservation of mass and momentum which have been discussed in Chapter 6. In a similar way, temperature distribution in a fluid stream is obtained from the solution of the equation of conservation of energy in the flow. This equation is known as energy equation.

Let us consider a small fluid element in motion as a closed system (Figure 7.1) appropriate to a rectangular Cartesian frame of coordinates. The first law of thermodynamics for the elemental closed system can be written as

$$\frac{\delta Q}{\delta t} = \frac{DE}{Dt} + \frac{\delta W}{\delta t} \tag{7.1}$$

where δQ and δW are respectively the amount of heat added to and work done from the system during a time interval of δt. The term DE/Dt represents the rate of change of internal energy which consists of local and convective changes.

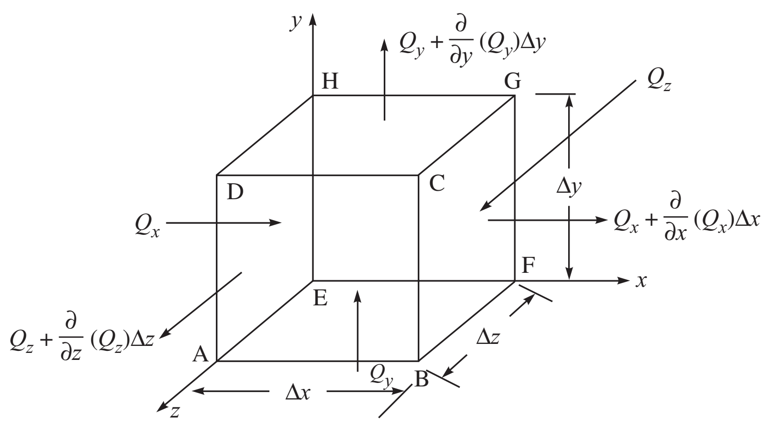


Figure 7.1 A fluid element as a closed system.

If the transfer of heat by radiation is neglected, it then occurs only through conduction and is determined by Fourier's law. The heat conducted into the element from the surface ADHE can be written as

$$Q_x = -\left(k\frac{\partial T}{\partial x}\right)\Delta y\,\Delta z$$

The rate of heat conducted out of the element from the surface BCGF becomes

$$Q_{x+\Delta x} = Q_x + \frac{\partial}{\partial x}(Q_x)\Delta x \quad \text{(neglecting the higher-order terms of } \Delta x)$$

Therefore, the net rate of heat conducted into the fluid element due to these two surfaces becomes

$$Q_x - \left[Q_x + \frac{\partial}{\partial x}(Q_x)\Delta x\right]$$

$$= -\frac{\partial}{\partial x}(Q_x)\Delta x$$

$$= \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)\Delta\mathbb{V}$$

$$\left(\text{Since, } Q_x = -k\frac{\partial T}{\partial x}\Delta y\ \Delta z\right)$$

where $\Delta\mathbb{V}(=\Delta x\,\Delta y\,\Delta z)$ is the volume of the fluid element. In a similar fashion, the net rate of heat conducted into the fluid element due to the surfaces DHGC and AEFB becomes $\dfrac{\partial}{\partial y}\left(k\dfrac{\partial T}{\partial y}\right)\Delta\mathbb{V}$ and that due to the surfaces EHGF and DCBA equals $\dfrac{\partial}{\partial z}\left(k\dfrac{\partial T}{\partial z}\right)\Delta\mathbb{V}$.

Therefore, the net rate of heat conducted into the element due to all its surfaces becomes

$$\frac{\delta Q}{\delta t} = \left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right]\Delta\mathbb{V} \tag{7.2}$$

The internal energy E comprises the intermolecular energy and kinetic energy of the element. The gravitational potential energy is neglected since its change is negligible.

Therefore,

$$\begin{aligned}\frac{DE}{Dt} &= \rho\Delta\mathbb{V}\left[\frac{De}{Dt} + \frac{1}{2}\frac{D}{Dt}(u^2 + v^2 + w^2)\right]\\ &= \rho\Delta\mathbb{V}\left[\frac{De}{Dt} + \left(u\frac{Du}{Dt} + v\frac{Dv}{Dt} + w\frac{Dw}{Dt}\right)\right]\end{aligned} \tag{7.3}$$

where e is the specific intermolecular energy and u, v, w are respectively the x, y and z components of fluid velocity. The work done δW, under the situation, is due to the surface forces acting on the fluid element. Let us determine the work done against the component σ_{xx}, the normal stress in x-direction, which then becomes

$$\begin{aligned}\frac{\delta W_{\sigma_{xx}}}{\delta t} &= -(dy\,dz)\left[-u\sigma_{xx} + \left(u + \frac{\partial u}{\partial x}dx\right)\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}dx\right)\right]\\ &= -\frac{\partial}{\partial x}(u\sigma_{xx})\Delta\mathbb{V} \qquad \text{(neglecting the higher-order terms)}\end{aligned}$$

The negative sign implies the sign convention of Eq. (7.1) according to which the work added to a fluid element from outside is negative. Following the similar way in evaluating the work done for each stress component, we can write for the total work done by the normal and shearing stresses per unit time as

$$\frac{\delta W}{\delta t} = -\Delta\mathbb{V}\left[\frac{\partial}{\partial x}(u\sigma_{xx} + v\tau_{xy} + w\tau_{zx}) + \frac{\partial}{\partial y}(u\tau_{xy} + v\sigma_{yy} + w\tau_{yz}) + \frac{\partial}{\partial z}(u\tau_{zx} + v\tau_{yz} + w\sigma_{zz})\right] \tag{7.4}$$

where, σ_{xx}, σ_{yy}, σ_{zz}, τ_{xy}, τ_{yz}, τ_{zx} denote the normal and shearing stresses as shown in Figure 6.5 of Chapter 6. Equation (7.4) can be rearranged in the form

$$\begin{aligned}\frac{\delta W}{\delta t} = -\Delta\mathbb{V}\Bigg[&u\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + v\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right) + w\left(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)\\ &+ \sigma_{xx}\frac{\partial u}{\partial x} + \sigma_{yy}\frac{\partial v}{\partial y} + \sigma_{zz}\frac{\partial w}{\partial z} + \tau_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \tau_{yz}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + \tau_{zx}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\Bigg]\end{aligned}$$

By making use of Eqs. (6.29a), (6.29b) and (6.29c), we can write the preceding equation as

$$\frac{\delta W}{\delta t} = -\Delta \mathbb{V}\left[\rho u \frac{Du}{Dt} + \rho v \frac{Dv}{Dt} + \rho w \frac{Dw}{Dt} + M\right] \tag{7.5}$$

where $M = \sigma_{xx}\frac{\partial u}{\partial x} + \sigma_{yy}\frac{\partial v}{\partial y} + \sigma_{zz}\frac{\partial w}{\partial z} + \tau_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \tau_{yz}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + \tau_{zx}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$ (7.6)

Substituting the stress components in terms of the rate of strain components from Eqs. (6.30a) to (6.30f) in Eq. (7.6), it becomes

$$M = -p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \mu\varphi \tag{7.7a}$$

where

$$\varphi = \left[2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 \right.$$
$$\left. -\frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2\right] \tag{7.7b}$$

The term φ as given by Eq. (7.7b) is known as dissipation function and the term $\mu\varphi$ implies physically the mechanical energy which is being dissipated by fluid friction (viscosity) into intermolecular energy. Substituting Eqs. (7.2), (7.3), (7.5) and (7.7a) in Eq. (7.1), we get

$$\rho\frac{De}{Dt} + p\nabla\cdot\mathbf{V} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \mu\varphi \tag{7.8}$$

where $\mathbf{V} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$.

From the thermodynamic property relation

$$e = h - \frac{p}{\rho}$$

where h is the specific enthalpy.

$$\frac{De}{Dt} = \frac{Dh}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} + \frac{p}{\rho^2}\frac{D\rho}{Dt} \tag{7.9}$$

From the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{V}) = 0$$

or

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{V} = 0$$

or

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{V}$$

Therefore,

$$\frac{De}{Dt} = \frac{Dh}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} - \frac{p}{\rho}\nabla\cdot\mathbf{V} \tag{7.10}$$

Substituting for $\frac{De}{Dt}$ from Eq. (7.10) in Eq. (7.8), we finally get

$$\rho \frac{Dh}{Dt} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \frac{Dp}{Dt} + \mu\varphi \tag{7.11}$$

where φ is given by Eq. (7.7b).

Equation (7.11) is the generalized form of energy equation.

If we consider the fluid to be an ideal gas or a liquid, then

$$\frac{Dh}{Dt} = c_p \frac{DT}{Dt}$$

and Eq. (7.11) becomes

$$\rho c_p \frac{DT}{Dt} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \frac{Dp}{Dt} + \mu\varphi \tag{7.12}$$

Equation (7.12) can be written in vector form as

$$\rho c_p \frac{DT}{Dt} = \nabla\cdot(k\,\nabla T) + \frac{Dp}{Dt} + \tau : \Delta \tag{7.13}$$

where the stress tension τ includes the normal deviatoric stresses and the shear stresses and Δ is the rate of strain tensor as follows.

$$\tau = \begin{bmatrix} (\sigma_{xx} - p) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_{yy} - p) & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & (\sigma_{zz} - p) \end{bmatrix}$$

$$\Delta = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & \left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & \dfrac{\partial v}{\partial y} & \left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) \\ \left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) & \left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & \dfrac{\partial w}{\partial z} \end{bmatrix}$$

For a perfect gas, the equation of state can be written as

$$p = \rho RT \tag{7.14}$$

For a compressible flow, Eqs. (6.5), (6.34a), (6.34b), (6.34c), (7.12), and (7.14) form a system of six simultaneous equations for the six variables u, v, w, p, ρ, T.

The viscous dissipation term '$\mu\varphi$' becomes important when the free-stream velocity is comparable with that of sound, or in case of very high viscous flow and flow with very high strain rates. For a steady flow, $\dfrac{\partial p}{\partial t} = 0$, $\dfrac{\partial T}{\partial t} = 0$, and hence

$$\frac{Dp}{Dt} = \mathbf{V}\cdot\nabla p = \left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right)$$

$$\frac{DT}{Dt} = \mathbf{V}\cdot\nabla T = \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right)$$

Again, if the flow is incompressible the term $\mathbf{V}\cdot(\nabla p)$, which represents the compression work, will vanish.

Therefore for a steady incompressible flow with a constant thermal conductivity, the energy equation becomes

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{7.15}$$

For an incompressible flow, Eqs. (6.10), (6.34a), (6.34b), (6.34c) and (7.15) form a system of five simultaneous equations in u, v, w, p and T.

The energy equation in cylindrical and spherical coordinate systems can be derived (i) either by expanding the vectorial form of Eq. (7.13), (ii) or by considering a fluid element appropriate to the reference frame of coordinates and then applying the principle of conservation of energy. The counterparts of Eq. (7.15) in cylindrical and spherical coordinate systems are

$$\rho c_p\left(v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z\frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{7.16}$$

$$\rho c_p\left(v_R\frac{\partial T}{\partial R} + \frac{v_\varphi}{R}\frac{\partial T}{\partial \varphi} + \frac{v_\theta}{r\sin\varphi}\frac{\partial T}{\partial \theta}\right) = k\left[\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\cot\varphi}{r^2}\frac{\partial T}{\partial \varphi} + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2 T}{\partial \theta^2}\right] \tag{7.17}$$

(For the nomenclature of coordinate systems, refer to Figures 6.3(a) and 6.4)

7.2 NON-DIMENSIONALIZATION OF ENERGY EQUATION AND RECOGNITION OF PERTINENT DIMENSIONLESS TERMS GOVERNING ITS SOLUTION FOR TEMPERATURE FIELD

Let us consider a steady two-dimensional flow in the rectangular Cartesian coordinate system. The energy equation becomes

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + \mu\varphi$$

where
$$\varphi = 2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2 \tag{7.18}$$

The reference quantities taken for nondimensionalizing the variables are (i) the representative length L, (ii) the free stream velocity U_∞, (iii) the free stream density ρ_∞, and (iv) the reference temperature difference $(T_w - T_\infty)$, the difference between the temperature at the solid surface and that at free stream. The dimensionless quantities are defined by a star as the superscript. Thus

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad u^* = \frac{u}{U_\infty}, \quad v^* = \frac{v}{U_\infty},$$

$$\rho^* = \frac{\rho}{\rho_\infty}, \quad p^* = \frac{p}{\rho_\infty U_\infty^2}, \quad T^* = \frac{T - T_\infty}{T_w - T_\infty}$$

By making use of the above relations, Eq. (7.18) is reduced to non-dimensional form as

$$\rho^*\left(u^*\frac{\partial T^*}{\partial x^*} + v^*\frac{\partial T^*}{\partial y^*}\right) = \frac{k}{\rho_\infty c_p U_\infty L}\left(\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}}\right) + \frac{U_\infty^2}{c_p \Delta T_0}\left(u^*\frac{\partial p^*}{\partial x^*} + v\frac{\partial p^*}{\partial y^*}\right)$$

$$+ \frac{\mu U_\infty}{\rho_\infty c_p L \Delta T_0}\varphi^* \tag{7.19}$$

where
$$\varphi^* = 2\left\{\left(\frac{\partial u^*}{\partial x^*}\right)^2 + \left(\frac{\partial v^*}{\partial y^*}\right)^2\right\} + \left(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}\right)^2 - \frac{2}{3}\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*}\right)^2$$

It is apparent that the solution of Eq. (7.19) depends on the following dimensionless groups:

$$\frac{k}{\rho_\infty c_p U_\infty L}, \qquad \frac{U_\infty^2}{c_p \Delta T_0}, \qquad \frac{\mu U_\infty}{\rho_\infty c_p L \Delta T_0}$$

The first group can be written as

$$\frac{k}{\rho_\infty c_p U_\infty L} = \frac{\mu}{\rho_\infty U_\infty L}\cdot\frac{k}{\mu c_p} = \frac{1}{\mathrm{Re}}\cdot\frac{1}{\mathrm{Pr}} \tag{7.20a}$$

where Re and Pr are the familiar Reynolds number and Prandtl number respectively. The Prandtl number depends only on the properties of the medium.

$$\mathrm{Pr} = \frac{\mu c_p}{k} = \frac{\mu/\rho}{k/\rho c_p} = \frac{\nu}{\alpha} \text{ (the ratio of momentum diffusivity to thermal diffusivity)}$$

The second group is

$$\frac{U_\infty^2}{c_p \Delta T_0} = \mathrm{Ec} \tag{7.20b}$$

where Ec is known as the Eckert number. The term $U_\infty^2/2c_p$ is recognized as the temperature increase through adiabatic compression. Therefore, we can write

$$\mathrm{Ec} = 2\frac{(\Delta T)_{\text{adiabatic}}}{\Delta T_0} \tag{7.20c}$$

The third group can be written as

$$\frac{\mu U_\infty}{\rho_\infty c_p L \Delta T_0} = \frac{U_\infty^2}{c_p \Delta T_0}\cdot\frac{\mu}{\rho_\infty U_\infty L} = \frac{\mathrm{Ec}}{\mathrm{Re}} \tag{7.20d}$$

Therefore, we can write Eq. (7.19) as

$$\rho^*\left(u^*\frac{\partial T^*}{\partial x^*} + v^*\frac{\partial T^*}{\partial y^*}\right) = \frac{1}{\mathrm{Re}\cdot\mathrm{Pr}}\left(\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}}\right) + \mathrm{Ec}\left(u^*\frac{\partial p^*}{\partial x^*} + v^*\frac{\partial p^*}{\partial y^*}\right) + \frac{\mathrm{Ec}}{\mathrm{Re}}\varphi^* \tag{7.21}$$

Hence, there are, in all, three independent dimensionless parameters, namely the Re (Reynolds number), the Pr (Prandtl number) and the Ec (Eckert number) which govern the solution of energy equation to obtain the temperature field.

It is now possible to state from Eqs. (7.19), (7.20b) and (7.20c), that viscous dissipation of energy and the work of compression are important in determining the temperature field when the free stream velocity U_∞ is so large that the adiabatic temperature increase is of the same order of magnitude as the prescribed temperature difference between the solid surface and the free stream.

It has already been discussed in Chapter 5 that the dimensionless form of convective heat transfer coefficient is expressed by a term known as Nusselt number 'Nu' which equals the non-dimensional temperature gradient at the solid surface (Eq. (5.20)). Therefore, it is expected that Nusselt number will depend on all the above-mentioned dimensionless parameters, namely the Reynolds number, the Prandtl number and the Eckert number. Thus, we can write

$$\mathrm{Nu}_S^* = \mathrm{F}_1(\mathrm{Re},\ \mathrm{Pr},\ \mathrm{Ec},\ \mathrm{S}^*) \tag{7.22a}$$

$$\overline{\mathrm{Nu}} = \mathrm{F}_1(\mathrm{Re},\ \mathrm{Pr},\ \mathrm{Ec}) \tag{7.22b}$$

where S* denotes the dimensionless space coordinate and Nu_S^* is the local Nusselt number while $\overline{\mathrm{Nu}}$ is the surface averaged Nusselt number over a given surface.

7.3 THERMAL BOUNDARY LAYER

It has been stated earlier in Chapter 5 that like hydrodynamic boundary layer, there is a thin region in the immediate neighbourhood of a solid surface within which the temperature varies from surface temperature to that of the free stream. This region is known as thermal boundary layer in which the temperature gradient in a direction normal to the solid surface is very large and the heat transfer due to conduction is of the same order of magnitude as that due to convection. This means that the temperature field which spreads from a solid surface extends essentially only over a thin region known as thermal boundary layer. It is therefore possible to take advantage of this fact to simplify the energy equation as applicable within the thermal boundary layer in a similar manner as done for the momentum equations (Navier–Stokes equations) in case of obtaining the hydrodynamic boundary layer equations.

Derivation of thermal boundary layer equation

The simplified form of energy equation as applicable within the thermal boundary layer is known as the thermal boundary layer equation. The simplification is made by an order-of-magnitude analysis of the energy equation. For this purpose, the non-dimensional energy equation (Eq (7.21)) for a two-dimensional steady flow is considered. The order of magnitude of each and every term of Eq. (7.21) has been estimated in a similar way as done before in deriving the hydrodynamic boundary layer Eq. (6.57). This is shown as follows:

$$
\begin{aligned}
&\rho^*\left(u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}\right) = \frac{1}{\mathrm{Re}\,\mathrm{Pr}}\left(\cancelto{0}{\frac{\partial^2 T^*}{\partial x^{*2}}}+\frac{\partial^2 T^*}{\partial y^{*2}}\right)+\mathrm{Ec}\left(u^*\frac{\partial p^*}{\partial x^*}+v^*\cancelto{0}{\frac{\partial p^*}{\partial y^*}}\right)\\
&0(1)\left(0(1)\frac{0(1)}{0(1)}+0(\delta^*)\frac{0(1)}{(\delta_T^*)}\right) \quad 0(\delta_T^*)^2\left(\frac{0(1)}{0(1)}+\frac{0(1)}{0(\delta_T^*)^2}\right)\ 0(1)\left(0(1)\frac{0(1)}{0(1)}+0(\delta^*)0(\delta^*)\right)\\
&\qquad +\frac{\mathrm{Ec}}{\mathrm{Re}}\left[2\left\{\left(\cancelto{0}{\frac{\partial u^*}{\partial x^*}}\right)^2+\left(\cancelto{0}{\frac{\partial v^*}{\partial y^*}}\right)^2\right\}\right.\\
&\qquad +0(1)0(\delta^*)^2\left[\left\{\frac{0(1)}{0(1)}+\frac{0(\delta^*)^2}{0(\delta_T^*)^2}\right\}\right]\\
&\qquad +\left(\frac{\partial u^*}{\partial y^*}+\cancelto{0}{\frac{\partial v^*}{\partial x^*}}\right)^2\\
&\qquad \left(\frac{0(1)}{0(\delta_T^*)}+\frac{0(\delta^*)}{0(1)}\right)^2\\
&\qquad \left.-\frac{2}{3}\left(\cancelto{0}{\frac{\partial u^*}{\partial x^*}}+\cancelto{0}{\frac{\partial v^*}{\partial y^*}}\right)^2\right]\\
&\qquad \left(\frac{0(1)}{0(1)}+\frac{0(\delta^*)}{0(\delta_T^*)}\right)^2
\end{aligned}
\tag{7.23}
$$

where, $\delta^* = \delta/L$, $\delta_T^* = \delta_T/L$; L is the representative length scale.

We observe from the order-of-magnitude analysis that the term $\dfrac{\partial^2 T^*}{\partial x^{*2}}$ can be neglected against $\dfrac{\partial^2 T^*}{\partial y^{*2}}$ in the conduction term, which is now represented by $\dfrac{\partial^2 T^*}{\partial y^{*2}}$. The order of magnitude of the conduction term becomes the same as that of the convection terms (the terms in the left-hand side of Eq. (7.23)) provided that the thickness of the thermal boundary layer is of the order

$$0(\delta_T) \sim \frac{1}{\sqrt{\mathrm{Re}}\sqrt{\mathrm{Pr}}}$$

We have already established in Chapter 6,

$$0(\delta) \sim \frac{1}{\sqrt{\mathrm{Re}}}$$

$$0\left(\frac{\delta}{\delta_T}\right) \sim \sqrt{\mathrm{Pr}}$$

It is observed that the only term which remains in the expression for work of compression is $u^* \dfrac{\partial p^*}{\partial x^*}$, while that in the expression of dissipation function is $\left(\dfrac{\partial u^*}{\partial y^*}\right)^2$.

Reverting to dimensional quantities, we can write the simplified form of energy equation as applicable within the thermal boundary layer as

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial p}{\partial x} \tag{7.24}$$

Equation (7.24) is known as the two-dimensional thermal boundary layer equation in the rectangular Cartesian coordinate system.

In case of incompressible flow, we can neglect the effect of viscous dissipation and work of compression. Hence Eq. (7.24) becomes

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2} \tag{7.25}$$

The laminar two-dimensional hydrodynamic boundary layer equation (Eq. (6.57)) is written here again

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{dp}{dx} + \mu\frac{\partial^2 u}{\partial y^2} \tag{7.26}$$

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{7.27}$$

These three equations (Eqs. (7.25) to (7.27)) are solved for u, v and T in respect of a laminar incompressible boundary layer flow. The pressure p as a function of x is found from the solution of Eulers equation in the potential flow outside the boundary layer as

$$U_\infty \frac{dU_\infty}{dx} = -\frac{dp}{dx} \tag{7.28}$$

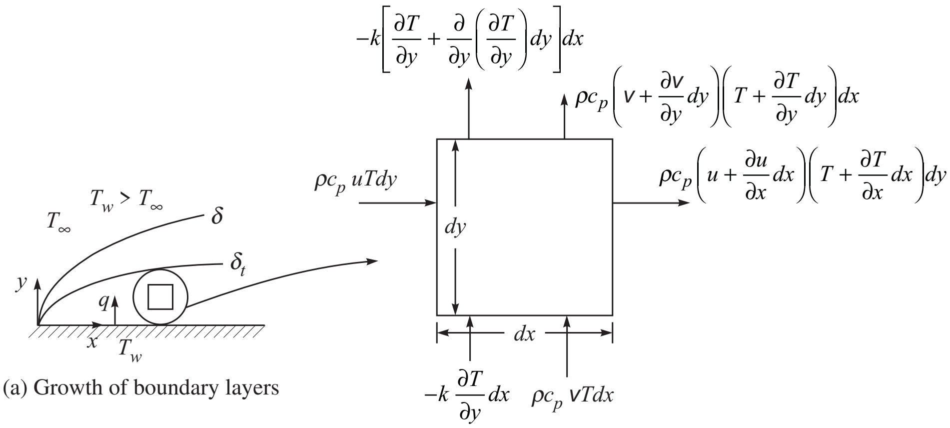
7.4 CONVECTIVE HEAT TRANSFER IN EXTERNAL FLOWS

7.4.1 Flat Plate in Parallel Flow

Let us consider a parallel flow over a flat plate (Figure (7.2a)).

The hydrodynamic boundary layer which develops from the leading edge ($x = 0$) is assumed to be laminar over the entire length of the plate. The expressions for the growth of hydrodynamic boundary layer and the velocity profile within the boundary layer have already been derived in Chapter 6 (Section 6.4.2).

Let us consider that the plate is kept at a constant temperature T_w over its entire length. The free stream temperature is T_∞, and $T_w > T_\infty$. Under the situation, a thermal boundary layer will also develop from the leading edge ($x = 0$) of the plate. The boundary layer flow is assumed to be incompressible. Our task is to determine the growth of the thermal boundary layer, the temperature distribution and finally the rate of heat transfer from the plate. For this purpose we have to solve Eq. (7.25). Before proceeding for the solution of Eq. (7.25), we like to mention that Eq. (7.25) can be derived independently by applying the principle of conservation of energy



(a) Growth of boundary layers

(b) A fluid element as a control volume within boundary layer

Figure 7.2 Flow over a flat plate with heat transfer.

to a control volume within the boundary layer. The derivation is shown here in a simple way for a clear understanding of the principle.

For a fluid element as the control volume as shown in Figure 7.2(b) the principle of conservation of energy for steady incompressible flow can be written as

Energy convected in through the left face and the bottom face + heat conducted in through the bottom face

=

Energy convected out through the right face and the top face + heat conducted out through the top face (7.29)

Since the thickness of the thermal boundary layer is very small compared to the length of the plate, the axial conduction, i.e. conduction in the x-direction is neglected compared to that in the y-direction. As the changes in kinetic energy and potential energy are neglected, the energy convection comprises only the enthalpy of the fluid which, in consideration of the fluid to be either a perfect gas or a liquid, is taken as c_pT. All the energy quantities crossing the surface of the fluid element are shown in Figure 7.2(b). The width of the element in a direction perpendicular to the plane of the figure is taken to be unity.

Substituting the energy quantities shown in Figure 7.2(b), in Eq. (7.29), we have

$$\rho c_p\left[u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + T\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] = k\frac{\partial^2 T}{\partial y^2}$$

Using the continuity relation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

we finally have

$$\rho c_p\left[u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right] = k\frac{\partial^2 T}{\partial y^2} \tag{7.30}$$

Similarity solution

In a similar way as done for the solution of hydrodynamic boundary layer over a flat plate, the solution of Eq. (7.30) scan be obtained by the same similarity transformation as given by Eq. (6.61b)

$$\eta = y\sqrt{U_\infty/\nu x} \tag{7.31a}$$

so that

$$T^* = \varphi(\eta) \tag{7.31b}$$

where the non-dimensional temperature $T^* = \dfrac{T - T_w}{T_\infty - T_w}$.

The temperature derivatives of Eq. (7.30) become

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}$$

$$= \frac{\partial \varphi}{\partial \eta}\cdot\left(-\frac{1}{2}y\sqrt{\frac{U_\infty}{\nu x}}\cdot\frac{1}{x}\right)(T_\infty - T_w)$$

$$= -\frac{1}{2}\cdot\frac{\eta}{x}\frac{\partial \varphi}{\partial \eta}(T_\infty - T_w)$$

$$\frac{\partial T}{\partial y} = \frac{\partial \varphi}{\partial \eta}\sqrt{\frac{U_\infty}{\nu x}}\,(T_\infty - T_w)$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 \varphi}{\partial \eta^2}\frac{U_\infty}{\nu x}(T_\infty - T_w)$$

Substituting the above relations and the relations for u and v from Eqs. (6.62b) and (6.62c) in Eq. (7.30), it becomes

$$\frac{d^2\varphi}{d\eta^2} + \frac{\Pr}{2}f\frac{d\varphi}{d\eta} = 0 \tag{7.32}$$

where the function f is defined by Eq. (6.62a).

It has to be noted that the dependence of the temperature solution on velocity field appears through the term f in Eq. (7.32). The boundary conditions for the solution of Eq. (7.32) are

$$T^*(0) = 0 \quad \text{and} \quad T^*(\infty) = 1$$

i.e.

$$\text{at } \eta = 0,\ \varphi = 0; \quad \text{at} \quad \eta \to \infty,\ \varphi = 1$$

Equation (7.32) can be solved numerically for different values of Prandtl number Pr. However, the solution procedure is excluded from the scope of this chapter. There are two limiting cases: One is for very small Prandtl numbers (Pr $\to$ 0) which occur in case of molten metals and the other is for very large Prandtl numbers (Pr $\to \infty$) which occur in case of heavy liquids and oils. For Pr $\to$ 0 (Figure 7.3(a)), it is possible to disregard the hydrodynamic boundary layer. Consequently, the velocity component u in thermal boundary layer is replaced by $U_\infty(x)$, and v, with the aid of continuity, by $-(dU_\infty/dx)y$. However, for a flat plate $dU_\infty/dx = 0$. The second limiting case, Pr $\to \infty$ (Figure 7.3(b)) was solved by M.A. Leveque. He introduced the reasonable assumption that the whole of the temperature field is confined inside that zone of the velocity field where the longitudinal velocity component, u, is still proportional to the transverse

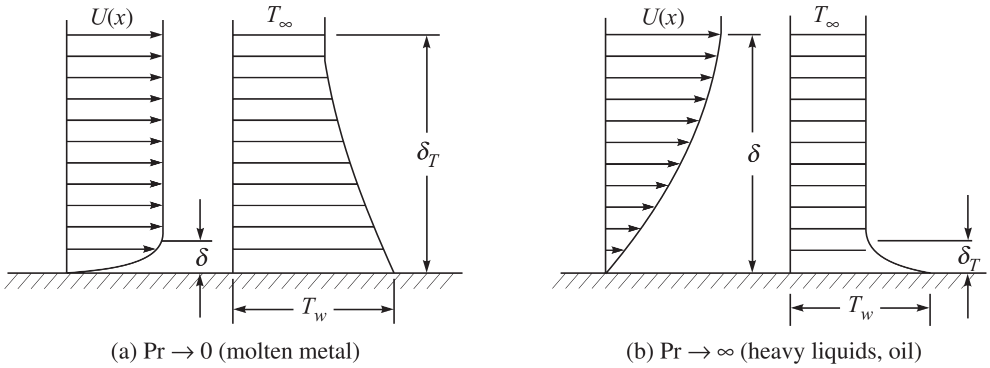


Figure 7.3 Temperature and velocity profiles for flow over flat plates for low and very high Prandtl numbers.

distance y. However, the important consequence of the solutions for all these cases pertains to the surface temperature gradient which can be written as

$$\varphi'(0) = \left(\frac{dT^*}{d\eta}\right)_{\eta=0}$$

$$= 0.332\ \mathrm{Pr}^{1/3} \qquad 0.6 < \mathrm{Pr} < 10 \tag{7.33a}$$

$$= 0.564\ \mathrm{Pr}^{1/2} \qquad \mathrm{Pr} \to 0 \tag{7.33b}$$

$$= 0.339\ \mathrm{Pr}^{1/3} \qquad \mathrm{Pr} \to \infty \tag{7.33c}$$

$(\varphi' = d\varphi/d\eta)$.

The local heat transfer coefficient h_x can be written as

$$h_x = \frac{-k\left(\dfrac{\partial T}{\partial y}\right)_{y=0}}{T_w - T_\infty}$$

$$= k\,\varphi'(0)\cdot\sqrt{\frac{U_\infty}{\nu x}}$$

Therefore, the local Nusselt number becomes

$$\mathrm{Nu}_x = \frac{h_x x}{k} = \varphi'(0)\cdot\sqrt{\frac{U_\infty x}{\nu}}$$

With the help of Eqs. (7.33a) to (7.33c), we can write the expressions for local Nusselt number as

$$\mathrm{Nu}_x = 0.332(\mathrm{Re}_x)^{1/2}\,\mathrm{Pr}^{1/3} \qquad 0.6 < \mathrm{Pr} < 10 \tag{7.34a}$$

$$= 0.564(\mathrm{Re}_x)^{1/2}\,\mathrm{Pr}^{1/2} \qquad \mathrm{Pr} \to 0 \tag{7.34b}$$

$$= 0.339(\mathrm{Re}_x)^{1/2}\,\mathrm{Pr}^{1/3} \qquad \mathrm{Pr} \to \infty \tag{7.34c}$$

where $\mathrm{Re}_x = U_\infty x/\nu$

It can be shown that the non-dimensional temperature gradient $(\partial T^*/\partial \eta)_{\eta=0}$ is proportional to the ratio of hydrodynamic boundary layer thickness to thermal boundary layer thickness. This is as follows:

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} \sim \frac{T_\infty - T_w}{\delta_T}$$

From Eq. (7.31a), we have

$$\frac{\partial y}{\partial \eta} = \sqrt{\frac{\nu x}{U_\infty}} \sim \delta$$

Then,

$$\left(\frac{\partial T^*}{\partial \eta}\right)_{\eta=0} = \frac{1}{T_\infty - T_w}\left(\frac{\partial T}{\partial y}\right)_{y=0}\left(\frac{\partial y}{\partial \eta}\right)$$

$$\sim \frac{\delta}{\delta_T}$$

Therefore, we can write with the help of Eqs. (7.33a) to (7.33c)

$$\frac{\delta}{\delta_T} \sim \mathrm{Pr}^{1/3} \qquad \text{(except for very low Prandtl numbers)} \tag{7.35a}$$

$$\frac{\delta}{\delta_T} = \mathrm{Pr}^{1/2} \qquad \text{(for very low Prandtl numbers)} \tag{7.35b}$$

Solution by energy integral method

The thermal boundary layesr equation, i.e. Eq. (7.30) can also be solved by the method of energy integral which appears to be relatively simple. To do this, we approach in a similar way as done for the solution of hydrodynamic boundary layer equation by momentum integral method. The first step is to derive an energy integral equation as follows:

Let us integrate Eq. (7.30) within the thermal boundary layer of thickness δ_T which is considered to be smaller than the hydrodynamic boundary layer thickness δ. Therefore, in consideration of the constant properties, we have

$$\rho c_p\left[\int_0^{\delta_T} u\frac{\partial T}{\partial x}dy + \int_0^{\delta_T} v\frac{\partial T}{\partial y}dy\right] = k\int_0^{\delta_T} \frac{\partial^2 T}{\partial y^2}dy \tag{7.36}$$

The second term on left-hand side of Eq. (7.36) can be written as

$$\int_0^{\delta_T} v\frac{\partial T}{\partial y}dy = \int_0^{\delta_T}\left[\frac{\partial}{\partial y}(vT) - T\frac{\partial v}{\partial y}\right]dy = T_\infty (v)_{\mathrm{at}\,\delta_T} - \int_0^{\delta_T} T\frac{\partial v}{\partial y}dy$$

Again from continuity,

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

$$(v)\delta_T = -\int_0^{\delta_T}\frac{\partial u}{\partial x}dy$$

Therefore, it becomes

$$\int_0^{\delta_T} v\frac{\partial T}{\partial y}dy = -T_\infty\int_0^{\delta_T}\frac{\partial u}{\partial x}dy + \int_0^{\delta_T} T\frac{\partial u}{\partial x}dy$$

With the substitution of the above relation, we have for the left-hand side of Eq. (7.36) as

$$\rho c_p \left[\int_0^{\delta_T} \frac{\partial}{\partial x}(uT)\,dy - \int_0^{\delta_T} \frac{\partial}{\partial x}(uT_\infty)\,dy \right]$$

According to the Leibnitz rule, the first and second terms of the above expression can be written as

$$\int_0^{\delta_T} \frac{\partial}{\partial x}(uT)dy = \frac{d}{dx}\int_0^{\delta_T}(uT)dy - u_{\delta_T} T_\infty \frac{d\delta_T}{dx}$$

$$\int_0^{\delta_T} \frac{\partial}{\partial x}(uT_\infty)dy = \frac{d}{dx}\int_0^{\delta_T}(uT_\infty)dy - u_{\delta_T} T_\infty \frac{d\delta_T}{dx}$$

Hence the left-hand side of Eq. (7.36) finally becomes

$$\rho c_p \frac{d}{dx}\int_0^{\delta_T} u(T - T_\infty)dy$$

The right-hand side of Eq. (7.36) becomes

$$-k\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

[Since $\left(\frac{\partial T}{\partial y}\right)_{y=\delta_T} = 0$, by the definition of thermal boundary layer]

Now, Eq. (7.36) becomes

$$\frac{d}{dx}\int_0^{\delta_T} u(T - T_\infty)dy = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{7.37}$$

This is the required energy integral equation.

The next task is to assume a temperature profile within the thermal boundary layer so that it satisfies the requisite boundary conditions as follows:

$$T = T_w \quad \text{at } y = 0 \tag{7.38a}$$

$$T = T_\infty \quad \text{at } y = \delta_T \tag{7.38b}$$

$$\frac{\partial T}{\partial y} = 0 \quad \text{at } y = \delta_T \tag{7.38c}$$

$$\frac{\partial^2 T}{\partial y^2} = 0 \quad \text{at } y = 0 \tag{7.38d}$$

A cubic temperature profile of the type

$$\frac{T - T_w}{T_\infty - T_w} = T^* = \frac{3}{2}\left(\frac{y}{\delta_T}\right) - \frac{1}{2}\left(\frac{y}{\delta_T}\right)^3 \tag{7.39}$$

satisfies all the boundary conditions as stated in Eqs. (7.38a) to (7.38d). Substituting the temperature profile from Eq. (7.39) and velocity profile from Eq. (6.79) in Eq. (7.37), we have

$$(T_\infty - T_w)U_\infty \frac{d}{dx}\int_0^{\delta_T}\left[1-\frac{3}{2}\left(\frac{y}{\delta_T}\right)+\frac{1}{2}\left(\frac{y}{\delta_T}\right)^3\right]\left[\frac{3}{2}\left(\frac{y}{\delta}\right)-\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]dy$$

$$= \frac{3\alpha(T_\infty - T_w)}{2\delta_T}$$

It has to be noted that the integration is carried out to $y = \delta_T$, since the integrand is zero for $y > \delta_T$. Performing the necessary algebraic manipulations and carrying out the integration, it becomes

$$U_\infty \frac{d}{dx}\left[\delta\left(\frac{3}{20}\zeta^2 - \frac{3}{280}\zeta^4\right)\right] = \frac{3}{2}\frac{\alpha}{\zeta\delta} \tag{7.40}$$

where $\zeta = \delta_T/\delta$. Since we have considered $\delta_T < \delta$, $\zeta < 1$, the term involving ζ^4 is small compared to the term involving ζ^2. Therefore, we neglect the ζ^4 term and write

$$\frac{3}{20}U_\infty \frac{d}{dx}(\delta\zeta^2) = \frac{3}{2}\frac{\alpha}{\zeta\delta}$$

or

$$\frac{1}{10}U_\infty\left(2\delta^2\zeta^2\frac{d\zeta}{dx} + \zeta^3\delta\frac{d\delta}{dx}\right) = \alpha \tag{7.41}$$

But we know from Eq. (6.81),

$$\delta\frac{d\delta}{dx} = \frac{140}{13}\frac{\nu}{U_\infty}$$

or

$$\delta^2 = \frac{280}{13}\frac{\nu x}{U_\infty}$$

Then Eq. (7.41) becomes

$$\zeta^3 + 4x\zeta^2\frac{d\zeta}{dx} = \frac{13}{14}\frac{\alpha}{\nu}$$

or

$$\zeta^3 + \frac{4}{3}x\frac{d}{dx}(\zeta^3) = \frac{13}{14}\frac{\alpha}{\nu}$$

Multiplying both sides by the integrating factor $\frac{3}{4}x^{-1/4}$, we have

$$\left(\frac{3}{4}x^{-1/4}\right)\zeta^3 + x^{3/4}\frac{d}{dx}(\zeta^3) = \frac{39}{56}\frac{\alpha}{\nu}x^{-1/4}$$

or

$$\frac{d}{dx}(\zeta^3\, x^{3/4}) = \frac{39}{56}\frac{\alpha}{\nu}x^{-1/4}$$

After integration, it becomes

$$\zeta^3 = \frac{13}{14}\frac{\alpha}{\nu} + C_1 x^{-3/4} \qquad \text{(where } C_1 \text{ is a constant)}$$

If the plate, under consideration is heated after a length of x_0 from the leading edge, then at $x = x_0$, $\zeta = 0$. This gives

$$\zeta = \left(\frac{39}{56}\frac{\alpha}{\nu}\right)^{1/3}\left[1-\left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3}$$

$$= \frac{1}{1.026}\mathrm{Pr}^{-1/3}\left[1-\left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3} \tag{7.42a}$$

If the plate is heated from the leading edge, then $x_0 = 0$ and we get

$$\zeta = \left(\frac{13}{14}\frac{\alpha}{\nu}\right)^{1/3} \tag{7.42b}$$

$$= \frac{1}{1.026}\mathrm{Pr}^{-1/3}$$

It is apparent from the above equation that the assumption $\zeta < 1$ in the present analysis is valid for Pr > 0.7. Fortunately, most gases and liquids fall within this category. Liquid metals form a notable exception, since they have Prandtl numbers of the order of 0·1.

The local heat transfer coefficient h_x becomes

$$h_x = \frac{-k(\partial T/\partial y)_{y=0}}{T_w - T_\infty} = \frac{3}{2}\frac{k}{\zeta\delta} \tag{7.43}$$

By making use of Eq. (6.83) for δ, and any one of the above two equations (Eqs. (7.42a) and (7.42b)) for ζ, depending upon the case, we have the following relations:

(a) For a plate heated after a length of x_0 from the leading edge,

$$\mathrm{Nu}_x = \frac{h_x x}{k} = 0.332(\mathrm{Re}_x)^{1/2}\mathrm{Pr}^{1/3}\left[1-\left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3} \tag{7.44a}$$

(b) For a plate heated over its entire length,

$$\mathrm{Nu}_x = \frac{h_x x}{k} = 0.332(\mathrm{Re}_x)^{1/2}\mathrm{Pr}^{1/3} \tag{7.44b}$$

For the case where $x_0 = 0$ (the plate is heated over its entire length), the average heat transfer coefficient may be obtained as

$$\bar{h}_L = \frac{1}{L}\int_0^L h_x\,dx \tag{7.45}$$

With the help of Eq. (7.44b) it can be shown that

$$\bar{h}_L = 2h_{x=L}$$

Hence,
$$\overline{\mathrm{Nu}_L} = 2\mathrm{Nu}_{x=L} = 0.664(\mathrm{Re}_L)^{1/2}\mathrm{Pr}^{1/3} \tag{7.46}$$

where
$$\mathrm{Re}_L = \rho U_\infty L/\mu.$$

The foregoing analysis was based on the assumption that the fluid properties were constant throughout the flow. When there is an appreciable difference between the free stream temperature and the plate temperature, the properties should be evaluated at the so called film temperature T_f, defined as the arithmetic mean between the plate and free stream temperature [$T_f = (T_w + T_\infty)/2$].

Constant heat flux at plate surface

In the above analysis we assumed the plate temperature to be constant. In many practical problems the plate may not be isothermal, rather a heat flux exists at the plate which may remain constant over the plate. It can be shown that for a situation of constant heat flux at plate surface,

$$\mathrm{Nu}_x = \frac{h_x x}{k} = 0.453\,(\mathrm{Re}_x)^{1/2}\mathrm{Pr}^{1/3}$$

Churchill and Ozee[1] developed a single relation of Nu with Re and Pr from their experimental data for a wide range of Prandtl numbers for a laminar flow over an isothermal flat plate as

$$\mathrm{Nu}_x = \frac{0.3387\,(\mathrm{Re}_x)^{1/2}\,\mathrm{Pr}^{1/3}}{\left[1+\left(\dfrac{0.0468}{\mathrm{Pr}}\right)^{2/3}\right]^{1/4}}$$

In case of constant heat flux at plate surface, the same relation can be used with the change in the values of the constants as

$$\mathrm{Nu}_x = \frac{0.4637\,(\mathrm{Re}_x)^{1/2}\,\mathrm{Pr}^{1/3}}{\left[1+\left(\dfrac{0.0207}{\mathrm{Pr}}\right)^{2/3}\right]^{1/4}}$$

Integral method for very low Prandtl number fluids (liquid metal heat transfer past a flat plate)

In practice, liquid metals are the fluids with very low Prandtl number. This is due to very high thermal conductivity of liquid metal. Therefore, the rate of heat conduction through liquid metals is very high compared to other fluids. As a consequence, the liquid metals are used where a large quantity of heat has to be removed from a small space, as in a nuclear reactor. The only difficulty in their use lies in handling them. They are corrosive and some may cause violent reactions when they come into contact with air or water.

Let us consider a parallel flow of a liquid metal past a flat plate. The thickness of the hydrodynamic boundary layer will be much smaller compared to that of the thermal boundary layer under the situation as shown in Figure 7.3(a). Therefore, one may expect that the temperature distribution and hence the heat transfer is not influenced by the fluid viscosity. As a first approximation, we may assume a slug-flow model for the calculation of the temperature profile. Such an approach for the exact solution of the thermal boundary layer equation has been discussed earlier. We will discuss the same approach for the solution of the thermal boundary layer equation by the integral method.

By assuming $u = U_\infty$ throughout the thermal boundary layer, we can write the energy integral equation as

$$\theta_\infty U_\infty \frac{d}{dx}\left[\delta_T \int_0^1 \left(1-\frac{3}{2}\eta+\frac{1}{2}\eta^3\right)d\eta\right] = \frac{3\alpha\theta_\infty}{2\delta_T} \tag{7.47}$$

where $$\frac{\theta}{\theta_\infty} = \frac{T - T_w}{T_\infty - T_w} = \frac{3}{2}\eta - \frac{1}{2}\eta^3$$

and $$\eta = y/\delta_T$$

From Eq. (7.47), we finally get

$$\delta_T \frac{d\delta_T}{dx} = \frac{4\alpha}{U_\infty}$$

The solution of the above equation with the boundary condition at $x = 0$, $\delta_T = 0$, gives

$$\delta_T = \sqrt{\frac{8\alpha x}{U_\infty}} \tag{7.48}$$

For local heat transfer coefficient,

$$h_x = \frac{-k(\partial T/\partial y)_{\text{wall}}}{T_w - T_\infty} = \frac{3k}{2\delta_T}$$

Substituting for δ_T from Eq. (7.48), we obtain

$$h_x = \frac{3\sqrt{2}}{8} k \sqrt{\frac{U_\infty}{\alpha x}}$$

$$\text{Nu}_x = \frac{h_x x}{k} = 0.530(\text{Re}_x \text{Pr})^{1/2}$$

$$= 0.530(\text{Pe}_x)^{1/2} \tag{7.49}$$

where $\text{Pe}_x (= \text{Re}_x \text{Pr})$ is the local Peclet number.

From Eq. (6.83), we can write for hydrodynamic boundary layer thickness

$$\delta = \frac{4.64}{\sqrt{\text{Re}_x}}$$

With the help of Eq. (7.48) and the above equation, we have

$$\frac{\delta}{\delta_T} = 1.64\sqrt{\text{Pr}}$$

The values of the Prandtl number for liquid metals vary from 0·02 to 0·003. For a Prandtl number of 0·02,

$$\frac{\delta}{\delta_T} = 0.232$$

The relation between fluid friction and heat transfer

We have already noticed that the temperature field and the flow field are closely related. The flow field directly influences the temperature field due to the presence of velocity components in the convective terms of the energy equation. On the other hand, though the Navier–Stokes equations do not contain any temperature term, the properties like viscosity and density are the functions of temperature. Therefore the temperature field influences the velocity field if the changes in properties due to temperature are significant and being accounted for, while the velocity field always influences the temperature field. We can find a relationship between fluid friction and heat transfer from similar type expressions for momentum transfer and heat transfer at the plate surface.

From an exact solution of hydrodynamic boundary layer equation, we have determined the skin friction coefficient as a function of the Reynolds number for a flow over a flat plate in Chapter 6 (Section 6.4.2). We can write, following Eq. (6.65)

$$\frac{c_{f_x}}{2} = 0.332(\mathrm{Re}_x)^{-1/2} \tag{7.50}$$

Again from an exact solution of the thermal boundary layer equation, as described in Section 7.4.1, we can write, following Eq. (7.34a)

$$\mathrm{Nu}_x = 0.332(\mathrm{Re}_x)^{1/2}\mathrm{Pr}^{1/3}$$

or

$$\frac{\mathrm{Nu}_x}{\mathrm{Re}_x\,\mathrm{Pr}}\cdot\mathrm{Pr}^{2/3} = 0.332(\mathrm{Re}_x)^{-1/2} \tag{7.51}$$

The group '$\mathrm{Nu}_x/\mathrm{Re}_x\mathrm{Pr}$' is known as Stanton number St_x. Since the right-hand side of Eqs. (7.50) and (7.51) are equal, we have

$$\mathrm{St}_x\,\mathrm{Pr}^{2/3} = \frac{c_{f_x}}{2} \tag{7.52}$$

The relation given by Eq. (7.52) is known as Reynolds–Colburn analogy. Though it appears to be valid only for a laminar boundary layer over a flat plate, it has been found that the relation is fairly accurate for its use in turbulent boundary layer. However Eq. (7.52) is applicable only for parallel flow over flat plates, not for other geometries. This simple analogy stems from the fact that the transport of momentum due to fluid viscosity and transport of heat due to thermal diffusivity are similar in nature and the hydrodynamic and thermal boundary layer equations look almost identical. In case of Pr = 1, the two equations become exactly identical and their solutions acquire identical algebraic forms.

A general form for the analogy between heat transfer and skin friction

For boundary-layer flows, there exists a remarkable relationship between skin friction and heat transfer. The simplest form in case of a flat plate, as given by Eq. (7.52), was derived by Osborn Reynolds.

In general, the solution of the two-dimensional boundary-layer equation for an incompressible flow past a solid surface takes the form

$$\frac{u}{U_\infty} = F_1\left(\frac{x}{L},\frac{y}{L}\sqrt{\mathrm{Re}_L}\right) \tag{7.53}$$

$$\frac{v}{U_\infty}\sqrt{\mathrm{Re}_x} = F_2\left(\frac{x}{L},\frac{y}{L}\sqrt{\mathrm{Re}_L}\right) \tag{7.54}$$

With the same reasoning, the solution of the thermal boundary-layer equation (Eq. (7.25)) for incompressible flows takes the form

$$T^* = \frac{T-T_\infty}{T_w-T_\infty} = F_3\left(\frac{x}{L},\frac{y}{L}\sqrt{\mathrm{Re}_L},\mathrm{Pr}\right) \tag{7.55}$$

where, x is the coordinate measured from the leading edge along the surface in the direction of flow, y is the transverse coordinate measured from the surface, and L is a representative length scale.

By making use of Eq. (7.53), we can write the expression for the local skin friction coefficient as follows:

Wall shear stress: $\tau_w = \mu \left(\dfrac{\partial u}{\partial y} \right)_{y=0} = \dfrac{\mu U_\infty \sqrt{\mathrm{Re}_L}}{L} F_1' \left(\dfrac{x}{L} \right)$ (where, $F_1' = (\partial F_1/\partial y)_{y=0}$)

$$= \frac{\mu U_\infty \sqrt{\mathrm{Re}_L}}{L} \varphi_1 \left(\frac{x}{L} \right)$$

Local skin friction coefficient: $c_f = \dfrac{\tau_w}{\dfrac{1}{2}\rho U_\infty^2}$

$$= \frac{2}{\sqrt{\mathrm{Re}_L}} \varphi_1 \left(\frac{x}{L} \right) \tag{7.56}$$

The local Nusselt number is found out with the help of Eq. (7.55) as

$$\mathrm{Nu}_x = \frac{hL}{k} = \frac{\left| -k \left(\dfrac{\partial T}{\partial y} \right)_{y=0} \right| L}{k (T_w - T_\infty)} = \sqrt{\mathrm{Re}_L}\, F_3' \left(\frac{x}{L}, \mathrm{Pr} \right) \quad \text{(where, } F_3' = (\partial F_3/\partial y)_{y=0}\text{)}$$

$$= \sqrt{\mathrm{Re}_L}\, \varphi_2 \left(\frac{x}{L}, \mathrm{Pr} \right) \tag{7.57}$$

On combining Eqs. (7.56) and (7.57), we obtain

$$\mathrm{Nu}_x = \frac{1}{2} c_f \mathrm{Re}_L \varphi \left(\frac{x}{L}, \mathrm{Pr} \right) \tag{7.58}$$

where
$$\varphi \left(\frac{x}{L}, \mathrm{Pr} \right) = \frac{\varphi_2 \left(\dfrac{x}{L}, \mathrm{Pr} \right)}{\varphi_1 \left(\dfrac{x}{L} \right)}$$

Equation (7.58) is the most general form of Reynolds analogy which is valid for all laminar boundary layers.

The explicit form of the functional relation $\varphi \left(\dfrac{x}{L}, \mathrm{Pr} \right)$ in any situation of boundary layer flow has to be generated from the solution of both hydrodynamic and thermal boundary layer equations for the situation. For a flat plate, as discussed earlier, the local Nusselt number and the skin friction coefficient are formed with coordinate x and $\varphi \left(\dfrac{x}{L}, \mathrm{Pr} \right) = \varphi(\mathrm{Pr}) = \mathrm{Pr}^{1/3}$. Therefore, the general equation (Eq. (7.58)) reduces as a special case to Eq. (7.52) for a flat plate.

Turbulent boundary layer over a flat plate

In Chapter 6, we discussed the turbulent boundary layer over a flat plate. The expressions for the growth of boundary layer thickness and the local skin friction coefficient were also developed. The analytical treatment for heat transfer studies requires the solution of thermal boundary layer equation with the aid of appropriate velocity profile. The analysis with universal velocity profile has met the purpose with a good success. The simplest way of predicting the heat transfer coefficient in a turbulent boundary layer is to use the Reynolds–Colburn analogy (Eq. (7.52)) and to substitute the appropriate value of c_{f_x} for turbulent flow.

The suitability of Eq. (7.52) in turbulent boundary layer has been verified by experiments. In a turbulent flow, thermal diffusivity is increased many times over that due to molecular diffusion because of energy transport by macroscopic fluid eddies. In analogy to momentum transport, we define an eddy thermal diffusivity ε_{th} which is much larger than its molecular counterpart ($\varepsilon_{\text{th}} >> \alpha$). Accordingly. a turbulent Prandtl number is defined as $\text{Pr}_{\text{th}} = \varepsilon_{\text{M}}/\varepsilon_{\text{th}}$, where ε_{M} is the eddy momentum diffusivity. If we can expect that the eddy momentum transport and eddy energy transport both increase in the same proportion compared with their molecular values, we might anticipate that the heat transfer coefficient can be calculated by Eq. (7.52) with the use of molecular Prandtl number ($\text{Pr} = \nu/\alpha$).

In consideration of the boundary layer over the entire plate to be turbulent, we can very well use Eq. (6.121) for the skin friction coefficient and can write with the aid of Eq. (7.52)

$$\text{St}_x \, \text{Pr}^{2/3} = 0.0296(\text{Re}_x)^{-1/5} \tag{7.59a}$$

or

$$\text{Nu}_x = 0.0296(\text{Re}_x)^{4/5}\,\text{Pr}^{1/3} \tag{7.59b}$$

For average Nusselt number, we can write

$$\overline{\text{Nu}_L} = 0.037(\text{Re}_L)^{4/5}\text{Pr}^{1/3} \tag{7.58c}$$

The range of applicability of the above equation is $5 \times 10^5 < \text{Re} < 10^7$.

If we consider, in accordance with reality, that the transition occurs from laminar to turbulent boundary layer in course of flow over the plate, we have to use Eq. (6.123) for the skin friction coefficient. Then, we have to develop the expression for average Nusselt number as follows:

The average Nusselt number over a length L is written as

$$\overline{\text{Nu}_L} = \frac{\bar{h}_L L}{k}$$

Under the situation,

$$\bar{h}_L = \frac{1}{L}\left(\int_0^{x_c} h_{x_{\text{lam}}}\, dx + \int_{x_c}^{L} h_{x_{\text{turb}}}\, dx \right) \tag{7.60}$$

From Eq. (7.44b),

$$\text{Nu}_{x\,\text{lam}} = \frac{h_{x_{\text{lam}}}\, x}{k} = 0.332\left(\frac{U_\infty x}{\nu}\right)^{1/2} \text{Pr}^{1/3}$$

or

$$h_{x\,\text{lam}} = 0.332\; k\left(\frac{U_\infty}{\nu x}\right)^{1/2} \text{Pr}^{1/3} \tag{7.61a}$$

Again from Eq. 7.59(b),

$$\text{Nu}_{x\,\text{turb}} = \frac{h_{x_{\text{turb}}}}{k} = 0.296\left(\frac{U_\infty x}{\nu}\right)^{4/5} \text{Pr}^{1/3}$$

or

$$h_{x\,\text{turb}} = 0.296\, k\left(\frac{U_\infty}{\nu}\right)^{4/5} \frac{1}{x^{1/5}}\, \text{Pr}^{1/3} \tag{7.61b}$$

Substituting the values of $h_{x\,\text{lam}}$ and $h_{x\,\text{turb}}$ from Eqs. (7.61a) and (7.61b) into Eq. (7.60), we have

$$\bar{h}_L = \left(\frac{k}{L}\right)\left[0.032\left(\frac{U_\infty}{\nu}\right)^{1/2}\int_0^{x_c}\frac{dx}{x^{1/2}} + 0.0296\left(\frac{U_\infty}{\nu}\right)^{4/5}\int_{x_c}^{L}\frac{dx}{x^{1/5}}\right]\text{Pr}^{1/3}$$

$$\overline{\text{Nu}_L} = \frac{\bar{h}_L L}{k} = [0.664\,(\text{Re}_{x,c})^{1/2} + 0.037\{(\text{Re}_L)^{4/5} - (\text{Re}_{x,c})^{4/5}\}\text{Pr}^{1/3} \tag{7.62}$$

where $\quad \text{Re}_{x,c}$ (the critical Reynolds number) $= \dfrac{U_\infty x_c}{\nu}$

and $\quad \text{Re}_L$ (the local Reynolds number) $= \dfrac{U_\infty L}{\nu}$

If we take $\text{Re}_{x,c} = 5 \times 10^5$, then Eq. (7.62) becomes

$$\overline{\text{Nu}_L} = 0.037\{(\text{Re}_L)^{4/5} - 871)\}\text{Pr}^{1/3} \tag{7.63}$$

The range of applicability of Eq. (7.63) is

$$0{\cdot}6 < \text{Pr} < 60$$
$$5 \times 10^5 < \text{Re}_L \le 10^8$$
$$\text{Re}_{x,c} = 5 \times 10^5$$

It has been mentioned earlier that for all correlations of Nusselt number with Reynolds number and Prandtl number, the property values have to be evaluated at film condition, i.e. at a temperature which is the arithmetic mean of the plate temperature and the free stream temperature. Whitaker [2] suggested a correlation that gives better results for some liquids whose viscosity is a strong function of temperature. This is,

$$\overline{\text{Nu}_L} = 0.036\,\text{Pr}^{0{\cdot}43}\,[(\text{Re}_L)^{0.8} - 9200]\left(\frac{\mu_\infty}{\mu_w}\right)^{1/4}$$

All properties are evaluated at the free stream temperature, while μ_w is the viscosity at the plate temperature.

Flat plate with constant surface heat flux

The Nusselt number for a parallel flow over a flat plate with constant surface heat flux is 36 per cent larger than the constant surface temperature result for laminar flow, while it is 4 per cent larger in case of turbulent flow. Therefore, we can write for laminar flow with constant surface heat flux

$$\text{Nu}_x = 0.453\,(\text{Re}_x)^{1/2}\,\text{Pr}^{1/3} \tag{7.64a}$$
$$(0.6 < \text{Pr} < 10)$$

and for turbulent flow with constant heat flux

$$\text{Nu}_x = 0.0308\,(\text{Re}_x)^{4/5}\,\text{Pr}^{1/3} \tag{7.64b}$$
$$\begin{pmatrix} 0.6 < \text{Pr} < 60 \\ 5 \times 10^5 < \text{Re} \le 10^7 \end{pmatrix}$$

7.4.2 Heat Transfer from Cylinder in Cross Flow

The behaviour of hydrodynamic boundary layer for flow past a circular cylinder in a direction perpendicular to the cylinder axis has been discussed in Chapter 6 (Section 6.4). The theoretical treatment for the solution of hydrodynamic boundary layer and thermal boundary layer even for a laminar flow is very complicated and is beyond the scope of the book. A few popular empirical correlations at different ranges of their applicability will be discussed here.

An expression for the average Nusselt number for flow of fluids past a circular cylinder was proposed by Whitekar [3] as

$$\overline{\mathrm{Nu}_D} = (0.4\mathrm{Re}^{1/2} + 0.06\mathrm{Re}^{2/3})\mathrm{Pr}^{0.4}\left(\frac{\mu_\infty}{\mu_w}\right)^{0.25} \tag{7.65}$$

The range of applicability is

$$40 < \mathrm{Re} < 10^5$$
$$0.67 < \mathrm{Pr} < 300$$
$$0.25 < \frac{\mu_\infty}{\mu_w} < 5.2$$

The physical properties are evaluated at the free-stream temperature except for μ_w, which is evaluated at the wall temperature. For gases, the viscosity correction is neglected and for such cases the properties are evaluated at the film temperature. It is found from Eq. (7.65) that Nusselt number has two different forms of functional dependence with Reynolds number. The functional dependence of $\mathrm{Re}^{0.5}$ characterizes the contribution from the undetached laminar boundary layer region, and that of $\mathrm{Re}^{2/3}$ characterizes the contribution from the wake region around the cylinder. A more general correlation is given by Churchill and Bernstein [4] as

$$\overline{\mathrm{Nu}}_D = 0.3 + \frac{0.62(\mathrm{Re}_D)^{1/2}\,\mathrm{Pr}^{1/3}}{[1+(0.4/\mathrm{Pr})^{2/3}]^{1/4}}\left[1+\left(\frac{\mathrm{Re}_D}{282{,}000}\right)^{5/8}\right]^{4/5} \tag{7.66}$$

The range of applicability of the above equation is $10^2 < \mathrm{Re} < 10^7$ and Pe(Peclet number) > 0.2, where $\mathrm{Pe} = \mathrm{Re}_D\,\mathrm{Pr}$, and all properties are evaluated at the film temperature. Here the subscript D in Nu and Re indicates that they are defined on the basis of the cylinder diameter.

The variation of local Nusselt number along the surface of the cylinder as found by Giedt [5] is shown in Figure 7.4. There are two distinct qualitative trends of variation of Nu_θ with θ. At a lower Reynolds number ($\mathrm{Re}_D < 105$), the curves show only one minimum point. The Nusselt number first decreases from the stagnation point due to the growth of laminar boundary layer and then reaches a minimum value at $\theta = 80°$ where separation occurs. The Nusselt number increases again due to enhanced mixing in the wake region. When $\mathrm{Re}_D > 10^5$, the curves show two minima—one corresponds to the point (θ between 80° and 100°) where the transition takes place from laminar to turbulent and the other ($\theta \approx 140°$) corresponds to the point of boundary layer separation. It must be remembered in this context that the separation is delayed due to the turbulent boundary layer.

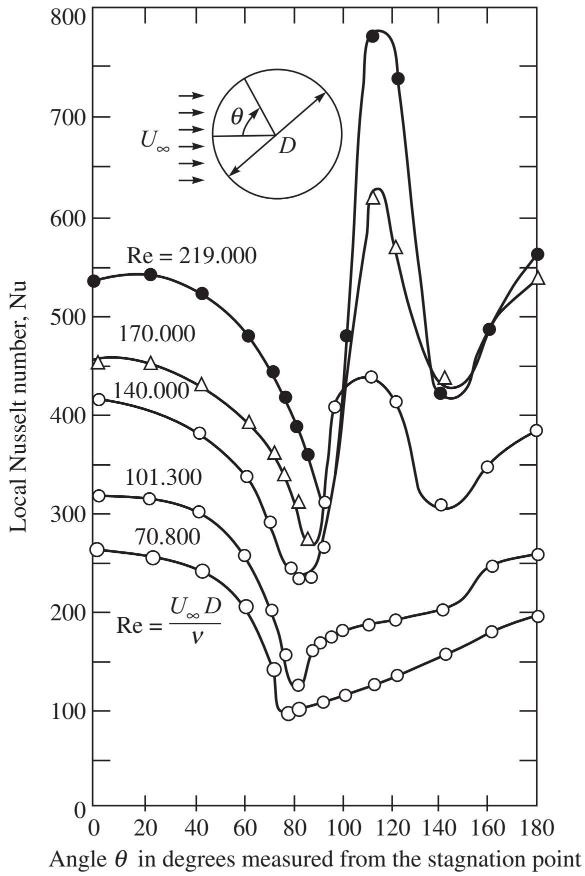


Figure 7.4 Variation of local Nusselt number around a circular cylinder in cross flow.

Hilpert [6] correlated from his experimental investigations the average Nusselt number for cross flow past a circular cylinder as

$$\overline{\mathrm{Nu}_D} = C(\mathrm{Re}_D)^n\,\mathrm{Pr}^{1/3} \tag{7.67}$$

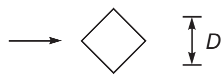
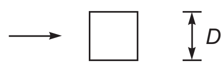
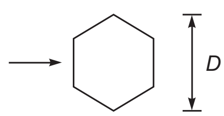
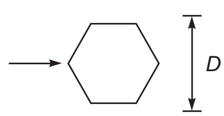
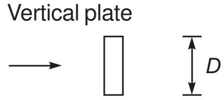
The constants C and n under different situations are given in Table 7. 1. The expression given by Eq. (7.67) is very simple. Jacob [7] recommended the use of the same equation even for non-circular cylinders. The values of C and n for different geometries and the characteristic dimension in defining Re_D are shown in Table 7.2. The values of C and n as shown in Tables 7.1 and 7.2 were generated from a curve fitting procedure of experimental data with air. Therefore, it is recommended that Eq. (7.67) be scaled for Prandtl number for its use in case of different fluids. What is to be done is to divide this equation by $(0.72)^{1/3}$ and multiply by the Prandtl number of the particular fluid raised to the power 1/3. Therefore, the modified form of Eq. (7.67) for fluids other than air is

$$\overline{\mathrm{Nu}_D} = 1.11\,C(\mathrm{Re}_D)^n\,\mathrm{Pr}^{1/3} \tag{7.68}$$

Table 7.1 The values of constants C and n of Eq. (7.67) applicable to circular cylinders

Re_D	C	n
0.4 – 4	0.989	0.330
4 – 40	0.911	0.385
40 – 4000	0.683	0.466
4000 – 40,000	0.193	0.618
40,000 – 400,000	0.027	0.805

Table 7.2 The values of constants C and n of Eq. (7.67) applicable to non-circular cylinders

Geometry	Re_D	C	n
Square			
	$5 \times 10^3 - 10^5$	0.246	0.588
	$5 \times 10^3 - 10^5$	0.102	0.675
Hexagon			
	$5 \times 10^3 - 1.95 \times 10^4$ $1.95 \times 10^4 - 10^5$	0.160 0.0385	0.638 0.782
	$5 \times 10^3 - 10^5$	0.153	0.638
Vertical plate			
	$4 \times 10^3 - 1.5 \times 10^4$	0.228	0.731

7.4.3 Heat Transfer in Flow Past a Sphere

The nature of boundary layer flow past a sphere is very similar to that past a circular cylinder, with transition and separation playing the prominent roles. Several correlations for average Nusselt number are available.

McAdams [8] recommended the following correlation for heat transfer from sphere to a flow of gas

$$Nu_D = 0.37\,C(Re_D)^{0.6} \tag{7.69}$$

for $17 < Re_D < 70{,}000$

and $Pr \approx 0.70$

The correlation of Whitakar [3], applicable to a much wider range, is of the form

$$Nu_D = 2 + [0.4(Re_D)^{1/2} + 0.06(Re_D)^{2/3}]\,Pr^{0.4}\left(\frac{\mu}{\mu_w}\right)^{1/4} \tag{7.70}$$

$$\text{for} \quad 0.71 < \Pr < 380$$
$$3.5 < \mathrm{Re}_D < 7.6 \times 10^4$$
$$1.0 < \left(\frac{\mu}{\mu_w}\right) < 3.2$$

All properties except μ_w are evaluated at T_∞, while μ_w is at T_w (the surface temperature). A very widely used correlation is due to Ranz and Marshall [9] and is given by

$$\overline{\mathrm{Nu}_D} = 2 + 0.66(\mathrm{Re}_D)^{1/2}\Pr^{1/3} \tag{7.71}$$

The above equation is valid in a much wider range of Reynolds number and Prandtl number. In the limit $\mathrm{Re}_D \to 0$. Equation (7.71) gives $\overline{\mathrm{Nu}_D} = 2$ which corresponds to heat conduction from a spherical surface to a stationary infinite medium surrounding the sphere. The result $\overline{\mathrm{Nu}_D} = 2$ can be found analytically by the straightforward application of one-dimensional heat conduction equation in a spherical coordinate system.

EXAMPLE 7.1 Engine oil at 60°C flows with a velocity of 1 m/s over a 5 m long flat plate whose temperature is 30°C. The flow of air is parallel to the length of the plate. Determine the rate of heat transfer per unit width of the entire plate. The properties of the engine oil at a film temperature of 45°C are as follows:

$$\rho = 870 \text{ kg/m}^3; \qquad \Pr = 2850;$$
$$k = 0.145 \text{ W/(m° C)}; \qquad \nu = 250 \times 10^{-6} \text{ m}^2\text{/s}.$$

Solution: First we check from the Reynolds number whether the flow is laminar or turbulent.

$$\mathrm{Re} = \frac{U_\infty L}{\nu} = \frac{1 \times 5}{250 \times 10^{-6}} = 2 \times 10^4$$

which is less than the critical Reynolds number. Thus we have laminar flow over the entire plate. The average Nusselt number over the entire length under the situation, is given by Eq. (7.46) as

$$\begin{aligned}\overline{\mathrm{Nu}_L} &= 0.664 \times \mathrm{Re}^{1/2}\Pr^{1/3} \\ &= 0.664 \times (2 \times 10^4)^{1/2}(2850)^{1/3} \\ &= 13.31 \times 10^2\end{aligned}$$

Then
$$h = \frac{k}{L}\cdot\overline{\mathrm{Nu}_L} = \frac{0.145}{5} \times 13.31 \times 10^2 = 38.60 \text{ W/(m}^2\text{ K)}$$

$$\begin{aligned}Q &= hA(T_\infty - T_w) \\ &= 38.60 \times 5 \times 1(60 - 30) \\ &= 5790 \text{ W} = 5.79 \text{ kW}\end{aligned}$$

EXAMPLE 7.2 Atmospheric air at $T_\infty = 300$ K and with a free stream velocity $U_\infty = 30$ m/s flows over a flat plate parallel to a side of length 2 m and is maintained at a uniform temperature of $T_w = 400$ K. The physical properties of air at the film temperature of 350 K are as follows:

$$k = 0.026 \text{ W/(m° C)}; \qquad \Pr = 0.705; \qquad \nu = 16.5 \times 10^{-6} \text{ m}^2\text{/s}$$

Determine

(a) the average heat transfer coefficient over the region where the boundary layer is laminar,
(b) the average heat transfer coefficient over the entire length $L = 2$ m of the plate, and
(c) the total heat transfer rate to the air from the entire plate for a width of 1 m.

Solution: We first find the location x (for $Re_x = 5 \times 10^5$) where the transition occurs

Thus,

$$x = \frac{\nu Re_x}{U_\infty} = \frac{16.5 \times 10^{-6} \times 5 \times 10^5}{30} = 0.275 \text{ m}$$

The average Nusselt number for the laminar zone is given by

$$\overline{Nu_x} = 0.664 \times (5 \times 10^5)^{1/2} (0.705)^{1/3} = 417.88$$

Therefore,

$$h_x = \frac{k}{x} \overline{Nu_x} = \frac{0.026}{0.275} \times 417.88 \text{ W/(m}^2\text{ K)} = 39.51 \text{ W/(m}^2\text{ K)}$$

The Reynolds number at $L = 2$ m is

$$Re_L = \frac{30 \times 2}{16.5 \times 10^{-6}} = 3.64 \times 10^6$$

Then the average heat transfer coefficient over $L = 2$ m is determined from Eq. (7.63) as

$$\bar{h}_L = \frac{0.026}{2}[0.037(3.64 \times 10^6)^{4/5} - 871](0.705)^{1/3} = 65.85 \text{ W/(m}^2\text{ K)}$$

The total heat transfer rate from the plate becomes

$$Q = (1)(2)(65.85)(400 - 300) \text{ W} = 13170 \text{ W} = 13.17 \text{ kW}$$

EXAMPLE 7.3 Air at a pressure of 101 kPa and 20°C flows with a velocity of 5 m/s over a 1 m × 5 m flat plate whose temperature is kept constant at 140°C. Determine the rate of heat transfer from the plate if the air flows parallel to the (a) 5 m long side and (b) the 1 m side. The properties of air at the film temperature of 80°C are as follows:

$$k = 0.03 \text{ W/(m K)}; \qquad Pr = 0.706; \qquad \nu = 2 \times 10^{-5} \text{ m}^2\text{/s}$$

Solution: (a) When the air flow is parallel to the long side we have $L = 5$ m, and the Reynolds number at the end of the plate becomes

$$Re_L = \frac{(5) \times (5)}{2 \times 10^{-5}} = 1.25 \times 10^6$$

which is greater than the critical Reynolds number. Thus, we have combined laminar and turbulent flow and the average Nusselt number of the entire plate is given by Eq. (7.63). Therefore, we have

$$\bar{h}_L = \frac{0.03}{5}[(0.037)(1.25 \times 10^6)^{4/5} - 871](0.706)^{1/3}$$
$$= 10.26 \text{ W/(m}^2\text{ K)}$$
$$Q = (1)(5)(10.26)(140 - 20)$$
$$= 6156 \text{ W} = 6.16 \text{ kW}$$

(b) When the air flow is parallel to the 1 m side, we have $L = 1$ m and the Reynolds number at the end of the plate is

$$\text{Re}_L = \frac{(5)\times(1)}{2\times10^{-5}} = 2.5 \times 10^5$$

which is less than the critical Reynolds number. Therefore, we have laminar flow over the entire plate and the average Nusselt number is given by Eq. (7.46). Then, we have

$$\bar{h}_L = \frac{0.03}{1} 0.664(2.5 \times 10^5)^{1/2}(0.706)^{1/3}$$
$$= 8.87 \text{ W/(m}^2\text{ K)}$$
$$Q = (1)(5)(8.87)(140 - 20)$$
$$= 5322 \text{ W} = 5.32 \text{ kW}$$

EXAMPLE 7.4 Castor oil at 36°C flows over a 6 m long and 1 m wide heated plate at 0·06 m/s. For a surface temperature of 96°C, determine (a) the thermal boundary layer thickness at the end of the plate, (b) the local heat transfer coefficient at the end of the plate, and (c) the rate of heat transfer from the entire plate. Assume the following properties of castor oil at the film temperature of 66°C.

$$\alpha = 7.22 \times 10^{-8} \text{ m}^2/\text{s}; \qquad \nu = 6.0 \times 10^{-5} \text{ m}^2/\text{s};$$
$$k = 0.21 \text{ W/(m K)}$$

Solution: (a) $\text{Re}_L = \dfrac{0.06\times6.0}{6.0\times10^{-5}} = 6 \times 10^3$

Therefore the boundary layer is laminar over the entire plate. From Eq. (6.68),

$$\delta = \frac{5.0\times L}{\sqrt{\text{Re}_L}}$$
$$= \frac{5.0\times6.0}{\sqrt{6000}}$$
$$= 0.387 \text{ m}$$
$$\text{Pr} = \frac{\nu}{\alpha} = \frac{6.0\times10^{-5}}{7.22\times10^{-8}}$$
$$= 831$$
$$\delta_T = \frac{\delta}{\text{Pr}^{1/3}} = \frac{0.387}{(831)^{1/3}} = 0.41 \text{ m}$$

(b) Since the Prandtl number is high, we use Eq. (7.34c) for Nusselt number. Therefore,

$$\begin{aligned}
\mathrm{Nu}_L &= 0.339 \times (6 \times 10^3)^{1/2}\,(831)^{1/3} \\
&= 246.87 \\
h_L &= \frac{k\,\mathrm{Nu}_L}{L} \\
&= \frac{0.21 \times 246.87}{6} \\
&= 8.64 \text{ W (m}^2\text{ K)}
\end{aligned}$$

(c) $$\bar{h}_L = 2\mathrm{h}_L = 2 \times 8.64 = 17.28 \text{ W/(m}^2\text{ K)}$$

$$\begin{aligned}
Q &= \bar{h}_L A(T_w - T_\infty) \\
&= 17.28 \times 6 \times 1 \times (96 - 36) \\
&= 6221 \text{ W} = 6.22 \text{ kW}
\end{aligned}$$

EXAMPLE 7.5 A flat plate of width 1 m is maintained at a uniform surface temperature of $T_w = 225°\mathrm{C}$ by using independently controlled, heat generating rectangular modules of thickness 10 mm and length 40 mm. Each module is insulated from its neighbours, as well as on its back side. Atmospheric air at 25°C flows over the plate at a velocity of 30 m/s. The thermophysical properties of the module are $k = 5.2$ W/(m K), $c_p = 320$ J/(kg K) and $\rho = 2300$ kg/m^3. Determine the required power generation in W/m^3 in a module positioned at a distance of 800 mm from the leading edge. Assume for air the following properties at the film temperature of 125°C.

$$k = 0.031 \text{ W/(m K)}; \qquad \nu = 22 \times 10^{-6} \text{ m}^2\text{/s}; \qquad \mathrm{Pr} = 0.7$$

Solution: The distance from the leading edge to the centre-line of the module, $L = 800 + 20 = 820$ mm $= 0.82$ m

$$\mathrm{Re}_L = \frac{30 \times 0.82}{22 \times 10^{-6}} = 1.12 \times 10^6$$

Therefore the flow is turbulent over the module. The local heat transfer coefficient at L is calculated using Eq. (7.59b) as

$$\begin{aligned}
h_L &= \left(\frac{k}{L}\right) 0.0296\,(\mathrm{Re}_L)^{4/5}\,\mathrm{Pr}^{1/3} \\
&= \frac{0.031}{0.82}(0.0296)(1.12 \times 10^6)^{4/5}(0.7)^{1/3} \\
&= 68.64 \text{ W/(m}^2\text{ K)}
\end{aligned}$$

We consider that the local heat transfer coefficient at $L = 0.82$ m remains the same over the module which extends from $L = 0.80$ m to 0.84 m. If q_m be the power generation in W/m^3 within the module, we can write from an energy balance on the module surface

$$q_m(0.04) \times (1) \times (0.01) = 68.64 \times (0.04) \times (1) \times (225 - 25)$$

which gives $$\begin{aligned} q_m &= 1.37 \times 10^6 \text{ W/m}^3 \\ &= 1.37 \text{ MW/m}^3 \end{aligned}$$

EXAMPLE 7.6 An aircraft is moving at a velocity of 150 m/s in air at an altitude where the pressure is 0.7 bar and the temperature is -5°C. The top surface of the wing absorbs solar radiation at a rate of 900 W/m^2. Considering the wing as a flat plate of length 2 m and to be of solid construction with a single uniform surface temperature, determine the steady-state temperature of the wing. (Use the properties of air at 268 K and 0.7 bar as k = 0.024 W/(m K), Pr = 0.72, and $\nu = 2 \times 10^{-5}$ m^2/s.)

Solution: From an energy balance of the aerofoil at steady state, we can write

$$q_r A_s = 2\bar{h}_L A_s (T_w - T_\infty)$$

where

q_r = radiation heat flux = 900 W/m^2 (given)
A_s = upper or lower surface area.
$\bar{h}_L$ = average heat transfer coefficient of the wing of length L
T_w = surface temperature of the wing

It has to be noted here that the solar radiation is received only on the upper surface while the convective heat loss takes place from both the upper and lower surfaces.

Therefore, it becomes

$$T_w = T_\infty + \frac{q_r}{2\bar{h}_L}$$

$$\mathrm{Re}_L = \frac{U_\infty L}{\nu} = \frac{150 \times 2}{2 \times 10^{-5}} = 1.5 \times 10^7$$

Since $\mathrm{Re}_L > \mathrm{Re}_c (= 5 \times 10^5)$ the flow is approximated as turbulent over the entire surface of the wing. Therefore, we use Eq. (7.64b), though the value of Re_L of 1.5×10^7 is little beyond the range of applicability of the equation as mentioned in the text.

Since,

$$\mathrm{Nu}_x = 0.0308(\mathrm{Re}_x)^{4/5}\,\mathrm{Pr}^{1/3}$$

$$\overline{\mathrm{Nu}_L} = \frac{\bar{h}_L L}{k}$$
$$= \frac{5}{4}\mathrm{Nu}_L$$
$$= \frac{5}{4}\,0.0308 \times (1.5 \times 10^7)^{4/5}(0.72)^{1/3}$$
$$= 1.9 \times 10^4$$

$$\bar{h}_L = \frac{1.9 \times 10^4 \times 0.024}{2} = 228 \text{ W/(m}^2\text{ K)}$$

Therefore,

$$T_w = (273 - 5) + \frac{900}{2 \times 228}$$
$$= 270 \text{ K}$$

EXAMPLE 7.7 A fine wire having a diameter of 0.04 mm is placed in an air stream at 25°C having a flow velocity of 60 m/s perpendicular to the wire. An electric current is passed through the wire, raising its surface temperature to 50°C. Calculate the heat loss per unit length. Use both

Eqs. (7.67) and (7.66) and then compare the results. For air at the film temperature of 37.5°C, k = 0.027 W/(m K), $\nu = 17 \times 10^{-6}$ m^2/s, and Pr = 0.71.

Solution: We first use the simple expression given by Eq. (7.67)

$$\text{Re} = \frac{60 \times (4 \times 10^{-5})}{17 \times 10^{-6}} = 141$$

The values of C and n as found from Table 7.1, for Re = 141 are C = 0.683, n = 0.466. Therefore, from Eq. (7.67),

$$\begin{aligned}\overline{\text{Nu}}_D &= (0.683)(141)^{0.466}(0.71)^{1/3}\\ &= 6.11\end{aligned}$$

The average heat transfer coefficient, $\bar{h} = \overline{\text{Nu}}_D \dfrac{k}{d}$

$$\begin{aligned}&= 6.11 \times \left(\frac{0.027}{4 \times 10^{-5}}\right)\\ &= 4124 \text{ W/(m}^2\text{ K)}\end{aligned}$$

Heat transfer per unit length,

$$\begin{aligned}q/L &= \pi d h (T_w - T_\infty)\\ &= \pi \times (4 \times 10^{-5})(4124)(50 - 25)\\ &= 12.95 \text{ W/m}\end{aligned}$$

If we use Eq. (7.66), we have

$$\begin{aligned}\overline{\text{Nu}}_D &= 0.3 + \frac{0.62 \times (141)^{1/2} \times (0.71)^{1/3}}{[1 + (0.4/0.71)^{2/3}]^{1/4}} \left[1 + \left(\frac{141}{2.82 \times 10^5}\right)^{5/8}\right]^{4/5}\\ &= 6.08\\ h &= \frac{6.08 \times 0.027}{4 \times 10^{-5}}\\ &= 4104 \text{ W/(m}^2\text{ K)}\\ q/L &= \pi \times (4 \times 10^{-5})(4104)(50 - 25)\\ &= 12.89 \text{ W/m}\end{aligned}$$

We thus find that the results from the two correlations of Eqs. (7.67) and (7.66) differ by 0.5 per cent.

7.5 CONVECTIVE HEAT TRANSFER IN INTERNAL FLOWS—BASIC CONCEPTS

7.5.1 Bulk Mean Temperature

In case of an internal flow with heat transfer, the fluid temperature cannot be uniform over the entire cross-section of the duct at any location. For example, if the stream is to be heated by the wall, the fluid layer situated closer to the wall will necessarily be warmer than a layer situated farther from the wall. Now the question that arises is, what fluid stream temperature will be used in defining the heat transfer coefficient. We have seen earlier that the local or average heat transfer coefficient in case of an external flow is defined on the basis of a difference in surface temperature and free stream temperature. In case of an internal flow, a bulk mean temperature is defined which is used in place of free stream temperature to define the heat transfer coefficient.

The bulk mean temperature T_m at a given cross-section is defined, on the basis of the thermal energy transported by the fluid stream which passes through that cross-section, as

$$T_m = \frac{\iint_A \rho c_p u T dA}{\iint_A \rho c_p u dA} \tag{7.72}$$

where u is the flow velocity in the axial direction and T is the temperature. Both u and T vary over the entire cross-section of area A.

7.5.2 Thermally Fully Developed Flow

Figure 7.5 shows the development of thermal boundary layer in a heated pipe flow. We know that for a hydrodynamically fully developed flow, the flow velocity u is not a function of space coordinate in the direction of flow, i.e. $\frac{\partial u}{\partial x} = 0$. But this cannot be so for a temperature profile. This is because $\frac{\partial T}{\partial x}$ can never become zero, otherwise heat transfer between the fluid stream and the wall will not be justified physically. For a thermally fully developed flow, the condition is imposed on a dimensionless temperature defined as $(T_w - T)/(T_w - T_m)$. Therefore, an internal flow is said to be thermally fully developed if

$$\frac{\partial}{\partial z}\left(\frac{T_w - T}{T_w - T_m}\right) = 0 \tag{7.73}$$

where T_s and T_m are the surface or wall temperature and bulk mean temperature respectively and z is the coordinate in the direction of flow as shown in Figure 7.5.

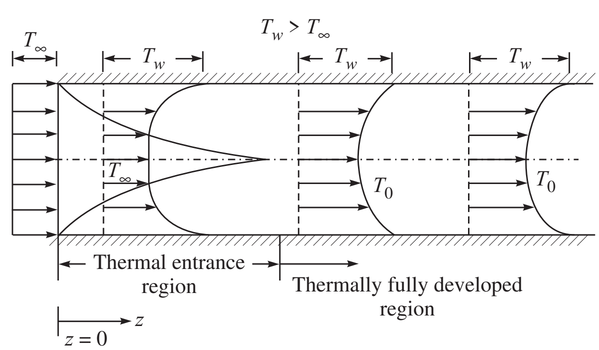


Figure 7.5 The growth of thermal boundary layer in a pipe flow.

Important consequences of a thermally fully developed flow

Let us consider a fully developed flow through a pipe. The condition given by Eq. (7.73) is eventually reached in the tube for which there is either a uniform surface heat flux (q_w = constant) or a constant surface temperature (T_w = constant). We come across these boundary conditions in many engineering applications. A constant surface heat flux is usually associated with a tube wall heated electrically. A constant surface temperature exists when there is a phase change (boiling or condensation) at the outer surface of a tube. We can write from Eq. (7.72)

$$\frac{\partial}{\partial z}\left(\frac{T_w - T}{T_w - T_m}\right) = \frac{(T_w - T_m)\left(\dfrac{dT_w}{dz} - \dfrac{\partial T}{\partial z}\right) - (T_w - T)\left(\dfrac{dT_w}{dz} - \dfrac{dT_m}{dz}\right)}{(T_w - T_m)^2} = 0$$

which gives

$$(T - T_m)\frac{dT_w}{dz} + (T_m - T_w)\frac{\partial T}{\partial z} + (T_w - T)\frac{dT_m}{\partial z} = 0 \tag{7.74}$$

Equation (7.74) has to be satisfied under all situations for a thermally fully developed flow. From Eq. (7.73) we find that the ratio $(T_w - T)/(T_w - T_m)$ is independent of z, and hence the derivative of the ratio with respect to r is also independent of z. Therefore, we can write

$$\frac{\partial}{\partial r}\left(\frac{T_w - T}{T_w - T_m}\right)_{\text{at}\, r=R} = \frac{-(\partial T/\partial r)_{r=R}}{T_w - T_m} \neq f(z) \tag{7.75}$$

The derivative is taken at the pipe surface and hence R is the pipe radius. We can write for the local heat transfer coefficient

$$h_x = \frac{k\left(\dfrac{\partial T}{\partial r}\right)_{r=R}}{T_w - T_m} \tag{7.76}$$

It has to be noted that in defining the heat transfer coefficient we use the bulk mean temperature as the reference temperature of the fluid stream. This is the convention and has been discussed earlier.

From Eqs. (7.75) and (7.76) we conclude that the 'local heat transfer coefficient is constant, i.e. independent of z (the direction of flow) in a thermally fully developed flow.'

In case of a constant surface heat flux boundary condition, we can write

$$q_w = h(T_w - T_m) = \text{constant}$$

This gives for a constant heat transfer coefficient

$$\frac{dT_w}{dz} = \frac{dT_m}{dz} \tag{7.77}$$

Substituting Eq. (7.77) in Eq. (7.74), we get

$$\frac{\partial T}{\partial z} = \frac{dT_m}{dz} \tag{7.78}$$

Therefore, the axial temperature gradient becomes independent of radial coordinate in a thermally fully developed flow with constant surface heat flux. However, this is not so for a constant surface temperature condition.

Again in consideration of energy balance of a fluid element flowing through a duct with uniform wall heat flux (Figure 7.6), we can write, neglecting axial conduction,

$$\frac{d}{dz}(\dot{m}c_p T_m) = q_w \cdot P$$

or

$$\frac{dT_m}{dz} = \frac{q_w P}{\dot{m}c_p} \tag{7.79}$$

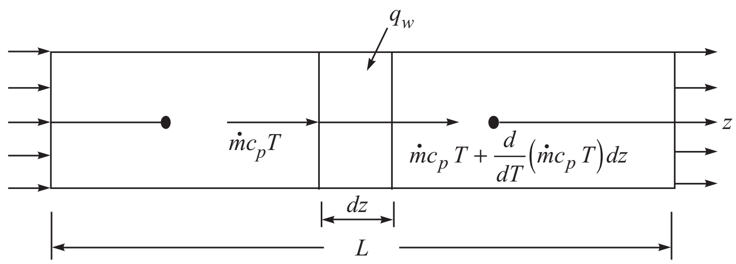


Figure 7.6 Energy balance of a fluid element in a flow through a duct with uniform wall heat flux.

where P is the perimeter of the surface. In case of a pipe of circular cross-section, $P = \pi D$, where D is the diameter of the pipe and q_w is the constant heat flux at the pipe wall. Since all the quantities on the right-hand side of Eq. (7.79) are fixed, we have

$$\frac{dT_m}{dz} = \text{constant}$$

With the help of Eq. (7.78), we can write

$$\frac{\partial T}{\partial z} = \frac{dT_m}{dz} = \text{constant}$$

The above relation implies that in a thermally fully developed flow with constant heat flux at the surface, the bulk mean temperature and also the temperature at any radial location change linearly with the space coordinate in the direction of flow.

In case of a laminar flow, the thermal entrance length is given by

$$\frac{L_e}{D} = 0.05\,\text{Re}\,\text{Pr}$$

where Re is the Reynolds number of pipe flow and Pr is the Prandtl number of the fluid. In contrast, for turbulent flow, the entrance length is almost independent of the Prandtl number and is approximately given by

$$\left(\frac{L_e}{D}\right)_{\text{turbulent}} = 10.$$

In the entrance region, the local heat transfer coefficient decreases along the direction of flow. This is because of the fact that the temperature gradient at the wall decreases due to an increase in the thermal boundary thickness. Near the inlet to the pipe, the thermal boundary layer thickness is very small and hence the local heat transfer coefficient h is extremely large. However,

h decays rapidly as the thermal boundary layer develops and finally reaches a constant value for the fully developed region as shown in Figure 7.7.

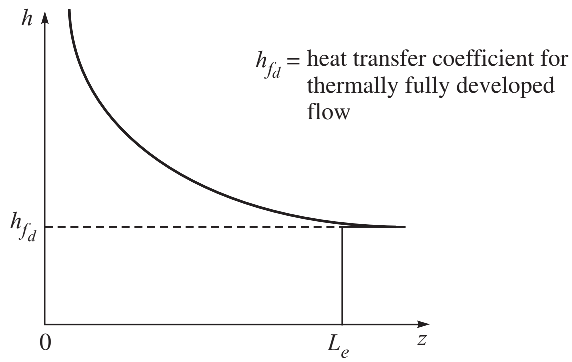


Figure 7.7 Axial variation of convective heat transfer coefficient in a pipe flow.

Variation of bulk mean temperature for a flow through a heated duct

We start with Eq. (7.79) as

$$\frac{dT_m}{dz} = \frac{q_w P}{\dot{m}c_p} = \frac{hP(T_w - T_m)}{\dot{m}c_p} \tag{7.80}$$

Constant surface heat flux: In this case, q_w is constant and hence we get upon integration

$$T_m = \frac{q_w P}{\dot{m}c_p} z + A$$

If the bulk mean temperature at inlet ($z = 0$) is denoted by T_{m_i}, we can then write

$$T_m = T_{m_i} + \frac{q_w P}{\dot{m}c_p} z \tag{7.81}$$

Constant surface temperature: In this case, following Eq. (7.80), we write

$$\frac{dT_m}{T_w - T_m} = \frac{hP}{\dot{m}c_p} dz$$

or
$$\ln \frac{T_w - T_m}{T_w - T_{m_i}} = -\frac{hP}{\dot{m}c_p}$$

or
$$T_m = T_w - (T_w - T_{m_i})\, e^{-\frac{hP}{\dot{m}c_p}} \tag{7.82}$$

The variations of surface temperature and bulk mean temperature in the direction of flow are shown in Figure 7.8(a) and 7.8(b). It is interesting to note that in case of constant surface heat flux, both T_w and T_m, in fully developed region, vary linearly with z and become parallel to each other. When T_w = constant, the bulk mean temperature T_m reaches asymptotically the value of T_w.

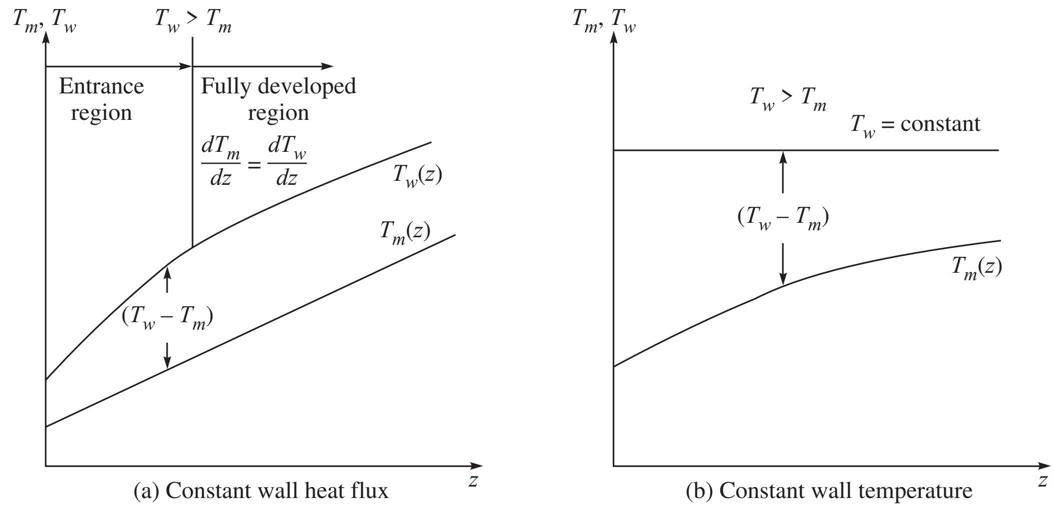


Figure 7.8 Axial variations of temperature for heat transfer in a pipe.

7.6 CONVECTIVE HEAT TRANSFER IN A HAGEN–POISEUILLE FLOW (STEADY LAMINAR INCOMPRESSIBLE AND FULLY DEVELOPED FLOW THROUGH A PIPE)

7.6.1 Constant Surface Heat Flux

Let us consider a steady, incompressible laminar fully developed flow through a pipe. An important corollary for a thermally fully developed flow under the situation is

$$\frac{\partial T}{\partial z} = \text{constant} \tag{7.83}$$

Since $v_r = v_\theta = 0$, the energy equation (7.16) under the situation can be written as

$$v_z \frac{\partial T}{\partial z} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right)$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial z} v_z r \tag{7.84}$$

Substituting for v_z from Eq. (6.44) in Eq. (7.84), we get

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{R^2}{4\mu\alpha}\left(-\frac{dp}{dz}\right)\left(\frac{\partial T}{\partial z}\right)\left(1 - \frac{r^2}{R^2}\right) r \tag{7.85}$$

or

$$r\left(\frac{\partial T}{\partial r}\right) = \frac{R^2}{4\mu\alpha}\left(-\frac{dp}{dz}\right)\left(\frac{\partial T}{\partial z}\right)\left(\frac{r^2}{2} - \frac{r^4}{4R^2}\right) + c_1 \tag{7.86}$$

Integrating again, it gives

$$T = \frac{R^2}{4\mu\alpha}\left(-\frac{dp}{dz}\right)\left(\frac{\partial T}{\partial z}\right)\left(\frac{r^2}{4} - \frac{r^4}{16R^2}\right) + c_1 \ln r + c_2$$

The boundary conditions are

$$\text{at } r = 0, \qquad \left(\frac{\partial T}{\partial r}\right) = 0 \quad \text{(axisymmetric condition)}$$

$$\text{at } r = 0, \qquad k\left(\frac{\partial T}{\partial r}\right) = q_w = \text{constant (a given value)}$$

The first boundary condition gives $c_1 = 0$. It is intended to express the temperature distribution in terms of the surface temperature T_w. Therefore, instead of using the second boundary condition we use the condition

$$\text{at } r = R, \qquad T = T_w(z)$$

This gives

$$c_2 = T_w - \frac{3R^4}{64\mu\alpha}\left(-\frac{dp}{dz}\right)\left(\frac{\partial T}{\partial z}\right)$$

Finally, we have for the temperature profile

$$T - T_w = \frac{R^4}{4\mu\alpha}\left(-\frac{dp}{dz}\right)\left(\frac{\partial T}{\partial z}\right)\left[\frac{1}{4}\left(\frac{r}{R}\right)^2 - \frac{1}{16}\left(\frac{r}{R}\right)^4 - \frac{3}{16}\right] \tag{7.87}$$

The bulk mean temperature is found out, following Eq. (7.72), as

$$T_m = \frac{\int_0^R \rho 2\pi r dr v_z c_p T}{\int_0^R \rho 2\pi r dr v_z c_p} \tag{7.88}$$

The radial distribution of v_z and T are substituted from Eqs. (6.44) and (7.87) respectively in Eq. (7.88) and then performing the integration, we have

$$T_w - T_m = \frac{11}{384}\left(\frac{R^4}{\mu\alpha}\right)\left(-\frac{dp}{dz}\right)\left(\frac{\partial T}{\partial z}\right) \tag{7.89}$$

The heat transfer coefficient h is determined as

$$h = \frac{k\left(\frac{\partial T}{\partial r}\right)_{r=R}}{T_w - T_m} \tag{7.90}$$

From Eq. (7.87),

$$\left(\frac{\partial T}{\partial r}\right)_{r=R} = \frac{R^3}{16\mu\alpha}\left(-\frac{dp}{dz}\right)\left(\frac{\partial T}{\partial z}\right) \tag{7.91}$$

By making use of Eqs. (7.89) and (7.91), Eq. (7.90) becomes

$$h = \frac{24}{11}\frac{k}{R} = \frac{48}{11}\frac{k}{D}$$

where $D(=2R)$ is the diameter of the pipe.

The Nusselt number, $\text{Nu} = \frac{hD}{k} = \frac{48}{11} = 4.364$

Therefore,

$$(\text{Nu})_{\text{fully developed flow with constant surface heat flux}} = 4.364 \tag{7.92}$$

7.6.2 Constant Surface Temperature

The energy equation for a fully developed flow can be written, under the situation, as

$$v_z \frac{\partial T}{\partial z} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right)$$

If we make a boundary layer type approximation that $\frac{\partial^2 T}{\partial z^2} << \frac{\partial^2 T}{\partial r^2}$ which is valid if the radius of the pipe is much less than its length, we can write

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha} v_z \frac{\partial T}{\partial z} \tag{7.93}$$

The boundary conditions for the solution of Eq. (7.93) are

$$\text{at} \quad r = 0 \quad \frac{\partial T}{\partial r} = 0 \text{ (condition of symmetry)}$$

$$r = 0, \quad T = T_w \text{ (constant)}$$

The integration of Eq. (7.93) is difficult as compared to that for constant surface heat flux condition. This is due to the fact that $\frac{\partial T}{\partial z}$ is not a constant, rather a function of both z and r. Let us now try to express Eq. (7.93) in a reduced form for a possible analytical or semi-analytical solution.

We can write from Eq. (7.74),

$$\frac{\partial T}{\partial z} = \frac{(T - T_w)}{(T_m - T_w)}\left(\frac{dT_m}{dz}\right) \tag{7.94}$$

Again, from an energy balance and following Eq. (7.80), we can write

$$\dot{m} c_p \frac{dT_m}{dz} = h(\pi D)\,(T_w - T_m)$$

or

$$\frac{dT_m}{dz} = \frac{4h(\pi D)}{\rho v_{z_{\text{av}}} \pi D^2 c_p}(T_w - T_m)$$

$$= \frac{4h}{\rho c_p D v_{z_{\text{av}}}}(T_w - T_m) \tag{7.95}$$

Substituting the expression of $\frac{dT_m}{dz}$ from Eq. (7.95) into Eq. (7.94), we have

$$\frac{\partial T}{\partial z} = \left(\frac{4h}{\rho c_p D}\right)\frac{T_w - T}{v_{z_{av}}} \tag{7.96}$$

The expression of $\frac{\partial T}{\partial z}$ is now substituted in Eq. (7.93). Finally, it becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \left(\frac{v_z}{v_{z_{av}}}\right)\frac{4\text{Nu}}{D^2}(T - T_w) \tag{7.97}$$

where the Nusselt number $\text{Nu} = \frac{hD}{k}$

The term $v_z/v_{z_{av}}$ in Eq. (7.97) is substituted with the help of Eqs. (6.44) and (6.46a) and we have

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = -8\left(1 - \frac{r^2}{R^2}\right)\frac{\text{Nu}}{\text{D}^2}(T - T_w) \tag{7.98}$$

If we adopt the non-dimensional variables as

$$T^* = \frac{T - T_w}{T_m - T_w}$$

and

$$r^* = \frac{r}{R}$$

Equation (7.98) is transformed into a non-dimensional form as

$$\frac{\partial^2 T^*}{\partial r^2} + \frac{1}{r^*}\frac{\partial T^*}{\partial r^*} = -2\,\text{Nu}(1 - r^{*2})T^* \tag{7.99}$$

where the Nusselt number Nu is given by

$$\text{Nu} = -2\left(\frac{\partial T^*}{\partial r}\right)_{r^*=1} \tag{7.100a}$$

The boundary conditions for the solution of Eq. (7.99) are

$$\text{at} \quad r^* = 0, \quad \frac{\partial T^*}{\partial r} = 0$$

$$\text{at} \quad r^* = 1, \quad T^* = 0$$

Equation (7.99) has to be solved with the above boundary conditions so that Eq. (7.100a) for the definition of Nusselt number is satisfied. The solution may be obtained by an iterative procedure which involves successive approximation to the temperature profile. It is difficult to describe the resulting temperature profile by a simple algebraic expression. However, the value of Nusselt number finally comes out to be 3.66. Therefore, we write

$$(\text{Nu})_{\text{fully developed pipe flow with constant surface temperature}} = 3.66 \tag{7.100b}$$

7.7 HEAT TRANSFER IN THE ENTRANCE REGION OF A PIPE FLOW

In the thermal entrance region, the flow is thermally developing which means that the growth of thermal boundary takes place along the direction of flow (Figure 7.5). Since the temperature outside the thermal boundary is uniform, the wall temperature gradient depends on the thermal boundary layer thickness δ_T but not on the pipe radius. Therefore it is expected that the heat transfer coefficient will go on decreasing in the direction of flow due to an increase in the thermal boundary layer thickness. It starts from an infinitely high value at the leading edge where $\delta_T = 0$, and then decreases finally to a constant value for the thermally fully developed region. The solution of energy equation for this region is much complicated. However, we can find out the pertinent dimensionless number on which does the heat transfer coefficient depend. This is found out from a scale analysis as follows:

We can write for local heat transfer coefficient in the entrance region

$$h_z \sim \frac{k\left(\dfrac{\partial T}{\partial r}\right)_{r=R}}{T_w - T_0}$$

Again,
$$\left(\frac{\partial T}{\partial r}\right)_{r=R} \sim \frac{T_w - T_0}{\delta_T}$$

Hence,
$$h_z \sim \frac{k}{\delta_T}$$

or
$$\mathrm{Nu}_z = \frac{hD}{k} \sim \frac{D}{\delta_T} \tag{7.101}$$

Equating the order of convection and conduction terms within the boundary layer, we have

$$\frac{U_\infty \Delta T}{z} \sim \frac{\alpha \Delta T}{\delta_T^2}$$

which gives
$$\delta_T / D \sim (zD)^{1/2}(\mathrm{Re}\mathrm{Pr})^{-1/2} \tag{7.102}$$

where
$$\mathrm{Re} = \frac{U_\infty D}{\nu},$$

By making use of the relation given by Eq. (7.102) in Eq. (7.101), we have

$$\mathrm{Nu}_z = \frac{hD}{k} \sim \left(\frac{z/D}{\mathrm{Re}\mathrm{Pr}}\right)^{-1/2}$$

Therefore, we see that the non-dimensional term influencing the local heat transfer coefficient is given by $\dfrac{z/D}{\mathrm{Re}\mathrm{Pr}}$. The reciprocal of this term is known as Graetz number $\mathrm{Gz} = (D/z)\mathrm{Re}\mathrm{Pr}$.

There are two types of entry length problems. In one, the thermal boundary layer develops in the presence of a fully developed velocity profile. Such a situation exists when the location of heat transfer begins after an unheated starting length of the surface or in case of a fluid with very large Prandtl number like oils heated from the duct entry. This is depicted in Figure 7.9. In another type, both the hydrodynamic and thermal boundary layers grow simultaneously. This is known as the combined entry length problem. The solution for combined entry length problem is more difficult. The thermal entry length is a function of both Reynolds number and Prandtl number.

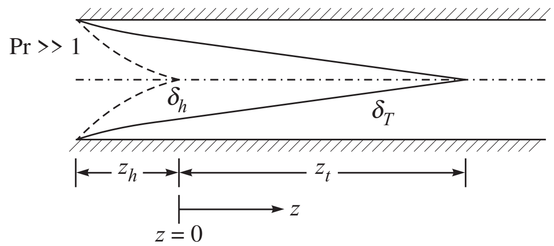


Figure 7.9 The growth of hydrodynamic boundary layer and thermal boundary layer in a tube for a fluid with very large Prandtl number.

7.7.1 Thermally Developing Hagen–Poiseuille Flow

Let us consider the first type of problem as stated above in a pipe flow The flow is fully developed hydrodynamically and hence we have a Hagen–Poiseuille flow. The physical model is shown in Figure 7.9. We consider a high Prandtl number limit and focus our attention on the tube section described by $z_h < z < z_t$. The location z_h corresponds to the section where the flow becomes hydrodynamically fully developed. The problem was treated first by Graetz [10] and is recognized in heat transfer literature as the Graetz problem. In the limit of $\Pr \to \infty$, $z \to 0$. Since δ_T/z is very small, we can neglect the axial conduction term $\left(\text{i.e. } \frac{\partial^2 T}{\partial z^2} << \frac{\partial^2 T}{\partial r^2}\right)$ and can write the energy equation as

$$v_z \frac{\partial T}{\partial z} = \alpha \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} \right) \tag{7.103}$$

By making use of Eq. (6.44) and Eq. (6.46a), we can write Eq. (7.103) as

$$2v_{z_{\text{av}}} \left(1 - \frac{r^2}{R^2}\right) \frac{\partial T}{\partial z} = \alpha \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} \right) \tag{7.104}$$

The boundary conditions for the solution of this equation are:

Constant wall temperature $T = T_w$ (constant) at $r = R$
(where R is the pipe radius)

Symmetry about the centre line, $\partial T/\partial r = 0$

Uniform fluid temperature at the entrance, $T = T_1$ for $z \le 0$ (where z is measured positive downstream from the location z_h (Figure 7.9))

Equation (7.104) is expressed in a non-dimensional form as

$$\frac{1}{2}(1 - r^{*2}) \frac{\partial T^*}{\partial z^*} = \frac{\partial^2 T^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial T^*}{\partial r^*} \tag{7.105}$$

The non-dimensional variables are

$$T^* = \frac{T - T_w}{T_1 - T_w}; \qquad r^* = \frac{r}{R}; \qquad z^* = \frac{z/D}{\text{RePr}}$$

where

$$\text{Re} = \frac{\rho v_{z_{av}} D}{\mu}$$

Equation (7.105) is linear and homogeneous.

The boundary conditions to be satisfied are

$$T^* = 0 \quad \text{at} \quad r^* = 1 \tag{7.106a}$$

$$\frac{\partial T^*}{\partial r} = 0 \quad \text{at} \quad r^* = 0 \tag{7.106b}$$

$$T^* = 1 \quad \text{at} \quad z^* = 0 \tag{7.106c}$$

The separation of variable method is adopted for which a product solution is assumed in the form

$$T^* = \Phi(r^*)\ \psi(z^*)$$

This yields two separate linear homogeneous equations as

$$\frac{d\psi}{dz^*} + 2\lambda^2\,\psi = 0 \tag{7.107a}$$

and

$$\frac{d^2\Phi}{dr^{*2}} + \frac{1}{r^*}\frac{d\Phi}{dr^*} + \lambda^2(1 - r^{*2})\Phi = 0 \tag{7.107b}$$

Equation (7.107a) gives

$$\psi = A\exp(-2\lambda^2 z^*)$$

Equation (7.107b) is of the Sturm-Liouville type and its solution is obtainable as infinite series. The details of the solution procedure is kept beyond the scope of this book. However, some results of interest are

$$T^*_{\text{mean}} = \frac{T_{\text{mean}} - T_w}{T_1 - T_w} = 8\sum_{n=0}^{\infty}\frac{\text{Gz}}{\lambda_n^2}\exp(-2\lambda_n^2 z^*) \tag{7.108a}$$

$$\text{Nu}_z = \frac{\displaystyle\sum_{n=0}^{\infty}\text{Gz}\exp(-2\lambda_n^2 z^*)}{\displaystyle 2\sum_{n=0}^{\infty}(\text{Gz}/\lambda_n^2)\exp(-2\lambda_n^2 z^*)} \tag{7.108b}$$

$$\overline{\text{Nu}}_z(\text{average Nu from 0 to } z) = \frac{1}{4z^*}\ln\left(\frac{1}{T^*_{\text{mean}}}\right) \tag{7.108c}$$

Selected results obtained by Kays and Crawford [11] for entry length heat transfer are shown in Figure 7.10.

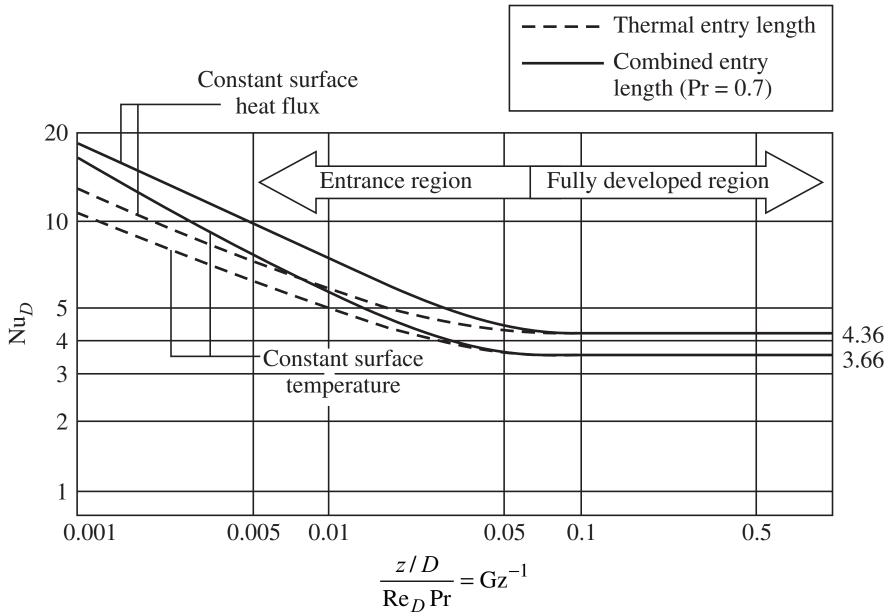


Figure 7.10 Local Nusselt number obtained from entry length solution in a pipe.

There are numerous correlations for Nusselt number in entry length of a pipe flow. For a combined entry length, a suitable correlation due to Sieder and Tate [12] is of the form

$$\overline{Nu_D} = 1.86\left(\frac{RePr}{L/D}\right)^{1/3}\left(\frac{\mu}{\mu_w}\right)^{0.14} \tag{7.109}$$

where $\overline{Nu_D}$ is the average Nusselt number over the entry length L. The pipe wall temperature is constant (T_w = constant). The ranges of applicability are

$$0.48 < Pr < 16{,}700$$

$$0.0044 < \left(\frac{\mu}{\mu_w}\right) < 9.75$$

All properties in Eq. (7.109) should be evaluated at the average temperature $T_m = (T_{m_i} + T_{m_o})/2$ except μ_w which is evaluated at T_w. Here T_{m_i} and T_{m_o} are the bulk mean temperatures at inlet to the pipe and at the end of the entry length.

7.8 HEAT TRANSFER IN TURBULENT FLOW IN A PIPE

The rate of heat transfer in a turbulent flow is much higher than that in a laminar flow. The enhancement in heat transfer is due to the turbulent eddies. Analogous to fluid flow situations, the additional heat conduction takes place through the eddies and is known as macroscopic conduction. In fact, heat is convected by the motion of macroscopic fluid particles comprising the turbulent eddies. This is conceived as turbulent heat conduction. The total heat conduction is the sum of molecular conduction and macroscopic or turbulent heat conduction and can be expressed as

$$\frac{q_t}{\rho c_p} = \alpha_t \frac{\partial T}{\partial y} \tag{7.110a}$$

where α_t is known as eddy thermal diffusivity. Therefore, the total heat transfer q becomes

$$\frac{q}{\rho c_p} = \frac{q_l + q_t}{\rho c_p} = (\alpha + \alpha_t)\frac{\partial T}{\partial y} \tag{7.110b}$$

Here q_l and q_t are the rates of laminar and turbulent heat fluxes in a transverse coordinate direction y (the direction normal to that of the average flow). With the help of Eq. (6.96b), we can write

$$\frac{q/c_p}{\tau} = \frac{(\alpha_l + \alpha_t)}{(\nu + \nu_t)}\frac{\partial T/\partial y}{\partial \bar{u}/\partial y} \tag{7.111}$$

One way of finding out the heat transfer coefficient is to use the Colburn analogy (the analogy between heat transfer coefficient and fluid friction) following Eq. (7.52) as

$$\frac{\overline{\text{Nu}}}{\text{RePr}} \times \text{Pr}^{2/3} = \frac{f}{8} \tag{7.112}$$

It has to be noted that c_f in Eq. (7.52) is replaced by f with the help of Eq. (6.54c). If we use the Blasius formula for f from Eq. (6.99) in Eq. (7.112), we have

$$\overline{\text{Nu}} = 0.0395\,\text{Re}^{3/4}\,\text{Pr}^{1/3} \tag{7.113}$$

The use of Reynolds–Colburn analogy in turbulent flow depends on the assumption that the ratio of heat flux to the turbulent shear stress at any radial location (as given by Eq. (7.111)) is constant. In a turbulent regime, we can use different expressions of f in different ranges of Re and accordingly obtain different empirical formulae for Nusselt number $\overline{\text{Nu}}$. A very widely used empirical relation for heat transfer coefficient in a fully developed turbulent flow through a pipe is due to Dittus and Boelter [13]. According to them,

$$\overline{\text{Nu}} = 0.023\,\text{Re}^{0.8}\,\text{Pr}^{n} \tag{7.114}$$

where $\overline{\text{Nu}} = \bar{h}D/k$ and $\text{Re} = \rho V_{\text{av}} D/\mu$

$n = 0.4$ for heating the fluid ($T_w > T_m$)

$n = 0.3$ for cooling the fluid ($T_w < T_m$)

Equation (7.114) with the values of n as prescribed above is valid within the range

$$0.7 \le \text{Pr} \le 160$$
$$\text{Re} \ge 10{,}000$$
$$L/D \ge 10$$

Equation (7.114) should be used only for small to moderate temperature difference $|(T_w - T_m)|$. All properties are evaluated at T_m. Equation (7.114) is known as Dittus–Boelter equation. For flows characterized by large property variations, Eq. (7.114) was modified empirically by Sieder and Tate [12] as

$$\mathrm{Nu} = 0.027\,\mathrm{Re}^{0.8}\,\mathrm{Pr}^{0.3}\left(\frac{\mu}{\mu_w}\right)^{0.14} \tag{7.115a}$$

Equation (7.115a) is valid for Prandtl numbers larger than those mentioned earlier for Eq. (7.114). To a good approximation, Eqs. (7.114) and (7.115a) may be applied for both uniform surface temperature and heat flux conditions.

Prandtl analogy

This analogy is based on the concept of two layers in a turbulent flow, a viscous sublayer where the molecular diffusivities are dominant, that is

$$\nu_t << \nu \qquad \text{and} \qquad \alpha_t << \alpha$$

and a turbulent zone where the turbulent diffusivities are dominant, that is,

$$\nu << \nu_t \qquad \text{and} \qquad \alpha << \alpha_t$$

These assumptions are utilized to simplify Eqs. (6.96b) and (7.110b) for each layer. The equations are integrated, and the definitions of friction factor and the heat transfer coefficient are introduced. The final result is as follows

$$\mathrm{St} = \frac{\mathrm{Nu}}{\mathrm{RePr}} = \frac{f}{8}\cdot\frac{1}{[1+5\sqrt{f/8}\,(\mathrm{Pr}-1)} \tag{7.115b}$$

This relationship is known as Prandtl analogy for momentum and heat transfer for fully developed turbulent flow in a pipe. It is to be noted that for Pr = 1, the Prandtl analogy reduces to Reynolds–Colburn analogy.

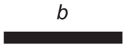
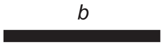
7.9 HEAT TRANSFER IN PLANE POISEUILLE FLOW

A plane Poiseuille flow refers to an incompressible, laminar hydrodynamically fully developed flow through a passage formed by two very large and parallel stationary plates. The dimensions of the plates are much larger compared to the spacing between them. The velocity distribution in such a flow is found from an exact solution of Navier–Stokes equation as done in Chapter 6 (Section 6.2.3). The velocity distribution for a plane Poiseuille flow is given by Eq. (6.41). For a thermally fully developed flow, the solution of energy equation for temperature distribution and Nusselt number in case of both constant surface heat flux and constant surface temperature can be made in exactly the similar way as done in case of a circular pipe. The final results for Nusselt number in both the cases are as follows:

$$(\mathrm{Nu})_{\text{constant surface heat flux}} = 8.23$$
$$(\mathrm{Nu})_{\text{constant surface temperature}} = 7.54$$

The Nusselt number for a fully developed internal flow through a non-circular pipe may be obtained from the solutions of Navier–Stokes equations and energy equation. The results relating to Nusselt number for such cases are shown in Table 7.3.

Table 7.3 Nusselt numbers for fully developed laminar flow in pipes of different cross-sections

Cross-section	$\frac{b}{a}$	$\mathrm{Nu}_D = \frac{hD_h}{k}$ *Uniform* q_w	*Uniform* T_w
	—	4.36	3.66
a  *b*	1.0	3.61	2.98
a  *b*	1.43	3.73	3.08
a  *b*	2.0	4.12	3.39
a  *b*	3.0	4.79	3.96
a  *b*	4.0	5.33	4.44
a  *b*	8.0	6.49	5.60
a  *b*	∞	8.23	7.54
	—	3.11	2.47

Note: (D_h is the hydraulic diameter)

Plane Poiseuille flow with generation of internal energy

A straightforward exact solution of energy equation is obtained in case of a plane Poiseuille flow with generation of internal energy. We consider a thermally fully developed plane Poiseuille flow (Figure 7.11).

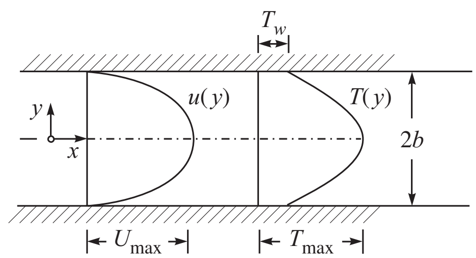


Figure 7.11 Velocity and temperature distributions in a plane Poiseuille flow with generation of internal energy due to friction.

The velocity distribution can be written following Eqs. (6.41) and (6.42a) as

$$u(y) = u_{max}\left(1 - \frac{y^2}{b^2}\right) \tag{7.116}$$

We consider the thermal boundary conditions as

$$T = T_w \qquad \text{for} \qquad y = \pm\, b \tag{7.117}$$

In this case, the dissipation function (Eq. (7.7b)) reduces to a simple expression $\Phi = (\partial u/\partial y)^2$, and hence the energy equation becomes

$$\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right) = k\left(u\frac{\partial^2 T}{\partial x^2}+v\frac{\partial^2 T}{\partial y^2}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{7.118}$$

With the boundary conditions given by Eq. (7.117) , the above equation has a solution which is independent of x. We assume that the resulting temperature distribution is due solely to the generation of internal energy through friction and to conduction of heat in the transverse direction. Thus both the axial convection and conduction are neglected. Since $v = 0$, the term $v\dfrac{\partial T}{\partial y}$ on the left-hand side also vanishes. Therefore, Eq. (7.117) is reduced to

$$k\frac{d^2 T}{dy^2} = \mu\left(\frac{du}{dy}\right)^2 \tag{7.119}$$

From Eq. (7.116),

$$\frac{du}{dy} = -u_{\max}\left(\frac{2y}{b^2}\right)$$

Therefore, Eq. (7.119) becomes

$$k\frac{d^2 T}{dy^2} = \frac{4\mu(u_{\max})^2}{b^4}y^2$$

The final solution satisfying the boundary conditions, given by Eq. (7.117), is

$$T(y) - T_w = \frac{1}{3}\frac{\mu(u_{\max})^2}{k}\left[1-\left(\frac{y}{b}\right)^4\right] \tag{7.120}$$

The temperature distribution is a parabola of fourth degree and is shown in Figure 7.11. The maximum temperature rise takes place at the centre of channel ($y = 0$) and is given by

$$T_{\max} - T_w = \frac{1}{3}\frac{\mu(u_{\max})^2}{k} \tag{7.121}$$

7.10 HEAT TRANSFER TO LIQUID METALS IN PIPE FLOW

We have discussed earlier, in connection with convection heat transfer in external flows, the advantages of heat transfer in liquid metals. Because of their high thermal conductivity, a large amount of heat can be transferred at high temperatures with a relatively low temperature difference between the fluid and the pipe wall. We will reproduce some empirical and theoretical correlations for heat transfer to liquid metals for a fully developed and entrance region turbulent flow in a circular pipe with uniform wall heat flux and uniform wall temperature boundary conditions. For all correlations described below, the physical properties are evaluated at the bulk mean temperature of the fluid.

Developed flow with uniform wall heat flux

The empirical relation due to Lubarsky and Kaufman [14] is of the form given by

$$\mathrm{Nu} = 0.625\,\mathrm{Pe}^{0.4} \tag{7.122}$$

$$\text{range of validity: } 10^2 < \mathrm{Pe} < 10^4,\ L/D > 60$$

The empirical relation due to Skupinski, Torel and Vautrey [15] is

$$\mathrm{Nu} = 4.82 + 0.0185\,\mathrm{Pe}^{0.827} \tag{7.123}$$

The range of applicability is same as above.

Developed flow with constant wall temperature

A semi-empirical equation developed by Seban and Shimazaki [16] with the aid of analogy between momentum and heat transfer is of the form given by

$$\mathrm{Nu} = 5.0 + 0.025\,\mathrm{Pe}^{0.8}$$

$$\text{for Pe} >> 100,\ L/D > 60.$$

Some other semi-empirical relations of Nusselt number investigated by different workers are as follows:

Sleicher and Tribus [17]

$$\mathrm{Nu} = 4.8 + 0.015\,\mathrm{Pe}^{0.91}\,\mathrm{Pr}^{0.3} \qquad \text{for Pr} < 0.05$$

Azer and Chao [18]

$$\mathrm{Nu} = 5.0 + 0.05\,\mathrm{Pe}^{0.77}\,\mathrm{Pr}^{0.25}$$

$$\text{for Pr} < 0.01,\ \mathrm{Pe} < 15 \times 10^3$$

Notter and Sleicher [19]

$$\mathrm{Nu} = 4.8 + 0.0156\,\mathrm{Pe}^{0.85}\,\mathrm{Pr}^{0.08}$$

$$\text{for } 0.004 < \mathrm{Pr} < 0.1,\ \mathrm{Re} < 5 \times 10^5$$

All the expressions are applicable for $L/D > 60$.

Thermal entrance region flow

Notter and Sleicher [19] also presented an approximate empirical relation for Nusselt number for entrance region as

$$\mathrm{Nu}_x = \mathrm{Nu}\left(1 + \frac{2}{x/D}\right) \qquad \text{for } \frac{x}{D} > 4$$

where

$$\mathrm{Nu} = 6.3 + 0.0167\,\mathrm{Pe}^{0.85}\,\mathrm{Pr}^{0.08} \qquad \text{for uniform wall heat flux}$$

$$\mathrm{Nu} = 4.8 + 0.156\,\mathrm{Pe}^{0.85}\,\mathrm{Pr}^{0.08} \qquad \text{for constant wall temperature}$$

The range of applicability is

$$0.004 < \mathrm{Pr} < 0.1$$

EXAMPLE 7.8 Compare the hydrodynamic and thermal entry lengths of mercury and a light oil flowing at 4 mm/s in a 25 mm diameter smooth tube at a bulk mean temperature of 80°C. The pertinent properties of the fluid at that temperature are $\nu_{Hg} = 1.0 \times 10^{-7}$ m^2/s, $\nu_{oil} = 6.5 \times 10^{-6}$ m^2/s, $Pr_{Hg} = 0.019$, $Pr_{oil} = 85$.

Solution: Reynolds numbers based on tube diameters are

$$Re_{Hg} = \frac{(4.0\times 10^{-3})(25\times 10^{-3})}{1\times 10^{-7}} = 1000$$

$$Re_{oil} = \frac{(4.0\times 10^{-3})(25\times 10^{-3})}{6.5\times 10^{-6}} = 15.38$$

The hydrodynamic entry lengths are

$$L_{e,Hg} = (0.05) \times (1000) \times (25 \times 10^{-3}) = 1.25 \text{ m}$$
$$L_{e,oil} = (0.05) \times 15.38 \times (25 \times 10^{-3}) = 0.02 \text{ m}$$

The thermal entry lengths are

$$L_{t,Hg} = (0.05) \times (1000) \times (0.019) \times (25 \times 10^{-3}) = 0.024 \text{ m}$$
$$L_{t,oil} = (0.05) \times 15.38 \times (85) \times (25 \times 10^{-3}) = 1.7 \text{ m}$$

The short thermal entrance length of mercury compared to the hydrodynamic entrance length gives rise to the assumption made in solving the liquid metal problems—that the flow is uniform across the tube when solving for the temperature in thermal entry region.

EXAMPLE 7.9 Air at one atmospheric pressure and 75°C enters a tube of 4.0 mm internal diameter with an average velocity of 2 m/s. The tube length is 1.0 m and a constant heat flux is imposed by the tube surface on the air over the entire length. An exit bulk mean temperature of air of 125°C is required. Determine (a) the heat transfer coefficient at exit h_L, (b) the constant surface heat flux q_w, and (c) the exit tube surface temperature. The properties of air at the average temperature of inlet and outlet bulk mean temperatures, i.e. (75 + 125)/2 = 100°C are as follows:

$$\rho = 0.95 \text{ kg/m}^3; \qquad c_p = 1.01 \text{ kJ/(kg K)}$$
$$\mu = 2.18 \times 10^{-5} \text{ kg/(m s)}; \qquad k = 0.03 \text{ W/(m K)}; \qquad Pr = 0.70$$

Solution: (a) $Re = \dfrac{(0.95)\times(2)\times(4\times 10^{-3})}{2.18\times 10^{-5}}$

$= 349$

Therefore, the flow is laminar. The hydrodynamic entrance length is given by

$$L_{e,h} = 0.05 \times (349)(4 \times 10^{-3}) = 0.071 \text{ m}$$

The thermal entrance length is given by

$$L_{e,t} = 0.05 \times (349) \times (0.7) \times (4 \times 10^{-3})$$
$$= 0.05 \text{ m}$$

The length of tube is given as 1 m. A reasonable approach is to consider the flow to be fully developed for both velocity and temperature profiles over the entire tube length, since only about 7 per cent is experiencing entry effects.

For a fully developed flow with constant surface heat flux, Nu = 4.36.

Therefore,

$$h = \text{Nu}\left(\frac{k}{D}\right)$$

$$= 4.36 \times \left(\frac{0.03}{0.004}\right)$$

$$= 32.7 \text{ W/(m}^2 \text{ K)}$$

Here, $h = h_L = 32.7$ W/(m^2 K). (The heat transfer coefficient under the situation is constant over the entire tube length.)

From an overall energy balance,

$$q_w \, \pi DL = \dot{m} c_p (T_{b,o} - T_{b,i})$$

The mass flow rate of air is given by

$$\dot{m} = (0.95) \times \left(\frac{\pi}{4}\right) \times (4 \times 10^{-3})^2 \times 2$$

$$= 2.39 \times 10^{-5} \text{ kg/s}$$

Therefore,

$$q_w = \frac{(2.39 \times 10^{-5}) \times (1.01 \times 10^3) \times (125 - 75)}{\pi \times (4 \times 10^{-3}) \times 1.0}$$

$$= 96 \text{ W/m}^2$$

Let $T_{w,e}$ be the tube surface temperature at the exit plane. Then, we can write

$$h_L (T_{w,e} - T_{b,o}) = q_w$$

or

$$T_{w,e} = T_{b,o} + \frac{q_w}{h_L}$$

$$= 125 + \frac{96}{32.7}$$

$$= 128°\text{C}$$

EXAMPLE 7.10 If the tube heated length in Example 7.9 is shortened to 0.04 m, determine all the quantities as required in Example 7.9. All other parameters remain the same and assume a fully developed flow for the velocity profile before entering the heated length.

Solution: Since the Reynolds number and the Prandtl number are unchanged, the thermal entrance length remains unchanged from the earlier value of 0.05, which is now more than the tube heated length. Therefore, the flow is hydrodynamically developed but not thermally developed.

We have to use Figure 7.9 for calculation of the Nusselt number. For this purpose, we first calculate the inverse Graetz number at $x = L = 0.04$ m.

$$\text{Gr}^{-1} = \left(\frac{x}{D}\right)\left(\frac{1}{\text{RePr}}\right) = \left(\frac{0.04}{0.004}\right)\left(\frac{1}{349 \times 0.7}\right)$$

$$= 4.1 \times 10^{-2}$$

From Figure 7.9, we find for constant surface heat flux,

$$\text{Nu} = 4.7 \quad \text{at} \quad \text{Gr} = 4.1 \times 10^{-2}$$

Therefore, the local heat transfer coefficient at exit becomes

$$h_L = (4.70)\left(\frac{0.03}{0.004}\right)$$
$$= 35.25 \text{ W/(m}^2 \text{ K)}$$

The surface heat flux
$$q_w = \frac{\dot{m}c_p\,(T_{b,o} - T_{b,i})}{\pi DL}$$

The values of all quantities, except L, on the right-hand side of the above equation remain same as those of Example 7.9.

Therefore,
$$q_w = 96 \times \frac{1.0}{0.04}$$
$$= 2.4 \times 10^3 \text{ W/m}^2$$
$$= 2.4 \text{ kW/m}^2$$

For the surface temperature at exit,

$$T_{w,e} = 125 + \frac{2.4 \times 10^3}{35.25}$$
$$= 193°\text{C}$$

EXAMPLE 7.11 Liquid sulphur dioxide in a saturated state flows inside a 5 m long tube and 25 mm internal diameter with a mass flow rate of 0.15 kg/s. The tube is heated at a constant surface temperature of –10°C and the inlet fluid temperature is $T_{b,i} = -40°\text{C}$. Determine the exit fluid temperature by making use of Sieder–Tate equation (Eq (7.115)).

Solution: The properties to be used in Eq. (7.109) should be estimated at a temperature which is the arithmetic mean of $T_{b,i}$ and $T_{b,o}$. Since $T_{b,o}$ is not known a priori, the solution has to be based on an iterative method starting with a guess value of

$$T_b^1 = (\text{T}_{b,i} + T_{b,o})/2$$

(Here we denote the bulk mean temperature as T_b instead of T_m. The superscript refers to the trial number).

For a first trial, guess $T_{b,o}^1 = -20°\text{C}$ for which $T_b^1 = -30°\text{C}$. We have the property values as follows at a temperature of –30°C (Table A.2).

$$\rho_b^1 = 1520.64 \text{ kg/m}^3; \qquad \nu_b^1 = 0.371 \times 10^{-6} \text{ m}^2\text{/s}$$
$$k_b^1 = 0.23 \text{ W/(m °C)}; \qquad \text{Pr}_b^1 = 3.31$$
$$\mu_b^1 = \nu_b^1 \rho_b^1 = (0.371 \times 10^{-6})(1520.64)$$
$$= 5.64 \times 10^{-4} \text{ kg/(m s)}$$
$$c_{p_b}^1 = 1361.6 \text{ J/(kg K)}$$
$$\mu_w(\text{at } T_w = 10°\text{C}) = \nu_w\, \rho_w$$
$$= (0.288 \times 10^{-6})(1463.61) \quad \text{(from Table A.2)}$$
$$= 4.22 \times 10^{-4} \text{ kg/(m s)}$$

The Reynolds number is found as

$$\mathrm{Re}^1 = \frac{v_{\mathrm{av}} D}{\nu_b^1} = \frac{4\dot{m}}{(\pi D)\mu_b^1}$$

$$= \frac{4 \times 0.15}{\pi(0.025)(5.64 \times 10^{-4})}$$

$$= 1.35 \times 10^4$$

Hence the flow is turbulent.

Now using Eq. (7.115),

$$\overline{\mathrm{Nu}^1} = 0.027(\mathrm{Re}^1)^{0.08}\,(\mathrm{Pr1})^{1/3}\left(\frac{\mu_b^1}{\mu_w}\right)^{0.14}$$

$$= 0.027 \times (13500)^{0.8} \times (3.31)^{1/3} \times \left(\frac{5.64}{4.22}\right)^{0.14}$$

$$= 84.44$$

$$\bar{h}^1 = \frac{k_b^1}{D}(\overline{\mathrm{Nu}^1})$$

$$= \frac{0.230}{0.025}(84.44)$$

$$= 777 \text{ W/(m}^2\ ^\circ\text{C)}$$

The outlet fluid temperature is found by making use of Eq. (7.82) as

$$T_{bo}^2 = T_w - (T_w - T_{b,i})\exp\left[\frac{-\pi DL}{\dot{m}c_p}\bar{h}^1\right]$$

$$= (-10) - [(10) - (-40)] \times \exp\left[\frac{-\pi \times (0.025) \times (5) \times (777)}{(0.15) \times (1361.6)}\right]$$

$$= (-10 - 6.73)$$

$$= -16.73^\circ\text{C}$$

$$T_b^2 = (-40 - 16.73)/2 = -28.36^\circ\text{C}$$

Since this value differs from the assumed value of $T_b^1 = -30^\circ$C, we require further iteration. Therefore, we start the second trial with $T_b^2 = -28.36^\circ$C.

We have the property values (Table A.2) at a temperature of –28.36°C as follows:

$$\rho_b^2 = 1514 \text{ kg/m}^3; \qquad \nu_b^2 = 0.362 \times 10^{-6}$$

$$k_b^2 = 0.229 \text{ W(m }^\circ\text{C)}; \quad \mathrm{Pr}^2 = 3.23$$

$$\mu_b^2 = \nu_b^2\,\rho_b^2 = (0.362 \times 10^{-6})(1514)$$

$$= 5.48 \times 10^{-4} \text{ kg/(m s)}$$

$$c_{p_b}^2 = 1362 \text{ J/(kg K)}$$

$$\mu_w = \text{(unchanged)}$$

$$\text{Re}^2 = \frac{4\dot{m}}{(\pi D_2)\mu_b^2} = \frac{4\times(0.15)}{\pi(0.025)(5.48\times10^{-4})}$$
$$= 1.39 \times 10^4$$

$$\overline{\text{Nu}^2} = 0.027 \times (13900)^{0.8} \times (3.23)^{1/3} \times \left(\frac{5.48}{4.22}\right)^{0.14}$$
$$= 85.39$$

$$\bar{h}^2 = \frac{0.229}{0.025} \times 85.39$$
$$= 782 \text{ W/(m}^2\,°\text{C)}$$

$$T_{bo}^3 = (-10) - [(10) - (-40)] \times \exp\left[\frac{-\pi\times(0.025)\times(5)\times(782)}{(0.15)\times(1362)}\right]$$
$$= -16.67°\text{C}$$
$$T_b^3 = (-40 - 16.67)/2 = -28.33°\text{C}$$

We see that the difference between T_{bo}^2 and T_{bo}^3 and that between T_b^2 and T_b^3 is marginal. Therefore, we can stop the process of iteration and present the result as

$$T_{bo} = -16.67°\text{ C}$$

7.11 HEAT TRANSFER AUGMENTATION

The practical considerations of energy conservation and material savings have prompted the need of augmentation in heat transfer to produce more efficient heat exchange equipment. We have already learned that the rate of heat flux at a solid surface is given by

$$q = \frac{Q}{A} = -k\left(\frac{\partial T}{\partial y}\right)_{\text{surface}} = h(T_{\text{surface}} - T_{\text{ref}})$$

where T_{ref} is the reference temperature of fluid. Therefore, we find that the total rate in heat transfer can be increased either by increasing the surface area or by increasing the heat transfer coefficient which means an increase in the surface temperature gradient. An increase in heat transfer coefficient or surface temperature gradient is caused by changing the flow situation in a way so that there occurs an enhanced mixing due to fluid eddies created either by turbulence or by secondary flows. The usual ways of augmentation technique are stated below.

Use of roughened surface

The surface roughness protruding through the laminar sublayer creates small turbulent eddies which are responsible for better mixing and thus enhance the heat transfer coefficient.

Use of extended surfaces

This has already been discussed in Chapter 2 (Section 2). The use of fins (the extended surfaces) augments the rate of heat transfer by providing additional surfaces for heat transfer.

Use of twisted tapes or inlet swirl to the fluid

Under these situations, a swirling flow (a flow with tangential component of velocity) takes place in the tube. This creates zones of recirculating flows comprising fluid eddies and thus promotes the mixing and enhances the heat transfer coefficient.

Use of coiled tubes

In this case, the curvature of the tube results in a secondary flow. The large eddies associated with this secondary flow promote the mixing and enhance the heat transfer coefficient.

SUMMARY

- The temperature distribution in a fluid stream near a solid surface is obtained from the equation of conservation of energy known as energy equation.
- The general form of the energy equation in the Cartesian coordinate system is

$$\rho c_p \frac{DT}{Dt} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \frac{Dp}{Dt} + \mu\varphi$$

where φ (dissipation function) =

$$\left[2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 \right.$$

$$\left. \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2\right]$$

As observed from the energy equation, the temperature field in a fluid stream is greatly influenced by the velocity field.
- The pertinent dimensionless parameters governing the solution of energy equation are Reynolds number, Prandtl number, and Eckert number.
- The simplified form of the energy equation applicable within the thermal boundary layer is known as the thermal boundary layer equation. The simplification is made by discarding certain terms from an order of magnitude analysis. A two-dimensional thermal boundary layer equation in the Cartesian coordinate system is

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{dp}{dx}$$

- An exact solution of energy equation for a steady incompressible parallel flow over a flat plate can be made by the same similarity transformation as used for the Blasius solution of hydrodynamic boundary layer equation to obtain the velocity fields in the same situation. The solution by the energy integral method appears to be relatively simple. In this method, the energy integral equation is first obtained by integrating the thermal boundary layer equation within the thermal boundary layer having a finite thickness. The expressions for local Nusselt number for an isothermal plate are as follows:

From the exact solution:

$$\begin{aligned}\mathrm{Nu}_x &= 0.332(\mathrm{Re}_x)^{1/2}\,\mathrm{Pr}^{1/3} \qquad 0.6 < \mathrm{Pr} < 10\\ &= 0.564(\mathrm{Re}_x)^{1/2}\,\mathrm{Pr}^{1/2} \qquad \mathrm{Pr} \to 0\\ &= 0.339(\mathrm{Re}_x)^{1/2}\,\mathrm{Pr}^{1/3} \qquad \mathrm{Pr} \to \infty\end{aligned}$$

From the solution by the energy integral method:

$$\mathrm{Nu}_x = \frac{h_x x}{k} = 0.332(\mathrm{Re}_x)\mathrm{Pr}^{1/3}$$

It is recognized from both the solutions

$$\frac{\delta}{\delta_T} \approx \mathrm{Pr}^{1/3}$$

- A number of empirical equations relating the local Nusselt number with the Reynolds number and Prandtl number for a flat plate with constant surface heat flux are provided in literature.
- A relation between fluid friction and heat transfer coefficient exists because of similarity in hydrodynamic and thermal boundary layer equations and because of similar type expressions for momentum transfer and heat transfer at the solid surface. The typical relation for flow over a flat plate is

$$\mathrm{St}_x\,\mathrm{Pr}^{2/3} = \frac{c_{f_x}}{2}$$

The above relation is known as Reynolds–Colburn analogy. A general form of this relation for a boundary layer flow past a solid surface is given by

$$\mathrm{Nu} = \frac{1}{2} c_f \sqrt{\mathrm{Re}_L}\,\varphi\left(\frac{x}{L}, \mathrm{Pr}\right)$$

- The Reynolds–Colburn analogy can also be used for a turbulent boundary layer over a flat plate in accordance with the assumption that eddy momentum transport and eddy energy transport both increase in the same proportion compared with their molecular values. The average and local Nusselt number in a turbulent boundary layer over an isothermal flat plate are given as

$$\begin{aligned}\mathrm{Nu}_x &= 0.0296(\mathrm{Re}_x)^{4/5}\,\mathrm{Pr}^{1/3}\\ \overline{\mathrm{Nu}_\mathrm{L}} &= 0.037(\mathrm{Re}_L)^{4/5}\,\mathrm{Pr}^{1/3}\end{aligned}$$

The range of applicability of the above equation is $5 \times 10^5 < \mathrm{Re} < 10^7$.

- The Nusselt number for a parallel flow over a flat plate with constant surface heat flux is larger than that for constant plate temperature. It is 36% larger for laminar flow and 4% larger in case of turbulent flow.
- The convective heat transfer coefficients for both laminar and turbulent flow past cylinder and sphere have been provided in the form of empirical relations suggested by several workers.
- In case of an internal flow, a bulk mean temperature at a given cross-section of flow is defined on the basis of thermal energy transported by the fluid stream which passes through that cross-section. The bulk mean temperature T_m at any cross-section is used as the reference temperature to define the local heat transfer coefficient.

- For a thermally fully developed flow,

$$\frac{\partial}{\partial z}\left(\frac{T_w - T}{T_w - T_m}\right) = 0$$

  and the local heat transfer coefficient is constant in the direction of flow.
- For a thermally fully developed Hagen–Poiseuille flow,

$$(\text{Nu})_{\text{constant surface heat flux}} = 4.364$$
$$(\text{Nu})_{\text{constant surface temperature}} = 3.66$$

  The local heat transfer coefficient in thermal entrance region of pipe flow is influenced by a non-dimensional term known as Graetz number, Gz = (D/z) Re Pr. The theoretical treatment in case of thermally developing flow is quite complicated. However, empirical equations of heat transfer coefficient for a thermally developing Hagen–Poiseuille flow have been provided in literature by several workers.
- The heat transfer coefficient in a turbulent pipe flow is evaluated by making use of Reynolds–Colburn analogy. One of the popular equations in use is the Dittus–Boeltier equation which is

$$\text{Nu} = 0.023\,\text{Re}^{0.8}\,\text{Pr}^n$$
$$n = 0.4, \text{ for heating the fluid}$$
$$n = 0.3, \text{ for cooling the fluid}$$

  The range of applicability is

$$0.7 \le \text{Pr} \le 160$$
$$\text{Re} \ge 10{,}000$$
$$L/D \ge 10$$

- In a thermally fully developed plane Poiseuille flow,

$$(\text{Nu})_{\text{constant surface heat flux}} = 8.23$$
$$(\text{Nu})_{\text{constant surface temperature}} = 7$$

The augmentation in the rate of heat transfer is accomplished in practice by several ways by making use of roughened surface, extended surfaces, twisted tapes, inlet swirl, and coiled tubes.

REVIEW QUESTIONS

1. (a) What is meant by thermal boundary layer?
(b) What is the role of thermal boundary layer in convective heat transfer?
(c) What is (are) the mode (modes) of heat transfer (i) at a point on solid surface, (ii) at a point little away from the solid surface but within the thermal boundary layer, and (iii) at the edge of the thermal boundary layer?
(d) Will a thermal boundary layer develop in flow over a solid surface when both the fluid and the surface are at the same temperature?

2. Choose the correct answer:
For a small temperature difference within a flow:
(a) The velocity field is strongly influenced by the temperature field.
(b) The velocity field is very weakly influenced by the temperature field.
(c) The temperature field is very strongly influenced by the velocity field.
(d) The temperature field is very weakly influenced by the velocity field.

3. (a) The energy equation for a flow between two parallel plates is given in the form

$$\rho c_p u \frac{\partial T}{\partial x} = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2$$

where u is the velocity in the axial direction x. Discuss the assumptions made to simplify the energy equation to this form. Explain also the physical significance of each and every term.

(b) The energy equation for flow inside a circular pipe is written in the form

$$\rho c_p v_z \frac{\partial T}{\partial z} = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + \mu\left(\frac{\partial v_z}{\partial r}\right)^2$$

where v_z is the velocity in the axial z-direction. Discuss the assumptions made to simplify the energy equation to this form.

4. Consider a parallel flow over an isothermal flat plate. Explain why the local heat transfer coefficient decreases in the direction of flow.

5. (a) In a hydrodynamically fully developed internal flow, the velocity profile does not change in the direction of flow and we can write $\frac{\partial u}{\partial z} = 0$, where u is the velocity in the flow direction z. Can we write for a thermally fully developed flow, $\frac{\partial T}{\partial z} = 0$? If not, explain why?
(b) Write the condition imposed on the temperature profile for a thermally fully developed internal flow.
(c) Show that the local heat transfer coefficient in a thermally fully developed internal flow does not change in the direction of flow.
(d) Why is the local heat transfer coefficient in thermal entrance region higher than that in the developed region?

6. (a) Does the Prandtl number depend on the type of flow or flow geometry? What is the role of the Prandtl number in convective heat transfer?
(b) What is the pertinent dimensionless term that influences the heat transfer in thermal entrance region?

7. (a) Why is the rate of heat transfer higher in a turbulent flow than that in a laminar flow?
(b) The Reynolds–Colburn analogy relating to friction factor with heat transfer coefficient in case of laminar flow can also be used in turbulent flow. Explain the reason.

PROBLEMS

[For the problems take the property values at the appropriate temperature from Tables A.1 and A.5]

7.1 Consider a flat plate at a constant temperature of 70°C. A parallel flow takes place over the plate with a free stream velocity of 2 m/s and a free stream temperature of 30°C. Determine the hydrodynamic and thermal boundary layer thickness at a distance of 0.1 m from the leading edge of the plate if the flowing medium is (i) air and (ii) water.

[**Ans.** δ_{air} = 4.50 mm, $\delta_{T_{\text{air}}}$ = 4.87 mm; δ_{water} = 1 mm, $\delta_{T_{\text{water}}}$ = 0.55 mm]

7.2 Engine oil at 40°C flows with a free stream velocity of 1 m/s over a 2 m long flat plate whose surface is maintained at a uniform temperature of 90°C. Determine (i) the local heat transfer coefficient at the edge of the plate (L = 2 m), (ii) the average heat transfer coefficient over the 2 m length of the plate, and (ii) the rate of heat transfer from the plate.

[**Ans.** (i) 31.46 W/(m^2 °C), (ii) 62.93 W/(m^2 °C), (iii) 6.293 kW/m]

7.3 Air at a pressure of 1 atm and a temperature of 60°C flows past the top surface of a flat plate. The flat plate is kept at a constant temperature of 140°C throughout its length of 0.2 m (in the direction of flow). The width of the plate is 0.1 m. The Reynolds number based on the flat plate length is 20,000. What is the rate of heat transfer from the plate to the air? If the free stream velocity of the air is doubled and the pressure is increased to 5 atm, what is the rate of heat transfer?

[**Ans.** 18.91 W, 59.76 W]

7.4 Consider a parallel flow of a liquid metal with a free steam velocity U_∞, and a free stream temperature T_∞ over a flat plate maintained at a uniform temperature T_w. Taking the temperature profile within the thermal boundary layer in the form

$$\frac{T(x,y) - T_w}{T_\infty - T_w} = \sin\left(\frac{\pi}{2}\frac{y}{\delta_T}\right)$$

develop an expression for the local Nusselt number. You can use Eq. (7.48).

[**Ans.** $\text{Nu}_x = 0.555(\text{Re}_x \ \text{Pr})^{1/2}$]

7.5 A 0.1m × 0.1m circuit board dissipating 10 W of power uniformly, is cooled by air approaching the circuit board at 40°C with a velocity of 5 m/s. Disregarding the heat transfer from the back surface of the board, determine the surface temperature of the electronic components (a) at the leading edge, (b) at the end of the board. Assume the flow to be turbulent, since the electronic components act as the turbulators. Use Eq. (7.64b) with the property values at an appropriate mean temperature of 35°C.

[**Ans.** 20°C, 55.85°C]

7.6 The top surface of the compartment of a passenger train moving with a velocity of 60 km/h is 3 m wide and 10 m long. The top surface is absorbing solar radiation at a rate of 260 W/m^2, and the temperature of the ambient air is 30°C. Assuming the roof of the compartment to be perfectly insulated and the radiation heat exchange to the surroundings to be small relative to convection, determine the equilibrium temperature of the roof. In case of laminar flow, use Eq. (7.33a), and in case of combined laminar and turbulent flow, use Eq. (7.63).

[**Ans.** 37.82°C]

7.7 An array of power transistors, dissipating 5 W of power each, are to be cooled by mounting them on a 250 mm × 250 mm square aluminum plate and blowing air at 30°C over the plate with a fan at a velocity of 4 m/s. The average temperature of the plate is not to exceed 60°C. Assuming the heat transfer from the back side of the plate to be negligible and disregarding radiation, determine the number of transistors that can be placed on the plate. Neglect the influence of transistors in making the flow turbulent.

[**Ans.** 6]

7.8 Air at 10°C and 1 atm pressure flows over a flat plate at 4 m/s. A heater strip 20 mm long is placed on the plate at a distance of 0.10 m from the leading edge. Calculate the heat lost from the strip per unit depth of the plate for a heater surface temperature of 60°C.

[**Ans.** 16.62 W/metre width of the plate]

7.9 A 4 m long, 1.5 kW electrical resistance wire is made of 3 mm diameter stainless steel. The resistance wire operates in an environment at 30°C. Determine the surface temperature of the wire if it is cooled by a fan blowing air at a velocity of 5 m/s. Use Eq. (7.67) with property values at free stream temperature of 30°C.

[**Ans.** 340°C]

7.10 A person tries to keep cool on a hot summer day by turning a fan on and exposing his entire body to air flow. The air temperature is 25°C and the fan is blowing air at a velocity of 5 m/s. If the person does light work and generates heat at a rate of 100 W, determine the average temperature of the outer surface of the person. The average human body can be treated as a 0.3 m diameter cylinder with an exposed surface area of 1.6 m^2. Neglect any heat transfer by radiation. Use Eq. (7.67) with property values at free stream temperature of 25°C.

[**Ans.** 27.95°C]

7.11 An average person generates heat at a rate of 85 W when he takes rest. Assuming one-quarter of this heat is lost from the head and neglecting radiation, determine the average surface temperature of the head when it is not covered and is subjected to a wind at 15°C flowing with a velocity of 30 km/h. The head can be approximated by a 0.3 m diameter sphere. Use Eq. (7.71) with property values at free stream temperature of 15°C.

[**Ans.** 18.69°C]

7.12 Consider a fully developed flow of air through a horizontal smooth tube. The average bulk temperature is 30°C and the wall temperature is maintained at 90°C. The tube length is 2 m and the diameter is 10 mm. The flow velocity is 2 m/s. Determine the rate of heat transfer. The average bulk temperature is defined as $T_{b,\text{av}} = \int_0^L T_b dx/L$, where L is the length of the tube.

[**Ans.** 35.70 W]

7.13 Water flows in a duct of rectangluar cross-section of height of 6 mm and width 12 mm with a mean bulk temperature of 30°C. If the duct wall temperature is constant at 60°C and fully developed laminar flow is experienced, calculate the heat transfer per unit length.

[**Ans.** 281.46 W/m]

7.14 Water at the rate of 0.5 kg/s is forced through a tube of 20 mm internal diameter. The inlet water temperature is 20°C and the outlet water temperature is 60°C. The tube wall temperature is 15°C higher than the water temperature all along the length of the tube. Determine the length of the tube. Use Eq. (7.114).

[**Ans.** 12 m]

7.15 Cooling water at 25°C and with a velocity of 4 m/s enters a condenser tube and leaves the tube at 35°C. The inside diameter of the tube is 20 mm. Assuming a fully developed turbulent flow, determine the average heat transfer coefficient. Use Eq. (7.114).

[**Ans.** 13.86 kW/(m^2 °C)]

7.16 Compare the Nusselt numbers for the flow of water at an average bulk mean temperature of 20°C inside a smooth pipe of 40 mm diameter at Re = 40,000, determined by using (a) the Reynolds-Colburn analogy, (b) the Prandtl analogy and (c) the Dittus-Boelter equation. Take the value of $n = 0.4$ for the use of Dittus–Boelter equation.

[**Ans.** (a) 243, (b) 325, (c) 242]

7.17 Water at an average bulk temperature of 20°C flows at the rate of 0.25 kg/s through a tube of inner and outer diameter of 20 mm and 28 mm respectively. The tube material is Teflon with a thermal conductivity of $k = 0.30$ W/(m K). A thin electrical heating tape wrapped around the outer surface of the tube delivers a uniform heat flux of 2500 W/m^2. The heat is also convected from the outer surface to air at 35°C. The convection heat transfer coefficient remains constant over the outer surface at 30 W/(m^2 K). Determine the percentage of power dissipated by the tape which is transferred to water. Also, find out the average outer surface temperature of the tube. Use Eq. (7.114) with $n = 0.4$.

[**Ans.** 75.51 per cent, 50.41°C]

7.18 Consider a fully developed laminar flow of a viscous liquid between two parallel plates in which one plate is stationary and the other is moving with a velocity of U_∞. Such a flow is termed couette flow. The lower plate is maintained at a temperature T_0, while the upper plate at a temperature T_1, with $T_1 > T_0$. There is a generation of internal energy due to viscous dissipation by the liquid flow. Determine (a) the velocity distribution and (b) the temperature distribution in the liquid. (You may consider a pressure gradient imposed on the flow.)

7.19 For Problem 7.18, determine the criterion for which there is no heat transfer in the upper plate. Also develop an expression for heat transfer rate at the lower plate.

7.20 Mercury at a temperature of 150°C and with a velocity of 1 m/s enters a tube of 15 mm diameter. The inside surface of the tube is maintained at a constant temperature of 300°C. Determine the length of the tube required to raise the temperature of mercury to 250°C. If Re $\leq 10^5$, use Eq. (7.113), otherwise use Eq. (7.114).

[**Ans.** 0.14 m]

REFERENCES

[1] Churchill, S.W., and H. Ozee, Correlations of Laminar Forced Convection in Flow Over an Isothermal Flat Plate and in Developing and Fully Developed Flow in an Isothermal Tube, *J. Heat Transfer*, Vol. 95, p.46, 1973.

[2] Whitaker, S., *Elementary Heat Transfer Analysis,* Pergamon, New York, 1976.

[3] Whitaker, S., Forced Convection Heat Transfer Calculations for flow in Pipes, Past Flat Plates, Single Cylinders, and for Flow in Packed Beds and Tube Bundles, *AICHE J.*, 18, p. 361, 1972.

[4] Churchill, S.W., and M. Bernstein, A Correlating Equation for Forced Convection from Gases and Liquids to a Circular Cylinder in Cross Flow, *J. Heat Transfer,* 99, p. 300, 1977.

[5] Giedt, W.H., Investigation of Variation Point Unit – Heat Transfer Coefficients around a Cylinder Normal to an Air Stream, *Trans. ASME*, 71, p. 375, 1949.

[6] Hilpert, R., *Forsch. Geb. Ingenieurwes,* 4, p. 215, 1933.

[7] Jacob, M., *Heat Transfer,* Vol. 1, Wiley, New York, 1949.

[8] McAdams, W.H., *Heat Transmission*, 3rd ed., McGraw-Hill, New York, 1954.

[9] Ranz, W.E., and W.R. Marshall, Evaporation from Drops, *Chem. Engg. Prog.*, 48, 141, p. 173, 1952.

[10] Graetz L., *Uber die Warmeleitung fahigkeit von Flussigkeiten* (*On The Thermal Conductivity of Liquids*).

[11] Kays, W.M. and M.E. Crawford, *Convective Heat and Mass Transfer*, McGraw-Hill, New York, 1980.

[12] Sieder, E.N., and G.E. Tate, Heat Transfer and Pressure Drop of Liquids in Tubes, *Ind. Eng. Chem.,* 28, p. 1429, 1936.

[13] Dittus, F.W., and L.M.K. Boelter, University of California, Barkley, *Publications on Engineering,* Vol. 2, p. 443, 1930.

[14] Lubarsky, B., and S. J. Kaufman, Review of Experimental Investigation of Liquid Metal Heat Transfer, *NACA Tech. note* 3336, 1955.

[15] Skupinski, E.S., J. Torel, and L. Vautrey, Determination des coefficient de convection d'un allige Sodium-Potassium dans un Tube Circularie, *Int. J. Heat Mass Transfer,* 8, p. 937, 1965.

[16] Seban, R.A., and T.T. Shimazaki, Heat Transfer to Fluid Fowing Turbulently in a Smooth Pipe with Walls of Constant Temperature, *Trans ASME,* 73, p. 803, 1951.

[17] Sleicher, C.A. Jr., and M. Tribus, Heat Transfer in Pipe with Turbulent Flow and Arbitrary Wall Temperature Distribution, *Transaction ASME*, 79, p. 789, 1957.

[18] Azer, N.Z., and B.T. Chao, Turbulent Heat Transfer in Liquid Metals—Fully Developed Pipe Flow with Constant Wall Temperature, *Int. J. Heat Mass Transfer,* 3, p. 77, 1961.

[19] Notter, R.H., and C.A. Sleicher, A Solution to the Turbulent Graetz Problem III. Fully Developed and Entry Heat Transfer Rates, *Chem Engg Sci*, 27, p. 2073, 1972.

8

Principles of Free Convection

In the previous chapter we considered heat transfer by convection where the flow velocities were created by an external agency. This situation was referred to as 'forced convection'. The situation, where the fluid motion, responsible for convective heat transfer, is caused by buoyancy force originated from a spatial variation in fluid density, is referred to as 'free convection'. We have already made a brief introductory discussion on free and forced convection in both Chapter 1 and Chapter 5. The flow velocities in free convection are much smaller than those encountered in forced convection, and hence, the heat transfer rate in free convection is much smaller than that in forced convection.

Learning objectives

The reading of this chapter will enable the readers

- to understand the basic mechanism of free convection,
- to recognize the pertinent dimensionless parameters influencing free convection heat transfer,
- to recognize the criterion defined by relevant dimensionless parameters in gauging the relative importance of free convection over the forced convection,
- to analyze free convection problems under simple situations from the solution of governing conservation equations in determining the heat transfer coefficient,
- to get acquainted with different empirical equations for heat transfer coefficient in external free convection under different situations,
- to know about the laminar and turbulent free convection flows and the criterion imposed by the relevant dimensionless parameter for transition from laminar to turbulent flow under different situations,
- to know the mechanism and nature of free convection flow within fluid layers in an enclosed space,
- to get acquainted with empirical equations for free convection heat transfer coefficient and turbulent flow regimes in an enclosed space of different geometrical configurations, and
- to understand the concept of mixed convection.

8.1 PHYSICAL EXAMPLES OF FREE CONVECTION FLOWS

Free convection flows are observed in a number of physical situations. The buoyancy driven flow of air past a heated vertical plate has already been shown in Figure 1.4(b) of Chapter 1. Let us consider a stationary fluid, say air, contained between two parallel horizontal plates at different temperatures (Figure 8.1). A temperature gradient will be established in the vertical direction. If the lower plate is at a higher temperature (Figure 8.1(a)), then the upper layer of fluid being at a relatively lower temperature will have a relatively higher density. Thus the confined fluid mass becomes top heavy, and will be in a metastable static equilibrium condition so long as the fluid friction opposes the buoyancy force to establish any bulk fluid motion. Under this situation, the mode of heat transfer will be conduction. If the temperature difference is increased beyond a certain critical value, the viscous forces within the fluid fail to sustain the buoyancy forces, and a circulatory flow pattern is set up which is a free convection flow.

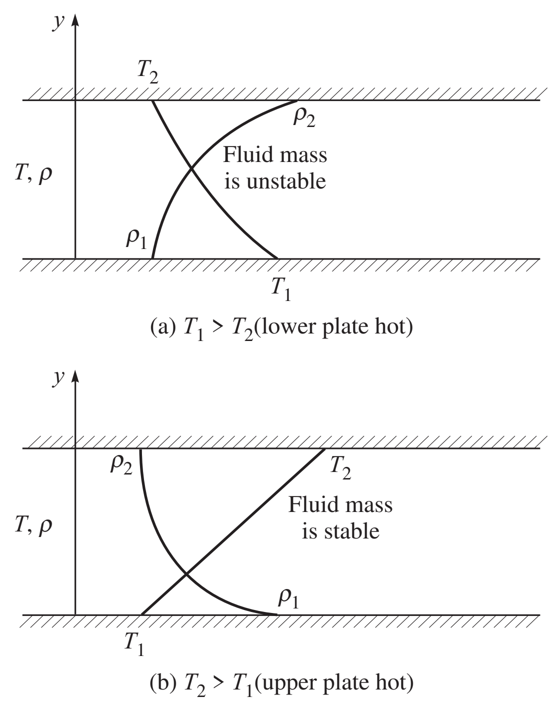


Figure 8.1 Fluid contained between two horizontal plates.

If the upper plate is at a higher temperature (Figure 8.1(b)), then the stationary fluid mass is bottom heavy and hence is in a condition of stable equilibrium. Therefore, no free convection flow is set up. The heat transfer takes place from the upper plate to the lower one solely by conduction through the entire fluid mass. The situation is referred to as *thermal stratification*.

The free convection flow past a vertical plate at a temperature different from that of its surrounding fluid is shown in Figure 8.2. The direction of flow depends upon whether the plate temperature is higher than the free stream temperature or vice versa.

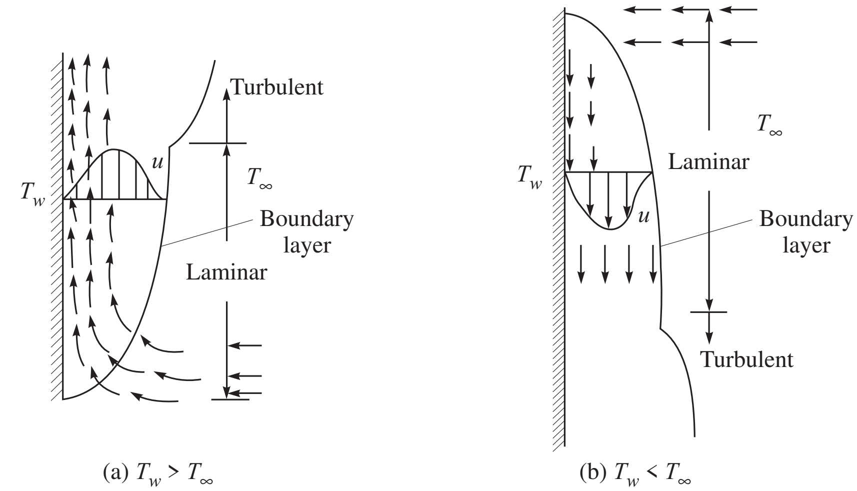


Figure 8.2 Free convection flow past a vertical plate.

Free convection flow may also appear in the form of a plume or buoyant jet. A plume is associated with the upward motion of fluid originated from a heated object immersed in an expanse of quiescent fluid as shown in Figure 8.3(a). The fluid that is heated up by the object rises due to buoyancy force. The width of the plume increases (Figure 8.3(a)) due to entrained surrounding air. Finally, the plume disappears since the upward fluid velocity is decreased due to viscous effects and the reduced buoyancy force caused by cooling of the entrainment fluid in the plume. A buoyant jet moves upwards due to the buoyancy force caused by a difference in fluid density associated with a temperature gradient. This happens when a heated fluid is discharged as a horizontal jet as shown in Figure 8.3(b).

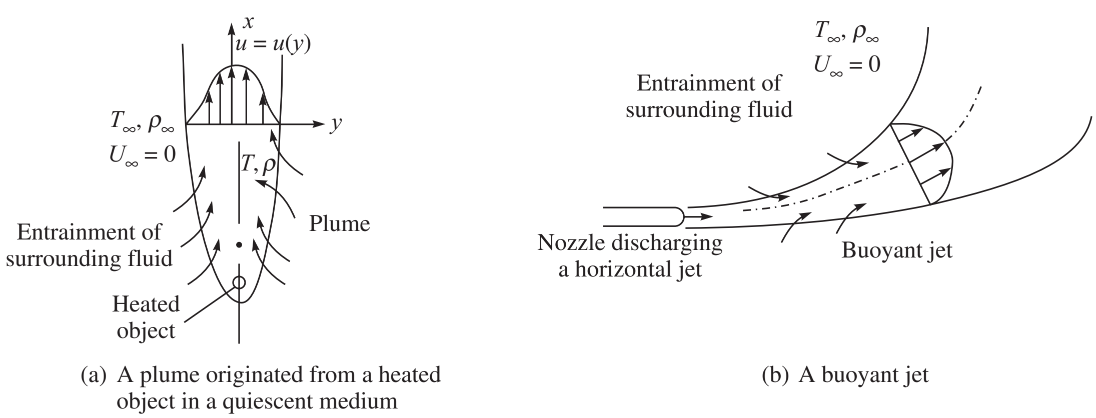


Figure 8.3 A plume and a buoyant jet.

Heat transfer by free convection arises in many engineering applications such as condenser tubes of a refrigerator, electric transformer and transmission lines, hot steam radiator for heating a room.

In Chapter 5, we have shown from a dimensional analysis that the pertinent dimensionless parameters governing the free convection heat transfer process are Nusselt number Nu, Grashoff number Gr, and Prandtl number Pr.

Nusselt number is the output parameter since it contains the heat transfer coefficient. Therefore, we always write

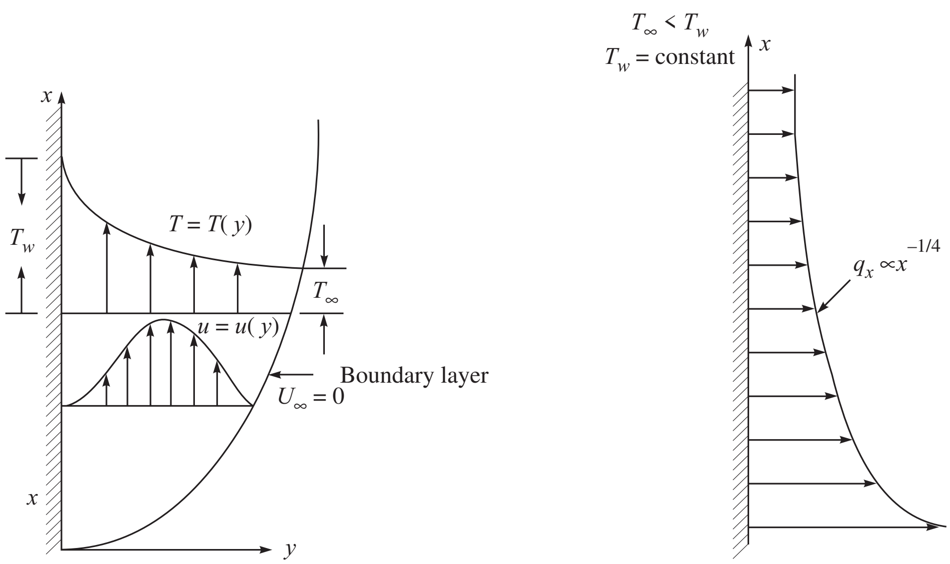
$$\text{Nu} = F(\text{Gr}, \text{Pr}) \tag{8.1}$$

It has also been mentioned in Chapter 5 that the sole objective of analysis of a free convection heat transfer phenomenon in any situation is to establish the exact relationship as shown in the functional form by Eq. (8.1).

We will first discuss a classical example of free convection heat transfer over a hot and isothermal vertical plate placed in a quiescent fluid medium.

8.2 LAMINAR FREE CONVECTION ON A VERTICAL PLATE AT CONSTANT TEMPERATURE

Let us consider a vertical plate of uniform temperature T_w placed in a large body of stationary fluid at a temperature of T_∞, where $T_\infty < T_w$ as shown in Figure 8.4.



(a) Temperature and velocity profile (b) Heat flux profile

Figure 8.4 Free convection from a vertical plate of uniform temperature.

The boundary layer grows along the plate due to the free convection flow. After a certain length of the plate, the boundary layer becomes turbulent depending upon the value of the local Grashoff number Gr_L. According to Bejan and Lage [1], the boundary layer will be turbulent if $\text{Gr}_L > 10^9$. However, we will assume the boundary layer to be laminar throughout for the present analysis.

Governing equations:

Continuity equation
(conservation of mass):
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{8.2}$$

Boundary layer equation
(conservation of momentum):
$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{dp}{dx} - \rho g + \mu\frac{\partial^2 u}{\partial y^2} \tag{8.3}$$

Energy equation
(conservation of energy):
$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{8.4}$$

The above governing differential equations are written in consideration of the flow to be steady and incompressible. The pressure p in Eq. (8.3) is the static pressure and hence we have included gravity as the body force which acts in the negative direction of x-axis. In accordance with the boundary layer approximation, the term dp/dx in Eq. (8.3) is determined from the pressure distribution in the adjoining potential core which is a stationary mass of fluid having a density of ρ_∞. Therefore, the term dp/dx is given by the principle of hydrostatics as

$$\frac{dp}{dx} = -\rho_\infty g \tag{8.5}$$

Substituting for dp/dx from Eq. (8.5) in Eq (8.3), we have

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = (\rho_\infty - \rho)g + \mu\frac{\partial^2 u}{\partial y^2} \tag{8.6}$$

The density ρ in Eq. (8.6) should essentially vary from point to point in the flow field because of the variation in temperature. For small temperature differences, the density ρ in the buoyancy term is considered to vary with temperature whereas the density appearing elsewhere in Eq. (8.6) is considered constant. This is known as *Boussinesq approximation*. It must be kept in mind that it is the variable density in the buoyancy force that induces fluid motion. As mentioned earlier in Chapter 5, the density difference in the buoyancy force is usually expressed in terms of the volumetric coefficient of thermal expansion of the fluid as

$$-\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p = \beta$$

or
$$(\rho_\infty - \rho) = \rho\beta(T - T_\infty) \tag{8.7}$$

(For a small temperature difference, $\partial\rho/\partial T$ is assumed to be linear)

Substituting for $(\rho_\infty - \rho)$ from Eq. (8.7) in Eq. (8.6), we have

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta(T - T_\infty) + \mu\frac{\partial^2 u}{\partial y^2} \tag{8.8}$$

Now we like to reduce Eqs. (8.2), (8.8) and (8.4) in non-dimensional forms by introducing the non-dimensional variables as

$$x^* = \frac{x}{L}, \qquad y^* = \frac{y}{L}, \qquad u^* = \frac{u}{U_0}$$

$$v^* = \frac{v}{U_0}, \qquad T^* = \frac{T - T_\infty}{T_w - T_\infty}$$

where L, the characteristic length, is the length of the plate and U_0 is an arbitrary reference velocity. Since the free stream conditions are quiescent here, therefore, there is no logical external reference velocity as in forced convection. With the help of the above defined non-dimensional variables, Eqs. (8.2), (8.8) and (8.4) become respectively

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{8.9}$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{g\beta(T_w - T_\infty)L}{U_0^2} T^* + \frac{1}{\mathrm{Re}_L} \frac{\partial^2 u^*}{\partial y^{*2}} \tag{8.10}$$

$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{\mathrm{Re}_L \,\mathrm{Pr}} \frac{\partial^2 T^*}{\partial y^{*2}} \tag{8.11}$$

where Re_L (Reynolds number) $= \dfrac{\rho U_0 L}{\mu}$

Pr (Prandtl number) $= \dfrac{\nu}{\alpha}$

The non-dimensional group in the first term on the right-hand side of Eq. (8.10) is the direct implication of buoyancy force. However, it is customary to rearrange this group by eliminating the term which is the square of an arbitrary velocity. This is done by multiplying Re_L^2 $(= U_0 L/\nu)^2$ with the group $g\beta(T_w - T_\infty)L/U_0^2$. The resulting non-dimensional group is referred to as the well known Grashoff number Gr.

Therefore,

$$\mathrm{Gr}_L = \left[\frac{g\beta(T_w - T_\infty)L}{U_0^2}\right]\left[\frac{U_0 L}{\nu}\right]^2$$

$$= \frac{g\beta(T_w - T_\infty)L^3}{\nu^2} \tag{8.12}$$

We can write for the original non-dimensional group in Eq. (8.10) as

$$\frac{g\beta(T_w - T_\infty)L}{\nu^2} = \frac{\mathrm{Gr}_L}{\mathrm{Re}_L^2} \tag{8.13}$$

The Grashoff number represents the ratio of buoyancy force to the viscous force acting on the fluid, and plays the same role in free convection as Reynolds number does in forced convection. In free convection, the transition from laminar to turbulent flow is governed by the critical value of the Grashoff number, while the same is being governed by the critical value of Reynolds number in forced convection. The parameter $\mathrm{Gr}_L/\mathrm{Re}_L^2$, defined by Eq. (8.13) is proportional to the ratio of buoyancy force to inertia force and hence is a measure of the relative importance of free convection over forced convection.

If $\dfrac{\mathrm{Gr}_L}{\mathrm{Re}_L^2} \approx 1$ free and forced convection are of comparable magnitude

Then $\mathrm{Nu}_L = f(\mathrm{Re}_L, \mathrm{Gr}_L, \mathrm{Pr})$ (Nusselt number depends on Reynolds number, Grashoff number and Prandtl number)

If $\dfrac{\mathrm{Gr}_L}{\mathrm{Re}_L^2} << 1$ free convection effects are neglected compared to effects of forced convection

Then $\mathrm{Nu}_L = f(\mathrm{Re}_L, \mathrm{Pr})$ (Nusselt number depends on Reynolds number and Prandtl number)

If $\dfrac{\mathrm{Gr}_L}{\mathrm{Re}_L^2} >> 1$ forced convection effects are neglected compared to dominant free convection effects

Then $\mathrm{Nu}_L = f(\mathrm{Gr}_L, \mathrm{Pr})$ (Nusselt number depends on Grashoff number and Prandtl number)

Another dimensionless parameter known as Rayleigh number Ra is often used and is defined by

$$\mathrm{Ra} = \mathrm{Gr}\cdot\mathrm{Pr}$$

Let us go back to the original governing equations (Eqs. (8.2), (8.3) and (8.4)) describing the laminar free convection from a vertical flat plate. The boundary conditions for the solution of these equations are:

$$\begin{aligned} &\text{At } y = 0, \quad u = v = 0, \quad T = T_w \\ &\text{At } y \to \infty, \quad u \to 0, \quad T \to T_\infty \end{aligned}$$

Equations (8.2), (8.6) and (8.4) along with the above mentioned boundary conditions are amenable to exact solution. Both an exact solution and an approximate solution are discussed below.

Exact solution by similarity transformation

The exact solution is due to Ostrach [2]. The solution involves the use of a similarity parameter η, the non-dimensional stream function $f(\eta)$ and the non-dimensional temperature T^* which are defined as follows:

$$\eta = \frac{y}{x}\left(\frac{\mathrm{Gr}_x}{4}\right)^{1/4} \tag{8.14}$$

$$\psi(x, y) = f(\eta)\left[4\nu\left(\frac{\mathrm{Gr}_x}{4}\right)^{1/4}\right] \tag{8.15}$$

$$T^* = \frac{T - T_\infty}{T_w - T_\infty} \tag{8.16}$$

The velocity components can be expressed as

$$\begin{aligned} u &= \frac{\partial \psi}{\partial y} \\ &= \frac{\partial \psi}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} \\ &= \frac{df}{d\eta}\left[4\nu\left(\frac{\mathrm{Gr}_x}{4}\right)^{1/4}\right]\frac{1}{x}\left(\frac{\mathrm{Gr}_x}{4}\right)^{1/4} \end{aligned}$$

$$= \frac{2\nu}{x}(\mathrm{Gr}_x)^{1/2}\frac{df}{d\eta}$$

$$v = -\frac{\partial\psi}{\partial x}$$

$$= -\frac{\partial\psi}{\partial x}\cdot\frac{\partial\eta}{\partial x}$$

$$= -\frac{df}{d\eta}\left[4\nu\left(\frac{\mathrm{Gr}_x}{4}\right)^{1/4}\right]\left[\frac{-y}{4x^2}\left(\frac{\mathrm{Gr}_x}{4}\right)^{1/4}\right]$$

$$= \frac{\nu y}{2x^2}(\mathrm{Gr}_x)^{1/2}\frac{df}{d\eta}$$

The momentum equation, i.e. Eq. (8.6) and the energy equation, i.e. Eq. (8.4) are transformed to

$$\frac{d^3f}{d\eta^3} + 3f\frac{d^2f}{d\eta^2} - 2\left(\frac{df}{d\eta}\right)^2 + T^* = 0 \tag{8.17}$$

$$\frac{d^2T^*}{d\eta^2} + 3\mathrm{Pr}\, f\,\frac{dT^*}{d\eta} = 0 \tag{8.18}$$

The continuity equation is identically satisfied. The boundary conditions in terms of dimensionless variables are:

$$\text{At } \eta = 0, \qquad f = \frac{df}{d\eta} = 0, \qquad T^* = 1 \tag{8.19a}$$

$$\text{At } \eta \to \infty, \qquad \frac{df}{d\eta} \to 0, \qquad T^* \to 0 \tag{8.19b}$$

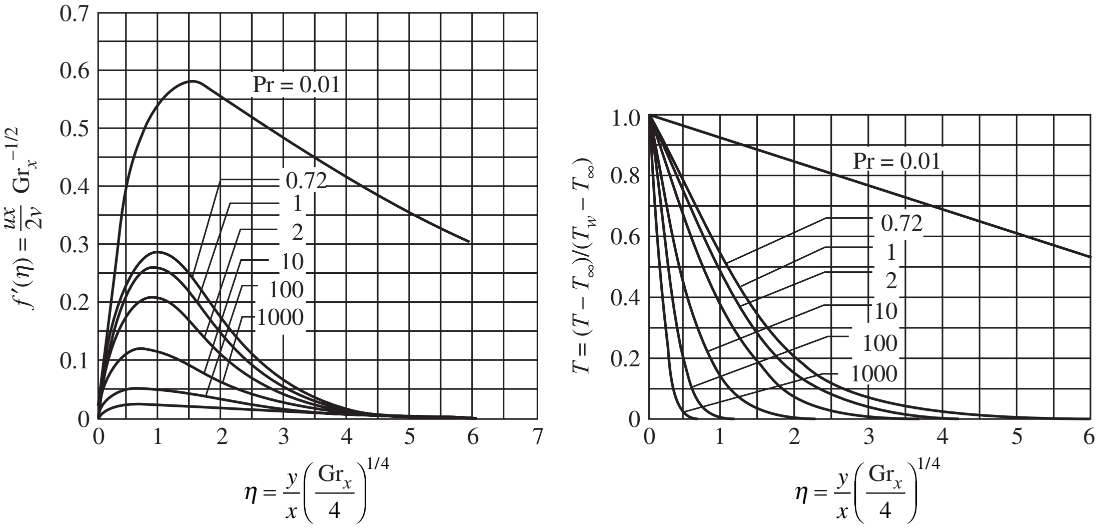
The coupled differential equations (Eqs. (8.17) and (8.18)) subject to boundary conditions given by Eqs. (8.19a) and (8.19b) were solved by Ostrach [2]. The results are shown in Figure 8.5. It is observed from Figure 8.5(a) that there is a decrease in the maximum velocity with an increase in Prandtl number, the temperature gradient at the surface becomes steeper which implies a higher rate of heat transfer.

For the determination of Nusselt number, we have to evaluate the heat transfer coefficient h as

$$h_x = \frac{-k\left(\dfrac{\partial T}{\partial y}\right)_{y=0}}{T_w - T_\infty} \tag{8.20}$$

$$= -\frac{k}{x}\left(\frac{\mathrm{Gr}_x}{4}\right)^{1/4}\left(\frac{\partial T^*}{\partial\eta}\right)_{\eta=0}$$

$$\mathrm{Nu}_x = \frac{h_x x}{k} = \left(\frac{\mathrm{Gr}_x}{4}\right)^{1/4}\left(\frac{\partial T^*}{\partial\eta}\right)_{\eta=0} \tag{8.21}$$



(a) Velocity profile

(b) Temperature profile

Figure 8.5 The velocity and temperature profiles in a free convection boundary layer over a flat isothermal vertical surface.

where the value of $(\partial T^*/\partial \eta)_{\eta=0}$ is a function of Prandtl number as observed in Figure 8.5(b). Let us write

$$\left(\frac{\partial T^*}{\partial \eta}\right)_{\eta=0} = g(\mathrm{Pr})$$

A numerical correlation with the help of an interpolation formula has been obtained for $g(\mathrm{Pr})$ as

$$g(\mathrm{Pr}) = \frac{0.75\,\mathrm{Pr}^{1/2}}{(0.609 + 1.221\,\mathrm{Pr}^{1/2} + 1.238\,\mathrm{Pr})^{1/4}} \tag{8.22}$$

The above relation is valid for the entire range of Prandtl numbers from 0 to ∞. Therefore,

$$\mathrm{Nu}_x = \frac{h_x x}{k} = \left(\frac{\mathrm{Gr}_x}{4}\right)^{1/4} g(\mathrm{Pr}) \tag{8.23a}$$

It is observed from Eq. (8.23a) that the local heat flux decreases with x as $h_x \sim x^{-1/4}$. This is shown in Figure 8.4(b).

$$\begin{aligned}\bar{h}_L &= \frac{1}{L}\int_0^L h_x\,dx \\ &= \frac{k}{L}\left[\frac{g\beta(T_w - T_\infty)}{4\nu^2}\right]^{1/4} g(\mathrm{Pr})\int_0^L \frac{dx}{x^{1/4}} \\ &= \frac{4}{3}\frac{k}{L}\left(\frac{\mathrm{Gr}_L}{4}\right)^{1/4} g(\mathrm{Pr})\end{aligned}$$

Hence,
$$\overline{\text{Nu}_L} = \frac{\bar{h}_L L}{k} = \frac{4}{3}\left(\frac{\text{Gr}_L}{4}\right)^{1/4} g(\text{Pr}) \tag{8.23b}$$

From a comparison of Eqs. (8.23a) and (8.23b), we have
$$\overline{\text{Nu}_L} = \frac{4}{3}\text{Nu}_L \tag{8.24}$$

The foregoing results apply irrespective of whether $T_w > T_\infty$ or $T_w < T_\infty$. If $T_w < T_\infty$, conditions are inverted from those of Figure 8.4. The leading edge is at the top of the plate, and positive x is defined in the direction of the gravity force. However, the above analysis is accurately valid in the range of $10^4 \le \text{Ra}_x \le 10^9$, where ($\text{Ra}_x = \text{Gr}_x$ Pr). This is because, if $\text{Ra}_x < 10^4$, the boundary layer assumptions do not hold good, and on the other hand if $\text{Ra}_x > 10^9$, the turbulence sets in the flow.

Approximate solution by the integral method

In this method, the momentum integral equation and the energy integral equation are solved with the assumed velocity and temperature profiles which satisfy the requisite boundary conditions. Substituting $U_\infty = 0$, and in consideration of the combined body force and pressure gradient term '$\rho g\beta(T - T_\infty)$', the momentum integral equation, i.e. Eq. (6.78) can be written for the present case of free convection as
$$\frac{d}{dx}\int_0^\delta u^2 dy = -\nu\left(\frac{\partial u}{\partial y}\right)_{y=0} + g\beta\int_0^\delta (T - T_\infty)dy \tag{8.25}$$

The energy integral equation is
$$\frac{d}{dx}\left[\int_0^\delta u(T - T_\infty)\,dy\right] = -\alpha\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{8.26}$$

Here, we assume $\delta_h \approx \delta_t \approx \delta$.

The boundary conditions for a temperature profile are:

At
$$\left.\begin{array}{ll} y = 0 & T = T_w \\ y = \delta & T = T_\infty \\ y = \delta & \dfrac{\partial T}{\partial y} = 0 \end{array}\right\} \tag{8.27}$$

The boundary conditions for a velocity profile are:

At
$$\left.\begin{array}{ll} y = 0 & u = 0 \\ y = \delta & u = 0 \\ y = \delta & \dfrac{\partial u}{\partial y} = 0 \\ y = 0 & \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{g\beta}{\nu}(T_w - T_\infty) \end{array}\right\} \tag{8.28}$$

Since there are three boundary conditions, we assume a polynomial of degree two for the temperature distribution as

$$T = a + by + cy^2$$

Utilizing the boundary conditions given by Eq. (8.27), we have

$$a = T_w$$

$$b = -\frac{2(T_w - T_\infty)}{\delta}$$

$$c = \frac{(T_w - T_\infty)}{\delta^2}$$

Finally, the temperature profile becomes

$$\frac{T - T_\infty}{T_w - T_\infty} = \left(1 - \frac{y}{\delta}\right)^2 \tag{8.29}$$

For the velocity profile, we have to assume a cubic polynomial since there are four boundary conditions

$$u = A + By + Cy^2 + Dy^3$$

Applying the boundary conditions, given by Eq. (8.28), to the above velocity profile, we have

$$\frac{u}{u_x} = \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2 \tag{8.30}$$

where, $u_x = g\beta\delta^2(T_w - T_\infty)/4\nu$ which is an arbitrary function of x with the dimension of velocity.

The temperature profile (Eq. (8.29)) and the velocity profile (Eq. (8.30)) are introduced into the momentum integral equation (Eq. (8.25)) and the energy integral equation (Eq. (8.26)) and the indicated operations, i.e. integrations and differentiations are performed. The momentum and energy equations then respectively become

$$\frac{1}{105}\frac{d}{dx}(u_x^2\,\delta) = \frac{1}{3}\,g\beta(T_w - T_\infty)\delta - \nu\frac{u_x}{\delta} \tag{8.31}$$

$$\frac{1}{30}(T_w - T_\infty)\frac{d}{dx}(u_x\,\delta) = 2\alpha\frac{(T_w - T_\infty)}{\delta} \tag{8.32}$$

From the expression of u_x as given earlier, we can write

$$u_x \sim \delta^2$$

Inserting the relation in Eq. (8.31), we can write

$$\delta \sim x^{1/4}$$

Therefore, we assume the following functions of u_x and δ for the solutions of Eqs. (8.31) and (8.32)

$$u_x = c_1 x^{1/2} \tag{8.33a}$$

$$\delta = c_2 x^{1/4} \tag{8.33b}$$

Substituting these relations into Eqs. (8.31) and (8.32), we have

$$\frac{5}{420} c_1^2 c_2 = g\beta(T_w - T_\infty)\frac{c_2}{3} - \frac{c_1}{c_2}\nu \tag{8.34a}$$

$$\frac{1}{40} c_1 c_2 = \frac{2\alpha}{c_2} \tag{8.34b}$$

The solution of these two simultaneous equations gives

$$c_1 = 5.17\nu\left(\frac{20}{21} + \frac{\nu}{\alpha}\right)^{-1/2}\left[\frac{g\beta(T_w - T_\infty)}{\nu^2}\right]^{1/2} \tag{8.35a}$$

$$c_2 = 3.93\left(\frac{20}{21} + \frac{\nu}{\alpha}\right)^{1/4}\left[\frac{g\beta(T_w - T_\infty)}{\nu^2}\right]^{-1/4}\left(\frac{\nu}{\alpha}\right)^{-1/2} \tag{8.35b}$$

The expression for boundary layer thickness becomes

$$\frac{\delta}{x} = 3.93\,\mathrm{Pr}^{-1/2}\,(0.952 + \mathrm{Pr})^{1/4}\,\mathrm{Gr}_x^{-1/4} \tag{8.36}$$

where the local Grashoff number Gr_x is defined as

$$\mathrm{Gr}_x = \frac{g\beta(T_w - T_\infty)x^3}{\nu^2} \tag{8.37}$$

The local heat transfer coefficient is evaluated from

$$h_x = \frac{-k\left(\dfrac{\partial T}{\partial y}\right)_{y=0}}{T_w - T_\infty}$$

Again from Eq. (8.29),

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{-2(T_w - T_\infty)}{\delta}$$

Hence,
$$h_x = \frac{2k}{\delta}$$

or
$$\mathrm{Nu}_x = \frac{h_x x}{k} = 2\cdot\frac{x}{\delta} \tag{8.38a}$$

With the help of Eq. (8.36), it becomes

$$\mathrm{Nu}_x = 0.508\,\mathrm{Pr}^{1/2}\,(0.952 + \mathrm{Pr})^{-1/4}\,\mathrm{Gr}_x^{1/4}$$

Average heat transfer coefficient over a length L becomes

$$\bar{h}_L = \frac{1}{L}\int_0^x h_x\,dx$$

$$= \frac{4}{3}h_L \tag{8.38b}$$

Therefore, $$\overline{\text{Nu}_L} = \frac{4}{3}\text{Nu}_L \tag{8.38c}$$

Criterion of turbulent free convection flow

The free convection flow, as mentioned earlier, is originated from a thermal instability when the viscous force cannot balance the buoyancy force. Similar to a forced flow, a disturbance in a free convection flow under a certain situation may lead to transition from a laminar to turbulent flow. Transition in a free convection flow depends on the relative magnitude of the buoyancy force and the viscous force in the fluid. The Rayleigh number which is the product of Grashoff number and Prandtl number is taken as the criterion to judge whether a flow is laminar or turbulent. For flow past a vertical plate the critical Rayleigh number is $\text{Ra}_c \approx 10^9$, regardless of the value of the Prandtl number.

8.3 EMPIRICAL RELATIONS FOR FREE CONVECTION UNDER DIFFERENT SITUATIONS

The mathematical analysis of free convection under different situations is quite complicated. Experiments have been performed for several years to correlate the free convection heat transfer coefficients with the governing parameters. For a variety of circumstances, the empirical relation is expressed in the form

$$\overline{\text{Nu}} = C(\text{Gr}\cdot\text{Pr})^n \tag{8.39}$$

where the empirical constants C and n vary from situation to situation. The properties in the dimensionless groups are evaluated at the film temperature $T_f = (T_\infty + T_w)/2$.

8.3.1 Correlations for Free Convection from Vertical Plates

Uniform wall temperature

The correlation proposed by McAdams [3] is of the form of Eq. (8.39) with the values of C and n as follows:

For laminar flow ($10^4 \le \text{Gr}_L\text{Pr} \le 10^9$)

$$C = 0.59, \qquad n = 1/3$$

For turbulent flow ($10^9 \le \text{Gr}_L\text{Pr} \le 10^{13}$)

$$C = 0.10, \qquad n = 1/3$$

Churchill and Chu [4] proposed two empirical relations. One is applicable to only laminar flow and holds good for all values of the Prandtl number. It is given by

$$\overline{\text{Nu}_L} = 0.68 + \frac{0.67(\text{Ra}_L)^{1/4}}{[1+(0.492/\text{Pr})^{9/16}]^{4/9}} \tag{8.40}$$

for $$10^{-1} < \text{Ra}_L < 10^9$$

The other expression which applies to both laminar and turbulent flow is given by

$$(\overline{\mathrm{Nu}_L})^{1/2} = 0.825 + \frac{0387(\mathrm{Ra}_L)^{1/6}}{[1+(0.492/\mathrm{Pr})^{9/16}]^{8/27}} \tag{8.41}$$

for $$10^{-1} < \mathrm{Ra}_L < 10^{12}$$

In a lower range of $10^{-1} < \mathrm{Gr\,Pr} < 10^4$, we can use Figure 8.6 given by McAdams [3]

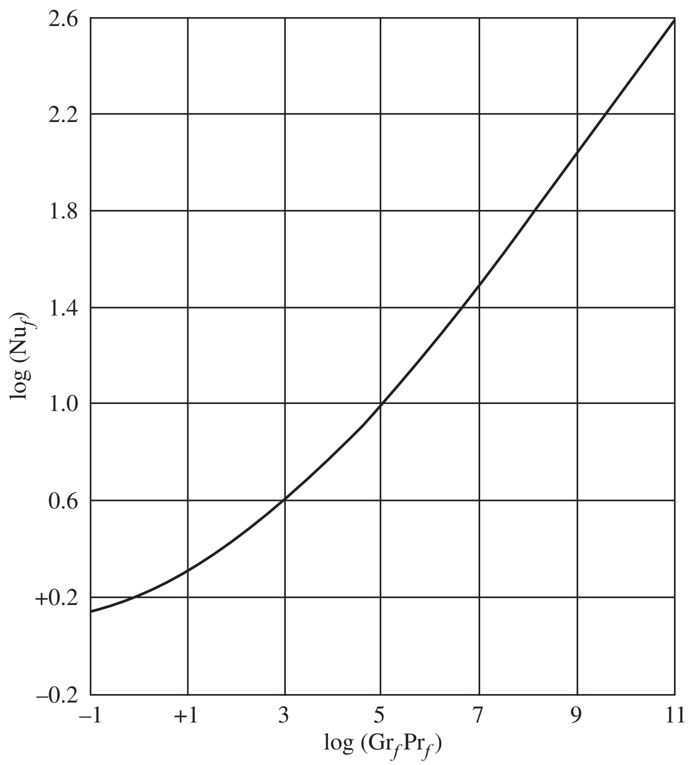


Figure 8.6 Free convection correlation for a heated vertical plate according to McAdams [3] (the subscript *f* refers to the property values being evaluated at film temperature).

Uniform wall heat flux

The empirical relation for free convection heat transfer coefficient is usually expressed as a function of the modified Grashoff number Gr_x^* defined by

$$\begin{aligned}\mathrm{Gr}_x^* &= \mathrm{Gr}_x\,\mathrm{Nu}_x \\ &= \left[\frac{g\beta(T_w - T_\infty)x^3}{\nu^2}\right]\left[\frac{q_w x}{(T_w - T_\infty)k}\right] \\ &= \frac{g\beta q_w x^4}{k\nu^2}\end{aligned} \tag{8.42}$$

A number of correlations are available in literature. However, we present two popular equations — one for laminar flow and the other for turbulent flow after Vilet and Liu [5] and Vilet [6] as follows:

$$\mathrm{Nu}_x = 0.60(\mathrm{Gr}_x^* \,\mathrm{Pr})^{1/5} \qquad \text{for } 10^5 < \mathrm{Gr}_x^* \,\mathrm{Pr} < 10^{11} \text{ (laminar)} \tag{8.43a}$$

$$\mathrm{Nu}_x = 0.568(\mathrm{Gr}_x^* \,\mathrm{Pr})^{0.22} \qquad \text{for } 2 \times 10^{13} < \mathrm{Gr}_x^* \,\mathrm{Pr} < 10^{16} \text{ (turbulent)} \tag{8.43b}$$

The average Nusselt number over a length L is based on average heat transfer coefficient $\bar{h}_L$ which is defined as

$$\bar{h}_L = \frac{1}{L}\int_0^L h_x \, dx$$

From Eq. (8.43a), we find that

$$h_x \sim \frac{x^{4/5}}{x} \sim x^{-0.2}$$

and from Eq. (8.43b)

$$h_x \sim \frac{x^{0.88}}{x} \sim x^{-0.12}$$

Therefore, in the laminar regime

$$\bar{h}_L = \frac{1}{1-0.2} h_L = 1.25\, h_L$$

and hence,

$$\overline{\mathrm{Nu}_L} = 1.25\, \mathrm{Nu}_L \tag{8.44a}$$

In the turbulent regime,

$$\bar{h}_L = \frac{1}{1-0.12} h_L = 1.136\, h_L$$

and hence,

$$\overline{\mathrm{Nu}_L} = 1.136\, \mathrm{Nu}_L \tag{8.44b}$$

All physical properties in the above correlations are evaluated at the film temperature.

8.3.2 Correlations for Free Convection from Horizontal Plates

The heat transfer characteristic from the upper surface of a hot horizontal plate in a cold environment is same as that to the lower surface of a cold plate in a hot environment. This is because the buoyancy driven flow pattern is same in both the situations as shown in Figures 8.7(a) and 8.7(b). Following the similar logic, we can say that the heat transfer characteristic from the lower surface of a hot horizontal plate in a cold environment is same as that from the upper surface of a cold plate in a hot environment. It is observed that a central plume flow is established in case of plate which is either hot upwards or cold downwards (Figures 8.7(a) and 8.7(b)). No plume flow is observed for plates which are either hot downwards or cold upwards (Figures 8.7(c) and 8.7(d)).

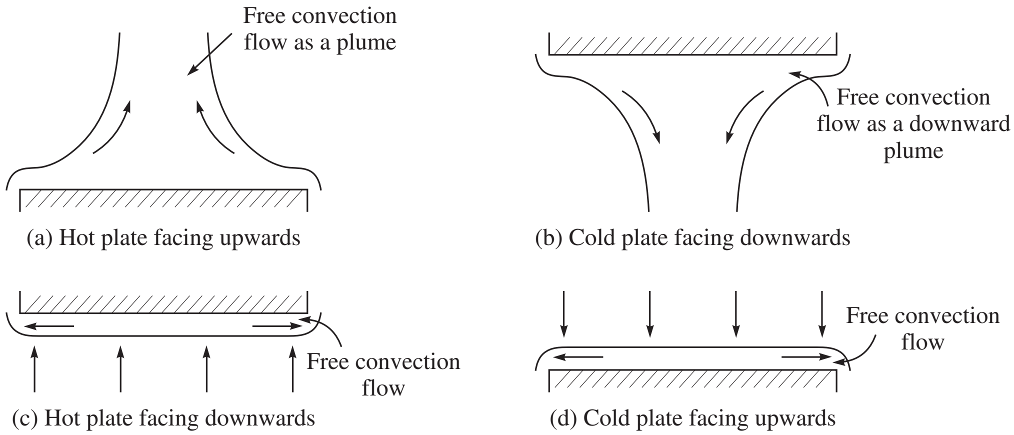


Figure 8.7 Free convection flow pattern for horizontal plates.

The average Nusselt number for horizontal plates with uniform temperature can be expressed by the Eq. (8.39) with the values of C and n as shown in Table 8.1.

Table 8.1 The values of C and n to be used in Eq. (8.39) for free convection from isothermal horizontal plates after McAdams [3]

Orientation of plate	*Range of* $Gr_L Pr$	*Flow regime*	C	n
Hot surface facing up or cold surface facing down	10^5 to 2×10^7 2×10^7 to 10^{10}	Laminar Turbulent	0.54 0.14	1/4 1/3
Hot surface facing down or cold surface facing up	3×10^5 to 3×10^{10}	Laminar	0.27	1/4

It is suggested that improved accuracy may be obtained if the characteristic length L for the plate is defined as

$$L = \frac{A}{P}$$

where A is the surface area of the plate and P is the perimeter which encompasses the area. The properties are evaluated at film temperature.

For uniform heat flux at plate surface, the following correlations are usually recommended.

For horizontal plate with heated surface facing upwards:

$$\overline{Nu_L} = 0.13(Gr_L Pr)^{1/3} \qquad \text{for } Gr_L Pr < 2 \times 10^8 \tag{8.45a}$$

$$\overline{Nu_L} = 0.16(Gr_L Pr)^{1/3} \qquad \text{for } 2 \times 10^8 < Gr_L Pr < 10^{11} \tag{8.45b}$$

For horizontal plate with heated surface facing downwards:

$$\overline{Nu_L} = 0.58(Gr_L Pr)^{1/5} \qquad \text{for } 10^6 < Gr_L Pr < 10^{11} \tag{8.46}$$

Equations (8.45a) and (8.45b) imply that the local heat transfer coefficient is independent of plate length.

8.3.3 Correlations for Free Convection from Inclined Plates

For an inclined plate, the angle θ the plate makes with the vertical is considered negative if the hot surface faces upwards as shown in Figure 8.8(a). Figure 8.8(b) shows the limiting cases of $\theta \to -19°$, the horizontal plate with hot surface facing upwards, and $\theta \to +90°$, the horizontal plate with hot surface facing downwards.

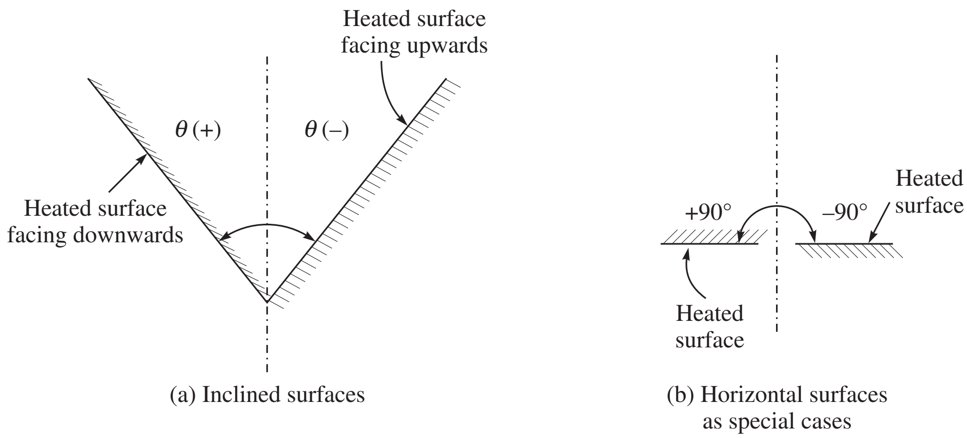


Figure 8.8 Nomenclature of inclined plates in free convection.

The following correlation was obtained for an inclined plate facing downwards with constant heat flux at the surface

By Fujii and Imura [7]:

$$\overline{\mathrm{Nu}_L} = 0.58(\mathrm{Gr}_L \mathrm{Pr} \cos\theta)^{1/4} \tag{8.47}$$
$$\text{for} \quad \theta < 88° \quad \text{and} \quad 10^5 < \mathrm{Gr}_L\mathrm{Pr} < 10^{11}$$

In Eqs. (8.45), (8.46) and (8.47) all properties except β (the volume expansion coefficient in Gr_L) are evaluated at a references temperature $T_r = T_w - 0.25(T_w - T_\infty)$, while β is evaluated at a temperature of $T_\infty + 025(T_w - T_\infty)$. Here T_w is the mean wall temperature.

Equation (8.46) is applicable for an almost horizontal plate ($88° < \theta < 90°$) with heated surface facing downwards. Again, Eq. (8.45) is applicable in the laminar range for a heated surface facing upwards. A suitable correlation in the turbulent range is recommended as

$$\overline{\mathrm{Nu}_L} = 0.145\,[(\mathrm{Gr}_L\mathrm{Pr})^{1/3} - (\mathrm{Gr}_c\mathrm{Pr})^{1/3}] + 0.56(\mathrm{Gr}_L\mathrm{Pr}\cos\theta)^{1/4} \tag{8.48}$$

for $10^5 < \mathrm{Gr}_L\mathrm{Pr}\cos\theta < 10^{11}$, $-15° < \theta < -75°$. Here Gr_c is the critical Grashoff number for the flow to be turbulent. The dependence of Gr_c on θ is shown in Table 8.2.

Table 8.2 Critical Grashoff number with θ

θ (*degrees*)	Gr_c
–15	5×10^9
–30	10^9
–60	10^8
–75	10^6

8.3.4 Correlations of Free Convection from Vertical and Horizontal Cylinders

Vertical cylinder

A vertical cylinder may be treated as a vertical plate when

$$\frac{L/D}{(\mathrm{Gr}_L)^{1/4}} < 0.025$$

If the boundary layer thickness is not large enough compared to the diameter of the cylinder, the correlations for vertical plates hold very good for vertical cylinders as well, and one can use Eqs. (8.40) and (8.41). The equations can be used satisfactorily for both the cases of constant surface temperature and constant surface heat flux. The properties should be evaluated at film temperature.

Horizontal cylinder

Morgan [8] recommended the use of Eq. (8.39) for horizontal cylinders with the values of C and n shown in Table 8.3.

Table 8.3 Values of C and n for the use of Eq. (8.39) for free convection from isothermal cylinders after Morgan [8]

C	n	Gr Pr
0.675	0.058	10^{-10} to 10^{-2}
1.02	0.148	10^{-2} to 10^{2}
0.850	0.188	10^{2} to 10^{4}
0.480	1/4	10^{4} to 10^{7}
0.125	1/3	10^{7} to 10^{12}

A more complicated relation is due to Churchill and Chu [9] which is as follows:

$$(\overline{\mathrm{Nu}_D})^{1/2} = 0.60 + \frac{0.387\,\mathrm{Ra}^{1/6}}{[1+(0.559/\mathrm{Pr})^{9/16}]^{8/27}} \tag{8.49}$$

$$\text{for} \quad 10^{-4} < \mathrm{Ra} < 10^{12}$$

It has to be noted here that both the Nusselt number Nu and Rayleigh number Ra are based on cylinder diameter. The properties are evaluated at film temperature.

8.3.5 Correlations of Free Convection from Spheres

The empirical equations for free convection from a single isothermal sphere which are in popular use are as follows:

$$\overline{\mathrm{Nu}} = 2 + 0.43\,\mathrm{Ra}^{1/4} \tag{8.50}$$

$$\text{for} \quad 1 < \mathrm{Ra} < 10^5 \quad \text{and} \quad \mathrm{Pr} \approx 1$$

The above relation was proposed by Yuge [10].

According to Amato and Tien [11],

$$\mathrm{Nu} = 2 + 0.50\,\mathrm{Ra}^{1/4} \tag{8.51}$$

$$\text{for} \quad 3 \times 10^5 < \mathrm{Ra} < 8 \times 10^8$$

According to Churchill [12]

$$\mathrm{Nu} = 2 + \frac{0.589\,\mathrm{Ra}^{1/4}}{[1 + (0.469/\mathrm{Pr})^{9/16}]^{4/9}} \tag{8.52}$$

for $\quad \mathrm{Ra} < 10^{11} \quad$ and $\quad \mathrm{Pr} > 0.5$

In all the above correlations, the Nusselt number Nu and Rayleigh number Ra are defined on the basis of sphere diameter and the properties are evaluated at film temperature.

EXAMPLE 8.1 Water is heated by a 200 mm by 200 mm vertical flat plate which is maintained at 60°C. Find the rate of heat transfer when the water is at 20°C. At mean film temperature of $T_f = (60 + 20)/2 = 40$°C, the relevant physical parameters can be taken as $k = 0.628$ W/(m K), Pr = 4.34, $\rho = 994.59$ kg/m^3, $\nu = 0.658 \times 10^{-6}$ m^2/s, $\beta = 3 \times 10^4$ K^{-1}. Use Eq. (8.38a) if the flow is laminar or Eq. (8.39) with $C = 0.10$ and $n = 1/3$, if the flow is turbulent.

Solution:

$$\mathrm{Gr}_L = \frac{g\beta(T_w - T_\infty)L^3}{\nu^2}$$

$$= \frac{9.81 \times 3 \times 10^{-4} \times (60 - 20) \times (0.2)^3}{(0.658 \times 10^{-6})^2}$$

$$= 21.75 \times 10^8$$

$$\mathrm{Ra}_L = \mathrm{Gr}_L\mathrm{Pr}$$

$$= 21.75 \times 10^8 \times 4.34 = 9.44 \times 10^9$$

Therefore, the flow is turbulent and we use Eq. (8.39).

$$\frac{\bar{h}_L L}{k} = 0.10(\mathrm{Gr}_L\mathrm{Pr})^{1/3}$$

or

$$\bar{h}_L = \frac{0.628}{0.2} \times (0.10) \times (9.44 \times 10^9)^{1/3}$$

$$= 663.62\ \mathrm{W/(m^2\ K)}$$

The rate of heat transfer,

$$q = \bar{h}_L \times A \times (T_w - T_\infty)$$

$$= 663.62 \times 0.2 \times 0.2(60 - 20)$$

$$= 1061.8\ \mathrm{W} = 1.06\ \mathrm{kW}$$

EXAMPLE 8.2 A number of thin plates are to be cooled by suspending them vertically in a water bath. The plates are kept at a uniform temperature of 60°C, while the temperature of water in the bath is 20°C. The plates are 90 mm long. The spacing between the plates is 9 mm. Verify whether interference between the free convection boundary layers of the plates will occur or not. In case of laminar flow, use the results of integral analysis. Take the properties of water at a mean temperature of 40°C the same as given in Example 8.1

Solution: The minimum spacing between the plates will be twice the thickness of the boundary layer at the trailing edge where $x = 0.09$ m

$$\mathrm{Gr}_L = \frac{g\beta(T_w - T_\infty)L^3}{\nu^2} = \frac{9.81 \times 3 \times 10^{-4} \times (60 - 20) \times (0.09)^3}{(0.658 \times 10^{-6})^2}$$

$$= 1.98 \times 10^8$$

$$\text{Ra}_L = \text{Gr}_L\text{Pr}$$
$$= 1.98 \times 10^8 \times 4.34$$
$$= 8.59 \times 10^8$$

Since Ra < 10^9, the flow is laminar.

From Eq. (8.36), we get

$$\delta = \frac{(0.09) \times (3.93) \times (0.952 + 4.34)^{1/4}}{(4.34)^{1/2} \times (1.98 \times 10^8)^{1/4}}$$
$$= 0.22 \times 10^{-2} \text{ m}$$
$$= 2.2 \text{ mm}$$

The minimum spacing = 2 × 2.2 = 4.4 mm.

Therefore, the boundary layer interference will not take place since the spacing between the plates is 9 mm which is more than the critical spacing of 4.4 mm.

EXAMPLE 8.3 Consider the problem of Example 5.7 (Chapter 5). What is the maximum velocity in the boundary layer at $x = 0.80$ m from the bottom of the wall? Determine the mass flow rate of nitrogen past the station $x = 0.8$ m. Use the property values as given in Example 5.7 and also use the results of integral analysis.

Solution: $$\text{Gr}_{x=0.80} = \frac{g\beta(T_w - T_\infty)x^3}{\nu^2} = \frac{9.81 \times 3.3 \times 10^{-3} \times (56 - 4) \times (0.8)^3}{(15.63 \times 10^{-6})^2} = 3.53 \times 10^9$$

From Eq. (8.36), we can write

$$\delta = \frac{0.8 \times 3.93 \times (0.952 + 0.713)^{1/4}}{(0.713)^{1/2} \times (3.53 \times 10^9)^{1/4}}$$
$$= 1.73 \times 10^{-2} \text{ m}$$

From Eq. (8.30),

$$\frac{u}{u_x} = \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2$$

$$u_x = \frac{g\beta\delta^2(T_w - T_\infty)}{4\nu}$$

$$= \frac{9.81 \times 3.3 \times 10^{-3} \times (1.73 \times 10^{-2})^2 \times (56 - 4)}{4 \times (15.63 \times 10^{-6})}$$

$$= 8.06 \text{ m/s}$$

Hence, $$u = \frac{8.06}{0.0173} y\left(1 - \frac{y}{0.0173}\right)^2$$

$$= 465.9y(1 - 57.8y)^2$$
$$= 465.9(y - 116y^2 + 3341y^3)$$

For maximum value of u

$$\frac{du}{dy} = 465.9(1 - 232\,y + 10023\,y^2) = 0$$

Hence,
$$y = \frac{232 \pm \sqrt{(232)^2 - 4 \times 10023}}{2 \times 10023}$$

which gives
$$y = 0.00573,\ 0.0173$$

The value of 0.0173 is at the edge of boundary layer, where $u = 0$.

Therefore, the maximum value of u occurs at $y = 0.00573$ m, i.e.

$$U_{\max} = 465.9 \times (0.00573)[1 - 57.8 \times (0.00573)^2]$$
$$= 2.66 \text{ m/s}$$

Mass flow rate at $x = 0.8$ m is given by

$$\dot{m} = \rho \times B \times \int_0^{\delta} u(y)\,dy$$
$$= 1.1421 \times 2.5 \times 465.9 \int_0^{0.0173} (y - 116y^2 + 3341y^3)\,dy$$
$$= 1.1421 \times 2.5 \times 465.9 \left[\frac{(0.0173)^2}{2} - 116 \times \frac{(0.0173)^3}{3} + 3341 \times \frac{(0.0173)^4}{4}\right]$$
$$= 0.032 \text{ kg}$$

EXAMPLE 8.4 A square plate 0.2 m × 0.2 m is suspended vertically in a quiescent atmospheric air at a temperature of 300 K. The temperature of the plate is maintained at 400 K.

Verify whether the free convection flow is laminar or not. In case of laminar flow, use the results of integral analysis and determine

(a) the boundary layer thickness at the trailing edge of the plate (at $x = 0.2$ m), and
(b) the average heat transfer coefficient over the entire length of the plate.

The required property values of air at the film temperature $T_f = (400 + 300)/2 = 350$ K are given as

$$\nu = 20.75 \times 10^{-6} \text{ m}^2\text{/s},\ \Pr = 0.69,\ k = 0.03 \text{ W/(m K)}$$

Solution:
$$\beta = \frac{1}{T_f} = \frac{1}{350} = 2.86 \times 10^{-3} \text{ K}^{-1}$$

$$\text{Gr} = \frac{g\beta(T_w - T_\infty)L^3}{\nu^2} = \frac{9.81 \times 2.86 \times 10^{-3} \times (400 - 300) \times (0.2)^3}{(20.75 \times 10^{-6})^2}$$
$$= 5.2 \times 10^7$$
$$\text{Ra} = \text{Gr} \cdot \Pr$$
$$= 5.2 \times 10^7 \times 0.69$$
$$= 3.59 \times 10^7$$

Hence, the flow is laminar.

(a) By making use of Eq. (8.36),

$$\delta_{x=0.2} = \frac{(3.93)\times(0.2)(0.952+0.69)^{1/4}}{(0.69)^{1/2}\times(5.2\times10^7)^{1/4}}$$
$$= 1.83 \times 10^{-2} \text{ m}$$
$$= 18.3 \text{ mm}$$

(c) From Eq. (8.38a),

$$\overline{h}_L = \frac{4}{3}\text{h}_L$$
$$= \frac{4}{3}\times\frac{2k}{(\delta)_{x=L}}$$
$$= \frac{8}{3}\frac{k}{(\delta)_{x=L}}$$
$$= \frac{8\times0.03}{3\times18.3\times10^{-3}}$$
$$= 4.37\,\text{W/(m}^2\text{ K)}$$

EXAMPLE 8.5 Consider a square plate of 0.5 m × 0.5 m in a room at 30°C. One side of the plate is kept at a uniform temperature of 74°C while the other side is insulated. Determine the rate of heat transfer from the plate by free convection under the following three situations:

(a) The plate is kept vertical. (Use the results of integral analysis and also the empirical relation, Eq. (8.39).)
(b) The plate is kept horizontal with hot surface facing up.
(c) The plate is kept horizontal with hot surface facing down.

The required physical properties of air at the film temperature of $T_f(74 + 30)/2 = 52$°C are

$$k = 0.28 \text{ W/(m °C)}, \qquad \text{Pr} = 0.71, \qquad \nu = 1.815 \times 10^{-5} \text{ m}^2\text{/s}$$

Solution:

$$\beta = \frac{1}{T_f} = \frac{1}{273+52} = 3.08 \times 10^{-3} \text{ K}^{-1}$$

$$\text{Gr}_L = \frac{g\beta(T_w - T_\infty)L^3}{\nu^2} = \frac{9.81\times3.08\times10^{-3}\times(74-30)\times(0.5)^3}{(1.815\times10^{-5})^2}$$
$$= 5.04 \times 10^8$$
$$\text{Ra}_L = \text{Gr}_L\text{Pr} = 5.04 \times 10^8 \times 0.71$$
$$= 3.58 \times 10^8$$

Therefore, the flow is laminar.

(a) We make use of Eqs. (8.38a) and (8.38c)

$$\overline{\text{Nu}_L} = \frac{4}{3}\times\frac{0.508\times(0.71)^{1/2}\times(5.04\times10^8)^{1/4}}{(0.952+0.71)^{1/4}}$$
$$= 75.31$$

$$\bar{h}_L = \frac{\overline{\text{Nu}}_L k}{L} = \frac{75.31 \times 0.028}{0.5}$$

$$= 4.22 \text{ W/(m}^2 \text{ °C)}$$

$$Q = \bar{h}_L \times A \times (T_w - T_\infty) = 4.22 \times (0.5)^2 \times (74 - 30)$$

$$= 46.42 \text{ W}$$

Now if we use Eq. (8.39) with the values of $C = 0.59$ and $n = ¼$ for laminar flow, we have

$$\overline{\text{Nu}}_L = 0.59 \times (3.58 \times 10^8)^{1/4}$$

$$= 81.15$$

$$\bar{h}_L = \frac{81.15 \times 0.028}{0.5}$$

$$= 4.54 \text{ W/(m}^2 \text{ °C)}$$

$$Q = 4.54 \times (0.5)^2 \times (74 - 30)$$

$$= 49.94 \text{ W}$$

(b) For horizontal plate with the hot surface facing up, we use Eq. (8.39) with the values of C and n from Table 8.1. Here, we have to use the characteristic length $L = A/P$

$$L = \frac{0.5 \times 0.5}{4 \times 0.5} = 0.125 \text{ m}$$

Then,

$$\text{Ra}_L = 3.58 \times 10^8 \times \left(\frac{0.125}{0.5}\right)^3$$

$$= 5.6 \times 10^6$$

The values of C and n at this value of Ra_L become (Table 8.1) 0.54 and 1/4 respectively.

Therefore,

$$\overline{\text{Nu}}_L = C(\text{Gr}_L\text{Pr})^n = 0.54 \times (5.60 \times 10^6)^{1/4}$$

$$= 26.27$$

$$\bar{h}_L = \frac{26.27 \times 0.028}{0.125}$$

$$= 5.88 \text{ W/(m}^2 \text{ °C)}$$

$$Q = 5.88 \times (0.5)^2 \times (74 - 30)$$

$$= 64.68 \text{ W}$$

(c) When the hot surface faces down,

$$\overline{\text{Nu}}_L = 0.27 \text{ Ra}^{1/4}$$

$$= 0.27 \times (5.6 \times 10^6)^{1/4}$$

$$= 13.13$$

$$\bar{h}_L = \frac{13.13 \times 0.028}{0.125}$$

$$= 2.94 \text{ W/m}^2$$

$$Q = 2.94 \times (0.5)^2 \times (74 - 30)$$

$$= 32.34 \text{ W}$$

EXAMPLE 8.6 Determine the electrical power that is required to maintain a 0.1 mm diameter, 0.5 m long vertical wire at 400 K in an atmosphere of quiescent air at 300 K. The resistance of the wire is 0.12 ohm per metre. Determine also the current flowing in the wire. The properties of air at the mean film temperature of 350 K are

$$\nu = 20.75 \times 10^{-6}\ \text{m}^2/\text{s}, \qquad \text{Pr} = 0.70, \qquad k = 0.03\ \text{W/(m K)}.$$

Use Eq. (8.39) for flat plate with proper values of C and n depending upon whether the flow is laminar or turbulent and then use a multiplying factor F with the heat transfer coefficient for the curvature effect as

$$F = 1.3[(L/D)/\text{Gr}_D]^{1/4} + 1.0$$

Solution: First consider the wire as a vertical flat plate of length 0.5 m and use Eq. (8.39).

$$\beta = \frac{1}{T_f} = \frac{1}{350} = 2.86 \times 10^{-3}\ \text{K}^{-1}$$

$$\text{Gr}_L = \frac{g\beta(T_w - T_\infty)L^3}{\nu^2} = \frac{9.81 \times 2.86 \times 10^{-3} \times (400 - 300) \times (0.5)^3}{(20.75 \times 10^{-6})^2}$$

$$= 8.14 \times 10^8$$

$$\text{Ra}_L = \text{Gr}_L\text{Pr} = 8.14 \times 10^8 \times 0.70$$

$$= 5.7 \times 10^8$$

Therefore, the flow is laminar.
Hence, we take $C = 0.59$ and $n = 1/4$ for the use of Eq. (8.39).

$$\overline{\text{Nu}}_L = 0.59(\text{Ra}_L)^{1/4}$$

$$= 0.59 \times (5.7 \times 10^8)^{1/4}$$

$$= 91.16$$

$$\bar{h}_L = \frac{91.16 \times 0.03}{0.5}$$

$$= 5.47\ \text{W/(m}^2\ \text{K)}$$

$$\text{Gr}_D = 8.14 \times 10^8 \times \left(\frac{0.1 \times 10^{-3}}{0.5}\right)^3$$

$$= 6.51 \times 10^{-3}$$

The correction factor

$$F = 1.3\left[\frac{0.5}{0.1 \times 10^{-3}} \times \frac{1}{6.51 \times 10^{-3}}\right]^{1/4} + 1.0$$

$$= 39.5$$

Therefore, the correct value of $\bar{h}_L = 5.47 \times 39.5$

$$= 216\ \text{W/(m}^2\ \text{K)}$$

The ohmic power loss is given by

$$I^2R = q = \bar{h}A\ (T_w - T_\infty)$$

$$= 216 \times \pi \times (0.1 \times 10^{-3})\,(0.5)\,(400 - 300) = 3.39\ \text{W}$$

The current flowing in the wire is

$$I = \sqrt{\frac{3.39}{0.12 \times 0.5}} = 7.52 \text{ A}$$

EXAMPLE 8.7 A long horizontal pressurized hot water pipe of 200 mm diameter passes through a room where the air temperature is 25°C. The pipe surface temperature is 130°C. Neglecting the radiation loss from the pipe, determine the rate of heat transfer to room air per metre of pipe length. The properties of air at the film temperature T_f (130 + 25)/2 = 77.5° C are

$$k = 0.03 \text{ W/(m K)}, \qquad \text{Pr} = 0.7, \qquad \nu = 21 \times 10^{-6} \text{ m}^2/\text{s}$$

Solution:

$$\beta = \frac{1}{T_f} = \frac{1}{273 + 77.5}$$

$$= 2.85 \times 10^{-3} \text{ K}^{-1}$$

$$\text{Gr}_D = \frac{(9.81) \times (2.85 \times 10^{-3}) \times (130 - 25) \times (0.2)^3}{(21 \times 10^{-6})^2}$$

$$= 5.32 \times 10^7$$

$$\text{Ra}_D = \text{Gr}_D\text{Pr} = (5.32 \times 10^7) \times (0.7)$$

$$= 3.72 \times 10^7$$

The flow is laminar over the entire cylinder. Using Eq. (8.49),

$$\overline{\text{Nu}}_D = \left[0.60 + \frac{(0.387) \times (3.72 \times 10^7)^{1/6}}{\{1 + (0.559/0.7)^{9/16}\}^{8/27}}\right]^2$$

$$= 42.63$$

$$\bar{h} = \frac{\overline{\text{Nu}}_D k}{D}$$

$$= \frac{42.63 \times 0.03}{0.2}$$

$$= 6.39 \text{ W/(m}^2 \text{ K)}$$

The heat loss per metre of length,

$$q = (6.39) \times (\pi \times 0.2) \times (130 - 25)$$

$$= 421.57 \text{ W}$$

EXAMPLE 8.8 An electric immersion heater 8 mm in diameter and 300 mm long is rated at 450 W. If the heater is horizontally positioned in a large tank of stationary water at 20°C, determine the steady state surface temperature of the heater.

Solution: At steady state,

The electrical power input = heat loss from the heater

Therefore, $$P = Q$$

$$= \bar{h}_D \times (\pi D) \times L \times (T_w - T_\infty)$$

$$T_w(\text{surface temperature}) = T_\infty + \frac{P}{\bar{h}_D \times \pi D \times L}$$

Here, $P = 450$ W, $D = 0.008$ m, $L = 0.3$, $T_\infty = 20°$C

Therefore we have to find $\bar{h}_D$.

To determine $\bar{h}_D$ from any correlation we require the properties of water at film temperature $T_f = (T_w + T_\infty)/2$. But T_w is unknown. In this type of problem we are to make a guess of T_w and iterate the solution for T_w till a desired convergence is achieved.

Let us take for the first trial $T_w = 64°$C.

Then, $$T_f = (64 + 20)/2 = 42°\text{C}$$

At this temperature of 42°C, the required properties of water are

$k = 0.634$ W/(m K), $\nu = 6.25 \times 10^7$ m^2/s, $\text{Pr} = 4.16$, $\beta = 4 \times 10^{-4}$ K^{-1}

Therefore,
$$\text{Gr}_D = \frac{(9.81) \times (4 \times 10^{-4}) \times (64 - 20) \times (0.008)^3}{(6.25 \times 10^{-7})^2}$$
$$= 2.26 \times 10^5$$
$$\text{Ra}_D = \text{Gr}_D\text{Pr} = 2.26 \times 10^5 \times 4.16$$
$$= 0.94 \times 10^6$$

The flow is laminar. We therefore use Eq. (8.49) to calculate:

$$\overline{\text{Nu}_D} = \left[0.60 + \frac{(0.387) \times (0.94 \times 10^6)^{1/6}}{\{1 + (0.559/4.16)^{9/16}\}^{8/27}}\right]^2$$
$$= 17.02$$
$$\bar{h}_D = \frac{17.02 \times 0.634}{0.008} = 1349 \text{ W/(m}^2\text{ K)}$$

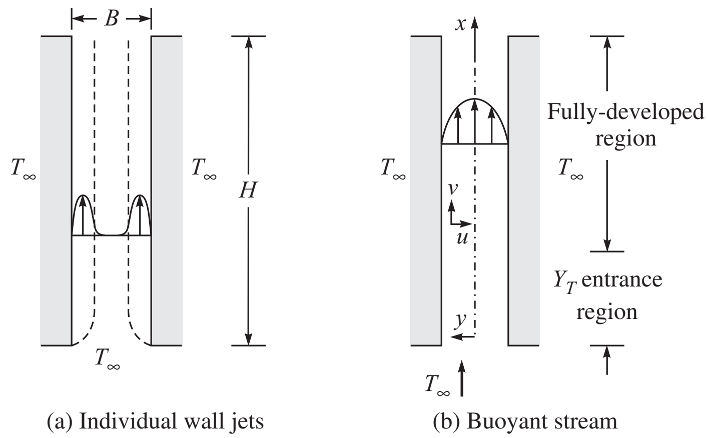
Hence,
$$T_w = 20 + \frac{450}{\pi \times (0.008) \times 0.3 \times 1349}$$
$$= 64.2°\text{C}$$

We see that our guess of T_w is in excellent agreement with the calculated value. Therefore, no further iteration is required and we can write

$$T_w = 64°\text{C}$$

8.4 FREE CONVECTION FLOW IN A VERTICAL CHANNEL

The interaction between the free convection boundary layers formed along the parallel walls are of practical importance. If the boundary layer thickness is much smaller than the spacing of the plates, then the flow along each wall will be unaffected by the other and will appear as individual wall jets with velocity distribution as shown in Figure 8.9(a). On the other hand, if the boundary layer thickness becomes comparable to the spacing between the plates, then the two boundary layers or wall jets merge into a single buoyant stream rising through the channel like a chimney flow. The velocity distribution in this situation is shown in Figure 8.9(b). This situation prevails in vertical fin-to-fin cooling channels in certain pieces of electronic equipment.



Fugure 8.9 Free convection in a channel formed by two vertical heated plates.

Let us consider two vertical flat plates of height H at a uniform temperature of T_w which is greater than the ambient temperature $T_\infty(T_w > T_\infty)$. The flow will take place, under the situation, in a vertically upward direction. The spacing between the plates B is considered to be small so that the boundary layers from each plate merge and the flow takes place as a single buoyant stream. Our objective is to solve for the free convection heat transfer coefficient in the present situation.

The momentum equation in the x-direction (Figure 8.9(b)) can be written as

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} - \rho g + \mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{8.53}$$

If the channel is long enough, the order of the velocity v will be very small as compared to that of u. This leads to the concept of fully developed flow, for which we have

$$\frac{\partial u}{\partial x} = 0 \tag{8.54}$$

In a fully developed flow, the pressure p is a function of x only. Moreover, both the ends of the channel are open to the ambient of density ρ_∞, and hence we can write

$$\frac{\partial p}{\partial x} = \frac{dp}{dx} = -\rho_\infty g \tag{8.55}$$

With the help of Eqs. (8.54) and (8.55) along with the usual Boussinesq approximation, Eq. (8.53) becomes

$$\frac{d^2 u}{dy^2} = \frac{g\beta}{\nu}(T - T_\infty) \tag{8.56}$$

where β is the coefficient of volumetric expansion as defined by Eq. (8.7).

(The term $\dfrac{\partial^2 u}{\partial x^2}$ is neglected in comparison to $\dfrac{\partial^2 u}{\partial y^2}$ since the scale of x is much larger than that of y.)

For integration of Eq. (8.56), we have to know the temperature T as a function of x which can be found from the solution of energy equation. Again, the solution of energy equation requires the knowledge of velocity distribution. Therefore, the two equations have to be solved simultaneously. However, a much simpler solution approach is possible, if we assume

$$(T_w - T) << (T_w - T_\infty)$$

so that $(T - T_\infty)$ in Eq. (8.56) can be substituted by $(T_w - T_\infty)$. Then, we have

$$\frac{d^2u}{dy^2} = -\frac{g\beta}{\nu}(T_w - T_\infty) \tag{8.57}$$

The integration of Eq. (8.57) with the boundary conditions $u = 0$ at $y = \pm \dfrac{B}{2}$, gives

$$u = \frac{g\beta B^2 (T_w - T_\infty)}{8\nu}\left[1 - \left(\frac{x}{B/2}\right)^2\right] \tag{8.58}$$

Mass flow rate per unit width (normal to the plane of Figure 8.9(b)) becomes

$$\dot{m} = \int_{-B/2}^{+B/2} \rho u \, dy$$

$$= \frac{\rho g\beta B^3 (T_w - T_\infty)}{12\nu} \tag{8.59}$$

Total rate of heat transfer in the channel can be written as

$$Q = \dot{m} c_p (T_w - T_\infty) \tag{8.60}$$

Equation (8.60) is written in consideration of the channel to be very long.

Then, we can write for the average Nusselt number

$$\overline{\mathrm{Nu}_H} = \frac{Q.H}{2H(T_w - T_\infty)k} \tag{8.61}$$

With the help of Eqs. (8.59) and (8.60), Eq. (8.61) becomes

$$\overline{\mathrm{Nu}_H} = \frac{\mathrm{Ra}}{24} \tag{8.62}$$

It has to be noted that the Rayleigh number Ra in Eq. (8.62) is based on plate-to-plate spacing B as

$$\mathrm{Ra} = \frac{g\beta B^3 (T_w - T_\infty)}{\nu^2}$$

The above analysis is valid provided the flow is fully developed which depends upon the aspect ratio of the channel, i.e. H/B.

8.5 FREE CONVECTION IN ENCLOSED SPACES

Horizontal fluid layers

The importance of free convection heat transfer in enclosed spaces stems from practical applications such as free convection in wall cavities, between window glazing, flat plate solar collectors, between concentric cylinders and spheres. Let us consider free convection across horizontal fluid layers contained in an enclosure made by two horizontal plates spaced one above another. The lower plate is maintained at a uniform temperature T_h which is higher than the uniform temperature T_c of the upper plate, i.e. $T_h > T_c$. Under the situation, heat flows upwards from lower-to-upper plate and a temperature profile is established which decreases linearly from a temperature T_h at the lower plate to a temperature T_c at the upper plate. Since the cold and denser layers lie above the hot and lighter layers, the system is not in a stable equilibrium state. However, for a very small temperature difference between the plates (i.e. for a very small value of $(T_h - T_c)$), the viscous forces overcome the buoyancy forces and no motion is set up in the fluid. Under the situation, heat flows purely by conduction and the rate of heat transfer is given by Fourier law as

$$q = k\frac{T_h - T_c}{B} = h(T_h - T_c)$$

The Nusselt number is given by $\quad \mathrm{Nu} = \dfrac{hB}{k} = 1$

where B is the gap between the two plates, i.e. the thickness of the fluid layer. This regime of conduction heat transfer without any motion in the fluid takes place up to a temperature difference $(T_1 - T_2)$ that corresponds to a Rayleigh number (based on B) of 1700. When the Rayleigh number exceeds this value of 1700, the fluid become unstable and a motion sets up in the enclosure. It has been observed that up to a value of Ra = 50,000, the recirculatory flow pattern is in the form of hexagonal cells as shown in Figure 8.10. These cells are called Benard cells after Benard who first observed these cells. The free convection heat transfer in this regime is termed Benard convection. Beyond a value of Rayleigh number, Ra = 50,000, the cellular convection pattern is destroyed and the turbulent free convection sets in. Therefore, we find that there are three flow regimes, depending upon the value of Ra, across horizontal fluid layers in an enclosure as follows:

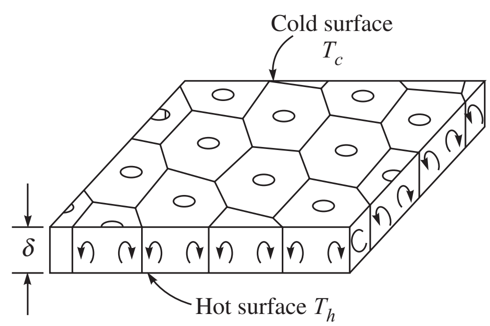


Figure 8.10 Benard convection in a horizontal fluid layer.

- Ra ≤ 1700; conduction heat transfer in a quiescent medium of fluid layers
- 1700 < Ra ≤ 50,000; cellular convection with recirculatory flow patterns in the form of orderly hexagonal cells
- Ra > 50,000; turbulent free convection

Graff and Heldt [13] proposed the following correlations across horizontal fluid layers in enclosed spaces with isothermal plates.

$$\begin{aligned}
\mathrm{Nu}_B &= 1, & \mathrm{Gr}_B &< 2000\\
\mathrm{Nu}_B &= 0.0507\ \mathrm{Gr}^{0.4}, & 2000 < \mathrm{Gr}_B &< 5\times10^4\\
\mathrm{Nu}_B &= 3.8, & 5\times10^4 < \mathrm{Gr}_B &< 2\times10^5\\
\mathrm{Nu}_B &= 0.426\ \mathrm{Gr}_B^{0.37}, & \mathrm{Gr}_B &> 2\times10^5
\end{aligned}$$

Vertical fluid layer

Let us consider a vertical fluid layer enclosed between two vertical isothermal plates maintained at two different temperatures and with top and bottom adiabatic plates. The system is referred to as side heated cavity. The different flow and heat transfer regimes as observed in this case are stated below.

- For a low temperature difference corresponding to Rayleigh numbers Ra < 10^3, heat transfer occurs by pure conduction without any appreciable motion in the fluid layer.
- For large temperature differences of $T_h - T_c$ corresponding to $10^3 < \text{Ra} < 3 \times 10^4$, a circulatory flow takes place with central layer stationary. The heat transfer through the central portion of the layer is by conduction while near the plates it is by convection. The flow in this regime is referred to as asymptotic flow.
- For Ra > 3×10^4, a boundary layer type flow takes place. The fluid moves upwards as a boundary layer flow along the hot wall and downwards along the cold wall with the central region remaining stationary. The heat transfer in this regime is primarily by convection in boundary layers and by conduction in the central stationary region.
- For Ra > 10^6 the transition from laminar to turbulent flow takes place giving rise to the vertical row of vortices. For Ra > 10^7, the flow becomes fully turbulent.

From experimental investigations, EI Shribiny et al. [14] proposed the following correlation for heat transfer in a vertical layer.

$$\text{Nu} = \text{Max of}[\text{Nu}_1, \text{Nu}_2, \text{Nu}_3]$$

which implies that one should choose the maximum of three Nusselt numbers Nu_1, Nu_2, Nu_3 which are expressed as

$$\text{Nu}_1 = 0.0605\,\text{Ra}^{1/3} \tag{8.63a}$$

$$\text{Nu}_2 = \left[1 + \left\{\frac{0.104\,\text{Ra}^{0.293}}{1 + (6310/\text{Ra})^{1.36}}\right\}^3\right]^{1/3} \tag{8.63b}$$

$$\text{Nu}_3 = 0.242\left(\frac{\text{Ra}}{A}\right)^{0.272} \tag{8.63c}$$

where A is the aspect ratio (= H/B). H and B represent the height and width of the enclosure. The Rayleigh number Ra is defined on the basis of width B of the enclosure as

$$\text{Ra} = \frac{g\beta(T_w - T_\infty)B^3}{\nu\alpha}$$

Equations (8.63a) to (8.63c) are valid in the range of

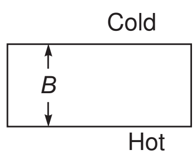
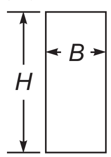
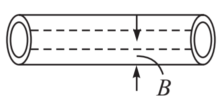
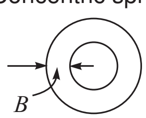
$$10^2 < \text{Ra}_B < 10^7$$
$$5 < A < 110$$

Empirical relations for free convection in enclosed spaces are sometimes put into a simple but general form with moderate accuracy as

$$\text{Nu}_B = C(\text{Ra}_B)^n\left(\frac{H}{B}\right)^m \tag{8.64}$$

where B is the thickness of the fluid layer. Table 8.4 shows the values of C, n and m under some selected circumstances. The property values are evaluated at film temperature.

Table 8.4 The values of C, n and m of Eq. (8.64)

Geometry	*Fluid*	*Aspect ratio H/B*	Range of Pr	Range of Ra_B	C	n	m
Horizontal enclosure (hot surface at top)	Gas or liquid	—	—	—	1	0	0
Horizontal rectangular enclosure (hot surface at the bottom) 	Gas or liquid	—	—	Ra < 1700	1	0	0
	Gas	—	0.5–2	1.7×10^3–7×10^3	0.059	0.4	0
		—	0.5–2	7×10^3–3.2×10^5	0.212	0.25	0
		—	0.5–2	$Ra > 3.2 \times 10^5$	0.061	1/3	0
		—	1–5000	1.7×10^3–6×10^3	0.012	0.6	0
		—	1–5000	6×10^3–3.7×10^4	0.375	0.2	0
		—	1–20	3.7×10^4– 10^8	0.130	0.3	0
		—	1–20	$Ra > 10^8$	0.057	1/3	0
Vertical rectangular enclosure 	Gas or liquid	—	—	Ra < 2000	$\perp$	0	0
	Gas	11–42	0.5–2	2×10^3–2×10^5	0.197	0.25	–1/9
		11–42	0.5–2	2×10^5–10^7	0.073	1/3	–1/9
	Liquid	10–40	1–20,000	10^4–10^7	$0.42Pr^{0.012}$	1/4	–0.3
		1–40	1–20	10^6–10^9	0.046	1/3	0
Concentric cylinders 	Gas or liquid	—	1–50,000	6×10^3–10^6	0.11	0.29	0
		—	1–50,000	10^6–10^8	0.40	0.20	0
Concentric sphere 	Gas or liquid	—	0.7–4000	10^2–10^9	0.228	0.226	0

Mixed convection

In Section 8.2, we recognized the fact that when $(Gr_L/Re_L^2) \approx 1$, both the forced convection and free convection are of equal importance. This regime of heat transfer is known as combined or mixed convection. The effect of buoyancy induced flow in free convection alters the Nusselt number of forced convection. Three special cases arise in this connection.

- When the direction of forced flow is same as that of buoyancy induced flow, the flow situation is termed *aiding* or *assisting* flow. This occurs in case of upward forced flow past a vertical flat plate, a horizontal cylinder or a sphere. The Nusselt number of forced convection is increased due to the presence of free convection.
- When the direction of forced flow is opposite to that of buoyancy driven flow, the flow situation is termed *opposing* flow. Downward forced flow past a vertical flat plate, cylinder and sphere are the examples of opposing flows. The Nusselt number of forced convection, in this case is reduced due to the presence of free convection.
- The flow situation in case of horizontal forced flow over heated cylinder sphere or horizontal plate is termed *transverse* flow. The Nusselt number of forced flow is also increased due to the presence of free convection in this case.

SUMMARY

- The situation where the fluid motion, responsible for convective heat transfer, is caused by the buoyancy force originated from a spatial variation in fluid density, is referred to as 'free convection'.
- The situation of free convection arises when either a hot solid surface is exposed in an ambient of quiescent cold air(or gas) or a cold surface is exposed in a quiescent hot air(or gas). Free convection may also appear in the form of a plume or buoyant jet when a heated solid body is immersed in an expanse of cold and quiescent fluid or a hot gas is discharged as a horizontal jet in quiescent cold air.
- The pertinent non-dimensional parameters describing a free convection heat transfer process are Nusselt number Nu, Grashoff number Gr and Prandtl number Pr. Therefore, for a free convection problem, $Nu = f(Gr, Pr)$.
- In analytical or numerical solution of a free convection heat transfer problem associated with a small temperature difference, the density ρ in the buoyancy force term in momentum equation is considered to be a variable while it is considered constant otherwise. This is known as Boussinesq approximation.
- The analytical solutions have been provided in determining the functional relation of Nu with Gr and Pr for laminar free convection from a heated vertical plate of uniform temperature in a quiescent air. Both the exact solution by the similarity transformation and the approximate solution by the integral method have been discussed. Free convection boundary layer flow past a flat surface becomes turbulent when Rayleigh number $Ra > 10^9$.
- The mathematical analysis of free convection under different situations is quite complicated. Experiments have been performed for several years to correlate the free

convection heat transfer coefficient with the governing parameters. For a variety of circumstances, the empirical relation is expressed in the form

$$\overline{\mathrm{Nu}} = C(\mathrm{Gr}\cdot\mathrm{Pr})^n$$

where the empirical constants C and n vary from situation to situation. The properties in the dimensionless groups are evaluated at the film temperature $T_f = (T_\infty + T_w)/2$.

- Three different regimes of heat transfer have been observed across horizontal fluid layers contained in an enclosure made by two horizontal plates spaced one above another, with the lower plate at a higher temperature than that of the upper one.
  - (i) Ra $\leq$ 1700; conduction heat transfer in a quiescent medium of fluid layers
  - (ii) 1700 < Ra $\leq$ 50,000; cellular convection with recirculatory flow patterns in the form of orderly hexagonal cells
  - (iii) Ra > 50,000; turbulent free convection
- The different free convection flow and heat transfer regimes are observed in a vertical fluid layer enclosed between two vertical isothermal plates maintained at two different temperatures and with top and bottom adiabatic plates as follows:
  - (i) For a low temperature difference corresponding to Rayleigh numbers Ra < 10^3, heat transfer occurs by pure conduction without any appreciable motion in the fluid layer.
  - (ii) For large temperature differences corresponding to $10^3 < \mathrm{Ra}_B < 3 \times 10^4$, a circulatory flow takes place with the central layer stationary. The heat transfer through the central portion of the layer is by conduction while near the plates it is by convection.
  - (iii) For Ra < 3×10^4, a boundary layer type flow takes place. The fluid moves upwards as a boundary layer flow along the hot wall and downwards along the cold wall with the central region remaining stationary. The heat transfer in this regime is primarily by convection in boundary layers and by conduction in the central stationary region
  - (iv) For Ra > 10^6 the transition from laminar to turbulent flow takes place giving rise to the vertical row of vortices. For Ra > 10^7, the flow becomes fully turbulent.

REVIEW QUESTIONS

1. What is the driving potential for free convection flow?
2. Consider two fluids, one with a large coefficient of volume expansion and the other with a small one. In which fluid will a hot surface produce a stronger free convection flow?
3. Define Grashof number. What is its physical significance?
4. How is modified Grashof number defined for a constant heat flux condition on a vertical plate?
5. Explain the criteria in terms of non-dimensional parameters to define the regime of (i) dominant free convection, (ii) dominant forced convection, and (iii) combined forced and free convection.

6. Define Rayleigh number. What is the approximate value of the Rayleigh number at which the transition from laminar to turbulent takes place in a free convection boundary layer past a vertical flat plate?
7. Consider laminar free convection from a vertical hot plate. Will the heat flux be higher at the top or will it be higher at the bottom of the plate? Discuss the situation when the plate is cold and the ambient air is hot.
8. Will a hot horizontal plate whose bottom surface is insulated cool faster or slower when its hot surface is facing down instead of up?
9. Under what condition can the outer surface of a vertical cylinder be considered as that of a vertical plate in calculation of free convection heat transfer?
10. What is Boussinesq approximation in solving the conservation equations in a free convection flow?

PROBLEMS

[For all problems determine the property values of air or other fluids as required from Tables A1 to A5.]

8.1 Water is heated by a 200 mm by 200 mm vertical plate which is maintained at 60°C. By making use of the results of similarity situation, find the heat transfer rate from one side of the plate when the water is at 20°C.

[**Ans.** 30.40 W]

8.2 Determine the heat loss from a vertical wall exposed to nitrogen at one atmospheric pressure and 20°C. The wall is 2 m high and 3 m wide. It is maintained at 80°C. Use Eq. (8.39) with the proper values of C and n depending upon whether the flow is laminar or turbulent.

[**Ans.** 1.04 W]

8.3 A vertical plate 0.5 m high is maintained at 160°C. The plate is exposed to air at 20°C. Determine the heat transfer coefficients (in W/(m^2 °C) at heights 0.1 m, 0.3 m and 0.5 m. Aso calculate the average heat transfer coefficient (in W/(m^2 °C)) for the plate. Use the results of similarity solution.

[**Ans.** 2.55, 1.94, 1.70, 2.27]

8.4 A 300 mm × 300 mm vertical plate fitted with an electrical heater that provides a constant heat flux of 500 W/m^2 is immersed in water at 20°C. Determine the average temperature of the plate. Use Eq. (8.43) with property values of water at 20°C.

[**Ans.** 22.15°C]

8.5 Two plates each at a uniform temperature of 100°C are immersed vertically and parallel to each other into a large tank containing water at 20°C. The height of the plates is 150 mm. Determine the minimum spacing between the plates to prevent the boundary layer interference. Use the result of integral analysis.

[**Ans.** 2.62 mm]

8.6 Consider a rectangular plate 0.2 m by 0.4 m which is maintained at a uniform temperature of 100°C. The plate is placed vertically in still water at 30°C. Find the rate of heat transfer.

(a) when 0.2 m side is vertical, and
(b) when 0.4 m side is vertical.

[**Ans.** (a) 6.43 kW, (b) 6.43 kW]

8.7 Determine the heat transfer rates from a plate 0.4 m by 0.4 m whose one surface is insulated and the other one is maintained at 150°C and exposed to quiescent atmospheric air at 30°C under the following situations:

(a) The plate is vertical.
(b) The plate is horizontal with heated surface facing up.
(c) The plate is horizontal with heated surface facing down.

[Use Eq. (8.39) with different values of C and n for different situations]
For horizontal plates, take the characteristic length defining $\mathrm{Gr_L}$ as $L = A/P$

[**Ans.** (a) 97.73 W, (b) 36.67 W, (c) 18.33 W]

8.8 Heat is transferred from one surface of an electrically heated plate of 0.2 m by 0.2 m in a quiescent air at 20°C. The other side of the plate is thermally insulated. The heat flux over the surface of the plate is uniform and results in an average temperature of 60°C. The plate is inclined, making an angle of 60°C with the vertical. Determine the heat loss from the plate

(a) when the heated surface is facing up, and
(b) when the heated surface facing down.

In determination of property values, use the appropriate reference temperatures as mentioned in the text.

[**Ans.** (a) 8.10 W, (b) 7.39 W]

8.9 Consider the velocity distribution in free convection boundary layer given by Eq. (8.30). At what position in the boundary layer does the maximum velocity occur? Write the expression for the maximum velocity.

8.10 A 0.4 m square vertical plate is heated electrically such that a uniform heat flux of 800 $\mathrm{W/m^2}$ at the surface is maintained. The heated surface is exposed in a quiescent atmospheric air at 25°C. Determine the values of heat transfer coefficients (in $\mathrm{W/(m^2\ °C)}$) at heights of 0.1 m, 0.2 m, 0.3 m and 0.4 m. Also find out the average temperature of the plate. Use Eq. (8.43a) with the property values evaluated at 25°C.

[**Ans.** 7.63, 6.64, 6.12, 5.78, 135.65°C]

8.11 The surface of a human body may be approximated by a vertical cylinder 0.3 m in diameter and 2 m in height. Assuming the surface temperature to be 30°C, determine the heat loss from a human body in a quiescent atmosphere of air at 20°C. Use the appropriate correlation.

[**Ans.** 56.55 W]

8.12 A horizontal rod 6 mm in diameter is immersed in water which is at a temperature of 20°C. If the surface of the rod is kept at a constant temperature of 60°C, determine the rate of heat transfer due to free convection per unit length of the rod. (Use Eq. (8.49).)

[**Ans.** 0.736 kW]

8.13 Condensing steam at 115°C is to be used inside a 60 mm diameter horizontal pipe to provide heating in a room where the ambient air is at 15°C. The total heating requirement is 3 kW. What length of pipe would be required to provide the necessary heating?

[**Ans.** 22 m]

8.14 Consider a 40 W incandescent bulb at 130°C in a quiescent atmospheric air at 30°C. Approximating the bulb as a 50 mm diameter sphere, determine what percentage of the power is lost by free convection. (Use Eq. (8.52).)

[**Ans.** 16.77 per cent]

8.15 Air at 1 atm pressure is contained between two spheres. The values of diameter of the inner and outer spheres are 400 mm and 600 mm respectively. The temperature of the inner sphere is 350°C while that of the outer sphere is 250°C. Determine the rate of heat transfer from the inner sphere to the outer sphere by free convection. Use Eq. (8.39) with the values of C, n and m from Table 8.4. The characteristic length defining Nu and Ra is the radial distance between the two spheres.

[**Ans.** 88.97 W]

8.16 A horizontal rectangular cavity consists of two parallel 0.6 m square plates separated by a distance of 60 mm, with the lateral boundaries insulated. The hot plate is maintained at 350°C and the cold plate at 250°C. Determine the rate of heat transfer between the plates (a) when the hot plate is at the top with cold plate at the bottom, (b) when the hot plate is at the bottom with the cold plate at the top.

[**Ans.** (a) 36.36 W, (b) 68.04 W]

REFERENCES

[1] Bejan, A., and J.L. Lage, The Prandtl number effect on the transition in natural convection along a vertical surface, *ASME J. Heat Transfer,* Vol. 112, 1990, p. 787.

[2] Ostrach, S., An analysis of laminar free convection flow and heat transfer about a flat plate parallel to the direction of the generating body force, *NACA Tech Report 2635*, 1952.

[3] McAdams, W.H., *Heat Transmission,* 3rd ed., McGraw-Hill, New York, 1954.

[4] Churchill, S.W., and H.H.S. Chu., correlating equations for laminar and turbulent free convection from a vertical plate, *Int. J. Heat Mass Transfer,* 18, p. 1323, 1975.

[5] Vilet, G.C, and C.K. Liu, An experimental study of natural convection boundary layers, *ASME J. Heat Transfer,* 91C, p. 517, 1969.

[6] Vilet, G.C., Natural convection local heat transfer coefficient on constant heat flux inclined surfaces, *ASME J. Heat Transfer,* 91C, p. 511, 1969.

[7] Fujii, T., and H. Imura, Natural convection heat transfer from a plate with arbitrary inclination, *Int. J. Heat Mass Transfer,* 15, p. 755, 1972.

[8] Morgan, V.T., The overall convective heat transfer from smooth circular cylinders, in T. F. Irvine and J.P. Hartnet (Eds.), *Advances in Heat Transfer,* Vol. 16, Academic, New York, p. 199, 1975.

[9] Churchill, S.W., and H.S. Chu, Correlating equations for laminar and turbulent free convection from a horizontal cylinder, *Int. J. Heat Mass Transfer,* 18, p. 1049, 1975.

[10] Yuge, T., Experiments on heat transfer from spheres including combined natural and forced convection. *ASME J. Heat Transfer,* 8.2c, p. 214, 1960.

[11] Amato, W.S., and S. Tien, Free convection heat transfer from isothermal spheres in water, *Int. J. Heat Mass Transfer,* 15, p. 327, 1972.

[12] Churchill, S.W., Free convection around immersed bodies p. 2.5.7–24, in G.F. Hewitt (Ed.), *Heat Exchanger Design Handbook*, Washington D.C., Hemisphere Publishing Corp., 1983.

[13] Graff, J.G.A., and Heldt van der, The relation between the heat transfer and convection phenomena in enclosed plain air layers, *Applied Scientific Research, Series A*, Vol. 3, p. 393, 1952.

[14] EI Shribiny, S.M., G.D. Raithly, and K.G.T. Hollands, Heat transfer by natural convection across vertical and inclined air layers, *ASME J. Heat Transfer,* Vol. 104, p. 96, 1982.

9

Heat Transfer in Condensation and Boiling

In previous chapters on forced and free convection, we considered the fluid medium through which heat transfer took place to be homogeneous and of single phase (either liquid or gas). But in many practical situations, the convective heat transfer processes are associated with a change of phase in the fluid. In a condenser of a steam power plant, steam is condensed to liquid water on the outer surfaces of a number of tubes, while cooling water flows inside the tubes. In a condenser of a refrigerator, the refrigerant vapour, while flowing through the condenser tubes, is condensed on the inner surfaces of the tubes. On the other hand, the phenomenon of boiling is observed in the inside surfaces of water tubes, while hot gases flow past the outer surfaces of the tubes in a boiler of a steam power plant. In all such situations, a phase change process is sustained by heat transfer to or from a solid surface. Since these processes involve fluid motion, the mode of heat transfer in condensation and boiling is convection. Because of the combined latent heat and buoyancy driven flow effect, a large amount of heat transfer takes place with a small temperature difference between the solid surface and the fluid. Hence the heat transfer coefficient becomes much larger under these situations. The pertinent parameters influencing the heat transfer process are latent heat (the enthalpy of condensation or vaporization), surface tension at the liquid–vapour interface and buoyancy due to change in densities of the two phases. In this chapter we shall discuss such heat transfer processes in condensation and boiling under different situations.

Learning objectives

The learning of this chapter will enable the students

- to understand the mechanism of heat transfer in condensation and boiling,
- to understand the principles of laminar film condensation on flat and curved surfaces,
- to analyse the problem of laminar film condensation and to recognize the physical parameters that influence the heat transfer rate in condensation,

- to know about the different regimes of boiling and their distinctive characteristics in boiling heat transfer,
- to get acquainted with the widely used empirical equations of heat transfer coefficient in boiling, and
- to solve independently the numerical problems on film condensation and boiling.

9.1 CONDENSATION HEAT TRANSFER

A vapour is condensed when it reaches the saturation temperature at its existing pressure and then heat is removed from it. In engineering applications, the removal of heat from the vapour is made by bringing it in contact with a cold solid surface. The phenomenon is usually referred to as *surface condensation*. During the process of condensation, the condensate (the condensed liquid) formed on the surface flows down the surface under the action of gravity. If the liquid wets the surface, a smooth liquid film is formed which flows down along the surface. This is called *film condensation*. If the liquid does not wet the surface, droplets are formed which fall down the surface in some random fashion. This process is called *drop-wise condensation*. In film condensation, the surface is blanketed by a liquid film which grows in thickness due to condensation of vapour as it moves down the plate. This film does not allow the vapour to come into contact with the bare solid surface and thus provides a thermal resistance to heat flow from the vapour to the solid surface. On the other hand, in case of a drop-wise condensation though the liquid droplets provide the thermal resistance, vapour gets an opportunity to come into contact with the unwetted part of the solid surface. Because of this, the heat transfer rate and hence the rate of condensation is relatively much higher in drop-wise condensation than that in film condensation. However, it is extremely difficult in practice to sustain a drop-wise condensation process. This is because of the fact that most of the surfaces become wet after being exposed to a condensing vapour over a period of time. Attempts have been made to treat the surface with coatings and vapour additives, but they have not met with general success. Therefore, we will focus our attention on film condensation.

9.1.1 Laminar Film Condensation on a Vertical Plate

Let us consider a flat vertical plate in an atmosphere of quiescent steam at a temperature T_g which is the saturation temperature at the ambient pressure. The condensate as a thin film of liquid flows past the plate in the direction of gravity and grows in its thickness along the flow (Figure 9.1). The plate is maintained at a constant temperature of T_w. The film thickness is represented by δ, and we choose the coordinate system with the positive direction of x measured downwards as shown in Figure 9.1.

First we have to find out the velocity distribution within the liquid film. Because of very slow motion, we neglect the influence of inertia force and therefore the fluid

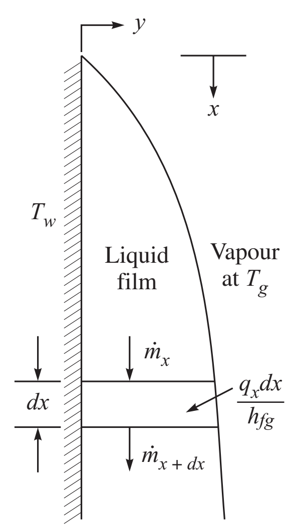


Figure 9.1 Laminar film condensation on a vertical plate.

is acted upon by viscous force and the body force that includes the weight and buoyancy force of a fluid element.

Since the thickness of the film is very small, the boundary layer type approximation (i.e. $\frac{\partial^2}{\partial y^2}$ (anything) >> $\frac{\partial^2}{\partial x^2}$ (anything) and $\frac{\partial p}{\partial y} \approx 0$) holds good, and we can write, in consideration of negligible influence of inertia terms, the equation of motion as

$$0 = -\frac{\partial p}{\partial x} + \rho g + \mu \frac{\partial^2 u}{\partial y^2}$$

The pressure gradient $\frac{\partial p}{\partial x}$ is evaluated from the vapour region outside the liquid film as

$$\frac{\partial p}{\partial x} = \rho_V g$$

where ρ_V is the density of the vapour.

Then, we have

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{-(\rho - \rho_V)}{\mu} g$$

or

$$\frac{\partial u}{\partial y} = -\frac{(\rho - \rho_V)}{\mu} g y + c_1$$

or

$$u = -\frac{(\rho - \rho_V)}{2\mu} g y^2 + c_1 y + c_2$$

The boundary conditions are:

$$\text{At} \quad y = 0, \qquad u = 0 \text{ (no slip)}$$

$$\text{At} \quad y = 0, \quad \frac{\partial u}{\partial y} = 0 \text{ (zero shear at the free surface)}$$

Hence, we get

$$u = \frac{(\rho - \rho_V) g}{\mu} \left(\delta y - \frac{1}{2} y^2 \right) \tag{9.1}$$

The mass flow of condensate at any section x is given by

$$\dot{m}_x = \int_0^\delta \rho u dy$$

$$= \int_0^\delta \rho \left[\frac{(\rho - \rho_V) g}{\mu} \left(\delta y - \frac{1}{2} y^2 \right) \right] dy$$

$$= \frac{\rho(\rho - \rho_V) g \delta^3}{3\mu} \tag{9.2}$$

Since the film thickness is small and the motion of the liquid is slow, it is expected that the mode of heat transfer through the film will be mainly the conduction. Therefore, we can assume a linear temperature profile with y within the liquid film and can write for the local heat flux q_x at a position x as

$$q_x = -k\frac{\partial T}{\partial y} = -k\frac{(T_g - T_w)}{\delta} \tag{9.3}$$

Since $T_g > T_w$, the value of q_x, as given by Eq. (9.3), is always negative which shows that the heat is flowing in a direction opposite to the positive direction of y.

From a mass balance of an element of fluid of thickness dx (Figure 9.1), we can write

$$\dot{m}_{x+dx} - \dot{m}_x = \frac{|q_x|\cdot dx}{h_{fg}}$$

or

$$\dot{m}_x + \frac{d}{dx}(\dot{m}_x)d_x - \dot{m}_x = \frac{|q_x|\cdot dx}{h_{fg}}$$

or

$$\frac{d\dot{m}_x}{dx} = \frac{|q_x|}{h_{fg}} \tag{9.4}$$

where h_{fg} is the enthalpy of condensation at a temperature T_g.

With the help of Eqs. (9.2) and (9.3), Eq. (9.4) becomes

$$\delta^3\frac{d\delta}{dx} = \frac{\mu k(T_g - T_w)}{gh_{fg}\rho(\rho - \rho_v)}$$

Integrating the above equation with the boundary condition at $x = 0$, $\delta = 0$, we have

$$\delta = \left[\frac{4\mu kx(T_g - T_w)}{gh_{fg}\rho(\rho - \rho_v)}\right]^{1/4} \tag{9.5}$$

The local heat transfer coefficient h_x can be written as

$$h_x = \frac{|q_x|}{T_g - T_w} = \frac{k}{\delta}$$

With the help of Eq. (9.5), we can write

$$h_x = \left[\frac{\rho(\rho - \rho_v)gh_{fg}k^3}{4\mu x(T_g - T_w)}\right]^{1/4} \tag{9.6}$$

Local Nusselt number, $$Nu_x = \frac{h_x x}{k}$$

$$= \left[\frac{\rho(\rho - \rho_v)gh_{fg}x^3}{4\mu k(T_g - T_w)}\right]^{1/4} \tag{9.7}$$

The average heat transfer coefficient over a length L of the plate can be written as

$$\bar{h}_L = \frac{1}{L}\int_0^L h_x\, d_x$$

$$= \frac{4}{3}h_L \quad \text{(since } h_x \sim x^{-1/4}\text{)}$$

$$= 0.943\left[\frac{\rho(\rho-\rho_v)gh_{fg}k^3}{\mu L(T_g-T_w)}\right]^{1/4} \tag{9.8}$$

Hence,
$$\overline{\mathrm{Nu}_L} = \frac{4}{3}\mathrm{Nu}_L$$

$$= 0.943\left[\frac{\rho(\rho-\rho_v)gh_{fg}L^3}{\mu k(T_g-T_w)}\right]^{1/4} \tag{9.9}$$

In all the above equations, the liquid properties should be evaluated at film temperature $T_f = (T_g + T_w)/2$. The classical treatment of condensation over a vertical flat plate as discussed above was first given by Nusselt [1]. Some significant modifications of the above analysis have been made which account for a nonlinear temperature profile in the liquid film due to convective effects, and an additional energy requirement to cool the film below the saturation temperature. These are incorporated by replacing h_{fg} by h'_{fg} which is defined as

$$h'_{fg} = h_{fg} + 0.68\,c(T_g - T_w) \tag{9.10}$$

where c is the specific heat of the liquid.

Condensation on inclined surfaces

The analysis of film condensation for a vertical surface can be extended for an inclined surface, making an angle φ with the horizontal, by simply replacing g representing the gravitational force in Eqs. (9.6) and (9.7) by $g \sin \varphi$, the component of the gravitational force along the heat transfer surface. Therefore, the local and average heat transfer coefficients become

$$h_x = \left[\frac{\rho(\rho-\rho_v)gh_{fg}k^3}{\mu x(T_g-T_w)}\sin\varphi\right] \tag{9.11}$$

$$\bar{h}_L = 0.943\left[\frac{\rho(\rho-\rho_v)gh_{fg}k^3}{\mu L(T_g-T_w)}\sin\varphi\right]^{1/4} \tag{9.12}$$

9.1.2 Condensation on Outer Surface of a Horizontal Tube

The analysis of film condensation over a cylindrical surface of a horizontal tube was provided by Nusselt [1]. The film of liquid as formed by the condensation of vapour at saturation state flows along the surface in the direction of gravity. The film grows on its thickness along the flow due to condensation of vapour as shown in Figure 9.2. In consideration of very slow motion of liquid film, we can neglect the convective effect and consider the mode of heat transfer through the liquid film by conduction. Therefore,

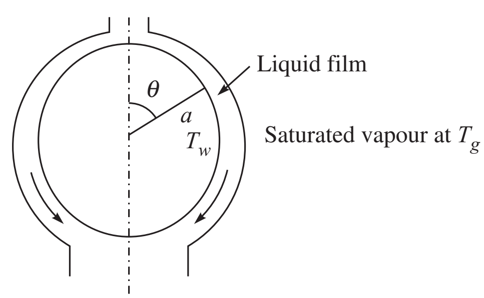


Figure 9.2 Laminar film condensation on the outer surface of a horizontal tube.

$$q_\theta = -k\left(\frac{\partial T}{\partial r}\right)_{r=a} = -\frac{k(T_g-T_w)}{\delta}$$

and hence
$$h_\theta = \frac{|q_\theta|}{T_g - T_w} = \frac{k}{\delta} \tag{9.13}$$

where δ is the liquid film thickness at a location given by θ, and T_w is the uniform surface temperature of the tube of radius a.

It is clear from Eq. (9.13) that the heat transfer coefficient is inversely proportional to the liquid film thickness. The next job is to calculate the liquid film thickness in a similar way as done in case of flat plate by finding out first the velocity distribution in the liquid film from momentum equation and then by making use of conservation of mass of a liquid element in the film. Nusselt [1] made this calculation and provided the relations as

$$h_\theta = \left[\frac{\rho(\rho - \rho_v) g h_{fg} k^3 \sin^{4/3}\theta}{4\mu(T_g - T_w) a \int_0^\theta \sin^{1/3}\theta d\theta}\right]^{1/4} \tag{9.14}$$

and
$$\bar{h} = 0.725\left[\frac{\rho(\rho - \rho_v) g k^3 h_{fg}}{\mu(T_g - T_w) D}\right]^{1/4} \tag{9.15}$$

where $D(=2a)$ is the diameter of the tube.

A comparison of Eq. (9.8) with (9.15) for condensation on a vertical plate of length L and that on a horizontal tube of diameter D, yields

$$\frac{\bar{h}_{\text{vertical plate}}}{\bar{h}_{\text{horizontal tube}}} = 1.30\left(\frac{D}{L}\right)^{1/4} \tag{9.16}$$

We should mention in this context that the expression given by Eq. (9.8) is well applicable for condensation on the outside surface of a vertical tube too, provided the tube radius is large compared to the thickness of the condensate film. Therefore, we can conclude from the expression given by Eq. (9.16) that for a given value of $T_g - T_w$, the average heat transfer coefficients for a vertical tube of length L and a horizontal tube of diameter D become equal when $L = 2.78D$. For any value of L greater than $2.78D$, the heat transfer coefficient for a horizontal tube will be greater than that for a vertical tube. For this reason, horizontal tube arrangements are generally preferred than vertical tube arrangements in condenser design.

It is clear from the above discussion and from Eq. (9.13) that an augmentation in the rate of heat transfer in condensation is achieved if the thickness of the condensate film is reduced. The film thickness is reduced if the condensate is drained out faster from the solid surface at which the condensation takes place. With this objective, the use of non-circular tube surface profile has been considered by various researches to enhance the flow of liquid film by increasing the effective gravity (the component of gravity in the direction of flow along the solid surface). Arijit et al. [2] in their recent work have investigated theoretically the phenomenon of film condensation on the outer surface of a tube whose profile is drawn in the direction of gravity with increasing radius of curvature. The combined effects of gravity force component and surface tension driven favourable pressure gradient in the direction of flow help in draining out the condensate from the surface much faster. This results in a reduction of liquid film thickness on tube surface, and hence in an augmentation in the rate of heat transfer in condensation.

Arijit et al. [2] considered the tube surface profile given by the polar curves of equiangular spiral type $R_p = ae^{m\theta}$ (where R_p is the radius of the tube) generated symmetrically about its vertical axis, and predicted that there was an enhancement in average heat transfer coefficient from 10 per cent to 20 per cent depending upon the values of m which were varied from 0.58 to 1.73 in their studies.

9.1.3 Reynolds Number for Condensate Flow

In case of condensation on a vertical plate or cylinder, if the plate or cylinder is sufficiently large or if there is a sufficient amount of condensate flow, turbulence may appear in the condensate film. The turbulence results in higher heat transfer rates. The criterion for determining whether the flow is laminar or turbulent is the Reynolds number which is defined under the situation as

$$\text{Re} = \frac{\rho v_{\text{av}} D_h}{\mu} \tag{9.17}$$

where

D_h is the hydraulic diameter
v_{av} is the average flow velocity of condensate.

The density ρ and the viscosity μ of the condensate should be evaluated at the film temperature, $T_f = (T_g + T_w)/2$.

We can write for D_h as

$$D_h = \frac{4A}{P} \tag{9.18}$$

where

A is the cross-sectional area for condensate flow
P is the wetted perimeter.

Again,

$$v_{\text{av}} = \frac{\dot{m}}{\rho A} \tag{9.19}$$

where $\dot{m}$ is the mass flow rate of condensate at a particular section. Substituting Eqs. (9.18) and (9.19) into Eq. (9.17), we can write

$$\text{Re} = \frac{4\dot{m}}{\mu P} \tag{9.20}$$

The wetted perimeter P for different geometries is given as

$P = \pi D$ for vertical tube of outside diameter D
$P = 2L$ for horizontal tube of length L
$P = B$ for vertical or inclined plate of width B.

It has to be mentioned in this context that to judge whether the flow is laminar or turbulent, we have to compare the critical Reynolds number with the Reynolds number at the trailing edge of the solid surface. In that case the quantity $\dot{m}$ in Eq. (9.20) becomes the mass flow rate of condensate at the end of the solid surface which, in turn, is the total rate of condensation $\dot{m}_c$ over the surface. Let us consider a vertical plate.

Then

$$\dot{m}_c = \frac{\bar{h}LB(T_g - T_w)}{h_{fg}}$$

If we substitute for $\dot{m}$ in Eq. (9.20) by $\dot{m}_c$ as given above, we have for a vertical plate

$$\text{Re}_L = \frac{4\bar{h}\,L(T_g - T_w)}{h_{fg}\,\mu} \tag{9.21}$$

It has been observed in practice that transition from laminar to turbulent condensation takes place at a Reynolds number of 1800. It can be mentioned in the present context that the theoretical results of laminar film condensation as discussed so far match with the experimental data if the film remains smooth. In practice, it has been found that ripples develop in the film beyond a Reynolds number of 30. When this occurs, the experimental values of average heat transfer coefficient can be 20 per cent higher than those predicted by Eqs. (9.8) and (9.15). Because of this, McAdams [3] suggested that the 20 per cent increase be adopted for design purposes. In consideration of this fact, Eqs. (9.8) and (9.15) can be written for the design purpose as

$$\bar{h}_{\text{vertical plate}} = 1.13\left[\frac{\rho(\rho - \rho_v)gh_{fg}k^3}{\mu L(T_g - T_w)}\right]^{1/4} \tag{9.22a}$$

$$\bar{h}_{\text{horizontal cylinder}} = 0.87\left[\frac{\rho(\rho - \rho_v)gk^3h_{fg}}{\mu(T_g - T_w)D}\right]^{1/4} \tag{9.22b}$$

9.1.4 Film Condensation Inside Horizontal Tubes

Film condensation on the inner surface of a tube is of much practical importance because of its applications to condensers in refrigeration and air conditioning systems. The analysis of such situations is complicated. This is because, the flow of vapour inside the tube influences the condensation, and again, the accumulation of condensate on the wall affects the flow of vapour. We shall present here only a few empirical relations.

Chato [4] recommended the following correlation from his experiments of condensation of refrigerant vapour at low velocities inside a tube

$$\bar{h} = 0.555\left[\frac{\rho(\rho - \rho_v)gh'_{fg}k^3}{\mu d(T_g - T_w)}\right]^{1/4} \tag{9.23}$$

where h'_{fg} is the modified enthalpy of vaporization and is given by

$$h'_{fg} = h_{fg} + \frac{3}{8}c_p(T_g - T_w) \tag{9.24}$$

The liquid properties ρ and μ in Eq. (9.23) are determined at film temperature while h_{fg} and ρ_v are evaluated at the saturation temperature T_g. Equation (9.23) is valid in the range of $\text{Re}_v <$ 35,000. The vapour Reynolds number Re_v is defined as

$$\text{Re}_v = \frac{\rho_v V_v D}{\mu_v}$$

where V_v is the average velocity of vapour at inlet, and ρ_v and μ_v are the density and viscosity of the vapour at inlet, D is the inside diameter of the tube.

For higher flow rates, Akers, Deans and Crosser [5] recommended an approximate correlation of Nusselt number as

$$\overline{\text{Nu}} = \frac{\bar{h}D}{k} = 0.026\ \text{Pr}^{1/3}\left[\text{Re} + \text{Re}_v\left(\frac{\rho}{\rho_v}\right)^{0.5}\right]^{0.8} \tag{9.25}$$

where $\text{Re} = \dfrac{4\dot{m}}{\pi D\mu}$, $\text{Re}_v = \dfrac{4\dot{m}_v}{\pi D\mu_v}$

Here $\dot{m}$ is the mass flow rate. The quantities with suffix v are for the vapour, while the quantities without any suffix refer to those of liquids. All properties for the liquid are evaluated at film temperature. Equation (9.25) correlates the experimental data within about 50 per cent for the range

$$\text{Re}_v >> 20{,}000 \quad \text{and} \quad \text{Re} < 5000$$

9.1.5 Condensation in Presence of Non-condensable Gas

In all the earlier discussions, the gas phase surrounding the liquid phase (the condensate) was a pure vapour. In many practical situations, non-condensable gases such as air are present in the condensing vapour. The rate of heat transfer, and hence the condensation rate is greatly reduced. This is because of the fact that the vapour has to diffuse through a mixture of vapour and air (non-condensable species) to reach the liquid–vapour interface for condensation. Due to this diffusion process, there is a drop in the partial pressure as well as in the temperature of the vapour from their values in the bulk of the mixture to the values at the liquid–vapour interface. Therefore, we find that the presence of a non-condensable gas acts as a barrier through which the vapour from its state of bulk mixture has to diffuse to reach the liquid–vapour interface for condensation. This picture is shown in case of condensation of steam in presence of air on a vertical plate in Figure 9.3. Since the interface temperature of the vapour Ts_2, is less than that in the bulk mixture Ts_1, it is apparent that the temperature gradient for heat flow through the condensate film is reduced. Even the presence of a small amount of non-condensable gas greatly reduces the rate of heat transfer in condensation. It has been observed by many researches that the vapour flow patterns in the vicinity of the condensing surface influences the reduction in heat transfer

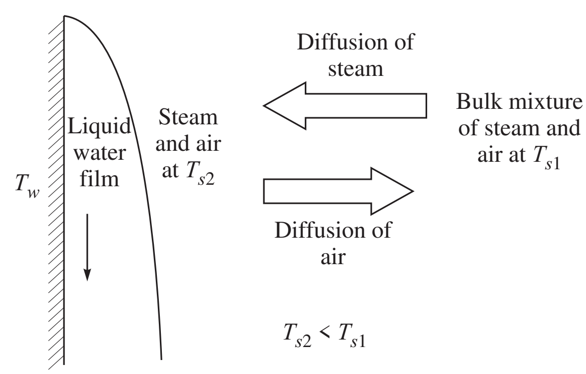


Figure 9.3 Condensation of steam in presence of air on a flat vertical plate.

coefficient. However the analysis is complicated. Empirical equations have been developed by many workers but the description of those equations and the analyses have been kept beyond the scope of this book. The general practice in the design of a condenser is to vent out non-condensable gases as much as possible.

EXAMPLE 9.1 A vertical cooling fin, approximating a flat plate 0.4 m high, is exposed to steam at atmospheric pressure. The fin is maintained at 90°C by cooling water. Determine the rate of heat transfer per unit width of the fin and the total rate of condensation. (Assume the condensate film to be laminar.)

Solution: The properties of condensate (liquid water) are evaluated at the mean film temperature $t_f = (100 + 90)/2 = 95°\text{C}$ as

$$\rho = 962 \text{ kg/m}^3 \qquad k = 0.677 \text{ W/(m K)} \qquad \mu = 3.0 \times 10^{-4} \text{ kg/(m s)}$$

The values of ρ_v and h_{fg} at 100°C are found from the steam table as

$$\rho_v = 0.598 \text{ kg/m}^3 \qquad h_{fg} = 2.27 \times 10^6 \text{ J/kg}$$

From Eq. (9.8), we get

$$\bar{h}_L = 0.943\left[\frac{962 \times (962 - 0.598) \times 9.81 \times 2.27 \times 10^6 \times (0.677)^3}{3.0 \times 10^{-4} \times 0.4 \times (100 - 90)}\right]^{1/4}$$
$$= 8.05 \times 10^3 \text{ W/(m}^2\text{ K)}$$

Rate of heat transfer per unit width,

$$Q = \bar{h}_L L(T_g - T_w)$$
$$= 8.05 \times 10^3 \times 0.4 \times (100 - 90)$$
$$= 32.20 \times 10^3 \text{ W/m}$$

The total rate of condensation, $\dot{m}_c = \dfrac{Q}{h_{fg}} = \dfrac{32.2 \times 10^3}{2.27 \times 10^6} = 0.014 \text{ kg/(s m)}$

We have to check whether the flow is laminar or not. By making use of Eq. (9.20),

$$\text{Re}_L = \frac{4 \times 0.014}{3.0 \times 10^{-4}} = 187$$

Therefore the flow is laminar, and hence the use of Eq. (9.8) is justified.

EXAMPLE 9.2 Steam at 100°C is being condensed on the outer surface of a horizontal tube of 3 m length and 50 mm outer diameter, while the tube surface is maintained at 90°C. Determine the average heat transfer coefficient and the total rate of condensation over the tube surface.

Solution: Since the saturation temperature of condensing steam and the temperature of solid surface are respectively the same as those of Example 9.1, all the required property values of condensate and steam remain same as those of Example 9.1.

We use Eq. (9.15),

$$\bar{h} = 0.725\left[\frac{962 \times (962 - 0.598) \times 9.81 \times (0.677)^3 \times 2.27 \times 10^6}{3.0 \times 10^{-4} \times (100 - 90) \times 0.05}\right]^{1/4}$$
$$= 10.4 \times 10^3 \text{ W/(m}^2\text{ K)}$$

The total rate of condensation is found out as

$$\dot{m}_c = \frac{\bar{h} \times (\pi DL) \times (T_g - T_w)}{h_{fg}}$$

$$= \frac{10.4 \times 10^3 \times (\pi \times 0.05 \times 3) \times (100 - 90)}{2.27 \times 10^6}$$

$$= 0.021 \text{ kg/s}$$

Check for Reynolds number:

$$\text{Re} = \frac{4\dot{m}_c}{\mu P}$$

For a horizontal tube of length L, $P = 2L$

Then,
$$\text{Re} = \frac{2\dot{m}_c}{\mu L}$$

$$= \frac{2 \times 0.021}{3.0 \times 10^{-4} \times 3}$$

$$= 47$$

The flow is laminar.

EXAMPLE 9.3 A vertical plate 1.5 m high is maintained at 60°C in the presence of saturated steam at atmospheric pressure. Determine (a) the heat transfer rate and (b) the average flow velocity of condensate from the trailing edge of the plate. Consider the width of the plate to be of 0.3 m.

Solution: (a) The film temperature is

$$t_f = \frac{100 + 60}{2} = 80\,°\text{C}$$

The relevant properties at 80°C are

$\rho = 972$ kg/m^3 $\quad c_p = 4.2$ J/(kg K) $\quad \mu = 3.54 \times 10^{-4}$ kg/(m s)

$k = 0.670$ W/(m K) $\quad \rho_v$(at 100°C) $= 0.598$ kg/m^3 $\quad h_{fg}$ (at 100°C) $= 2.27 \times 10^6$ J/kg

We use Eq. (9.8) and get

$$\bar{h} = 0.943 \left[\frac{972 \times (972 - 0.598) \times 9.81 \times 2.27 \times 10^6 \times (0.670)^3}{3.54 \times 10^{-4} \times 1.5 \times (100 - 60)} \right]^{1/4}$$

$$= 3.92 \times 10^3 \text{ W(m}^2\text{ K)}$$

Therefore,
$$Q = \bar{h} A (T_w - T_g)$$

$$= 3.92 \times 10^3 \times 1.5 \times 0.3(100 - 60)$$

$$= 70.56 \times 10^3 \text{ W}$$

$$= 70.56 \text{ kW}$$

(b) The film thickness at the trailing edge is found out by making use of Eq. (9.5) as

$$\delta_{\text{at}\,x=1.5\,\text{m}} = \left[\frac{4\times 3.54\times 10^{-4}\times 0.668\times 1.5\times (100-60)}{9.81\times 2.27\times 10^{6}\times 972\,(972-0.598)}\right]^{1/4}$$

$$= 0.228 \times 10^{-3} \text{ m} = 0.228 \text{ mm}$$

The total rate of condensation,

$$\dot{m}_c = \frac{70.56\times 10^{3}}{2.27\times 10^{6}}$$

$$= 31.08 \times 10^{-3} \text{ kg/s}$$

Hence the average flow velocity at the trailing edge becomes

$$v = \frac{31.08\times 10^{-3}}{972\times 0.228\times 10^{-3}\times 0.3}$$

$$= 0.467 \text{ m/s}$$

EXAMPLE 9.4 Saturated Freon–12 at 35°C is condensed inside a horizontal tube of 15 mm diameter at a low vapour velocity. The tube wall is maintained at 25°C. Determine the mass of Freon–12 condensed in one hour per metre of tube length. For Freon–12 at 35°C, h_{fg} = 131.33 kJ/kg and ρ_v = 42.68 kg/m^3.

Solution: We have to use Eq. (9.23).

The film temperature, $$t_f = \frac{35+25}{2} = 30°\text{C}$$

The relevant liquid properties at 30°C are

$$\rho = 1.29 \times 10^3 \text{ kg/m}^3 \qquad k = 0.071 \text{ W/(m K)}$$

$$\mu = 2.50 \times 10^{-4} \text{ kg/(m s)} \qquad c_p = 983 \text{ J/(kg °C)}$$

For the use of Eq. (9.23), we find out the modified enthalpy of vaporization h'_{fg} as

$$h'_{fg} = h_{fg} + \frac{3}{8}c_p(T_g - T_w) = 131.33 \times 10^3 + \frac{3}{8}(983)\,(35 - 25)$$

$$= 135.02 \times 10^3 \text{ J/kg}$$

Therefore,

$$\bar{h} = 0.555\left[\frac{1290\times(1290-42.68)\times 9.81\times 135.02\times 10^{3}\times(0.071)^{3}}{2.50\times 10^{-4}\times 0.015\times(35-25)}\right]^{1/4}$$

$$= 1178 \text{ W/(m}^2\text{ K)}$$

The total rate of condensation

$$\dot{m}_c = \frac{\bar{h}\pi DL(T_g - T_w)}{h_{fg}} = \frac{1178\times\pi\times 0.015\times 1\times(35-25)}{131.33\times 10^{3}}$$

$$= 4.23 \times 10^{-3} \text{ kg/s}$$

$$= 4.23 \times 10^{-3} \times 3600 \text{ kg/h}$$

$$= 15.23 \text{ kg/h}$$

9.1.6 Boiling Heat Transfer

Boiling occurs when a liquid comes into contact with a solid surface which is maintained at a temperature higher than the saturation temperature of the liquid at the existing pressure. Boiling may occur under various conditions as stated below. When the heated surface is submerged below the free surface of a quiescent fluid, the boiling phenomenon is referred to as *pool boiling*. In this situation, heat is transferred from the solid surface to the liquid by free convection. The formation of vapour bubbles take place at the surface and they grow while moving up and subsequently collapse near the free surface. The pool boiling may be of two types—subcooled boiling and saturated boiling.

Subcooled boiling

When the bulk temperature of the liquid is below the saturation temperature of the liquid, the process is called subcooled boiling.

Saturated boiling

If the bulk temperature of the liquid is maintained at the saturation temperature, the process is called saturated boiling. In both the cases, the difference between the solid surface temperature T_w and the saturation temperature T_s of the liquid is termed excess temperature. Therefore, excess temperature $\Delta T_e = T_w - T_s$.

The characteristic features in the process of boiling define different regimes of boiling as shown in Figure 9.4. The typical figure (Figure 9.4) describing the variation of surface heat flux with excess temperature is known as boiling curve. The curve has been obtained from experiments by many researches in the field. Let us first explain physically the boiling curve with reference to the different regimes of boiling.

Free convection boiling

When the excess temperature is $\Delta T_e \leq 5°C$, the formation of vapour is insufficient to cause a real boiling process. Free convection currents are responsible for fluid motion near the surface where the liquid is in a slight superheated metastable state and subsequently evaporates when it rises to the surface. In this regime, we can determine the heat flux from free convection correlations as given in Chapter 8. If the liquid motion is laminar, the heat transfer coefficient h is proportional to $\Delta T_e^{1/4}$ and accordingly heat flux q varies as $\Delta T_e^{5/4}$. In case of turbulent flow, $h \sim \Delta T_e^{1/3}$, and hence, $q \sim \Delta T_e^{4/3}$. This regime of boiling is referred to as free convection boiling as shown in Figure 9.4.

Nucleate boiling

When the excess temperature ΔT_e exceeds 5°C, the process of boiling starts with the generation of considerable vapour and is referred to as nucleate boiling. The onset of nucleate boiling is shown by the point A on the boiling curve (Figure 9.4). Initially in this regime, the formation of isolated bubbles takes place at the nucleation site and then the bubbles separate from the surface and rise up near the free surface where they are dissipated as shown in Figure 9.5. The growth or collapse of a vapour bubble depends on the temperature and pressure of the bulk liquid. In a subcooled or saturated liquid, the bubbles collapse due to condensation of vapour inside the

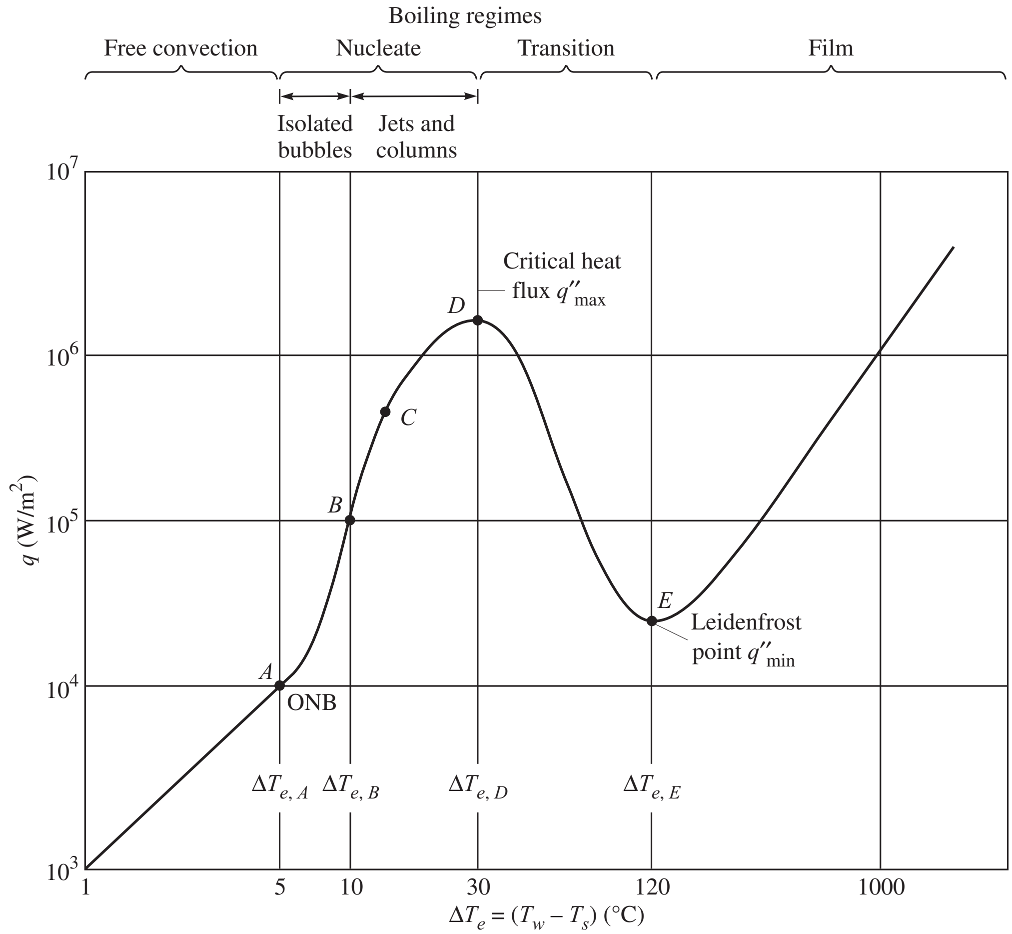


Figure 9.4 Typical boiling curve for saturated water at atmospheric pressure.

bubbles. This can be explained in a sense that the vapour pressure inside a bubble is higher than the liquid pressure outside because of surface tension. Hence the temperature inside the bubble is higher than that of the liquid outside and heat must be conducted out of the bubble resulting into the condensation of vapour inside it. In order for the bubbles to grow and escape from the free surface, they must receive heat from the liquid. This requires that the liquid be in a superheated condition so that the liquid temperature is higher than that of the vapour inside. This is a metastable thermodynamic state which is observed experimentally in some regions of nucleate boiling. This initial part of nucleate boiling associated with the formation of isolated bubbles is shown as the portion A–B on the boiling curve (Figure 9.4). The formation of vapour bubbles, their movement and final dissipation induce considerable fluid mixing near the surface and increase substantially the heat transfer coefficient h and the heat flux q.

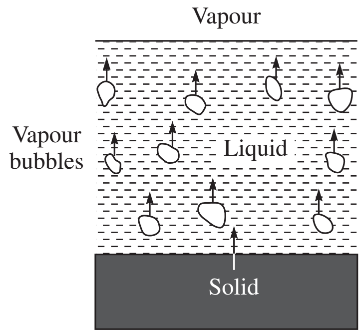


Figure 9.5 Nucleate boiling with the formation of vapour bubbles.

As the value of excess temperature is increased beyond $\Delta T_{e,B}$, more nucleation sites become active and the bubble formation is increased. This cause vapours to escape as jets or columns which subsequently merge into slugs of vapours. This regime of nucleate boiling is shown as the part *B–D* on the boiling curve (Figure 9.4). There is an inflection point *C* up to which the heat transfer coefficient *h* increases with ΔT_e and hence there occurs a steep rise in heat flux *q* with ΔT_e. At the point *C*, the heat transfer coefficient *h* attains its maximum value after which *h* decreases with ΔT_e and therefore the heat flux *q* continues to increase, but at a slower rate with ΔT_e. At the point *D*, the heat flux *q* attains its maximum value.

Transition boiling

This regime of boiling is shown as the portion *D–E* on the boiling curve (Figure 9.4) where heat flux *q* decreases with an increase in excess temperature ΔT_e. We have observed that the formation of vapour bubbles increases rapidly with ΔT_e in the latter part of the nucleate boiling regime. Beyond a value of ΔT_e corresponding to the point *D*, the bubble formation is so rapid that a vapour film or blanket begins to form on the surface. Since the thermal conductivity of vapour is much less than that of the liquid, the heat transfer coefficient is drastically reduced so that heat flux also reduces with an increase in ΔT_e. The situation oscillates between a continuous film and a number of bubbles adhering to the surface. Hence the regime is known as transition boiling.

Film boiling

At point *E* the heat flux reaches its minimum value where a stable vapour film is formed at the surface. The point *E* is referred to as Leidenfrost point. An increase in ΔT_e beyond the point *E* increases the heat flux. As the temperature increases, the radiation becomes prominent and hence a steep rise in heat flux with ΔT_e takes place.

Power control method of obtaining the boiling curve

In practice, the boiling curve is usually obtained from an experiment where the heat flux is provided by an electrically-heated nichrome or platinum wire submerged in a pool of saturated water at atmospheric pressure. The heat flux is determined from the measurement of current flowing through the wire and the potential drop across it. Since the heat flux is varied as the independent parameter and the wire temperature is measured as the dependent parameter, this type of experimental arrangement is referred to as ‘power-controlled heating’. If we gradually increase the heat flux from a low value by increasing the electrical power across the wire, we will obtain the heating curve as shown by arrows in Figure 9.6. The firm line represents the actual boiling curve as explained earlier. The heating curve of the experiment coincides with the typical boiling curve up to the point *D*. But it is observed that at point *D*, the temperature of the wire increases abruptly to a high value corresponding to the melting temperature of the wire with a marginal change or even without any change in the heat flux. Thus the wire is burnt out. This happens because of the fact that at point *D* the heat transfer coefficient starts decreasing rapidly due to the formation of vapour blanket while in the experiment we have control only over the power input to the wire and not on its temperature. Therefore the portions of transition and film boiling are missed out in the experimental heating curve. Possibly an experiment with an independent control of wire temperature can capture these portions of the boiling curve.

If the experiment is performed by decreasing the heat flux to the wire from a high temperature near its melting point, then a cooling curve is obtained as shown by arrows in Figure 9.6. When the heat flux reaches the minimum point, a further decrease in power causes the wire temperature to drop abruptly. The hysteresis effect as observed is also a consequence of power-controlled method of heating and cooling in the experiment.

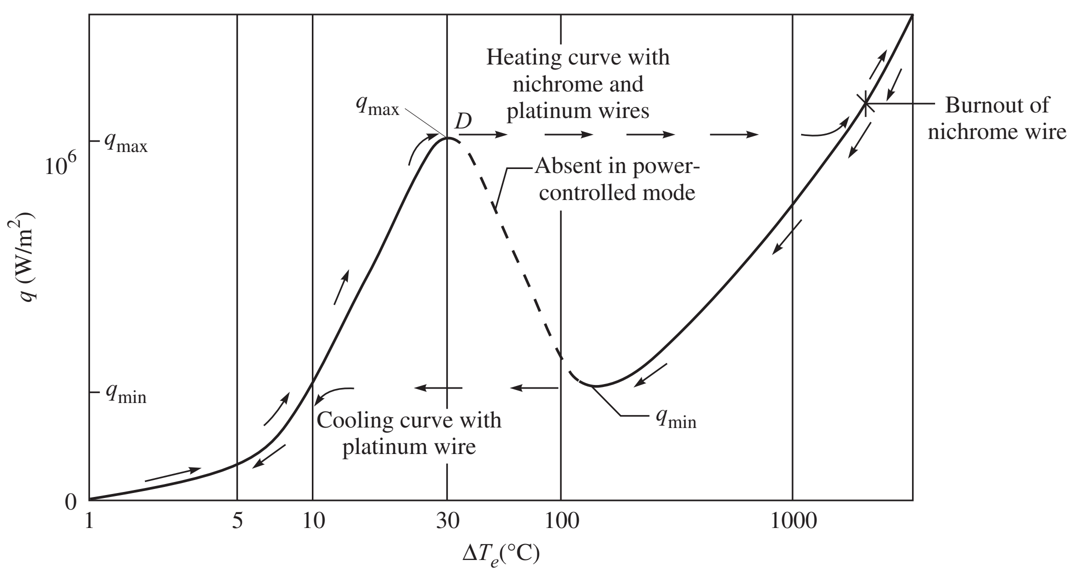


Figure 9.6 Typical characteristic curve of boiling vis-á-vis the curve obtained from power-controlled method.

Correlations in boiling heat transfer

The rate of heat transfer in nucleate boiling depends on the number of surface nucleation sites and the rate of formation of bubbles from those sides. Therefore, it is expected that the heat transfer coefficient in nucleate boiling is greatly influenced by the type of surface and the liquid. A number of experimental investigations have been made by several workers to relate the heat transfer rate with the number of nucleation sites. The most widely used empirical correlation in nucleate boiling has been developed by Rhosenow [6]. The equation is usually written for surface heat flux as

$$q = \mu_1 h_{fg}\left[\frac{(\rho_l - \rho_v)g}{\sigma}\right]^{1/2}\left[\frac{c_{p,1}\Delta T_e}{C_{s,f} h_{fg} \mathrm{Pr}_1^n}\right]^3 \tag{9.26}$$

where

$c_{p \cdot 1}$ is the specific heat of liquid at constant pressure
h_{fg} is the enthalpy of vaporization of the liquid at saturation temperature
Pr_1 is the Prandtl number of saturated liquid
ΔT_e is the excess temperature
μ_1 is the liquid viscosity
ρ_l is the density of saturated liquid
ρ_v is the density of saturated vapour
σ is the surface tension of liquid in its own vapour.

The coefficient $C_{s,f}$ and the exponent n depend on the surface–liquid combination. The representative values are shown in Table 9.1. The term '$c_{p \cdot 1}\, \Delta T_e / h_{fg}$' is referred to as Jacob number.

Table 9.1 Values of $C_{s,f}$ for various surface–fluid combinations

Surface–fluid combination	$C_{s,f}$	n
Water–Copper		
Scored	0.0068	1.0
Polished	0.0130	1.0
Water–stainless steel		
Chemically etched	0.0130	1.0
Mechanically polished	0.0130	1.0
Ground and polished	0.0060	1.0
Water–brass	0.0060	1.0
Water–nickel	0.006	1.0
Water–platinum	0.0130	1.0
n-Pentane–copper		
Polished	0.0154	1.7
Lapped	0.0149	1.7
Benzene–chromium	0.101	1.7
Ethyl alcohol–chromium	0.0027	1.7

Kutateladze [7] and Zuber [8] developed an expression for the critical heat flux (the peak heat flux in the boiling curve) as

$$q_c = \frac{\pi}{24} h_{fg}\, \rho_v \left[\frac{\sigma g(\rho_l - \rho_v)}{\rho_v^2}\right]^{1/4}\left[\frac{(\rho_l + \rho_v)}{\rho_l}\right]^{1/2} \tag{9.27}$$

The constant $\pi/24$ in Eq. (9.27) is replaced by an empirical constant 0.149 by Lienhard et al. [9].

The situation in film boiling beyond the leindenfrost point is very much similar to that in film condensation. While in film condensation, a liquid film adheres to the surface and the saturated vapour is condensed to liquid at the interface, in case of film boiling, a vapour film adheres to the surface and a saturated liquid is vaporized at the interface. The rate of heat transfer in film boiling depends on the surface geometry but not on the surface condition and one can use the similar expression as developed in the case of film condensation. One such equation for film boiling on cylindrical and spherical surfaces is

$$\overline{\mathrm{Nu}_D} = \frac{\bar{h}_{\mathrm{conv}} D}{k_v} = C\left[\frac{(\rho_l - \rho_v) g h'_{fg} D^3}{\nu_v k_v (T_w - T_s)}\right]^{1/4} \tag{9.28}$$

The constant C is 0.62 for horizontal cylinder and 0.67 for sphere.

EXAMPLE 9.5 A 0.1 m long, 1 mm diameter nickel wire submerged horizontally in water at one atmospheric pressure requires 150 A at 2.2 V to maintain the wire at 110°C. Determine the heat transfer coefficient.

Solution: The saturation temperature of water at one atmospheric pressure (101 kPa) is 100°C. Therefore, we can write from the energy balance of the wire at steady state

$$EI = hA\Delta T$$

or

$$h = \frac{EI}{A\Delta T}$$

$$= \frac{150 \times 2.2}{\pi (0.001)(0.1)(110 - 100)}$$

$$= 1.05 \times 10^5 \text{ W/m}^2$$

EXAMPLE 9.6 In a laboratory experiment, a current of 100 A burns out a 0.3 m long, 1 mm diameter nickel wire which is submerged horizontally in water at one atmospheric pressure. Determine the voltage across the wire at burnout. Consider that the wire is burnt out when the heat flux reaches its maximum value.

For saturated water at one atmospheric pressure, ρ_l = 960 kg/m^3, ρ_v = 0.60 kg/m^3, h_{fg} = 2.26 × 10^6 J/kg, σ = 0.055 N/m

Solution: The wire is burnt out when the heat flux reaches its peak. We use Eq. (9.27) with the empirical constant 0.149 instead of π/24.

$$q_c = 0.149\, h_{fg}\, \rho_v \left[\frac{\sigma g (\rho_l - \rho_v)}{\rho_v^2}\right]^{1/4} \left[\frac{\rho_l + \rho_v}{\rho_l}\right]^{1/2}$$

Therefore,

$$q_c = 0.149 (2.26 \times 10^6)\,(0.60) \left[\frac{0.055 \times 9.81 \times (960 - 0.60)}{(0.60)^2}\right]^{1/4} \times 1$$

$$= 1.24 \times 10^6 \text{ W/m}^2$$

The burnout voltage E must satisfy the relation

$$EI = q_c \times A$$

or

$$E = \frac{1.24 \times 10^6 \times \pi \times (0.001) \times (0.3)}{100}$$

$$= 11.67 \text{ V}$$

EXAMPLE 9.7 A heated nickel plate at 110°C is submerged in water at one atmospheric pressure. Determine the rate of heat transfer per unit area. For nucleate boiling, assume $C_{s,f} = 0.006$ and $n = 1$ for the use of Eq. (9.26).

Solution: The saturation temperature of water at one atmospheric pressure = 100°C.

Therefore, $$\Delta T_e = (110 - 100) = 10°\text{C}$$

At this value of ΔT_e, boiling is most likely nucleate. We use the values of ρ_l, ρ_v, h_{fg} and σ to be the same as those in Example 9.6. We take

$$c_{p,l} = 4.216 \text{ kJ/(kg K)}$$
$$\text{Pr}_l = 1.74$$
$$\mu_l = 2.82 \times 10^{-4} \text{ kg/(m s)}$$

By making use of Eq. (9.26), we have

$$q = (2.82 \times 10^{-4})\,(2.26 \times 10^{6})\left(\frac{9.81(960-0.60)}{0.055}\right)^{1/2}\left(\frac{4.216\times10^{3}\times(110-100)}{0.006\times2.26\times10^{6}\times1.74}\right)^{3}$$
$$= 15.04 \times 10^{5} \text{ W/m}^2$$

The peak heat flux for water at one atmospheric pressure was found in Example 9.6 to be 1.24×10^6 W/m^2. Since $q < q_c$, the regime of boiling is nucleate.

EXAMPLE 9.8 An experiment is performed to determine the boiling characteristics of a special coating applied to a surface exposed to boiling. A copper bar is used in the experiment, one end of which is exposed to boiling water while the other end is encapsulated by an electric heater as shown in Figure 9.7. Thermocouples are inserted in the bar to measure the temperatures at two locations A and B at distances $x_A = 10$ mm and $x_B = 30$ mm from the surface as shown in the figure. Under steady-state conditions, nucleate boiling is maintained in saturated water at atmospheric pressure and the measured values of temperatures are $T_A = 140$°C and $T_B = 180$°C. If Eq. (9.26) has to be valid for the heat flux in the present situation, what is the value of the coefficient $C_{s,f}$ if $n = 1$.

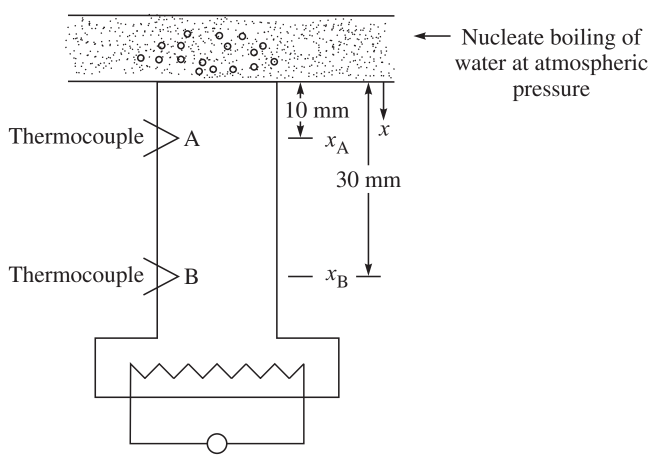


Figure 9.7 Experimental arrangement (Example 9.8).

Solution: The values of relevant properties of water and other parameters are taken to be the same as those in Example 9.7. They are

$$\rho_l = 960 \text{ kg/m}^3 \qquad \rho_v = 0.60 \text{ kg/m}^3 \qquad c_{p,1} = 4.216 \text{ kJ/(kg K)}$$

$$h_{fg} = 2.26 \times 10^6 \text{ J/kg} \qquad \text{Pr}_l = 1.74 \qquad \mu_1 = 2.82 \times 10^{-4} \text{ kg/(m s)} \qquad \sigma_1 = 0.055 \text{ N/m}$$

To determine the value of C_{sf} from Eq. (9.26), we have to know the value of heat flux q and the surface temperature T_w. Since we know the temperatures at locations A and B, the heat flux q is determined by writing the Fourier law of heat conduction in the bar at steady-state as

$$q = k\frac{T_\text{B} - T_\text{A}}{x_\text{B} - x_\text{A}}$$

We take for copper $k = 375$ W/(m K)

Hence
$$q = 375\frac{(180 - 140)}{(30 - 10)\times 10^{-3}}$$
$$= 7.5 \times 10^5 \text{ W/m}^2$$

The surface temperature
$$T_w = T_\text{A} - \frac{T_\text{B} - T_\text{A}}{x_\text{B} - x_\text{A}} x_\text{A}$$
$$= 140 - \frac{180 - 140}{30 - 10} \times 10$$
$$= 120°\text{C}$$

Now, we use Eq. (9.26)

$$7.5 \times 10^5 = (2.82 \times 10^{-4})(2.26 \times 10^6)\left(\frac{9.81(960 - 0.60)}{0.055}\right)^{1/2}\left(\frac{4.216(120 - 100)}{C_{sf} \times 2.26 \times 10^6 \times 1.74}\right)^3$$

which gives
$$C_{sf} = 0.007$$

SUMMARY

- The phenomenon of surface condensation takes place when a vapour is kept in continuous contact with a solid surface whose temperature is below the saturation temperature of the vapour at the existing pressure.
- The surface condensation is of two types: (i) drop-wise condensation and (ii) film condensation. In drop-wise condensation, the condensed liquid does not wet the solid surface, rather it forms liquid droplets which move down the surface due to gravity and surface tension. In film condensation, a continuous liquid film is formed on the surface.
- The rate of heat transfer and hence the rate of condensation is relatively lower in film condensation compared to that in drop-wise condensation. However it is extremely difficult in practice to sustain a drop-wise condensation process since all solid surfaces become wet after being exposed to a condensing vapour for a period of time.

- The heat transfer coefficient in condensation is very high since a large amount of heat is transferred with a small temperature difference across the fluid in contact with a solid surface. The average heat transfer coefficient in case of condensation over a flat surface inclined at an angle φ to the horizontal is given by

$$\bar{h}_L = 0.943\left[\frac{\rho(\rho-\rho_v)gh_{fg}k^3}{\mu L(T_g - T_w)}\sin\varphi\right]^{1/4}$$

  In case of a vertical plate, $\sin\varphi = 1$.
- The average heat transfer coefficient in condensation over a cylindrical surface of a horizontal tube is given by

$$\bar{h} = 0.725\left[\frac{\rho(\rho-\rho_v)gk^3h_{fg}}{\mu(T_g - T_w)D}\right]^{1/4}$$

- The Reynolds number in flow of a liquid film is defined as $\mathrm{Re} = 4\dot{m}/\mu P$. The flow of condensate film becomes turbulent beyond a Reynolds number of 1800. However, the ripples develop in liquid film beyond a Reynolds number of 30.
- An augmentation in the rate of heat transfer in condensation is achieved if the thickness of the condensate film is reduced. The film thickness is reduced if the condensate is drained out faster from the solid surface. The surface profile plays an important role in it. The rate of condensation is greatly reduced in the presence of non-condensable gases.
- Boiling occurs when a liquid comes into contact with a solid surface which is maintained at a temperature higher than the saturation temperature of the liquid at the existing pressure. When the heated surface is submerged below the free surface of a quiescent liquid, the boiling phenomenon is referred to as pool boiling. The pool boiling may be classified as subcooled boiling and saturated boiling depending upon whether the bulk temperature of the liquid is below or equal to the saturation temperature respectively.
- According to the characteristic features, the process of boiling is divided into different regimes, namely (i) free convection boiling, (ii) nucleate boiling, (iii) transition boiling, and (iv) film boiling.

  When the excess temperature ΔT_e (the difference in temperature between the solid surface and bulk of liquid) is less than 5°C, the formation of vapour is insufficient and the boiling takes place with free convection currents being responsible for fluid motion. When ΔT_e exceeds 5°C, the process of boiling starts with the generation of considerable vapour in the form of bubbles and it is referred to as nucleate boiling. The formation of vapour bubbles, their movement and final dissipation near the free surface induce considerable fluid mixing near the solid surface and increase substantially the heat transfer coefficient and heat flux. In this regime, heat flux reaches its maximum value and is then followed by a decreasing trend with ΔT_e in the transition mode of boiling. In this mode, the bubble formation is extremely rapid and the situation oscillates between a continuous vapour film and a number of bubbles adhering to the solid surface. At high values of ΔT_e, there exists the regime of film boiling where a stable vapour film is formed at the surface and the heat flux increases monotonically with ΔT_e.

- The heat transfer coefficient in boiling is very high since a large amount of heat is transferred with a small value of ΔT_e. The widely used empirical correlation in nucleate boiling is given Rhosenow [6] as

$$q = \mu_l \, h_{fg} \left[\frac{\rho_l - \rho_v}{\sigma} \right]^{1/2} \left[\frac{C_{p,1} \, \Delta T_e}{C_{s,f} \, h_{fg} \, \mathrm{Pr}_1^n} \right]^3$$

REVIEW QUESTIONS

1. What is the difference between drop-wise condensation and film condensation? Which of the two is the more effective way of condensation and why?
2. What is the influence of liquid film thickness on heat transfer in film condensation? On what factors does the film thickness depend?
3. How is the wetted perimeter defined in relation to condensate flow? How does the wetted perimeter differ from actual geometrical perimeter?
4. What is the criterion for turbulent film condensation?
5. Consider film condensation on a vertical plate. Will the heat flux be higher at the top or at the bottom of the plate? Why?
6. In case of condensation on the outer surface of a tube, why is a horizontal tube preferred over a vertical one in practice?
7. Explain physically how does the presence of a non-condensable gas in a vapour reduce the rate of condensation?
8. Why are the heat transfer coefficients in condensation and boiling very high compared to those in forced convection without phase change?
9. What is the difference between evaporation and boiling?
10. What is the difference between nucleate and film boiling? Why does nucleate boiling induce higher heat flux?
11. Why in a power control method of obtaining the boiling curve, can we not, capture the transition boiling regime and the Leidenfrost point?
12. How is the liquid motion induced in nucleate boiling?
13. Out of the two regimes (i) nucleate and (ii) film boiling, which one do you think is more difficult to be modelled theoretically?

PROBLEMS

[Use Table A.2 for the properties of saturated water.]

9.1 Saturated steam at one atmospheric pressure condenses on a 3 m high and 4 m wide vertical plate that is maintained at 90°C by circulating cooling water through the other side. Determine (a) the total rate of heat transfer by condensation to the plate, (b) the average heat transfer coefficient over the entire plate, and (c) the rate at which the condensate drips off the plate at the bottom.

[**Ans.** (a) 584 kW, (b) 4.87 kW/(m^2 °C), (c) 0.258 kg/s]

9.2 Repeat Problem 9.1 for the case of the plate being tilted 40° from the vertical and compare the results with that of a vertical plate.

[**Ans.** (a) 564 kW, (b) 4.70 kW/(m^2 °C), (c) 0.249 kg/s]

9.3 Saturated steam at 60°C is to be condensed at a rate of 15 kg/h on the outside of a 25 mm outer diameter vertical tube whose surface is maintained at 50°C by cooling water. Determine the tube length required. Use Eq. (9.8) for the purpose.

[**Ans.** 2.94 m]

9.4 For laminar film condensation, what is the ratio of heat transfer to a horizontal tube of large diameter to that due to a vertical tube of the same size for the same temperature difference?

9.5 What L/D ratio will produce the same laminar film condensation heat transfer rate to a tube in both vertical and horizontal orientations? Assume that the diameter is large compared with the condensate thickness.

[**Ans.** 2.86]

9.6 Following the similar line of deduction as made in case of a flat plate, obtain the expression as given by Eq. (9.14) in case of condensation over a horizontal tube.

9.7 Saturated steam at one atmospheric pressure condenses on a 2 m long vertical plate. What is the plate temperature T_w below which the condensate film throughout the plate will remain laminar? Use all property values at 100°C.

[**Ans.** 91.38°C]

9.8 Saturated vapour of refrigerant-134 at 30°C is to be condensed in a 4 m long 15 mm diameter horizontal tube which is maintained at a temperature of 20°C. If the refrigerant enters the tube at a rate of 2 kg/min, determine the fraction of the refrigerant that will be condensed at the end of the tube. Use Eq. (9.23). Take the following properties of the refrigerant:

$$\rho_1 = 1190 \text{ kg/m}^3, \qquad \mu_1 = 0.2 \times 10^{-3} \text{ kg/(m s)}$$
$$K_1 = 0.08 \text{ W/(m °C)}, \qquad \rho_v = 37 \text{ kg/m}^3, \; c_{p,1} = 1450 \text{ J/(kg °C)}$$
$$h_{fg} = 173 \text{ kJ/(kg)}$$

[**Ans.** 0.45]

9.9 Estimate the peak heat flux using Eq. (9.27) for boiling water at normal atmospheric pressure.

[**Ans.** 1.09 MW]

9.10 Determine the excess temperature $\Delta T_e = T_w - T_s$ for a 1 mm diameter and 0.3 m long horizontal polished copper wire submerged in water at normal atmospheric pressure. The wire is electrically heated with a power of 650 W. Nucleate boiling is observed. Use Eq. (9.26) with property values at 100°C.

[**Ans.** 16.78°C]

9.11 In an experiment of boiling, saturated water at 100°C with an electrically heating element, a heat flux of 5×10^5 W/m^2 is observed with a temperature difference $\Delta T = T_w - T_s = 10$°C. What is the value of C_{sf} of the heater surface for use of Eq. (9.26)? Take $n = 1$.

[**Ans.** 0.008]

9.12 If the coefficient C_{sf} in Eq. (9.26) is increased by a factor of 2, what should be the change in the heat flux q while the other quantities remain the same?

[**Ans.** reduced by a factor of 0.125]

9.13 Determine the critical heat flux for boiling water at one atmospheric pressure on the surface of the moon where the acceleration due to gravity is one-sixth that of the earth.

[**Ans.** 0.70 W]

9.14 The surface tension of water at 100°C is 0.059 N/m for the vapour in contact with the liquid. Assuming that the saturated vapour inside a bubble is at 101°C, while the surrounding liquid is saturated at 100°C, calculate the diameter of a spherical bubble.

[**Ans.** 0.056 mm]

9.15 Compare the heat transfer coefficients in nucleate boiling of water and in consideration of steam on a horizontal cylinder at normal atmospheric pressure and with same value of ΔT (the difference in saturation temperature and temperature of solid surface). Use Eq. (9.15) for condensation replacing h_{fg} by h'_{fg}, and Eq. (9.28) for boiling.

[**Ans.** $\bar{h}_b / \bar{h}_c = 0.025$]

9.16 A 6 mm diameter copper heater rod is submerged in water at normal atmospheric pressure. If the heater temperature is 10°C more than the saturation temperature of water, determine the heat loss per unit length of the rod. Use Eq. (9.26) with $C_{sf} = 0.013$ and $n = 1$.

[**Ans.** 2.75 kW/m]

REFERENCES

[1] Nusselt, W., Die Oberflachenkondensation *des Wasserdampfers,* VDI z, Vol. 60, p. 541, 1916.

[2] Datta Arijit, S.K. Som, and P.K. Das, Film Condensation of Saturated Vapour Over Horizontal Non-Circular Tubes with Progressively Increasing Radius of Curvature Drawn in the Direction of Gravity, *J. Heat Transfer,* ASME, Vol. 126, p. 906, 2004.

[3] McAdams, W.H., *Heat Transmission,* 3rd ed., McGraw-Hill, 1954.

[4] Chato, J.C., *J. Am. Soc. Refrig. Air Cond. Eng.,* February 1962, p. 52.

[5] Akers, W.W., H.A. Deans, and O.K. Crosser, Condensing Heat Transfer within Horizontal Tubes, *Chemical Eng. Prog. Symp. Ser,* Vol. 55, No. 29, p. 171, 1958.

[6] Rhosenow, W.M., and P. Griffith, Correlation of Maximum Heat Flux Data for Boiling of Saturated Liquids, *Trans ASME*, Vol. 74, p. 969, 1952.

[7] Kutateladze, S.S., On the Transition to Film Boiling Under Natural Circulation, *Kotloturbostroenie,* 3, 10, 1948.

[8] Zuber, N., On Stability of Boiling Heat Transfer, *Trans ASME,* 80, p. 711, 1958.

[9] Lienhard, J.H., V. K. Dhir, and D.M. Riherd, Peak Pool Boiling Heat Flux Measurements on Finite Horizontal Flat Plates, *J. Heat Transfer,* ASME, Vol. 95, p. 477, 1973.

10

Principles of Heat Exchangers

The process of heat exchange between two fluids is often required in many engineering applications, namely in power industries, air-conditioning and chemical process industries. The device used to execute this purpose is known as heat exchanger. The process of heat transfer between two fluids can be accomplished either by direct contact between the fluids or by indirect contact between them by a separating solid wall. The examples of direct contact type heat exchangers are cooling towers, jet condensers, etc. where water is used in the form of an atomized spray which comes in direct contact with a gas or vapour for heat transfer. However, we will discuss only the indirect contact type heat exchangers in this chapter.

Learning objectives

The reading of the present chapter will enable the readers

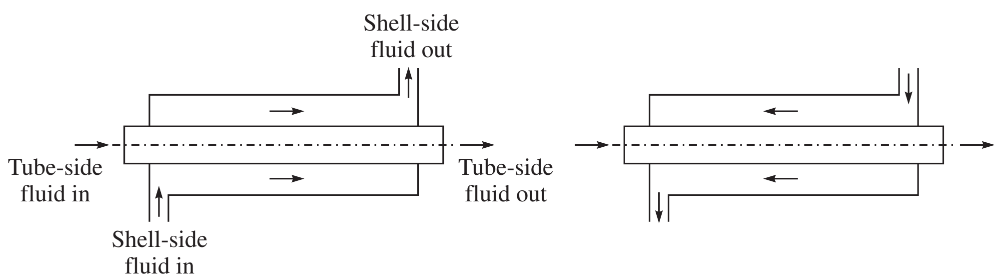
- to know the importance of heat exchanger in industrial applications and the different types of heat exchangers used for different purposes,
- to understand the mechanism of heat transfer in a shell-and-tube type heat exchanger, and get acquainted with the mathematical analyses for simple shell-and-tube type heat exchangers,
- to know how the results of mathematical analysis of simple double-pipe heat exchanger can be used for different complex types of shell-and-tube exchangers,
- to understand the concept of compact heat exchangers, and
- to know about the selection criteria of heat exchangers and how to select a heat exchanger for a specific purpose.

10.1 CLASSIFICATIONS OF HEAT EXCHANGERS

Various types of heat exchangers are made in practice which differ from one another in geometrical configuration, construction, flow arrangement, and heat transfer mechanism. In indirect contact type heat exchanger, as mentioned earlier, heat is transferred from one fluid to another through a solid wall usually by the mode of conduction and convection. This type of exchanger is classified depending upon the geometrical configuration and flow arrangement. The simplest one is known as shell-and-tube exchanger.

10.1.1 Shell-and-Tube Heat Exchanger

The simplest form of this type consists of two concentric tubes. While one fluid flows through the inner tube, the other one flows through the annulus. When the two fluids move in the same direction, the arrangement is known as parallel flow (Figure 10.1(a)) and when they move in the opposite direction, the arrangement is known as counterflow arrangement (Figure 10.1(b)). The fluid flowing through the inner tube is referred to as 'tube-side fluid' while the fluid flowing through the annulus is referred to as 'shell-side fluid'. When the directions of flow of tube and shell side fluids are perpendicular to each other, the arrangement is known as cross flow. Figure 10.2 shows a shell-and-tube heat exchanger with multiple tubes and with combined cross-flow and counterflow arrangements.



(a) Parallel flow (b) Counterflow

Figure 10.1 Simple shell-and-tube heat exchanger.

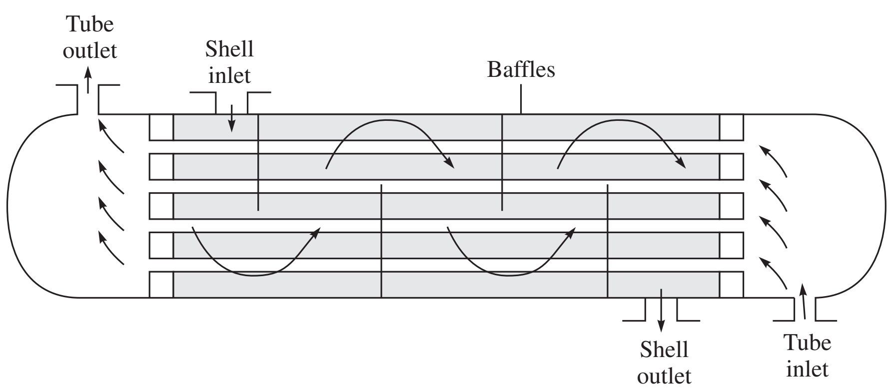
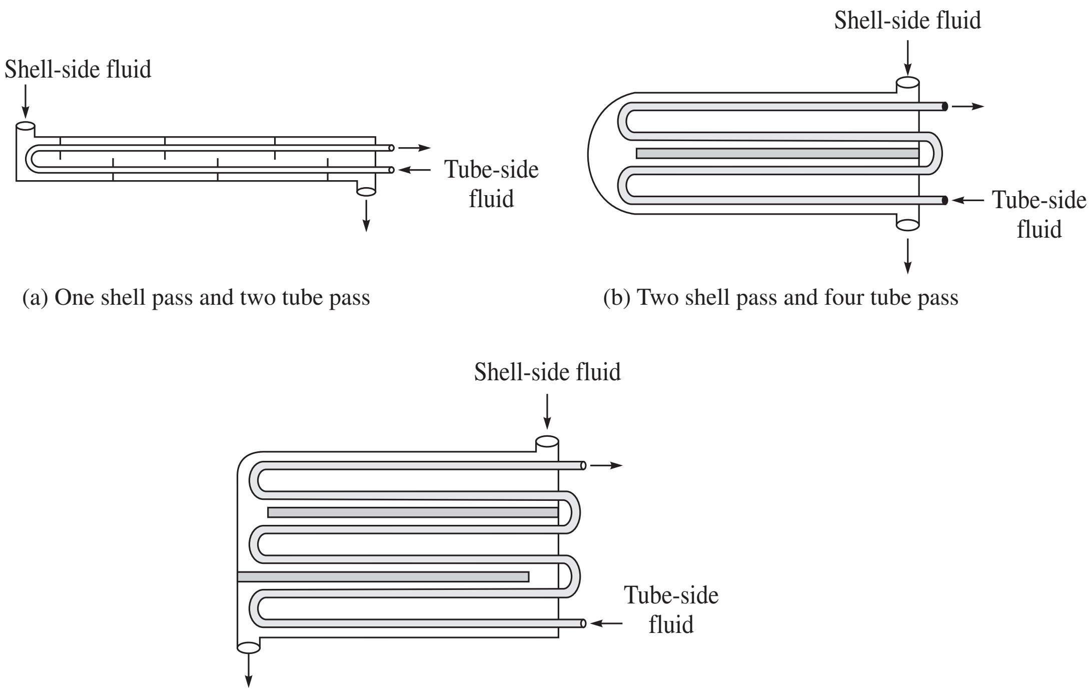


Figure 10.2 Shell-and-tube heat exchanger with one shell pass and one tube pass (cross–counterflow mode of operation).

In heat exchangers shown in Figures 10.1 and 10.2, both the shell-side and the tube-side fluids have only one pass. We can have in practice different combinations of multiple shell and tube passes as shown for examples, in Figures 10.3(a), 10.3(b), and 10.3(c).



(a) One shell pass and two tube pass

(b) Two shell pass and four tube pass

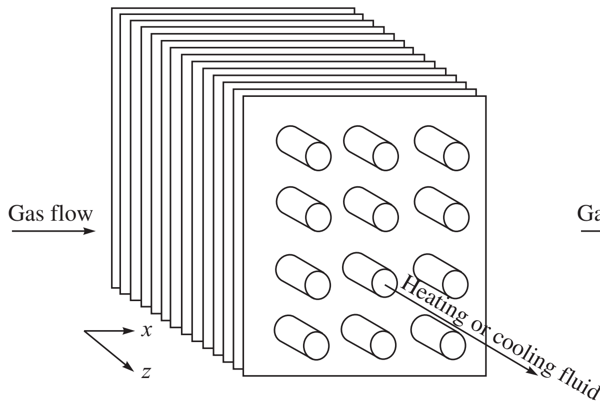
(c) Three shell pass and six tube pass

Figure 10.3 Multipass heat exchangers.

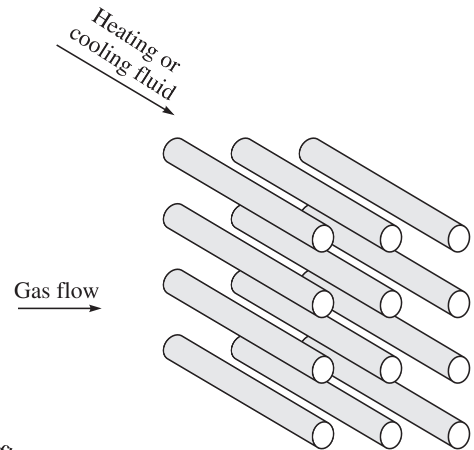
10.1.2 Compact Heat Exchangers

It is essential at this point to mention what is meant by compact heat exchangers. For a given temperature difference, the rate of heat transfer from one fluid to the other in a heat exchanger depends upon the available surface area and the heat transfer coefficients associated with the flow of fluids. When one of the fluids or both the fluids are gases, then a low heat transfer coefficient is encountered in course of their flows. Hence a large surface area of the heat exchanger is required for a relatively high heat transfer rate. Again the size of the heat exchanger cannot be big because of space constraints in specific applications. Therefore, special designs of heat exchangers are made to provide a large surface area per unit volume. The ratio of surface area to volume of a heat exchanger is called the area density or compactness factor. A heat exchanger whose area density is greater than 700 m^2/m^3 is classified as compact heat exchanger. A large surface area in compact heat exchangers is obtained by attaching closely spaced thin plates or fins to the walls separating the two fluids. A cross-flow arrangement is usually made where the two fluids move perpendicular to each other.

Heat exchangers with cross-flow arrangement are shown in Figures 10.4(a) and 10.4(b). The heat exchanger as shown in Figure 10.4(a) consists of tubes with plate fins where the outer fluid is forced to flow through interfin spacing and is thus prevented from moving in the transverse direction. This arrangement is known as unmixed type cross-flow heat exchanger. The cross-flow arrangement as shown in Figure 10.4(b) is said to be mixed in the transverse direction. Kays and London [1] have extensively studied the different types of compact heat exchangers. A few examples of compact heat exchangers are shown in Figures 10.5(a) to 10.5(c).

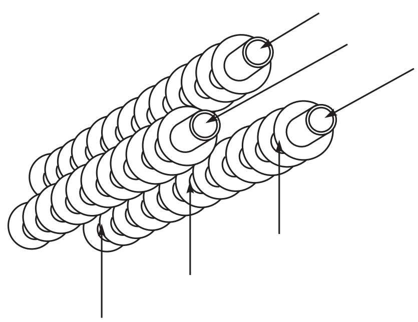


(a) Both fluids unmixed

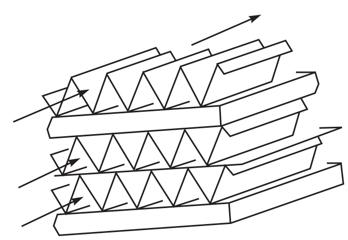


(b) One fluid mixed and one unmixed

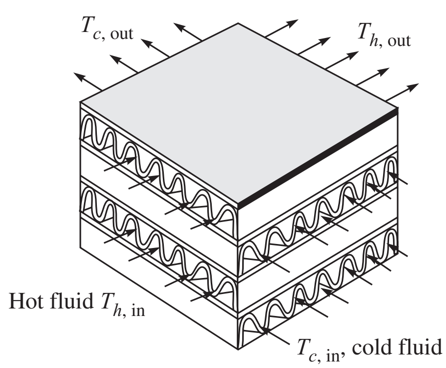
Figure 10.4 Cross-flow heat exchangers.



(a) Tube-fin type



(b) Plate-fin type



(c) Plate-fin type

Figure 10.5 Compact heat exchangers.

10.1.3 Regenerator

There is another type of heat exchanger where the process of heat exchange is performed by the alternate passage of the hot and cold fluid streams through the same flow area. The heat exchangers of this type are called regenerators. They are of two types: static and dynamic. The static type regenerator basically consists of a porous solid mass like balls, pebbles, powders, wire mesh, etc. which has a large storage capacity. The solid mass is known as *matrix*. Hot and cold fluids flow through this matrix alternately. Heat is transferred from the hot fluid to the matrix of the regenerator during the flow of the hot fluid and from the matrix to the cold fluid during the flow of cold fluid. Thus the matrix serves as a temporary heat storage medium. A flow-switching device regulates the periodic flow of hot and cold fluids. Static regenerators can be of both compact and non-compact types. The non-compact types are used as air preheaters for coke manufacturing and glass melting tank while compact types are used in refrigeration. The dynamic type regenerator usually consists of a rotating drum. Hot and cold fluids flow continuously through different portions of the drum so that any portion of the drum passes periodically through the hot stream storing heat, and then through the cold stream rejecting the stored heat.

10.2 MATHEMATICAL ANALYSES OF HEAT EXCHANGERS

We will discuss the mathematical analyses of only the simple double-pipe, shell-and-tube type heat exchanger as shown in Figure 10.1. There may be either a parallel flow or a counterflow situation. The temperature variations of the fluids in both the situations are shown in Figure 10.6. Here we consider the temperature of the fluids across a section to be uniform. The abscissa of Figure 10.6 may either be L (length of the heat exchanger) or be A (surface area of the heat exchanger), since A is directly proportional to L.

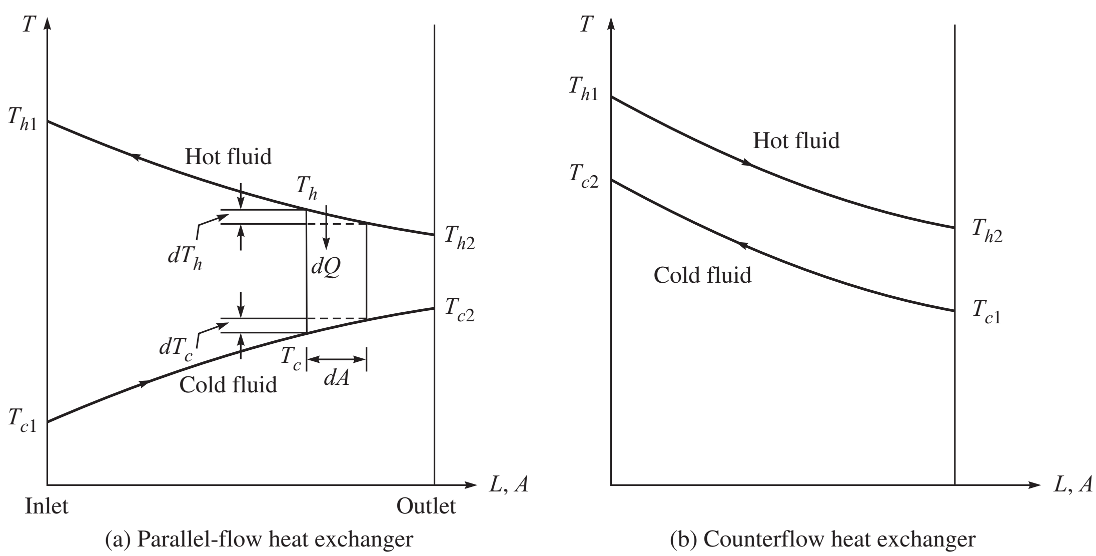


Figure 10.6 Temperature variation along the length of a shell-and-tube heat exchanger.

It has to be noted that in case of counterflow arrangement, hot and cold fluids enter the heat exchanger from opposite ends, and the outlet temperature of the cold fluid in this case may exceed the outlet temperature of the hot fluid. In the limiting case of an infinitely long counterflow heat exchanger, the cold fluid will be heated to the inlet temperature of the hot fluid. However, for an infinitely long parallel flow heat exchanger, the cold fluid will be heated to the outlet temperature of the hot fluid. Two special types of heat exchangers commonly used in practice are condensers and boilers. One of the fluids in a boiler or condenser undergoes a phase change process accompanied by a transfer of heat to or from the fluid without any change in fluid temperature. In a boiler, the fluid undergoing the process of phase change, is the cold fluid (Figure 10.7(a)), while in a condenser, it is the hot fluid (Figure 10.7(b)).

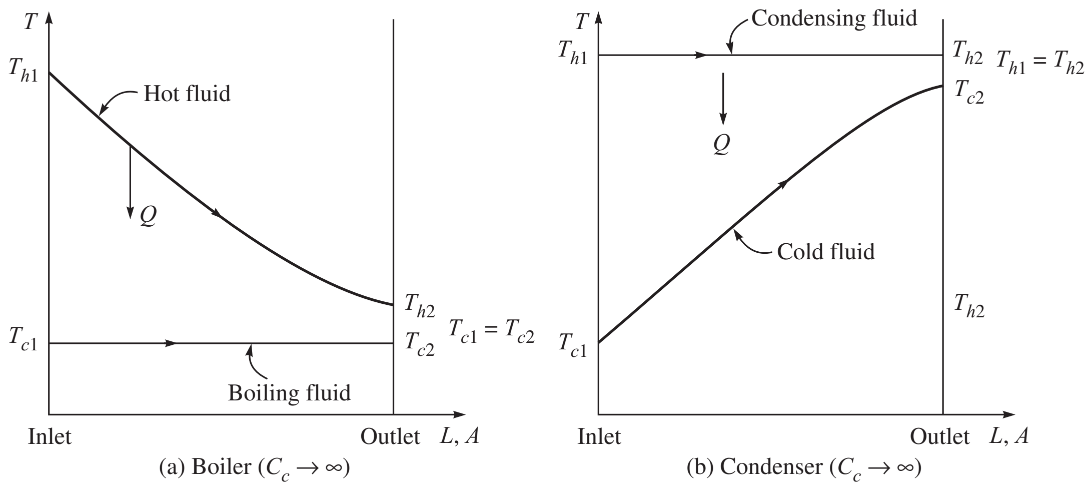


Figure 10.7 Variations of fluid temperatures in a heat exchanger when one of the fluids undergoes the process of phase change.

Overall heat transfer coefficient

Consider a simple double-pipe heat exchanger. Let h_i and h_o be the convective heat transfer coefficients at inside and outside of the inner tube. Then the overall heat transfer coefficient of the heat exchanger can be defined based on either inner or outer surface area of the inner tube as

$$\frac{1}{U_i A_i} = \frac{1}{U_o A_o} = \frac{1}{h_i A_i} + \frac{\ln(r_o/r_i)}{2\pi k L} + \frac{1}{A_o h_o} \tag{10.1a}$$

which gives

$$U_i = \frac{1}{\dfrac{1}{h_i} + \dfrac{A_i \ln(r_o/r_i)}{2\pi k L} + \dfrac{A_i}{A_o}\dfrac{1}{h_o}} \tag{10.1b}$$

$$U_o = \frac{1}{\dfrac{A_o}{A_i}\dfrac{1}{h_i} + \dfrac{A_o \ln(r_o/r_i)}{2\pi k L} + \dfrac{1}{h_o}} \tag{10.1c}$$

where, A_o, r_o are respectively the outer surface area and outer radius of the inner tube, and A_i, r_i are respectively the inner surface area and inner radius of the inner tube. k is thermal conductivity of the inner tube material and L is the tube length.

Fouling factor

In the course of operation of a heat exchanger, the heat transfer surfaces usually get coated with various deposits which are present as impurities in the fluids. This may result in corrosion of the surfaces. The deposition of materials in the form of a film on the surface acts as an additional thermal resistance and greatly reduces the rate of heat transfer between the fluids. This is represented by a fouling factor R_f which is simply a thermal resistance added in series to the process of heat transfer between the fluids. Therefore, we can write

$$\frac{1}{U_{\text{dirty}}} = \frac{1}{U_{\text{clean}}} + R_f$$

or

$$R_f = \frac{1}{U_{\text{dirty}}} - \frac{1}{U_{\text{clean}}} \tag{10.2}$$

The value of R_f depends upon the fluid, the operating temperature and the length of service of the heat exchanger. Representative values of fouling factors for various fluids are shown in Table 10.1.

Table 10.1 Typical fouling factors according to Standards of the Tubular Exchanger [2]

Fluid	R_f (m^2 K)/W
Seawater and treated boiler feedwater (below 50°C)	0.0001
Seawater and treated boiler feedwater (above 50°C)	0.0002
River water (below 50°C)	0.0002–0.001
Fuel oil	0.0009
Refrigerating liquids	0.0002
Steam (non-oil bearing)	0.0001

Equation (10.2) defines the fouling factor. If we consider the scale formation on both inner and outer surfaces of the inner tube, we can write from Eqs. (10.1a) and (10.2)

$$\frac{1}{UA_{\text{ref}}} = \frac{1}{h_i A_i} + \frac{\ln(r_o/r_i)}{2\pi k L} + \frac{1}{h_o A_o} + \frac{R_{fi}}{A_i} + \frac{R_{fo}}{A_o} \tag{10.3}$$

where R_{fi} and R_{fo} are respectively the fouling factors for deposits at inner and outer surfaces of the tube and A_{ref} is the reference area based on which the overall heat transfer coefficient is defined. Usually, the reference area is taken to be the outer surface area A_o.

Log mean temperature difference

The total rate of heat transfer between the two fluids in a shell-and-tube heat exchanger (Figure 10.1) can be found from a given value of overall heat transfer coefficient provided we know a suitable mean value of the temperature difference ΔT_m between the fluids so that

$$Q = UA\Delta T_m \tag{10.4}$$

where

U is the heat transfer coefficient

A is the reference surface area for the definition of heat transfer coefficient

ΔT_m is the suitable mean temperature difference between the fluids across the heat exchanger.

It is observed from Figure 10.6 that the temperatures of both the fluids vary along the length of the heat exchanger. For example, in case of a parallel flow, the temperature difference between the fluids is maximum at the inlet and minimum at the outlet. For a parallel flow heat exchanger (Figure 10.6(a)), the heat transfer through an element of area dA can be written as

$$dQ = -\dot{m}_h c_h \, dT_h = \dot{m}_c c_c \, dT_c \tag{10.5}$$

where $\dot{m}$ and c are respectively the mass flow rate and specific heats of the fluids, and the subscripts h and c designate the hot and cold fluid respectively. The rate of heat transfer can also be expressed as

$$dQ = U(T_h - T_c)dA \tag{10.6}$$

From Eq. (10.5),

$$dT_h = \frac{-dQ}{\dot{m}_h c_c} \tag{10.7a}$$

$$dT_c = \frac{dQ}{\dot{m}_c c_c} \tag{10.7b}$$

From Eqs. (10.7a) and (10.7b), we can write

$$dT_h - dT_c = d(T_h - T_c) = -dQ\left(\frac{1}{\dot{m}_h c_h} + \frac{1}{\dot{m}_c c_c}\right) \tag{10.8}$$

Substituting for dQ from Eq. (10.6) in Eq. (10.8), it becomes

$$\frac{d(T_h - T_c)}{(T_h - T_c)} = -U\left(\frac{1}{\dot{m}_h c_h} + \frac{1}{\dot{m}_c c_c}\right)dA$$

which gives

$$\ln\frac{T_{h2} - T_{c2}}{T_{h1} - T_{c1}} = -UA\left(\frac{1}{\dot{m}_h c_h} + \frac{1}{\dot{m}_c c_c}\right) \tag{10.9}$$

Again from Eq. (10.5), we have

$$Q = \dot{m}_h c_h (T_{h1} - T_{h2}) = \dot{m}_c c_c (T_{c2} - T_{c1})$$

Substituting the values of $\dot{m}_h c_h$ and $\dot{m}_c c_c$ from the above equation in Eq. (10.9), we finally have

$$Q = UA\frac{(T_{h2} - T_{c2}) - (T_{h1} - T_{c1})}{\ln\dfrac{T_{h2} - T_{c2}}{T_{h1} - T_{c1}}} \tag{10.10}$$

Comparing Eq. (10.5) with Eq. (10.4), we find that the mean temperature difference ΔT_m becomes

$$\Delta T_m = \frac{(T_{h2} - T_{c2}) - (T_{h1} - T_{c1})}{\ln \dfrac{T_{h2} - T_{c2}}{T_{h1} - T_{c1}}} \tag{10.11}$$

This mean temperature difference ΔT_m, as given by Eq. (10.11), is called the log mean temperature difference (LMTD). The similar expression can be obtained for a counterflow arrangement. The derivation for this is left as an exercise to the reader. The two main assumptions in deriving the above equation for LMTD are (i) the convective heat transfer coefficients are constant throughout the heat exchanger and (ii) the specific heats of the fluid do not vary with the temperature. If we discard the entrance effect and if the variations in temperatures of the fluid are not too large, these assumptions hold fairly good.

Equation (10.11) can be written in terms of the temperature differences at the two ends of the heat exchanger as

$$\text{LMTD} = \frac{\Delta T_2 - \Delta T_1}{\ln\left(\dfrac{\Delta T_2}{\Delta T_1}\right)} \tag{10.12}$$

where

$$\Delta T_1 = T_{h1} - T_{c1}$$
$$\Delta T_2 = T_{h2} - T_{c2}$$

In a counterflow heat exchanger, the temperature difference between the hot and the cold fluids will remain constant along the heat exchanger when the heat capacity rates of the two fluids are equal, that is, $\Delta T_1 = \Delta T_2 = \Delta T$ when $\dot{m}_h c_h = \dot{m}_c c_c$. The log mean temperature difference LMTD, under the situation, is found from Eq. (10.12) to be of the form $\dfrac{0}{0}$ which is indeterminate. In this case the value of LMTD is determined by the application of L'Hospital's rule to Eq. (10.12) which gives

$$\text{LMTD} = \Delta T_1 = \Delta T_2$$

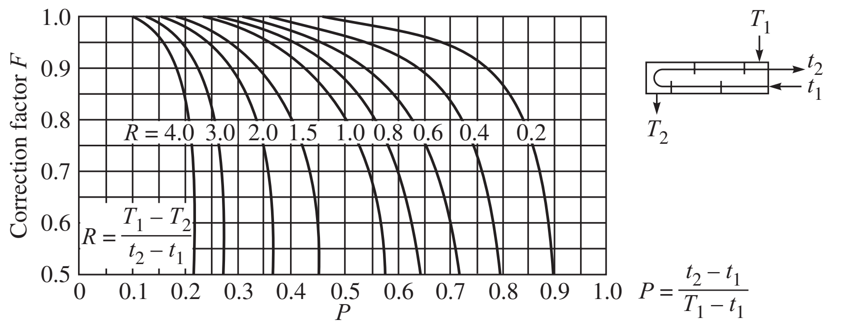
It has to be noted in this context that LMTD is always less than the arithmetic mean temperature difference ΔT_{av} $[=(\Delta T_1 + \Delta T_2)/2]$. Therefore, the use of ΔT_{av} instead of LMTD in Eq. (10.4) will overestimate the rate of heat transfer between the two fluid in a heat exchanger. It can be shown analytically that the arithmetic mean temperature difference ΔT_{av} gives a result to within 95 per cent accuracy when the end temperature differences ΔT_1 and ΔT_2 vary no more than a factor of 2.2.

Multipass and cross-flow heat exchangers: use of a correction factor

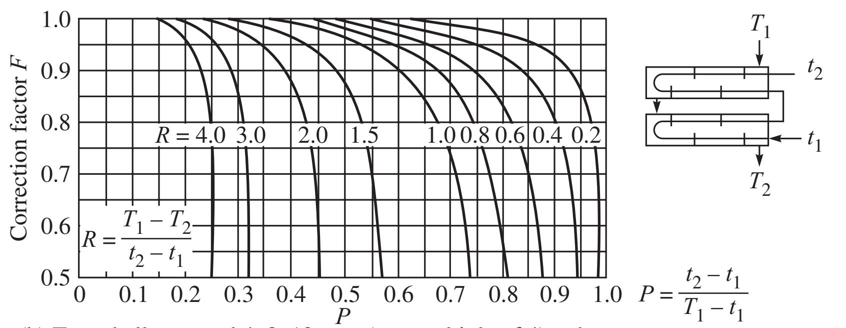
In case of heat exchangers other than a simple double-pipe heat exchanger, the heat transfer rate is calculated by using a correction factor which is multiplied by the LMTD for a counterflow double-pipe exchanger with the same hot and cold fluid temperatures. Equation (10.4) then takes the form

$$\begin{aligned} Q &= UAF\Delta T_m \\ &= UAF(\text{LMTD}) \end{aligned} \tag{10.13}$$

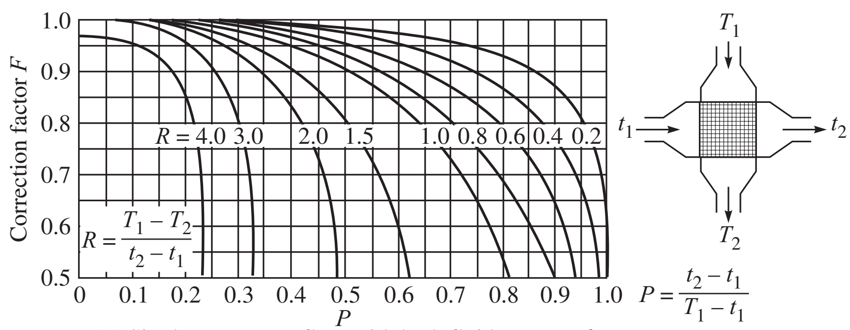
The values of the correction factor F under different situations are shown in Figure 10.8 after Bowman et al. [3].



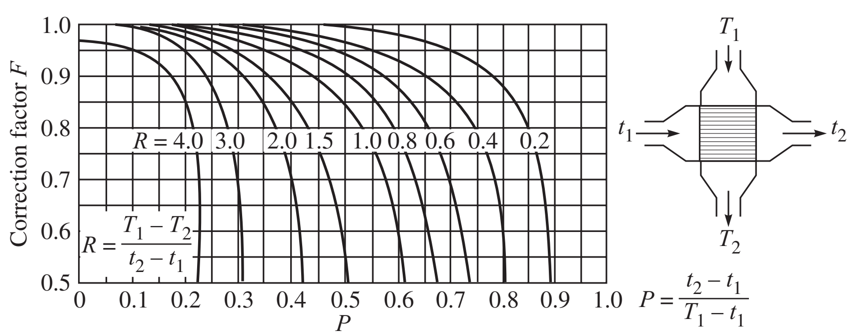
(a) One-shell pass and 2, 4, 6, etc. (any multiple of 2), tube passes



(b) Two-shell pass and 4, 8, 12, etc. (any multiple of 4), tube passes



(c) Single-pass cross-flow with both fluids *unmixed*



(d) Single-pass cross-flow with one fluid mixed and the other *unmixed*

Figure 10.8 Correction factor *F* for different shell-and-tube type heat exchangers after Bowman et al. [3].

EXAMPLE 10.1 In a food processing plant, a brine solution is heated from 8°C to 14°C in a double-pipe heat exchanger by water entering at 55°C and leaving at 40°C at the rate of 0.18 kg/s. If the overall heat transfer coefficient is 800 W/(m^2 K), determine the area of heat exchanger required (a) for a parallel flow arrangement, and (b) for a counterflow arrangement. Take c_p for water = 4.18 kJ/(kg K).

Solution: The rate of heat transfer from water is given by

$$Q = (0.18)(4.18 \times 10^3)(55 - 40) = 11.29 \times 10^3 \text{ W}$$

(a) For a parallel flow arrangement (Figure 10.9(a))

$$\Delta T_1 = 55 - 8 = 47°\text{C}$$
$$\Delta T_2 = 40 - 14 = 26°\text{C}$$

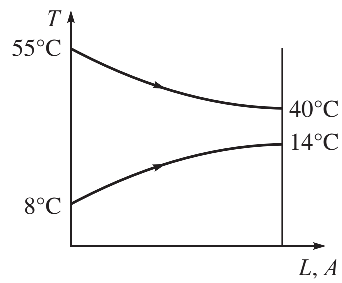


Figure 10.9(a) The temperature variation of the fluids in a parallel flow arrangement (Example 10.1).

Hence,
$$\text{LMTD} = \frac{\Delta T_2 - \Delta T_1}{\ln(\Delta T_2/\Delta T_1)} = \frac{26 - 47}{\ln(26/47)}$$
$$= 35.47°\text{C}$$

Therefore,
$$A = \frac{Q}{U.(\text{LMTD})}$$
$$= \frac{11.29 \times 10^3}{800 \times 35.47} = 0.398 \text{ m}^2$$

(b) In case of a counterflow arrangement (Figure 10.9(b))

$$\Delta T_1 = 55 - 14 = 41°\text{C}$$
$$\Delta T_2 = 40 - 8 = 32°\text{C}$$

Hence,
$$\text{LMTD} = \frac{32 - 41}{\ln(32/41)}$$
$$= 36.31°\text{C}$$

Therefore,
$$A = \frac{11.29 \times 10^3}{800 \times 36.31} = 0.389 \text{ m}^2$$

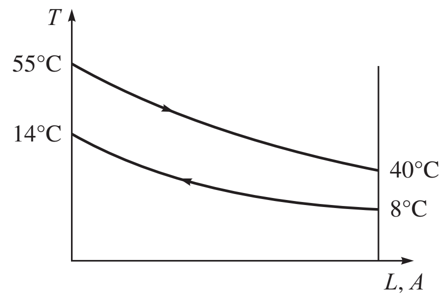


Figure 10.9(b) The temperature variation of the fluids in a counterflow arrangement (Example 10.1).

EXAMPLE 10.2 Hot oil [c_p = 2.09 kJ/(kg K)] flows through a counterflow heat exchanger at the rate of 0.7 kg/s. It enters at 200°C and leaves at 70°C. Cold oil [c_p = 1.67 kJ/(kg K)] exits at 150°C at the rate of 1.2 kg/s. Determine the surface area of the heat exchanger required for the purpose if the overall heat transfer coefficient is 650 W/(m^2 K).

Solution: The unknown inlet temperature of the cold oil may be found from an energy balance of the two fluids as

$$\dot{m}_c c_c (T_{c2} - T_{c1}) = \dot{m}_h c_h (T_{h1} - T_{h2})$$

i.e.

$$T_{c1} = T_{c2} - \frac{\dot{m}_h c_h}{\dot{m}_c c_c}(T_{h1} - T_{h2})$$

$$= 150 - \frac{(0.7)(2.09)}{(1.2)(1.67)}(200 - 70)$$

$$= 55.09°\text{C}$$

The rate of heat transfer can be calculated as

$$Q = (0.7)\,(2.09)\,(200 - 70) \times 10^3$$

$$= 190.19 \times 10^3 \text{ W}$$

$$\Delta T_1 = T_{h1} - T_{c2} = 200 - 150 = 50°\text{C}$$

$$\Delta T_2 = T_{h2} - T_{c1} = 70 - 55.09 = 14.91°\text{C}$$

Hence,

$$\text{LMTD} = \frac{14.91 - 50}{\ln(14.91/50)} = 29$$

$$A = \frac{Q}{U.(\text{LMTD})}$$

$$= \frac{190.19 \times 10^3}{650 \times 29} = 10.09 \text{ m}^2$$

EXAMPLE 10.3 A cross-flow heat exchanger with both fluids unmixed is used to heat water [c_p = 4.18 kJ/(kg K)] from 50°C to 90°C, flowing at the rate of 1.0 kg/s. Determine the overall heat transfer coefficient if the hot engine oil [c_p = 1.9 kJ/(kg K)] flowing at the rate of 3 kg/s enters at 100°C. The heat transfer area is 20 m^2.

Solution: The outlet temperature of the oil is first determined from an energy balance as

$$\dot{m}_h c_h (T_{hi} - T_{ho}) = \dot{m}_c c_c (T_{c,o} - T_{c,i})$$

i.e.

$$T_{ho} = T_{hi} - \frac{\dot{m}_c c_c}{\dot{m}_h c_h}(T_{c,o} - T_{c,i})$$

$$= 100 - \frac{(1.0)(4.18)}{(3)(1.9)}(90 - 50)$$

$$= 70.67°\text{C}$$

For a counterflow heat exchanger,

$$\Delta T_1 = 100 - 90 = 10°\text{C}$$

$$\Delta T_2 = 70.67 - 50 = 20.67°\text{C}$$

$$(\text{LMTD})_{\text{counterflow}} = \frac{20.67 - 10}{\ln\left(\dfrac{20.67}{10}\right)} = 14.69°\text{C}$$

We have to employ a correction factor F for the cross-flow arrangement. We evaluate the dimensionless parameters P and R as follows (see Figure 10.8).

$$P = \frac{90 - 50}{100 - 50} = 0.8$$

$$R = \frac{100 - 70.67}{90 - 50} = 0.73$$

We find from Figure 10.8(c),

$$F = 0.75$$

The overall heat transfer coefficient U is calculated from the relation

$$Q = UAF(\text{LMTD})_{\text{counterflow}}$$

Hence,

$$U = \frac{(1.0)(4.18 \times 10^3)(90 - 50)}{(20)(0.75)(14.69)}$$

$$= 0.76 \times 10^3 \text{ W/(m}^2 \text{ K)}$$

$$= 0.76 \text{ kW/(m}^2 \text{ K)}$$

EXAMPLE 10.4 A heat exchanger, when it is new, transfers 15% more heat than it does after being in service for six months. Assuming that it operates between the same temperature differentials and that there is insufficient scale build up to change the effective surface area, determine the fouling factor in terms of its clean (new) overall heat transfer coefficient.

Solution: The ratio of heat transfer can be written as

$$\frac{Q_{\text{clean}}}{Q_{\text{dirty}}} = \frac{U_{\text{clean}} A(\text{LMTD})}{U_{\text{dirty}} A(\text{LMTD})} = 1.15$$

or

$$\frac{U_{\text{clean}}}{U_{\text{dirty}}} = 1.15$$

$$R_f = \frac{1}{U_{\text{dirty}}} - \frac{1}{U_{\text{clean}}}$$
$$= \frac{1.15}{U_{\text{clean}}} - \frac{1}{U_{\text{clean}}}$$
$$= \frac{0.15}{U_{\text{clean}}}$$

The effectiveness-NTU method

The log mean temperature difference method as described above is very simple and explicit in its use when the inlet and outlet temperatures of the hot and cold fluids are known or can be determined straightforward from an energy balance. The LMTD method is very suitable for determining the size of a heat exchanger to meet the prescribed heat transfer requirements. Here all the inlet and outlet temperatures of the fluids are known along with the value of overall heat transfer coefficient. We simply use Eqs. (10.11) and (10.13) to determine the heat transfer surface area. When more than one of the inlet or the outlet temperatures of the heat exchanger are unknown, the simplicity of the LMTD method is lost, since it requires a trial-error iterative approach.

A second kind of problem encountered in heat exchanger analysis is the determination of the heat transfer rate and the outlet temperatures of the hot and cold fluids when the type and size of the heat exchanger are specified. Thus we ascertain the suitability of a type of heat exchanger for a specified duty and can compare the various types of heat exchangers for a given duty. The LMTD method could still be used in this type of problem, but the procedure will require tedious iterations. A relatively simple method is adopted for the purpose after Kays and London[1]. It is known as the effectiveness-NTU method. The method is based on a dimensionless parameter called the heat transfer effectiveness ε which is defined as

$$\varepsilon = \frac{Q}{Q_{\text{max}}} = \frac{\text{actual heat transter rate}}{\text{maximum possible heat transfer rate}}$$

The actual heat transfer rate in a heat exchanger is determined from the energy balance of hot fluid or cold fluid as shown below.

$$Q = m_h c_h (T_{hi} - T_{ho}) = m_c c_c (T_{c,i} - T_{c,o}) \tag{10.14}$$

Here we use the subscripts i and o for inlet and outlet of the fluids respectively.

In a heat exchanger, the heat transfer rate will be maximum when, either the cold fluid is heated to the inlet temperature of the hot fluid or the hot fluid is cooled to the inlet temperature of the cold fluid. These two limiting conditions cannot be reached simultaneously unless the heat capacity rates of the hot and cold fluids are identical (i.e. $m_h c_h = m_c c_c$). Usually this is not so and in practice, $m_h c_h \neq m_c c_c$. The fluid with smaller heat capacity will experience the larger temperature change, and thus it will be the first to experience the maximum temperature change, i.e. $(T_{hi} - T_{ci})$, at which point the process of heat transfer will stop. Therefore, the maximum possible heat transfer rate in a heat exchanger can be written as

$$Q_{\text{max}} = (\dot{m}c)_{\text{min}} \, (T_{h,i} - T_{c,i}) \tag{10.15}$$

where $(\dot{m}c)_{\min}$ is the smaller of $m_h c_h$ and $m_c c_c$. We can write

$$Q = \varepsilon\, Q_{\max} \tag{10.16}$$

Therefore, we find that the actual heat transfer rate can be found out without knowing the outlet temperatures of the fluids, provided we know the effectiveness ε. The effectiveness of a heat exchanger depends on the geometry and heat transfer area of the heat exchanger as well as the flow arrangement. The different heat exchangers have different relations for effectiveness with the governing parameters. We shall develop such relations for simple double-pipe parallel and counterflow heat exchangers.

Let us consider a parallel flow arrangement. The corresponding nomenclature for the temperatures is shown in Figure 10.6(a).

We can write

$$\varepsilon_h = \frac{\dot{m}_h c_h (T_{h1} - T_{h2})}{\dot{m}_h c_h (T_{h1} - T_{c1})} = \frac{T_{h1} - T_{h2}}{T_{h1} - T_{c1}} \quad \text{when hot fluid is the minimum capacity fluid} \tag{10.17a}$$

$$\varepsilon_h = \frac{\dot{m}_c c_c (T_{c2} - T_{c1})}{\dot{m}_c c_c (T_{h1} - T_{c1})} = \frac{T_{c2} - T_{c1}}{T_{h1} - T_{c1}} \quad \text{when cold fluid is the minimum capacity fluid} \tag{10.17b}$$

where the subscripts h and c for ε are used to indicate the situations where the hot fluid is the minimum capacity fluid and the cold fluid is the minimum capacity fluid respectively. It is observed from Eqs. 10.17(a) and 10.17(b) that the effectiveness can be expressed in a general way as

$$\varepsilon = \frac{\text{temperature difference of minimum capacity fluid}}{\text{maximum temperature difference in heat exchanger}}$$

Rewriting Eq. (10.9), we have

$$\ln\left(\frac{T_{h2} - T_{c2}}{T_{h1} - T_{c1}}\right) = -\frac{UA}{\dot{m}_c c_c}\left(1 + \frac{\dot{m}_c c_c}{\dot{m}_h c_h}\right) \tag{10.18}$$

or

$$\frac{T_{h2} - T_{c2}}{T_{h1} - T_{c1}} = \exp\left[\frac{-UA}{\dot{m}_c c_c}\left(1 + \frac{\dot{m}_c c_c}{\dot{m}_h c_h}\right)\right] \tag{10.19}$$

From an energy balance relation, we can write

$$\dot{m}_h c_h (T_{h1} - T_{h2}) = \dot{m}_c c_c (T_{c2} - T_{c1})$$

or

$$T_{h2} = T_{h1} + \frac{\dot{m}_c c_c}{\dot{m}_h c_h}(T_{c1} - T_{c2}) \tag{10.20}$$

[It has to be mentioned here, that the nomenclature for temperatures is used in accordance with the parallel flow arrangement shown in Figure 10.6(a).]

Using the expression of T_{h2} given by Eq. (10.20), the temperature ratio on the left-hand side of Eq. (10.19) becomes

$$\frac{T_{h2} - T_{c2}}{T_{h1} - T_{c1}} = \frac{T_{h1} + (\dot{m}_c c_c / \dot{m}_h c_h)(T_{c1} - T_{c2}) - T_{c2}}{T_{h1} - T_{c1}} \tag{10.21}$$

If cold fluid is minimum capacity fluid, then

$$\varepsilon = \frac{T_{c2} - T_{c1}}{T_{h1} - T_{c1}}$$

With the expression of ε, we can write Eq. (10.21) as

$$\frac{T_{h2} - T_{c2}}{T_{h1} - T_{c1}} = \frac{T_{h1} + T_{c1} + (\dot{m}_c c_c/\dot{m}_h c_h)(T_{c1} - T_{c2}) + (T_{c1} - T_{c2})}{T_{h1} - T_{c1}}$$

$$= 1 - \left(1 + \frac{\dot{m}_c c_c}{\dot{m}_h c_h}\right)\varepsilon \tag{10.22}$$

Therefore, Eq. (10.19) becomes

$$1 - \left(1 + \frac{\dot{m}_c c_c}{\dot{m}_h c_h}\right)\varepsilon = \exp\left[\frac{-UA}{\dot{m}_c c_c}\left(1 + \frac{\dot{m}_c c_c}{\dot{m}_h c_h}\right)\right]$$

or

$$\varepsilon = \frac{1 - \exp[(-UA/\dot{m}_c c_c)(1 + \dot{m}_c c_c/\dot{m}_h c_h)}{(1 + \dot{m}_c c_c/\dot{m}_h c_h)} \tag{10.23}$$

In case of hot fluid being the minimum capacity fluid, we obtain the same expression (10.23) except that $\dot{m}_c c_c$ and $\dot{m}_h c_h$ are interchanged. Therefore, the effectiveness ε for a parallel flow can be expressed in a general form as

$$\varepsilon_{\text{parallel flow}} = \frac{1 - \exp[(-UA/C_{\min})(1 + C_{\min}/C_{\max})]}{1 + C_{\min}/C_{\max}} \tag{10.24}$$

where $C = \dot{m}c$ and is known as the capacity rate.

Following the similar way as done for a parallel flow, the expression of ε in case of a double-pipe counterflow heat exchanger can be developed as

$$\varepsilon_{\text{counterflow}} = \frac{1 - \exp[(-UA/C_{\min})(1 - C_{\min}/C_{\max})]}{1 - (C_{\min}/C_{\max})\exp[(-UA/C_{\min})(1 - C_{\min}/C_{\max})]} \tag{10.25}$$

We observe from Eqs. (10.24) and (10.25) that the effectiveness is a function of two non-dimensional quantities, namely the capacity ratio $C = C_{\min}/C_{\max}$ and the parameter $UA/C_{\min}$. The parameter $UA/C_{\min}$ is called the number of transfer units NTU. Therefore,

$$\text{NTU} = \frac{UA}{C_{\min}} = \frac{UA}{(\dot{m}c)_{\min}}$$

where U is the overall heat transfer coefficient and A is the heat transfer surface area of the heat exchanger. The parameter NTU is proportional to A. Therefore, for specified values of U and $C_{\min}$, the value of NTU is a measure of the heat transfer area A.

Boilers and condensers: We discussed earlier that boilers and condensers are the heat exchangers where one of the two fluids undergoes the process of phase change. The variations in the fluid temperatures along the flow in this type of exchangers are shown in Figure 10.7. Under these situations, we can write mathematically

$$C_{\max} \to \infty$$

or

$$C(= C_{\min}/C_{\max}) \to 0$$

If we put this condition in Eqs. (10.24) and (10.25), we get the same expression for ε as

$$\varepsilon = 1 - \exp^{-\text{NTU}}$$

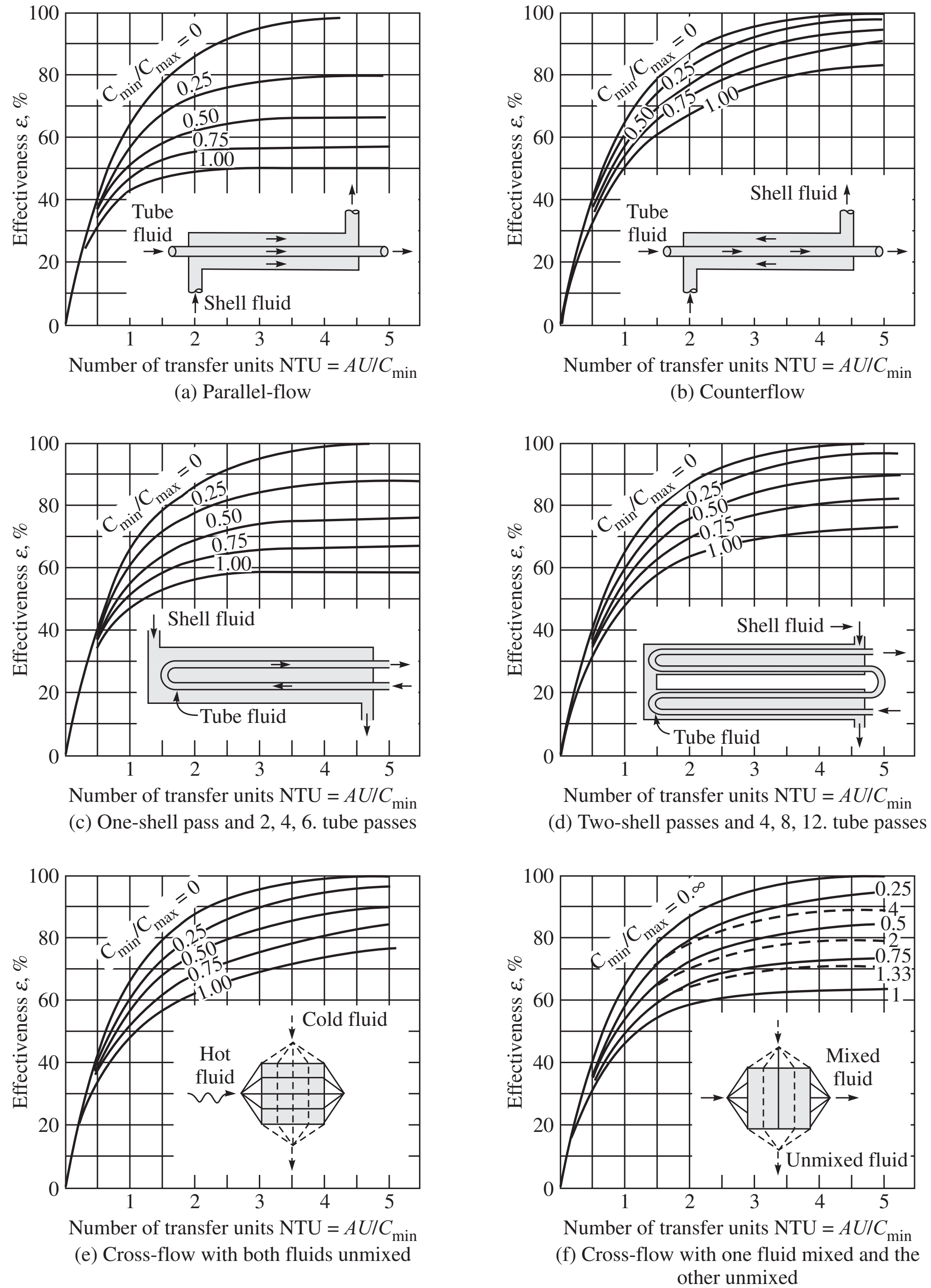
The effectiveness-NTU relations, i.e. $\varepsilon = f(\text{NTU}, C)$ for different heat exchangers are given in Table 10.2. The effectiveness relations of some common types of heat exchangers are shown in Figure 10.10. For convenience, the expressions given in Table 10.2 are sometimes expressed as the relations in the form of NTU as a function of ε and C as shown in Table 10.3.

Table 10.2 Effectiveness relations for heat exchangers: $\varepsilon = f(\text{NTU}, C)$

Heat exchanger type	*Effectiveness relation*
1 *Double pipe:*	
Parallel flow	$\varepsilon = \dfrac{1-\exp[-\text{NTU}(1+C)]}{1+C}$
Counterflow	$\varepsilon = \dfrac{1-\exp[-\text{NTU}(1-C)]}{1-C\exp[-\text{NTU}(1-C)]}$
2 *Shell and tube:* One-shall pass 2, 4,... tube passes	$\varepsilon = 2\left\{1+C+\sqrt{1+C^2}\,\dfrac{1+\exp[-\text{NTU}\sqrt{1+C^2}]}{1-C\exp[-\text{NTU}\sqrt{1+C^2}]}\right\}^{-1}$
3 *Cross-flow:* (*single-pass*)	
Both fluids unmixed	$\varepsilon = 1-\exp\left\{\dfrac{\text{NTU}^{0.22}}{C}[\exp(-C\,\text{NTU}^{0.78})-1]\right\}$
C_{max} mixed, C_{min} unmixed	$\varepsilon = \dfrac{1}{C}(1-\exp\{1-C[1-\exp(-\text{NTU})\}\})$
C_{min} mixed, C_{max} unmixed	$\varepsilon = 1-\exp\left\{-\dfrac{1}{C}[1-\exp(-C\,\text{NTU})]\right\}$
4 All heat exchangers with $C = 0$	$\varepsilon = 1-\exp(-\text{NTU})$

Table 10.3 Effectiveness relations for heat exchangers: $\text{NTU} = f(\varepsilon, C)$

Heat exchanger type	NTU *relation*
1 *Double pipe:*	
Parallel flow	$\text{NTU} = -\dfrac{\ln[1-\varepsilon(1+C)]}{1+C}$
Counterflow	$\text{NTU} = \dfrac{1}{C-1}\ln\left(\dfrac{\varepsilon-1}{\varepsilon C-1}\right)$
2 *Shell and tube:* One-shall pass 2, 4,... tube passes	$\text{NTU} = -\dfrac{1}{\sqrt{1+C^2}}\ln\left(\dfrac{2/\varepsilon-1-C-\sqrt{1+C^2}}{2/\varepsilon-1-C+\sqrt{1+C^2}}\right)$
3 *Cross-flow:* (*single-pass*)	
C_{max} mixed, C_{min} unmixed	$\text{NTU} = -\ln\left[1+\dfrac{\ln(1-\varepsilon C)}{C}\right]$
C_{min} mixed, C_{max} unmixed	$\text{NTU} = -\dfrac{\ln(C\ln(1-\varepsilon)+1)}{C}$
4 All heat exchangers with $C = 0$	$\text{NTU} = -\ln(1-\varepsilon)$



(a) Parallel-flow

(b) Counterflow

(c) One-shell pass and 2, 4, 6. tube passes

(d) Two-shell passes and 4, 8, 12. tube passes

(e) Cross-flow with both fluids unmixed

(f) Cross-flow with one fluid mixed and the other unmixed

Figure 10.10 Effectiveness for heat exchangers after Kays and London [1].

The following observations are made from Figure 10.10 for ε-NTU relations.

(a) The value of effectiveness lies between 0 to 1. The increase in ε with NTU is very sharp for small values of NTU (up to about NTU = 1.5) and then it becomes slow for larger values of NTU. It signifies physically that the use of a heat exchanger of large size is not justified from an economic point of view, since only a marginal gain in effectiveness is obtained at a much higher manufacturing cost of the heat exchanger.
(b) For a given NTU and capacity ratio $C = C_{\text{min}}/C_{\text{max}}$, the counterflow heat exchanger has a higher effectiveness followed closely by the cross-flow heat exchangers with both fluids unmixed.
(c) The effectiveness of a heat exchanger is independent of the capacity ratio C for NTU values less than about 0.3.
(d) For a given value of NTU, the effectiveness ε becomes a maximum for $C = 0$ and a minimum for $C = 1$.

EXAMPLE 10.5 Water is heated from 30°C to 90°C in a counterflow double-pipe heat exchanger. Water flows at the rate of 1.2 kg/s. The heating is accomplished by a geothermal fluid which enters the heat exchanger at 160°C at a mass flow rate of 2 kg/s. The inner tube is thin-walled and has a diameter of 15 mm. If the overall heat transfer coefficient of the heat exchanger is 600 W/(m^2 °C), determine the length of the heat exchanger required for the purpose. The specific heats of water and geothermal fluid are 4.18 and 4.31 kJ/(kg °C) respectively. Use both (a) the LMTD method and (b) the NTU-ε method.

Solution: **(a) LMTD method:** The rate of heat transfer in the heat exchanger is found as

$$\begin{aligned} Q &= [\dot{m} c_p (T_{\text{out}} - T_{\text{in}})]_{\text{water}} \\ &= 1.2(4.18)(90 - 30) \\ &= 301 \text{ kW} \end{aligned}$$

The outlet temperature of geothermal fluid is determined as

$$\begin{aligned} T_{\text{out}} &= 160 - \frac{301}{2(4.31)} \\ &= 125.10°\text{C} \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta T_1 &= 160 - 90 = 70°\text{C} \\ \Delta T_2 &= 125.10 - 30 = 95.10°\text{C} \end{aligned}$$

and

$$\begin{aligned} \text{LMTD} &= \frac{70 - 95.10}{\ln(70/95.10)} \\ &= 81.91°\text{C} \end{aligned}$$

Hence

$$\begin{aligned} A &= \frac{Q}{U(\text{LMTD})} \\ &= \frac{301 \times 10^3}{(600)(81.91)} \\ &= 6.12 \text{ m}^2 \end{aligned}$$

To provide this surface area, the length of the tube required is found as

$$L = \frac{A}{\pi D}$$

$$= \frac{6.12}{\pi(0.015)} = 129.87 \text{ m}$$

(b) **NTU method:** We first determine the heat capacity rates of the hot and cold fluids to identify the smaller value of the two.

$$C_h = \dot{m}_h c_h = 2(4.31) = 8.62 \text{ kW/°C}$$

$$C_c = \dot{m}_c c_c = 1.2(4.18) = 5.02 \text{ kW/°C}$$

Therefore, $$C_{\min} = C_c = 5.02 \text{ kW/°C}$$

and $$C = C_{\min}/C_{\max} = 5.02/8.62 = 0.583$$

$$\varepsilon = \frac{Q}{C_{\min}(T_{h,\text{in}} - T_{c,\text{in}})}$$

$$= \frac{301.00}{5.02(160-30)}$$

$$= 0.461$$

Now we determine the value of NTU by making use of the expression of NTU for a counterflow heat exchanger from Table 10.3.

$$\text{NTU} = \frac{1}{C-1}\ln\left(\frac{\varepsilon - 1}{\varepsilon C - 1}\right)$$

$$= \frac{1}{0.583-1}\ln\left(\frac{0.461-1}{0.461\times 0.583-1}\right)$$

$$= 0.731$$

We know $$\text{NTU} = \frac{UA}{C_{\min}}$$

or $$A = \frac{\text{NTU}\cdot C_{\min}}{U}$$

$$= \frac{0.731(5.02\times 10^3)}{600} = 6.12 \text{ m}^2$$

Hence, $$L = 129.87 \text{ m.}$$

Therefore, we find that the same result is obtained in both the methods.

EXAMPLE 10.6 Water enters a counterflow double-pipe heat exchanger at 35°C flowing at the rate of 0.8 kg/s. It is heated by oil [c_p = 1.88 kJ/(kg K)] flowing at the rate of 1.5 kg/s from an inlet temperature of 120°C. For an area of 15 m^2, and an overall heat transfer coefficient of 350 W/(m^2 K), determine the total heat transfer rate. Take the specific heat of water as 4.18 kJ/(kg K).

Solution: Only the inlet temperatures are known. Therefore, we will use the NTU-ε method.

$$C_{\text{water}} = \dot{m}_w c_w = 0.8(4.18) = 3.34 \text{ kW/K}$$

$$C_{\text{oil}} = \dot{m}_o c_o = 1.5(1.88) = 2.82 \text{ kW/K}$$

Therefore,

$$C = \frac{C_{\min}}{C_{\max}} = \frac{2.82}{3.34} = 0.84$$

$$\text{NTU} = \frac{UA}{C_{\min}} = \frac{350(15)}{2.82 \times 10^3}$$
$$= 1.86$$

Equation (10.25) is applicable for the present case.

Hence,

$$\varepsilon = \frac{1 - \exp[-\text{NTU}(1 - C)]}{1 - C\exp[-\text{NTU}(1 - C)]}$$

$$= \frac{1 - \exp[-1.86(1 - 0.84)]}{1 - 0.84\exp[-1.86(1 - 0.84)]}$$

$$= 0.684$$

Hence,

$$Q = \varepsilon C_{\min}(T_{h,i} - T_{c,i})$$
$$= 0.684\ (2.82 \times 10^3)\ (120 - 35)$$
$$= 163.95 \times 10^3 \text{ W} = 163.95 \text{ kW}.$$

EXAMPLE 10.7 Water enters a cross-flow heat exchanger (both fluids unmixed) at 20°C and flows at the rate of 7 kg/s to cool 10 kg of air per second from 125°C. If the overall heat transfer coefficient of the heat exchanger and its surface area are 220 W/(m^2 K) and 250 m^2 respectively, determine the exit air temperature. Take the specific heat of air at constant pressure to be 1.01 kJ/(kg K) and that of water to be 4.18 kJ/(kg K).

Solution:

$$(\dot{m}c_p)_{\text{air}} = 10(1.01) = 10.1 \text{ kW/K}$$

$$(\dot{m}c_p)_{\text{water}} = 7(4.18) = 29.26 \text{ kW/K}$$

which gives

$$C = \frac{C_{\min}}{C_{\max}} = \frac{10.1}{29.26} = 0.345$$

$$\text{NTU} = \frac{(250)(220)}{10.1 \times 10^3} = 5.44$$

To determine the effectiveness of the heat exchanger we have to find out the suitable expression of ε from Table 10.2 for the type of heat exchanger as stated in the problem

Thus, we can write for the present case

$$\varepsilon = 1 - \exp\left[\frac{\text{NTU}^{0.22}}{C}\{\exp(-C \cdot \text{NTU}^{0.78}) - 1\}\right]$$

$$= 1 - \exp\left[\frac{(5.44)^{0.22}}{0.345}\{\exp(-0.345)(5.44)^{0.78}) - 1\}\right]$$

$$= 0.95$$

The rate of heat transfer $\dot{Q} = \varepsilon C_{\min} (T_{h,i} - T_{c,i})$

$$= (0.95)(10.10 \times 10^3)(125 - 20)$$

$$= 1 \times 10^6 \text{ W}$$

The exit temperature of the air is given by

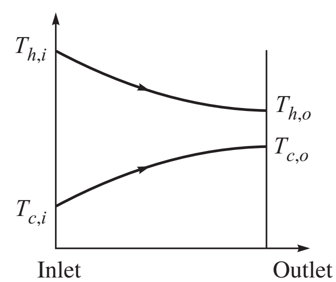
$$T_{h,o} = T_{h,i} = \frac{\dot{Q}}{(\dot{m}c_p)_{\text{air}}}$$

$$= 125 - \frac{1 \times 10^6}{10.1 \times 10^3} = 26°\text{C}$$

EXAMPLE 10.8 A double-pipe heat exchanger of 0.30 m length is to be used to heat water [c_p = 4.18 kJ/(kg K)] from 25°C to 50°C at a flow rate of 2 kg/s. The water flows through the inner tube, while the oil [c_p = 1.88 kJ/(kg K)] as the hot fluid flows in the annulus with an inlet temperature of 100°C.

(a) Considering a parallel-flow arrangement, determine the minimum flow rate required for the oil.
(b) Determine the overall heat transfer coefficient required for the conditions of part (a). Explain whether it is possible to achieve the condition prescribed in part (a)
(c) Considering a counterflow arrangement, determine the minimum flow rate required for oil. What is the effectiveness of the heat exchanger under this situation?

Solution: (a) Let us consider a parallel-flow arrangement as shown in the figure below:



From an energy balance in the heat exchanger,

$$\dot{m}_h c_h (T_{h,i} - T_{h,o}) = \dot{m}_c c_c (T_{c,o} - T_{c,i})$$

All quantities except $\dot{m}_h$ and $T_{h,o}$ in the above equations are prescribed. Therefore, for a minimum value of $\dot{m}_h$, the value of $T_{h,o}$ has to be minimum. The theoretical minimum value of $T_{h,o}$, under the situation, is $T_{c,o}$.

Hence, $$T_{h,o} = T_{c,o} = 50°\text{C}$$

Then, $$\dot{m}_{h_{\text{minimum}}} = \frac{\dot{m}_c c_c (T_{c,o} - T_{c,i})}{c_h (T_{h,i} - T_{h,o})}$$

$$= \frac{2 \times (4.18)(50 - 25)}{1.88(100 - 50)} = 2.22 \text{ kg/s}$$

(b) $$\text{LMTD} = \frac{\Delta T_1 - \Delta T_2}{\ln(\Delta T_1/\Delta T_2)}$$

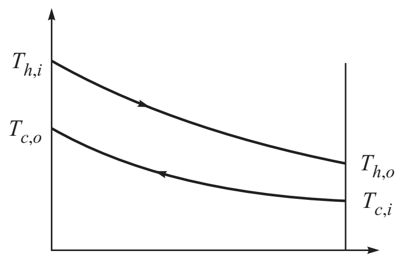
Here, $$\Delta T_2 = T_{h,o} - T_{c,i} = 0$$

As $$\Delta T_2 \to 0, \quad \text{LMTD} \to 0$$

We know $$Q = UA(\text{LMTD})$$

if LMTD $\to 0$, then for a finite value of heat transfer rate, $UA \to \infty$. For a given finite length, this implies an infinite value of U which is not possible.

(b) Let us consider a counterflow arrangement as shown below.



In this case, the minimum value of $T_{h,o}$ is $T_{c,i}$.

Therefore, $$\dot{m}_{h_{\text{minimum}}} = \frac{\dot{m}_c c_c (T_{c,o} - T_{c,i})}{c_h (T_{h,i} - T_{c,i})}$$

$$= \frac{2(4.18)(50 - 25)}{1.88(100 - 25)}$$

$$= 1.48 \text{ kg/s}$$

$$C_h = \dot{m}_h c_h = (1.48)(1.88)$$

$$C_c = \dot{m}_c c_c = (2)(4.18)$$

Hence, $$C_{\text{minimum}} = C_h$$

Therefore, $$\varepsilon = \frac{\dot{m}_h c_h (T_{h,i} - T_{h,o})}{\dot{m}_h c_h (T_{h,i} - T_{c,i})}$$

For $$T_{h,o} = T_{c,i}, \ \varepsilon = 1.$$

Mathematical approach in the analysis of compact heat exchangers

Heat transfer and flow characteristics of compact heat exchangers vary from one type to the other. The results are usually presented, for a specific type, in the form of a correlation of Colburn J factor $J(= \text{St Pr}^{2/3})$ and the Reynolds number Re, and the correlation of friction factor f with Reynolds number Re. The Reynolds number is expressed as

$$\text{Re} = \frac{GD_h}{\mu}$$

where D_h is the hydraulic diameter. The parameter G is related to mass flow rate as

$$G = \frac{\dot{m}}{A_{ff}} = \frac{\dot{m}}{\sigma A_{ft}}$$

where, $\dot{m}$ is the mass flow rate, A_{ff} and A_{ft} are the free-flow area of the finned passage (cross-sectional area perpendicular to flow direction) and the frontal area respectively.

$$\sigma = A_{ff}/A_{ft}$$

In a compact heat exchanger calculation, the average convective heat transfer coefficient of the finned surfaces is found out first from the suitable empirical relations and then the overall heat transfer coefficient is determined using LMTD or the ε-NTU method.

10.3 SELECTION CRITERIA OF HEAT EXCHANGERS

The selection of a heat exchanger for a particular application depends upon several factors as follows:

Heat transfer requirement: This is an important parameter in the selection of a heat exchanger. A heat exchanger selected for a purpose must transfer heat at the specific rate in order to achieve the desired temperature change of the fluid at the prescribed mass flow rate.

Cost: It is another important parameter in the selection of heat exchangers. An off-the-shelf heat exchanger has a definite cost advantage over those made to order. In situations where none of the existing heat exchangers suits the purpose, one has to undertake the expensive task of designing and manufacturing a heat exchanger to suit the needs.

Pressure drop characteristics and pumping power: In a heat exchanger, both the fluids are usually forced to flow by pumps or fans. The pumping power depends upon the pressure drop in course of flow of the fluids at specified flow rates, and is given by

$$\text{Pumping power} = \text{pressure drop} \times \text{volumetric flow rate}$$

The annual cost of electricity depends upon the pumping power and the duration of operation of the pumps and fans.

Size and weight: The heat exchanger is normally expected to be smaller and lighter. These are the essential requirements for automotive and aerospace industries where size and weight are of great concern. The space available for the heat exchanger, in some cases, limits the length of the tubes

Materials: The materials of construction for the heat exchanger is also an important consideration for the selection. At high temperatures and pressures the thermal and structural stresses become important for their considerations and the materials should sustain these stresses without deformation. The higher values of temperature difference between the tube and shell may pose differential thermal expansion which should be considered in selecting the material of construction. In the case of corrosive fluids, we have to select expensive corrosion-resistant materials such as stainless steel.

SUMMARY

- The device that serves the purpose of heat exchange between two fluids is known as heat exchanger. The process of heat transfer between the fluids is accomplished either by direct contact or by indirect contact between them.

- Various types of heat exchangers, used in practice, differ from one another in geometrical configuration, construction, flow arrangement and heat transfer mechanism. In indirect contact type heat exchangers, heat is transferred from one fluid to another through a solid wall separating them. The simplest form of this type of heat exchanger is known as shell-and-tube heat exchanger.
- A simple shell-and-tube heat exchanger consists of two concentric tubes. While one fluid flows through the inner tube, the other one flows through the annulus. In a parallel flow arrangement, both the fluids flow in the same direction, while in a counterflow arrangement, they move in opposite directions. When the directions of flow of fluids are perpendicular to each other, the arrangement is known as cross-flow. There may be more than one tube or one shell and multiple passes of both tube-side fluid and shell-side fluid in a shell-and-tube heat exchanger.
- The overall heat transfer coefficient in a simple double-pipe heat exchanger is given by

$$\frac{1}{U_i A_i} = \frac{1}{U_o A_o} = \frac{1}{h_i A_i} + \frac{\ln(r_o/r_i)}{2\pi k L} + \frac{1}{A_o h_o}$$

The additional thermal resistance to heat transfer due to coatings and various deposits of impurities in the fluids in course of operation of a heat exchanger is known as fouling factor. It is given by

$$R_f = \frac{1}{U_{\text{dirty}}} - \frac{1}{U_{\text{clean}}}$$

- The mathematical analysis of heat exchangers is based on either the LMTD method or the ε-NTU method.
- The LMTD method is employed when the inlet and the outlet fluid temperatures are prescribed and one has to find out the heat exchanger area or overall heat transfer coefficient.
- In the LMTD method, the total rate of heat transfer in a double-pipe heat exchanger is given by

$$Q = UA(\text{LMTD})$$

where LMTD is known as the log mean temperature difference and is given by

$$\text{LMTD} = \frac{\Delta T_1 - \Delta T_2}{\ln\left(\dfrac{\Delta T_1}{\Delta T_2}\right)}$$

in which ΔT_1 and ΔT_2 are the temperature differences between the two fluids at the two terminals of the heat exchanger.
In case of multi-pass, multi-tube, multi-shell and cross-flow heat exchangers, a correction factor F is employed in LMTD to determine the rate of heat transfer as

$$Q = UAF(\text{LMTD})$$

- The ε-NTU method is employed in the analysis of a heat exchanger when the type and size of the exchanger are specified and one has to find out the heat transfer rate and the outlet temperature of the fluids.

The effectiveness ε of a heat exchanger is defined as

$$\varepsilon = \frac{\text{actual heat transfer rate}}{\text{maximum possible heat transfer rate}}$$

The number of transfer units NTU is defined as NTU = $-UA/(\dot{m}c)_{\min}$. The relationships of ε with NTU for a double-pipe heat exchanger are as follows:
For a parallel flow,

$$\varepsilon_{\text{parallel-flow}} = \frac{1-\exp[(-UA/C_{\min})(1+C_{\min}/C_{\max})]}{1+C_{\min}/C_{\max}}$$

For a counterflow,

$$\varepsilon_{\text{counterflow}} = \frac{1-\exp[(-UA/C_{\min})(1-C_{\min}/C_{\max})]}{1-(C_{\min}/C_{\max})\exp[(-UA/C_{\min})(1-C_{\min}/C_{\max})]}$$

- The heat exchangers which have a large heat transfer area per unit volume, typically above 700 m^2/m^3, are referred to as compact heat exchangers. The large heat transfer area is usually provided by the arrangement of fins.
- The selection of a heat exchanger for a particular application depends upon several factors like heat transfer requirement, cost, pumping power, size and weight and materials.

REVIEW QUESTIONS

1. What are the different modes of heat transfer from hot fluid to cold fluid in a double-pipe heat exchanger?
2. Under what conditions can the overall heat transfer coefficient be written as $U = \left(\frac{1}{h_i} + \frac{1}{h_o}\right)^{-1}$?
3. How does the fouling effect rate of heat transfer and pressure drop in a heat exchanger?
4. How does the log mean temperature difference for a heat exchanger differ from the arithmetic mean temperature difference? For specified inlet and outlet temperatures, which one of these two quantities is larger?
5. Can the outlet temperature of a cold fluid in a heat exchanger be higher than the outlet temperature of hot fluid in a parallel flow heat exchanger? Can it be in a counterflow heat exchanger?
6. Consider two heat exchangers. One has a parallel-flow arrangement and another has a counterflow arrangement. For the same inlet and outlet temperatures for the two exchangers, which one will have the higher value of LMTD?
7. Under what conditions is the effectiveness-NTU method definitely preferred over the LMTD method in the analysis of a heat exchanger?
8. Can the effectiveness of a heat exchanger be greater than one?
9. Consider a heat exchanger in which both fluids have the same specific heats but different mass flow rates. Which fluid will experience a larger temperature change—the one with the lower or the higher mass flow rate?

10. If the length of a counterflow heat exchanger is increased, what will happen to the effectiveness of the exchanger—will it increase or decrease?

11. How do you define the NTU of a heat exchanger? What does it represent? Is it necessary to buy a heat exchanger with a very large value of NTU?

PROBLEMS

10.1 A double-pipe heat exchanger is constructed of a copper inner tube of internal diameter $D_i = 10$ mm and external diameter $D_o = 15$ mm and an outer tube diameter of 35 mm. The connective heat transfer coefficient is reported to be h_i = 700 W/(m^2 K) on the inner surface of the tube and h_o = 1200 W/(m^2 K) on its outer surface. For a fouling factor R_{fi} = 0.0004 (m^2 K)/W on the tube side and R_{fo} = 0.0002 (m^2 K)/W on the shell side, determine (a) the thermal resistance of the heat exchanger per unit length and (b) the overall heat transfer coefficients U_i and U_o based on the inner and outer surface areas of the inner tube respectively. Take the thermal conductivity of copper to be 400 W/(m K).

[**Ans.** (a) 0.064 (m^2 K)/W, (b) 497.61 W/(m^2 K), (c) 331.74 W/(m^2 K)]

10.2 In a double-pipe, counterflow heat exchanger, water flows at the rate of 0.45 kg/s and is heated from 20°C to 35°C by an oil having a specific heat of 1.5 kJ/(kg °C). The oil enters the exchanger at 95°C and exits at 60°C. Determine the heat exchanger area for an overall heat transfer coefficient of U = 290 W/(m^2 °C). The specific heat of water is 4.18 kJ/(kg °C).

[**Ans.** 1.97 m^2]

10.3 In a one-shell pass, one-tube pass (double-pipe) heat exchanger, one fluid enters at 50°C and leaves at 200°C. The other fluid enters at 400°C and leaves at 250°C. Determine the log mean temperature difference for (a) parallel-flow arrangement and (b) counterflow arrangement.

[**Ans.** (a) 154.17°C, (b) 200°C]

10.4 A shell-and-tube heat exchanger cools oil which flows at a rate of 5 kg/s with an inlet temperature of 80°C and the outlet temperature of 40°C. The cold fluid is water which flows at the rate of 10 kg/s with an inlet temperature of 20°C. The overall heat transfer coefficient is 550 W/(m^2 °C). Determine the heat transfer surface area required for (a) a parallel-flow heat exchanger and (b) a counterflow heat exchanger. Take $c_{p,\text{oil}}$ = 2 kJ/(kg °C), $c_{p,\text{water}}$ = 4.2 kJ/(kg °C).

[**Ans.** (a) 25.63 m^2, (b) 22.09 m^2]

10.5 Air flowing at the rate of 1.5 kg/s is to be heated from a temperature of 25°C to 55°C with hot water entering at 90°C and leaving at 60°C. The overall heat transfer coefficient is 400 W/(m^2 °C) with water flowing through the titles. Determine the surface area required by using (a) a one-shell pass and two-tube pass heat exchanger and (b) a two-shell pass and four-tube pass heat exchanger.

[**Ans.** (a) 5.40 m^2, (b) 4.80 m^2]

10.6 It is required to cool 0.5 kg/s of oil from 110°C to 50°C using a counterflow double-pipe heat exchanger. The cooling water is available at 25°C and can leave at 35°C. The oil flows through the copper tube of 12 mm diameter, 1.5 m length and water flows through the annulus. The internal diameter of the outer tube is 30 mm. The inner tube of 3 mm thickness is made of copper [$k = 400$ W/(m K)]. Use the Dittus-Boelter equation to estimate the convective heat transfer coefficients on both sides of the inner tube, and determine the overall heat transfer coefficient of the heat exchanger. Take the following properties of oil and water at their respective bulk temperatures:

$$\rho_{\text{oil}} = 8.64 \text{ kg/m}^3, \qquad c_{p,\text{oil}} = 2.5 \text{ kJ/(kg K)}$$
$$\nu_{\text{oil}} = 0.85 \times 10^{-4} \text{ m}^2\text{/s}, \qquad k_{\text{oil}} = 0.16 \text{ W/(m K)}$$
$$\rho_{\text{water}} = 1000 \text{ kg/m}^3, \qquad c_{\text{water}} = 4.18 \text{ kJ/(kg K)}$$
$$\nu_{\text{water}} = 0.15 \times 10^{-6} \text{ m}^2\text{/s}, \qquad k_{\text{water}} = 0.16 \text{ W/(m K)}$$

[**Ans.** 3.03 kW/(m^2 K)]

10.7 It is desired to cool 0.5 kg/s of oil from 105°C by using an equal flow rate of cooling water. The cooling water is available at 20°C. The specific heat of oil and water are 2.8 kJ/(kg K) and 4.2 kJ/(kg K) respectively. The two double-pipe heat exchangers are available

Heat exchanger 1: $U = 500$ W/(m^2 K); $A = 4.5$ m^2
Heat exchanger 2: $U = 800$ W/(m^2 K); $A = 2$ m^2

Which heat exchanger should be used for a parallel-flow arrangement?

[**Ans.** Exchanger 1]

10.8 A double-pipe counterflow heat exchanger is to cool ethylene glycol [$c_p = 2.56$ kJ/(kg K)] flowing at the rate of 2 kg/s from 80°C to 40°C by water [$c_p = 4.18$ kJ/(kg K)] that enters at 20°C and leaves at 60°C. The overall heat transfer coefficient based on the inner surface area of the tube is 250 W/(m^2 K). Determine (a) the rate of heat transfer, (b) the mass flow rate of water, and (c) the heat transfer surface area on the inner side of the tube.

[**Ans.** (a) 205 kW, (b) 1.22 kg/s, (c) 41 m^2]

10.9 Saturated water vapour leaves a steam turbine at a flow rate of 2 kg/s and a pressure of 50 kPa. The vapour is to be completely condensed to saturated liquid in a shell-and-tube heat exchanger that uses city water as the cold fluid. The water enters the thin-walled tube at 20°C and is to leave at 60°C. Assuming an overall heat transfer coefficient of 2000 W/(m^2 K), determine the required heat exchanger surface area and the water flow rate. After extended operation, fouling causes the overall heat transfer coefficient to decrease to 1000 W/(m^2 K) and therefore for complete condensation of vapour, there must be a reduction in vapour flow rate if the water flow rate remains fixed. Determine this new vapour flow rate.

[**Ans.** 61 m^2, 27.57 kg/s, 1.25 kg/s]

10.10 A counterflow heat exchanger has an overall heat transfer coefficient of 225 W/(m^2 K) and a surface area of 33 m^2. The hot fluid [$c_p = 3.56$ kJ/(kg K)] enters at 94°C and flows at the rate of 2.52 kg/s. The cold fluid [$c_p = 1.67$ kJ/(kg K)] enters at 16°C and flows at the rate of 2.27 kg/s. Determine the rate of heat transfer.

[**Ans.** 231 kW]

10.11 Water enters a crossflow single-pass heat exchanger (both fluids unmixed) at 20°C and flows at the rate of 8 kg/s to cool 10 kg/s of air from 120°C. From an overall heat transfer coefficient of 230 W/(m^2 K), and an exchanger surface area of 250 m^2, what is the exit air temperature? Take $c_{p,\text{water}}$ = 4.18 kJ/(kg K), $c_{p,\text{air}}$ = 1.0 kJ/(kg K).

[**Ans.** 23.4°C]

10.12 Consider an oil-to-oil double-pipe heat exchanger whose flow arrangement is not known. The temperature measurements indicate that the cold oil enters at 20°C and leaves at 55°C, while the hot oil enters at 80°C and leaves at 45°C. Is it a parallel-flow or counter-flow heat exchanger? Assuming the mass flow rate of both fluids to be the same, determine the effectiveness of the heat exchanger [$c_{p,\text{oil}}$ = 2.2 kJ/(kg K)].

[**Ans.** Counterflow, 0.583]

10.13 Ethanol is vaporized at 78°C [h_{fg} = 846 kJ/(kg K)] in a double-pipe parallel-flow heat exchanger at a rate of 0.03 kg/s by hot oil [c_p = 2.2 kJ/(kg K)] that enters at 120°C. If the heat transfer surface area and the overall heat transfer coefficient are 8 m^2 and 200 W/(m^2 k) respectively, determine the outlet temperature and the mass flow rate of the oil using the ε-NTU method.

[**Ans.** 0.3 kg/s, 81.54°C]

10.14 A counterflow double-pipe heat exchanger is currently used to heat 2.5 kg/s of water from 25°C to 65°C by cooling an oil [c_p = 2.2 kJ/(kg K)] from 150°C to 95°C. It is desired to 'bleed off' 0.6 kg/s of water at 45°C so that the single exchanger will be replaced by a two-exchanger arrangement which will permit this. The overall heat transfer coefficient is 400 W/(m^2 K) for the single exchanger and may be taken as this same value for each of the two smaller exchangers. The same oil flow is used for the two-exchanger arrangement, except that the flow is split between the two exchangers. Determine the areas of each of the smaller exchangers and the oil flow through each. Assume that the water flows in series through the two exchangers with the bleed-off taking place between them. Assume that the two smaller exchangers have the same areas. Take $c_{p,\text{water}}$ = 4.18 kJ/(kg K).

[**Ans.** 5.75 m^2, 2.01 kg/s, 1.44 kg/s]

REFERENCES

[1] Kays, W.M., and A.L. London, *Compact Heat Exchangers*, 2nd ed., McGraw-Hill, New York, 1964.

[2] *Standards of the Tubular Exchanger Manufacturers Association*, 6th ed., Tubular Exchange Manufacturers Association, New York, 1978.

[3] Bowman, R.A., A.C. Mueller, and W.M. Nagle, Mean Temperature Difference in Design, *Transactions of the ASME*, 62 (1940), p. 283.

11

Radiation Heat Transfer

In preceding chapters we observed that heat is transferred by two different modes—conduction and convection. The transfer of heat by these two modes takes place essentially through a medium (a material body) in the direction of decreasing temperature. On the other hand, our experience tells us that when a hot object is placed in an evacuated chamber whose walls are at room temperature, the object eventually cools down and reaches thermal equilibrium with its surroundings. Heat transfer between the object and the chamber could not have taken place by conduction or convection, since these two mechanisms cannot occur in a vacuum. Another experience tells us that if we place an object in front of a heat source (say a flame), the object gets heated in spite of the fact that the ambient air separating the object and the heat source remains cold.

The above two examples tell us that there is a mode of heat transfer which can take place even without a medium, and if a medium is present it is not necessary that the heat flow has to be accompanied by a continuous decrease in temperature of the medium from the hot body temperature to that of the colder one. This mode of heat transfer is known as radiation heat transfer. This chapter discusses the basic principles of radiation heat transfer.

Learning Objectives

The reading of this chapter will enable the students

- to know the basic mechanism of radiation heat transfer,
- to know and identify the different radiation properties of a body (or surface) and their role in radiation heat transfer,
- to understand the basic qualitative and quantitative principles governing the rate of radiation heat transfer between different bodies (or surfaces),
- to understand the concept of radiation shield, and
- to have the basic information about solar radiation.

11.1 PHYSICAL MECHANISM

All matters above a temperature of absolute zero emit their sensible internal energy in all directions. This emission of energy is referred to as radiation. There are two theories which explain the phenomenon of radiation from matters. One theory views radiation as the propagation of a collection of particles termed *photons* or *quanta*. In another view, radiation is considered as the propagation of electromagnetic waves and hence it is called electromagnetic radiation. While the theory of photons was proposed by Max Planck, the theory of electromagnetic wave was given by James Clerk Maxwell. Both concepts have been utilized to describe the emission and propagation of radiation.

When radiation is treated as an electromagnetic wave, the radiation from a body at an absolute temperature T is considered to have been emitted at all wavelengths from 0 to ∞. The wave nature of thermal radiation implies that the wavelength λ should be related to the frequency of radiation ν as

$$\lambda = \frac{c}{\nu} \tag{11.1}$$

where c is the speed of propagation in the medium. If the medium of radiation is vacuum, the speed c equals the speed of light, that is, $2{\cdot}9979 \times 10^8$ m/s. Figure 11.1 shows the subdivisions of radiation on the electromagnetic wave spectrum.

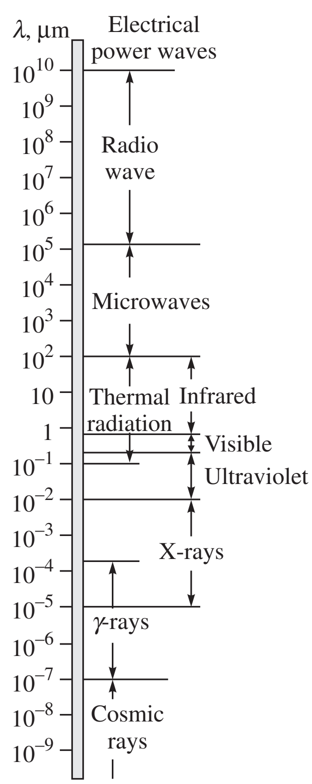


Figure 11.1 Electromagnetic wave spectrum.

The portion of electromagnetic radiation that extends from about 0·1 μm to 100 μm in wavelength is known as thermal radiation. The radiation emitted by bodies because of their temperature falls almost entirely into this wavelength range. Light or visible portion of the electromagnetic spectrum lies in a narrow band of 0·40 μm to 0·76 μm. The radiation emitted within this band makes the sensation to human eye to see the light. The body that emits radiation in the visible range is called a light. The electromagnetic radiation emitted by the sun is known as solar radiation and falls into the wavelength band of 0·3 μm to 3 μm.

The radiation emitted by bodies at room temperature falls into infrared region of the spectrum which extends from 0·76 μm to 100 μm. The emission in the wavelength band of 0·01 μm to 0·4 μm is ultraviolet radiation which kills microorganism and causes serious damage to human and other living organism. About 12 per cent of solar radiation is in the ultraviolet range, and it would be devastating if it were to reach the surface of the earth. Fortunately the ozone (O_3) layer in the upper atmosphere acts as a blanket and absorbs most of this ultraviolet radiation. The thermal radiation (0·1 μm to 100 μm) includes the entire visible and infrared radiation as well as a portion of the ultraviolet radiation (Figure 11.1).

The radiation is, in general, a volumetric phenomenon. This is because the electrons, atoms and molecules of all solids, liquids and gases above absolute zero temperature are in constant motion and hence the energy is constantly emitted, absorbed and transmitted throughout the entire volume of the matter. However for some materials like metals, wood and rocks, radiation is considered to be a surface phenomenon, since the radiation emitted by the interior regions can never reach the surface, and the radiation incident on such bodies is usually absorbed within a few microns from the surface. These materials or bodies are termed opaque to the radiation.

11.2 CONCEPT OF RADIATION INTENSITY AND EMISSIVE POWER

The radiation from a surface is emitted in all possible directions. It is of interest to know the amount of radiation emitted by a body streaming into a given direction. Let us consider the emission from an elemental area dA_1, in a particular direction $\vec{\Omega}$ specified by polar and azimuthal angles θ and ϕ respectively (Figure 11.2) with respect to a spherical polar coordinate system. The spectral radiation intensity is defined as the rate of radiation energy emitted by the body at a temperature T, streaming through a unit area perpendicular to the direction of propagation, per unit wave length interval $d\lambda$ about the wavelength λ per unit solid angle about the direction of the beam. Here the term 'solid angle' has to be understood clearly. If we consider an elemental area dA_Ω perpendicular to the direction $\vec{\Omega}$ and at a radial distance r from the point O on the surface dA (Figure 11.2), then the solid angle $d\omega$ at point O about the direction $\vec{\Omega}$ is defined as

$$d\omega = \frac{dA_\Omega}{r^2} \tag{11.2}$$

Thus solid angle implies the sense of a region which contains all the radiation beams emitted from point O and passing through the area dA_Ω. If δq is the amount of radiation energy from the surface dA_1, then we can write

$$I(\lambda,\ \theta,\ \phi,\ T) = \frac{\delta q(T)}{(dA_1 \cos\theta)(d\omega)(d\lambda)} \tag{11.3}$$

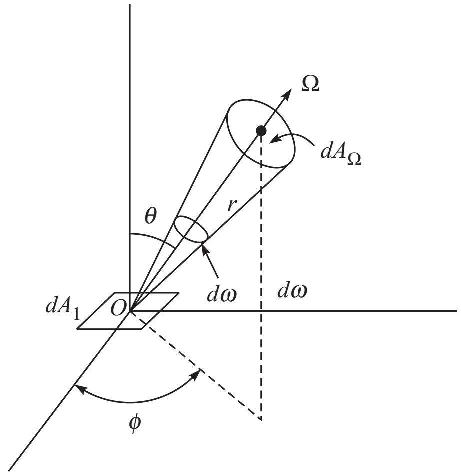


Figure 11.2 Directional nature of radiation.

where $I(\lambda, \theta, \phi, T)$ is the spectral radiation intensity. The arguments λ, θ, ϕ and T imply the dependence of radiation intensity on wavelength, direction and temperature. Denoting $\delta q_\lambda(T)$ (= $\delta q(T)/d\lambda$) as the emission of radiation energy per unit wavelength interval about the wavelength λ, we can write from Eq. (11.3)

$$\delta q_\lambda(T) = I(\lambda, \theta, \phi, T) dA_1 \cos\theta \cdot d\omega \tag{11.4}$$

The solid angle $d\omega$ can be written in a spherical polar coordinate system (Figure 11.3) as

$$\begin{aligned} d\omega &= \frac{dA_\Omega}{r^2} \\ &= \frac{r^2 \sin\theta \, d\theta \, d\phi}{r^2} \\ &= \sin\theta \, d\theta \, d\phi \end{aligned} \tag{11.5}$$

Substituting for $d\omega$ from Eq. (11.5) in Eq. (11.4), we have

$$\delta q_\lambda(T) = I(\lambda, \theta, \phi, T) dA_1 \sin\theta \cos\theta d\theta \, d\phi \tag{11.6}$$

Now we introduce the concept of spectral emissive power E_λ as the total rate of radiation energy of wavelength λ leaving a surface in all directions per unit surface area per unit interval $d\lambda$ of the wavelength λ. Therefore, if we integrate Eq. (11.6) with respect to the direction of emission, we get the expression for emissive power. Equation (11.6) is integrated for this purpose over a hypothetical hemisphere above the elemental area dA_1 and we can write

$$E_\lambda = \frac{1}{dA_1} \int\int \delta Q_1(T) = \int_0^{2\pi} \int_0^{\pi/2} I(\lambda, \theta, \phi, T) \sin\theta \cos\theta \, d\theta \, d\phi \tag{11.7}$$

The term E_λ is thus called spectral hemispherical emissive power. The total hemispherical emissive power is the rate at which the radiation energy is emitted per unit area in all possible directions and at all possible wavelengths.

Therefore,
$$E = \int_0^{\infty} E_\lambda d_\lambda \tag{11.8}$$

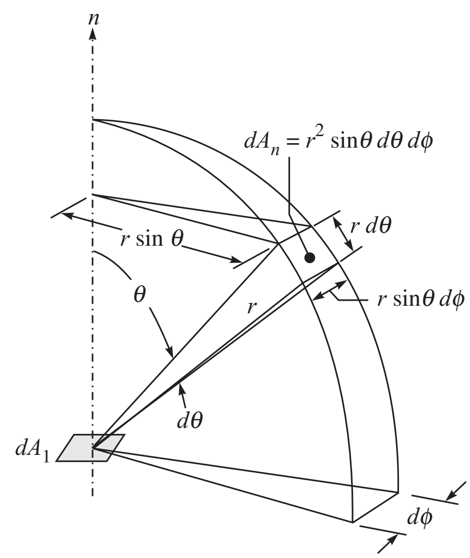


Figure 11.3 The solid angle subtended by dA_Ω at a point on dA_1 in a spherical coordinate system.

11.3 BLACKBODY RADIATION

A body at a temperature above absolute zero emits radiation in all directions over a wide range of wavelength. The amount of radiation from the surface of a body at a given temperature and at a given wavelength depends much on the material of the body and the nature of its surface. Therefore, different bodies emit different amount of radiation even under identical temperature condition. For the purpose of comparison of radiative properties of real surfaces, a concept of an idealized surface has been made which is a perfect emitter and absorber of radiation. This ideal surface is known as a blackbody which has the following characteristic features:

(i) At a specified temperature and wavelength a blackbody emits more radiation energy than any real body.
(ii) A blackbody absorbs all incident radiation regardless of wavelength and direction.
(iii) A blackbody emits radiation energy uniformly in all directions. That is, a blackbody is a diffuse emitter (Figure 11.4). The term 'diffuse' means 'independent of direction'.

Since a blackbody emits radiation uniformly in all direction, we can write for spectral emissive power of a blackbody from Eq. (11.7) as

$$\begin{aligned} E_{b\lambda} &= I_b(\lambda, T) \int_0^{2\pi} \int_0^{\pi/2} \sin\theta \cos\theta\, d\theta\, d\phi \\ &= \pi I_b(\lambda, T) \end{aligned} \tag{11.9}$$

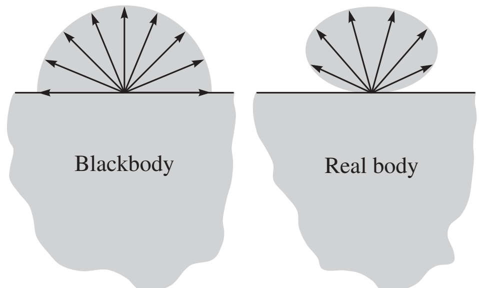


Figure 11.4 Emission from a blackbody and a real body.

The expression for $I_b(\lambda, T)$ for a blackbody emitting into a vacuum was first determined by Planck [1] and is given by

$$I_b(\lambda, T) = \frac{2hc^2}{\lambda^5\{\exp(hc/\lambda kT) - 1\}} \tag{11.10}$$

where

c is the speed of light (= 2·9979 × 10^8 m/s)
h is the Planck constant (= 6·6256 × 10^{-34} J s)
k is the Boltzmann constant (= 1·38054 × 10^{-23} J/K)

Substituting the expression for $I_b(\lambda, T)$ in Eq. (11.9), we have

$$E_{b\lambda} = \frac{2\pi hc^2}{\lambda^5\{\exp(hc/\lambda kT) - 1\}} \tag{11.11}$$

The variation of spectral emissive power of a blackbody $E_{b\lambda}$ with wavelength λ and temperature T, as given by Eq. (11.11), is shown in Figure 11.5.

It is found from Figure 11.5 that at a given temperature, there is a wavelength at which the spectral emissive power becomes maximum. The value of this wavelength decreases with an increase in temperature in a fashion given by

$$\lambda_{\max} T = 2897.6 \ \mu\text{m K} \tag{11.12}$$

The above relationship is known as Wien's displacement law. The subscript 'max' is used to denote the wavelength at which $E_{b\lambda}$ becomes maximum. It is further observed from Figure 11.5 that as the temperature of a blackbody increases, the maximum emission becomes close to the visible region. The radiation emitted by the sun, which is considered to be a blackbody at roughly 5800 K reaches its peak in the visible region of the spectrum. On the other hand, surfaces having temperature less than 800 K emit almost entirely in the infrared region and thus are not visible to the eye unless they reflect light coming from other sources.

The Stefan–Boltzmann law

The total emissive power of a blackbody E_b, i.e. the rate of radiation energy emitted by a blackbody at an absolute temperature T over all wavelengths per unit area, is given by integrating Eq. (11.11) as

$$E_b = \int_0^\infty \frac{2\pi hc^2}{\lambda^5\{\exp(hc/\lambda kT) - 1\}} d\lambda \tag{11.13}$$

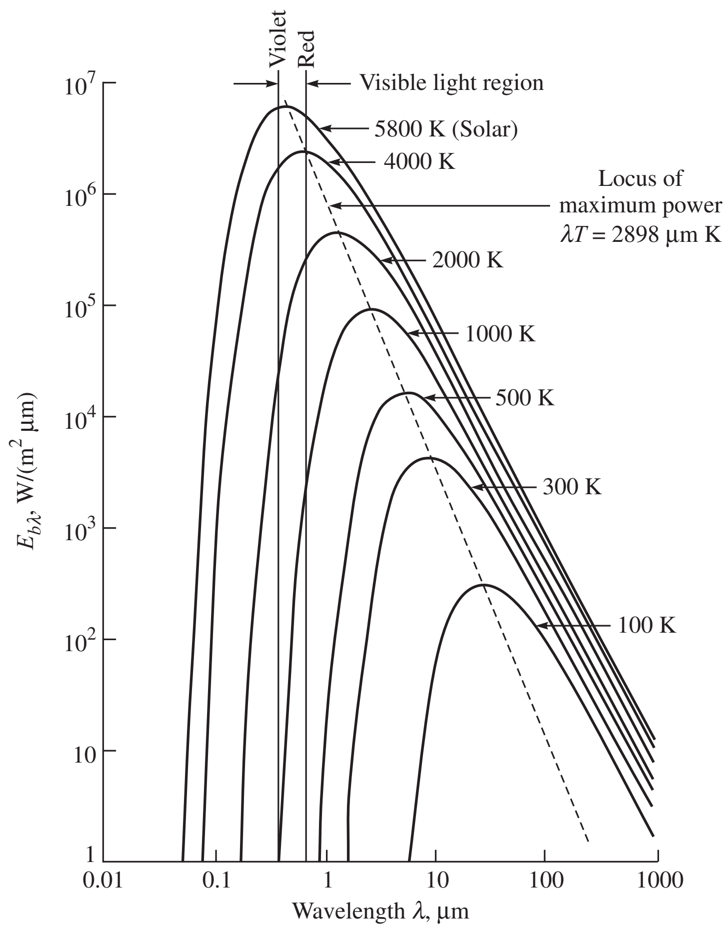


Figure 11.5 The variation of blackbody emissive power with wavelength for different temperatures.

If we change the variable λ as

$$\lambda T = x$$

then we can write Eq. (11.13) as

$$E_b = T^4 \int_0^\infty \frac{2\pi hc^2}{x^5\{\exp(hc/kx) - 1\}} dx$$

After performing the integration, the final result can be written as

$$E_b = \sigma T^4 \tag{11.14}$$

where σ is a constant which is a function of c and k. It is known as Stefan–Boltzmann constant and its value is

$$\sigma = 5.67 \times 10^{-8} \text{ W/(m}^2\text{ K}^4)$$

Equation (11.14) is known as Stefan–Boltzmann law.

Blackbody radiation functions

In practical applications, we are interested in the amount of energy radiated by a blackbody at a given temperature in a certain specified range of wavelength. For this purpose, we introduce the concept of radiation function $f_{0-\lambda}(T)$ which is defined as the ratio of the blackbody emission from $\lambda = 0$ to λ to the total blackbody emission from $\lambda = 0$ to $\lambda = \infty$.

Hence,

$$f_{0-\lambda}(T) = \frac{\int_0^\lambda E_{b\lambda}\, d\lambda}{\int_0^\infty E_{b\lambda}\, d\lambda}$$

$$= \frac{1}{\sigma T^4}\int_0^\lambda E_{b\lambda} d\lambda \tag{11.15}$$

With the help of Eq. (11.11), the above expression (i.e. Eq (11.15)) can be written as

$$f_{0-\lambda}(T) = \frac{c_1}{\sigma}\int_0^{\lambda T} \frac{d\,(\lambda T)}{(\lambda T)^5\,[\exp(c_2/\lambda T) - 1]} \tag{11.16}$$

where

$$c_1 = 2\pi hc^2$$

$$c_2 = \frac{hc}{k}$$

The value of $f_{0-\lambda}(T)$ for a given value of λT can be evaluated by performing the integral of Eq. (11.16). The results of such calculations, i.e. the values of blackbody radiation function $f_{0-\lambda}(T)$ for various values of λT are shown in Table 11.1 and Figure 11.6.

Table 11.1 Blackbody radiation functions

λT	$E_{b\lambda}/T^5$	$\frac{E_{b0-\lambda T}}{\sigma T^4}$	λT	$E_{b\lambda}/T^5$	$\frac{E_{b0-\lambda T}}{\sigma T^4}$
μm·K	$\frac{\text{W}}{\text{m}^2\cdot\text{K}^5\cdot\mu\text{m}\times 10^{11}}$		μm·K	$\frac{\text{W}}{\text{m}^2\cdot\text{K}^5\cdot\mu\text{m}\times 10^{11}}$	
1000	0.02110	0.00032	1900	0.77736	0.05210
1100	0.04846	0.00091	2000	0.87858	0.06672
1200	0.09329	0.00213	2100	0.96994	0.08305
1300	0.15724	0.00432	2200	1.04990	0.10088
1400	0.23932	0.00779	2300	1.11768	0.12002
1500	0.33631	0.01285	2400	1.17314	0.14025
1600	0.44359	0.01972	2500	1.21659	0.16135
1700	0.55603	0.02853	2600	1.24868	0.18311
1800	0.66872	0.03934	2700	1.27029	0.20535

(*Contd.*)

Table 11.1 Blackbody radiation functions (*Contd.*)

λT	$E_{b\lambda}/T^5$	$\dfrac{E_{b0-\lambda T}}{\sigma T^4}$	λT	$E_{b\lambda}/T^5$	$\dfrac{E_{b0-\lambda T}}{\sigma T^4}$
μm·K	$\dfrac{W}{m^2 \cdot K^5 \cdot \mu m \times 10^{11}}$		μm·K	$\dfrac{W}{m^2 \cdot K^5 \cdot \mu m \times 10^{11}}$	
2800	1.28242	0.22788	6900	0.33940	0.80219
2900	1.28612	0.25055	7000	0.32679	0.80807
3000	1.28245	0.27322	7100	0.31471	0.81373
3100	1.27242	0.29576	7200	0.30315	0.81918
3200	1.25702	0.31809	7300	0.29207	0.82443
3300	1.23711	0.34009	7400	0.28146	0.82949
3400	1.21352	0.36172	7500	0.27129	0.83436
3500	1.18695	0.38290	7600	0.26155	0.83906
3600	1.15B06	0.40359	7700	0.25221	0.84359
3700	1.12739	0.42375	7800	0.24326	0.84796
3800	1.09544	0.44336	7900	0.23468	0.85218
3900	1.06261	0.46240	8000	0.22646	0.85625
4000	1.02927	0.48085	8100	0.21857	0.86017
4100	0.99571	0.49872	8200	0.21101	0.86396
4200	0.96220	0.51599	8300	0.20375	0.86762
4300	0.92892	0.53267	8400	0.19679	0.87115
4400	0.89607	0.54877	8500	0.19011	0.87456
4500	0.86376	0.56429	8600	0.18370	0.87786
4600	0.83212	0.57925	8700	0.17755	0.88105
4700	0.80124	0.59366	8800	0.17164	0.88413
4800	0.77117	0.60753	8900	0.16596	0.88711
4900	0.74197	0.62088	9000	0.16051	0.88999
5000	0.71366	0.63372	9100	0.15527	0.89277
5100	0.68628	0.64606	9200	0.15024	0.89547
5200	0.65983	0.65794	9300	0.14540	0.89807
5300	0.63432	0.66935	9400	0.14075	0.90060
5400	0.60974	0.68033	9500	0.13627	0.90304
5500	0.58608	0.69087	9600	0.13197	0.90541
5700	0.54146	0.71076	9700	0.127B3	0.90770
5800	0.52046	0.72012	9800	0.123B4	0.90992
5900	0.50030	0.72913	9900	0.12001	0.91207
6000	0.48096	0.73778	10,000	0.11632	0.91415
6100	0.46242	0.74610	10,200	0.10934	0.91813
6200	0.44464	0.75410	10,400	0.10287	0.92188
6300	0.42760	0.76180	10,600	0.09685	0.92540
6400	0.41128	0.76920	10,800	0.09126	0.92872
6500	0.39564	0.77631	11,000	0.08606	0.93184
6600	0.38066	0.78316	11,200	0.08121	0.93479
6700	0.36631	0.78975	11,400	0.07670	0.93758
6800	0.35256	0.79609	11,600	0.07249	0.94021

(*Contd.*)

Table 11.1 Blackbody radiation functions (*Contd.*)

λT	$E_{b\lambda}/T^5$	$\dfrac{E_{b0-\lambda T}}{\sigma T^4}$	λT	$E_{b\lambda}/T^5$	$\dfrac{E_{b0-\lambda T}}{\sigma T^4}$
μm·K	$\dfrac{W}{m^2 \cdot K^5 \cdot \mu m \times 10^{11}}$		μm·K	$\dfrac{W}{m^2 \cdot K^5 \cdot \mu m \times 10^{11}}$	
11,800	0.06856	0.94270	19,200	0.01285	0.98387
12,000	0.06488	0.94505	19,400	0.01238	0.98431
12,200	0.06145	0.94728	19,600	0.01193	0.98474
12,400	0.05823	0.94939	19,800	0.01151	0.98515
12,600	0.05522	0.95139	20,000	0.01110	0.98555
12,800	0.05240	0.95329	21,000	0.00931	0.98735
13,000	0.04976	0.95509	22,000	0.00786	0.98886
13,400	0.04494	0.95843	23,000	0.00669	0.99014
13,600	0.04275	0.95998	24,000	0.00572	0.99123
13,800	0.04069	0.96145	25,000	0.00492	0.99217
14,000	0.03875	0.96285	26,000	0.00426	0.99297
14,200	0.03693	0.96418	27,000	0.00370	0.99367
14,400	0.03520	0.96546	28,000	0.00324	0.99429
14,600	0.03358	0.96667	29,000	0.00284	0.99482
14,800	0.03205	0.96783	30,000	0.00250	0.99529
15,000	0.03060	0.96893	31,000	0.00221	0.99571
15,200	0.02923	0.96999	32,000	0.00196	0.99607
15,400	0.02794	0.97100	33,000	0.00175	0.99640
15,600	0.02672	0.97196	34,000	0.00156	0.99669
15,800	0.02556	0.97288	35,000	0.00140	0.99695
16,000	0.02447	0.97377	36,000	0.00126	0.99719
16,200	0.02343	0.97461	37,000	0.00113	0.99740
16,400	0.02245	0.97542	38,000	0.00103	0.99759
16,600	0.02152	0.97620	39,000	0.00093	0.99776
16,800	0.02063	0.97694	40,000	0.00084	0.99792
17,000	0.01979	0.97765	41,000	0.00077	0.99806
17,200	0.01899	0.97834	42,000	0.00070	0.99819
17,400	0.01823	0.97899	43,000	0.00064	0.99831
17,600	0.01751	0.97962	44,000	0.00059	0.99842
17,800	0.01682	0.98023	45,000	0.00054	0.99851
18,000	0.01617	0.98081	46,000	0.00049	0.99861
18,200	0.01555	0.98137	47,000	0.00046	0.99869
18,400	0.01496	0.98191	48,000	0.00042	0.99877
18,600	0.01439	0.98243	49,000	0.00039	0.99884
18,800	0.01385	0.98293	50,000	0.00036	0.99890
19,000	0.01334	0.98340			

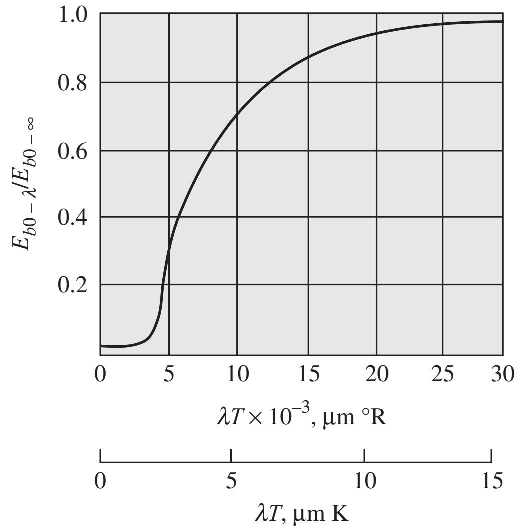


Figure 11.6 The variation of blackbody radiation function with λT.

11.4 RADIATION PROPERTIES OF SURFACES

The radiation energy may be incident on a surface from a source. In such case a part of the energy incident on the surface is reflected, a part is absorbed and a part is transmitted (Figure 11.7). The relative proportions of these parts depend on the nature of the surface. The

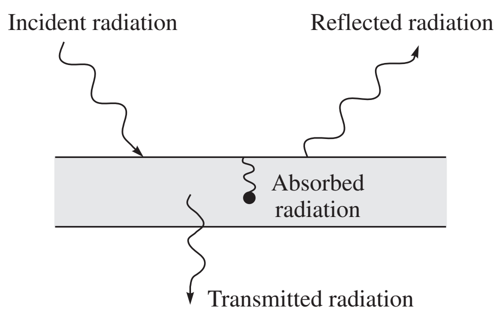


Figure 11.7 The absorption, reflection and transmission of incident radiation by a material.

reflection of energy may be specular or diffuse, as shown in Figure 11.8, depending upon the type of surface. However, no real surface is either specular or diffuse. A smooth and polished surface is more specular while a rough surface exhibits mostly the diffuse radiation.

The fraction of incident energy absorbed by the surface is called the absorptivity α, the fraction reflected by the surface is known as reflectivity ρ, and the fraction transmitted is called the transmissivity τ. Thus,

Absorptivity, $$\alpha = \frac{\text{Absorbed radiation energy}}{\text{Incident radiation energy}} \tag{11.17a}$$

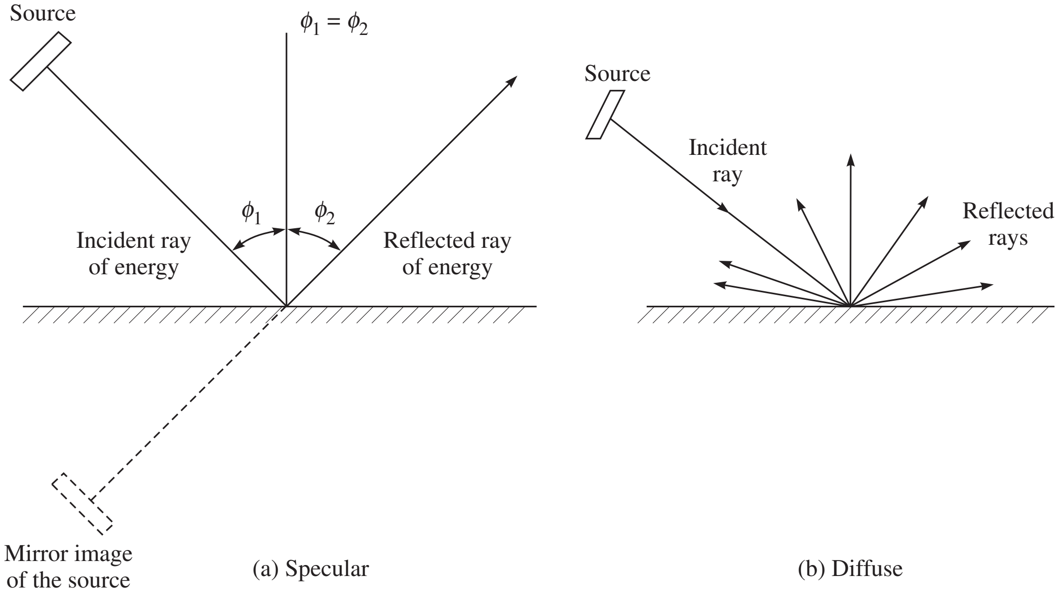


Figure 11.8 Reflection of incident radiation energy by a surface.

Reflectivity,
$$\rho = \frac{\text{Reflected radiation energy}}{\text{Incident radiation energy}} \tag{11.17b}$$

Transmissivity,
$$\tau = \frac{\text{Transmitted radiation energy}}{\text{Incident radiation energy}} \tag{11.17c}$$

It is apparent from the above definition that all the quantities α, ρ and τ lie between 0 and 1, and $\alpha + \rho + \tau = 1$

For a blackbody,

$$\alpha = 1$$

and
$$\rho = \tau = 0$$

No real surface has the properties of blackbody. The closest approximation is achieved by a cavity with a small aperture and internal surface at uniform temperature. If any radiation enters the cavity through the small aperture, then the incident ray gets reflected back and forth by the internal surface (Figure (11.9)) before coming out from the cavity. Hence it is almost entirely absorbed by the cavity, and the blackbody behaviour is approximated.

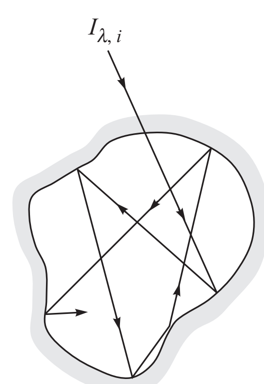


Figure 11.9 An isothermal blackbody cavity.

Kirchhoff's identity

The emissive power of a body E is defined as the energy emitted by the body per unit area per unit time.

The emissive power thus depends upon the temperature of the body and wavelength of radiation. Therefore, we specify the emissive power for a given wavelength λ and at a given temperature T as $E_{\lambda T}$ which is known as spectral hemispherical emissive power. The word hemispherical bears the sense that the radiations in all directions are taken care of by integrating the radiation with respect to direction over a hypothetical hemisphere above the surface. The ratio of spectral hemispherical emissive power of a body to that of a black body is defined as spectral emissivity $\varepsilon_{\lambda,T}$ of the body.

Therefore,

$$\varepsilon_{\lambda,T} = \frac{E_{\lambda,T}}{E_{b\lambda,T}} \tag{11.18}$$

If we integrate the emissive power over all wavelength, we get the total emissive power E_T and the emissivity ε_T is then termed total emissivity. Thus,

$$\varepsilon_T = \frac{E_T}{E_{b_T}} = \frac{\int_0^\infty E_{\lambda,T}\, d\lambda}{\int_0^\infty E_{b\lambda,T}\, d\lambda} \tag{11.19}$$

We know that the emissive power of a blackbody is more than any other body, and therefore emissivity of any body (or surface) is always less than unity.

A body (or surface) whose radiation properties are independent of wavelength is known as gray body. Therefore the emissivity of a gray body is same at all wavelengths, and hence the spectral emissivity is equal to the total emissivity.

Let us consider a perfectly black enclosure as shown in Figure 11.10. Suppose that a sample body is placed at a given position in the enclosure and is allowed to come to thermal equilibrium with the enclosure. At equilibrium, there will be no net energy inflow to or outflow from the body which will keep the temperature of the body at a constant value to that of the enclosure. Therefore, we can write

$$E_T A = q_{i,T} A \alpha \tag{11.20}$$

where $q_{i,T}$ is the incident radiation flux on the body, A is the surface area of the body, and α is the absorptivity.

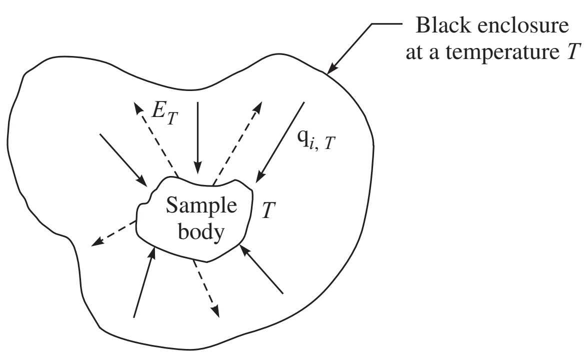


Figure 11.10 Physical model for the derivation of Kirchhoff's identity.

If the sample body is replaced by a blackbody of same surface area A and at the same temperature T, then we can write for thermal equilibrium

$$E_{b_T} A = q_{i,T} A \tag{11.21}$$

From Eqs. (11.20) and (11.21), we find

$$\frac{E_T}{E_{b_T}} = \alpha$$

The left-hand side of the above equation is the emissivity of the body ε.

Therefore, we have

$$\varepsilon = \alpha \tag{11.22}$$

Equation (11.22) is known as Kirchhoff's identity and is stated as 'at thermal equilibrium, the emissivity of a body (or surface) is equal to its absorptivity.

11.5 VIEW FACTOR

We have so far discussed the radiation energy emitted by a surface at a given temperature. Now if two such surfaces at different temperatures are separated by a medium or are in a vacuum, then there will be a net exchange of radiation energy between them. The magnitude of this energy transfer depends upon two important factors. One is the nature of the medium, i.e. whether the medium is participating or non-participating one. A non-participating medium does not absorb, emit or scatter radiation and hence the radiation exchange between the surfaces is unaffected by the medium. A vacuum, for example, is a perfect non-participating medium.

The second important factor on which the radiation energy exchange between surfaces depends is the view factor. It is our common experience that to make the most use of heat from a fire in a cold night we have to stand as close to the fire as possible blocking as much of the radiation coming from the fire by turning our front to the fire and we do the reverse to avoid heat from a fire if felt. This gives the concept of view factor that determines the amount of energy which leaves a surface and reaches the other. The view factor of a surface i with respect to a surface j is defined as the fraction of energy leaving surface i which reaches surface j, and is denoted as F_{ij}. The view factor is also termed radiation shape factor.

Let us consider two black surfaces of areas A_1 and A_2 as shown in Figure 11.11. If they are at temperatures T_1 and T_2, ($T_1 \neq T_2$) then there will be a net radiation heat exchange between them. The concept of view factor is required to determine this net energy exchange between the surfaces. This is because both the surfaces are diffuse emitters, i.e. emit energy in all directions. Therefore, all the energy emitted by one surface will not reach the other.

The energy leaving surface A_1 and reaching surface $A_2 = E_{b1} A_1 F_{12}$

The energy leaving surface A_2 and reaching surface $A_1 = E_{b2} A_2 F_{21}$

where E_{b1} and E_{b2} are the emissive powers of the two black surfaces and are given by

$$E_{b1} = \sigma T_1^4$$

and

$$E_{b2} = \sigma T_2^4$$

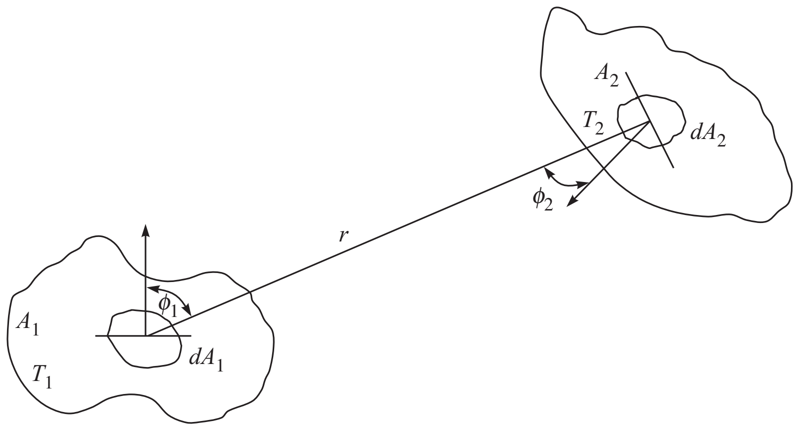


Figure 11.11 The diagram for the derivation of view factor between two elemental surfaces.

Since the surfaces are black, all the incident radiation will be absorbed, and the net energy exchange Q_{1-2} is given by

$$\begin{aligned} Q_{1-2} &= E_{b1} A_1 F_{12} - E_{b2} A_2 F_{21} \\ &= \sigma (T_1^4 A_1 F_{12} - T_2^4 A_2 F_{21}) \end{aligned} \tag{11.23}$$

If both surfaces are at the same temperature ($T_1 = T_2$), then there will be no heat exchange between the surfaces.

Then, we get from Eq. (11.23)

$$Q_{1-2} = \sigma T_1^4 (A_1 F_{12} - A_2 F_{21}) = 0$$

which gives

$$A_1 F_{12} = A_2 F_{21} \tag{11.24}$$

The relationship given by Eq. (11.24) is known as reciprocity relation. Although this relation is derived for black surfaces, it holds for other surfaces too, as long as diffuse radiation is involved.

A general expression for view factor

Let us consider two elemental surfaces dA_1 and dA_2 separated by a distance r as shown in Figure 11.11. Let ϕ_1 and ϕ_2 be the polar angles made by the line r (between the centres of two area elements) with the normal to the elements dA_1 and dA_2 respectively as shown in Figure 11.11. Let I_1 be the intensity of radiation leaving the surface element dA_1 diffusely in all directions, then the amount of radiation energy leaving the surface element dA_1 that strikes the surface element dA_2 can be written as

$$\begin{aligned} \delta Q_{1-2} &= I_1 (dA_1 \cos \phi_1)\, \omega_{1-2} \\ &= \frac{E_{b1} (dA_1 \cos\phi_1)\, \omega_{1-2}}{\pi} \end{aligned} \tag{11.25}$$

In Eq. (11.25), the term $dA_1 \cos \phi_1$ represents the projected area of dA_1 normal to the direction r (joining the two area elements) and the term ω_{1-2} is the solid angle under which an observer at dA_1 sees the surface element dA_2 Therefore, ω_{1-2} can be expressed as

$$\omega_{1-2} = \frac{dA_2 \cos \phi_2}{r^2} \tag{11.26}$$

The numerator $dA_2 \cos \phi_2$ on the right-hand side of above expression is the projection of the area dA_2 normal to direction r.

By making use of Eq. (11.26), we can write Eq. (11.25) as

$$\delta Q_{1-2} = E_{b1} \cos \phi_1 \cos \phi_2 \frac{dA_1\, dA_2}{\pi r^2} \tag{11.27}$$

In a similar way, we can write that the energy leaving the area element dA_2 that reaches dA_1 is

$$\delta Q_{2-1} = E_{b2} \cos \phi_1 \cos \phi_2 \frac{dA_1\, dA_2}{\pi r^2} \tag{11.28}$$

Therefore the net energy exchange between dA_1 and dA_2 becomes

$$\delta Q_{\text{net},1-2} = (E_{b1} - E_{b2}) \cos \phi_1 \cos \phi_2 \frac{dA_1\, dA_2}{\pi r^2} \tag{11.29}$$

The net energy exchange between the finite surfaces A_1 and A_2 then becomes

$$Q_{\text{net},1-2} = (E_{b1} - E_{b2}) \int_{A_2} \int_{A_1} \cos \phi_1 \cos \phi_2 \frac{dA_1\, dA_2}{\pi r^2} \tag{11.30}$$

Following Eqs. (11.23) and (11.24), we conclude that the integral on the right-hand side of Eq. (11.30) is either A_1F_{12} or A_2F_{21}. Hence, we can write

$$F_{12} = \frac{1}{A_1} \int_{A_2} \int_{A_1} \cos \phi_1 \cos \phi_2 \frac{dA_1\, dA_2}{\pi r^2} \tag{11.31}$$

$$F_{21} = \frac{1}{A_2} \int_{A_2} \int_{A_1} \cos \phi_1 \cos \phi_2 \frac{dA_1\, dA_2}{\pi r^2} \tag{11.32}$$

View factor relations

We have already discussed the reciprocity relation given by Eq. (11.24). This is one of the important view factor relations.

The radiative energy leaving a surface may or may not reach the same surface. This depends on the geometry of the surface. Therefore a view factor F_{ii} is defined in this regard which represents the fraction of energy leaving a surface i that reaches the same surface i. In case of a plane or a convex surface (Figure 11.12), F_{ii} becomes zero, while for a concave surface it is nonzero.

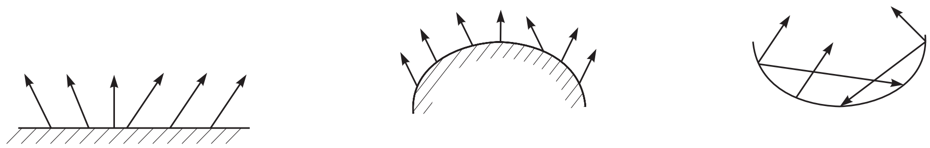


Figure 11.12 The energy emitted from different surfaces.

Therefore, we can write

$$F_{ii} = 0, \qquad \text{for a plane or a convex surface}$$
$$F_{ii} \neq 0, \qquad \text{for a concave surface}$$

We now proceed to determine the relation of view factors between the surfaces forming an enclosure. Let us consider for the purpose, an enclosure formed by N number of surfaces as illustrated in Figure 11.13. It is assumed that the surfaces may be plane, convex or concave and behave as diffuse emitters.

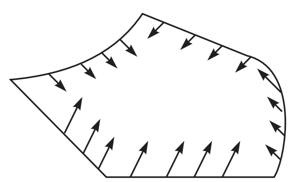


Figure 11.13 An *N*-surface enclosure.

Each surface of the zone has view factors with respect to all the surfaces of the enclosure including itself. Therefore, for an enclosure with N surfaces, we have N^2 view factors which can be expressed in the form of a matrix as

$$F_{i,j} = \begin{bmatrix} F_{11}\,F_{12} & \cdots & F_{1N} \\ F_{21}\,F_{22} & \cdots & F_{2N} \\ \vdots & & \vdots \\ F_{N1}\,F_{N2} & \cdots & F_{NN} \end{bmatrix}$$

However, out of these N^2 view factors to be determined, we have a number of constraining relations which reduce the number of unknown view factors to be determined. One set of such relations comes from the reciprocity relations which can be written as

$$A_i F_{ij} = A_j F_{ji} \tag{11.33}$$

The expression given by Eq. (11.33) gives N (N – 1)/2 relations.

Another set of constraining relations comes from the fact that the view factor of a surface, say A_i, of the enclosure, including to itself, when summed up, must be equal to unity by the definition of the view factor. This can be written mathematically as

$$\sum_{j=1}^{N} F_{ij} = 1 \qquad \text{(for all values of } i \text{ from 1 to } N) \tag{11.34}$$

Equation (11.34) is known as the summation rule and gives N additional relations amongst the view factors. Therefore, the total number of view factors that are to be determined for N surface enclosures becomes

$$N^2 - \frac{1}{2} N(N-1) - N = \frac{1}{2} N(N-1)$$

For example, in an enclosure of 5 surfaces ($N = 5$), there will be $5^2 = 25$ view factors. However, we have to determine $\frac{1}{2}(5)(5-1) = 10$ view factors as unknowns from the geometrical configuration, while the remainder 15 view factors can be determined from reciprocity relations and summation rule.

EXAMPLE 11.1 Consider two concentric spheres as shown in Figure 11.14. The outer surface of inner sphere is designated 1 while the inner surface of the outer sphere is designated 2. Determine all the view factors associated with the enclosure

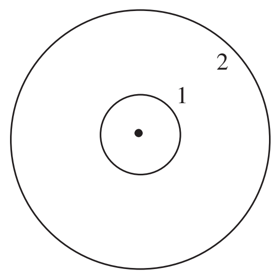


Figure 11.14 Two concentric spheres (Example 11.1).

Solution: Here the enclosure comprises 2 surfaces, and hence, there are $2^2 = 4$ view factors. They are F_{11}, F_{12}, F_{22}, F_{21}.

We know for convex surface i, $F_{ii} = 0$

Therefore we can write, $F_{11} = 0$.

We write the relations from summation rule as

$$F_{11} + F_{12} = 1 \tag{11.35a}$$

$$F_{21} + F_{22} = 1 \tag{11.35b}$$

From reciprocity relation,

$$A_2 F_{21} = A_1 F_{12} \tag{11.36}$$

We have already known $F_{11} = 0$, then from Eq. (11.35a), we have $F_{12} = 1$. This result may also be arrived at from the physical consideration that the surface 1 is completely enclosed by surface 2. From Eq. (11.36), we get

$$F_{21} = \frac{A_1}{A_2}(1)$$

$$= \frac{A_1}{A_2}$$

Again from Eq. (11.35b), we have

$$F_{22} = 1 - F_{21} = 1 - \frac{A_1}{A_2}$$

EXAMPLE 11.2 Consider an elemental surface of area dA_1 and a circular flat disk of area A_2 which are parallel to each other and separated by a distance H as shown in Figure 11.15. The surface dA_1 is designated 1 and surface A_2 is designated 2. Determine F_{12}.

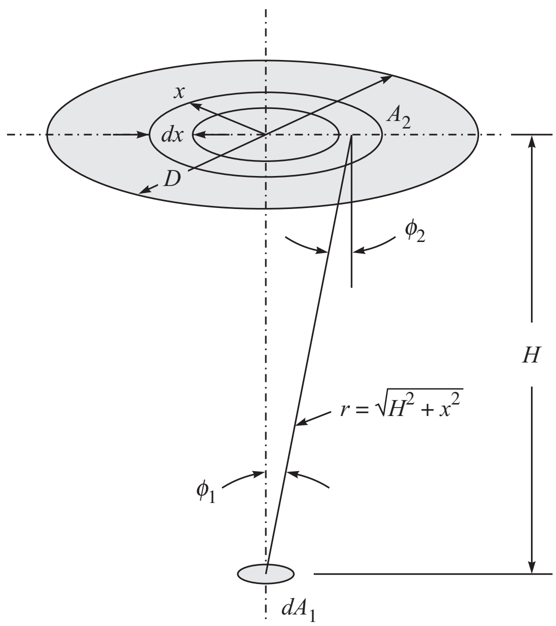


Figure 11.15 Radiation from an elemental surface area to a circular flat disk (Example 11.2).

Solution: A circular ring of radius x and thickness dx is chosen as an element of area dA_2 as shown in Figure 11.15 for the use of Eq. (11.31).

Thus, $$dA_2 = 2\pi x dx$$

Here $$\phi_1 = \phi_2$$

Therefore, we have from Eq. (11.31)

$$F_{12} = \frac{1}{dA_1} dA_1 \int_{A_2} \cos^2 \phi_1 \frac{2\pi x\, dx}{\pi r^2} \tag{11.37}$$

Again we have from the geometry,

$$\cos \phi_1 = \frac{H}{(H^2 + x^2)^{1/2}}$$

and $$r = (H^2 + x^2)^{1/2}$$

Substituting the value of cos ϕ_1 and r from the above equations in Eq. (11.37), we can write

$$F_{12} = H^2 \int_0^{D/2} \frac{2x}{(H^2 + x^2)}\, dx; \text{ where } D \text{ is the diameter of the circular disk.}$$

$$= H^2 \int_0^{D/2} \frac{d(H^2 + x^2)}{(H^2 + x^2)} dx$$

$$= H^2 \left[-\frac{1}{(H^2 + x^2)} \right]_0^{D/2}$$

$$= \frac{D^2}{4H^2 + D^2}$$

Therefore, we have

$$F_{12} = \frac{D^2}{4H^2 + D^2}$$

The determination of view factors between two surfaces involves integration which sometimes becomes difficult to be evaluated analytically. The references [2] to [6] have compiled view factors along with the description of analytical techniques. The view factors for a few configurations are shown in Figures 11.16 to 11.20.

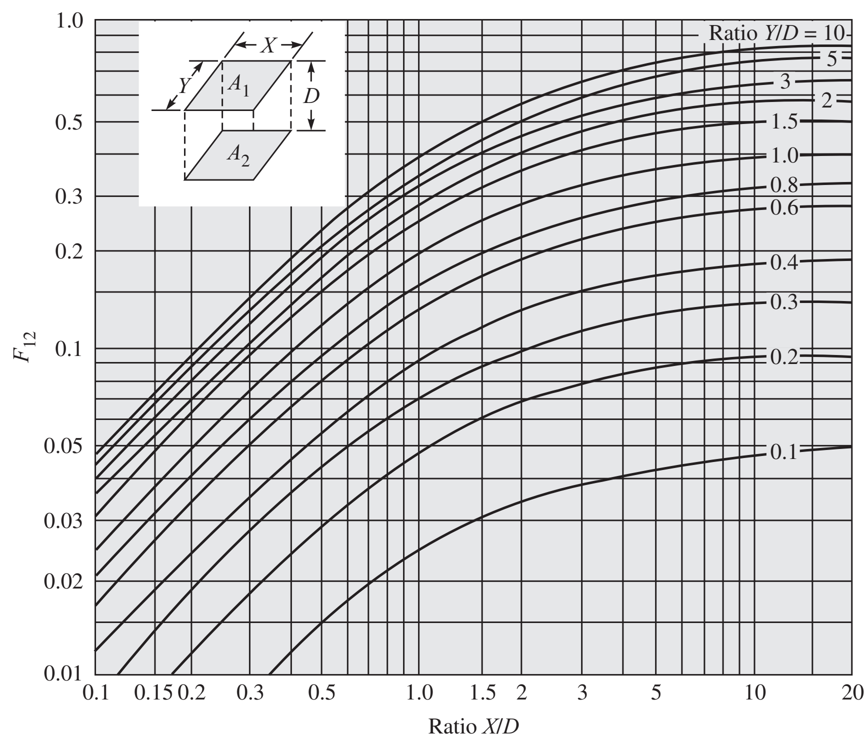


Figure 11.16 Radiation shape factor between parallel rectangles.

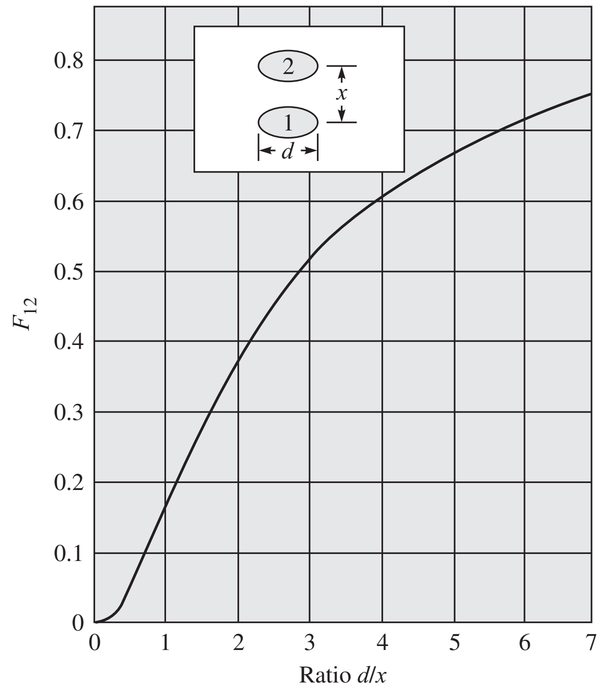


Figure 11.17 Radiation shape factor between two parallel coaxial disks of equal diameter.

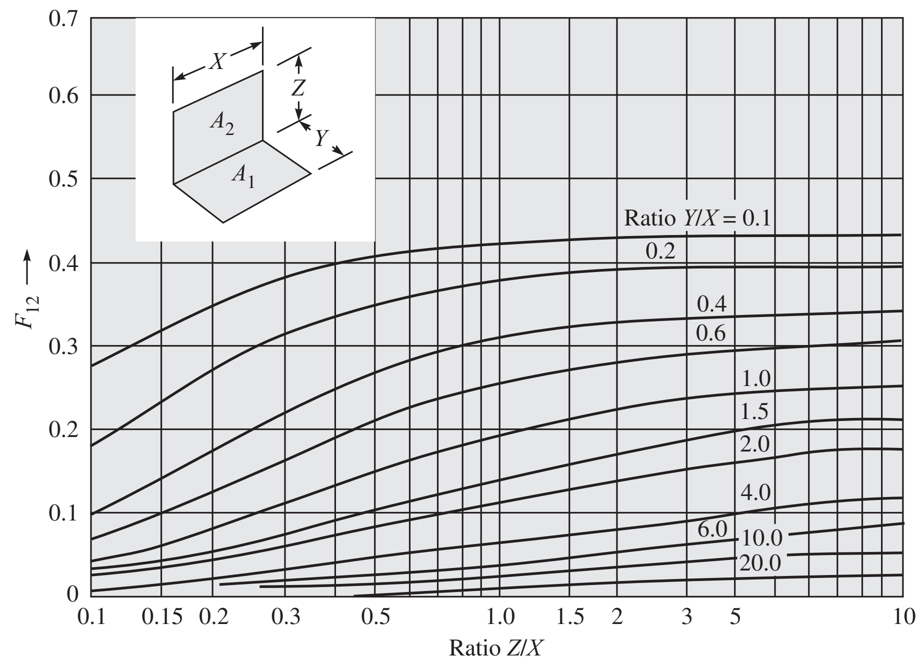
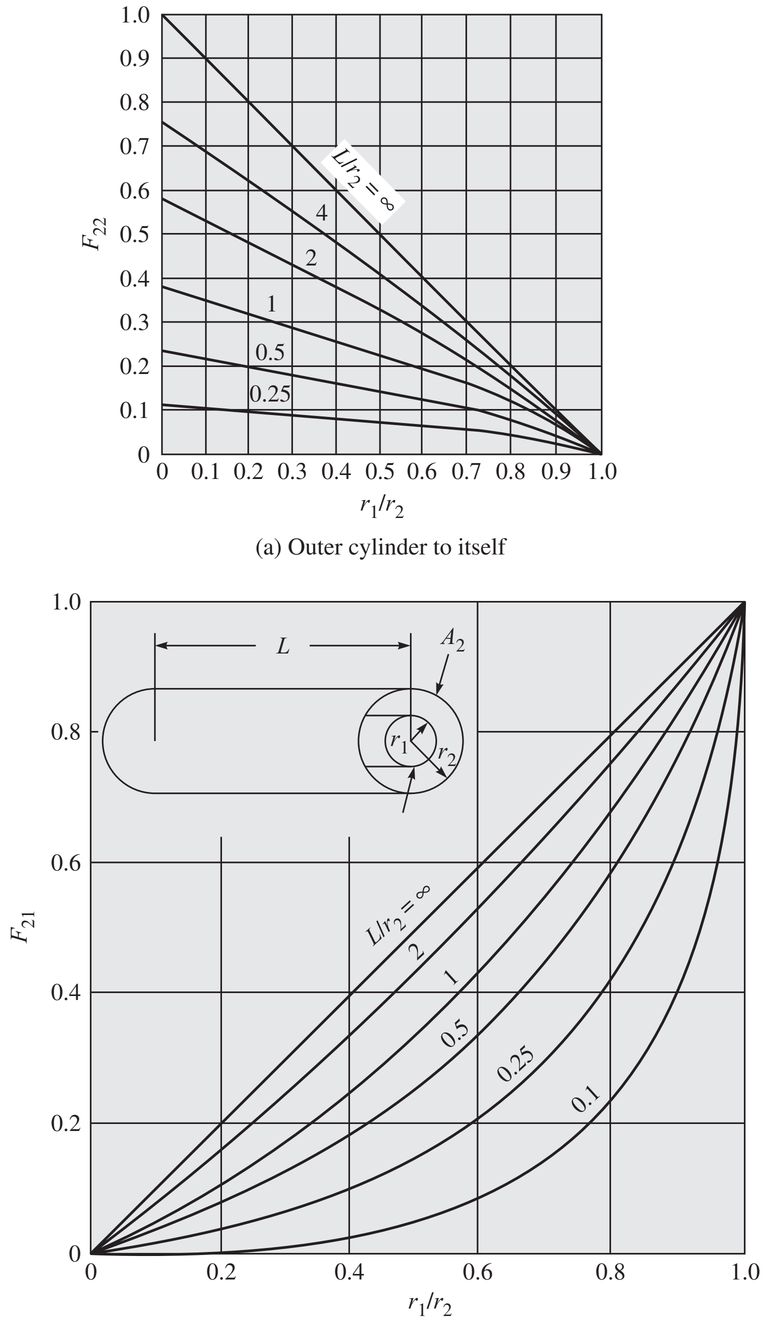


Figure 11.18 Radiation shape factor between two perpendicular rectangles with a common edge.



(a) Outer cylinder to itself

(b) Outer cylinder to inner cylinder

Figure 11.19 Radiation shape factors for two concentric cylinders of finite length.

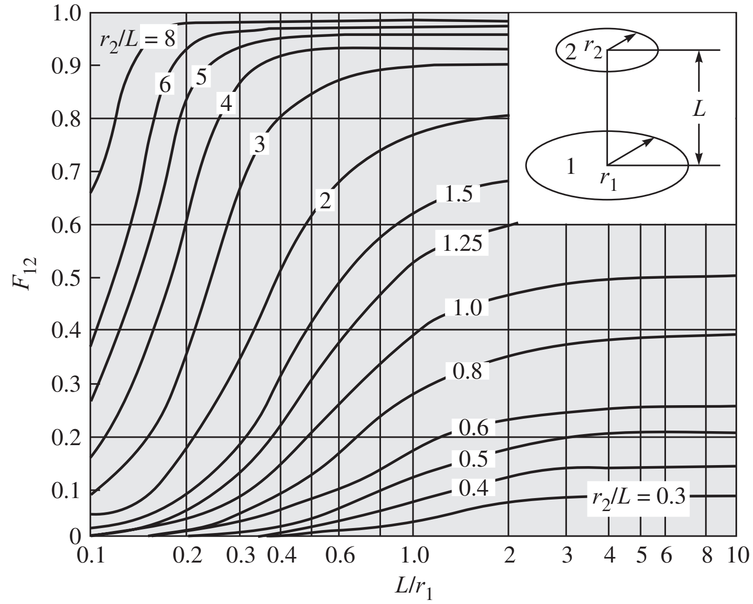


Figure 11.20 Radiation shape factor between two parallel coaxial disks.

The standard relations, figures or charts of view factors are available for only a limited number of simple configurations. However, it is possible to split up the configuration of a complicated geometric arrangement into a number of simple configurations in such a manner that the required view factor may be determined by making use of the standard relations. Such an approach is known as view factor algebra. Let us consider the view factor from an area A_1 to a combined area A_2 and A_3 and vice versa as shown in Figure 11.21.

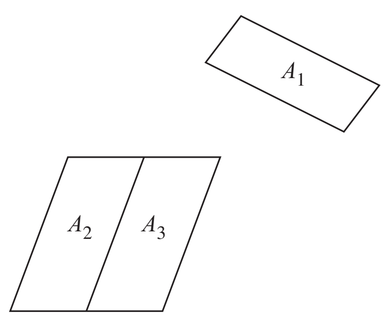


Figure 11.21 Illustration for view factor algebra.

The combined surface is designated as $A_{2,3}$. From the definition of view factor, we can write

$$F_{1-2,3} = F_{12} + F_{13} \tag{11.38}$$

Multiplying both sides of Eq. (11.38) by A_1, we get

$$A_1 F_{1-2,3} = A_1 F_{12} + A_1 F_{13} \tag{11.39}$$

Again from the reciprocity relation, we can write

$$A_1 F_{1-2,3} = A_{2,3} F_{2,3-1}$$
$$A_1 F_{12} = A_2 F_{21}$$
$$A_1 F_{13} = A_3 F_{31}$$

With the help of the above relations, Eq. (11.39) can be written as

$$F_{2,3-1} = \frac{A_2 F_{21} + A_3 F_{31}}{A_{2,3}} \tag{11.40}$$

EXAMPLE 11.3 Determine the view factors F_{13} and F_{31} between the surfaces 1 and 3 as shown in Figure 11.22.

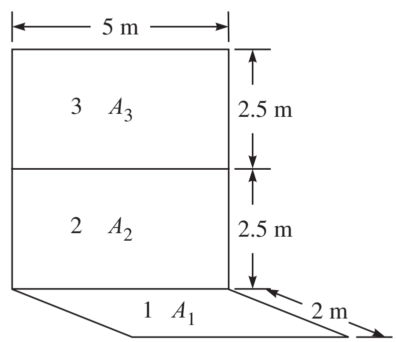


Figure 11.22 Composite surface (Example 11.3).

Solution:

$$F_{1-2,3} = F_{12} + F_{13}$$
$$F_{13} = F_{1-2,3} - F_{12}$$

From Figure 11.18,

$$F_{1-2,3} = 0.31$$
$$F_{12} = 0.27$$

Therefore,

$$F_{13} = 0.31 - 0.27 = 0.04$$

From reciprocity relation,

$$F_{31} = \frac{A_1}{A_3} F_{13}$$
$$= \frac{2}{2.5}(0.04)$$
$$= 0.032$$

EXAMPLE 11.4 Determine the view factor F_{14} for the composite surface shown in Figure 11.23.

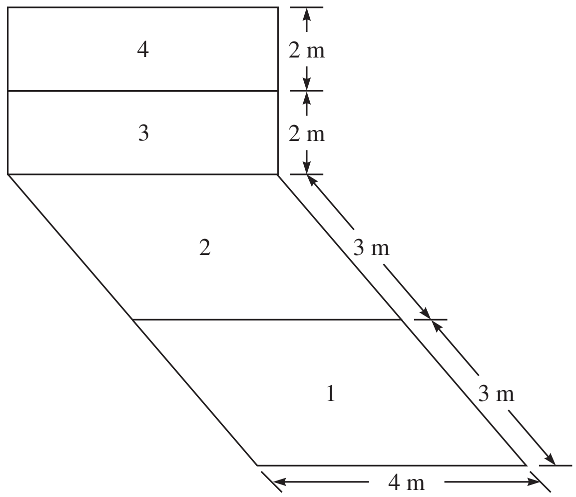


Figure 11.23 Composite surface (Example 11.4).

Solution: From Figure 11.18, $F_{1,2-3,4} = 0.14$

and $F_{1,2-3} = 0.1$

By subdivision of the receiving surface (3 and 4) vide Eq. (11.38),

$$\begin{aligned} F_{1,2-4} &= F_{1,2-3,4} - F_{1,2-3} \\ &= 0.14 - 0.1 \\ &= 0.04 \end{aligned}$$

Again from Figure 11.18,

$$\begin{aligned} F_{2-3,4} &= 0.24 \\ F_{23} &= 0.18 \end{aligned}$$

By subdivision of the receiving surfaces vide Eq. (11.38),

$$\begin{aligned} F_{24} &= F_{2-3,4} - F_{23} \\ &= 0.24 - 0.18 \\ &= 0.06 \end{aligned}$$

Now by subdivision of the emitting surface vide Eq. (11.40),

$$F_{1,2-4} = \frac{1}{(A_1 + A_2)} [A_1 F_{14} + A_2 F_{24}]$$

$$0.04 = \frac{1}{(12 + 12)} [12 F_{14} + 12(0.6)]$$

which gives

$$\begin{aligned} F_{14} &= \frac{24(0.04) - 12(0.06)}{12} \\ &= 0.02 \end{aligned}$$

EXAMPLE 11.5 Consider the cylinder shown in Figure 11.24. Determine the view factor of the cylindrical surface with respect to the base, when $L = 2r$.

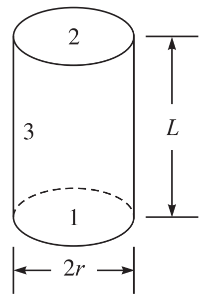


Figure 11.24 The cylinder (Example 11.5).

Solution: Let 1, 2 and 3 denote the base, the top and the cylindrical surfaces respectively. For $L = 2r$, we find from Figure 11.17

$$F_{12} = 0.16$$

By the summation rule of an enclosure,

$$F_{11} + F_{13} + F_{12} = 1$$

But $\quad F_{11} = 0 \quad$ (since the base surface is flat)

Therefore,

$$F_{13} = 1 - F_{12} = 1 - 0.16 = 0.84$$

By making use of the reciprocity theorem, we have

$$F_{31} = \frac{A_1}{A_3} F_{13} = \frac{\pi r^2}{2\pi r(2r)} F_{13} = \frac{1}{4} F_{13} = \frac{0.84}{4} = 0.21$$

EXAMPLE 11.6 Two blackbody rectangles 1·5 m by 3·0 m are parallel and directly opposed as shown in Figure 11.25. They are 3 m apart. If surface 1 is at 127°C and surface 2 at 327°C, determine:

(a) the rate of heat transfer Q_{1-2}
(b) the net rate of energy loss from the surface at 127°C (side facing surface 2 only) if the surrounding other than the two surfaces behaves as a blackbody at (i) 0 K, and (ii) 300 K.

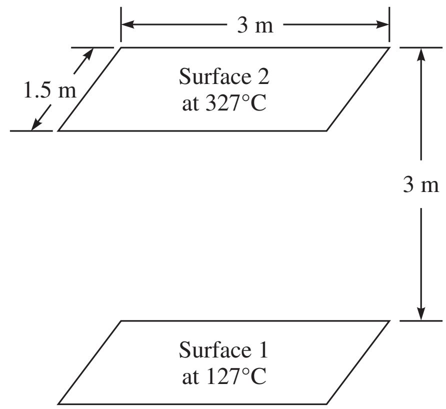


Figure 11.25 Two parallel rectangular surfaces as explained in Example 11.6.

Solution: (a) $Q_{1-2} = A_1F_{12}\sigma(T_1^4 - T_2^4)$

From Figure 11.16, we find

$$F_{12} = 0.11$$

Hence,

$$Q_{12} = (1.5)(3)(0.11)(5.67 \times 10^{-8})[(400)^4 - (600)^4]$$
$$= -2919 \text{ W}$$

The minus sign indicates that the net heat transfer is from surface 2 to surface 1.

(b) (i) Surface 1 receives energy only from surface 2, since the surrounding is at 0 K. Therefore,

$$Q_1 = A_1E_{b1} - A_2F_{21}E_{b2}$$

Since $A_1 = A_2$, we have from reciprocity theorem

$$F_{21} = F_{12}$$

Hence,

$$Q_1 = A_1E_{b1} - A_2F_{12}E_{b2}$$
$$= A_1\sigma(T_1^4 - F_{12}T_2^4)$$
$$= (1.5)(3)(5.67 \times 10^{-8})[(400)^4 - 0.11(600)^4]$$
$$= 2894 \text{ W}$$

(ii) In this case the surrounding is at 300 K and hence the energy received from the surrounding by the surface 1 has to be considered.

Application of the summation rule of view factor gives

$$F_{11} + F_{12} + F_{1s} = 1$$

or

$$F_{1s} = 1 - F_{11} - F_{12}$$
$$= 1 - 0 - 0.11$$
$$= 0.89$$

The subscript s denotes the surrounding.

$$Q_1 = A_1E_{b1} - A_2F_{21}E_{b2} - A_sF_{s1}E_{bs}$$

With the help of reciprocity relations,

$$A_2F_{21} = A_1F_{12}$$
$$A_sF_{s1} = A_1F_{1s}$$

Therefore, we can write

$$\begin{aligned} Q_1 &= A_1(E_{b1} - F_{12}E_{b2} - F_{1s}E_{bs}) \\ &= A_1\sigma\,(T_1^4 - F_{12}\,T_2^4 - F_{1s}\,T_s^4) \\ &= (1.5)\,(3)\,(5.67 \times 10^{-8})\,[(400)^4 - 0.11(600)^4 - 0.89(300)^4] \\ &= 1055\text{ W} \end{aligned}$$

11.6 RADIATION ENERGY EXCHANGE BETWEEN NONBLACK SURFACES

We have considered so far the radiation energy exchange between black bodies. If the bodies exchanging energy are nonblack, then the situation becomes complex. This is because all the energy striking a surface will not be absorbed, part will be reflected back to another heat transfer surface, and part may be reflected out of the system. We have to define in this context, two important terms, namely the *radiosity* and *irradiation*.

Radiosity is defined as the total radiation energy leaving a surface per unit time and per unit area. This is usually expressed by the symbol J.

Irradiation is defined as the total radiation energy incident on a surface per unit time per unit area. This is usually expressed by the symbol G.

Concept of surface resistance to net radiation energy transfer to (or from) a surface

Let us consider a nonblack surface as shown in Figure 11.26. It is assumed that the surface is diffuse and uniform in temperature and the radiation properties are constant over the entire surface. The radiation properties are also considered to be independent of wavelength which means the surface is gray. The radiosity J is the sum of the original emission of the surface due to its temperature and the energy reflected by the surface. Therefore, we can write

$$J = \varepsilon E_b + \rho G$$

where ε and ρ are respectively the emissivity and reflectivity of the surface. If the surface is considered to be opaque, then

$$\rho = 1 - \varepsilon \qquad \text{(since } \tau\text{, the transmissivity} = 0)$$

Hence,

$$J = \varepsilon E_b + (1 - \varepsilon)G \tag{11.41}$$

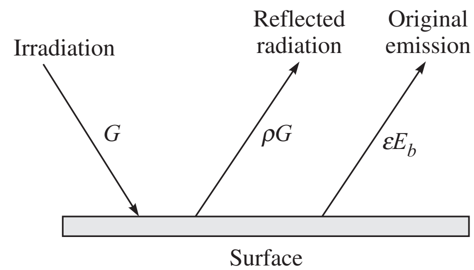


Figure 11.26 The radiation energy interaction by a nonblack surface with its surroundings.

The net energy leaving the surface Q is the difference between the radiosity and the irradiation.

Thus,

$$\begin{aligned}\frac{Q}{A} &= q = J - G \\ &= \varepsilon E_b + (1 - \varepsilon)G - G \end{aligned} \tag{11.42}$$

Eliminating G from Eqs. (11.41) and (11.42), we have

$$Q = \frac{\varepsilon A}{1-\varepsilon}(E_b - J)$$

or

$$Q = \frac{E_b - J}{(1-\varepsilon)/\varepsilon A} \tag{11.43}$$

Now a useful interpretation of Eq. (11.43) is made. The numerator of Eq. (11.43) can be viewed as a potential difference at the surface and the denominator as the surface resistance to the net radiation heat flux from the surface. The thermal network system and the analogues electrical network system are shown in Figure 11.27.

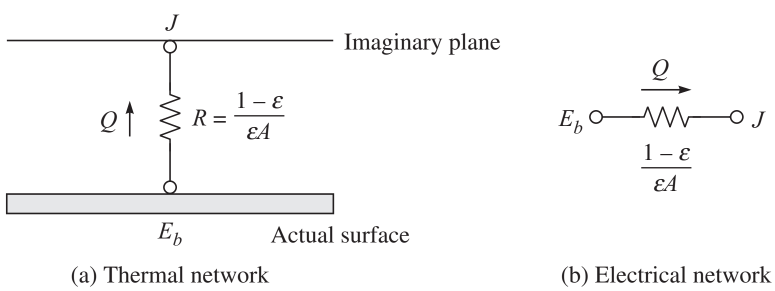


Figure 11.27 Radiation network for net energy transfer from a surface.

Special cases: If the surface is black, then $\varepsilon = 1$, and hence the surface resistance $R = 0$. Then Eq. (11.43) reduces to

$$J = E_b = \sigma T^4$$

In this case, the radiosity is equal to the blackbody emissive power of the surface. If the surface has a very large area ($A \to \infty$), then also the surface resistance becomes zero, and $J = E_b$. If convection effects are negligible, then a perfectly insulated surface (the surface whose back side is well insulated) at steady state will have a zero radiation heat transfer at the surface. This means that the surface must lose as much radiation energy as it gains. Therefore the surface reradiates all the energy incident upon it and is called a reradiating surface. Setting $Q = 0$ in Eq. (11.43), we get $J = E_b = \sigma T^4$. One thing has to be remembered that the surface resistance in this case is not zero, but there is no need to consider the resistance since no current (heat flux) is flowing through it.

Two surface enclosure: Let us consider an enclosure made by two gray surfaces A_1 and A_2 as shown in Figure 11.28. This means that there is energy exchange only between the two surfaces and nothing else.

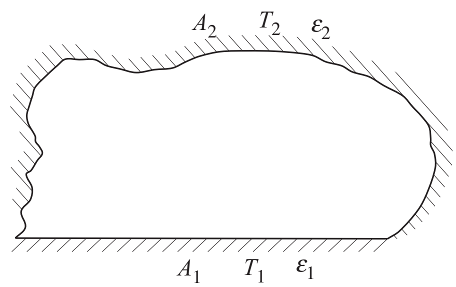


Figure 11.28 A two surface enclosure.

The surface A_1 is maintained at a temperature T_1 while the surface A_2 at T_2. Let us assume $T_1 > T_2$, so that there will be a net radiation heat transfer from A_1 to A_2. If J_1 and J_2 are respectively the radiosities of surfaces A_1 and A_2, then the net energy exchange between them becomes

$$\begin{aligned} Q_{1-2} &= J_1 A_1 F_{12} - J_2 A_2 F_{21} \\ &= A_1 F_{12}(J_1 - J_2) \qquad \text{(since by reciprocity theorem } A_1F_{12} = A_2F_{21}\text{)} \end{aligned} \tag{11.44}$$

Again, the net energy exchange Q_{1-2} can be expressed in terms of the surface resistances following Eq. (11.43) as

$$Q_{1-2} = \frac{E_{b1} - J_1}{(1-\varepsilon_1)/\varepsilon_1 A_1} \tag{11.45}$$

and

$$Q_{1-2} = \frac{J_2 - E_{b2}}{(1-\varepsilon_2)/\varepsilon_2 A_2} \tag{11.46}$$

Combining Eqs. (11.44) to (11.46), we can write

$$\begin{aligned} Q_{1-2} &= \frac{E_{b1} - E_{b2}}{\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1} + \dfrac{1}{A_1 F_{12}} + \dfrac{1-\varepsilon_2}{\varepsilon_2 A_2}} \\ &= \frac{\sigma(T_1^4 - T_2^4)}{\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1} + \dfrac{1}{A_1 F_{12}} + \dfrac{1-\varepsilon_2}{\varepsilon_2 A_2}} \end{aligned} \tag{11.47}$$

Equation (11.47) implies that the heat exchange between two surfaces can be represented by a network as shown in Figure 11.29. When the surfaces are black, the surface

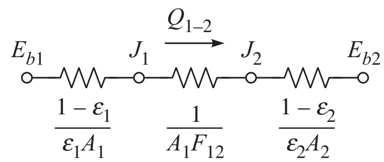


Figure 11.29 Radiation network for energy exchange between two surfaces.

resistances $(1 - \varepsilon_1)/\varepsilon_1 A_1$ and $(1 - \varepsilon_2)/\varepsilon_2 A_2$ become zero and Eq. (11.47) gives

$$\begin{aligned} Q_{1-2} &= A_1 F_{12}(E_{b1} - E_{b2}) \\ &= A_1 F_{12}\sigma(T_1^4 - T_2^4) \end{aligned}$$

Three surface enclosure: In case of a three surface enclosure as shown in Figure 11.30, the radiation network is as shown in Figure 11.31.

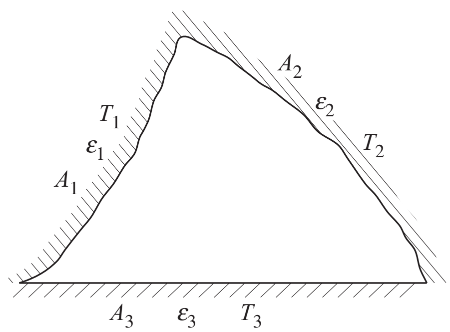


Figure 11.30 A three surface enclosure.

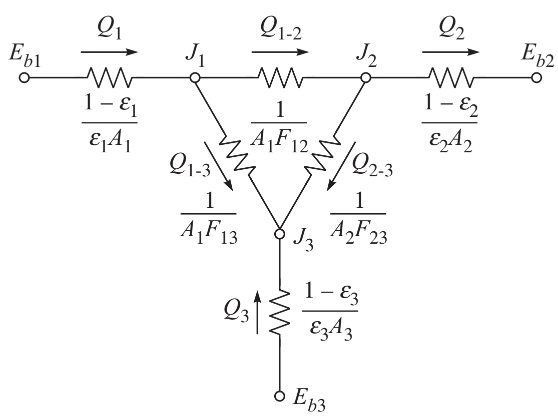


Figure 11.31 The radiation network for a three surface enclosure.

If one of the surfaces, say surface A_3 of the three surface enclosure, is a reradiating surface, then the net energy flow to or from that surface is zero as discussed earlier. Therefore, there is no need of considering the surface resistance of the surface and under the situation $J_3 = E_{b3}$. The network is shown in Figure 11.32.

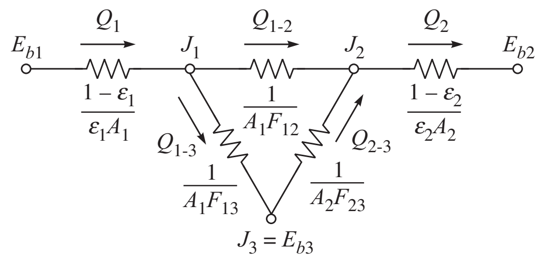


Figure 11.32 The radiation network for a three surface enclosure with one reradiating surface.

It is observed from the three-zone network shown in Figure 11.32, where one of the zones is reradiating, that the network is simplified to a series-parallel arrangement. The solution for heat flow is given by one explicit expression as

$$Q_1 = Q_2 = \frac{E_{b1} - E_{b2}}{R} \tag{11.47a}$$

The equivalent resistance R in the above equation is given by

$$R = R_1 + \left(\frac{1}{R_{12}} + \frac{1}{R_{13} + R_{23}}\right)^{-1} + R_2 \tag{11.47b}$$

where

$$R_1 = \frac{1-\varepsilon_1}{\varepsilon_1 A_1}, \qquad R_2 = \frac{1-\varepsilon_2}{\varepsilon_2 A_2} \tag{11.47c}$$

$$R_{12} = \frac{1}{A_1 F_{12}}, \qquad R_{13} = \frac{1}{A_1 F_{13}}, \qquad R_{23} = \frac{1}{A_2 F_{23}} \tag{11.47d}$$

Special cases of radiation heat exchange between two surfaces

Infinite parallel surfaces: In case of two infinitely long parallel plane surfaces (Figure 11.33) exchanging radiation energy between them, we can use Eq. (11.47) with $A_1 = A_2 = A$ and $F_{12} = 1$ as

$$\frac{Q}{A} = \frac{\sigma (T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} - 1} \tag{11.48}$$

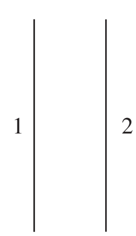


Figure 11.33 Two infinite parallel plane surfaces.

Concentric cylinders: In case of two long concentric cylinders as shown in Figure 11.34, we can write Eq. (11.47) with $F_{12} = 1$ as

$$\frac{Q}{A_1} = \frac{\sigma(T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + \left(\dfrac{A_1}{A_2}\right)\left(\dfrac{1}{\varepsilon_2} - 1\right)} \tag{11.49}$$

where A_1 and A_2 are the surface areas of the cylinders (Figure 11.34).

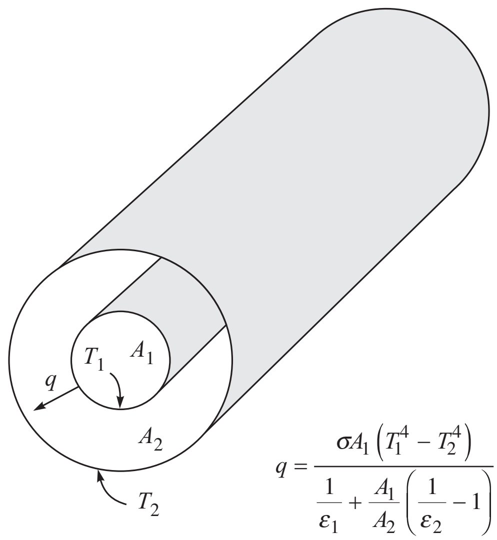


Figure 11.34 Two long concentric cylinders.

Concentric spheres: In case of two concentric spheres (Figure 11.35), $F_{12} = 1$ and hence the same equation for concentric cylinders, i.e. Eq. (11.49) is applicable for the radiation energy exchange between them.

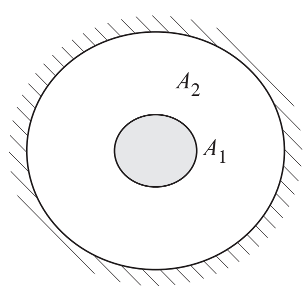


Figure 11.35 Concentric spheres.

In case of a small convex object within a large enclosure (Figure 11.36), $F_{12} = 1$ and $A_1/A_2 \rightarrow 0$. Therefore, we can write, following Eq. (11.47) or Eq. (11.49), as

$$Q = A_1\varepsilon_1\sigma(T_4^1 - T_4^2)$$

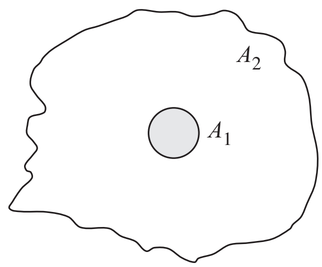


Figure 11.36 A small convex object in a large enclosure.

Concept of radiation shields

Let us consider the net radiative heat exchange between two infinite plane parallel surfaces (Figure 11.33) as given by Eq. (11.48). For fixed surface temperatures, the net radiation heat transfer can be reduced if we increase the thermal resistance (the denominator of Eq. (11.48)) between the surfaces. One way of doing this is to use surface materials which are highly reflective. A highly reflective surface will have a poor absorptivity and hence a poor emissivity too. Low values of ε_1 and ε_2 will increase the denominator of Eq. (11.48). Another way of increasing the thermal resistance for decreasing the radiation heat transfer is to use radiation shields between the heat exchange surfaces. A radiation shield is a third surface placed between the two surfaces (Figure 11.37) so that it does not deliver or remove any heat from the overall system.

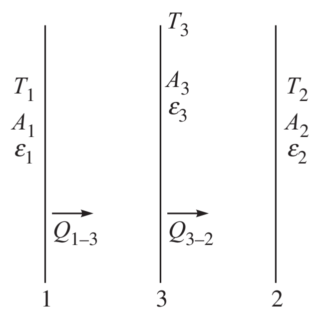


Figure 11.37 A radiation shield between two infinite parallel planes.

Since the shield does not deliver or remove any heat from the system, the heat transfer between surface 1 to shield (surface 3 in Figure 11.37) must be the same as that from the shield to surface 2, i.e. $Q_{1-3} = Q_{3-2}$. Therefore, we can write, following Eq. (11.48), as

$$\frac{Q_{1-3}}{A} = \frac{\sigma(T_1^4 - T_3^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_3} - 1} \tag{11.50}$$

$$\frac{Q_{3-2}}{A} = \frac{\sigma(T_3^4 - T_2^4)}{\dfrac{1}{\varepsilon_3} + \dfrac{1}{\varepsilon_2} - 1} \tag{11.51}$$

where $$A_1 = A_2 = A_3 = A$$

Since
$$Q_{1-3} = Q_{3-2}$$
$$\frac{T_1^4 - T_3^4}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_3} - 1} = \frac{T_3^4 - T_2^4}{\dfrac{1}{\varepsilon_3} + \dfrac{1}{\varepsilon_2} - 1}$$
or
$$T_3^4 = \frac{X_1 T_1^4 + X_2 T_2^4}{C} \tag{11.52}$$
where
$$X_1 = \frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_2} - 1$$
$$X_2 = \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_3} - 1$$

Equation (11.52) thus determines the temperature of the shield and hence we can determine the radiation heat exchange from any one of the two Eqs. (11.50) and (11.51). If the emissivities of all the three surfaces are equal, i.e. $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$, we obtain from Eq. (11.52)
$$T_3^4 = \frac{T_1^4 + T_2^4}{2}$$
and
$$\frac{Q_{1-3}}{A} = \frac{Q_{3-2}}{A} = \frac{Q}{A} = \frac{1}{2}\left[\frac{\sigma(T_1^4 - T_2^4)}{\dfrac{2}{\varepsilon} - 1}\right]$$
where
$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon \tag{11.53}$$

The heat exchange between the surfaces 1 and 2 without a shield can be written as
$$\frac{Q_{1-2}}{A} = \left[\frac{\sigma(T_1^4 - T_2^4)}{\dfrac{2}{\varepsilon} - 1}\right]$$

Therefore, we observe that the radiation heat transfer in the presence of a shield is just one-half that without a shield. The radiation network of this situation with a shield is shown in Figure 11.38.

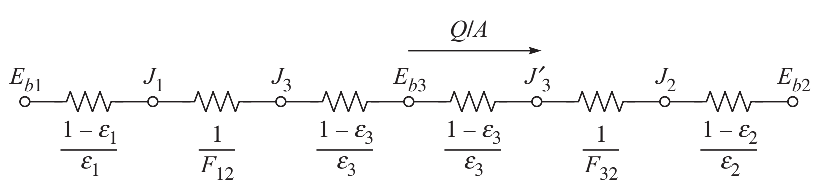


Figure 11.38 Radiation network for two parallel plane surfaces separated by a radiation shield.

From the radiation network we find that due to the presence of a shield, the heat transfer is impeded by the insertion of three additional resistances from that which would be present with two surfaces without the shield. These additional resistances comprise two surface resistances and one space resistance. The higher the reflectivity of the shield, the lower is the emissivity and the higher is the surface resistance inserted, and hence, the lower is the heat transfer.

A number of shields may be provided to reduce radiation heat transfer to a much lower level. Let us consider two infinitely long parallel plane surfaces exchanging radiation heat transfer with n number of radiation shields placed between them. If we consider that the emissivities of all surfaces including the shields are equal, then we have from the viewpoint of radiation network

R_n (the total thermal resistance with n shields)

$$= \underbrace{2n\left(\frac{1-\varepsilon}{\varepsilon}\right)}_{\substack{\text{Total surface}\\ \text{resistance for}\\ n \text{ number of}\\ \text{shields}}} + \underbrace{2\left(\frac{1-\varepsilon}{\varepsilon}\right)}_{\substack{\text{Surface resistance}\\ \text{for two heat}\\ \text{exchange surface}}} + \underbrace{(n+1)(1)}_{\substack{\text{Total space}\\ \text{resistance}\\ \text{(in case of very}\\ \text{large plates)}}}$$

$$= (n+1)\left(\frac{2}{\varepsilon}-1\right)$$

Again, R (the total thermal resistance without the shields)

$$= \frac{1}{\varepsilon}+\frac{1}{\varepsilon} - 1$$

$$= \frac{2}{\varepsilon} - 1$$

Then, we can write

$$\left(\frac{Q}{A}\right)_{\text{with } n \text{ number of shields}} = \frac{1}{n+1}\left(\frac{Q}{A}\right)_{\text{without any shields}} \tag{11.54}$$

EXAMPLE 11.7 Two parallel infinite plane surfaces are maintained at 200°C and 300°C. Determine the net rate of radiation heat transfer per unit area when (a) the two surfaces are gray having emissivity of 0·7 (b) the two surfaces are black.

Solution: (a) In this case we use Eq. (11.48) with $\varepsilon_1 = \varepsilon_2 = 0.7$. Therefore,

$$\frac{Q}{A} = \frac{\sigma(T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1}+\dfrac{1}{\varepsilon_2}-1}$$

$$= \frac{5.67\times10^{-8}[(573.15)^4-(473.15)^4]}{\dfrac{1}{0.7}+\dfrac{1}{0.7}-1}$$

$$= 1764 \text{ W}$$

(b) Under this situation, $\varepsilon_1 = \varepsilon_2 = 1$, and hence we have

$$\frac{Q}{A} = \sigma(T_1^4 - T_2^4)$$

$$= 5.67 \times 10^{-8}[(573.15)^4 - (473.15)^4]$$

$$= 3277 \text{ W}$$

EXAMPLE 11.8 Two concentric spheres of diameters $D_1 = 0{\cdot}5$ m and $D_2 = 1$ m are separated by an air space as shown in Figure 11.39 and have surface temperatures of 400 K and 300 K respectively.

(a) If the surfaces are black, what is the net rate of radiation exchange between the spheres?

(b) (i) What is the net rate of radiation exchange between the surfaces if they are diffuse and gray with $\varepsilon_1 = 0{\cdot}5$ and $\varepsilon_2 = 0{\cdot}5$? (ii) What error would be introduced by assuming blackbody behaviour for the outer surface ($\varepsilon_2 = 1$) with all other conditions remaining the same?

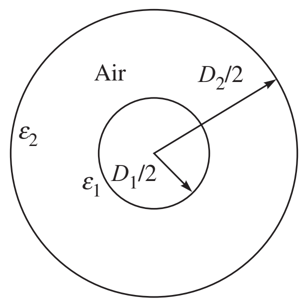


Figure 11.39 Two concentric spheres (Example 11.8).

Solution: The radiation heat exchange in case of two concentric spheres is given by Eq. (11.49) as

$$Q = \frac{A_1\sigma(T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + (A_1/A_2)\left(\dfrac{1}{\varepsilon_2} - 1\right)}$$

(a) When the spheres are black, $\varepsilon_1 = \varepsilon_2 = 1$.

Hence,
$$\begin{aligned} Q &= A_1\sigma(T_1^4 - T_2^4) \\ &= \pi(0.5)^2(5.67 \times 10^{-8})[(400)^4 - (300)^4] \\ &= 779 \text{ W} \end{aligned}$$

(b) (i)
$$Q = \frac{\pi(0.5)^2(5.67\times10^{-8})[(400)^4-(300)^4]}{\dfrac{1}{0.5}+\left(\dfrac{0.5}{1}\right)^2\left(\dfrac{1}{0.5}-1\right)}$$
$$= 346 \text{ W}$$

(ii) When $\varepsilon_2 = 1$,
$$\begin{aligned} Q &= A_1\,\sigma\varepsilon_1(T_1^4 - T_2^4) \\ &= \frac{\pi}{4}(0.5)^2(5.67 \times 10^{-8})(0.5)[(400)^4 - (300)^4] \\ &= 389.5 \text{ W} \end{aligned}$$

$$\text{Error induced} = \frac{389.5 - 346}{346} \times 100 = 12.57 \text{ per cent}$$

EXAMPLE 11.9 Two infinite parallel plates are maintained at temperatures T_1 and T_2 with $T_1 > T_2$. To reduce the rate of radiation heat transfer between the plates, they are separated by a thin radiation shield which has different emissivities on opposite surfaces. One surface has an emissivity of ε_s and the other surface of $2\varepsilon_s$ where $\varepsilon_s < 0{\cdot}5$. Determine the orientation of the shield, i.e. whether the surface of emissivity ε_s or the surface of emissivity $2\varepsilon_s$ would be facing towards the plate at temperature T_1, for

(a) the largest reduction in heat transfer between the plates, and
(b) the larger value of shield temperature T_s.

Solution: (a) Let the surface of emissivity ε_s face the surface at temperature T_1. The thermal network for this arrangement is shown below in Figure 11.40.

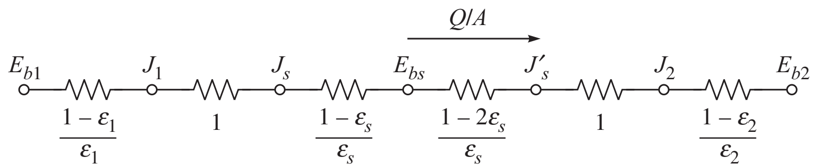


Figure 11.40 Thermal network (Example 11.9).

Here ε_1 and ε_2 are the emissivities of the plates at temperatures T_1 and T_2 respectively.

$$\text{The net resistance} = \frac{1-\varepsilon_1}{\varepsilon_1} + \frac{1-\varepsilon_s}{\varepsilon_s} + \frac{1-2\varepsilon_s}{2\varepsilon_s} + 1 + \frac{1-\varepsilon_2}{\varepsilon_2}$$

$$= \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + \frac{3}{2\varepsilon_s} - 3$$

Therefore,

$$\frac{Q}{A} = \frac{E_{b1} - E_{b2}}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} + \dfrac{3}{2\varepsilon_s} - 3}$$

$$= \frac{\sigma(T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} + \dfrac{3}{2\varepsilon_s} - 3}$$

We see that the net resistance to heat flow remains same for any orientation of the shield. Therefore the net heat exchange between the plates is independent of the orientation of the shield.
(b) Let us consider the part of the network circuit from potential E_{b1} to E_{bs}. Then, we can write

$$\frac{Q}{A} = \frac{E_{b1} - E_{bs}}{\dfrac{1-\varepsilon_1}{\varepsilon_1} + 1 + \dfrac{1-\varepsilon_s}{\varepsilon_s}}$$

$$= \frac{\sigma(T_1^4 - T_s^4)}{\dfrac{1}{\varepsilon_1} + \left(\dfrac{1-\varepsilon_s}{\varepsilon_s}\right)}$$

or

$$T_s^4 = T_1^4 - \frac{Q}{A\sigma}\left[\frac{1}{\varepsilon_1} + f(\varepsilon_s)\right] \tag{11.55}$$

where
$$f(\varepsilon_s) = \frac{1-\varepsilon_s}{\varepsilon_s}$$

Now if the shield surface of emissivity $2\varepsilon_s$ faces the plate at T_1 then $f(\varepsilon_s)$ in Eq. (11.55) becomes

$$f(\varepsilon_s) = \frac{1-2\varepsilon_s}{2\varepsilon_s}$$

It is apparent from Eq. (11.55) that for given values of T_1, Q, and ε_1, the value of T_s will be larger when $f(\varepsilon_s)$ will be smaller. Since $(1 - 2\varepsilon_s)/2\varepsilon_s$ is smaller than $(1 - \varepsilon_s)/\varepsilon_s$, we can conclude that the larger value of shield temperature will be attained when the shield surface having emissivity of $2\varepsilon_s$ faces the surface at temperature T_1.

EXAMPLE 11.10 The configuration of a furnace can be approximated as an equilateral triangular duct (Figure 11.41) which is sufficiently long so that the end effects are negligible. The hot wall is maintained at 1000 K and has an emissivity of $\varepsilon_1 = 0{\cdot}75$. The cold wall is at 350 K and has an emissivity of $\varepsilon_2 = 0{\cdot}7$. The third wall is a reradiating zone for which $Q_3 = 0$. Determine the radiation heat flux leaving the hot wall.

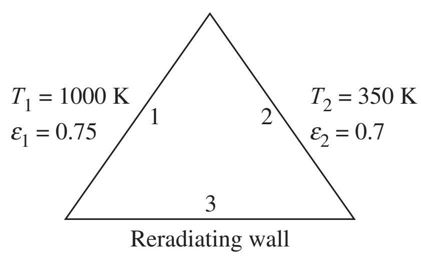


Figure 11.41 The triangular duct (Example 11.10).

Solution: This is an example of straightforward application of Eq. (11.47a)
Following Eq. (11.47a), we can write

$$\frac{Q}{A} = \frac{\sigma(T_1^4 - T_2^4)}{AR}$$

Here $A_1 = A_2 = A$, and R is the equivalent resistance of the thermal network.

$$\varepsilon_1 = 0{\cdot}75, \qquad \varepsilon_2 = 0{\cdot}70$$

By summation rule of view factors,

$$F_{33} + F_{31} + F_{32} = 1$$
$$F_{33} = 0 \qquad \text{(in consideration of furnace surfaces to be plane)}$$

From symmetry,

$$F_{31} = F_{32}$$

Hence

$$F_{31} = F_{32} = 0{\cdot}5$$

Again from the reciprocity relation,

$$F_{13} = F_{31} = 0{\cdot}5$$

$$F_{23} = F_{32} = 0{\cdot}5 \qquad (\text{since } A_1 = A_2 = A_3 = A)$$

Again,

$$F_{11} + F_{12} + F_{13} = 1$$

$$F_{11} = 0 \quad \text{and} \quad F_{13} = 0{\cdot}5$$

Hence

$$F_{12} = 0{\cdot}5$$

Now we can use Eqs. (11.47b) to (11.47d) to determine R,

$$F_{12} = F_{13} = F_{23} = 0{\cdot}5$$

Therefore,

$$R = \frac{1 - 0.75}{0.75} + \frac{1}{0.5 + (2+2)^{-1}} + \frac{0.3}{0.7}$$

$$= 2.09$$

$$\frac{Q}{A} = \frac{5.67 \times 10^{8}(1000^{4} - 350^{4})}{2.09}$$

$$= 26.72 \times 10^{3}\ \text{W/m}^{2} = 26.72\ \text{kW/m}^{2}$$

11.7 RADIATION IN AN ABSORBING EMITTING MEDIUM

We have considered so far the radiation energy between surfaces which are separated in a medium that does not participate in the process of radiation exchange. Radiation propagated through such a medium remains unchanged. The medium under the situation is termed transparent medium. But it is not always so in practice. For example, gases such as CO, NO, CO_2, SO_2, H_2O and various hydrocarbons absorb and emit radiation over certain wavelength regions. In numerous other applications, the absorption and emission of radiation take place within a body through which the radiation energy is propagated. The analysis of radiation heat transfer under the situation is very much complicated. A simple analysis is presented below to derive the basic governing equation relating the spatial variation of radiation intensity in such a system.

Let us consider an absorbing, emitting semitransparent plate of thickness L as shown in Figure 11.42

The plate is maintained at a uniform temperature T. A beam of thermal radiation having a spectral intensity $I_\lambda(0)$ and propagating along a direction x, normal to the surface, is incident on the boundary surface at $x = 0$. As the beam propagates through the plate, its intensity is attenuated due to absorption by the plate and augmented as a result of emission of radiation by the plate due to its temperature. From an energy balance of a control volume of length dx, we can write

$$\frac{dI_\lambda(x)}{dx} = \begin{pmatrix}\text{emission per}\\ \text{unit volume}\end{pmatrix} - \begin{pmatrix}\text{absorption per}\\ \text{unit volume}\end{pmatrix} \tag{11.56}$$

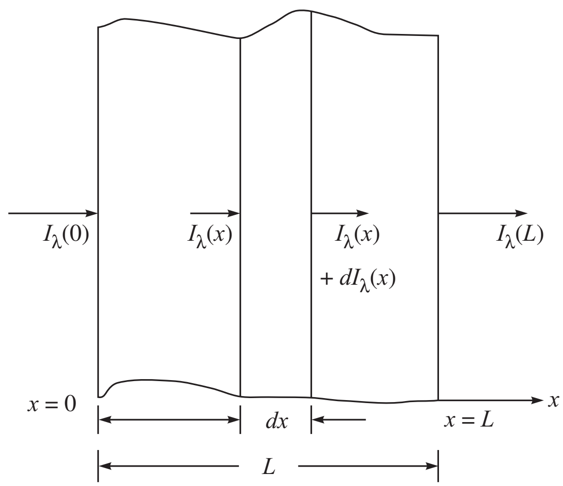


Figure 11.42 Radiation energy passing through an absorbing, emitting plate.

The absorption per unit volume is given by $k_\lambda I_\lambda(x)$, where k_λ is the absorption coefficient.

If the medium, i.e. the plate under consideration, is in local thermodynamic equilibrium, then according to Kirchhoff's law, the absorption coefficient becomes equal to emission coefficient. Therefore, we can write emission per unit volume = $k_\lambda I_{b\lambda}(T)$

Now Eq. (11.56) becomes

$$\frac{dI_\lambda(x)}{dx} + k_\lambda I_\lambda(x) = k_\lambda I_{b\lambda}(T) \tag{11.57}$$

The above equation is called the equation of radiative transfer for an absorbing, emitting medium. The boundary condition for the solution of Eq. (11.57) is specified as, at $x = 0$, $I_\lambda(x) = I_\lambda(0)$. With the assumptions k_λ and $I_{b\lambda}$ to be constant, the solution of Eq. (11.57) with the above mentioned boundary condition is

$$I_\lambda(x) = I_\lambda(0)e^{-k_\lambda x} + (1 - e^{-k_\lambda x})I_{b\lambda}(T) \tag{11.58}$$

Equation (11.58) physically signifies that the radiation intensity at any location x is contributed by the two factors. The first term of the right-hand side of the equation represents the contribution due to the beam of radiation entering the medium at $x = 0$. and the second term is the contribution due to the emission of the medium along the path from $x = 0$ to $x = L$. The intensity at the other end of the plate can be obtained by putting $x = L$ in Eq. (11.58).

Let us consider a special case where the emission from the medium is negligible compared to its absorption. Then, we have according to Eq. (11.56),

$$\frac{dI_\lambda(x)}{dx} = -k_\lambda I_\lambda(x) \tag{11.59}$$

$$I_\lambda(x) = I_\lambda(0)e^{-k_\lambda x}$$

Hence,

$$I_\lambda(L) = I_\lambda(0)e^{-k_\lambda L}$$

Therefore, special transmitivity of the medium over a length L is given by

$$\tau_\lambda = \frac{I_\lambda(L)}{I_\lambda(0)} = e^{-k_\lambda L} \tag{11.60}$$

If we assume the medium to be nonreflecting, then we can write

$$\tau_\lambda + \alpha_\lambda = 1$$

where α_λ is the spectral absorptivity of the medium over the length L.

Then,

$$\alpha_\lambda = 1 - \tau_\lambda = 1 - e^{-k_\lambda L} \tag{11.61}$$

By the application of Kirchhoff's law, we can write

$$\varepsilon_\lambda = \alpha_\lambda = 1 - e^{-k_\lambda L} \tag{11.62}$$

11.8 SOLAR RADIATION

The energy coming out of the sun is called solar energy. This energy reaches the earth in the form of electromagnetic waves after experiencing considerable interactions with the atmosphere. The sun is nearly a spherical body with an appropriate diameter of 1.4×10^9 m and a mass of 2×10^{30} kg. The mean distance of sun from the earth is 1.50×10^{11} m. It emits radiation energy continuously at a rate of 3.8×10^{26} W. However, the rate of energy striking the earth is about $1{\cdot}7 \times 10^{17}$ W. The energy is generated continuously in the sun due to a continuous fusion reaction during which two hydrogen atoms fuse to form one atom of helium. Therefore, the sun can be considered as a nuclear reactor. The temperature in the core region of the sun is as high as 4×10^7 K. However due to radiation of energy, the temperature drops to about 6000 K at the outer surface of the sun. The solar energy reaching the earth's atmosphere is determined by measurements using high-altitude aircraft, balloons and spacecraft.

The intensity of solar radiation incident on a surface normal to the sun's rays at the outer edge of the atmosphere when the earth is at a mean distance from the sun is called **solar constant**. The solar constant is usually designated by the symbol G_s and its value is 1353 W/m^2. Since earth moves in an elliptical orbit around the sun, the amount of solar radiation received by the earth changes during a year. However the variation is less and it makes a change within ±3.4 per cent from the value at the mean distance. Hence the amount of radiation energy flux received at the outer atmosphere of earth is taken to be constant during the whole year and equals the solar constant (1353 W/m^2). The measured value of solar constant is used to determine approximately the surface temperature of the sun . From the principle of conservation of energy, we can write

$$(4\pi L)^2\, G_s = (4\pi r^2)\sigma T_{\text{sun}}^4 \tag{11.63}$$

In the above equation, L is the mean distance between the sun's centre and the earth's atmosphere and the sun is considered as a spherical blackbody of radius r (Figure 11.43). The left-hand side of the equation represents the total energy passing through a spherical surface whose radius is the mean distance between the sun and the earth's outer atmosphere. The right-hand side represents the total energy that leaves the sun's outer surface. The equation implies that the solar energy leaving the sun's surface experiences neither attenuation nor enhancement while passing through the vacuum (the space between the sun and the earth's outer atmosphere). The effective temperature of the sun as determined by Eq. (11.63) comes out to be 5762 K.

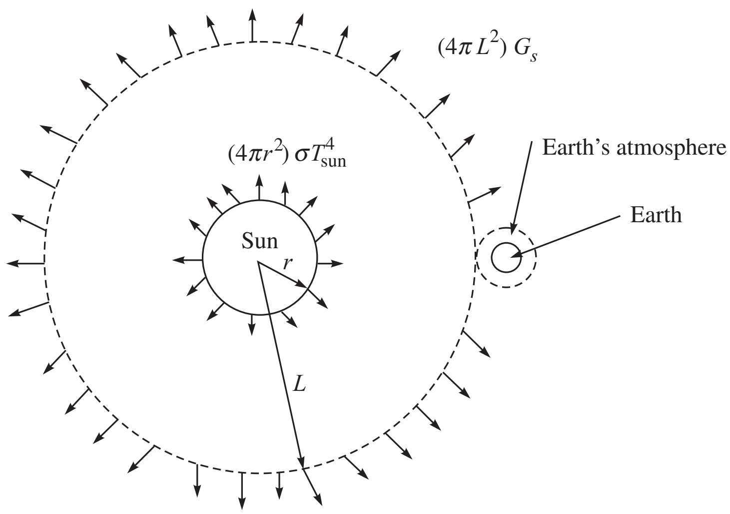


Figure 11.43 The solar energy coming from the sun's surface reaching the earth's atmosphere.

The solar radiation undergoes considerable attenuation as it passes through the earth's atmosphere as a result of absorption and scattering. The most of the earth's atmosphere is contained within a distance of 30 km from the earth's surface. The presence of gases like O_2, O_3 (ozone), CO_2 and H_2O in atmosphere causes a considerable absorption of the radiation energy passing through them. Absorption by oxygen occurs in a narrow band about $\lambda = 0.76$ μm. The ozone absorbs ultraviolet radiation at wavelengths below 0·3 μm completely, and radiation in the range of 0·3 – 0·4 μm considerably. The carbon dioxide gas and water vapour absorbs mostly the infrared radiation. The dust particles and other pollutants in the atmosphere also absorb radiation at various wavelengths. As a result of these absorptions, the solar energy reaching the surface is attenuated considerably to a value of about 950 W/m^2 on a clear day and much less on cloudy and smoggy days, Practically, all the solar energy reaching the earth's surface falls in wavelength band from 0·3 to 2·5 μm.

The gases like CO_2 and water vapour in the atmosphere transmit bulk of the solar radiation to the earth while absorbing the infrared radiation emitted by the surface of the earth. This one way passage causes the energy to be trapped on earth which eventually results in global warming. This is known as greenhouse effect. In actual greenhouses, glass windows are used which are entirely transparent to the solar radiation (wavelength region of 0·3 to 3 μm), while opaque to the entire infrared radiation emitted by the surfaces in the house. Another important feature of the atmosphere that attenuates the intensity of solar energy is scattering or reflection by air molecules and many other kinds of particles like dust, smog, water droplets suspended in air. The oxygen and nitrogen molecules primarily scatter radiation at very short wavelengths comparable to the size of the molecules. Therefore the radiation at wavelengths corresponding to violet and blue colours is scattered most in all directions, and it is for this reason that the sky appears blue to our eyes. The solar energy striking the earth's surface is considered to comprise direct and diffuse parts. The part of solar radiation that reaches the earth's surface without being absorbed or scattered by the atmosphere is called direct solar radiation. The scattered radiation which is assumed to reach the earth's surface uniformly from all directions is called the diffuse solar radiation. It is found convenient in radiation calculations to treat the atmosphere as a blackbody at some fictitious temperature that emits an equivalent amount of radiation energy reaching the earth's surface. This fictitious temperature is called the effective sky temperature, T_{sky}. The value of T_{sky} varies from 230 K for cold clear sky to 285 K for warm, cloudy-sky conditions.

Effect of radiation on temperature measurements

The error occurs in the measurement of temperature since the sensor of a temperature measuring device cannot attain the temperature of the medium to be measured in spite of direct contact because of radiation heat exchange between the sensor and the surrounding surfaces. Let us consider a thermometer which is used to measure the temperature of a fluid flowing through a duct whose walls are at a lower temperature than that of the fluid as shown in Figure 11.44

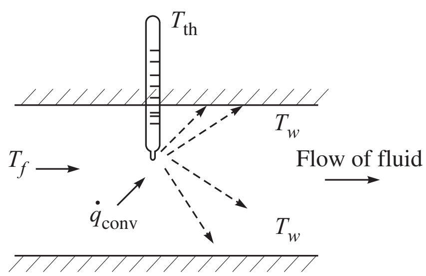


Figure 11.44 A thermometer used to measure the temperature of fluid flowing through a duct.

The thermometer bulb receives heat from the fluid by convection, while it loses heat by radiation with the duct wall. At steady state, we can write

$$h(T_f - T_{th}) = \varepsilon\sigma\,(T_{th}^4 - T_w^4)$$

or

$$T_f = T_{th} + \frac{\varepsilon\sigma(T_{th}^4 - T_w^4)}{h} \tag{11.64}$$

where

- T_f is the actual temperature of the fluid
- T_{th} is the temperature measured by the thermometer
- T_w is the temperature of duct wall
- h is the convection heat transfer coefficient
- ε is the emissivity of the thermometer.

The last term in Eq. (11.64) is due to the radiation effect and thus represents the radiation correction term in temperature measurement. It is observed that the radiation correction term is most significant when the convection heat transfer coefficient is small or the emissivity of the sensor is high. Therefore the sensor should be coated with a material of high reflectivity (low emissivity) to reduce the radiation effect. Sometimes a radiation shield is used without interfering the fluid flow in order to reduce the radiation heat transfer.

SUMMARY

- All matters above a temperature of absolute zero emit their sensible internal energy in all directions. In electromagnetic wave theory, the radiation from a body at an absolute temperature is considered to be emitted at all wavelengths from 0 to ∞. The portion of electromagnetic radiation that extends from 0·1 μm to 100 μm in wavelength is known as thermal radiation. The radiation emitted by bodies because of their temperature, falls almost entirely into this wavelength range.
- Light or visible portion of the electromagnetic radiation lies in a narrow band of 0·40 μm to 76 μm. The electromagnetic radiation emitted by the sun is known as solar radiation and falls into the wavelength band of 0·3 μm to 3 μm. The radiation emitted by bodies at room temperature falls into the infrared region of the spectrum which extends from 0·76 μm to 100 μm. The emission in the wavelength band of 0·01 μm to 0·4 μm is ultraviolet radiation which kills microorganism and causes serious damage to human and other living organisms.
- The rate of radiation energy emitted by a body at a temperature T, streaming through a unit area perpendicular to the direction of propagation, per unit wavelength interval $d\lambda$ about the wavelength λ per unit solid angle about the direction of the beam is defined as spectral radiation intensity. Spectral emissive power is defined as the total rate of radiation energy of wavelength λ leaving a surface in all directions per unit surface area per interval $d\lambda$ of wavelength λ.
- An ideal emitter which emits, at any specified temperature and wavelength, more than any real body and absorbs all incident radiation regardless of wavelength and direction is known as a blackbody. A blackbody emits radiation energy uniformly in all directions. The total emissive power of a blackbody E_b, i.e. the rate of radiation energy emitted by a blackbody at an absolute temperature T over all wavelengths per unit area is given by $E_b = \sigma T^4$. The relationship is known as Stefan–Boltzmann law. The value of σ, the Stefan–Boltzmann constant, is 5.67×10^{-8} W/(m^2 K^4). The blackbody radiation function is defined as the ratio of blackbody emission from $\lambda = 0$ to λ to the total blackbody emission from $\lambda = 0$ to $\lambda = \infty$.
- The radiation energy incident on a surface is partly reflected, partly absorbed and partly transmitted. The fraction of incident energy absorbed by the surface is called the absorptivity α, the fraction reflected by the surface is known as reflectivity ρ and the fraction transmitted is the transmissivity τ. Hence, $\alpha + \rho + \tau = 1$.
- The ratio of spectral hemispherical emissive power of a body to that of a blackbody at a given temperature is defined as spectral emissivity ε_λ of the body. At thermal equilibrium the spectral emissivity of a body equals its spectral absorptivity. This principle is known as Kirchhoff's identity.
- The view factor of a surface i with respect to a surface j is defined as the fraction of energy leaving surface i which reaches surface j, and is denoted as F_{ij}. The net radiation energy exchange between two black surfaces of areas A_1 and A_2 and at temperatures T_1 and T_2 is given by

$$\begin{aligned} Q_{1-2} &= E_{b1} A_1 F_{12} - E_{b2} A_2 F_{21} \\ \sigma &= (T_1^4 A_1 F_{12} - T_2^4 A_2 F_{21}) \end{aligned}$$

- Radiosity is defined as the total radiation energy leaving a surface per unit time per unit area, while irradiation is defined as the total radiation energy incident on a surface per unit time per unit area. The net exchange of radiation energy between two gray surface enclosure with surfaces at temperatures T_1 and T_2 is given by

$$Q_{1-2} = \frac{E_{b1} - E_{b2}}{\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1} + \dfrac{1}{A_1 F_{12}} + \dfrac{1-\varepsilon_2}{\varepsilon_2 A_2}}$$

$$= \frac{\sigma (T_1^4 - T_2^4)}{\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1} + \dfrac{1}{A_1 F_{12}} + \dfrac{1-\varepsilon_2}{\varepsilon_2 A_2}}$$

  The first and third terms in the denominator of the right-hand side represent the surface resistances, while the middle term implies a space resistance.
- A surface which radiates all the energy incident upon it is known as a reradiating surface.
- In case of two infinite non-black parallel plane surfaces at temperatures T_1 and T_2, The net radiation energy exchange is given by

$$\frac{Q}{A} = \frac{\sigma (T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} - 1}$$

- In case of concentric cylinders or spheres, the net radiation energy exchange is given by

$$\frac{Q}{A_1} = \frac{\sigma (T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + \left(\dfrac{A_1}{A_2}\right) \dfrac{1}{\varepsilon_2 - 1}}$$

- A radiation shield is a third surface placed between the two surfaces so that it does not deliver or remove any heat from the overall system. The incorporation of a radiation shield provides additional resistances to the thermal network and hence reduces the net heat transfer between the surfaces.
- The change in radiation intensity of a beam of thermal radiation propagating through an absorbing-emitting medium is partly due to absorption of the incident radiation and partly due to original emission and its absorption by the medium. For a non-reflecting but absorbing-emitting medium, the spectral emissivity ε_λ or absorbitivity α_λ is given by

$$\varepsilon_\lambda = \alpha_\lambda = 1 - e^{-k\lambda L}$$

  where k_λ and L, are the absorption coefficient of the medium and the total length covered by the beam along its direction of propagation.
- The energy coming out of the sun is called solar energy. The sun is nearly a spherical body with an approximate diameter of 1.4×10^9 m and mass of 2×10^{30} kg. The mean distance between the sun from the earth is approximately 1.50×10^{11}m. The sun radiates energy continuously at the rate of 3.8×10^{26} W, while the rate at which energy reaches the earth surface after considerable interactions with the atmosphere is 1.7×10^{17} W.

The intensity of solar radiation incident on a surface normal to the sun's rays at the outer edge of the atmosphere when the earth is at a mean distance from the sun is called the solar constant and its value is 1353W/m^2

REVIEW QUESTIONS

1. What is thermal radiation? How does it differ from other forms of electromagnetic radiation?

2. What is visible light? Is the colour of a surface at room temperature related to the radiation that it emits?

3. Consider two identical blackbodies, one at 100 K and the other at 1500 K. Which body emits more radiations at a wavelength of 20 μm?

4. What is meant by blackbody radiation function?

5. What is a gray body?

6. What is meant by view factor and why is it so important in calculation of radiation heat transfer?

7. What is greenhouse effect? Why is it a matter of great concern among atmospheric scientists?

8. What is solar constant? How is it used to determine the effective surface temperature of the sun?

9. Explain why does the sky appear blue in colour?

10. What do you mean by effective sky temperature?

PROBLEMS

11.1 A tungsten filament is heated to 3000 K. What is the maximum radiative heat flux from the filament, and what fraction of this energy is in the visible range (λ = 0·4 μm to 0·7 μm)? Consider the filament as a blackbody.

[**Ans.** 4.59 MW/m^2, 0.081]

11.2 Consider two identical blackbodies, one at 1000 K and the other at 2000 K. Which body emits more radiation at a wavelength of 20 μm?

11.3 The spectral emissive power E_λ for a diffusely emitting surface is

$$E_\lambda = \begin{cases} 0 & \text{for } \lambda < 4\ \mu\text{m} \\ 200\ \text{W/(m}^2\ \mu\text{m)} & \text{for } 4 < \lambda < 16\ \mu\text{m} \\ 350\ \text{W/(m}^2\ \mu\text{m)} & \text{for } 16 < \lambda < 30\ \mu\text{m} \\ 0 & \text{for } \lambda > 30\ \mu\text{m} \end{cases}$$

[**Ans.** 7.30 kW/m^2, 2.32 kW/m^2]

(a) Calculate the total emissive power of the surface over the entire wavelength.

(b) Calculate the intensity of radiation, assuming that I is independent of direction.

11.4 Consider a small surface of area $A_1 = 10^{-4}$ m^2, which emits diffusely with a total, hemispherical emissive power of $E_1 = 15 \times 10^4$ W/m^2.

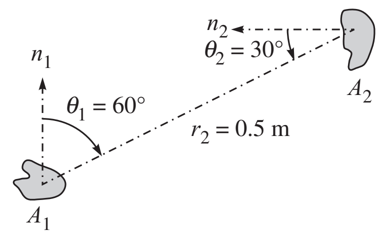


Figure 11.45 Two elemental surfaces (Problem 11.4).

At what rate is this emission intercepted by a small surface of area $A_2 = 6 \times 10^{-4}$m^2, which is oriented as shown? [**Ans.** 4.96 mW]

11.5 A red-hot surface is at 3000 K. What fraction of the total radiation emitted is in the following wavelength bands (assume blackbody radiation)?

(a) $\Delta\lambda_1$ = 1 to 5 μm (c) $\Delta\lambda_3$ = 10 to 15 μm
(b) $\Delta\lambda_2$ = 5 to 10 μm (d) $\Delta\lambda_4$ = 15 to 20 μm

[**Ans.** (a) 0.696, (b) 0.026, (c) 0.003, (d) 0.001]

11.6 Consider a blackbody at 1449 K emitting into air.

(a) Determine the wavelength at which the blackbody spectral emissive power $E_{\lambda b}(T)$ is maximum.
(b) Calculate the corresponding spectral emissive power and the spectral blackbody radiation intensity.

[**Ans.** (a) 2 μm, 82.60 kW/(m^2 μm), 26.29 kW/(m^2 μm sr)]

11.7 A 3 mm thick glass window transmits 90 per cent of the radiation between λ = 0·3 μm and 3·0 μm and is essentially opaque for radiation at other wavelengths. Determine the rate of radiation transmitted through a 1 m × 1 m glass window from blackbody sources at (a) 6000 K and (b) 1000 K. [**Ans.** (a) 69.18 MW, (b) 1.25 MW]

11.8 Consider a hemispherical furnace with a flat circular base of diameter D. Determine the view factor of the dome of this furnace with respect its base. [**Ans.** 1/2]

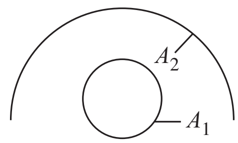
11.9 Determine the view factors from the base of a cube to each of the other five surfaces. [**Ans.** 0.2]

11.10 Consider a conical enclosure of height h and base diameter D. Determine the view factor from the conical side surface to a hole of diameter d located at the centre of the base.

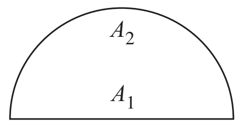
$$\left[\textbf{Ans.}\ \frac{3}{h}\frac{d^2}{D^2}\right]$$

11.11 Determine F_{12} and F_{21} for the following configurations using the reciprocity theorem and other basic shape factor relations. Do not use tables or charts.

(a) Small sphere of area A_1 under a concentric hemisphere of area $A_2 = 2A_1$



(b) Long duct. What is F_{22} for this case?



[**Ans.** (a) $F_{12} = \dfrac{1}{2}$, $F_{21} = \dfrac{1}{4}$ (b) $F_{12} = 1$, $F_{21} = \dfrac{2}{\pi}$, $F_{22} = (\pi - 2)/\pi$]

11.12 Two very large parallel plates are maintained at uniform temperatures of $T_1 = 600$ K and $T_2 = 400$ K and emissivities $\varepsilon_1 = 0{\cdot}5$ and $\varepsilon_2 = 0{\cdot}9$, respectively. Determine the net rate of radiation heat transfer between the two surfaces per unit area of the plates.

[**Ans.** 2.79 kW/m^2]

11.13 Two large parallel planes having emissivities of 0·25 and 0·5 are maintained at temperatures of 1000 K and 500 K, respectively. A radiation shield having an emissivity of 0·1 on both sides is placed between the two planes. Calculate (a) the heat-transfer rate per unit area if the shield were not present, (b) the heat-transfer rate per unit area with the presence of the shield, and (c) the temperature of the shield.

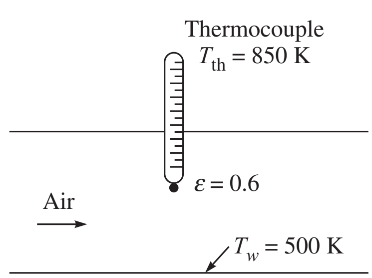
[**Ans.** (a) 10.63 kW/m^2, (b) 2.21 kW/m^2, (c) 837.59 K]

11.14 A square room 4 m by 4 m has the floor heated to 320 K, the ceiling to 300 K, and the walls are assumed to be perfectly insulated. The height of the room is 3 m. The emissivity of all surfaces is 0.8. Using the network method, find the net interchange between the floor and the ceiling and the wall temperature.

[**Ans.** 364.96 W, 310.48 K]

11.15 A thermocouple used to measure the temperature of hot air flowing in a duct whose walls are maintained at $T_w = 500$ K, shows a temperature reading of $T_{th} = 850$ K. Assuming the emissivity of the thermocouple junction to be $\varepsilon = 0{\cdot}6$ and the convection heat transfer coefficient to be $h = 60$ W/(m^2 °C), determine the actual temperature of the air.

[**Ans.** 1110 K]



REFERENCES

[1] Planck, M., *The Theory of Heat Radiation,* Dover, New York, 1959.

[2] Hamilton, D.C., and W.R. Morgan, Radiant Interchange Configuration Factors, NACA Tech Note, 2836, 1952.

[3] Siegel, R., and J.R. Howell, *Thermal Radiation Heat Transfer*, 2nd ed., New York, McGraw-Hill, 1980.

[4] Howell, J.R.A., *Catalog of Radiation Configuration Factors,* New York, McGraw-Hill, 1982.

[5] Ozisk, M.N., Radiative Transfer and Interactions with Convection and Conduction, Wily, New York, 1973.

[6] Leuenberger, H. and R.A. Pearson, Complication of Radiant Shape Factors for Cylindrical Assemblies, ASME Paper 56-A-144, 1956.

12

Principles of Mass Transfer

The mass transfer is a phenomenon of movement of a chemical species from a region of high concentration to a region of low concentration in a heterogeneous multicomponent system. The process of mass transfer due to concentration gradient takes place by diffusion at molecular level. We cite below a few examples of mass transfer by diffusion.

Most of us must have observed by naked eye that if a small crystal of potassium permanganate is dropped into a beaker of water, the $KMnO_4$ begins to dissolve in the water. We see a dark purple concentrated solution of $KMnO_4$ very near to the crystal. Because of this concentration gradient established, the $KMnO_4$ then diffuses away from the crystal into the solution. The progress of diffusion is seen by watching the growth of the purple region in the entire solution, there being dark purple region where the $KMnO_4$ concentration is high and light purple where it is low. Finally, the $KMnO_4$ concentration becomes uniform throughout the solution, making it uniformly purple coloured.

In another example, let us consider a tank that is divided into equal parts by a partition. The left-half of the tank contains oxygen while the right-half contains nitrogen, and both of them are at same pressure and temperature as shown in Figure 12.1. When the partition is removed, the nitrogen starts diffusing into the oxygen and oxygen diffuses into the nitrogen by the net movement of the respective molecules. This is because the oxygen gas is subjected to a

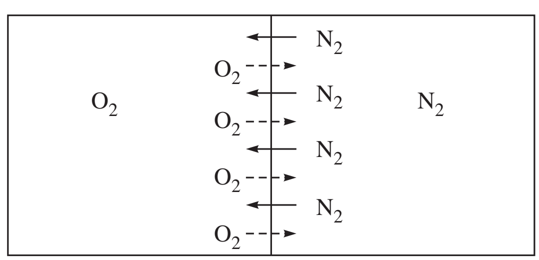


Figure 12.1 A tank contains N_2 and O_2 in two compartments and the diffusion takes place when the partition is removed.

concentration gradient due to the fact that its mass concentration is 1 at the left-half of the tank while it is zero at the right-half. Similarly the reverse is the case for the nitrogen gas. After a long time when equilibrium will be reached, we will have a homogeneous mixture of N_2 and O_2 in the tank, i.e. the concentrations of the species will be uniform throughout the tank.

Another common example of mass transfer phenomenon is the diffusion of liquid vapour from an interface of an evaporating liquid into ambient. A cup of water left in a room will eventually evaporate because of the diffusion of vapour molecules into air resulting in a diffusion of water molecules into a vapour–air mixture adhering to the free water surface as shown in Figure 12.2.

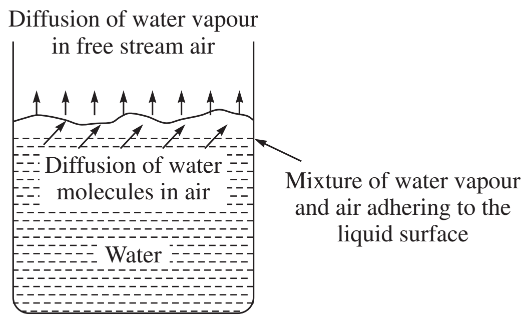


Figure 12.2 A cup of water evaporating into atmospheric air.

The primary driving potential of mass diffusion is the concentration gradient, and the mass diffusion due to concentration gradient is known as the *ordinary diffusion*. There may be mass diffusion in a medium due to temperature gradient or pressure gradient and these are respectively referred to as thermal diffusion (Soret effect) or pressure diffusion. Both of these diffusions are usually negligible, unless the gradients are very large. We shall discuss in this chapter only the ordinary diffusion.

Learning objectives

The reading of the present chapter will enable the readers

- to know the basic mechanism and principles of mass transfer,
- to identify the components of mass transfer due to diffusion and those due to advection,
- to know about the conservation of mass of individual species in a mass transfer phenomenon in a heterogeneous system,
- to recognize the similarities and dissimilarities between the phenomena of mass transfer and heat transfer, and
- to develop the mathematical formulation and obtain the solution of simple mass transfer problems.

12.1 DEFINITIONS OF CONCENTRATIONS

In a multicomponent system, the concentrations of the various species may be expressed in numerous ways. We shall discuss here four conventional ways of defining these concentrations. The system comprising a number of species will be referred to as the mixture.

mass concentration (ρ_i): The mass of a species i per unit volume of the mixture.

molar concentration (c_i): The number of moles of a species i per unit volume of the mixture.

mass fraction (w_i): The ratio of mass of a species i to the total mass of the mixture. The mass fraction w_i can be expressed as $w_i = \rho_i/\rho$, where ρ is the mass density of the mixture.

mole fraction (y_i): The ratio of moles of a species i to the total number of moles in the mixture. The mole fraction y_i can be expressed as $y_i = c_i/c$, where c is the molar density of the solution, i.e. the ratio of the number of moles to volume of the mixture.

Some basic relations among the different concentrations as defined above are given below.

$$\sum_{i=1}^{r} \rho_i = \rho_i(\text{mass density of the mixture}) \tag{12.1a}$$

$$\sum_{i=1}^{r} c_i = c(\text{molar density of the mixture}) \tag{12.1b}$$

$$\sum_{i=1}^{r} w_i = 1 \tag{12.1c}$$

$$\sum_{i=1}^{r} y_i = 1 \tag{12.1d}$$

where r is the number of chemical species in the mixture.

Again,

$$c_i = \frac{\rho_i}{M_i} \tag{12.2a}$$

$$w_i = \frac{\rho_i}{\rho} \tag{12.2b}$$

$$y_i = \frac{c_i}{c} \tag{12.2c}$$

$$\sum_{i=1}^{r} \frac{w_i}{M_i} = M \tag{12.2d}$$

$$\sum_{i=1}^{r} y_i M_i = M \tag{12.2e}$$

$$w_i = \frac{y_i M_i}{M}$$

or

$$y_i = \frac{w_i}{M_i} M \tag{12.2f}$$

where M_i and M are respectively the molecular weight of species i and the molecular weight of mixture.

12.2 FICK'S LAW OF DIFFUSION

Let us consider a binary system, i.e. a mixture which consists of two components, say A and B. The composition of A and B is assumed to vary only in one direction of space coordinates, say x, because of which diffusion of both A and B will take place in the mixture. The molar fluxes of A and B are related to the concentration gradients by Fick's law as

$$J_A = -cD_{AB}\frac{dy_A}{dx} \tag{12.3a}$$

$$J_B = -cD_{BA}\frac{dy_B}{dx} \tag{12.3b}$$

where J_A and J_B are respectively the molar fluxes of species A and B relative to the mixture, c is the molar concentration of the mixture, D_{AB} is the mass diffusivity (or the diffusion coefficient) of A in B and D_{BA} is the mass diffusivity of B in A; they are equal to each other, i.e. $D_{AB} = D_{BA.}$

Moreover, $y_A + y_B = 1$, and hence $dy_A/dx = -dy_B/dx$. This gives $J_A = J_B$, i.e. the diffusion of A and the diffusion B take place in the opposite direction and they are equal. In Eqs. (12.3a) and (12.3b), J_A and J_B are expressed in mole/(m^2 s), while c (the molar density) is in mole/m^3. Therefore the unit of mass diffusivity D_{AB} is m^2/s which is same as units of thermal diffusivity α and momentum diffusivity ν (also called kinematic viscosity). The Fick's law can be expressed in terms of mass flux by using the relations $c = \rho/M$ and $y_A = \dfrac{w_A M}{M_A}$ in Eq. (12.3a) as

$$j_A = -\rho D_{AB}\frac{dw_A}{dx} \tag{12.4}$$

The mass flux j_A in Eq. (12.4) is expressed in kg/(m^2 s). The Fick's law as stated by Eqs. (12.3) and (12.4) actually defines the transport property D_{AB} (the mass diffusivity). The Fick's law (Eqs. (12.3) and (12.4)) can be written in a vector form for a multidimensional diffusion as

$$J_A = -cD_{AB}\,\nabla\, y_A \tag{12.5a}$$
$$j_A = -\rho D_{AB}\,\nabla w_A \tag{12.5b}$$

The primary driving potential of mass diffusion is the concentration gradient, and mass diffusion due to concentration gradient is known as the ordinary diffusion. However, diffusion may also be caused by other effects like temperature gradient, pressure gradient, imposition of external electrical or magnetic force field, etc. The theory of diffusion is very complex and is described by the kinetic theory of matter. Therefore it is difficult to evaluate the mass diffusivities or diffusion coefficients theoretically, and hence they are usually determined experimentally. It is predicted from kinetic theory of gases that the diffusivity for dilute gases at ordinary pressure is essentially independent of mixture composition and tends to increase with temperature while decreasing with pressure as

$$D_{AB} \propto \frac{T^{3/2}}{p} \tag{12.6}$$

This relation is useful in determining the diffusion coefficient for gases at different temperatures and pressures from a knowledge of the diffusion coefficient at a specified temperature and pressure. Several expressions for diffusivity of a binary mixture have been developed from kinetic theory of matter. For gases at low density, one popular relation is the Chapman–Enskong formula which can be expressed as

$$D_{AB} = 2.2646 \times 10^{-5} \frac{\left[T\left(\frac{1}{M_A} + \frac{1}{M_B}\right)\right]^{1/2}}{c\sigma_{AB}^2 \Omega_{D_{AB}}} \tag{12.7}$$

For ideal gases, the molar concentration c can be replaced by $p/\overline{R}\,T$ where $\overline{R}$ is the universal gas constant, M_A and M_B are respectively the molecular weights of the species A and B. The parameters σ_{AB} and $\Omega_{D_{AB}}$ represent the characteristic features of molecular interactions. σ_{AB} is known as collision diameter which is a characteristic diameter of a molecule. The parameter $\Omega_{D_{AB}}$ is known as collision integral which displays the deviation in the behaviour of real molecules from that of rigid spheres considered in the model of kinetic theory. Interested readers may consult the references [1,2] for further and detailed information in this regard.

The diffusion coefficients, in general, are highest in gases and lowest in solids. The diffusion coefficients of gases are several orders of magnitude greater than those of liquids. The binary diffusion coefficients for several binary gas mixtures and solid and liquid solutions are shown in Tables 12.1 and 12.2

Table 12.1 Binary diffusion coefficients of dilute gas mixtures at 1 atm

Substance A	*Substance B*	*T* (K)	D_{AB} or D_{BA} (m^2/s)
Air	Acetone	273	1.1×10^{-5}
Air	Ammonia, NH_3	298	2.6×10^{-5}
Air	Benzene	298	0.88×10^{-5}
Air	Carbon dioxide	298	1.6×10^{-5}
Air	Chlorine	273	1.2×10^{-5}
Air	Ethyl alcohol	298	1.2×10^{-5}
Air	Ethyl ether	298	0.93×10^{-5}
Air	Helium, He	298	7.2×10^{-5}
Air	Hydrogen, H_2	298	7.2×10^{-5}
Air	Iodine, I_2	298	0.83×10^{-5}
Air	Methanol	298	1.6×10^{-5}
Air	Mercury	614	4.7×10^{-5}
Air	Napthalene	300	0.62×10^{-5}
Air	Oxygen, O_2	298	2.1×10^{-5}
Air	Water vapour	298	2.5×10^{-5}
Argon, Ar	Nitrogen, N_2	293	1.9×10^{-5}
Carbon dioxide, CO_2	Benzene	318	0.72×10^{-5}
Carbon dioxide, CO_2	Hydrogen, H_2	273	5.5×10^{-5}
Carbon dioxide, CO_2	Nitrogen, N_2	293	1.6×10^{-5}
Carbon dioxide, CO_2	Oxygen, O_2	273	1.4×10^{-5}
Carbon dioxide, CO_2	Water vapour	298	1.6×10^{-5}
Hydrogen, H_2	Nitrogen, N_2	273	6.8×10^{-5}
Hydrogen, H_2	Oxygen, O_2	273	7.0×10^{-5}
Oxygen, O_2	Ammonia	293	2.5×10^{-5}

(*Contd.*)

Oxygen, O_2	Benzene	296	0.39×10^{-5}
Oxygen, O_2	Nitrogen, N_2	273	1.8×10^{-5}
Oxygen, O_2	Water vapour	298	2.5×10^{-5}
Water vapour	Argon, Ar	298	2.4×10^{-5}
Water vapour	Helium, He	298	9.2×10^{-5}
Water vapour	Nitrogen, N_2	298	2.5×10^{-5}

Table 12.2 Binary diffusion coefficients of dilute liquid solutions and solid solutions at 1 atm

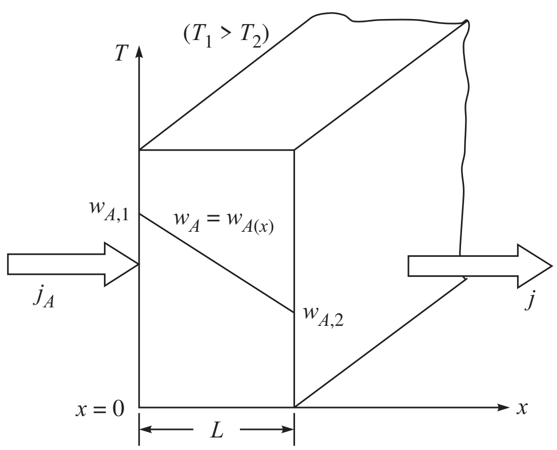
Substance A (solute)	*Substance B (solvent)*	*T* (K)	D_{AB} (m^2/s)
	Diffusion through liquids		
Ammonia	Water	285	1.6×10^{-9}
Benzene	Water	293	1.0×10^{-9}
Carbon dioxide	Water	298	2.0×10^{-9}
Chlorine	Water	285	1.4×10^{-9}
Ethanol	Water	283	0.84×10^{-9}
Ethanol	Water	288	1.0×10^{-9}
Ethanol	Water	298	1.2×10^{-9}
Glucose	Water	298	0.69×10^{-9}
Hydrogen	Water	298	6.3×10^{-9}
Methane	Water	275	0.85×10^{-9}
Methane	Water	293	1.5×10^{-9}
Methane	Water	333	3.6×10^{-9}
Methanol	Water	288	1.3×10^{-9}
Nitrogen	Water	298	2.6×10^{-9}
Oxygen	Water	298	2.4×10^{-9}
Water	Ethanol	298	1.2×10^{-9}
Water	Ethylene glycol	298	0.18×10^{-9}
Water	Methanol	298	1.8×10^{-9}
Chloroform	Methanol	288	2.1×10^{-9}
	Diffusion through solids		
Carbon dioxide	Natural rubber	298	1.1×10^{-10}
Nitrogen	Natural rubber	298	1.5×10^{-10}
Oxygen	Natural rubber	298	2.1×10^{-10}
Helium	Pyrex	773	2.0×10^{-12}
Helium	Pyrex	293	4.5×10^{-15}
Helium	Silicon dioxide	298	4.0×10^{-14}
Hydrogen	Iron	298	2.6×10^{-13}
Hydrogen	Nickel	358	1.2×10^{-12}
Hydrogen	Nickel	438	1.0×10^{-11}
Cadmium	Copper	293	2.7×10^{-19}
Zinc	Copper	773	4.0×10^{-18}
Zinc	Copper	1273	5.0×10^{-13}
Antimony	Silver	293	3.5×10^{-25}
Bismuth	Lead	293	1.1×10^{-20}
Mercury	Lead	293	2.5×10^{-19}
Copper	Aluminum	1273	1.0×10^{-10}
Copper	Aluminum	773	4.0×10^{-14}
Carbon	Iron (fcc)	773	5.0×10^{-15}
Carbon	Iron (fcc)	1273	3.0×10^{-11}

12.3 STEADY STATE MASS DIFFUSION THROUGH A WALL

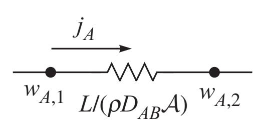
Steady-state one-dimensional mass diffusion of a species through a wall in absence of any chemical reaction occurs in many practical situations. Such mass transfer problems are analogous to the steady one-dimensional heat conduction problems without heat generation and can be analyzed in exactly the similar way as done in case of heat transfer.

Let us consider a solid plane wall (Figure 12.3(a)) of area $\mathcal{A}$, thickness L and density ρ. The wall is subjected on both sides to different concentrations of a species A to which it is permeable. The mass fractions of species A are specified as $w_{A,1}$ on boundary surface at $x = 0$, and $w_{A,2}$ on boundary surface at $x = L$, with $w_{A,1} > w_{A,2}$. Hence diffusion of species A will take place in the positive direction of x. At steady state, the concentration of species A at any point will remain invariant with time, and there will be no production or destruction of species A since no chemical reaction takes place in the medium. Therefore, we can write from the principle of conservation of mass,

$$\dot{m}_{\text{diff}, A} = j_A \mathcal{A} = \text{constant}$$



(a) Mass diffusion through a plane wall.



(b) Analogous electrical circuit.

Figure 12.3 Steady-state one-dimensional mass diffusion.

With the help of Ficks law (Eq. (12.4)), it becomes

$$j_A = \frac{\dot{m}_{\text{diff}, A}}{\mathcal{A}} = -\rho D_{AB} \frac{dw_A}{dx}$$

or

$$\frac{\dot{m}_{\text{diff}, A}}{\mathcal{A}} \int_0^L dx = -\int_{w_{A,1}}^{w_{A,2}} \rho D_{AB}\, dw_A$$

If the density ρ and mass diffusivity D_{AB} remain constant in the medium, then we have from the above equation

$$\dot{m}_{\text{diff}, A} = \frac{w_{A,1} - w_{A,2}}{[L/(\rho D_{AB}\mathcal{A})]} \tag{12.8}$$

The denominator of the right-hand side of Eq. (12.8) represents the diffusion resistance. Therefore we conclude that Eq. (2.13) can be used for the present purpose provided the temperatures are replaced by mass (or mole) fractions, thermal conductivity by ρD_{AB} (or cD_{AB}) and heat flux by mass (or mole) flux. The analogous electrical circuit is shown in Figure 12.3(b).

The analogy between heat and mass diffusion also applies to cylindrical and spherical geometries. Repeating the approach outlined in Chapter 2 for heat conduction, the following analogous relations for steady one-dimensional mass diffusion through non-reacting cylindrical and spherical layers are obtained as follows:

For cylindrical wall (Figure 12.4),

Mass basis: $$\dot{m}_{\text{diff}, A} = \frac{w_{A,1} - w_{A,2}}{\dfrac{\ln(r_2/r_1)}{2\pi L(\rho D_{AB})}} \tag{12.9a}$$

Molar basis: $$\dot{N}_{\text{diff}, A} = \frac{y_{A,1} - y_{A,2}}{\dfrac{\ln(r_2/r_1)}{2\pi L(cD_{AB})}} \tag{12.9b}$$

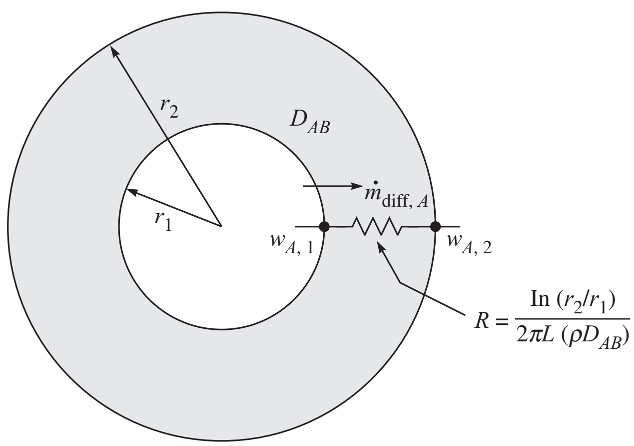


Figure 12.4 Mass diffusion through a cylindrical wall.

For spherical wall (Figure 12.5),

Mass basis: $$\dot{m}_{\text{diff}, A} = \frac{w_{A,1} - w_{A,2}}{(r_2 - r_1)/4\pi r_1 r_2(\rho D_{AB})} \tag{12.10a}$$

Molar basis: $$\dot{N}_{\text{diff}, A} = \frac{y_{A,1} - y_{A,2}}{(r_2 - r_1)/4\pi r_1 r_2(cD_{AB})} \tag{12.10b}$$

12.4 DIFFUSION IN A MOVING MEDIUM

Many practical problems, such as the evaporation of water from a lake under the influence of the wind, evaporation of water spray in a cooling tower, mixing of two fluids as they flow in

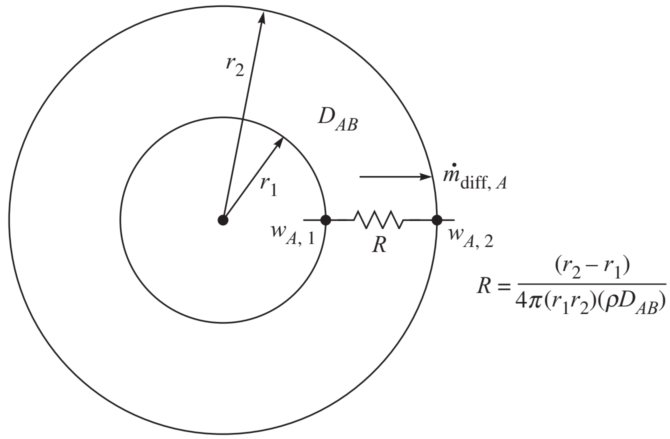


Figure 12.5 Mass diffusion through a spherical cell.

a pipe, involve diffusion in a moving medium where the bulk motion is caused by an external power. Even for a initially stationary medium, the diffusion of species induces a bulk motion in the medium. The velocities and mass (or mole) flow rates consist of two components: one due to molecular diffusion and the other due to convection.

The diffusional molar (or mass) flux which we obtain from Ficks law (Eq. (12.3) and Eq. (12.4)) is the flux relative to the medium. Therefore, the flux relative to a stationary coordinate for a convective medium becomes equal to the sum of diffusion flux and the convective flux. For a one-dimensional diffusion and convection in a binary system, we can write for species A

$$\begin{aligned}\dot{m}_A &= w_A \dot{m} - \rho D_{AB}\,\mathcal{A}\frac{dw_A}{dx}\\ &= w_A(\dot{m}_A + \dot{m}_B) - \rho D_{AB}\,\mathcal{A}\frac{dw_A}{dx}\end{aligned} \tag{12.11a}$$

Similarly,

$$\begin{aligned}\dot{N}_A &= y_A \dot{N} - cD_{AB}\,\mathcal{A}\frac{dy_A}{dx}\\ &= y_A(\dot{N}_A + \dot{N}_B) - cD_{AB}\,\mathcal{A}\frac{dy_A}{dx}\end{aligned} \tag{12.11b}$$

In the above equations, the first term represents the flow (mass or molar) due to convection while the second term is that due to diffusion.

For a multidimensional diffusion and convection, Eqs. (12.11a) and (12.11b) have to be written in a vector form in terms of fluxes as follows:

$$\vec{j}_A = w_A \vec{j} - \rho D_{AB}\,\nabla w_A \tag{12.12a}$$

$$\vec{J}_A = y_A \vec{J} - cD_{AB}\,\nabla y_A \tag{12.12b}$$

12.5 DIFFUSION OF VAPOUR THROUGH A STATIONARY GAS: CONCEPT OF STEFAN FLOW

The diffusion of vapour in a stationary or moving gaseous medium is associated with the phenomena of evaporation and condensation which are involved in many engineering applications such as heat pipe, cooling ponds, cooling towers, etc. To have an understanding of diffusion of vapour in such processes, let us consider a layer of water in a tank surrounded by air at atmospheric pressure and temperature (Figure 12.6).

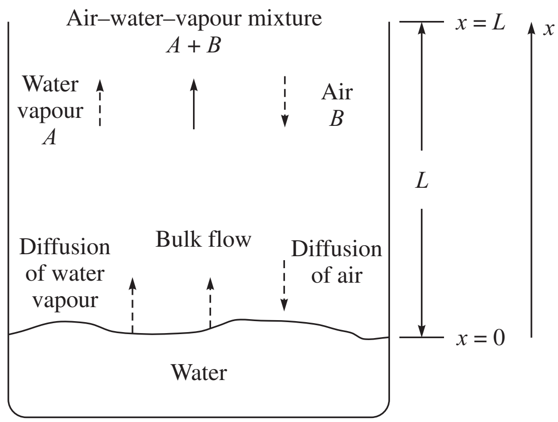


Figure 12.6 Diffusion of water vapour in a stagnant air–water vapour column.

The air is considered to be insoluble to liquid water. Thermodynamic equilibrium exists between the liquid–vapour interface. Therefore the vapour pressure, i.e. the partial pressure of water vapour at the interface is the saturation pressure of water at the temperature of the interface, i.e. the atmospheric temperature in the present case. If the air at the top of the tank is not saturated with water vapour, the vapour pressure and hence the concentration of water vapour at the interface will be greater than that at the top of the tank, and this vapour pressure (or concentration) difference will cause the diffusion of water vapour upwards from the interface which will be sustained by the evaporation of liquid water into vapour. This induces a bulk upward flow of gas (vapour and air mixture) at the interface. Under steady conditions, the molar (or mass) flow rate of any species throughout the column of the gas remains constant. The pressure and temperature of air–vapour mixture is considered to be constant. The concentration of air is more at the top of the tank compared to that at the interface, and hence the air diffuses downwards through the mixture. Since air is considered insoluble in water, the air–water interface will be impermeable to air, and hence the net mass flow rate of air at the interface will be zero. Therefore, at steady state there can be no net molar (or mass) flow rate of air throughout the gas column. This implies physically that the upward convection of air due to bulk motion is balanced by the downward diffusion of air due to concentration difference. Let us describe water as species A and air as species B. Following Eqs. (12.11a) and (12.11b), we can write

$$\dot{m}_B = w_B \mathcal{A} j - \rho \mathcal{A} D_{AB} \frac{dw_B}{dx} = 0 \tag{12.13a}$$

$$\dot{N}_B = y_B \mathcal{A} J - c\mathcal{A}D_{AB} \frac{dy_B}{dx} = 0 \tag{12.13b}$$

If $\mathcal{A}_1$, the cross-sectional area of the tank, is constant, the average velocity of the gas mixture remains constant throughout the tank, and we can write

$$j = \rho V$$

and

$$J = cV$$

Substituting the values of j and J in terms of V, we obtain from Eqs. (12.13a) and (12.13b)

$$V = \frac{D_{AB}\dfrac{dw_B}{dx}}{w_B} = \frac{D_{AB}\dfrac{dy_B}{dx}}{y_B} \tag{12.14}$$

The upward average velocity given by Eq. (12.14) is known as 'Stefan flow velocity'. For water vapour (species A), we can write

$$J_A = \frac{\dot{N}_A}{\mathcal{A}} = y_A J - cD_{AB}\frac{dy_A}{dx}$$

$$= y_A(J_A + J_B) - cD_{AB}\frac{dy_A}{dx}$$

Since $J_B = \dfrac{\dot{N}_B}{\mathcal{A}} = 0$, the above relation becomes

$$J_A = y_A J_A - cD_{AB}\frac{dy_A}{dx}$$

Solving for J_A gives

$$J_A = -\frac{cD_{AB}}{1 - y_A}\frac{dy_A}{dx}$$

$$\frac{J_A}{cD_{AB}}dx = -\frac{dy_A}{1 - y_A} \tag{12.15}$$

At steady state J_A and c are constant throughout the tank and hence we get upon integration of the above equation from $x = 0$ where $y_A = y_{A,0}$ to $x = L$ where $y_A = y_{A,L}$

$$\ln\left[\frac{1 - y_{A,L}}{1 - y_{A,0}}\right] = \frac{J_A}{cD_{AB}}L$$

or

$$J_A = \frac{cD_{AB}}{L}\ln\left[\frac{1 - y_{A,L}}{1 - y_{A,0}}\right] \tag{12.16}$$

This relation is known as Stefan's law. The net mass flow rate (or molar flow rate) can also be expressed in terms of Stefan flow velocity as

$$\left.\begin{aligned} \dot{m} &= \rho \mathcal{A} V \\ \dot{m}_A &= w_A \dot{m} = \rho_A \mathcal{A} V \\ \dot{m}_B &= w_B \dot{m} = \rho_B \mathcal{A} V \end{aligned}\right\} \tag{12.17a}$$

$$\left.\begin{aligned} \dot{N} &= c \mathcal{A} V \\ \dot{N}_A &= y_A \dot{N} = c_A \mathcal{A} V \\ \dot{N}_B &= y_B \dot{N} = c_B \mathcal{A} V \end{aligned}\right\} \tag{12.17b}$$

An expression for the variation of mole fraction of water vapour (species A) along the direction x can be obtained in the following way.

If we integrate Eq. (12.15) from $x = 0$ to the upper limit of x where $y_A(x) = y_A$, we have

$$J_A = \frac{cD_{AB}}{x} \ln \frac{1 - y_A}{1 - y_{A,0}} \tag{12.18}$$

Comparing Eqs. (12.16) and (12.18), we get

$$\frac{1 - y_A}{1 - y_{A,0}} = \left[\frac{1 - y_{A,L}}{1 - y_{A,0}}\right]^{x/L} \tag{12.19a}$$

or

$$\frac{y_B}{y_{B,0}} = \left[\frac{y_{B,L}}{y_{B,0}}\right]^{x/L} \tag{12.19b}$$

Equimolar counterdiffusion

Let us consider a system as shown in Figure 12.7. Two large reservoirs containing a mixture of two gases A and B are connected by a channel. The mixture compositions are different in two reservoirs. In the left reservoir the concentration of A is more while that of B is more in the right

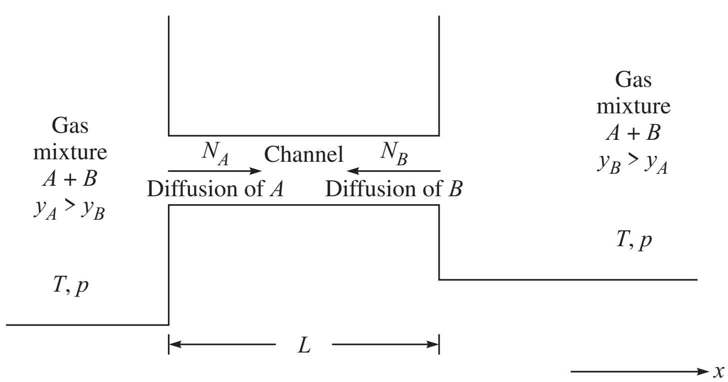


Figure 12.7 Equimolar counterdiffusion of two gases.

reservoir. Hence the species A will diffuse through the channel from left to right and species B will diffuse from right to left. The entire system comprising the binary mixture of gases A and B is at uniform pressure p and temperature T throughout. The concentrations of species at each reservoir are maintained constant. If the gases behave as ideal gases, then the molar concentration c of the mixture remains constant at constant values of p and T. Therefore, we can write

$$c = c_A + c_B = \text{constant}$$

This requires that the molar rate of species A from left to right must be balanced by the molar rate of B from right to left which means that N_A and N_B are equal in magnitude but opposite in direction. This process is called *equimolar counterdiffusion*.

Again,

$$\dot{N} = \dot{N}_A + \dot{N}_B = c\mathcal{A}V = 0 \tag{12.20a}$$

which gives

$$V = 0 \tag{12.20b}$$

It means that the mixture is stationary.

Therefore,

$$\frac{\dot{N}_A}{\mathcal{A}} = J_A = -cD_{AB}\frac{dy_A}{dx} \tag{12.21a}$$

$$\frac{\dot{N}_B}{\mathcal{A}} = J_B = -cD_{AB}\frac{dy_B}{dx} \tag{12.21b}$$

At steady state,

$$\frac{dy_A}{dx} = \frac{y_{A,\mathrm{R}} - y_{A,\mathrm{L}}}{L} \tag{12.22a}$$

$$\frac{dy_B}{dx} = \frac{y_{B,\mathrm{R}} - y_{B,\mathrm{L}}}{L} \tag{12.22b}$$

where the second subscripts L and R represent the state at left and right reservoirs respectively.

The equation of continuity for a binary mixture

In this section we will drive the law of conservation of mass of individual species (known as species continuity equation) in a binary mixture. Let us consider an elementary volume element of $\Delta x\ \Delta y\ \Delta z$ in space (Figure 12.8) through which a binary mixture of A and B is flowing.

The rate of mass flow of species A into the elementary volume through the face AEHD can be written with the help of Eq. (12.13a) as

$$\dot{m}_{Ax} = w_A\,\rho V_x(dy\ dz) - \rho D_{AB}(\Delta y\ \Delta z)\frac{dw_A}{dx}$$

The first term on right-hand side of the above equation represents the convection of species A, while the second term represents the diffusion of species A.

The rate of mass flow of species A out of the elementary volume through the face BFGC becomes $\dot{m}_{Ax} + \dfrac{\partial}{\partial x}(\dot{m}_{Ax})\Delta x$ (neglecting the higher-order terms)

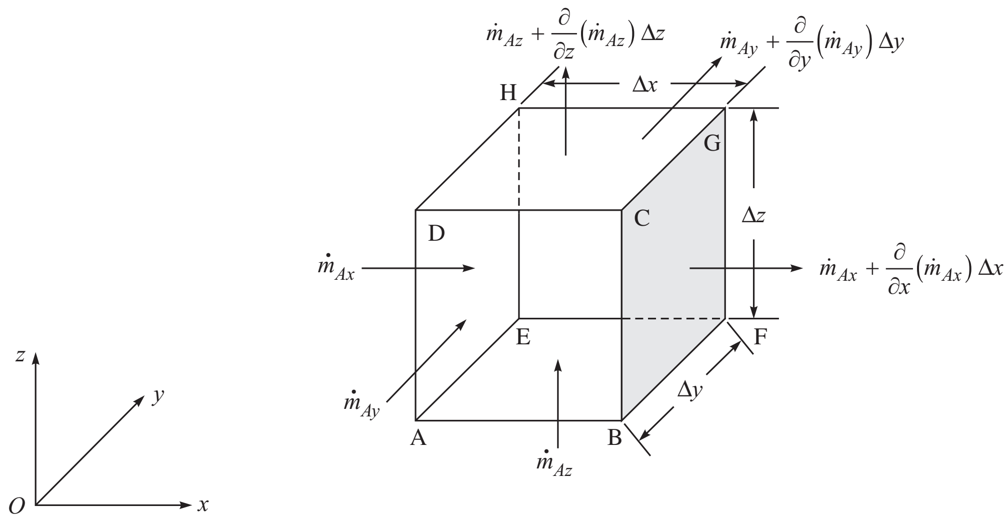


Figure 12.8 A control volume appropriate to a rectangular Cartesian coordinate system.

Therefore the net rate of mass inflow to the elementary volume due to the two faces perpendicular to x direction becomes

$$\dot{m}_{Ax} - \left[\dot{m}_{Ax} + \frac{\partial}{\partial x}(\dot{m}_{Ax})\Delta x\right]$$

$$= -\frac{\partial}{\partial x}(\dot{m}_{Ax})\Delta x$$

$$= -\left[\frac{\partial}{\partial x}(w_A \rho V_x) - \frac{\partial}{\partial x}\left(\rho D_{AB}\frac{dw_A}{dx}\right)\right]\Delta x\,\Delta y\,\Delta z$$

In a similar fashion, the net rate of mass inflow of species A to the elementary volume through the faces perpendicular to y direction

$$= -\left[\frac{\partial}{\partial y}(w_A \rho V_y) - \frac{\partial}{\partial y}\left(\rho D_{AB}\frac{dw_A}{dy}\right)\right]\Delta x\,\Delta y\,\Delta z$$

and the net rate of mass inflow of species A to the elementary volume through the faces perpendicular to z direction

$$= -\left[\frac{\partial}{\partial z}(w_A \rho V_z) - \frac{\partial}{\partial z}\left(\rho D_{AB}\frac{dw_A}{dz}\right)\right]\Delta x\,\Delta y\,\Delta z$$

Therefore, the net rate of mass inflow of species A to the elementary control volume

$$= \left[\frac{\partial}{\partial x}(w_A \rho V_x) + \frac{\partial}{\partial y}(w_A \rho V_y) + \frac{\partial}{\partial z}(w_A \rho V_z)\right.$$
$$\left. - \frac{\partial}{\partial x}\left(\rho D_{AB}\frac{dw_A}{dx}\right) - \frac{\partial}{\partial y}\left(\rho D_{AB}\frac{dw_A}{dy}\right) - \frac{\partial}{\partial z}\left(\rho D_{AB}\frac{dw_A}{dz}\right)\right]\Delta x\,\Delta y\,\Delta z$$

Here, V_x, V_y, V_z are the x, y and z components of the mass average velocity V.

The rate of accumulation of mass of species A within the elementary volume

$$= \frac{\partial}{\partial t}(\rho w_A \,\Delta x\,\Delta y\,\Delta z)$$

$$= \frac{\partial}{\partial t}(\rho w_A)\Delta x\,\Delta y\,\Delta z$$

Let r_A be the rate of generation of species A in mass per unit volume by a chemical reaction.

Then from the principle of conservation of mass of species A within the elementary control volume, we can write

$$\frac{\partial}{\partial t}(\rho w_A)\Delta x\,\Delta y\,\Delta z = \left[\frac{\partial}{\partial x}(w_A \rho V_x) + \frac{\partial}{\partial y}(w_A \rho V_y) + \frac{\partial}{\partial z}(w_A \rho V_z)\right.$$

$$\left. - \frac{\partial}{\partial x}\left(\rho D_{AB}\frac{dw_A}{dx}\right) - \frac{\partial}{\partial y}\left(\rho D_{AB}\frac{dw_A}{dy}\right) - \frac{\partial}{\partial z}\left(\rho D_{AB}\frac{dw_A}{dz}\right) + r_A\right]\Delta x\,\Delta y\,\Delta z$$

The above equation is valid for any size, i.e. $d\forall$ ($= dx\,dy\,dz$) of the elementary control volume. Therefore, it becomes

$$\frac{\partial}{\partial t}(\rho\, w_A) + \frac{\partial}{\partial x}(w_A \rho V_x) + \frac{\partial}{\partial y}(w_A \rho V_y) + \frac{\partial}{\partial z}(w_A \rho V_z)$$

$$= \frac{\partial}{\partial x}\left(\rho D_{AB}\frac{dw_A}{dx}\right) + \frac{\partial}{\partial y}\left(\rho D_{AB}\frac{dw_A}{dy}\right) + \frac{\partial}{\partial z}\left(\rho D_{AB}\frac{dw_A}{dz}\right) + r_A \tag{12.23}$$

The above equation can be written in a vector form as

$$\frac{\partial}{\partial t}(\rho w_A) + \nabla\cdot(w_A \rho \vec{V}) = \nabla\cdot(\rho D_{AB}\,\nabla w_A) + r_A \tag{12.24a}$$

Equation (12.24a) is the continuity equation of species A in a convective binary mixture. Similarly the continuity equation of species B is written as

$$\frac{\partial}{\partial t}(\rho w_B) + \nabla\cdot(w_B \rho \vec{V}) = \nabla\cdot(\rho D_{AB}\,\nabla w_B) + r_B \tag{12.24b}$$

where r_B is the rate of generation of species B in mass per unit volume by a chemical reaction.

Adding the two Eqs. (12.24a) and (12.24b), and noting that $\nabla w_A = -\nabla w_B$ and $r_A + r_B = 0$, we have

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \vec{V}) = 0 \tag{12.25}$$

which is the equation of continuity for the mixture.

Equation (12.24a) can also be expressed (or developed) in terms of species mole fraction y_A, molar concentration in c_A and mass concentration ρ_A. The simplest way is to use any of the required relations given by Eqs. (12.2a) to (12.2f) in expressing Eq. (12.24a) in terms of any of the variables specifying the composition of species A as mentioned above. The different forms of species continuity equation in a binary mixture are shown in Table 12.3, where r_A and r_B are the rate of generation of species A and B respectively in mass per unit time per unit volume, while R_A ($= r_A/M_A$) and R_B ($= r_B/M_B$) are the rate of generation of species A and B respectively in moles per unit time per unit volume.

Table 12.3 Different forms of species continuity equation in a binary mixture of species A and B

In terms of mass fraction w	Species A:	$\frac{\partial}{\partial t}(\rho w_A) + \nabla\cdot(w_A \rho \vec{V}) = \nabla\cdot(\rho D_{AB} \nabla w_A) + r_A$
	Species B:	$\frac{\partial}{\partial t}(\rho w_B) + \nabla\cdot(w_B \rho \vec{V}) = \nabla\cdot(\rho D_{AB} \nabla w_B) + r_B$
In terms of mass concentration ρ	Species A:	$\frac{\partial}{\partial t}(\rho_A) + \nabla\cdot(\rho_A \vec{V}) = \nabla\cdot(\rho D_{AB} \nabla(\rho_A/\rho)) = r_A$
	Species B:	$\frac{\partial}{\partial t}(\rho_B) + \nabla\cdot(\rho_B \vec{V}) = \nabla\cdot(\rho D_{AB} \nabla(\rho_B/\rho)) = r_B$
In terms of mole fraction y	Species A:	$\frac{\partial}{\partial t}(cy_A) + \nabla\cdot(cy_A \rho \vec{V}) = \nabla\cdot[cMD_{AB} \nabla(y_A/M)] + R_A$
	Species B:	$\frac{\partial}{\partial t}(cy_B) + \nabla\cdot(cy_B \rho \vec{V}) = \nabla\cdot[cMD_{AB} \nabla(y_B/M)] + R_B$
In terms of molar concentration c	Species A:	$\frac{\partial}{\partial t}(c_A) + \nabla\cdot(c_A \vec{V}) = \nabla\cdot[cMD_{AB} \nabla(c_A/cM)] + R_A$
	Species B:	$\frac{\partial}{\partial t}(c_B) + \nabla\cdot(c_B \vec{V}) = \nabla\cdot[cMD_{AB} \nabla(c_B/cM)] + R_B$

Assumption of constant ρ and D_{AB}

Under this situation the species continuity equation gets simplified. Let us consider the equation for species A in terms of w_A. For constant ρ and D_{AB}, the species continuity equation becomes

$$\frac{\partial w_A}{\partial t} + \nabla\cdot(w_A \vec{V}) = D_{AB}\nabla^2 w_A + \frac{r_A}{\rho}$$

Again the term $\nabla\cdot(w_A \vec{V})$ can be written as

$$\nabla\cdot(w_A \vec{V}) = \vec{V}\cdot\nabla w_A + w_A \nabla\cdot\vec{V} = 0$$

for constant ρ, $\quad \nabla\cdot\vec{V} = 0$

Then $\quad \nabla\cdot(w_A\cdot\vec{V}) = \vec{V}\cdot\nabla w_A$

Therefore the continuity equation for species A finally becomes

$$\frac{\partial w_A}{\partial t} + \vec{V}\cdot\nabla w_A = D_{AB}\nabla^2 w_A + \frac{r_A}{\rho} \tag{12.26}$$

Equation (12.26) is used in case of incompressible flow and dilute liquid solutions.

Table 12.4 shows the continuity equation for species A with constant ρ and D_{AB} in different coordinate systems.

Assumption of zero velocity and non-reacting mixture with constant ρ and D_{AB}

If there is no chemical reaction and the medium is stationary, r_A, r_B, R_A and R_B and V are all zero. Then we have from the equations in Table 12.3,

for mass fraction $\quad \frac{\partial w_A}{\partial t} = D_{AB}\nabla^2 w_A \qquad (12.27a)$

Table 12.4 Species continuity equation at constant ρ and D_{AB} in different coordinate systems

Rectangular Cartesian coordinates:

$$\frac{\partial w_A}{\partial t} + v_x \frac{\partial w_A}{\partial x} + v_y \frac{\partial w_A}{\partial y} + v_z \frac{\partial w_A}{\partial z} = D_{AB}\left(\frac{\partial^2 w_A}{\partial x^2} + \frac{\partial^2 w_A}{\partial y^2} + \frac{\partial^2 w_A}{\partial z^2}\right) + \frac{r_A}{\rho}$$

Cylindrical coordinates: (Figure 2.2(a))

$$\frac{\partial w_A}{\partial t} + v_r \frac{\partial w_A}{\partial r} + \frac{v_\theta}{r}\frac{\partial w_A}{\partial \theta} + v_z \frac{\partial w_A}{\partial z} = D_{AB}\left(\frac{\partial^2 w_A}{\partial r^2} + \left(\frac{1}{r}\frac{\partial w_A}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w_A}{\partial \theta^2} + \frac{\partial^2 w_A}{\partial z^2}\right) + \frac{r_A}{\rho}$$

Spherical coordinates: (Figure 2.3(a)

$$\frac{\partial w_A}{\partial t} + v_R \frac{\partial w_A}{\partial R} + \frac{v_\varphi}{R}\frac{\partial w_A}{\partial \varphi} + \frac{v_\varphi}{R\sin\varphi}\frac{\partial w_A}{\partial \theta} =$$

$$D_{AB}\left[\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial w_A}{\partial R}\right) + \frac{1}{R^2\sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi\frac{\partial w_A}{\partial \varphi}\right) + \frac{1}{R^2\sin^2\varphi}\frac{\partial^2 w_A}{\partial \theta^2}\right] + \frac{r_A}{\rho}$$

for mass concentration $$\frac{\partial \rho_A}{\partial t} = D_{AB}\nabla^2 \rho_A \qquad (12.27b)$$

for mole fraction $$\frac{\partial y_A}{\partial t} = D_{AB}\nabla^2 y_A \qquad (12.27c)$$

for molar fraction $$\frac{\partial c_A}{\partial t} = D_{AB}\nabla^2 c_A \qquad (12.27d)$$

Any one of the above equations is known as diffusion equation

Mass diffusion with homogeneous chemical reaction in a stationary medium

Let us illustrate a situation of mass diffusion with a homogeneous chemical reaction. We consider the system shown in Figure 12.9. Here gas A dissolves in liquid B and diffuses into the liquid phase. As it diffuses, the species A also undergoes an irreversible reaction given by

$$A + B \rightarrow AB$$

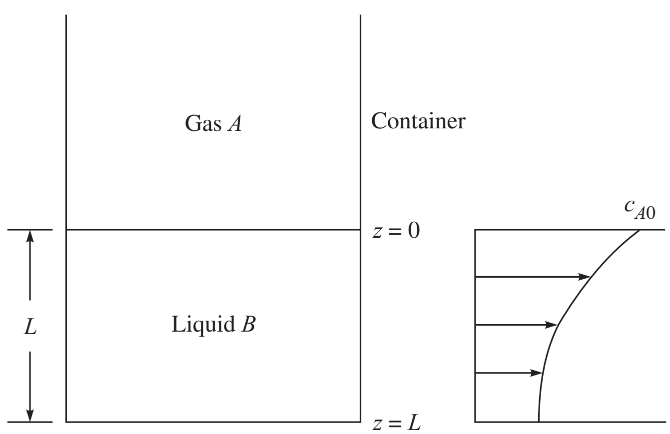


Figure 12.9 Absorption with homogeneous chemical reaction.

The reaction is said to be irreversible since it takes place in one direction. Here the species A is depleted. The equation which describes the rate of depletion of species A is given by the chemical kinetics of the above reaction. If the reaction is considered to be a first-order reaction, then the rate of reaction, i.e. the rate of depletion of species A in moles per unit volume per unit time can be written as

$$R_A = K_1 c_A$$

We choose the species conservation equation in terms of molar concentration (Table 12.1) and write it for the present case of a stationary, one-dimensional steady state diffusion with constant values of D_{AB} and c, as

$$D_{AB}\frac{d^2 c_A}{dz^2} = k_1 c_A \tag{12.28}$$

This is to be solved with the boundary conditions

$$\text{at} \quad z = 0, \quad c_A = c_{A0}$$

and

$$\text{at} \quad z = L, \quad \frac{dc_A}{dz} = 0$$

The first boundary condition states that the surface concentration is maintained at a fixed value c_{A0}. The second condition states that the bottom surface of the container is impermeable to species A, i.e. species cannot diffuse through the bottom of the container. The solution of Eq. (12.28) subject to the above mentioned boundary condition is

$$\frac{c_A}{c_{A0}} = \frac{\cosh[m(L-z)]}{\cosh(mL)} \tag{12.29}$$

where
$$m = \sqrt{\frac{k_1}{D_{AB}}}$$

Quantities of special interest are the concentration of species A at the bottom and the flux of A across the gas–liquid interface. Substituting $z = L$ in Eq. (12.29), we get

$$c_A(L) = \frac{c_{A0}}{\cosh(mL)} \tag{12.30}$$

Moreover,

$$J_A(0) = -D_{AB}\left(\frac{dc_A}{dz}\right)_{z=0}$$

The value of $\left(\frac{dc_A}{dz}\right)_{z=0}$ is obtained from Eq. (12.29), and we have

$$J_A(0) = D_{AB}\, c_{A0}\, m \tanh(mL) \tag{12.31}$$

Mass convection and concentration boundary layer

While diffusion of mass takes place because of a concentration gradient, the advection of mass occurs due to bulk motion of fluid medium. Like heat transfer, the transfer of mass between a surface and a moving fluid due to the combined effect of both diffusion and advection is termed *mass convection*.

Consider the flow of air over the free surface of a water body such as a lake under isothermal conditions as shown in Figure 12.10.

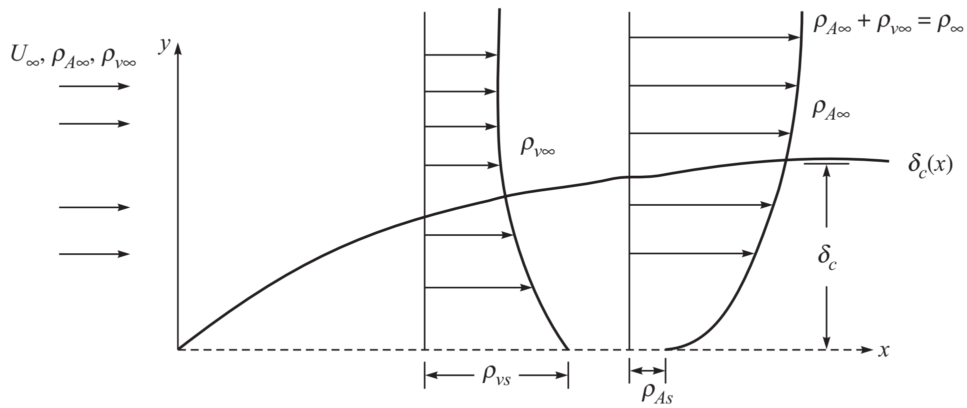


Figure 12.10 Concentration boundary layer of air–vapour mixture flowing over a flat water surface.

In convection of heat, we defined the region in which the temperature gradient exists as the thermal boundary layer. Similarly, in mass convection, we define the region of the fluid in which the concentration gradients exist as the concentration boundary layer (Figure 12.10).

In internal flows, there exists a concentration entrance region where the concentration profile develops, in addition to the hydrodynamic and thermal entrance regions as shown in Figure 12.11.

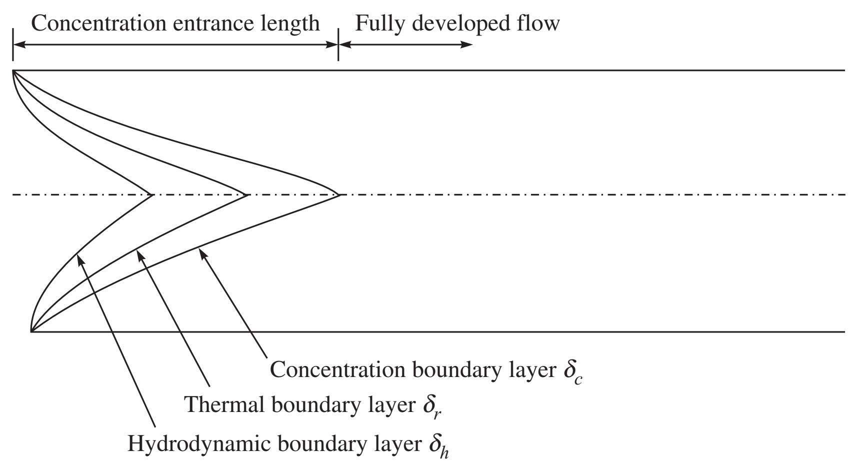


Figure 12.11 The development of hydrodynamic, thermal and concentration boundary layers in internal flow.

Similar to the fully-developed temperature profile beyond the thermal entrance region, the fully-developed concentration profile beyond the concentration entry region is characterized by the relation

$$\frac{\partial}{\partial z}\left(\frac{\rho_{As} - \rho_A}{\rho_{As} - \rho_{AB}}\right) = 0 \tag{12.32}$$

where ρ_{As} is the mass concentration (or density) of species A at the surface and ρ_{AB} is the bulk mean density which is defined as

$$\rho_{AB} = \frac{1}{A V_{av}} \iint \rho_A \, v dA \tag{12.33}$$

In mass convection, a mass transfer coefficient h_{mass} is defined as

$$\dot{m}_{conv} = h_{mass} A(\rho_{As} - \rho_{A,ref}) = h_{mass} \, \rho_\rho (w_{As} - w_{A,ref}) \tag{12.34}$$

In Eq. (12.34),

$$\left.\begin{aligned} \rho_{A,ref} &= \rho_{A,\infty} \\ w_{A,ref} &= w_{A,\infty} \end{aligned}\right\} \text{for external flows}$$

$$\left.\begin{aligned} \rho_{A,ref} &= \rho_{A,B} \\ w_{A,ref} &= w_{A,B} \end{aligned}\right\} \text{for internal flows}$$

Again we can write at the surface where mass transfer is purely by diffusion because of the no-slip boundary condition,

$$j_A(\text{mass flux}) = \left(\frac{\dot{m}_A}{\mathcal{A}}\right) = -\rho D_{AB} \frac{\partial w_A}{\partial y}$$

Then, we have

$$h_{mass} = \frac{-\rho D_{AB} \dfrac{\partial w_A}{\partial y}}{w_{As} - w_{A,ref}} \tag{12.35}$$

Equation (12.35) defines the local mass transfer coefficient.

Like Nusselt number in case of heat transfer coefficient, a non-dimensional mass transfer coefficient known as Sherwood number is defined as

$$\text{Sh} = \frac{h_{mass} L_{ref}}{D_{AB}}$$

where

$L_{ref} = D_{hydraulic}$ (the hydraulic diameter) in case of internal flows
$= L$ (the distance along a surface in the direction of flow) in case of external flows.

The relative magnitudes of mass diffusion to that of momentum and heat diffusion are given by two non-dimensional numbers as follows:

Schmidt number: $$\text{Sc} = \frac{\nu}{D_{AB}} = \frac{\text{momentum diffusivity}}{\text{mass diffusivity}} \tag{12.36a}$$

Lewis number: $$\text{Le} = \frac{\alpha}{D_{AB}} = \frac{\text{thermal diffusivity}}{\text{mass diffusivity}} \tag{12.36b}$$

Therefore, we have $$\text{Le} = \frac{\text{Sc}}{\text{Pr}} \tag{12.36c}$$

The relative thicknesses of hydrodynamic, thermal and concentration boundary layer are given by

$$\frac{\delta_h}{\delta_t} = \text{Pr}^n, \qquad \frac{\delta_h}{\delta_c} = \text{Sc}^n, \quad \frac{\delta_t}{\delta_c} = \text{Le}^n$$

where n, in the above relations, is almost equal to $\frac{1}{3}$ for most of the applications.

Following the analogy between the Nusselt number and the Sherwood number, it is observed that for a given geometry, while the average Nusselt number in forced convection depends on Reynolds number and Prandtl number, the average Sherwood number depends on Reynolds number and Schmidt number. Therefore, we can write

$$\text{Nu} = f(\text{Re, Pr}) \tag{12.37a}$$

$$\text{Sh} = f(\text{Re, Sc}) \tag{12.37b}$$

The functional form is the same for both the above relations in a given geometry provided that the thermal and concentration boundary conditions are of the same type. Therefore, the Sherwood number can be obtained from the Nusselt number expression by simply replacing the Prandtl number by the Schmidt number.

In natural convection mass transfer, we can write

$$\text{Sh} = f(\text{Gr, Sc}) \tag{12.38a}$$

Here Grashoff number Gr should be determined directly from

$$\text{Gr} = \frac{(\rho_\infty - \rho_s)\, g L_{\text{ref}}^3}{\rho \nu^2} \tag{12.38b}$$

which is applicable to both temperature and/or concentration driven natural convection flows. In homogeneous fluids (i.e. fluids within which there is no concentration gradient), the density difference is solely due to temperature gradient and we replace $(\rho_\infty - \rho_s)/\rho$ with $\beta(T_\infty - T_s)$ as we did in natural convection heat transfer (Chapters 5 and 8). However in nonhomogeneous and nonisothermal fluids, density differences are due to the combined effects of temperature and concentration gradients and hence $(\rho_\infty - \rho_s)/\rho$ cannot be replaced by $\beta \Delta T$.

Analogy between friction coefficient, heat transfer coefficient and mass transfer coefficient

Similar to Reynolds–Colburn analogy, Chilton and Colburn suggested the analogy between friction, heat and mass transfer coefficients in combined heat and mass convection in a parallel flow over a flat surface as

$$\frac{c_f}{2} = \text{St}\,\text{Pr}^{2/3} = \text{St}_{\text{mass}}\,\text{Sc}^{2/3} \tag{12.39}$$

where the mass transfer Stanton number St_{mass} is defined as

$$\text{St}_{\text{mass}} = \frac{\text{Sh}}{\text{Re Sc}} \tag{12.40}$$

The relation given by Eq. (12.39) is valid for $0.6 < \text{Pr} < 60$ and $0.6 < \text{Sc} < 3000$, and is known as Chilton–Colburn analogy. Using the definition of heat transfer and mass transfer Stanton numbers, the analogy given Eq. (12.39) can be written in terms of heat transfer and mass transfer coefficients as

$$h_{\text{mass}} = \frac{h_{\text{heat}}}{\rho c_p}\left(\frac{D_{AB}}{\alpha}\right)^{2/3} \tag{12.41}$$

EXAMPLE 12.1 The pressure in a pipeline that transports helium gas at a rate of 4 kg/s is maintained at 1 atm by venting helium to the atmosphere through a tube of 6 mm internal diameter which extends 20 m into the air as shown in Figure 12.12. Assuming both the helium and the atmospheric air to be at 25°C, determine (a) the mass flow rate of helium lost to the atmosphere through the venting tube and (b) the mass flow rate of air that infiltrates into the pipeline. Assume steady conditions, helium and atmospheric air are ideal gases, no chemical reaction occurs in the tube. Neglect air concentration in pipeline and helium concentration in atmospheric air. The diffusion coefficient of helium in air at normal atmosphere is D_{AB} = 7.20 × 10^{-5} m²/s. The molecular weights of air and helium are 29 and 4 kg/kmol respectively (1 atm = 101 kPa)

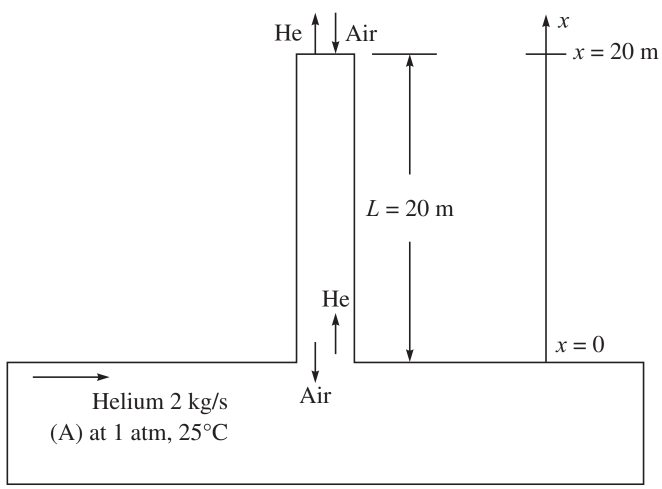


Figure 12.12 Transportation of helium through a pipe as stated in Example 12.1.

Solution: This is a typical equimolar counterdiffusion process through a tube whose two ends are connected with two large reservoirs of ideal gas mixtures of fixed species concentrations. Here one reservoir is the pipeline and the other is the atmosphere.

(a) With the help of Eqs. (12.21) and (12.22), we can write

$$\dot{N}_{\text{helium}} = \dot{N}_{\text{air}} = \mathcal{A}cD_{AB}\frac{(y_{A0} - y_{AL})}{L}$$

Here, flow area $$A = \frac{\pi \times (0.006)^2}{4} = 2.83 \times 10^{-5} \text{ m}^2$$

molar concentration of the mixture, (which is constant throughout)

$$c = \frac{p}{RT} = \frac{101 \times 10^3}{8.31 \times 10^3 \times 298}$$
$$= 0.04$$

Helium has been considered as species A

$$Y_{A0}\begin{pmatrix}\text{helium mole fraction at the bottom}\\ \text{of the tube, i.e. in the pipeline}\end{pmatrix} = 1$$

$$y_{A0}\begin{pmatrix}\text{helium mole fraction at the top open end}\\ \text{of the tube, i.e. in the atmospheric air}\end{pmatrix} = 0$$

Therefore, we have

$$\dot{N}_{\text{helium}} = \dot{N}_{\text{air}} = (2.83 \times 10^{-5})(0.04)\ (7.20 \times 10^{-5})\frac{1-0}{20}$$
$$= 4.07 \times 10^{-12}\ \text{kmol/s}$$

Mass flow rate of helium, $\dot{m}_{\text{helium}} = M_{\text{helium}} \times \dot{N}_{\text{helium}}$

$$= 4 \times (4.07 \times 10^{-12})$$
$$= 16.28 \times 10^{-12}\ \text{kg/s}$$

Mass flow rate of air, $\dot{m}_{\text{air}} = 29 \times (4.07 \times 10^{12})$

$$= 11.80 \times 10^{-11}\ \text{kg/s}$$

EXAMPLE 12.2 Atmospheric air at 40°C flows over a wet-bulb thermometer (i.e. a thermometer the bulb of which is always wrapped by a wet cloth). The reading of the thermometer which is called the wet-bulb temperature is $T_w = 20$°C. Determine the concentration of water vapour c_∞ at free stream. Also determine the relative humidity of the air stream. The properties of air at film temperature

$$T_f = (40 + 20)/2 = 30\text{°C}$$

are as follows:

$$\rho = 1.13\ \text{kg/m}^3 \qquad c_p = 1.007\ \text{kJ/(kg K)} \qquad \alpha = 0.241 \times 10^{-4}\ \text{m}^2\text{/s}$$

The diffusivity $D_{AB} = 0.26 \times 10^{-4}$ m^2/s. The enthalpy of vaporization of water at 20°C is $h_{fg} = 2407$ kJ/kg.

Solution: At steady state,

$$\begin{pmatrix}\text{Rate of heat transfer from air to the}\\ \text{wet cover of thermometer bulb}\end{pmatrix} = \begin{pmatrix}\text{Heat removed by evaporation of water}\\ \text{from the wet cover of thermometer bulb}\end{pmatrix}$$

or

$$h_{\text{heat}}(T_\infty - T_s) = h_{\text{mass}}(\rho_s - \rho_\infty)h_{fg} \tag{12.42}$$

The subscripts ∞ and s in the above equation designate the properties at free stream air and bulb surface respectively. The density ρ refers to that of the water vapour. The water vapour at bulb surface is at saturated state. Therefore, the partial pressure of water vapour is the saturation pressure corresponding to 20°C and this is found out from steam table as

$$p_s = 2.34\ \text{kPa}$$

The mass concentration (or density) of water vapour at bulb surface

$$\rho_s = \frac{p_s M}{R T_s}$$

$$= \frac{(2.34 \times 10^3)(18)}{(8.31 \times 10^3)(293)}$$

$$= 0.0173 \text{ kg/m}^3$$

We use Eq. (12.41) to determine $h_{\text{heat}}/h_{\text{mass}}$ as

$$\frac{h_{\text{heat}}}{h_{\text{mass}}} = \rho c_p \left(\frac{\alpha}{D_{AB}} \right)^{2/3}$$

Therefore, we can write from Eq. (12.42)

$$\rho_\infty = \rho_s - \rho c_p \left(\frac{\alpha}{D_{AB}} \right)^{2/3} \frac{(T_\infty - T_s)}{h_{fg}}$$

$$= 0.0173 - 1.13\,(1.007 \times 10^3) \left(\frac{0.241 \times 10^{-4}}{0.26 \times 10^{-4}} \right)^{2/3} \frac{(40 - 20)}{2407 \times 10^3}$$

$$= 0.0083 \text{ kg/m}^3$$

The mass concentration (or density) of saturated water vapour at 40°C (as found from steam table) is 0.051 kg/m^3.

Therefore, relative humidity in free stream air

$$= \frac{0.0083}{0.051} \times 100$$

$$= 16.27 \text{ per cent}$$

EXAMPLE 12.3 A tube of 35 mm is used to measure the binary diffusion coefficient of water-vapour in air at 25°C at an elevation of 1500 m where the atmospheric pressure is 80 kPa. The tube is partially filled with water and the distance from the water surface to the open end of the tube is 500 mm. Dry air is blown over the open end so that the water vapour rising to the top is removed immediately to maintain zero concentration of water vapour at the top of the tube. After 12 days of continuous operation at constant pressure and temperature the amount of water evaporated was measured to be 1.2×10^{-3} kg. Determine the diffusion coefficient of water vapour in air at 25°C and 80 kPa. Assume air and water vapour to behave as ideal gases and that the air is not dissolved in liquid water. Heat is transferred to the water from the surroundings to make up for the latent heat of vaporization so that the temperature of water remains constant at 25°C.

Solution: The example refers to the straightforward application of Eq. (12.16). The vapour pressure of water at the air–water interface is the saturation pressure corresponding to 25°C and is found from the steam table as $p_{\text{vapour}} = 3.17$ kPa.

The water vapour is designated as species A.

The mole fraction of water vapour at the interface is

$$y_{A,0} = \frac{p_{\text{vapour}}}{p} = \frac{3.17}{80} = 0.0396$$

According to the problem, the mole fraction of water vapour at the top end of the tube is

$$y_{A,L} = 0$$

The total molar concentration c throughout the tube remains constant since the pressure and temperature remain constant.

Thus,
$$c = \frac{p}{RT} = \frac{80 \times 10^3}{(8.31 \times 10^3)(298)}$$
$$= 0.032 \text{ kmol/m}^3$$

The cross-sectional area of the tube, $\mathcal{A} = \frac{\pi}{4}(35 \times 10^{-3})^2$
$$= 9.62 \times 10^{-4} \text{ m}^2$$

The molar flow rate of water vapour is given by

$$\dot{N} = \frac{\dot{m}}{M}$$
$$= \frac{1.2 \times 10^{3}}{(12 \times 24 \times 3600)(18)}$$
$$= 6.43 \times 10^{-11} \text{ kmol/s}$$

By making use of Eq. (12.20), we can write

$$\dot{N} = \dot{J} \cdot A = \frac{c D_{AB} \mathcal{A}}{L} \ln\left[\frac{1 - y_{AL}}{1 - y_{A0}}\right]$$

or

$$6.43 \times 10^{-11} = \frac{(0.032) D_{AB} (9.62 \times 10^{-4})}{0.5} \ln\left[\frac{1 - 0}{1 - 0.0396}\right]$$

which gives

$$D_{AB}\begin{pmatrix}\text{the diffusion coefficient of} \\ \text{water vapour in air}\end{pmatrix} = 2.58 \times 10^{-5} \text{ m/s}$$

SUMMARY

- The mass transfer is a phenomenon of movement of a chemical species from a region of high concentration to a region of low concentration in a heterogeneous multicomponent system.
- In a multicomponent system, the concentration of various species may be expressed in different ways such as, mass concentration, molar concentration, mass fraction and

mole fraction. Ficks law defines a property known as mass diffusivity and can be expressed in terms of both mass or molar flux of specis A in a binary system of A and B as

$$J_A = -cD_{AB}\frac{dy_A}{dx}$$

$$j_A = -\rho D_{AB}\frac{dw_A}{dx}$$

The primary driving potential of mass diffusion is the concentration gradient. However, diffusion may also be caused by other effects like temperature gradient, pressure gradient, imposition of external force field.

- The mass diffusivities are usually determined from experiments. However, diffusivity in a binary system can be predicted from kinetic theory of matter. Diffusivity for dilute gases at ordinary pressure is essentially independent of mixture composition and tends to increase with temperature while decreasing with pressure as $D_{AB} \propto T^{3/2}/p$. For gases at low density, one popular relation developed from kinetic theory of matter is the Chapman–Enskong formula which is as follows:

$$D_{AB} = 2.2646 \times 10^{-5}\,\frac{\left[T\left(\dfrac{1}{M_A}+\dfrac{1}{M_B}\right)\right]^{1/2}}{c\sigma_{AB}^2\Omega_{D_{AB}}}$$

The diffusion coefficients, in general, are highest in gases and lowest in slids with those of liquids being in the middle. The diffusion coefficients of gases are several orders of magnitude greater than those of liquids.

- In a binary system, the rates of mass transfer of a species through walls of plane, cylindrical and spherical geometry with two faces maintaining fixed concentrations of the species are given as follows:

Plane wall : $\dot{m}_{\text{diff}, A} = \dfrac{w_{A,1} - w_{A,2}}{[L/\rho D_{AB})\mathcal{A}]}$

Cylindrical wall: $\dot{m}_{\text{diff}, A} = \dfrac{w_{A,1} - w_{A,2}}{\dfrac{\ln(r_2/r_1)}{2\pi L(\rho D_{AB})}}$

Spherical wall: $\dot{m}_{\text{diff}, A} = \dfrac{w_{A,1} - w_{A,2}}{(r_2 - r_1)/4\pi r_1 r_2(\rho D_{AB})}$

- The mass (or molar) flux relative to a stationary coordinate in a convective medium becomes equal to the sum of diffusional flux and the convective flux. For a one-dimensional diffusion and convection in a binary system, the rate of mass transfer of species A can be written as

$$\dot{m}_A = w_A\dot{m} - \rho D_{AB}\,\mathcal{A}\frac{dw_A}{dx}$$

- In case of evaporation(or condensation), the outward (or inward) velocity of liquid vapour in gas phase induced by the diffusion of vapour at the liquid–gas interface is known as stefan flow
- The equation for conservation of mass of a species A in a multicomponent system can be written as

$$\frac{\partial}{\partial t}(\rho w_A) + \nabla\cdot(w_A \rho \vec{V}) = \nabla\cdot(\rho D_{AB} \nabla w_A) + r_A$$

- Similar to velocity and temperature boundary layer, the concentration boundary layer is defined as the region in which there exists the concentration gradient responsible for mass (or mole) diffusion. A fully developed concentration profile in internal flows is characterized by the relation

$$\frac{\partial}{\partial z}\left(\frac{\rho_{As} - \rho_A}{\rho_{As} - \rho_{AB}}\right) = 0$$

- In convection of mass, a mass transfer coefficient h_{mass} is defined as

$$\dot{m}_{\text{conv}} = h_{\text{mass}} \mathcal{A}(\rho_{As} - \rho_{A,\text{ref}}) = h_{\text{mass}} \rho_A (w_{As} - w_{A,\text{ref}})$$

  Like Nusselt number in case of heat transfer coefficient, a non-dimensional mass transfer coefficient known as Sherwood number Sh is defined as

$$\text{Sh} = \frac{h_{\text{mass}} L_{\text{ref}}}{D_{AB}}$$

  The ratio of momentum diffusivity to mass diffusivity is known as Schmidt number and the ratio of thermal diffusivity to mass diffusivity is known as Lewis number Le. The phenomenon of mass transfer in a given geometry are analytically described as

$$\text{Sh} = f(\text{Re Sc})$$

- The analogy between friction coefficient and mass transfer coefficient is given by

$$\frac{c_f}{2} = \text{St Pr}^{2/3} = \text{St}_{\text{mass}} \text{ Sc}^{2/3}$$

  and is known as the Chilton–Colburn analogy

REVIEW QUESTIONS

1. What is the driving force for (a) heat transfer, (b) electric current flow, (c) fluid flow, and (d) mass transfer?

2. Explain the similarity between the Fick's law of diffusion and the Fourier's law of heat conduction.

3. Fick's law of diffusion can be expressed on the basis of both mass and molar fluxes as

$$j_A = -\rho D_{AB} \frac{dw_A}{dx}$$

$$J_A = -c D_{AB} \frac{dy_A}{dx}$$

respectively. Are the diffusion coefficients, D_{AB}, in the two relations the same or different?

4. Someone claims that the mass and the mole fractions for a mixture of CO_2 and N_2O gases are identical. Do you support the claim? Explain.
5. What is Stefan flow? How is the Stefan flow velocity related to the concentration of evaporating species at the liquid–air interface of a system of liquid mass evaporating to open atmosphere?

PROBLEMS

12.1 The molar analysis of a gas mixture at 300 K and 300 kPa is 60 per cent N_2, 30 per cent O_2, and 10 per cent CO_2. Determine the mass fraction and partial pressure of each gas.

[**Ans.** $w_{N_2} = 0.545$, $w_{O_2} = 0.312$, $w_{CO_2} = 0.143$, $p_{N_2} = 180$ kPa, $p_{O_2} = 90$ kPa, $p_{CO_2} = 30$ kPa]

12.2 Consider a carbonated drink in a bottle at 25°C and 120 kPa. Assuming that the gas space above the liquid consist of a saturated mixture of CO_2 and water vapour and treating the drink as water, determine (a) the mole fraction of water vapour in CO_2 gas and (b) the mass of dissolved CO_2 in a 250 ml drink.

[**Ans.** (a) 0.0264, (b) 0.519×10^{-3} kg]

12.3 A thin plastic membrane separates hydrogen from air. The molar concentrations of hydrogen in the membrane at the inner and outer surfaces are determined to be 0.070 and 0.003 kmol/m^3 respectively. The binary diffusion coefficient of hydrogen in plastic at the working temperature is 5×10^{-10} m^2/s. Determine the mass flow rate of hydrogen by diffusion through the membrane under steady conditions if the thickness of the membrane is 4 mm.

[**Ans.** 0.42×10^{-8} kg/(m^2 s)]

12.4 Consider a system containing water in a tube as shown in Figure 12.13. Determine the rate of evaporation (in kg/h) of water into air at 25°C. The following data are given:

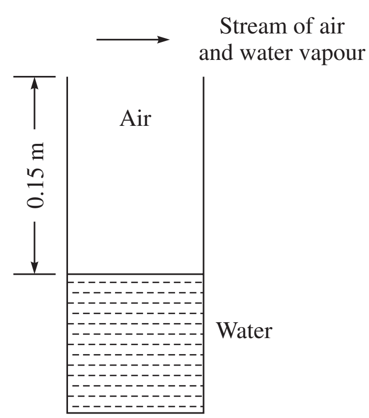


Figure 12.13 Diffusion of water vapour in air as stated in Problem 12.4.

Total pressure = 101 kPa
Diffusivity of water vapour in air at 25°C = 2.5×10^{-5} m^2/s
Vapour pressure of water at 25°C = 3.17 kPa
Surface area of water exposed for evaporation = 300 mm^2

[**Ans.** 4.2×10^{-6} kg/h]

12.5 A natural gas mixture is contained in a pyrex tube having dimensions as shown in Figure 12.14. Show that the rate at which helium (a constituent of the gas mixture) will leak through the tube is given by

$$J = 2\pi L \frac{D_{\text{He-pyrex}}(c_{\text{He},1} - c_{\text{He},2})}{\ln(R_2/R_1)}$$

where L is the length of the tube and $c_{\text{He},1}$ and $c_{\text{He},2}$ are the interfacial molar concentrations of helium inside and outside the pyrex tube respectively

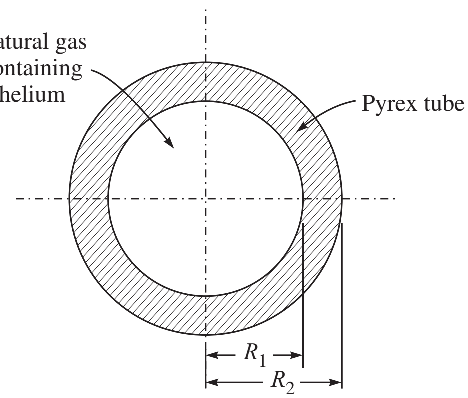


Figure 12.14 The pyrex tube (Problem 5).

12.6 A droplet of substance A is suspended in a stream of gas B. The droplet radius r_1. We postulate that there is a spherical stagnant gas film of radius r_2 as shown in Figure 12.15.

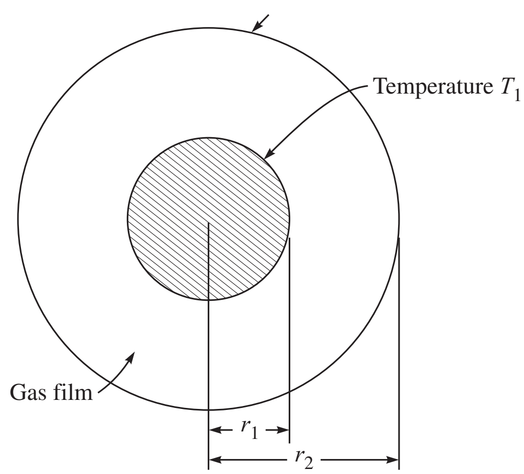


Figure 12.15 Diffusion through a spherical film as described in Problem 6.

The mole fractions of A in the gas phase is y_{A1} at $r = r_1$ and y_{A2} at $r = r_2$. Show that the rate of diffusion of molar flux of A at the surface $r = r_1$ is given by

$$(J_A)_{\text{at}\, r=r_1} = \frac{cD_{AB}}{r_2 - r_1}\left(\frac{r_2}{r_1}\right)\ln\left[\frac{1-y_{A2}}{1-y_{A1}}\right]$$

where c is the molar concentration of the mixture.

12.7 Two large vessels contain homogeneous mixtures of oxygen and nitrogen of 101 kPa and 300 K. Vessel 1 contains 70 mole per cent oxygen and 30 mole per cent nitrogen, while vessel 2 contains 20 mole per cent oxygen and 80 mole per cent nitrogen. Determine the rate of mass transfer of oxygen between these two vessels at steady state, if the vessels are connected by a pipe of 0.15 m diameter and 2 m long. The concentrations of the constituents of the gas mixtures in the vessels are kept constant. The diffusivity of oxygen in nitrogen at 101 kPa and 300 K is 2×10^{-5} m^2/s.

[**Ans.** 0.41×10^{-6} kg/h]

12.8 Dry air at 40°C and 101 kPa flows over a 2 m long wet surface with a free stream velocity of 1 m/s. Determine the average mass transfer coefficient. The temperature of the surface is kept constant at 20°C [Apply Chilton-Colburn analogy. Use either Eq. (7.46) or (7.59c) for average heat transfer coefficient depending upon the situation.] The properties of air at film temperature of 30°C are:

$$\rho = 1.12 \text{ kg/m}^3,\ \nu = 1.66 \times 10^{-5} \text{ m}^2/\text{s},\ \alpha = 2.40 \times 10^{-5} \text{ m}^2/\text{s},$$
$$c_p = 1007 \text{ J/(kg K)},\ k = 0.0265 \text{ W/(m K)}.$$

The diffusivity of water vapour in air at 30°C is 2.60×10^{-5} m^2/s.

Appendix A

Thermophysical Properites of Materials After[1]

Table A1.1 Solid metals

Composition	Melting Point K	Properties at 300 K ρ kg/m^3	C_p J/kg·K	k W/m·K	$\alpha \times 10^6$ m^2/s	Properties at Various Temperatures (K) k(W/m · K)/C_p(J/kg·K) 100	200	400	600	800	1000
Aluminum:											
Pure	933	2702	903	237	97.1	302	237	240	231	218	
						482	798	949	1033	1146	
Alloy 2024-T6 (4.5% Cu, 1.5% Mg. 0.6% Mn)	775	2770	875	177	73.0	65	163	186	186		
						473	787	925	1042		
Alloy 195, Cast (4.5% Cu)		2790	883	168	68.2			174	185		
Beryllium	1550	1850	1825	200	59.2	990	301	161	126	106	90.8
						203	1114	2191	2604	2823	3018
Bismuth	545	9780	122	7.86	6.59	16.5	9.69	7.04			
						112	120	127			
Boron	2573	2500	1107	27.0	9.76	190	55.5	16.8	10.6	9.60	9.85
						128	600	1463	1892	2160	2338
Cadmium	594	8650	231	96.8	48.4	203	99.3	94.7			
						198	222	242			

[1] Reproduced with permission (Yunus A. Cengel. *Heat Transfer–A Practical Approach*, Second Edition, Tata McGraw-Hill, 2004) from McGraw-Hill Education.

(*contd.*)

Table A1.1 Solid metals (*contd.*)

Composition	Melting Point K	*Properties at* 300 K: ρ kg/m^3	C_p J/kg·K	k W/m·K	$\alpha \times 10^6$ m^2/s	*Properties at Various Temperatures* (K) k(W/m · K)/C_p(kg·K): 100	200	400	600	800	1000
Chromium	2118	7160	449	93.7	29.1	159	111	90.9	80.7	71.3	65.4
						192	384	484	542	581	616
Cobalt	1769	8862	421	99.2	26.6	167	122	85.4	67.4	58.2	52.1
						236	379	450	503	550	628
Copper:											
Pure	1358	8933	385	401	117	482	413	393	379	366	352
						252	356	397	417	433	451
Commercial bronze	1293	8800	420	52	14		42	52	59		
(90% Cu, 10% Al)							785	160	545		
Phosphor gear bronze	1104	8780	355	54	17		41	65	74		
(89% Cu, 11 % Sn)							—	—	—		
Cartridge brass	1188	8530	380	110	33.9	75	95	137	149		
(70% Cu, 30% Zn)							360	395	425		
Constantan	1493	8920	384	23	6.71	17	19				
(55% Cu, 45% Nil						237	362				
Germanium	1211	5360	322	59.9	34.7	232	96.8	43.2	27.3	19.8	17.4
						190	290	337	348	357	375
Gold	1336	19,300	129	317	127	327	323	311	298	284	270
						109	124	131	135	140	145
Iridium	2720	22,500	130	147	50.3	172	153	144	138	132	126
						90	122	133	138	144	153
Iron:											
Pure	1810	7870	447	80.2	23.1	134	94.0	69.5	54.7	43.3	32.8
						216	384	490	574	680	975
Armco											
(99.75% pure)		7870	447	72.7	20.7	95.6	80.6	65.7	53.1	42.2	32.3
						215	384	490	574	680	975

(*contd.*)

Table A1.1 Solid metals (*contd.*)

		Properties at 300 K				Properties at Various Temperatures (K) k(W/m · K)/C_p(kg · K)					
Composition	Melting Point K	ρ kg/m^3	C_p J/kg · K	k W/m · K	$\alpha \times 10^6$ m^2/s	100	200	400	600	800	1000
Carbon steels:											
Plain carbon (Mn ≤ 5 1%,		7854	434	60.5	17.7			56.7	48.0	39.2	30.0
Si ≤ 5 0.1%)								487	559	685	1169
AISI1010		7832	434	63.9	18.8			58.7	48.8	39.2	31.3
								487	559	685	1168
Carbon-silicon (Mn ≤ 1%,		7817	446	51.9	14.9			49.8	44.0	37.4	29.3
0.1% < Si ≤ 5 0.6%)								501	582	699	971
Carbon-manganese-silicon		8131	434	41.0	11.6			42.2	39.7	35.0	27.6
(1% < Mn ≤ 5 1.65%								487	559	685	1090
0.1% < Si ≤ 5 0.6%)											
Chromium (low) steels:											
$\frac{1}{2}$Cr–$\frac{1}{4}$ Mo–Si (0.18% C.		7822	444	37.7	10.9			38.2	36.7	33.3	26.9
0.65% Cr. 0.23% Mo. 0.6% Si)								492	575	688	969
1Cr–$\frac{1}{2}$Mo		7858	442	42.3	12.2			42.0	39.1	34.5	27.4
(0.16%C, 1% Cr,											
0.54% Mo.											
0.39% Si)								492	575	688	969
1Cr–V		7836	443	48.9	14.1			46.8	42.1	36.3	28.2
(0.2% C, 1.02% Cr.											
0.15% V)								492	575	688	969
Stainless steels:											
AISI 302		8055	480	15.1	3.91			17.3	20.0	22.8	25.4
								512	559	585	606
AISI 304	1670	7900	477	14.9	3.95	9.2	12.6	16.6	19.8	22.6	25.4
						272	402	515	557	582	611

(*contd.*)

Table A1.1 Solid metals (*contd.*)

Composition	Melting Point K	*Properties at* 300 K: ρ kg/m^3	C_p J/kg·K	k W/m·K	$\alpha \times 10^6$ m^2/s	*Properties at Various Temperatures* (K) k(W/m · K)/C_p(J/kg·K): 100	200	400	600	800	1000
AISI 316		8238	468	13.4	3.48			15.2	18.3	21.3	24.2
								504	550	576	602
AISI 347		7978	480	14.2	3.71			15.8	18.9	21.9	24.7
								513	559	585	606
Lead	601	11,340	129	35.3	24.1	39.7	36.7	34.0	31.4		
						118	125	132	142		
Magnesium	923	1740	1024	156	87.6	169	159	153	149	146	
						649	934	1074	1170	1267	
Molybdenum	2894	10,240	251	138	53.7	179	143	134	126	118	112
						141	224	261	275	285	295
Nickel:											
Pure	1728	8900	444	90.7	23.0	164	107	80.2	65.6	67.6	71.8
						232	383	485	592	530	562
Nichrome	1672	8400	420	12	3.4			14	16	21	
(80% Ni, 20% Cr)								480	525	545	
Inconel X-750	1665	8510	439	11.7	3.1	8.7	10.3	13.5	17.0	20.5	24.0
(73% Ni, 15% Cr,											
6.7% Fe)						—	372	473	510	546	626
Niobium	2741	8570	265	53.7	23.6	55.2	52.6	55.2	58.2	61.3	64.4
						188	249	274	283	292	301
Palladium	1827	12,020	244	71.8	24.5	76.5	71.6	73.6	79.7	86.9	94.2
						168	227	251	261	271	281
Platinum:											
Pure	2045	21,450	133	71.6	25.1	77.5	72.6	71.8	73.2	75.6	78.7
						100	125	136	141	146	152
Alloy 60Pt-40Rh	1800	16,630	162	47	17.4			52	59	65	69
(60% Pt, 40% Rh)								—	—	—	—

(*contd.*)

Table A1.1 Solid metals (*contd.*)

Composition	Melting Point K	ρ kg/m^3	C_p J/kg·K	k W/m·K	$\alpha \times 10^6$ m^2/s	100	200	400	600	800	1000
		Properties at 300 K				*Properties at Various Temperatures* (K) k(W/m · K)/C_p(J/kg·K)					
Rhenium	3453	21,100	136	47.9	16.7	58.9	51.0	46.1	44.2	44.1	44.6
						97	127	139	145	151	156
Rhodium	2236	12,450	243	150	49.6	186	154	146	136	127	121
						147	220	253	274	293	311
Silicon	1685	2330	712	148	89.2	884	264	98.9	61.9	42.4	31.2
						259	556	790	867	913	946
Silver	1235	10,500	235	429	174	444	430	425	412	396	379
						187	225	239	250	262	277
Tantalum	3269	16,600	140	57.5	24.7	59.2	57.5	57.8	58.6	59.4	60.2
						110	133	144	146	149	152
Thorium	2023	11,700	118	54.0	39.1	59.8	54.6	54.5	55.8	56.9	56.9
						99	112	124	134	145	156
Tin	505	7310	227	66.6	40.1	85.2	73.3	62.2			
						188	215	243			
Titanium	1953	4500	522	21.9	9.32	30.5	24.5	20.4	19.4	19.7	20.7
						300	465	551	591	633	675
Tungsten	3660	19,300	132	174	68.3	208	186	159	137	125	118
						87	122	137	142	146	148
Uranium	1406	19.070	116	27.6	12.5	21.7	25.1	29.6	34.0	38.8	43.9
						94	108	125	146	176	180
Vanadium	2192	6100	489	30.7	10.3	35.8	31.3	31.3	33.3	35.7	38.2
						258	430	515	540	563	597
Zinc	693	7140	389	116	41.8	117	118	III	103		
						297	367	402	436		
Zirconium	2125	6570	278	22.7	12.4	33.2	25.2	21.6	20.7	21.6	23.7
						205	264	300	332	342	362

(*contd.*)

Table A1.2 Solid nonmetals

Composition	Melting Point, K	*Properties at* 300 K ρ kg/m^3	C_p J/kg·K	k W/m·K	$\alpha \times 10^6$ m^2/s	*Properties at Various Temperatures* (K), k(W/m · K)/C_p(J/kg·K) 100	200	400	600	800	1000
Aluminum oxide,	2323	3970	765	46	15.1	450	82	32.4	18.9	13.0	10.5
sapphire						—	—	940	1110	1180	1225
Aluminum oxide,	2323	3970	765	36.0	11.9	133	55	26.4	15.8	10.4	7.85
polycrystalline						—	—	940	1110	1180	1225
Beryllium oxide	2725	3000	1030	272	88.0			196	111	70	47
								1350	1690	1865	1975
Boron	2573	2500	1105	27.6	9.99	190	52.5	18.7	11.3	8.1	6.3
						—	—	1490	1880	2135	2350
Boron fi ber epoxy	590	2080									
(30% vol.) composite											
k, II to fibers				2.29		2.10	2.23	2.28			
k, ⊥ to fibers				0.59		0.37	0.49	0.60			
C_p			1122			364	757	1431			
Carbon											
Amorphous	1500	1950	—	1.60	—	0.67	1.18	1.89	21.9	2.37	2.53
						—	—	—	—	—	—
Diamond,											
type IIa	—	3500	509	2300		10,000	4000	1540			
insulator						21	194	853			
Graphite, pyrolytic	2273	2210									
k, II to layers				1950		4970	3230	1390	892	667	534
k, ⊥. to layers				5.70		16.8	9.23	4.09	2.68	2.01	1.60
C_p			709			136	411	992	1406	1650	1793
Graphite fiber	450	1400									
epoxy (25% vol.)											
composite											
k, heat flow II to fibers				11.1		5.7	8.7	13.0			
k, heat flow ⊥ to fibers				0.87		0.46	0.68	1.1			
C_p			935			337	642	1216			

(*contd.*)

Table A1.2 Solid nonmetals (*contd.*)

Composition	Melting Point, K	*Properties at* 300 K: ρ kg/m^3	C_p J/kg·K	k W/m·K	$\alpha \times 10^6$ m^2/s	*Properties at Various Temperatures* (K), k(W/m · K)/C_p(J/kg·K): 100	200	400	600	800	1000
Pyroceram,	1623	2600	808	3.98	1.89	5.25	4.78	3.64	3.28	3.08	2.96
Corning 9606						—	—	908	1038	1122	1197
Silicon carbide	3100	3160	675	490	230			—	—	—	87
								880	1050	1135	1195
Silicon dioxide,	1883	2650									
crystalline (quartz)											
k, II to c-axis				10.4		39	16.4	7.6	5.0	4.2	
k, ⊥. to c-axis				6.21		20.8	9.5	4.70	3.4	3.1	
C_p		745				—	—	885	1075	1250	
Silicon dioxide,	1883	2220	745	1.38	0.834	0.69	1.14	1.51	1.75	2.17	2.87
polycrystalline											
(fused silica)						—	—	905	1040	1105	1155
Silicon nitride	2173	2400	691	16.0	9.65	—	—	13.9	11.3	9.88	8.76
						—	578	778	937	1063	1155
Sulfur	392	2070	708	0.206	0.141	0.165	0.185				
						403	606				
Thorium dioxide	3573	9110	235	13	6.1			10.2	6.6	4.7	3.68
								255	274	285	295
Titanium dioxide,	2133	4157	710	8.4	2.8			7.01	5.02	3.94	3.46
polycrystalline								805	880	910	930

Table A1.3 Building materials (at a mean temperature of 24°C)

Material	*Thickness, L* mm	*Density.* ρ kg/m^3	*Thermal Conductivity, k* W/m·°C	*Specific Heat,* C_p kJ/kg·°C	*R-value (for listed thickness, L/k).* °C·m^2/W
Building Boards					
Asbestos-cement board	6 mm	1922	—	1.00	0.011
Gypsum of plaster board	10 mm	800	—	1.09	0.057
	13 mm	800	—	—	0.078
Plywood (Douglas fir)	—	545	0.12	1.21	—
	6mm	545	—	1.21	0.055
	10mm	545	—	1.21	0.083
	13 mm	545	—	1.21	0.110
	20mm	545	—	1.21	0.165
Insulated board and sheating	13 mm	288	—	1.30	0.232
(regular density)	20mm	288	—	1.30	0.359
Hardboard (high density, standard tempered)	—	1010	0.14	1.34	—
Particle board:					
Medium density	—	800	0.14	1.30	
Underlayment	16 mm	640	—	1.21	0.144
Wood subfloor	20mm	—	—	1.38	0.166
Building Membrane					
Vapor-permeable felt	—	—	—	—	0.011
Vapor-seal (2 layers of mopped 0.73 kglm^2 felt)	—	—	—	—	0.021
Flooring Materials					
Carpet and fibrous pad	—	—	—	1.42	0.367
Carpet and rubber pad	—	—	—	1.38	0.217
Tile (asphalt, linoleum, vinyl)	—	—	—	1.26	0.009
Masonry Materials					
Masonry *units:*					
Brick, common		1922	0.72	—	—
Brick, face		2082	1.30	—	—
Brick, fire clay		2400	1.34	—	—
		1920	0.90	0.79	—
		1120	0.41	—	—
Concrete blocks (3 oval cores, sand and gravel	100 mm	—	0.77	—	0.13
aggregate)	200 mm	—	1.0	—	0.20
	300 mm	—	1.30	—	0.23
Concretes:					
Lightweight aggregates (including		1920	1.1	—	—

(contd.)

Table A1.3 Building materials (at a mean temperature of 24°C) (*contd.*)

Material	*Thickness, L* mm	*Density. ρ* kg/m^3	*Thermal Conductivity, k* W/m·°C	*Specific Heat, C_p* kJ/kg·°C	*R-value (for listed thickness, L/k).* °C·m^2/W
expanded shale, clay, or					
slate;		1600	0.79	0.84	—
expanded slags; cinders;		1280	0.54	0.84	—
pumice; and scoria)		960	0.33		
		940	0.18	—	—
Cement/lime, mortar, and		1920	1.40	—	—
		1280	0.65	—	—
Stucco		1857	0.72	—	—
Roofing					
Asbestos-cement shingles		1900	—	1.00	0.037
Asphalt roll roofing		1100	—	1.51	0.026
Asphalt shingles.		1100	—	1.26	0.077
Built-in roofing	10 mm	1100	—	1.46	0.058
Slate	13 mm	—	—	1.26	0.009
Wood shingles (plain and plastic/film faced)		—	—	1.30	0.166
Plastering Materials					
Cement plaster, sand aggregate	19 mm	1860	0.72	0.84	0.026
Gypsum plaster:					
Lightweight aggregate	13 mm	720	—	—	0.055
Sand aggregate	13 mm	1680	0.81	0.84	0.016
Perlite aggregate	—	720	0.22	1.34	—
Siding Material (on flat surfaces)					
Asbestos-cement shingles	—	1900	—	—	0.037
Hardboard siding	11 mm	—	—	1.17	0.12
Wood (drop) siding	25 mm	—	—	1.30	0.139
Wood (plywood) siding, lapped	10 mm	—	—	1.21	0.111
Aluminum or steel siding (over sheeting):					
Hollow backed	10 mm	—	—	1.22	0.11
Insulating-board backed	10 mm	—	—	1.34	0.32
Architectural glass	—	2530	1.0	0.84	0.018
Woods					
Hardwoods (maple, oak, etc.)	—	721	0.159	1.26	—
Softwoods (fir, pine, etc.)	—	513	0.115	1.38	—
Metals					
Aluminum (1100)	—	2739	222	0.896	—
Steel, mild	—	7833	45.3	0.502	—
Steel, Stainless	—	7913	15.6	0.456	—

Table A1.4 Insulating materials (at a mean temperature of 24°C)

Material	*Thickness, L* mm	*Density, p* kg/m^3	*Thermal Conductivity, k* W/m·°C	*Specific Heat, C_p* kJ/kg·°C	*R-value* (*for listed thickness, L/k*), °C·m^2/W
Blanket and Batt					
Mineral fiber (fibrous form	50 to 70 mm	4.8–32	—	0.71–0.96	1.23
processed from rock, slag,	75 to 90 mm	4.8–32	—	O.71–D.96	1.94
or glass)	135 to 165 mm	4.8–32	—	0.71–D.96	3.32
Board and Slab					
Cellular glass		136	0.055	1.0	—
Glass fiber (organic bonded)		64–144	0.036	0.96	—
Expanded polystyrene (molded beads)		16	0.040	1.2	—
Expanded polyurethane (*R*-ll expanded)		24	0.023	1.6	—
Expanded perlite (organic bonded)		16	0.052	1.26	—
Expanded rubber (rigid)		72	0.032	1.68	—
Mineral fiber with resin binder		240	0.042	0.71	—
Cork		120	0.039	1.80	—
Sprayed or Formed in Place					
Polyurethane foam		24–40	O.023–D.026.	—	—
Glass fiber		56–72	0.038–0.039	—	—
Urethane, two-part mixture (rigid foam)		70	0.026	1.045	—
Mineral wool granules with asbestos/inorganic binders (sprayed)		190	0.046	—	—
Loose Fill					
Mineral fiber (rock, slag, or glass)	~ 75 to 125 mm	9.6–32	—	0.71	1.94
	~ 165 to 222 mm	9.6–32	—	0.71	3.35
	~ 191 to 254 mm	—	—	0.71	3.87
	~ 185 mm	—	—	0.71	5.28
Silica aerogel		122	0.025	—	—
Vermiculite (expanded)		122	0.068	—	—
Perlite, expanded		32–66	0.039–0.045	1.09	—
Sawdust or shavings		128–240	0.065	1.38	—
Cellulosic insulation (milled paper or wood pulp)		37–51	0.039–0.046	—	—

(*contd.*)

Table A1.4 Insulating materials (at a mean temperature of 24°C) (*contd.*)

Material	*Thickness, L* mm	*Density, ρ* kg/m^3	*Thermal Conductivity, k* W/m·°C	*Specific Heat, C_p* kJ/kg·°C	*R-value* (*for listed thickness, L/k*), °C·m^2/W
Roof Insulation					
Cellular glass	—	144	0.058	1.0	—
Preformed, for use above deck	13 mm	—	—	1.0	0.24
	25 mm	—	—	2.1	0.49
	50 mm	—	—	3.9	0.93
Reflective Insulation					
Silica powder (evacuated)		160	0.0017	—	—
Aluminum foil separating fluffy glass mats; 10–12 layers (evacuated); for cryogenic applications (150 K)		40	0.00016	—	—
Aluminum foil and glass paper laminate; 75–150 layers (evacuated); for cryogenic applications (150 K)		120	0.000017	—	—

Table A1.5 Miscellaneous materials
(Values are at 300 K unless indicated otherwise)

Material	*Density, ρ* kg/m^3	*Thermal Conductivity, k* W/m·K	*Specific Heat, C_p* J/kg·K
Asphalt	2115	0.062	920
Bakelite	1300	1.4	1465
Brick, refractory			
Chrome brick			
473 K	3010	2.3	835
823 K	—	2.5	—
1173 K	—	2.0	—
Fire clay, burnt 1600 K			
773 K	2050	1.0	960
1073 K	—	1.1	—
1373 K	—	1.1	—
Fire clay, burnt 1725 K			
773 K	2325	1.3	960
1073 K	—	1.4	—
1373 K	—	1.4	—
Fire clay brick			
478 K	2645	1.0	960
922 K	—	1.5	—
1478 K	—	1.8	—
Magnesite			
478 K	—	3.8	1130
922 K	—	2.8	—
1478 K	—	1.9	—
Chicken meat, white (74.4% water content)			
198 K	—	1.60	—
233 K	—	1.49	—
253 K	—	1.35	—
273 K	—	0.48	—
293 K	—	0.49	—

Material	*Density, ρ* kg/m^3	*Thermal Conductivity, k* W/m·K	*Specific Heat, C_p* J/kg·K
Clay, dry	1550	0.930	—
Clay, wet	1495	1.675	—
Coal, anthracite	1350	0.26	1260
Concrete (stone mix)	2300	1.4	880
Cork	86	0.048	2030
Cotton	80	0.06	1300
Fat	—	0.17	—
Glass			
Window	2800	0.7	750
Pyrex	2225	1–1.4	835
Crown	2500	1.05	—
Lead	3400	0.85	—
Ice			
273 K	920	1.88	2040
253 K	922	2.03	1945
173 K	928	3.49	1460
Leather, sole	998	0.159	—
Linoleum	535	0.081	—
	1180	0.186	—
Mica	2900	0.523	—
Paper	930	0.180	1340
Plastics			
Plexiglass	1190	0.19	1465
Teflon			
300 K	2200	0.35	1050
400 K	—	0.45	—
Lexan	1200	0.19	1260
Nylon	1145	0.29	—
Polypropylene	910	0.12	1925
Polyester	1395	0.15	1170
PVC, vinyl	1470	0.1	840

(*contd.*)

Table A1.5 Miscellaneous materials (*contd.*)
(Values are at 300 K unless indicated otherwise)

Material	Density, ρ kg/m^3	Thermal Conductivity, k W/m·K	Specific Heat, C_p J/kg·K	Material	Density, ρ kg/m^3	Thermal Conductivity, k W/m·K	Specific Heat, C_p J/kg·K
Porcelain	2300	1.5	—	Muscle	—	0.41	—
Rubber, natural	1150	0.28	—	Vaseline	—	0.17	—
Rubber, vulcanized				Wood, cross-grain			
Soft	1100	0.13	2010	Balsa	140	0.055	—
Hard	1190	0.16	—	Fir	415	0.11	2720
Sand	1515	0.2–1.0	800	Oak	545	0.17	2385
Snow, fresh	100	0.60	—	White pine	435	0.11	—
Snow, 273 K	500	2.2	—	Yellow pine	640	0.15	2805
Soil, dry	1500	1.0	1900	Wood, radial			
Soil, wet	1900	2.0	2200	Oak	545	0.19	2385
Sugar	1600	0.58	—	Fir	420	0.14	2720
Tissue, human				Wool, ship	145	0.05	—
Skin	—	0.37	—				
Fat layer	—	0.2	—				

Table A2 Saturated Water

Temp., T °C	Saturation Pressure, P_{sat} kPa	Density, ρ kg/m^3 Liquid	Density, ρ kg/m^3 Vapor	Enthalpy of Vapori-zation, h_{fg} kJ/kg	Specific Heat, C_p J/kg·°C Liquid	Specific Heat, C_p J/kg·°C Vapor	Thermal Conductivity, k W/m·°C Liquid	Thermal Conductivity, k W/m·°C Vapor	Dynamic Viscosity, μ kg/m·s Liquid	Dynamic Viscosity, μ kg/m·s Vapor	Prandt! Number, Pr Liquid	Prandt! Number, Pr Vapor	Volume Expansion Coefficient, β 1/K Liquid
0.01	0.6113	999.8	0.0048	2501	4217	1854	0.561	0.0171	1.792×10^{-3}	0.922×10^{-5}	13.5	1.00	$^{-}0.068 \times 10^{-3}$
5	0.8721	999.9	0.0068	2490	4205	1857	0.571	0.0173	1.519×10^{-3}	0.934×10^{-5}	11.2	1.00	0.015×10^{-3}
10	1.2276	999.7	0.0094	2478	4194	i862	0.580	0.0176	1.307×10^{-3}	0.946×10^{-5}	9.45	1.00	0.733×10^{-3}
15	1.7051	999.1	0.0128	2466	4186	1863	0.589	0.0179	1.138×10^{-3}	0.959×10^{-5}	8.09	1.00	0.138×10^{-3}
20	2.339	998.0	0.0173	2454	4182	1867	0.598	0.0182	1.002×10^{-3}	0.973×10^{-5}	7.01	1.00	0.195×10^{-3}
25	3.169	997.0	0.0231	2442	4180	1870	0.607	0.0186	0.891×10^{-3}	0.987×10^{-5}	6.14	1.00	0.47×10^{-3}
30	4.246	996.0	0.0304	2431	4178	1875	0.615	0.0189	0.798×10^{-3}	1.001×10^{-5}	5.42	1.00	0.294×10^{-3}
35	5.628	994.0	0.0397	2419	4178	1880	0.623	0.0192	0.720×10^{-3}	1.016×10^{-5}	4.83	1.00	0.337×10^{-3}
40	7.384	992.1	0.0512	2407	4179	1885	0.631	0.0196	0.653×10^{-3}	1.031×10^{-5}	4.32	1.00	0.377×10^{-3}
45	9.593	990.1	0.0655	2395	4180	1892	0.637	0.0200	0.596×10^{-3}	1.046×10^{-5}	3.91	1.00	0.415×10^{-3}
50	12.35	988.1	0.0831	2383	4181	1900	0.644	0.0204	0.547×10^{-3}	1.062×10^{-5}	3.55	1.00	$.0.451 \times 10^{-3}$
55	15.76	985.2	0.1045	2371	4183	1908	0.649	0.0208	0.504×10^{-3}	1.077 × 10 5	3.25	1.00	0.484×10^{-3}
60	19.94	983.3	0.1304	2359	4185	1916	0.654	0.0212	0.467×10^{-3}	1.093×10^{-5}	2.99	1.00	0.517×10^{-3}
65	25.03	980.4	0.1614	2346	4187	1926	0.659	0.0216	0.433×10^{-3}	1.110×10^{-5}	2.75	1.00	0.548×10^{-3}
70	31.19	977.5	0.1983	2334	4190	1936	0.663	0.0221	0.404×10^{-3}	1.126×10^{-5}	2.55	1.00	0.578×10^{-3}
75	38.58	974.7	0.2421	2321	4193	1948	0.667	0.0225	0.378×10^{-3}	1.142×10^{-5}	2.38	1.00	0.607×10^{-3}
80	47.39	971.8	0.2935	2309	4197	1962	0.670	0.0230	0.355×10^{-3}	1.159×10^{-5}	2.22	1.00	0.653×10^{-3}
85	57.83	968.1	0.3536	2296	4201	1977	0.673	0.0235	0.333×10^{-3}	1.176×10^{-5}	2.08	1.00	0.670×10^{-3}
90	70.14	965.3	0.4235	2283	4206	1993	0.675	0.0240	0.315×10^{-3}	1.193×10^{-5}	1.96	1.00	0.702×10^{-3}
95	84.55	961.5	0.5045	2270	4212	2010	0.677	0.0246	0.297×10^{-3}	1.210×10^{-5}	1.85	1.00	0.716×10^{-3}
100	101.33	957.9	0.5978	2257	4217	2029	0.679	0.0251	0.282×10^{-3}	1.227×10^{-5}	1.75	1.00	0.750×10^{-3}
110	143.27	950.6	0.8263	2230	4229	2071	0.682	0.0262	0.255×10^{-3}	1.261×10^{-5}	1.58	1.00	0.798×10^{-3}
120	198.53	943.4	1.121	2203	4244	2120	0.683	0.0275	Q.232 × 10^{-3}	1.296×10^{-5}	1.44	1.00	0.858×10^{-3}
130	270.1	934.6	1.496	2174	4263	2177	0.684	0.0288	0.213×10^{-3}	1.330×10^{-5}	1.33	1.01	0.913×10^{-3}
140	361.3	921.7	1.965	2145	4286	2244	0.683	0.0301	0.197×10^{-3}	1.365×10^{-5}	1.24	1.02	0.970×10^{-3}
150	475.8	916.6	2.546	2114	4311	2314	0.682	0.0316	0.183×10^{-3}	1.399×10^{-5}	1.16	1.02	1.025×10^{-3}
160	617.8	907.4	3.256	2083	4340	2420	0.680	0.0331	0.170×10^{-3}	1.434×10^{-5}	1.09	1.05	1.145×10^{-3}

(*contd.*)

Table A2 Saturated Water (*contd.*)

Temp., T °C	Saturation Pressure, P_{sat} kPa	Density, ρ kg/m^3 Liquid	Vapor	Enthalpy of Vaporization, h_{fg} kJ/kg	Specific Heat, C_p J/kg·°C Liquid	Vapor	Thermal Conductivity, k W/m·°C Liquid	Vapor	Dynamic Viscosity, μ kg/m·s Liquid	Vapor	Prandtl Number, Pr Liquid	Vapor	Volume Expansion Coefficient, β 1/K Liquid
170	791.7	897.7	4.119	2050	4370	2490	0.677	0.0347	0.160×10^{-3}	1.468×10^{-5}	1.03	1.05	1.178×10^{-3}
180	1002.1	887.3	5.153	2015	4410	2590	0.673	0.0364	0.150×10^{-3}	1.502×10^{-5}	0.983	1.07	1.210×10^{-3}
190	1254.4	876.4	6.388	1979	4460	2710	0.669	0.0382	0.142×10^{-3}	1.537×10^{-5}	0.947	1.09	1.280×10^{-3}
200	1553.8	864.3	7.852	1941	4500	2840	0.663	0.0401	0.134×10^{-3}	1.571×10^{-5}	0.910	1.11	1.350×10^{-3}
220	2318	840.3	11.60	1859	4610	3110	0.650	0.0442	0.122×10^{-3}	1.641×10^{-5}	0.865	1.15	1.520×10^{-3}
240	3344	813.7	16.73	1767	4760	3520	0.632	0.0487	0.111×10^{-3}	1.712×10^{-5}	0.836	1.24	1.720×10^{-3}
260	4688	783.7	23.69	1663	4970	4070	0.609	0.0540	0.102×10^{-3}	1.788×10^{-5}	0.832	1.35	2.000×10^{-3}
280	6412	750.8	33.15	1544	5280	4835	0.581	0.0605	0.094×10^{-3}	1.870×10^{-5}	0.854	1.49	2.380×10^{-3}
300	8581	713.8	46.15	1405	5750	5980	0.548	0.0695	0.086×10^{-3}	1.965×10^{-5}	0.902	1.69	2.950×10^{-3}
320	11,274	667.1	64.57	1239	6540	7900	0.509	0.0836	0.078×10^{-3}	2.084×10^{-5}	1.00	1.97	—
340	14,586	610.5	92.62	1028	8240	11,870	0.469	0.110	0.070×10^{-3}	2.255×10^{-5}	1.23	2.43	—
360	18,651	528.3	144.0	720	14,690	25,800	0.427	0.178	0.060×10^{-3}	2.571×10^{-5}	2.06	3.73	—
374.14	22,090	317.0	317.0	0	∞	∞	∞	∞	0.043×10^{-3}	4.313×10^{-5}	—	—	—

Table A3 Saturated Refrigerant-134A

Temp., T	*Saturation Pressure, P*	*Density, ρ* kg/m^3		*Enthalpy of Vapori-zation, h_{fg}*	*Specific Heat, C_p* J/kg·°C		*Thermal Conductivity, k* W/m·°C		*Dynamic Viscosity, μ* kg/m·s		*Prandtl Number, Pr*		*Volume Expansion Coefficient, β* 1/K,	*Surface Tension,*
°C	*kPa*	*Liquid*	*Vapor*	kJ/kg	*Liquid*	*Vapor*	*Liquid*	*Vapor*	*Liquid*	*Vapor*	*Liquid*	*Vapor*	*Liquid*	N/m
−40	51.2	1418	2.773	225.9	1254	748.6	0.1101	0.00811	4.878×10^{-4}	2.550×10^{-6}	5.558	0.235	0.00205	0.01760
−35	66.2	1403	3.524	222.7	1264	764.1	0.1084	0.00862	4.509×10^{-4}	3.003×10^{-6}	5.257	0.266	0.00209	0.01682
−30	84.4	1389	4.429	219.5	1273	780.2	0.1066	0.00913	4.178×10^{-4}	3.504×10^{-6}	4.992	0.299	0.00215	0.01604
−25	106.5	1374	5.509	216.3	1283	797.2	0.1047	0.00963	3.882×10^{-4}	4.054×10^{-6}	4.757	0.335	0.00220	0.01527
−20	132.8	1359	6.787	213.0	1294	814.9	0.1028	0.01013	3.614×10^{-4}	4.651×10^{-6}	4.548	0.374	0.00227	0.01451
−15	164.0	1343	8.288	209.5	1306	833.5	0.1009	0.01063	3.371×10^{-4}	5.295×10^{-6}	4.363	0.4i5	0.00233	0.01376
−10	200.7	1327	10.04	206.0	1318	853.1	0.0989	0.01112	3.150×10^{-4}	5.982×10^{-6}	4.198	0.459	0.00241	0.01302
−5	243.5	1311	12.07	202.4	1330	873.8	0.0968	0.01161	2.947×10^{-4}	6.709×10^{-6}	4.051	0.505	0.00249	0.01229
0	293.0	1295	14.42	198.7	1344	895.6	0.0947	0.01210	2.761×10^{-4}	7.471×10^{-6}	3.919	0.553	0.00258	0.01156
5	349.9	1278	17.12	194.8	1358	918.7	0.0925	0.01259	2.589×10^{-4}	8.264×10^{-6}	3.802	0.603	0.00269	0.01084
10	414.9	1261	20.22	190.8	1374	943.2	0.0903	0.01308	2.430×10^{-4}	9.081×10^{-6}	3.697	0.655	0.00280	0.01014
15	488.7	1244	23.75	186.6	1390	969.4	0.0880	0.01357	2.281×10^{-4}	9.915×10^{-6}	3.604	0.708	0.00293	0.00944
20	572.1	1226	27.77	182.3	1408	997.6	0.0856	0.01406	2.142×10^{-4}	1.075×10^{-5}	3.521	0.763	0.00307	0.00876
25	665.8	1207	32.34	177.8	1427	1028	0.0833	0.01456	2.012×10^{-4}	1.160×10^{-5}	3.448	0.819	0.00324	0.00808
30	770.6	1188	37.53	173.1	1448	1061	0.0808	0.01507	1.888×10^{-4}	1.244×10^{-5}	3.383	0.877	0.00342	0.00742
35	887.5	1168	43.41	168.2	1471	1098	0.0783	0.01558	1.772×10^{-4}	1.327×10^{-5}	3.328	0.935	0.00364	0.00677
40	1017.1	1147	50.08	163.0	1498	1138	0.0757	0.01610	1.660×10^{-4}	1.408×10^{-5}	3.285	0.995	0.00390	0.00613
45	1160.5	1125	57.66	157.6	1529	1184	0.0731	0.01664	1.554×10^{-4}	1.486×10^{-5}	3.253	1.058	0.00420	0.00550
50	1318.6	1102	66.27	151.8	1566	1237	0.0704	0.01720	1.453×10^{-4}	1.562×10^{-5}	3.231	1.123	0.00456	0.00489
55	1492.3	1078	76.11	145.7	1608	1298	0.0676	0.01777	1.355×10^{-4}	1.634×10^{-5}	3.223	1.193	0.00500	0.00429
60	1682.8	1053	87.38	139.1	1659	1372	0.0647	0.01838	1.260×10^{-4}	1.704×10^{-5}	3.229	1.272	0.00554	0.00372
65	1891.0	1026	100.4	132.1	1722	1462	0.0618	0.01902	1.167×10^{-4}	1.771×10^{-5}	3.255	1.362	0.00624	0.00315
70	2118.2	996.2	115.6	124.4	1801	1577	0.0587	0.01972	1.077×10^{-4}	1.839×10^{-5}	3.307	1.471	0.00716	0.00261
75	2365.8	964	133.6	115.9	1907	1731	0.0555	0.02048	9.891×10^{-5}	1.908×10^{-5}	3.400	1.612	0.00843	0.00209
80	2635.2	928.2	155.3	106.4	2056	1948	0.0521	0.02133	9.011×10^{-5}	1.982×10^{-5}	3.558	1.810	0.01031	0.00160
85	2928.2	887.1	182.3	95.4	2287	2281	0.0484	0.02233	8.124×10^{-5}	2.071×10^{-5}	3.837	2.116	0.01336	0.00114
90	3246.9	837.7	217.8	82.2	2701	2865	0.0444	0.02357	7.203×10^{-5}	2.187×10^{-5}	4.380	2.658	0.01911	0.00071
95	3594.1	772.5	269.3	64.9	3675	4144	0.0396	0.02544	6.190×10^{-5}	2.370×10^{-5}	5.746	3.862	0.03343	0.00033
100	3975.1	651.7	376.3	33.9	7959	8785	0.0322	0.02989	4.765×10^{-5}	2.833×10^{-5}	11.77	8.326	0.10047	0.00004

Table A4 Liquids

Temp., T °C	*Density, ρ* kg/m^3	*Specific Heat, C_p* J/kg·°C	*Thermal Conductivity, k* W/m·°C	*Thermal Diffusivity, α* m^2/s	*Dynamic Viscosity, μ* kg/m·s	*Kinematic Viscosity, ν* m^2/s	*Prandtl Number, Pr*	*Volume Expan. Coeff· β* 1/K
				Methane (CH_4)				
–160	420.2	3492	0.1863	1.270×10^{-7}	1.133×10^{-4}	2.699×10^{-7}	2.126	0.00352
–150	405.0	3580	0.1703	1.174×10^{-7}	9.169×10^{-5}	2.264×10^{-7}	1.927	0.00391
–140	388.8	3700	0.1550	1.077×10^{-7}	7.551×10^{-5}	1.942×10^{-7}	1.803	0.00444
–130	371.1	3875	0.1402	9.749×10^{-8}	6.288×10^{-5}	1.694×10^{-7}	1.738	0.00520
–120	351.4	4146	0.1258	8.634×10^{-8}	5.257×10^{-5}	1.496×10^{-7}	1.732	0.00637
–110	328.8	4611	0.1115	7.356×10^{-8}	4.377×10^{-5}	1.331×10^{-7}	1.810	0.00841
–100	301.0	5578	0.0967	5.761×10^{-8}	3.577×10^{-5}	1.188×10^{-7}	2.063	0.01282
–90	261.7	8902	0.0797	3.423×10^{-8}	2.761×10^{-5}	1.055×10^{-7}	3.082	0.02922
				Methanol [$C_3H_3(OH)$]				
20	788.4	2515	0.1987	1.002×10^{-7}	5.857×10^{-4}	7.429×10^{-7}	7.414	0.00118
30	779.1	2577	0.1980	9.862×10^{-8}	5.088×10^{-4}	6.531×10^{-7}	6.622	0.00120
40	769.6	2644	0.1972	9.690×10^{-8}	4.460×10^{-4}	5.795×10^{-7}	5.980	0.00123
50	760.1	2718	0.1965	9.509×10^{-8}	3.942×10^{-4}	5.185×10^{-7}	5.453	0.00127
60	750.4	2798	0.1957	9.320×10^{-8}	3.510×10^{-4}	4.677×10^{-7}	5.018	0.00132
70	740.4	2885	0.1950	9.128×10^{-8}	3.146×10^{-4}	4.250×10^{-7}	4.655	0.00137
				Isobutane (R600a)				
–100	683.8	1881	0.1383	1.075×10^{-7}	9.305×10^{-4}	1.360×10^{-6}	12.65	0.00142
–75	659.3	1970	0.1357	1.044×10^{-7}	5.624×10^{-4}	8.531×10^{-7}	8.167	0.00150
–50	634.3	2069	0.1283	9.773×10^{-8}	3.769×10^{-4}	5.942×10^{-7}	6.079	0.00161
–25	608.2	2180	0.1181	8.906×10^{-8}	2.688×10^{-4}	4.420×10^{-7}	4.963	0.00177
0	580.6	2306	0.1068	7.974×10^{-8}	1.993×10^{-4}	3.432×10^{-7}	4.304	0.00199
25	550.7	2455	0.0956	7.069×10^{-8}	1.510×10^{-4}	2.743×10^{-7}	3.880	0.00232
50	517.3	2640	0.0851	6.233×10^{-8}	1.155·x 10^{-4}	2.233×10^{-7}	3.582	0.00286
75	478.5	2896	0.0757	5.460×10^{-8}	8.785×10^{-5}	1.836"X 10^{-7}	3.363	0.00385
100	429.6	3361	0.0669	4.634×10^{-8}	6.483×10^{-5}	1.509×10^{-7}	3.256	0.00628

(*contd.*)

Table A4 Liquids (*contd.*)

Temp., T °C	*Density*, ρ kg/m^3	*Specific Heat*, C_p J/kg·°C	*Thermal Conductivity, k* W/m·°C	*Thermal Diffusivity,* α m^2/s	*Dynamic Viscosity,* μ kg/m·s	*Kinematic Viscosity,* ν m^2/s	*Prandtl Number, Pr*	*Volume Expan. Coeff·* β 1/K
				Glycerin				
0	1276	2262	0.2820	9.773×10^{-8}	10.49	8.219×10^{-3}	84101	
5	1273	2288	0.2835	9.732×10^{-8}	6.730	5.287×10^{-3}	54327	
10	1270	2320	0.2846	9.662×10^{-8}	4.241	3.339×10^{-3}	34561	
15	1267	2354	0.2856	9.576×10^{-8}	2.496	1.970×10^{-3}	20570	
20	1264	2386	0.2860	9.484×10^{-8}	1.519	1.201×10^{-3}	12671	
25	1261	2416	0.2860	9.388×10^{-8}	0.9934	7.878×10^{-4}	8392	
30	1258	2447	0.2860	9.291×10^{-8}	0.6582	5.232×10^{-4}	5631	
35	1255	2478	0.2860	9.195×10^{-8}	0.4347	3.464×10^{-4}	3767	
40	1252	2513	0.2863	9.101×10^{-8}	0.3073	2.455×10^{-4}	2697	
				Engine Oil (unused)				
0	899.0	1797	0.1469	9.097×10^{-8}	3.814	4.242×10^{-3}	46636	0.00070
20	888.1	1881	0.1450	8.680×10^{-8}	0.8374	9.429×10^{-4}	10863	0.00070
40	876.0	1964	0.1444	8.391×10^{-8}	0.2177	2.485×10^{-4}	2962	0.00070
60	863.9	2048	0.1404	7.934×10^{-8}	0.07399	8.565×10^{-5}	1080	0.00070
80	852.0	2132	0.1380	7.599×10^{-8}	0.03232	3.794×10^{-5}	499.3	0.00070
100	840.0	2220	0.1367	7.330×10^{-8}	0.01718	2.046×10^{-5}	279.1	0.00070
120	828.9	2308	0.1347	7.042×10^{-8}	0.01029	1.241×10^{-5}	176.3	0.00070
140	816.8	2395	0.1330	6.798×10^{-8}	0.006558	8.029×10^{-6}	118.1	0.00070
150	810.3	2441	0.1327	6.708×10^{-8}	0.005344	6.595×10^{-6}	98.31	0.00070

Table A5 Liquids Metals

Temp., T °C	Density, ρ kg/m^3	Specific Heat, C_p J/kg·°C	Thermal Conductivity, k W/m·°C	Thermal Diffusivity, α m^2/s	Dynamic Viscosity, μ kg/m·s	Kinematic Viscosity, ν m^2/s	Prandtl Number, Pr	Volume Expan. Coeff. β 1/K
Mercury (Hg) Melting point: – 39°C								
0	13595	140.4	8.18200	4.287×10^{-6}	1.687×10^{-3}	1.241×10^{-7}	0.0289	1.810×10^{-4}
25	13534	139.4	8.51533	4.514×10^{-6}	1.534×10^{-3}	1.133×10^{-7}	0.0251	1.810×10^{-4}
50	13473	138.6	8.83632	4.734×10^{-6}	1.423×10^{-3}	1.056×10^{-7}	0.0223	1.810×10^{-4}
75	13412	137.8	9.15632	4.956×10^{-6}	1.316×10^{-3}	9.819×10^{-8}	0.0198	1.810×10^{-4}
100	13351	137.1	9.46706	5.170×10^{-6}	1.245×10^{-3}	9.326×10^{-8}	0.0180	1.810×10^{-4}
150	13231	136.1	10.07780	5.595×10^{-6}	1.126×10^{-3}	8.514×10^{-8}	0.0152	1.810×10^{-4}
200	13112	135.5	10.65465	5.996×10^{-6}	1.043×10^{-3}	7.959×10^{-8}	0.0133	1.815×10^{-4}
250	12993	135.3	11.18150	6.363×10^{-6}	9.820×10^{-4}	7.558×10^{-8}	0.0119	1.829×10^{-4}
300	12873	135.3	11.68150	6.705×10^{-6}	9.336×10^{-4}	7.252×10^{-8}	0.0108	1.854×10^{-4}
Bismuth (Bi) Melting point: 271°C								
350	9969	146.0	16.28	$1..118\times10^{-5}$	1.540×10^{-3}	1.545×10^{-7}	0.01381	
400	9908	148.2	16.10	1.096×10^{-5}	1.422×10^{-3}	1.436×10^{-7}	0.01310	
500	9785	152.8	15.74	1.052×10^{-5}	1.188×10^{-3}	1.215×10^{-7}	0.01154	
600	9663	157.3	15.60	1.026×10^{-5}	1.013×10^{-3}	1.048×10^{-7}	0.01022	
700	9540	161.8	15.60	1.010×10^{-5}	8.736×10^{-4}	9.157×10^{-8}	0.00906	
Lead (Pb) Melting point: 327°C								
400	10506	158	15.97	9.623×10^{-6}	2.277×10^{-3}	2.167×10^{-7}	0.02252	
450	10449	156	15.74	9.649×10^{-6}	2.065×10^{-3}	1.976×10^{-7}	0.02048	
500	10390	155	15.54	9.651×10^{-6}	1.884×10^{-3}	1.814×10^{-7}	0.01879	
550	10329	155	15.39	9.610×10^{-6}	1.758×10^{-3}	1.702×10^{-7}	0.01771	
600	10267	155	15.23	9.568×10^{-6}	1.632×10^{-3}	1.589×10^{-7}	0.01661	
650	10206	155	15.07	9.526×10^{-6}	1.505×10^{-3}	1.475×10^{-7}	0.01549	
700	10145	155	14.91	9.483×10^{-6}	1.379×10^{-3}	1.360×10^{-7}	0.01434	

(*contd.*)

Table A5 Liquids Metals (*contd.*)

Temp., T °C	*Density, ρ* kg/m³	*Specific Heat, C_p* J/kg·°C	*Thermal Conductivity, k* W/m·°C	*Thermal Diffusivity, α* m²/s	*Dynamic Viscosity, μ* kg/m·s	*Kinematic Viscosity, ν* m²/s	*Prandtl Number, Pr*	*Volume Expan. Coeff. β* 1/K
Sodium (Na) Melting point: 98°C								
100	927.3	1378	85.84	6.718×10^{-5}	6.892×10^{-4}	7.432×10^{-7}	0.01106	
200	902.5	1349	80.84	6.639×10^{-5}	5.385×10^{-4}	5.967×10^{-7}	0.008987	
300	877.8	1320	75.84	6.544×10^{-5}	3.878×10^{-4}	4.418×10^{-7}	0.006751	
400	853.0	1296	71.20	6.437×10^{-5}	2.720×10^{-4}	3.188×10^{-7}	0.004953	
500	828.5	1284	67.41	6.335×10^{-5}	2.411×10^{-4}	2.909×10^{-7}	0.004593	
600	804.0	1272	63.63	6.220×10^{-5}	2.101×10^{-4}	2.614×10^{-7}	0.004202	
Potassium (K) Melting point: 64°C								
200	795.2	790.8	43.99	6.995×10^{-5}	3.350×10^{-4}	4.213×10^{-7}	0.006023	
300	771.6	772.8	42.01	7.045×10^{-5}	2.667×10^{-4}	3.456×10^{-7}	0.004906	
400	748.0	754.8	40.03	7.090×10^{-5}	1.984×10^{-4}	2.652×10^{-7}	0.00374	
500	723.9	750.0	37.81	6.964×10^{-5}	1.668×10^{-4}	2.304×10^{-7}	0.003309	
600	699.6	750.0	35.50	6.765×10^{-5}	1.487×10^{-4}	2.126×10^{-7}	0.003143	
Sodium-Potassium (%22Na-%78K) Melting point: –11°C								
100	847.3	944.4	25.64	3.205×10^{-5}	5.707×10^{-4}	6.736×10^{-7}	0.02102	
200	823.2	922.5	26.27	3.459×10^{-5}	4.587×10^{-4}	5.572×10^{-7}	0.01611	
300	799.1	900.6	26.89	3.736×10^{-5}	3.467×10^{-4}	4.339×10^{-7}	0.01161	
400	775.0	879.0	27.50	4.037×10^{-5}	2.357×10^{-4}	3.041×10^{-7}	0.00753	
500	751.5	880.1	27.89	4.217×10^{-5}	2.108×10^{-4}	2.805×10^{-7}	0.00665	
600	728.0	881.2	28.28	4.408×10^{-5}	1.859×10^{-4}	2.553×10^{-7}	0.00579	

Table A6 Air at One Atmospheric Pressure

Temp., T °C	Density, ρ kg/m^3	Specific Heat, C_p J/kg·°C	Thermal Conductivity, k W/m·°C	Thermal Diffusivity, α m^2/s	Dynamic Viscosity, μ kg/m·s	Kinematic Viscosity, ν m^2/s	Prandtl Number, Pr
−150	2.866	983	0.01171	4.158×10^{-6}	8.636×10^{-6}	3.013×10^{-6}	0.7246
−100	2.038	966	0.01582	8.036×10^{-6}	1.189×10^{-6}	5.837×10^{-6}	0.7263
−50	1.582	999	0.01979	1.252×10^{-5}	1.474×10^{-5}	9.319×10^{-6}	0.7440
−40	1.514	1002	0.02057	1.356×10^{-5}	1.527×10^{-5}	1.008×10^{-5}	0.7436
−30	1.451	1004	0.02134	1.465×10^{-5}	1.579×10^{-5}	1.087×10^{-5}	0.7425
−20	1.394	1005	0.02211	1.578×10^{-5}	1.630×10^{-5}	1.169×10^{-5}	0.7408
−10	1.341	1006	0.02288	1.696×10^{-5}	1.680×10^{-5}	1.252×10^{-5}	0.7387
0	1.292	1006	0.02364	1.818×10^{-5}	1.729×10^{-5}	1.338×10^{-5}	0.7362
5	1.269	1006	0.02401	1.880×10^{-5}	1.754×10^{-5}	1.382×10^{-5}	0.7350
10	1.246	1006	0.02439	1.944×10^{-5}	1.778×10^{-5}	1.426×10^{-5}	0.7336
15	1.225	1007	0.02476	2.009×10^{-5}	1.802×10^{-5}	1.470×10^{-5}	0.7323
20	1.204	1007	0.02514	2.074×10^{-5}	1.825×10^{-5}	1.516×10^{-5}	0.7309
25	1.184	1007	0.02551	2.141×10^{-5}	1.849×10^{-5}	1.562×10^{-5}	0.7296
30	1.164	1007	0.02588	2.208×10^{-5}	1.872×10^{-5}	1.608×10^{-5}	0.7282
35	1.145	1007	0.02625	2.277×10^{-5}	1.895×10^{-5}	1.655×10^{-5}	0.7268
40	1.127	1007	0.02662	2.346×10^{-5}	1.918×10^{-5}	1.702×10^{-5}	0.7255
45	1.109	1007	0.02699	2.416×10^{-5}	1.941×10^{-5}	1.750×10^{-5}	0.7241
50	1.092	1007	0.02735	2.487×10^{-5}	1.963×10^{-5}	1.798×10^{-5}	0.7228
60	1.059	1007	0.02808	2.632×10^{-5}	2.008×10^{-5}	1.896×10^{-5}	0.7202
70	1.028	1007	0.02881	2.780×10^{-5}	2.052×10^{-5}	1.995×10^{-5}	0.7177
80	0.9994	1008	0.02953	2.931×10^{-5}	2.096×10^{-5}	2.097×10^{-5}	0.7154
90	0.9718	1008	0.03024	3.086×10^{-5}	2.139×10^{-5}	2.201×10^{-5}	0.7132
100	0.9458	1009	0.03095	3.243×10^{-5}	2.181×10^{-5}	2.306×10^{-5}	0.7111
120	0.8977	1011	0.03235	3.565×10^{-5}	2.264×10^{-5}	2.522×10^{-5}	0.7073
140	0.8542	1013	0.03374	3.898×10^{-5}	2.345×10^{-5}	2.745×10^{-5}	0.7041
160	0.8148	1016	0.03511	4.241×10^{-5}	2.420×10^{-5}	2.975×10^{-5}	0.7014
180	0.7788	1019	0.03646	4.593×10^{-5}	2.504×10^{-5}	3.212×10^{-5}	0.6992
200	0.7459	1023	0.03779	4.954×10^{-5}	2.577×10^{-5}	3.455×10^{-5}	0.6974
250	0.6746	1033	0.04104	5.890×10^{-5}	2.760×10^{-5}	4.091×10^{-5}	0.6946
300	0.6158	1044	0.04418	6.871×10^{-5}	2.934×10^{-5}	4.765×10^{-5}	0.6935
350	0.5664	1056	0.04721	7.892×10^{-5}	3.101×10^{-5}	5.475×10^{-5}	0.6937
400	0.5243	1069	0.05015	8.951×10^{-5}	3.261×10^{-5}	6.219×10^{-5}	0.6948
450	0.4880	1081	0.05298	1.004×10^{-4}	3.415×10^{-5}	6.997×10^{-5}	0.6965
500	0.4565	1093	0.05572	1.117×10^{-4}	3.563×10^{-5}	7.806×10^{-5}	0.6986
600	0.4042	1115	0.06093	1.352×10^{-4}	3.846×10^{-5}	9.515×10^{-5}	0.7037
700	0.3627	1135	0.06581	1.598×10^{-4}	4.111×10^{-5}	1.133×10^{-4}	0.7092
800	0.3289	1153	0.07037	1.855×10^{-4}	4.362×10^{-5}	1.326×10^{-4}	0.7149
900	0.3008	1169	0.07465	2.122×10^{-4}	4.600×10^{-5}	1.529×10^{-4}	0.7206
1000	0.2772	1184	0.07868	2.398×10^{-4}	4.826×10^{-5}	1.741×10^{-4}	0.7260
1500	0.1990	1234	0.09599	3.908×10^{-4}	5.817×10^{-5}	2.922×10^{-4}	0.7478
2000	0.1553	1264	0.11113	5.664×10^{-4}	6.630×10^{-5}	4.270×10^{-4}	0.7539

Table A7 Gases at 1 Atmospheric Pressure

Temp., T °C	*Density, ρ* kg/m^3	*Specific Heat, C_p* J/kg·°C	*Thermal Conductivity, k* W/m·°C	*Thermal Diffusivity, α* m^2/s	*Dynamic Viscosity, μ* kg/m·s	*Kinematic Viscosity, ν* m^2/s	*Prandtl Number,* Pr
			Carbon dioxide, CO_2				
–50	2.4035	746	0.01051	5.860×10^{-6}	1.129×10^{-5}	4.699×10^{-6}	0.8019
0	1.9635	811	0.01456	9.141×10^{-6}	1.375×10^{-5}	7.003×10^{-6}	0.7661
50	1.6597	866.6	0.01858	1.291×10^{-5}	1.612×10^{-5}	9.714×10^{-6}	0.7520
100	1.4373	914.8	0.02257	1.716×10^{-5}	$1{:}841 \times 10^{-5}$	1.281×10^{-5}	0.7464
150	1.2675	957.4	0.02652	2.186×10^{-5}	2.063×10^{-5}	1.627×10^{-5}	0.7445
200	1.1336	995.2	0.03044	2.698×10^{-5}	2.276×10^{-5}	2.008×10^{-5}	0.7442
300	0.9358	1060	0.03814	3.847×10^{-5}	2.682×10^{-5}	2.866×10^{-5}	0.7450
400	0.7968	1112	0.04565	5.151×10^{-5}	$3{,}061 \times 10^{-5}$	3.842×10^{-5}	0.7458
500	0.6937	1156	0.05293	6.600×10^{-5}	3.416×10^{-5}	4.924×10^{-5}	0.7460
1000	0.4213	1292	0.08491	1.560×10 .	4.898×10^{-5}	1.162×10^{-4}	0.7455
1500	0.3025	1356	0.10688	2.606×10^{-4}	6.106×10^{-5}	2.019×10^{-4}	0.7745
2000	0.2359	1387	0.11522	3.521×10^{-4}	7.322×10^{-5}	3.103×10^{-4}	0.8815
			Carbon monoxide, CO				
–50	1.5297	1081	0.01901	1.149×10^{-5}	1.378×10^{-5}	9.012×10^{-6}	0.7840
0	1.2497	1048	0.02278	1.739×10^{-5}	1.629×10^{-5}	1.303×10^{-5}	0.7499
50	1.0563	1039	0.02641	2.407×10^{-5}	1.863×10^{-5}	1.764×10^{-5}	0.7328
100	0.9148	1041	0.02992	3.142×10^{-5}	2.080×10^{-5}	2.274×10^{-5}	0.7239
150	0.8067	1049	0.03330	3.936×10^{-5}	2.283×10^{-5}	2.830×10^{-5}	0.7191
200	0.7214	1060	0.03656	4.782×10^{-5}	2.472×10^{-5}	3.426×10^{-5}	0.7164
300	0.5956	1085	0.04277	6.619×10^{-5}	2.812×10^{-5}	4.722×10^{-5}	0.7134
400	0.5071	1111	0.04860	8.628×10^{-5}	3.111×10^{-5}	6.136×10^{-5}	0.7111
500	0.4415	1135	0.05412	1.079×10^{-4}	3.379×10^{-5}	7.653×10^{-5}	0.7087
1000	0.2681	1226	0.07894	2.401×10^{-4}	4.557×10^{-5}	1.700×10^{-4}	0.7080
1500	0.1925	1279	0.10458	4.246×10^{-4}	6.321×10^{-5}	3.284×10^{-4}	0.7733
2000	0.1502	1309	0.13833	7.034×10^{-4}	9.826×10^{-5}	6.543×10^{-4}	0.9302
			Methane, CH_4				
–50	0.8761	2243	0.02367	1.204×10^{-5}	8.564×10^{-6}	9.774×10^{-6}	0.8116
0	0.7158	2217	0.03042	1.917×10^{-5}	1.028×10^{-5}	1.436×10^{-5}	0.7494
50	0.6050	2302	0.03766	2.704×10^{-5}	1.191×10^{-5}	1.969×10^{-5}	0.7282
100	0.5240	2443	0.04534	3.543×10^{-5}	1.345×10^{-5}	2.567×10^{-5}	0.7247
150	0.4620	2611	0.05344	4.431×10^{-5}	1.491×10^{-5}	3.227×10^{-5}	0.7284
200	0.4132	2791	0.06194	5.370×10^{-5}	1.630×10^{-5}	3.944×10^{-5}	0.7344
300	0.3411	3158	0.07996	7.422×10^{-5}	1.886×10^{-5}	5.529×10^{-5}	0.7450
400	0.2904	3510	0.09918	9.727×10^{-5}	2.119×10^{-5}	7.297×10^{-5}	0.7501
500	0.2529	3836	0.11933	1.230×10^{-4}	2.334×10^{-5}	9.228×10^{-5}	0.7502
1000	0.1536	5042	0.22562	2.914×10^{-4}	3.281×10^{-5}	2.136×10^{-4}	0.7331
1500	0.1103	5701	0.31857	5.068×10^{-4}	4.434×10^{-5}	4.022×10^{-4}	0.7936
2000	0.0860	6001	0.36750	7.120×10^{-4}	6.360×10^{-5}	7.395×10^{-4}	1.0386

(contd.)

Table A7 Gases at 1 Atmospheric Pressure (*contd.*)

Temp., T °C	*Density, ρ* kg/m^3	*Specific Heat, C_p* J/kg·°C	*Thermal Conductivity, k* W/m·°C	*Thermal Diffusivity, α* m^2/s	*Dynamic Viscosity, μ* kg/m·s	*Kinematic Viscosity, ν* m^2/s	*Prandtl Number,* Pr
				Hydrogen, H_2			
−50	0.11010	12635	0.1404	1.009×10^{-4}	7.293×10^{-6}	6.624×10^{-5}	0.6562
0	0.08995	13920	0.1652	1.319×10^{-4}	8.391×10^{-6}	9.329×10^{-5}	0.7071
50	0.07603	14349	0.1881	1.724×10^{-4}	9.427×10^{-6}	1.240×10^{-4}	0.7191
100	0.06584	14473	0.2095	2.199×10^{-4}	1.041×10^{-5}	1.582×10^{-4}	0.7196
150	0.05806	14492	0.2296	2.729×10^{-4}	1.136×10^{-5}	1.957×10^{-4}	0.7174
200	0.05193	14482	0.2486	3.306×10^{-4}	1.228×10^{-5}	2.365×10^{-4}	0.7155
300	0.04287	14481	0.2843	4.580×10^{-4}	1.403×10^{-5}	3.274×10^{-4}	0.7149
400	0.03650	14540	0.3180	5.992×10^{-4}	1.570×10^{-5}	4.302×10^{-4}	0.7179
500	0.03178	14653	0.3509	7.535×10^{-4}	1.730×10^{-5}	5.443×10^{-4}	0.7224
1000	0.01930	15577	0.5206	1.732×10^{-3}	2.455×10^{-5}	1.272×10^{-3}	0.7345
1500	0.01386	16553	0.6581	2.869×10^{-3}	3.099×10^{-5}	2.237×10^{-3}	0.7795
2000	0.01081	17400	0.5480	2.914×10^{-3}	3.690×10^{-5}	3.414×10^{-3}	1.1717
				Nitrogen, N_2			
−50	1.5299	957.3	0.02001	1.366×10^{-5}	1.390×10^{-5}	9.091×10^{-6}	0.6655
0	1.2498	1035	0.02384	1.843×10^{-5}	1.640×10^{-5}	1.312×10^{-5}	0.7121
50	1.0564	1042	0.02746	2.494×10^{-5}	1.874×10^{-5}	1.774×10^{-5}	0.7114
100	0.9149	1041	0.03090	3.244×10^{-5}	2.094×10^{-5}	2.289×10^{-5}	0.7056
150	0.8068	1043	0.03416	4.058×10^{-5}	2.300×10^{-5}	2.851×10^{-5}	0.7025
200	0.7215	1050	0.03727	4.921×10^{-5}	2.494×10^{-5}	3.457×10^{-5}	0.7025
300	0.5956	1070	0.04309	6.758×10^{-5}	2.849×10^{-5}	4.783×10^{-5}	0.7078
400	0.5072	1095	0.04848	8.727×10^{-5}	3.166×10^{-5}	6.242×10^{-5}	0.7153
500	0.4416	1120	0.05358	1.083×10^{-4}	3.451×10^{-5}	7.816×10^{-5}	0.7215
1000	0.2681	1213	0.07938	2.440×10^{-4}	4.594×10^{-5}	1.713×10^{-4}	0.7022
1500	0.1925	1266	0.11793	4.839×10^{-4}	5.562×10^{-5}	2.889×10^{-4}	0.5969
2000	0.1502	1297	0.18590	9.543×10^{-4}	6.426×10^{-5}	4.278×10^{-4}	0.4483
				Oxygen, O_2			
−50	1.7475	984.4	0.02067	1.201×10^{-5}	1.616×10^{-5}	9.246×10^{-6}	0.7694
0	1.4277	928.7	0.02472	1.865×10^{-5}	1.916×10^{-5}	1.342×10^{-5}	0.7198
50	1.2068	921.7	0.02867	2.577×10^{-5}	2.194×10^{-5}	1.818×10^{-5}	0.7053
100	1.0451	931.8	0.03254	3.342×10^{-5}	2.451×10^{-5}	2.346×10^{-5}	0.7019
150	0.9216	947.6	0.03637	4.164×10^{-5}	2.694×10^{-5}	2.923×10^{-5}	0.7019
200	0.8242	964.7	0.04014	5.048×10^{-5}	2.923×10^{-5}	3.546×10^{-5}	0.7025
300	0.6804	997.1	0.04751	7.003×10^{-5}	3.350×10^{-5}	4.923×10^{-5}	0.7030
400	0.5793	1025	0.05463	9.204×10^{-5}	3.744×10^{-5}	6.463×10^{-5}	0.7023
500	0.5044	1048	0.06148	1.163×10^{-4}	4.114×10^{-5}	8.156×10^{-5}	0.7010
1000	0.3063	1121	0.09198	2.678×10^{-4}	5.732×10^{-5}	1.871×10^{-4}	0.6986
1500	0.2199	1165	0.11901	4.643×10^{-4}	7.133×10^{-5}	3.243×10^{-4}	0.6985
2000	0.1716	1201	0.14705	7.139×10^{-4}	8.417×10^{-5}	4.907×10^{-4}	0.6873

(*contd.*)

Table A7 Gases at 1 Atmospheric Pressure (*contd.*)

Temp., T °C	*Density,* ρ kg/m^3	*Specific Heat,* C_p J/kg·°C	*Thermal Conductivity, k* W/m·°C	*Thermal Diffusivity,* α m^2/s	*Dynamic Viscosity,* μ kg/m·s	*Kinematic Viscosity,* ν m^2/s	*Prandtl Number,* Pr
Water vapor, H_2O							
−50	0.9839	1892	0.01353	7.271×10^{-6}	7.187×10^{-6}	7.305×10^{-6}	1.0047
0	0.8038	1874	0.01673	1.110×10^{-5}	8.956×10^{-6}	1.114×10^{-5}	1.0033
50	0.6794	1874	0.02032	1.596×10^{-5}	1.078×10^{-5}	1.587×10^{-5}	0.9944
100	0.5884	1887	0.02429	2.187×10^{-5}	1.265×10^{-5}	2.150×10^{-5}	0.9830
150	0.5189	1908	0.02861	2.890×10^{-5}	1.456×10^{-5}	2.806×10^{-5}	0.9712
200	0.4640	1935	0.03326	3.705×10^{-5}	1.650×10^{-5}	3.556×10^{-5}	0.9599
300	0.3831	1997	0.04345	5.680×10^{-5}	2.045×10^{-5}	5.340×10^{-5}	0.9401
400	0.3262	2066	0.05467	8.114×10^{75}	2.446×10^{-5}	7.498×10^{-5}	0.9240
500	0.2840	2137	0.06677	1.100×10^{-4}	2.847×10^{-5}	1.002×10^{-4}	0.9108
1000	0.1725	2471	0.13623	3.196×10^{-4}	4.762×10^{-5}	2.761×10^{-4}	0.8639
1500	0.1238	2736	0.21301	6.288×10^{-4}	6.411×10^{-5}	5.177×10^{-4}	0.8233
2000	0.0966	2928	0.29183	1.032×10^{-3}	7.808×10^{-5}	8.084×10^{-4}	0.7833

Table A8 Emissivity of Surfaces

Material	*Temperature, K*	*Emissivity,* ε	*Material*	*Temperature, K*	*Emissivity,* ε
Aluminum			Lead		
Polished	300–900	0.04–0.06	Polished	300–500	0.06–0.08
Commercial sheet	400	0.09	Unoxidized, rough	300	0.43
Heavily oxidized	400–800	0.20–0.33	Oxidized	300	0.63
Anodized	300	0.8	Magnesium, polished	300–500	0.07–0.13
Bismuth, bright	350	0.34	Mercury	300–400	0.09–0.12
Brass			Molybdenum		
Highly polished	50~50	0.03–0.04	Polished	300–2000	0.05–0.21
Polished	350	0.09	Oxidized	600–800	0.80–0.82
Dull plate	30~OO	0.22	Nickel		
Oxidized	450–800	0.6	Polished	500–1200	0.07–0.17
Chromium, polished	300–1400	0.08–0.40	Oxidized	450–1000	0.37–0.57
Copper			Platinum, polished	500–1500	0.06–0.18
Highly polished	300	0.02	Silver, polished	300–1000	0.02–0.07
Polished	300–500	0.04–Q.05	Stainless steel		
Commercial sheet	300	0.15	Polished	300–1000	0.17–0.30
Oxidized	600–1000	0.5–0.8	Lightly oxidized	600–1000	0.30–0.40
Black oxidized	300	0.78	Highly oxidized	600–1000	0.70–0.80
Gold			Steel		
Highly polished	300–1000	0.03–0.06	Polished sheet	300–500	0.08–0.14
Bright foil	300	0.07	Commercial sheet	500–1200	0.20–0.32
Iron			Heavily oxidized	300	0.81
Highly polished	300–500	0.05–0.07	Tin, polished	300	0.05
Case iron	300	0.44	Tungsten		
Wrought iron	300–500	0.28	Polished	300–2500	0.03–0.29
Rusted	300	0.61	Filament	3500	0.39
Oxidized	500–900	0.64–0.78	Zinc		
			Polished	300–800	0.02–0.05
			Oxidized	300	0.25

(*contd.*)

Table A8 Emissivity of Surfaces (*contd.*)

Material	*Temperature, K*	*Emissivity, ε*
Alumina	800–1400	0.65–0.45
Aluminum oxide	600–1500	0.69–0.41
Asbestos	300	0.96
Asphalt pavement	300	0.85–0.93
Brick		
Common	300	0.93–0.96
Fireclay	1200	0.75
Carbon filament	2000	0.53
Cloth	300	0.75–0.90
Concrete	300	0.88–0.94
Glass		
Window	300	0.90–0.95
Pyrex	300–1200	0.82–0.62
Pyroceram	300–1500	0:85–0.57
Ice	273	0.95–0.99
Magnesium oxide	400–800	0.69–0.55
Masonry	300	0.80
Paints		
Aluminum	300	0.40–0.50
Black, lacquer, shiny	300	0.88
Oils, all colors	300	0.92–0.96
Red primer	300	0.93
White acrylic	300	0.90
White enamel	300	0.90
Paper, white	300	0.90
Plaster, white	300	0.93
Porcelain, glazed	300	0.92
Quartz, rough, fused	300	0.93
Rubber		
Hard	300	0.93
Soft	300	0.86
Sand	300	0.90
Silicon carbide	600–1500	0.87–0.85
Skin, human	300	0.95
Snow	273	0.80–0.90
Soil, earth	300	0.93–0.96
Soot	300–500	0.95
Teflon	300–500	0.85–0.92
Water, deep	273–373	0.95–0.96
Wood		
Beech	300	0.94
Oak	300	0.90

Appendix B

Derivation for the Expression of Similarity Parameter η Given by Eq. (4.67)

Equation (4.65) is written here again

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{B.1}$$

The initial and boundary conditions are

$$T(x,\ 0) = T_i$$
$$T(0,\ t) = T_0 \qquad \text{for } t > 0$$

If the dependent variable T is expressed as $\theta = T - T_0$, then Eq. (B.1) becomes

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2} \tag{B.2}$$

The partial differential equation, Eq. (B.2.), can be transformed into an ordinary differential equation by the principle of similarity. This principle allows the temperature profile to be expressed as

$$\frac{\theta}{\theta_i} = f(\eta)$$

where $\theta = T - T_0$, $\theta_i = T_i - T_0$, and η is a non-dimensional similarity parameter which is a combination of x, t and α.

The parameter η can be written as

$$\eta = \frac{1}{2} \alpha^a x^b t^c$$

where the indices a, b, c are the dimensionless numbers to be determined, and the factor 1/2 is inserted for convenience.

Now, we have

$$\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial \eta}\frac{\partial \eta}{\partial t} = \theta_i \frac{df}{d\eta}\left(\frac{c}{2}\alpha^a x^b t^{c-1}\right) = \theta_i c t^{-1}\eta \frac{df}{d\eta}$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial \eta}\frac{\partial \eta}{\partial x} = \theta_i \frac{df}{d\eta}\left(\frac{b}{2}\alpha^a x^{b-1} t^{c}\right)$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial \eta}\left(\frac{\partial \theta}{\partial x}\right)\frac{\partial \eta}{\partial x}$$

$$= \frac{\partial}{\partial \eta}\left[\theta_i \frac{df}{d\eta}\left(\frac{b}{2}\alpha^a x^{b-1} t^{c}\right)\right]\left(\frac{b}{2}\alpha^a x^{b-1} t^{c}\right)$$

$$= \theta_i\left[\left(\frac{b^2}{4}\alpha^{2a} x^{2(b-1)} t^{2c}\right)\frac{d^2 f}{d\eta^2} + \frac{b}{2}(b-1)\,\alpha^a x^{b-2} t^c \frac{df}{d\eta}\right]$$

Substituting the values of $\frac{\partial \theta}{\partial t}$ and $\frac{\partial^2 \theta}{\partial x^2}$ in Eq. (B.2), we have

$$\theta_i c t^{-1}\ \eta \frac{df}{d\eta} = \theta_i\left[\left(\frac{b^2}{4}\alpha^{2a+1} x^{2(b-1)} t^{2c}\right)\frac{d^2 f}{d\eta^2} + \frac{b}{2}(b-1)\,\alpha^{a+1} x^{b-2} t^c \frac{df}{d\eta}\right] \quad \text{(B.3)}$$

For Eq. (B.3) to become an ordinary differential equation, the power of t and x on both sides of the equation must be identical. This can happen if

$$a = -1/2, \qquad b = 1, \qquad c = -1/2$$

and Eq. (B.3) reduces to

$$\frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} = 0$$

where

$$\eta = \frac{x}{2\sqrt{\alpha t}}$$

Alternative method

The solution of Eq. (B.2) with the initial and boundary conditions $\theta(x, 0) = \theta_i$ and $\theta(0, t) = 0$, can be expressed by an implicit functional relation as

$$\theta = f(\theta_i,\ \alpha,\ x,\ t)$$

or

$$F(\theta,\ \theta_i,\ \alpha,\ x,\ t) = 0 \quad \text{(B.4)}$$

There are five variables defining the problem and they can be expressed in terms of three fundamentals dimensions, namely temperature, length and time.

By the use of Buckingham's Π theorem or Rayleish's indicial method of dimensional analysis in relation to the principle of similarity, the five variables of the problem can be reduced to two non-dimensional variables as $\frac{\theta}{\theta_i}$ and $\frac{x}{\sqrt{\alpha t}}$.

Hence the functional relationship given by Eq. (B.4) is reduced to

$$\Phi\left(\frac{\theta}{\theta_i}, \frac{x}{\sqrt{\alpha t}}\right) = 0$$

or

$$\frac{\theta}{\theta_i} = \varphi\left(\frac{x}{\sqrt{\alpha t}}\right)$$

Appendix C

Thomas' Algorithm for the Solution of Tridiagonal System of Equations

Let us consider a system of m linear simultaneous algebric equations with m unknowns, x_1, x_2, …, x_m as follows.

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \text{(C.1)}$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \quad \text{(C.2)}$$

$$a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3 \quad \text{(C.3)}$$

$$\begin{array}{ccccc} . & . & . & & . \\ & . & . & . & . \\ & & . & . & . \end{array}$$

$$a_{m-1\,m-2}\,x_{m-2} + a_{m-1\,m-1}\,x_{m-1} + a_{m-1\,m}\,x_m = b_{m-1} \quad \text{(C.4)}$$

$$a_{m\,m-1}\,x_{m-1} + a_{mm}\,x_m = b_m \quad \text{(C.5)}$$

This is a tridiagonal system of equations with finite coefficients. Thomas' algorithm is essentially the Gaussian elimination method for solving the tridiagonal system of equations.

In this method the lower diagonal terms (a_{21}, a_{32}, a_{43}, …, $a_{m\,m-1}$) are eliminated as given below.

Multiplying Eq. (C.1) by a_{21} and Eq. (C.2) by a_{11}, we have

$$a_{21}a_{11}x_1 + a_{21}a_{12}x_2 = a_{21}\,b_1 \quad \text{(C.6)}$$

$$a_{11}a_{21}x_1 + a_{11}a_{22}x_2 + a_{11}a_{23}x_3 = a_{11}\,b_2 \quad \text{(C.7)}$$

Subtracting Eq. (C.6) from Eq. (C.7), it becomes

$$(a_{11}a_{22} - a_{21}a_{12})x_2 + a_{11}a_{23}x_3 = a_{11}b_2 - a_{21}\,b_1 \quad \text{(C.8)}$$

Dividing Eq. (C.8) by a_{11}, we get

$$\left(a_{22} - \frac{a_{21}\,a_{12}}{a_{11}}\right)x_2 + a_{23}\,x_3 = b_2 - \frac{a_{21}\,b_1}{a_{11}} \quad \text{(C.9)}$$

It has to be noted that Eq. (C.9) has no longer a lower lower-diagonal term. Equation (C.9) is written in a simple form as

$$\boxed{a'_{22}\, x_2 + a_{23}\, x_3 = b'_2} \tag{C.10}$$

where

$$\boxed{a'_{22} = a_{22} - \frac{a_{21}\, a_{12}}{a_{11}}} \tag{C.11}$$

and

$$\boxed{b'_2 = b_2 - \frac{a_{21}}{a_{11}}\, b_1} \tag{C.12}$$

Let us replace Eq. (C.2) by Eq. (C.10) and continue the process of elimination. Now the lower diagonal term, i.e. the first term of Eq. (C.3) is eliminated with the help of Eq. (C.10).

Equation (C.3) is multiplied by a'_{22} and we have

$$a'_{22}a_{32}x_2 + a'_{22}\, a_{33}x_3 + a'_{22}a_{34}x_4 = a'_{22}\, b_3 \tag{C.13}$$

Equation (C.10) is multiplied by a_{32} as

$$a_{32}\, a'_{22}x_2 + a_{32}\, a_{23}x_3 = a_{32}\, b'_2 \tag{C.14}$$

Subtracting Eq. (C.14) from Eq. (C.13), we get

$$(a'_{22}a_{33} - a_{32}\, a_{23})x_3 + a'_{22}a_{34}x_4 = a'_{22}\, b_3 - a_{32}\, b'_2$$

Dividing by a'_{22},

$$\left(a_{33} - \frac{a_{32}\, a_{23}}{a'_{22}}\right)x_3 + a_{34}x_4 = b_3 - \frac{a_{32}}{a'_{22}}\, b'_2 \tag{C.15}$$

Equation (C.15) is written in a simpler form as

$$\boxed{a'_{33}\, x_3 + a_{34}\, x_4 = b'_3} \tag{C.16}$$

$$\boxed{a'_{33} = a_{33} - \frac{a_{32}\, a_{23}}{a'_{22}}} \tag{C.17}$$

where

$$\boxed{b'_3 = b_3 - \frac{a_{32}}{a'_{22}}\, b'_2} \tag{C.18}$$

We replace Eq. (C.3) by Eq. (C.16).

We now observe that a pattern is formed in this process of elimination. Before proceeding further with the elimination process, let us discuss the pattern obtained.

Equation (C.1) is left as it is. Equation (C.2) is replaced by Eq. (C.10) which is obtained by dropping the first term (the term involving x_1) of Eq. (C.2), replacing the main diagonal coefficient a_{22} (the coefficient of the second term) by $a_{22} - \dfrac{a_{21}\, a_{12}}{a_{11}}$ (as given by Eq. (C.11)), keeping the third term unchanged and replacing the term on the right-hand side of the equation by $b_2 - \dfrac{a_{21}}{a_{11}}\, b_1$ (as given by Eq. (C.12))

Equation (C.3) is replaced by Eq (C.16). Comparing these two equations we see exactly the same pattern as described above.

The first term of Eq. (C.3) is dropped. The main diagonal coefficient (coefficient of second term) a_{33} is replaced by

$$a'_{33} = a_{33} - \frac{a_{32}\,a_{23}}{a'_{22}} \qquad \text{(as given by Eq. (C.16))}$$

The term b_3 on the right-hand side is replaced by

$$b'_3 = b_3 - \frac{a_{32}}{a'_{22}}\,b'_2 \qquad \text{(as given by Eq. (C.18))}$$

If this process of elimination is continued in which each and every equation (except Eq. (C.1) of the tridiagonal system of equations (Eqs. (C.1) to (C.5)) is replaced by a new one as described above, we arrive at a final result in obtaining an upper bidiagonal form of equations given by

$$a_{11}x_1 + a_{12}x_2 = b_1 \tag{C.19}$$

$$a'_{22}x_2 + a_{23}x_3 = b'_2 \tag{C.20}$$

$$a'_{33}x_2 + a_{34}x_4 = b'_3 \tag{C.21}$$

$$\vdots$$

$$a'_{m-1\,m-1}\,x_{m-1} + a_{m-1\,m}\,x_m = b'_{m-1} \tag{C.22}$$

$$a'_{mm}\,x_m = b'_m \tag{C.23}$$

where we can write for the diagonal coefficients and the terms on the right-hand sides of the above system of equations in a generic form as

$$a'_{ii} = a_{ii} - \frac{a_{i\,i-1}\,a_{i-1\,i}}{a'_{i-1\,i-1}} \qquad i = 2, 3, \ldots, m \tag{C.24}$$

$$b'_i = b_i - \frac{a_{i\,i-1}}{a'_{i-1\,i-1}}\,b'_{i-1} \qquad i = 2, 3, \ldots, m \tag{C.25}$$

We note that the last equation of the bidiagonal system, i.e. Eq. (C.23) contains only one unknown, namely x_m, and hence

$$x_m = \frac{b'_m}{a'_{mm}} \tag{C.26}$$

The solution of the remaining unknowns is obtained by using sequentially the equations upward in the system. For example, after x_m is obtained from Eq. (C.23), the value of x_{m-1} is determined from Eq. (C.22) as

$$x_{m-1} = \frac{b'_{m-1} - a'_{m-1\,m}\,x_m}{a'_{m-1\,m-1}} \tag{C.27}$$

By inspection, we can replace Eq. (C.27) by a general recursion formula as

$$x_i = \frac{b_i' - a_{i\,i+1}' x_{i+1}}{a_{ii}'} \qquad \text{for } i = (m-1), (m-2), \ldots, 3, 2, 1 \tag{C.28}$$

Therefore, we see that Thomas' algorithm reduces a system of linear, algebraic equations in tridiagonal form to a system of upper bidiagonal form by dropping the first term in each of tridiagonal systems, replacing the coefficient of the main diagonal term by Eq. (C.24) and replacing the right-hand side by Eq. (C.25). This results in the last equation of the system in having only one unknown, namely, x_m. We solve for x_m from Eq. (C.26). Then all other unknowns are found in sequence from Eq. (C.28), starting with $x_i = x_{m-1}$ and ending with $x_i = x_1$.

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