SOLUTIONS MANUAL FOR

Heat Conduction
Fourth Edition

by

Yemen Yener
Sadik Kakac

CRC Press
Taylor & Francis Group

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CHAPTER 1

FOUNDATIONS OF HEAT TRANSFER

PROB. 1.1: Since the flow is steady, the first law (1.31) reduces to

$$\int_{c.s.} \rho \left(h+\frac{1}{2}V^2+gz\right)\vec{V}\cdot\hat{n}\,dA = q_{c.s.} - \dot{W}_{shear} - \dot{W}_{shaft} \int_{c.v.} \dot{q}_e\, d\forall \qquad (1)$$

Assume that all the properties are constant over cross-sections ① and ②. Thus,

$$\int_{c.s.} \rho \left(h+\frac{1}{2}V^2+gz\right)\vec{V}\cdot\hat{n}\,dA = \rho_2\left(h_2+\frac{1}{2}V_2^2+gz_2\right)V_2A_2 - \rho_1\left(h_1+\frac{1}{2}V_1^2+gz_1\right)V_1A_1$$

The continuity condition yields

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 = \dot{m}$$

Thus,

$$\int_{c.s.} \rho \left(h+\frac{1}{2}V^2+gz\right)\vec{V}\cdot\hat{n}\,dA = \dot{m}\left[\left(h_2+\frac{1}{2}V_2^2+gz_2\right)-\left(h_1+\frac{1}{2}V_1^2+gz_1\right)\right]$$

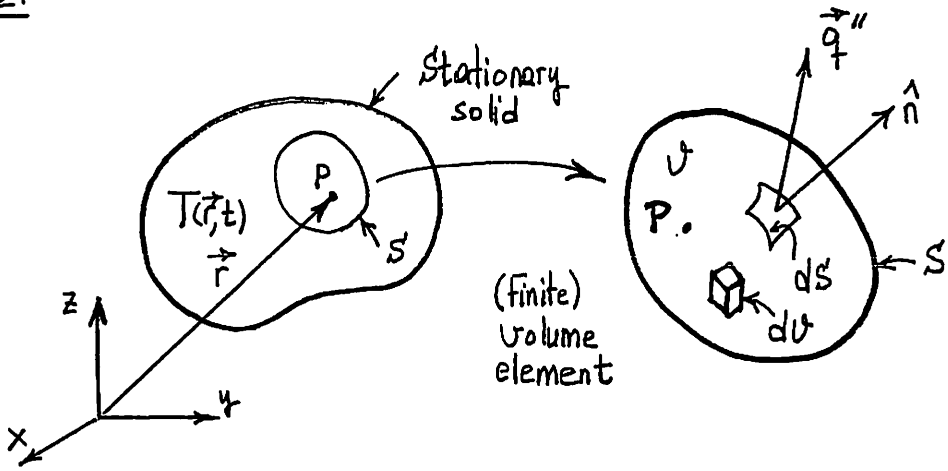
Furthermore, if it is further assumed that the flow is frictionless, then $\dot{W}_{shear}=0$. In addition,

$$q_{c.s.} = q\,, \quad \dot{W}_{shaft} = \dot{W} \quad \text{and} \quad \dot{q}_e = 0$$

Therefore, the steady-flow energy equation (1) can be rewritten as

$$\boxed{h_1+\frac{1}{2}V_1^2+gz_1+\frac{q}{\dot{m}} = h_2+\frac{1}{2}V_2^2+gz_2+\frac{\dot{W}}{\dot{m}}}$$

PROB. 1.2:



Since $\vec{V}=0$, $\dot{W}_{shear}=\dot{W}_{shaft}=0$, Eq.(1.30) reduces to

$$\underbrace{\int_{\vartheta} \rho \frac{\partial e}{\partial t} d\vartheta}_{} = q_s + \int_{\vartheta} \dot{q}_e d\vartheta \qquad \text{—— (1)}$$

$$\int_{\vartheta} \rho \frac{\partial u}{\partial t} d\vartheta = \int_{\vartheta} \rho c \frac{\partial T}{\partial t} d\vartheta \qquad \text{—— (2)}$$

for solids $du = c\,dT$, c = specific heat

Furthermore,

$$q_s = \int_S \vec{q}.\hat{n}\, dS = \int_{\vartheta} \vec{\nabla}.q'' d\vartheta \qquad \text{—— (3)}$$

Here, $\vec{q}''$ is the heat flux vector at dS, where $\hat{n}$ represents the outward-drawn unit vector normal to dS. Then, by substituting Eqs.(2) and (3) into (1), we obtain

$$\int_{\vartheta} \left[-\vec{\nabla}.\vec{q}'' + \dot{q}_e - \rho c \frac{\partial T}{\partial t}\right] d\vartheta = 0$$

Since this volume integral vanishes for every volume element ϑ, its integrand must vanish everywhere, thus yielding

$$\boxed{-\vec{\nabla}\cdot\vec{q}'' + \dot{q}_e = \rho c \frac{\partial T}{\partial t}}$$

PROB. 1.3: Under the conditions stated the 1st law gives:

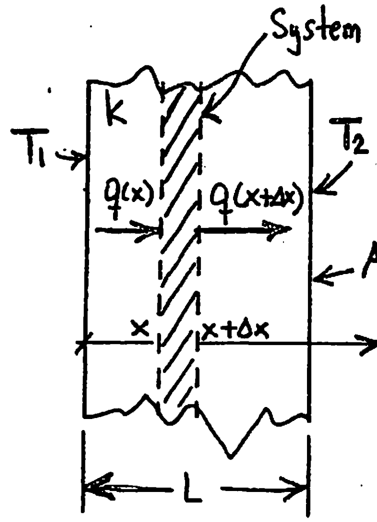
$$q(x) = q(x+\Delta x)$$

As $\Delta x \to 0$,

$$q(x+\Delta x) = q(x) + \frac{dq}{dx}\Delta x$$

Hence, as $\Delta x \to 0$, the first law reduces to

$$\frac{dq}{dx} = 0 \quad \text{or} \quad q(x) = \text{const.}$$



PROB. 1.4:

$$\underset{\text{First law}}{\frac{dq}{dx} = 0} \quad \& \quad \underset{\text{Fourier's law}}{q = -kA\frac{dT}{dx}} \Rightarrow \underbrace{\frac{d}{dx}\left(-kA\frac{dT}{dx}\right) = 0}_{\frac{d^2T}{dx^2} = 0}$$

$$\frac{d^2T}{dx^2} = 0 \Rightarrow T(x) = Ax + B$$

$$\left.\begin{aligned} T(0) = T_1 &\Rightarrow B = T_1 \\ T(L) = T_2 &\Rightarrow A = \frac{T_2 - T_1}{L} \end{aligned}\right\} \quad \therefore \boxed{T(x) = T_1 - \frac{T_1 - T_2}{L}x}$$

PROB. 1.5: The first law gives (see Prob. 1.3)

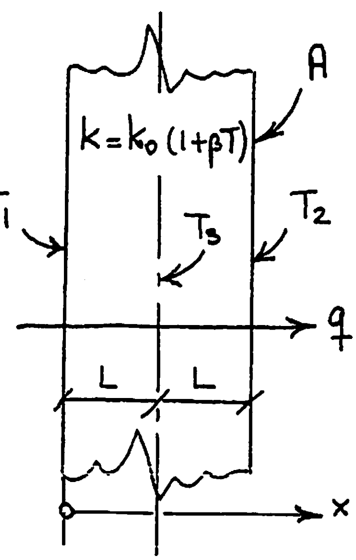
$$q = -kA\frac{dT}{dx} = \text{Const.} \Rightarrow k\frac{dT}{dx} = -\frac{q}{A}$$

$$\text{or} \quad k_0(1+\beta T)dT = -\frac{q}{A}dx$$

$$\therefore [T(x_1) - T(x_2)] + \beta\frac{1}{2}[T^2(x_1) - T^2(x_2)] = -\frac{q}{Ak_0}(x_1 - x_2)$$

$$T_3 - T_1 + \frac{\beta}{2}(T_3^2 - T_1^2) = -\frac{qL}{Ak_0} \quad \text{— (1)}$$

$$T_2 - T_3 + \frac{\beta}{2}(T_2^2 - T_3^2) = -\frac{qL}{Ak_0} \quad \text{— (2)}$$



These two equations yield:

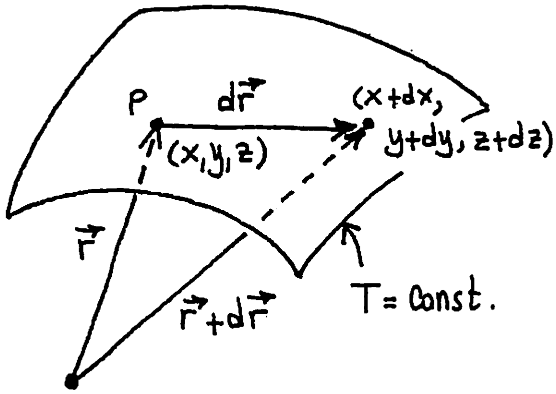
$$\left.\begin{aligned} \beta &= -2\,\frac{T_1+T_2-2T_3}{T_1^2+T_2^2-2T_3^2} \\ k_0 &= \frac{2(q/A)\times L}{(T_2-T_1)\left[1+\frac{\beta}{2}(T_1+T_2)\right]} \end{aligned}\right\} \Rightarrow k(T)=k_0[1+\beta T]$$

PROB. 1.6:

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz$$

$$= \vec{\nabla} T \cdot \underbrace{(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)}_{d\vec{r}}$$

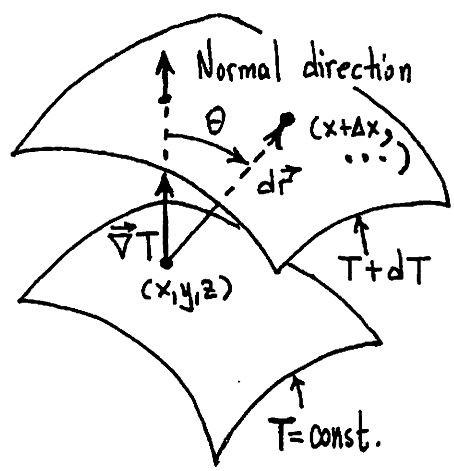
$$= \vec{\nabla} T \cdot d\vec{r}$$



On the T = Const. surface $dT = 0$. Thus, $\vec{\nabla}T$ is a vector normal to all such $d\vec{r}$. On the other hand, $d\vec{r}$ is a vector tangent to the isothermal surface passing through P. Therefore, $\vec{\nabla}T$ is a vector normal to the isothermal surface.

Again, $dT = \vec{\nabla}T \cdot d\vec{r}$

In order for $dT > 0$, $\cos\theta > 0$; that is, $\vec{\nabla}T$ has to point in the direction of increasing temperature as shown.



PROB. 1.7:

$$\vec{\nabla}T \cdot \hat{s} = \left(\hat{i}\frac{\partial T}{\partial x} + \hat{j}\frac{\partial T}{\partial y} + \hat{k}\frac{\partial T}{\partial z}\right) \cdot \left(\hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma\right)$$

Direction Cosines of $\hat{s}$

$$= \frac{\partial T}{\partial x}\underbrace{\cos\alpha}_{\frac{dx}{ds}} + \frac{\partial T}{\partial y}\underbrace{\cos\beta}_{\frac{dy}{ds}} + \frac{\partial T}{\partial z}\underbrace{\cos\gamma}_{\frac{dz}{ds}}$$

$$= \frac{\partial T}{\partial s}$$

PROB. 1.8:

$$T(^\circ C) = 150 - 400x^2$$

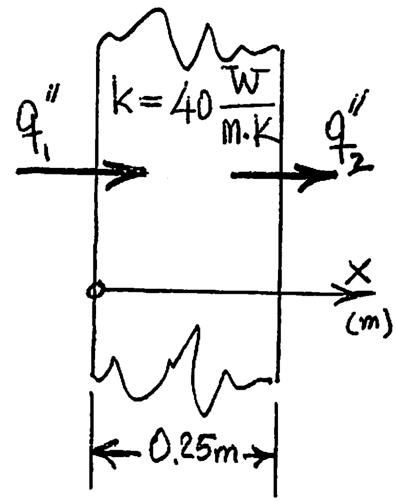
$$q_1'' = -k\left(\frac{\partial T}{dx}\right)_{x=0} = -k(-800x)_{x=0}$$

$$= 0 \ W/m^2$$

$$q_2'' = -k\left(\frac{dT}{dx}\right)_{x=L} = -k(-800L)$$

$$= 40 \times 800 \times 0.25$$

$$= 8{,}000 \ W/m^2$$



PROB. 1.9:

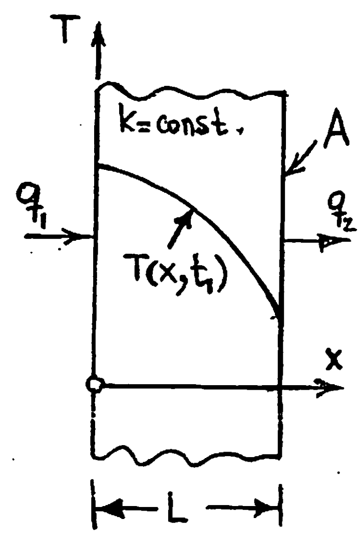
$$q_1 = -kA\left(\frac{\partial T}{\partial x}\right)_{x=0} \quad \text{and} \quad q_2 = -kA\left(\frac{\partial T}{\partial x}\right)_{x=L}$$

$$-\left(\frac{\partial T}{\partial x}\right)_{x=0} < -\left(\frac{\partial T}{\partial x}\right)_{x=L}$$

$$\Downarrow$$

$$q_1 < q_2$$

$\therefore$ The wall is being cooled at t_1.



PROB.1.10:

$$h = \frac{-k\left(\frac{\partial T}{\partial y}\right)_{y=0}}{T_W - T_\infty} = \frac{-(0.62)\frac{W}{m.K} \times (-80 \times 800)\frac{K}{m}}{(100-20)K}$$

$$= 496 \frac{W}{m^2.K}$$

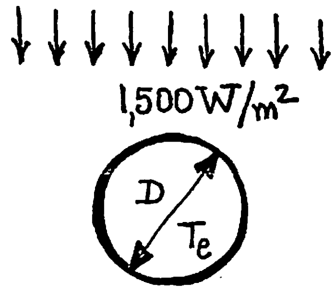
PROB.1.11:

Energy balance yields

$$\alpha_s (1{,}500) \frac{W}{m^2} \times (D \times L)\ m^2$$

$$= \varepsilon (\pi DL)\ m^2 \times \sigma (T_e^4)\ K^4$$

$\uparrow$ $5.67 \times 10^{-8}\ W/(m^2.K^4)$



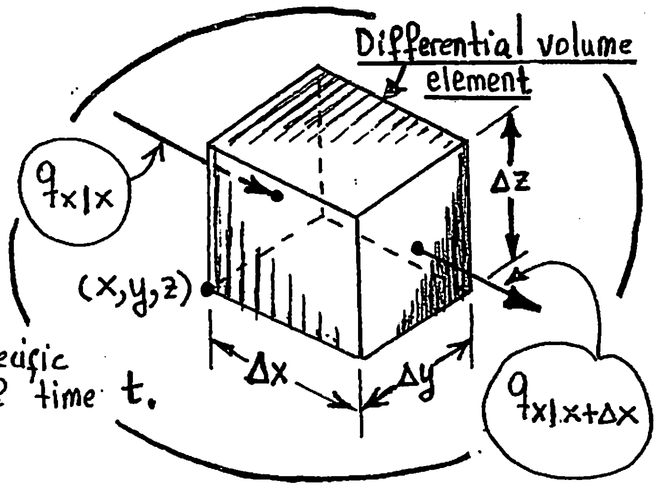
L = Length of cylinder

$$T_e = \frac{1500}{\pi \times 5.67 \times 10^{-8}} = 303K = 30^\circ C.$$

CHAPTER 2

GENERAL HEAT CONDUCTION EQUATION

PROB. 2.1:



- Homogeneous and isotropic solid, i.e., $k = \text{const}$.
- Let there be no heat sources or sinks, i.e., $\dot{q} = 0$.
- Assume the density ρ and specific heat c are independent of time t.

$$\left\{\begin{array}{l}\text{Net rate of heat conduction}\\ \text{into the C.V. in the}\\ x\text{- direction}\end{array}\right\} = q_{x|x} - q_{x|x+\Delta x} = q_{x|\Delta x} - \left\{ q_{x|x} + \frac{\partial q_{x|x}}{\partial x}\Delta x \right\}$$

$$= -\frac{\partial q_{x|x}}{\partial x}\Delta x$$

where

$$q_{x|x} = -k\,(\Delta y \Delta z)\frac{\partial T}{\partial x} \longleftarrow \underline{\text{Rate of heat conduction in the } x\text{-direction at } x \text{ into the C.V.}}$$

$$\therefore\quad q_{x|x} - q_{x|x+\Delta x} = k\frac{\partial^2 T}{\partial x^2}\Delta V, \qquad \Delta V = \Delta x \Delta y \Delta z$$

Similarly,

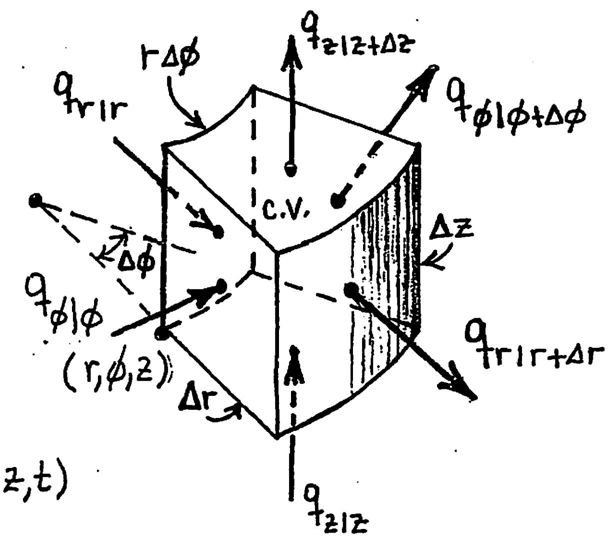
$$q_{y|y} - q_{y|y+\Delta y} = k\frac{\partial^2 T}{\partial y^2}\Delta V \quad \text{and} \quad q_{z|z} - q_{z|z+\Delta z} = k\frac{\partial^2 T}{\partial z^2}\Delta V$$

Thus, energy balance on the C.V. gives

$$k\left\{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right\}\cancel{\Delta V} = \left\{\begin{array}{l}\text{Net rate of increase}\\ \text{of internal energy}\\ \text{in C.V.}\end{array}\right\} = \frac{\partial}{\partial t}\left\{Tc\rho\cancel{\Delta V}\right\}$$

$$\therefore\quad \boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}\;, \quad \alpha = \frac{k}{\rho c}$$

PROB. 2.2: (a)



$T = T(r,\phi,z,t)$

$\Delta V = r\,\Delta r\,\Delta\phi\,\Delta z$

$$q_{r|r} - q_{r|r+\Delta r} = -\frac{\partial q_{r|r}}{\partial r}\Delta r = \frac{\partial}{\partial r}\left[k r\Delta\phi\Delta z\frac{\partial T}{\partial r}\right]\Delta r = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\Delta V$$

$$q_{\phi|\phi} - q_{\phi|\phi+\Delta\phi} = -\frac{\partial q_{\phi|\phi}}{\partial(r\phi)} r\Delta\phi = \frac{\partial}{\partial\phi}\left[k\Delta r\Delta z\frac{\partial T}{\partial(r\phi)}\right]\Delta\phi = \frac{k}{r^2}\frac{\partial^2 T}{\partial\phi^2}\Delta V$$

$$q_{z|z} - q_{z|z+\Delta z} = -\frac{\partial q_{z|z}}{\partial z}\Delta z = \frac{\partial}{\partial z}\left[k r\Delta\phi\Delta r\frac{\partial T}{\partial z}\right]\Delta z = k\frac{\partial^2 T}{\partial z^2}\Delta V$$

$$\therefore \underbrace{k\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2}\right\}\cancel{\Delta V}}_{\text{Net rate of heat conduction into c.v.}} = \underbrace{\frac{\partial}{\partial t}\left\{Tc\rho\cancel{\Delta V}\right\}}_{\text{Net rate of increase of internal energy in c.v.}}$$

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}, \quad \alpha = \frac{k}{\rho c}$$

PROB. 2.3: General heat conduction equation in the rectangular coordinates:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

a) The cylindrical coordinates:

$$\left.\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \\ z &= z \end{aligned}\right\} \quad \begin{aligned} r &= (x^2+y^2)^{1/2} \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \\ z &= z \end{aligned}$$

Thus,

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial x} = \cos\phi\frac{\partial T}{\partial r} - \frac{\sin\phi}{r}\frac{\partial T}{\partial \phi}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial r}\left(\frac{\partial T}{\partial x}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \phi}\left(\frac{\partial T}{\partial x}\right)\frac{\partial \phi}{\partial x}$$

$$= \cos^2\phi\frac{\partial^2 T}{\partial r^2} + 2\frac{\cos\phi\sin\phi}{r^2}\frac{\partial T}{\partial \phi} - 2\frac{\sin\phi\cos\phi}{r}\frac{\partial^2 T}{\partial r\partial \phi}$$

$$+ \frac{\sin^2\phi}{r}\frac{\partial T}{\partial r} + \frac{\sin^2\phi}{r^2}\frac{\partial^2 T}{\partial \phi^2}$$

Similarly,

$$\frac{\partial T}{\partial y} = \sin\phi\frac{\partial T}{\partial r} + \frac{\cos\phi}{r}\frac{\partial T}{\partial \phi}$$

$$\frac{\partial^2 T}{\partial y^2} = \sin^2\phi\frac{\partial^2 T}{\partial r^2} + 2\frac{\sin\phi\cos\phi}{r}\frac{\partial^2 T}{\partial \phi\partial r} - 2\frac{\sin\phi\cos\phi}{r^2}\frac{\partial T}{\partial \phi}$$

$$+ \frac{\cos^2\phi}{r}\frac{\partial T}{\partial r} + \frac{\cos^2\phi}{r^2}\frac{\partial^2 T}{\partial \phi^2}$$

In addition, in both coordinate systems: $\frac{\partial^2 T}{\partial z^2} = \frac{\partial^2 T}{\partial z^2}$

Therefore,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

and the general heat conduction equation in the cylindrical coordinate is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

b) The spherical coordinates:

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \qquad \begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \\ \theta &= \cos^{-1}\frac{z}{[x^2+y^2+z^2]^{1/2}} \end{aligned}$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \sin\theta\cos\phi\frac{\partial T}{\partial r} - \frac{\sin\phi}{r\sin\theta}\frac{\partial T}{\partial \phi} + \frac{\cos\theta\cos\phi}{r}\frac{\partial T}{\partial \theta}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial r}\left(\frac{\partial T}{\partial x}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \phi}\left(\frac{\partial T}{\partial x}\right)\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial \theta}\left(\frac{\partial T}{\partial x}\right)\frac{\partial \theta}{\partial x}$$

$$= \sin^2\theta\cos^2\phi\frac{\partial^2 T}{\partial r^2} + 2\frac{\cos\phi\sin\phi}{r^2\sin^2\theta}\frac{\partial T}{\partial \phi} - 2\frac{\sin\phi\cos\phi}{r}\frac{\partial^2 T}{\partial r\partial \phi}$$

$$+ \frac{\cos\theta}{r^2}\left(-2\sin\theta\cos^2\phi + \frac{\sin^2\phi}{\sin\theta}\right)\frac{\partial T}{\partial \theta} + 2\frac{\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2 T}{\partial r\partial \theta}$$

$$+ \left(\frac{\cos^2\theta\cos^2\phi}{r} + \frac{\sin^2\phi}{r}\right)\frac{\partial T}{\partial r} + \frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}$$

$$- 2\frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 T}{\partial \phi\partial \theta} + \frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2 T}{\partial \theta^2}$$

Similarly,

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= \sin\theta\sin\phi\frac{\partial T}{\partial r} + \frac{\cos\phi}{r\sin\theta}\frac{\partial T}{\partial \phi} + \frac{\cos\theta\sin\phi}{r}\frac{\partial T}{\partial \theta}$$

$$\frac{\partial^2 T}{\partial y^2} = \sin^2\theta \sin^2\phi \frac{\partial^2 T}{\partial r^2} - 2\frac{\sin\phi\cos\phi}{r^2\sin^2\theta} + 2\frac{\sin\phi\cos\phi}{r}\frac{\partial^2 T}{\partial r\partial\theta}$$

$$+ \frac{\cos\theta}{r^2}\left(-2\sin\theta\sin^2\phi + \frac{\cos^2\phi}{\sin\theta}\right)\frac{\partial T}{\partial\theta} + 2\frac{\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial^2 T}{\partial r\partial\theta}$$

$$+ \frac{\cos^2\theta\sin^2\phi + \cos^2\phi}{r}\frac{\partial T}{\partial r} + \frac{\cos^2\phi}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}$$

$$+ 2\frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 T}{\partial\phi\partial\theta} + \frac{\cos^2\theta\sin^2\phi}{r^2}\frac{\partial^2 T}{\partial\theta^2}$$

and

$$\frac{\partial T}{\partial z} = \cos\frac{\partial T}{\partial r} - \frac{\sin\theta}{r}\frac{\partial T}{\partial\theta}$$

$$\frac{\partial^2 T}{\partial z^2} = \cos^2\theta\frac{\partial^2 T}{\partial r^2} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial T}{\partial\theta} - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 T}{\partial r\partial\theta}$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial T}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 T}{\partial\theta^2}$$

Thus,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \overbrace{\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}}^{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)} + \overbrace{\frac{\cos\theta}{r^2\sin\theta}\frac{\partial T}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2}}^{\frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)}$$

$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}$$

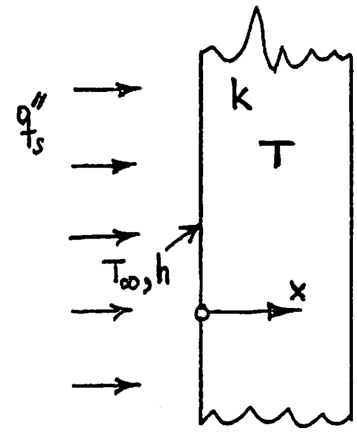
and the general heat conduction equation in the spherical coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

PROB. 2.4:

B.C. at $x=0$:

$$-k\left(\frac{\partial T}{\partial x}\right)_{x=0} = h\left[T_\infty - T\right]_{x=0} + q_s''$$



$$\boxed{\left[-k\frac{\partial T}{\partial x} + hT\right]_{x=0} = hT_\infty + q_s''}$$

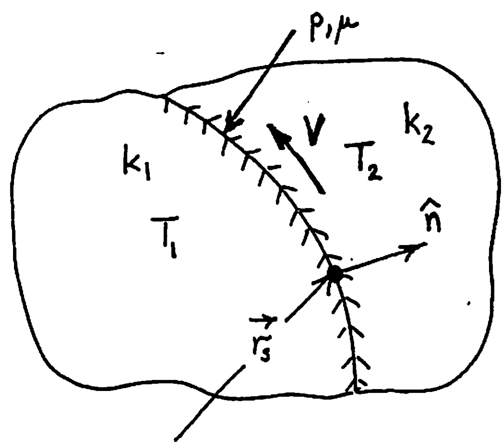
PROB. 2.5:

1) Since the contact is perfect,

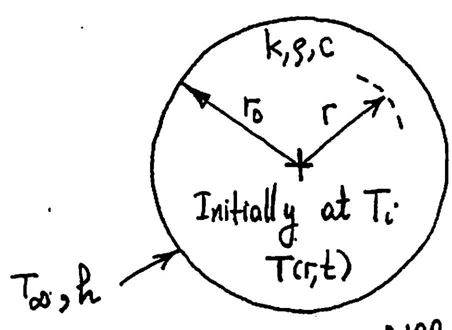
$$T_1\big|_{\vec{r}_s} = T_2\big|_{\vec{r}_s}$$

2)

$$-k_1\frac{\partial T_1}{\partial n}\bigg|_{n=0} + \mu p V = -k_2\frac{\partial T_2}{\partial n}\bigg|_{n=0}$$



PROB. 2.6:



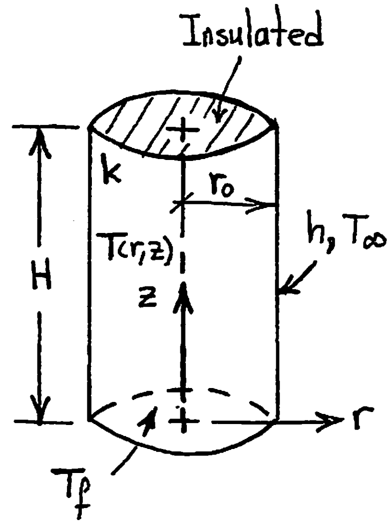
The temperature distribution will be one-dimensional, that is, $T=T(r,t)$. Thus, the formulation of the problem for $T(r,t)$ can be given as

Diff. Eq. $\longrightarrow$ $\dfrac{\partial^2 T}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial T}{\partial r} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$, $\alpha = \dfrac{k}{\rho c}$

Initial Cond. $\longrightarrow$ $T(r,0) = T_i$

Two boundary conds. $\longrightarrow$ $T(0,t) = \text{finite}$; $-k\dfrac{\partial T}{\partial r}\bigg|_{r=r_0} = h\left[T(r_0,t) - T_\infty\right]$

PROB. 2.7: Because of the symmetry in the ϕ-direction, under steady-state conditions $T = T(r,z)$. In addition, the temperature of the lower surface will be equal to the temperature of the boiling water, T_f (due to very large h). Thus, the formulation is given by

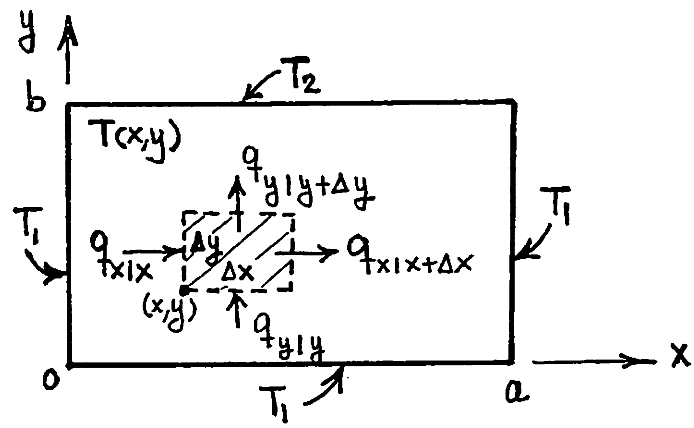


$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0$$

$$T(0,z) = \text{finite}; \quad -k\left(\frac{\partial T}{\partial r}\right)_{r=r_o} = h\left[T(r_o,z) - T_\infty\right]$$

$$T(r,0) = T_f; \quad \left(\frac{\partial T}{\partial z}\right)_{z=H} = 0$$

PROB. 2.8:



$k_x = 2k_y$

For unit depth:

$$q_{x|x} - q_{x|x+\Delta x} = -\frac{\partial q_{x|x}}{\partial x}\Delta x = \frac{\partial}{\partial x}\left(k_x \Delta y \frac{\partial T}{\partial x}\right)\Delta x$$

$$= k_x \frac{\partial^2 T}{\partial x^2}\Delta x \Delta y$$

and

$$q_{y|y} - q_{y|y+\Delta y} = -\frac{\partial q_{y|y}}{\partial y}\Delta y = \frac{\partial}{\partial y}\left(k_y \Delta x \frac{\partial T}{\partial y}\right)\Delta y$$
$$= k_y \frac{\partial^2 T}{\partial y^2}\Delta x \Delta y$$

Thus, under steady-state conditions and in the absence of heat sources or sinks, energy balance on the c.v. shown yields

$$k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} = 0 \quad \text{or} \quad 2\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Now, the formulation of the problem for $T = T(x,y)$ can be written as

$$2\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$
$$T(0,y) = T_1 \; ; \; T(a,y) = T_1$$
$$T(x,0) = T_1 \; ; \; T(x,b) = T_2$$

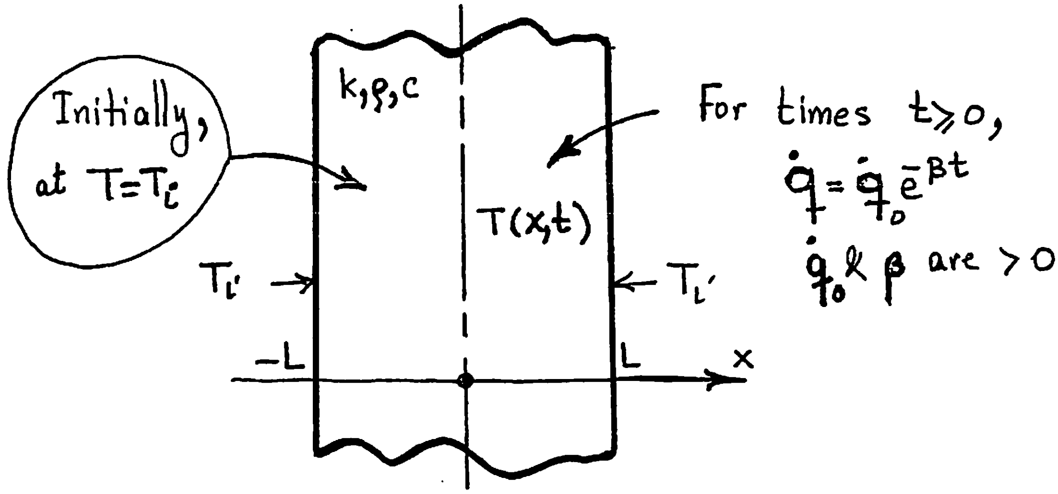
If we introduce a new temperature function as

$$\theta(x,y) = T(x,y) - T_1$$

then the formulation can be rewritten as

$$\boxed{\begin{array}{l} 2\dfrac{\partial^2 \theta}{\partial x^2} + \dfrac{\partial^2 \theta}{\partial y^2} = 0 \\ \theta(0,y) = 0 \; ; \; \theta(a,y) = 0 \\ \theta(x,0) = 0 \; ; \; \theta(x,b) = T_2 - T_1 \end{array}}$$

PROB. 2.9:



Formulation of the problem for $T(x,t)$ can be written as

D.E. $\longrightarrow \dfrac{\partial^2 T}{\partial x^2} + \dfrac{\dot{q}_0}{k} e^{-\beta t} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}, \quad \alpha = \dfrac{k}{\rho c}$

I.C. $\longrightarrow \quad T(x,0) = T_i$

B.C.'s $\longrightarrow \dfrac{\partial T(0,t)}{\partial x} = 0; \quad T(L,t) = T_i$

If we define a new temperature function as

$$\theta(x,t) = T(x,t) - T_i$$

then the formulation can be rewritten as

$$\boxed{\begin{gathered} \frac{\partial^2 \theta}{\partial x^2} + \frac{\dot{q}_0}{k} e^{-\beta t} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \\ \theta(x,0) = 0 \\ \frac{\partial \theta(0,t)}{\partial x} = 0; \quad \theta(L,t) = 0 \end{gathered}}$$

PROB. 2.10:

$$\left\{\begin{array}{c} \dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t} \\ T(x,0) = T_i(x) \\ T(0,t) = T_1 \; ; \; T(L,t) = T_2 \end{array}\right\}$$

Assume that there are more than one solution. Let $T^{(1)}(x,t)$ and $T^{(2)}(x,t)$ be two different solutions. They both satisfy the formulation of the problem:

$$\boxed{\begin{array}{c} \dfrac{\partial^2 T^{(1)}}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial T^{(1)}}{\partial t} \\ T^{(1)}(x,0) = T_i(x) \\ T^{(1)}(0,t) = T_1 \; ; \; T^{(1)}(L,t) = T_2 \end{array}} \qquad \boxed{\begin{array}{c} \dfrac{\partial^2 T^{(2)}}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial T^{(2)}}{\partial t} \\ T^{(2)}(x,0) = T_i(x) \\ T^{(2)}(0,t) = T_1 \; ; \; T^{(2)}(L,t) = T_2 \end{array}}$$

Define a new function $\phi = T^{(1)} - T^{(2)}$. Since the problem is linear, ϕ will satisfy the following problem:

$$\boxed{\begin{array}{c} \dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial \phi}{\partial t} \\ \phi(x,0) = 0 \\ \phi(0,t) = 0 \; ; \; \phi(L,t) = 0 \end{array}}$$

Physical interpretation: The solution of this problem, $\phi(x)$, represents the unsteady-state temperature distribution in a slab with zero initial condition and zero boundary conditions. Hence, it will always be zero, which implies $T^{(1)} = T^{(2)}$.

<u>Mathematical interpretation</u>: Multiply the differential equation by ϕ and then integrate over x from 0 to L:

$$\underbrace{\int_0^L \phi \frac{\partial^2\phi}{\partial x^2}\, dx}_{\phi \frac{\partial\phi}{\partial x}\Big|_0^L - \int_0^L \left(\frac{\partial\phi}{\partial x}\right)^2 dx} = \underbrace{\frac{1}{\alpha}\int_0^L \phi \frac{\partial\phi}{\partial t}\, dx}_{\frac{1}{2\alpha}\frac{d}{dt}\int_0^L \phi^2 dx}$$

$$-\int_0^L \left(\frac{\partial\phi}{\partial x}\right)^2 dx = \frac{1}{2\alpha}\frac{d}{dt}\int_0^L \phi^2 dx$$

Now, integrate this result over t from $t=0$ to any t:

$$\underbrace{-\int_0^t \left[\int_0^L \left(\frac{\partial\phi}{\partial x}\right)^2 dx\right] dt}_{\text{Always negative}} = \underbrace{\frac{1}{2\alpha}\left\{\int_0^L \phi^2(x,t)\,dx - \int_0^L \cancelto{0}{\phi^2(x,0)\,dx}\right\}}_{\text{Always positive}}$$

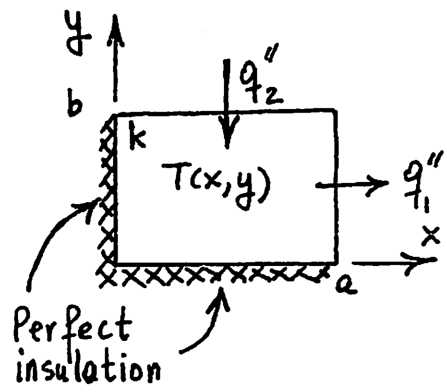
Hence, the left- and right-hand sides can be equal to each other only if they are both zero.

$$\therefore \int_0^L \phi^2(x,t)\,dx = 0$$

This, in turn, implies

$$\phi(x,t) \equiv 0 \quad \text{in} \quad 0 < x < L.$$

PROB. 2.11:



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0, \quad -k\left.\frac{\partial T}{\partial x}\right|_{x=a} = q_1''$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0, \quad k\left.\frac{\partial T}{\partial y}\right|_{y=b} = q_2''$$

a) Under steady-state conditions:

$$q_2'' \cdot a = q_1'' \cdot b$$

$$\therefore \quad \boxed{q_1'' = \frac{a}{b} q_2''}$$

b) Note that if $T(x,y)$ is a solution, then $T(x,y) + C$ is also a solution, where C is any constant. This can be proved by direct substitution in the formulation of the problem. That is, let $\theta = T + c$. Then,

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=0} = 0, \quad -k\left.\frac{\partial \theta}{\partial x}\right|_{x=a} = q_1''$$

$$\left.\frac{\partial \theta}{\partial y}\right|_{y=0} = 0, \quad +k\left.\frac{\partial \theta}{\partial y}\right|_{y=b} = q_2''$$

Thus, both $T(x,y)$ and $\theta(x,y)$ satisfy the same problem.

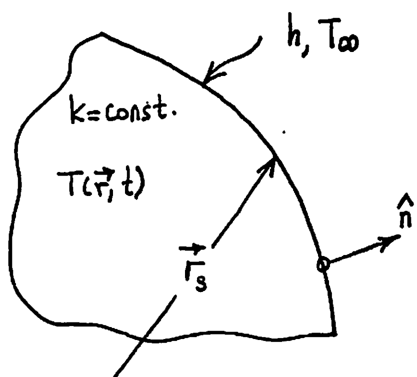
PROB. 2.12:

$$\left\{\begin{aligned} &\frac{d}{dx}\left[k(T)\frac{dT}{dx}\right] + \dot{q} = 0 \\ &\frac{dT(0)}{dx} = 0 \\ &T(L) = T_w \end{aligned}\right\} \rightarrow \quad \theta(x) = \frac{1}{k_w}\int_{T_w}^{T(x)} k(T')\,dT' \qquad \frac{d\theta}{dx} = \frac{k(T)}{k_w}\frac{dT}{dx} \quad \rightarrow \left\{\begin{aligned} &\frac{d^2\theta}{dx^2} + \frac{\dot{q}}{k_w} = 0 \\ &\frac{d\theta(0)}{dx} = 0 \\ &\theta(L) = 0 \end{aligned}\right\}$$

Linear in $\theta(x)$.

provided that $\dot{q}$ is not a non-linear function of the temperature T.

PROB. 2.13:



$$-k\left.\frac{\partial\theta}{\partial n}\right|_{n=0} = h\left[T(\vec{r}_s,t) - T_\infty\right] \qquad ①$$

Let

$$\theta(\vec{r},t) = \frac{1}{k_R}\int_{T_R}^{T(\vec{r},t)} k(T')\,dT' \quad , \quad k_R = k(T_R)$$

Then, the boundary condition ① would be rewritten as

$$-k_R\left.\frac{\partial\theta}{\partial n}\right|_{n=0} = h^*(\vec{r}_s,t)\,\theta(\vec{r}_s,t) \qquad ②$$

provided that the heat transfer coefficient h in ① is given by

$$h = \frac{h^*(\vec{r}_s,t)}{k_R[T(\vec{r}_s,t)-T_\infty]}\int_{T_R}^{T(\vec{r}_s,t)} k(T')dT' \qquad (3)$$

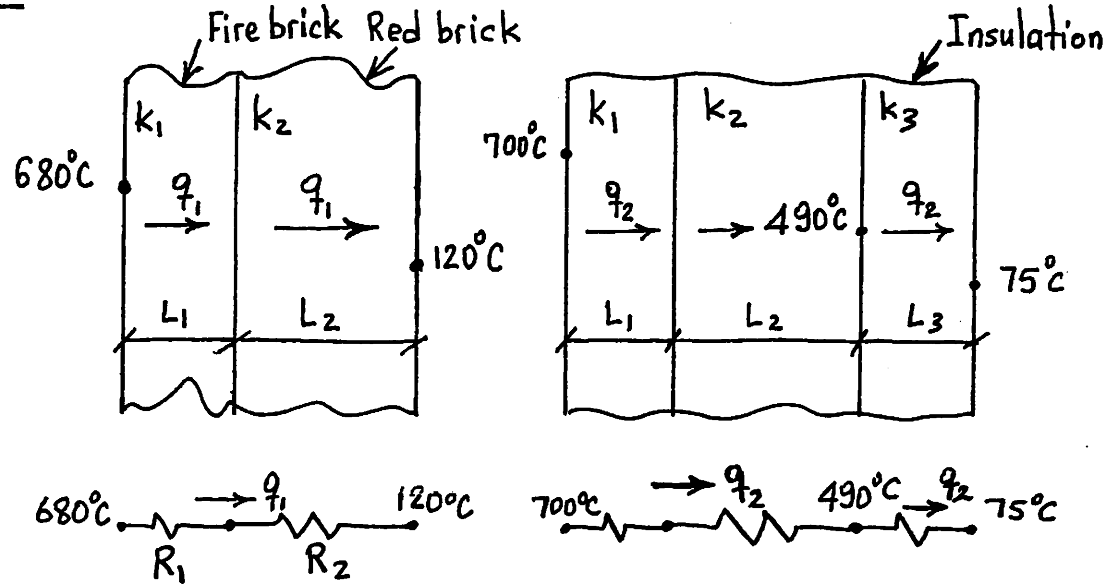
where $h^*(\vec{r}_s,t)$ is any function of $\vec{r}_s$ and t.

Note that (a) the condition (2) is a linear and homogeneous boundary condition of the third kind, and (b) for this transformation to be possible the heat transfer coefficient h has to be a function of the surface temperature $T(\vec{r}_s,t)$ as specificed by the relation (3).

CHAPTER 3

ONE-DIMENSIONAL STEADY-STATE HEAT CONDUCTION

PROB. 3.1:



$$\left.\begin{array}{l} \dfrac{680-120}{R_1+R_2} = q_1 \\[2ex] \dfrac{700-490}{R_1+R_2} = q_2 \end{array}\right\} \quad \frac{q_2}{q_1} = \frac{210}{560} = 0.375 \quad \text{or} \quad 37.5\%$$

PROB. 3.2: a) Under steady-state conditions,

$$q = -k\,2\pi r L \frac{dT}{dr} = \text{const.}, \qquad k = 0.137 + 0.002T(°C)$$

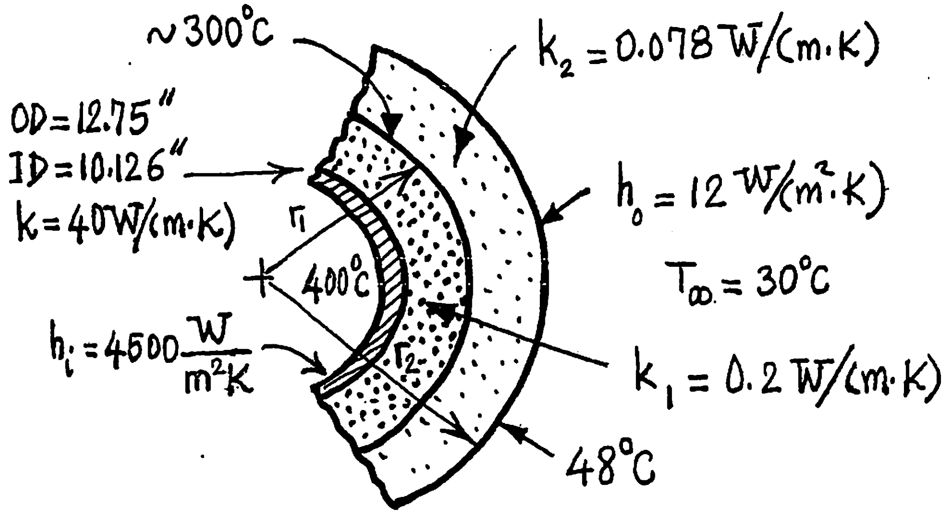
$$\int_{T_o}^{T_i} k(T)\,dT = \frac{q}{2\pi L} \ln\frac{r_o}{r_i} \qquad [k] = \frac{W}{m.K}$$

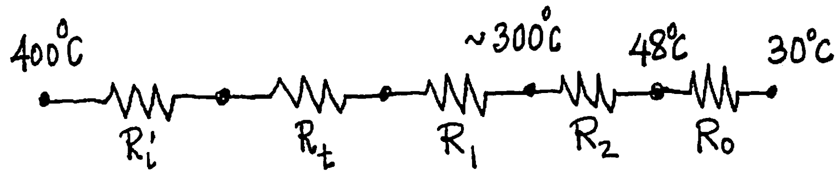
$$0.137(500-80) + \frac{0.002}{2}(500^2 - 80^2) = \frac{q}{2\pi L} \ln\frac{25}{12.5}$$

$12\,W/(m^2 \cdot K)$ $\quad \therefore \quad \underline{\dfrac{q}{L} = 1{,}311.5 \ \dfrac{W}{m}}$

b) $\dfrac{q}{L} = 1{,}311.5 = h \times 2\pi r_o \times (80 - T_\infty) \quad \Rightarrow \quad \underline{T_\infty = 10.4°C}$

PROB. 3.31:





$$R_i' = \frac{1}{h_i \pi ID} = \frac{1}{4500 \times \pi \times 10.126 \times 0.0254} = 2.75 \times 10^{-4}\ \frac{m\cdot K}{W}$$

$$R_t = \frac{1}{2\pi k} \ln\frac{OD}{ID} = \frac{1}{2\pi \times 40} \ln\frac{12.75}{10.126} = 9.17 \times 10^{-4}\ \frac{m\cdot K}{W}$$

$$R_o = \frac{1}{h_o \pi\, OD} = \frac{1}{12 \times \pi \times 2 \times r_2} = \frac{0.01326}{r_2}$$

$$R_1 = \frac{1}{2\pi k_1} \ln\frac{r_1}{(OD/2)} = 0.7958\left[1.8206 + \ln r_1\right];\quad R_2 = \frac{1}{2\pi k_2}\ln\frac{r_2}{r_1} = 2.04 \ln\frac{r_2}{r_1}$$

$$\left.\begin{aligned} \frac{400-300}{R_i + R_t + R_1} &= \frac{48-30}{R_o} \rightarrow \ln r_1 = \frac{0.0926}{r_2} - 1.816 \\ \frac{300-48}{R_2} &= \frac{48-30}{R_o} \rightarrow \ln\frac{r_2}{r_1} = \frac{0.091}{r_2} \end{aligned}\right\} \Rightarrow \boxed{\ln r_2 = \frac{0.1836}{r_2} - 1.816}$$

Assume r_2 (m)	$\rightarrow \ln r_2$	$\rightarrow$ Find r_2 (m)
0.35	−1.2914	0.275
0.275	−1.1483	0.317
0.317	−1.237	0.29
0.3	−1.204	0.2999

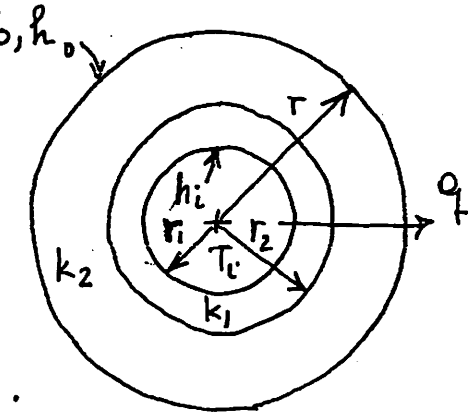
$\therefore\ r_2 = 0.3\,m \rightarrow r_1 = 0.22\,m$

High-temp. insulation thickness: $r_1 - \frac{OD}{2} = \underline{0.058\,m}$

Magnesia insulation thickness $r_2 - r_1 = 0.3 - 0.22 = \underline{0.08\,m}$

PROB. 3.4:

$$q = \frac{T_i - T_o}{\sum R_{th}}$$



$$\sum R_{th} = \frac{1}{4\pi}\left[\frac{1}{r_1^2 h_i} + \frac{r_2 - r_1}{r_2 r_1 k_1} + \frac{r - r_2}{r r_2 k_2} + \frac{1}{r^2 h_o}\right]$$

$$\frac{d}{dr}\left(\sum R_{th}\right) = \frac{1}{4\pi r^2}\left[\frac{1}{k_2} - \frac{2}{h_o r}\right] = 0$$

This equation is satisfied when $r = \frac{2k_2}{h_o}$.
This value of r is termed the critical radius of insulation for a sphere, i.e., $r_{cr} = 2k_2/h_o$. One more differentiation yields

$$\left.\frac{d^2}{dr^2}\left(\sum R_{th}\right)\right|_{r=r_{cr}} = \frac{h_o^3}{32\pi k_2^4} > 0$$

Hence, at $r = r_{cr} = 2k_2/h_o$, the total thermal resistance is minimum and therefore q is maximum.

PROB. 3.5: (Refer to Fig. 3.11 in text)

Since $r_{cr} = \frac{0.151}{6} = 0.0252\,m = 2.52\,cm > r_2 = 0.5\,in = 1.27\,cm$, adding insulation will increase the heat loss until r becomes equal to r_{cr}, and then the heat loss will start decreasing.

$$\left.\begin{aligned} &\text{Heat loss without insulation: } q_1 = \frac{T_f - T_\infty}{R_1} \\ &\text{Heat loss with insulation: } q_2 = \frac{T_f - T_\infty}{R_2} \end{aligned}\right\} \Rightarrow \begin{gathered}\text{If } q_1 = q_2\text{, then} \\ R_1 = R_2\end{gathered}$$

$$\therefore \quad \frac{1}{h r_2} = \frac{1}{k_2}\ln\frac{r}{r_2} + \frac{1}{hr}$$

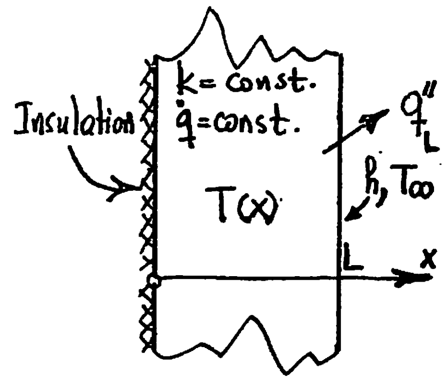
This relation is satisfied when $r = 6.1\,cm$. This is the value of r at which the loss with and without the insulation will be the same.

PROB. 3.6: Formulation of the problem:

$$\frac{d^2T}{dx^2}+\frac{\dot{q}}{k}=0$$

$$\frac{dT(0)}{dx}=0$$

$$-k\frac{dT(L)}{dx}=h\left[T(L)-T_\infty\right]$$



Sol. to DE: $T(x)=-\frac{\dot{q}x^2}{2k}+C_1x+C_2$

B.C. at $x=0 \Rightarrow C_1=0$ and B.C. at $x=L \Rightarrow C_2=T_\infty+\frac{\dot{q}L^2}{2k}+\frac{\dot{q}L}{h}$

$$\therefore \quad T(x)=T_\infty+\frac{\dot{q}L^2}{2k}\left[1-\left(\frac{x}{L}\right)^2\right]+\frac{\dot{q}L}{h}$$

Heat flux from the surface at $x=L$: $q''_L=\dot{q}L$

PROB. 3.7: $L=3$ in, $\dot{q}=5\times10^4$ W/m³, $T_\infty=30°C$, $h=600$ W/(m²·K), and $k=17$ W/(m·K). From Prob. 3.6, the max. temp. in the wall will be given by

$$T_{max}=T_\infty+\frac{\dot{q}L^2}{2k}+\frac{\dot{q}L}{h}=T_\infty+\frac{\dot{q}L^2}{2k}\left(1+2\frac{k}{hL}\right)$$

$$=30+\frac{5\times10^4\times3\times2.54\times10^{-2}}{2\times17}\left(1+2\times\frac{17}{3\times2.54\times10^{-2}\times600}\right)$$

$$=225.4°C$$

PROB. 3.8: Formulation of the problem:

$$\left\{\begin{array}{l}\frac{d^2T}{dx^2}+\frac{\dot{q}}{k}=0\\ T(0)=T_1\,;\ T(L)=T_2\end{array}\right\}$$

Sol. to DE: $T(x)=-\frac{\dot{q}x^2}{2k}+C_1x+C_2$

$T(0)=T_1 \Rightarrow C_2=T_1$ & $T(L)=T_2 \Rightarrow C_1=\frac{T_2-T_1}{L}+\frac{\dot{q}L}{2k}$

$$T(x)-T_1=\frac{T_2-T_1}{L}x+\frac{\dot{q}x}{2k}\left[L-x\right]$$

PROB. 3.9:

Formulation

$$\left\{\begin{array}{l} \frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}+\frac{\dot{q}}{k}=0 \\ k\frac{dT(r_i)}{dr}=h\left[T(r_i)-T_f\right] \\ \frac{dT(r_o)}{dr}=0 \end{array}\right\}$$

Sol. to D.E.

$$T(r)=-\frac{\dot{q}r^2}{4k}+c_1\ln r+c_2$$

B.C. at r_o $\Rightarrow$ $c_1=\frac{\dot{q}r_o^2}{2k}$

B.C. at r_i $\Rightarrow$ $C_2=T_f+\frac{\dot{q}r_i^2}{4k}\left\{1-2\left(\frac{r_o}{r_i}\right)^2\ln r_i+\frac{2}{Bi}\left[1-\left(\frac{r_o}{r_i}\right)^2\right]\right\}$, $Bi=\frac{hr_i}{k}$

Substituting C_1 & C_2 into the solution of DE one gets the temp. dist.

Rate of Heat Transfer to the gas / length $=\pi(r_o^2-r_i^2)\dot{q}$

PROB. 3.10:

Formulation

$$\left\{\begin{array}{l} \frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}+\frac{\dot{q}}{k}=0 \\ -k\frac{dT(r_o)}{dr}=h_2\left[T(r_o)-T_{f_2}\right] \\ k\frac{T(r_i)}{dr}=h_1\left[T(r_i)-T_{f_1}\right] \end{array}\right\}$$

Sol. to DE:

$$T(r)=-\frac{\dot{q}r^2}{4k}+c_1\ln r+c_2$$

B.C.'s yield:

$$C_1=\frac{1}{\Delta_1}\left\{\frac{\dot{q}}{2}\left[\frac{r_o}{h_2}+\frac{r_i}{h_1}\right]+\frac{\dot{q}}{4k}\left[r_o^2-r_i^2\right]+(T_{f_2}-T_{f_1})\right.$$

$$C_2=-\frac{\Delta_2}{\Delta_1}\left\{\frac{\dot{q}}{2}\left[\frac{r_o}{h_2}+\frac{r_i}{h_1}\right]+\frac{\dot{q}}{4k}\left[r_o^2-r_i^2\right]+(T_{f_2}-T_{f_1})\right\}+\frac{\dot{q}r_o}{2h_2}+\frac{\dot{q}r_o^2}{4k}+T_{f_2}$$

where

$$\Delta_1=k\left(\frac{1}{r_oh_2}+\frac{1}{r_ih_1}\right)+\ln\frac{r_o}{r_i} \quad \& \quad \Delta_2=\frac{k}{r_oh_2}+\ln r_o$$

Substituting C_1 & C_2 into the solution of DE gives the temp. dist.
Then, one would obtain the surface temperatures from

$$T(r_o)=-\frac{\dot{q}r_o^2}{4k}+c_1\ln r_o+c_2 \quad \& \quad T(r_i)=-\frac{\dot{q}r_i^2}{4k}+c_1\ln r_i+c_2$$

Max. Temp.? $\quad \frac{dT}{dr}=-\frac{\dot{q}r}{2k}+\frac{c_1}{r}=0 \Rightarrow r^*=\sqrt{2kc_1/\dot{q}}$

If $r_i\le r^*\le r_o$ then $T_{max}=T(r^*)$

If $r^*<r_i$ then $T_{max}=T(r_i)$

If $r^*>r_o$ then $T_{max}=T(r_o)$

PROB. 3.11:

Formulation of the problem

$$\left\{\begin{array}{l} \dfrac{d^2T}{dx^2}+\dfrac{\dot{q}_0}{k}\left[1-\left(\dfrac{x}{L}\right)^2\right]=0 \\ \dfrac{dT(0)}{dx}=0 \\ T(L)=T_W \end{array}\right\} \begin{array}{l} \rightarrow T(x)=-\dfrac{\dot{q}_0}{k}\left[\dfrac{x^2}{2}-\dfrac{x^4}{12L^2}\right]+c_1x+c_2 \\ \longrightarrow c_1=0 \\ \longrightarrow c_2=T_W+\dfrac{5}{12}\dfrac{\dot{q}_0L^2}{k} \end{array}$$

$$\therefore\ T(x)-T_W=\frac{\dot{q}_0L^2}{2k}\left[\left(\frac{x}{L}\right)^2-\frac{1}{6}\left(\frac{x}{L}\right)^4+\frac{5}{6}\right]$$

PROB. 3.12:

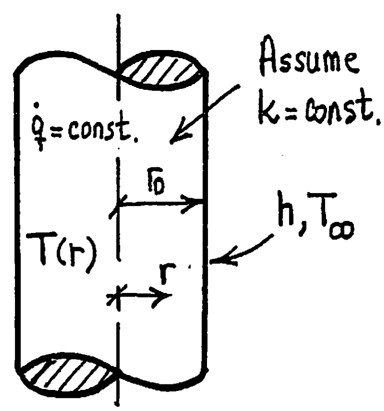
Formulation of the problem

$$\left\{\begin{array}{l} \dfrac{d^2T}{dr^2}+\dfrac{1}{r}\dfrac{dT}{dr}+\dfrac{\dot{q}_0}{k}\left(1-\dfrac{r}{r_0}\right)=0 \\ T(0)=\text{finite} \\ T(r_0)=T_W \end{array}\right\} \begin{array}{l} \rightarrow T(r)=-\dfrac{\dot{q}_0}{k}\left(\dfrac{r^2}{4}-\dfrac{r^3}{9r_0}\right)+c_1\ln r+c_2 \\ \rightarrow c_1=0 \\ \rightarrow c_2=T_W+\dfrac{\dot{q}_0r_0^2}{k}\dfrac{5}{36} \end{array}$$

$$\therefore\ \frac{T(r)-T_W}{\dot{q}_0r_0^2/k}=\frac{5}{36}-\left(\frac{r}{r_0}\right)^2\left(\frac{1}{4}-\frac{1}{9}\frac{r}{r_0}\right)$$

PROB. 3.13: Formulation of the problem:

$$\left\{\begin{array}{l} \dfrac{d^2T}{dr^2}+\dfrac{1}{r}\dfrac{dT}{dr}+\dfrac{\dot{q}}{k}=0 \\ T(0)=\text{finite} \\ -k\dfrac{dT(r_0)}{dr}=h\left[T(r_0)-T_\infty\right] \end{array}\right\}$$



Sol. to DE: $T(r)=-\dfrac{\dot{q}r^2}{4k}+c_1\ln r+c_2$

B.C. at $r=0 \Rightarrow c_1=0$ & B.C. at $r=r_0 \Rightarrow c_2=T_\infty+\dfrac{\dot{q}r_0}{2h}+\dfrac{\dot{q}r_0^2}{4k}$

$$\therefore\ T(r)=T_\infty+\frac{\dot{q}r_0^2}{4k}\left[1-\left(\frac{r}{r_0}\right)^2\right]+\frac{\dot{q}r_0}{2h}$$

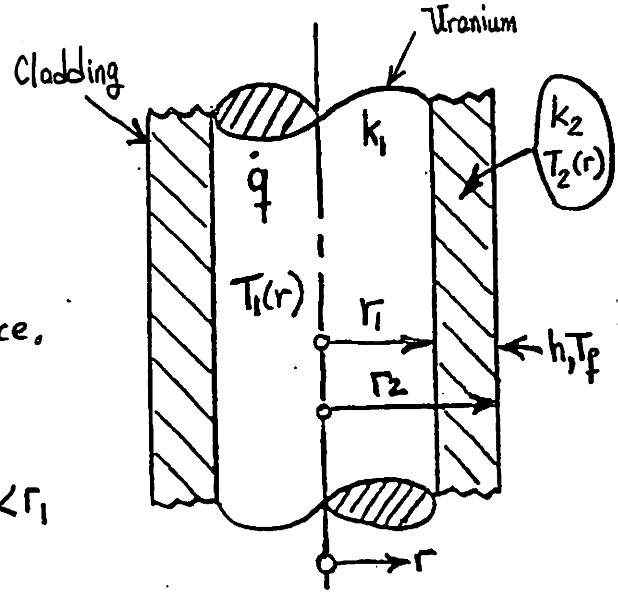
PROB. 3.14: Assumptions:

1- steady-state.

2- k_1 & k_2 are constant.

3- One-dimensional.

4- Perfect contact at the interface.



Formulation:

$$\frac{d^2T_1}{dr^2}+\frac{1}{r}\frac{dT_1}{dr}+\frac{\dot{q}_0}{k}\left[1-\left(\frac{r}{r_1}\right)^2\right]=0, \quad 0\le r<r_1$$

$$\frac{d^2T_2}{dr^2}+\frac{1}{r}\frac{dT_2}{dr}=0, \quad r_1<r<r_2$$

$$\frac{dT_1(0)}{dr}=0 \quad \text{or} \quad T_1(0)=\text{finite}$$

$$-k_1\frac{dT_1(r_1)}{dr}=-k_2\frac{dT_2(r_1)}{dr}$$

$T_1(r_1)=T_2(r_1)$ ← Assuming perfect contact.

$$-k_2\frac{dT_2(r_2)}{dr}=h\left[T_2(r_2)-T_f\right]$$

Solutions to DE's:

$$T_1(r)=-\frac{\dot{q}_0 r}{4k_1}\left[1-\left(\frac{r}{2r_1}\right)^2\right]+C_1\ln r+C_2 \quad \& \quad T_2(r)=C_3\ln r+C_4$$

Boundary conditions give:

$$C_1=0, \quad C_2=\frac{3\dot{q}_0 r_1^2}{16k_1}+\frac{\dot{q}_0 r_1^2}{4k_2}\left[\ln\frac{r_2}{r_1}+\frac{k_2}{r_2 h}\right]+T_f$$

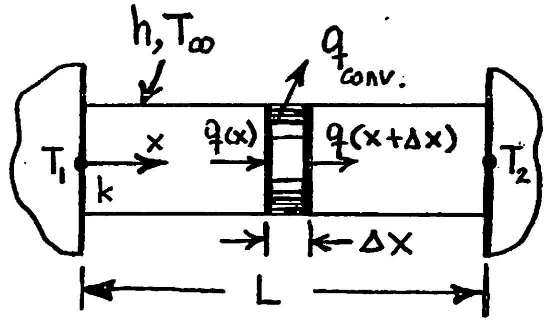
$$C_3=-\frac{\dot{q}_0 r_1^2}{4k_2}, \quad C_4=\frac{\dot{q}_0 r_1^2}{4k_2}\left[\frac{k_2}{r_2 h}+\ln r_2\right]+T_f$$

Thus,

$$T_1(r)-T_f=-\frac{\dot{q}_0 r^2}{4k_1}\left[1-\left(\frac{r}{2r_1}\right)^2\right]+\frac{3}{16}\frac{\dot{q}_0 r_1^2}{k_1}+\frac{\dot{q}_0 r_1^2}{4k_2}\left[\ln\frac{r_2}{r_1}+\frac{k_2}{r_2 h}\right], \quad 0\le r\le r_1$$

$$T_2(r)-T_f=\frac{\dot{q}_0 r_1^2}{4k_2}\left[\frac{k_2}{r_2 h}-\ln\left(\frac{r}{r_2}\right)\right], \quad r_1\le r\le r_2$$

PROB. 3.15: (a) An energy balance on the system shown yields



A = cross-section area

P = perimeter

$$\cancel{q(x)} = q(x+\Delta x) + q_{conv.}$$

As $\Delta x \to 0$

$$= \cancel{q(x)} + \frac{dq}{dx}\Delta x + P\Delta x\, h[T(x) - T_\infty]$$

$$\left.\begin{aligned} &\frac{dq}{dx} + hP[T(x) - T_\infty] = 0 \\ &q = -kA\frac{dT}{dx} \end{aligned}\right\} \Rightarrow \frac{d^2T}{dx^2} - \frac{hP}{kA}[T(x) - T_\infty] = 0$$

Let $\left\{\begin{aligned} &\theta(x) = T(x) - T_\infty \\ &m^2 = \frac{hP}{kA} \end{aligned}\right\} \rightarrow \frac{d^2\theta}{dx^2} - m^2\theta = 0$

$$\therefore \theta(x) = c_1 \sinh mx + c_2 \cosh mx$$

$$T(0) = T_1 \Rightarrow \theta(0) = T_1 - T_\infty = \theta_1 \Rightarrow c_2 = \theta_1$$

$$T(L) = T_2 \Rightarrow \theta(L) = T_2 - T_\infty = \theta_2 \Rightarrow c_1 = \frac{\theta_2 - \theta_1 \cosh mL}{\sinh mL}$$

$$\therefore \theta(x) = \frac{\theta_2 - \theta_1 \cosh mL}{\sinh mL}\sinh mx + \theta_1 \cosh mx$$

$$\boxed{\theta(x) = \theta_2 \frac{\sinh mx}{\sinh mL} + \theta_1 \frac{\sinh[m(L-x)]}{\sinh mL}}$$

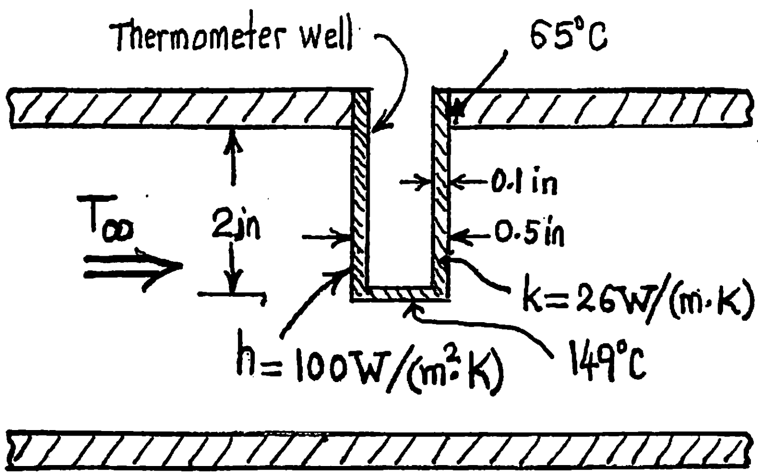
b)

$$q_L = -kA\frac{dT(0)}{dx} + kA\frac{dT(L)}{dx} = kA\left[\frac{dT(L)}{dx} - \frac{dT(0)}{dx}\right]$$

$$= kAm(\theta_2 + \theta_1)\left[\coth mL - \frac{1}{\sinh mL}\right]$$

$$\boxed{q_L = (\theta_2 + \theta_1)\sqrt{hPkA}\,\frac{\cosh mL - 1}{\sinh mL}}$$

PROB. 3.16:



From Eq. (3.132) in text:

$$\frac{T(L)-T_\infty}{T_b - T_\infty} = \frac{149-T_\infty}{65-T_\infty} = \frac{1}{\cosh mL + N \sinh mL}$$

where

$$m = \sqrt{\frac{hP}{kA}}, \quad P = \pi \times 0.5 \text{ in} \quad \& \quad A = \frac{\pi}{4}\left(0.5^2 - 0.3^2\right) = \pi \times 0.04 \text{ in}^2$$

$$\frac{A}{P} = \frac{\pi \times 0.04}{\pi \times 0.5} = 0.8 \text{ in} = 0.8 \times 2.54 \times 10^{-2} = 0.02032 \text{ m}$$

$$m = \sqrt{\frac{100}{26 \times 0.02032}} = 13.76 \text{ m}^{-1} \Rightarrow mL = 13.76 \times 2 \times 2.54 \times 10^{-2}$$
$$= 0.7$$

$$N = \frac{h}{mk} = \frac{100}{13.76 \times 26} = 0.28$$

$$\frac{149-T_\infty}{65-T_\infty} = \frac{1}{\cosh(0.7) + 0.28 \times \sinh(0.7)} = 0.682$$

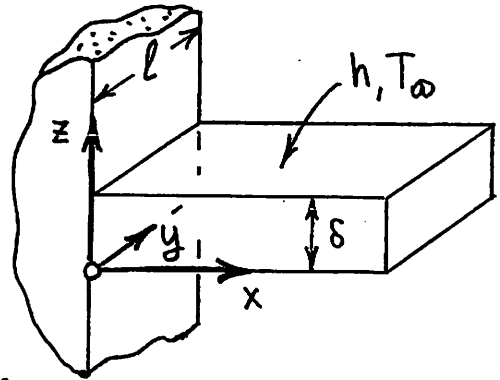
$$149 - T_\infty = 44.33 - 0.682 \times T_\infty$$

$$\Downarrow$$

$$\underline{T_\infty = 329^\circ\text{C}}$$

PROB. 3.17: For steady-state problems with no internal energy generation Eq. (2.16) reduces to

$$\nabla^2 T = 0$$



In rectangular coordinates,

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}=0, \quad \theta = T - T_\infty$$

Integrate over y from $y=0$ to $y=\ell$:

$$\int_{y=0}^{\ell}\left\{\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}\right\}dy = \int_{y=0}^{\ell}\left\{\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial z^2}\right\}dy + \left.\frac{\partial\theta}{\partial y}\right|_{y=0}^{y=\ell} = 0$$

Integrate over z from $z=0$ to $z=\delta$:

$$\int_{z=0}^{\delta}\int_{y=0}^{\ell}\frac{\partial^2\theta}{\partial x^2}\,dy\,dz + \underbrace{\int_{y=0}^{\ell}\left.\frac{\partial\theta}{\partial z}\right|_{z=0}^{z=\delta}dy + \int_{z=0}^{\delta}\left.\frac{\partial\theta}{\partial y}\right|_{y=0}^{y=\ell}dz}_{} = 0$$

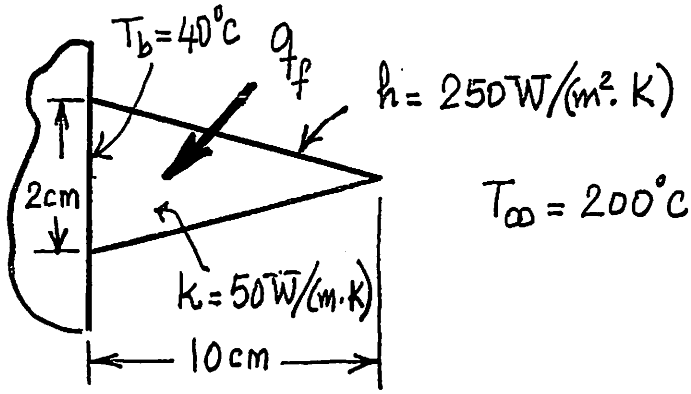
$$-\frac{h}{k}\left\{\int_{y=0}^{\ell}\left[\theta(x,y,\delta)+\theta(x,y,0)\right]dy + \int_{z=0}^{\delta}\left[\theta(x,\ell,z)+\theta(x,0,z)\right]dz\right\}$$

For one-dimensional extended surfaces $\theta(x,y,z) \cong \theta(x)$. Thus,

$$(\delta\cdot\ell)\frac{d^2\theta}{dx^2} - \frac{h}{k}\,2(\ell+\delta)\,\theta(x) = 0$$

$$\boxed{\frac{d^2\theta}{dx^2} - \frac{hP}{kA}\theta(x) = 0} \qquad A = \delta\cdot\ell, \; P = 2(\ell+\delta)$$

PROB. 3.18:



For unit depth:

$$q'_f = \sqrt{2hkb}\ \theta_b\ \frac{I_1(2m\sqrt{L})}{I_0(2m\sqrt{L})}$$

$$m = \sqrt{\frac{2hL}{kb}} = \sqrt{\frac{2\times 250\times 0.1}{50\times 0.02}} = 7.07\ m^{-1/2}$$

$$2m\sqrt{L} = 2\times 7.07\sqrt{0.1} = 4.47$$

$$\therefore\ q'_f = \sqrt{2\times 250\times 50\times 0.02}\times(200-40)\times\frac{I_1(4.47)}{I_0(4.47)} = \underline{3147\ W/m}$$

<u>Alternative solution:</u>

$$q'_f = h A_f \eta_f (T_\infty - T_b)$$

Refer to Fig. 3.25:

$L_c = 0.1\ m$ & $A_m = 0.1\times\frac{0.02}{2} = 1\times 10^{-3}\ m^2$

$$L_c^{3/2}\left(\frac{h}{kA_m}\right)^{1/2} = (0.1)^{3/2}\left(\frac{250}{50\times 10^{-3}}\right)^{1/2} = 2.24 \Rightarrow \eta_f \cong 41\%$$

$$\therefore\ q'_f = 250\times 2\times[0.1^2 + 0.01^2]^{1/2}\times 0.41\times(200-40) \cong \underline{3280\ W/m}$$

PROB. 3.19:

Pipe: $ID = 1$ in

Fins: $L = 0.75$ in

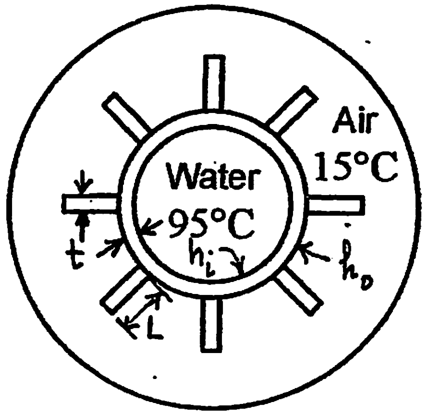
Pipe & Fins:

$t = 0.25$ cm (assumed)

$k = 41\ W/(m.K)$

$h_i = 500\ W/(m^2.K)$

$h_o = 12\ W/(m^2.K)$



Heat transfer rate from hot water to air per unit length of the tube:

$$q' = \frac{95-15}{\frac{1}{h_i A_i} + \frac{1}{2\pi k}\ln\frac{OD}{ID} + \frac{1}{8 \times \sqrt{h_o P k A}\tanh mL + h_o A_{unf}}}$$

where

$$A_i = \pi ID = \pi \times 2.54 \times 10^{-2} = 7.98 \times 10^{-2} m^2, \quad P = 2m, \quad A = 0.25 \times 10^{-2} m^2$$

$$m = \sqrt{\frac{h_o P}{kA}} = \left(\frac{12 \times 2}{41 \times 0.25 \times 10^{-2}}\right)^{1/2} = 15.3\, m^{-1}$$

$$mL = 15.3 \times 0.75 \times 2.54 \times 10^{-2} = 0.2915, \quad \tanh mL = 0.2835$$

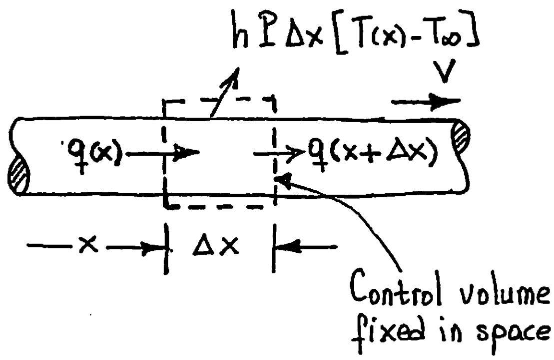
$$A_{unf} = [\pi \times (\underbrace{2.54 + 0.5}_{OD}) - 8 \times 0.25] \times 10^{-2} = 7.55 \times 10^{-2}\ m^2$$

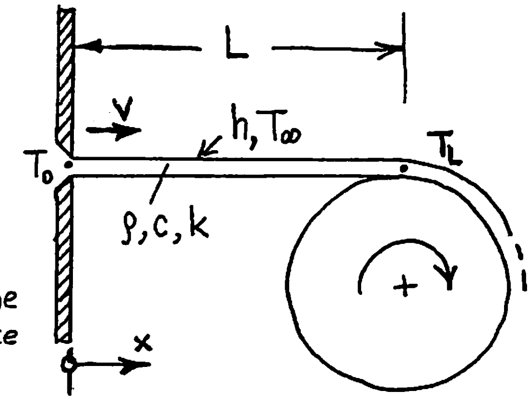
$$\frac{80}{q'} = \frac{100}{500 \times 7.98} + \frac{1}{2\pi \times 41}\ln\frac{3.04}{2.54} + \frac{1}{8 \times \sqrt{12 \times 2 \times 41 \times 0.25 \times 10^{-2}} \times 0.2835 + 12 \times 7.55 \times 10^{-2}}$$

$$\Rightarrow \underline{q' = 320.2\ W/m}$$

PROB. 3.20: a) Differential equation:

Method-I: Control-volume approach.





$$q(x) = Tc\rho VA + \left(-kA\frac{dT}{dx}\right)$$

P = Perimeter
A = Cross-section area

$$q(x+\Delta x) = q(x) + \frac{dq}{dx}\Delta x$$

As $\Delta x \to 0$

Energy (thermal) balance on the C.V. gives

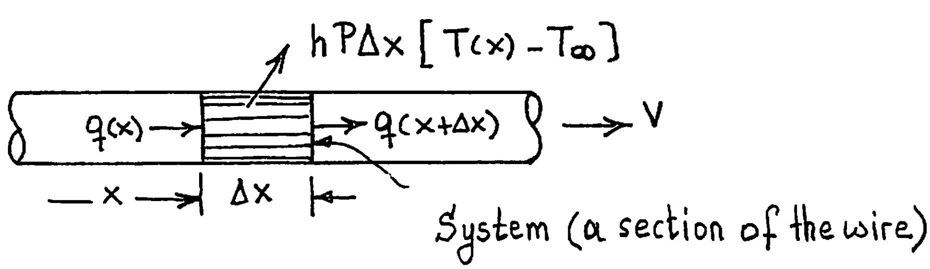
$$q(x) = q(x+\Delta x) + hP\Delta x\,[T(x) - T_\infty]$$

$$\frac{dq}{dx} + hP\,[T(x) - T_\infty] = 0$$

$$\rho cVA\frac{dT}{dx} + \frac{d}{dx}\left(-kA\frac{dT}{dx}\right) + hP\,[T(x) - T_\infty] = 0$$

$$\boxed{\frac{d^2T}{dx^2} - \frac{V}{\alpha}\frac{dT}{dx} - \frac{hP}{kA}[T(x) - T_\infty] = 0}\ ,\quad \alpha = \frac{k}{\rho c}$$

Method-II: System approach.



Energy balance on the system:

$$\cancel{q(x)} = \underbrace{q(x+\Delta x)}_{\cancel{q(x)} + \frac{dq}{dx}\Delta x} + hP\Delta x\,[T(x)-T_\infty] + \underbrace{\left\{\begin{matrix}\text{Rate of change}\\ \text{of internal energy}\\ \text{of the system}\end{matrix}\right\}}$$

$$\frac{d}{dt}(\rho c A \Delta x T) = \rho c A \Delta x \frac{dT}{dx}\underbrace{\frac{dx}{dt}}_{V}$$

$$\frac{dq}{dx} + hP[T-T_\infty] + \rho c A V \frac{dT}{dx} = 0$$

In this case: $\quad q(x) = -kA\dfrac{dT}{dx}$

$$\therefore \boxed{\frac{d^2T}{dx^2} - \frac{V}{\alpha}\frac{dT}{dx} - \frac{hP}{kA}[T-T_\infty] = 0}$$

Boundary conditions: $\quad T(0) = T_0 \quad$ and $\quad T(L) = T_L$.

b) Let $\quad \theta(x) = T(x) - T_\infty$.

$$\boxed{\begin{gathered}\frac{d^2\theta}{dx^2} - \frac{V}{\alpha}\frac{d\theta}{dx} - m^2\theta = 0\\ \theta(0) = T_0 - T_\infty\,;\ \theta(L) = T_L - T_\infty\end{gathered}} \qquad m^2 = \frac{hP}{kA}$$

Solution is given by

$$\boxed{\theta(x) = T(x) - T_\infty = e^{ax}\left\{\frac{e^{-aL}(T_L - T_\infty)\sinh bx + (T_0 - T_\infty)\sinh b(L-x)}{\sinh bL}\right\}}$$

where $\quad a = \dfrac{V}{2\alpha} \quad$ and $\quad b = \sqrt{a^2 + m^2}$

PROB. 3.21: From Prob. 3.20,

$$\theta(x) = e^{ax}\left[\frac{e^{-aL}(T_L - T_\infty)\sinh bx + (T_0 - T_\infty)\sinh b(L-x)}{\sinh bL}\right]$$

where

$$a = \frac{V}{2\alpha}\,,\quad b = \sqrt{a^2+m^2}\,,\quad m^2 = \frac{hP}{kA}\,.$$

In this solution, V and L are considered to be known. On the other hand, if we are seeking a relationship between V and L (and the other parameters $T_\infty, h, P, A, k, \rho, c, T_0$, and T_L), then V becomes an unknown for given L (and the other parameters). Therefore, to obtain such a relationship, a third condition has to be imposed on this solution. Since at the spool the temperature is T_L, it can be assumed that

$$\frac{dT(L)}{dx} \cong 0 \rightarrow \frac{d\theta(L)}{dx} \cong 0 \xrightarrow{\text{which yields}} \boxed{\frac{e^{aL}(T_0 - T_\infty) - (T_L - T_\infty)\cosh bL}{(T_L - T_\infty)\sinh bL} = \frac{a}{b}}$$

This is a transcendental equation for V, which can be solved, for example, numerically to find V as a function of the parameters of the problem.

An Approximate Solution for V: Neglect heat conduction along the wire. Then,

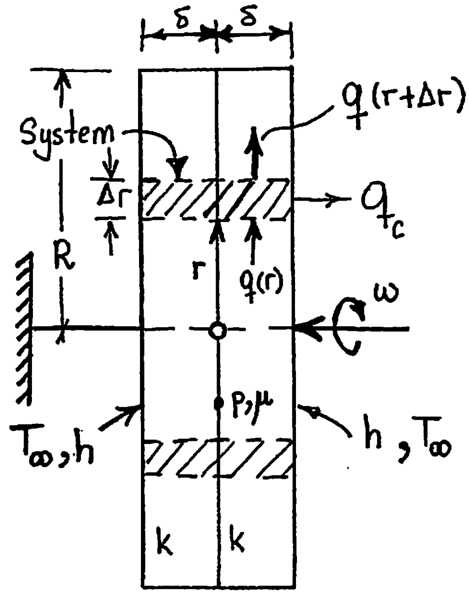
$$\left.\begin{aligned}&\frac{d\theta}{dx} + \frac{hP}{\rho c A V}\theta = 0\\ &\theta(0) = T_0 - T_\infty\end{aligned}\right\}\quad \theta(x) = \theta(0)\exp\left\{-\frac{hP}{\rho c A V}x\right\}$$

$$\theta(L) = T_L - T_\infty = (T_0 - T_\infty)\,e^{-\frac{hP}{\rho c A V}L}$$

$$\therefore\quad \boxed{V = \frac{hPL}{\rho c A}\left\{\ln\frac{T_0 - T_\infty}{T_L - T_\infty}\right\}^{-1}}$$

PROB. 3.22: Assumptions: 1) Steady-state (i.e., ω = const.); 2) h = const.; 3) T_∞ = Const.; 4) k = const. (given); 5) $p r$ = const. (given); 6) μ = const.; 7) Since $\delta \ll R$, the temp. distribution can be assumed to be one-dimensional; that is, $T = T(r)$; 8) Negligible heat loss from the edges at $r = R$; that is

$$\frac{dT(R)}{dr} \cong 0$$



An energy balance on the system shown yields

$$\cancel{q(r)} + \mu p r \omega \, 2\pi r \Delta r = \underbrace{q(r+\Delta r)}_{\cancel{q(r)} + \frac{dq}{dr}\Delta r \;\leftarrow\; \text{As } \Delta r \to 0} + 2h\, 2\pi r \Delta r \left[T(r) - T_\infty\right]$$

$$q(r) = -k\, 2\pi r (2\delta) \frac{dT}{dr} \;\rightarrow\; \Downarrow \;\leftarrow\; \theta(r) = T(r) - T_\infty$$

$$r^2 \frac{d^2\theta}{dr^2} + r\frac{d\theta}{dr} - m^2 r^2 \theta(r) = -\frac{\mu \overbrace{(pr)}^{\text{Const.}} \omega}{2k\delta} r^2, \quad m^2 = \frac{h}{k\delta}$$

$$\therefore \theta(r) = A I_0(mr) + B K_0(mr) + \frac{\mu (pr) \omega}{2h} \leftarrow \text{Particular solution}$$

Boundary conditions:

$$T(0) = \text{finite} \rightarrow \theta(0) = \text{finite} \rightarrow B \equiv 0$$

$$\frac{dT(R)}{dr} = 0 \rightarrow \frac{d\theta(R)}{dr} = 0 \rightarrow A = 0$$

Thus,

$$\boxed{\theta(r) = T(r) - T_\infty = \frac{\mu p \omega}{2h} r}$$

PROB. 2.23: If ρ = const., then the formulation of the problem becomes (See Prob. 2.22)

$$r^2\frac{d^2\theta}{dr^2}+r\frac{d\theta}{dr}-m^2r^2\theta(r)=a r^3,\quad \theta(r)=T(r)-T_\infty,\quad m^2=\frac{h}{k\delta}\ \&\ a=-\frac{\mu\rho\omega}{2k\delta}$$

with the same boundary conditions as in Prob. 2.22.

$$\theta(r)=C_1 I_0(mr)+C_2K_0(mr)+\theta_p(r)$$

Let $\quad \theta_p(r)=A(r)I_0(mr)+B(r)K_0(mr)$

$$\frac{d\theta_p}{dr}=\underbrace{\frac{dA}{dr}I_0(mr)+\frac{dB}{dr}K_0(mr)}_{\text{Set }=0}+A(r)\,mI_1(mr)-B(r)\,mK_1(mr)$$

$$\frac{dA}{dr}I_0(mr)+\frac{dB}{dr}K_0(mr)=0\qquad\text{— (1)}$$

$$\frac{d^2\theta_p}{dr^2}=\frac{dA}{dr}mI_1(mr)-\frac{dB}{dr}mK_1(mr)+A(r)m\left[mI_0(mr)-\frac{1}{r}I_1(mr)\right]$$
$$-B(r)m\left[-mK_0(mr)-\frac{1}{r}K_1(mr)\right]$$

Now, substitute θ_p, $\frac{d\theta_p}{dr}$ and $\frac{d^2\theta}{dr^2}$ into the differential equation:

$$r^2\left[\frac{dA}{dr}mI_1(mr)-\frac{dB}{dr}mK_1(mr)+\cancel{A(r)m^2I_0(mr)}-\cancel{\frac{A(r)m}{r}I_1(mr)}\right.$$
$$\left.+\cancel{B(r)m^2K_0(mr)}+\cancel{\frac{B(r)m}{r}K_1(mr)}\right]+r\left[\cancel{A(r)mI_1(mr)}-\cancel{B(r)mK_1(mr)}\right]$$
$$-m^2r^2\left[\cancel{A(r)I_0(mr)}+\cancel{B(r)K_0(mr)}\right]=ar^3$$

$$\frac{dA}{dr}I_1(mr)-\frac{dB}{dr}K_1(mr)=\frac{a}{m}r\qquad\text{— (2)}$$

From Eqs. (1) and (2), we obtain

$$\frac{dA}{dr} = \frac{a}{m}\frac{r K_0(mr)}{\Delta(mr)} \quad \text{and} \quad \frac{dB}{dr} = -\frac{a}{m}\frac{r I_0(mr)}{\Delta(mr)}$$

where

$$\Delta(mr) = I_1(mr)K_0(mr) + K_1(mr)I_0(mr) = \frac{1}{mr}$$

See Hildebrand, Adv. Calculus for Applications, 2nd ed., p. 178, Prob. 44.4c.

Thus,

$$\frac{dA}{dr} = a r^2 K_0(mr) \quad \text{and} \quad \frac{dB}{dr} = -a r^2 I_0(mr)$$

$$A(r) = a\int_0^r K_0(m\xi)\xi^2 d\xi, \quad B(r) = -a\int_0^r I_0(m\xi)\xi^2 d\xi \leftarrow$$

Note that we let $A(0) = B(0) = 0$ (Why?)

The solution can now be written as

$$\theta(r) = C_1 I_0(mr) + C_2 K_0(mr) + a\int_0^r \left[I_0(mr)K_0(m\xi) - I_0(m\xi)K_0(mr)\right]\xi^2 d\xi \quad \text{— (3)}$$

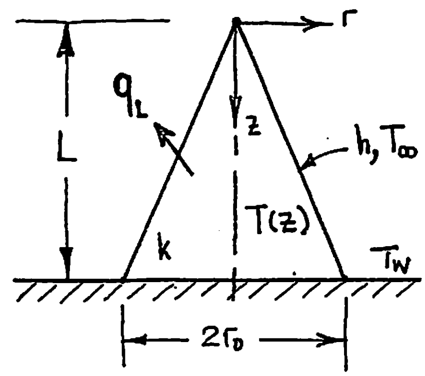
$$\theta(0) = \text{finite} \implies C_2 = 0 \quad \text{— (4)}$$

$$\frac{d\theta(R)}{dr} = C_1 m I_1(mR) + a\left[I_0(mR)\cancel{K_0(mR)} - I_0(\cancel{mR})K_0(mR)\right]R^2$$
$$+ am\int_0^R \left[I_1(mR)K_0(m\xi) + I_0(m\xi)K_1(mR)\right]\xi^2 d\xi = 0$$

$$\therefore C_1 = -\frac{a}{I_1(mR)}\int_0^R \left[I_1(mR)K_0(m\xi) + I_0(m\xi)K_1(mR)\right]\xi^2 d\xi \quad \text{— (5)}$$

Now, substitute C_1 and C_2 from (5) and (4) into (3) to obtain the final form of the solution $\theta(r)$.

PROB. 3.24: (a) Differential equation:



$$\frac{d}{dz}\left[A(z)\frac{d\theta}{dz}\right] - \frac{h P(z)}{k}\theta(z) = 0$$

where

$$\theta(z) = T(z) - T_\infty$$

$$A(z) = \pi r^2 = \pi\left(\frac{r_0}{L}z\right)^2 = \pi\left(\frac{r_0}{L}\right)^2 z^2$$

$$P(z) = 2\pi\frac{r_0}{L}z$$

$$\Downarrow$$

$$z^2\frac{d^2\theta}{dz^2} + 2z\frac{d\theta}{dz} - m^2 z\theta = 0, \quad m^2 = \frac{2hL}{kr_0}$$

Solution of this equation is

$$\theta(z) = z^{-1/2}\left[A I_1(2m z^{1/2}) + B K_1(2m z^{1/2})\right]$$

Boundary conditions:

$$\theta(0) = \text{finite} \quad \Rightarrow \quad B = 0^*$$

$$\theta(L) = T_w - T_\infty = \theta_w \quad \Rightarrow \quad A = \frac{\sqrt{L}\,\theta_w}{I_1(2m\sqrt{L})}$$

$$\therefore \boxed{\frac{\theta(z)}{\theta_w} = \frac{T(z)-T_\infty}{T_w - T_\infty} = \sqrt{\frac{L}{z}}\,\frac{I_1(2m\sqrt{z})}{I_1(2m\sqrt{L})}}$$

b)

$$q_L = +k\pi r_0^2\frac{dT(L)}{dz} = k\pi r_0^2\frac{d\theta(L)}{dz} = \frac{k\pi r_0^2\theta_w\sqrt{L}}{I_1(2m\sqrt{L})}\frac{d}{dz}\left\{\frac{I_1(2m\sqrt{z})}{\sqrt{z}}\right\}_{z=L}$$

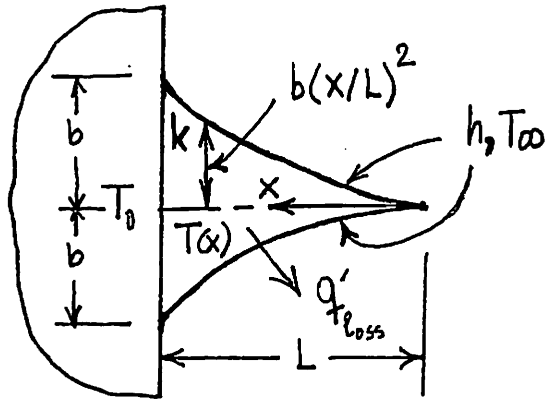
$$= \frac{\pi r_0^2 k\,\theta_w\sqrt{L}}{I_1(2m\sqrt{L})}\left\{\frac{m\,I_2(2m\sqrt{z})}{z}\right\}_{z=L} = \frac{\pi r_0^2 k\,(T_w - T_\infty)}{\sqrt{L}}\,\frac{m I_2(2m\sqrt{L})}{I_1(2m\sqrt{L})}$$

---

* Because, $\lim\limits_{z\to 0}\left\{\frac{1}{\sqrt{z}}I_1(2m\sqrt{z})\right\} \to 0$ and $\lim\limits_{z\to 0}\left\{\frac{1}{\sqrt{z}}K_1(2m\sqrt{z})\right\} \to \infty$

PROB. 3.25: The fin equation:

$$\frac{d}{dx}\left[A(x)\frac{d\theta}{dx}\right]-\frac{hP}{k}\theta(x)=0$$



where

$$\theta(x)=T(x)-T_\infty$$

and

$$\begin{cases} A(x)=2b\,(x/L)^2 \\ P(x)=2 \end{cases}$$

For unit depth

$$\Downarrow$$

$$\frac{d}{dx}\left[x^2\frac{d\theta}{dx}\right]-m^2\theta(x)=0\,,\qquad m^2=\frac{hL^2}{b^2k}$$

$\Downarrow$ Let $z=\ln x$ or $x=e^z$

$$\frac{d^2\theta}{dz^2}+\frac{d\theta}{dz}-m^2\theta(z)=0\;\rightarrow\;\theta(z)=e^{-\frac{1}{2}z}\left[c_1e^{-az}+c_2e^{az}\right]$$

$$a=\frac{1}{2}\sqrt{1+4m^2}$$

$$\therefore\ \theta(x)=x^{-1/2}\left[c_1x^{-a}+c_2x^{a}\right]$$

At $x=0$: $\theta=$ finite $\Rightarrow$ $C_1=0$ (Note $a>\frac{1}{2}$)

At $x=L$: $\theta=T_0-T_\infty$ $\Rightarrow$ $C_2=\dfrac{T_0-T_\infty}{L^{a-1/2}}$

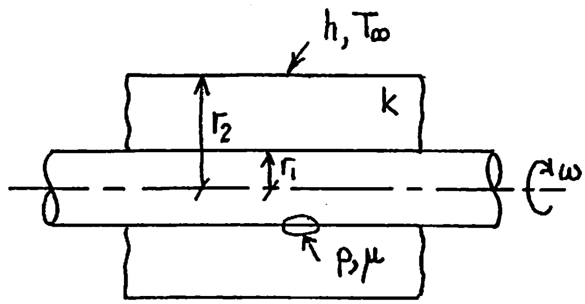
$$\therefore\quad \boxed{\frac{T(x)-T_\infty}{T_0-T_\infty}=\left(\frac{x}{L}\right)^{a-\frac{1}{2}}}$$

Rate of heat loss per unit depth:

$$q'_{loss}=k\,2b\left(\frac{dT}{dx}\right)_{x=L}=\cdots=\frac{2kb}{L}(T_0-T_\infty)\left(a-\frac{1}{2}\right)$$

PROB. 3.26:

Under steady-state conditions the rod will have a uniform temperature, say T_s. Thus,



$$(p\mu)(2\pi r_1)(r_1\omega) = \frac{T_s - T_\infty}{\frac{1}{2\pi k}\ln\frac{r_2}{r_1} + \frac{1}{2\pi r_2 h}}$$

Note that we have taken the interface temperature $= T_s$. (Perfect Contact)

$$\therefore\ T_s = T_\infty + p\mu r_1^2\omega\left\{\frac{1}{k}\ln\frac{r_2}{r_1} + \frac{1}{r_2 h}\right\}$$

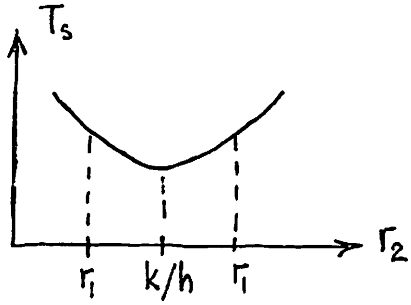
$$\frac{dT_s}{dr_2} = p\mu r_1^2\omega\left\{\frac{1}{kr_2} - \frac{1}{r_2^2 h}\right\} = 0 \quad\Rightarrow\quad r_2 = \frac{k}{h}$$

Thus, the rod (and therefore the interface) temperature will be the lowest for $r_2 = \frac{k}{h}$, provided that $r_1 < \frac{k}{h}$.

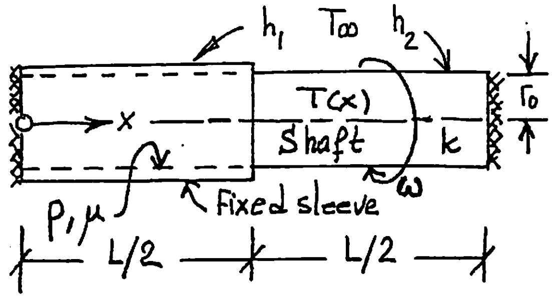
Check: $\left.\frac{d^2T_s}{dr_2^2}\right|_{r_2=\frac{k}{h}} = (\quad)\left.\left\{-\frac{1}{kr_2^2} + \frac{2}{hr_2^3}\right\}\right|_{r_2=\frac{k}{h}} = (\quad)\frac{h^2}{k^3} > 0$

Thus, at $r_2 = \frac{k}{h}$, T_s is a minimum, if $r_1 < \frac{k}{h}$.

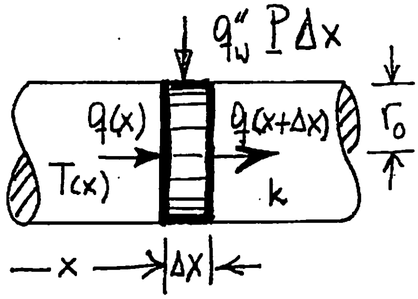
<u>What happens</u> if $r_1 > \frac{k}{h}$? In that case for T_s to be minimum $r_2 \sim r_1$. See the plot below:



PROB. 3.27:



- Because of symmetry, consider only one half of the system.
- Since the shaft is thin, $T = T(x)$.
- Since the sleeve is of negligible thickness, its temperature at any x will be the same as that of the shaft's surface at the same x.
- Assume k = const.



$A = \pi r_o^2, \; P = 2\pi r_o$

Energy balance on the system shown:

$$q(x) + q''_w P\Delta x = q(x+\Delta x)$$

As $\Delta x \to 0$

$$= q(x) + \frac{dq}{dx}\Delta x$$

$$\therefore \left.\begin{aligned} \frac{dq}{dx} &= q''_w P \\ q(x) &= -kA\frac{dT}{dx} \end{aligned}\right\} \quad \frac{d^2T}{dx^2} = -\frac{q''_w P}{kA}$$

When $0 < x < L/2$:

$$q''_w = \mu p r_o \omega - h[T(x) - T_\infty] \longrightarrow \frac{d^2\theta}{dx^2} - m_1^2\theta(x) = -\frac{\mu p r_o \omega P}{kA}$$

Because of the fact that the sleeve has negligible thickness.

where $\theta(x) = T(x) - T_\infty, \; m_1^2 = \frac{h_1 P}{kA}$

$$\theta(x) = C_1 \sinh m_1 x + C_2 \cosh m_1 x + \theta_p, \quad \theta_p = \frac{\mu p r_o \omega \cancel{P}}{\cancel{kA}}\frac{\cancel{kA}}{h_1 \cancel{P}} = \frac{\mu p r_o \omega}{h_1}$$

At $x=0$: $\frac{dT}{dx} = 0 \Rightarrow \frac{d\theta}{dx} = 0 \Rightarrow C_1 = 0$

$$\theta(x) = C_2 \cosh m_1 x + \frac{\mu p r_o \omega}{h_1} \quad \text{—— (1)}$$

When $L/2 < x < L$: $\quad q''_{w} = h_2 \left[T_\infty - T(x) \right]$

$$\therefore \frac{d^2\theta}{dx^2} - m_2^2\,\theta(x) = 0, \qquad m_2^2 = \frac{h_2 P}{kA}$$

$$\theta(x) = C_3 \sinh m_2 (L-x) + C_4 \cosh m_2 (L-x)$$

At $x=L$: $\quad \frac{dT}{dx} = \frac{d\theta}{dx} = 0 \Rightarrow C_3 = 0$

$$\theta(x) = C_4 \cosh m_2 (L-x) \qquad \text{———} \; (2)$$

At $x = L/2$: From Eqs. (1) & (2) we obtain

$$C_2 \cosh m_1 \frac{L}{2} + \frac{\mu P r_0 \omega}{h_1} = C_4 \cos m_2 \frac{L}{2} \quad \text{(Continuity of temp.)}$$

$$C_2 m_1 \sinh m_1 \frac{L}{2} = -C_4 m_2 \sinh m_2 \frac{L}{2} \quad \text{(Continuity of heat flux)}$$

$$C_1 = -\frac{\mu P r_0 \omega}{h_1} \; \frac{m_2 \sinh m_2 \frac{L}{2}}{m_1 \sinh m_1 \frac{L}{2} \cosh m_2 \frac{L}{2} + m_2 \cosh m_1 \frac{L}{2} \sinh m_2 \frac{L}{2}}$$

$$C_2 = \frac{\mu P r_0 \omega}{h_1} \; \frac{m_1 \sinh m_1 \frac{L}{2}}{m_1 \sinh m_1 \frac{L}{2} \cosh m_2 \frac{L}{2} + m_2 \cosh m_1 \frac{L}{2} \sinh m_2 \frac{L}{2}}$$

Thus,

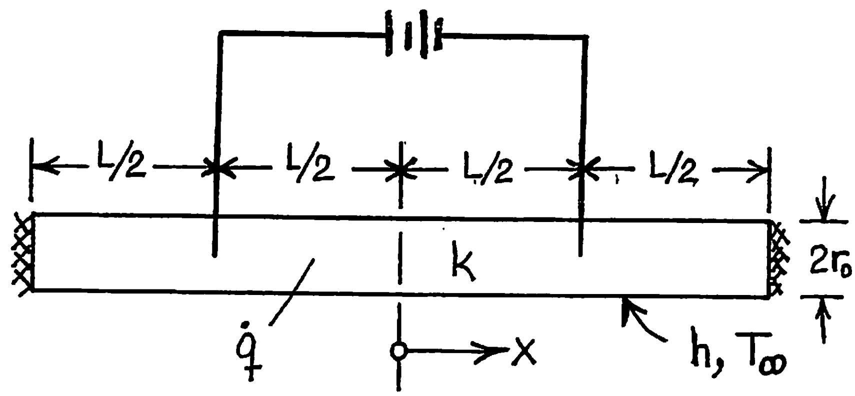
$$T(x) - T_\infty = \frac{\mu P r_0 \omega}{h_1} \left[1 - \frac{m_2 \sinh m_2 \frac{L}{2} \cosh m_1 x}{m_1 \sinh m_1 \frac{L}{2} \cosh m_2 \frac{L}{2} + m_2 \cosh m_1 \frac{L}{2} \sinh m_2 \frac{L}{2}} \right]$$

when $0 < x < L/2$

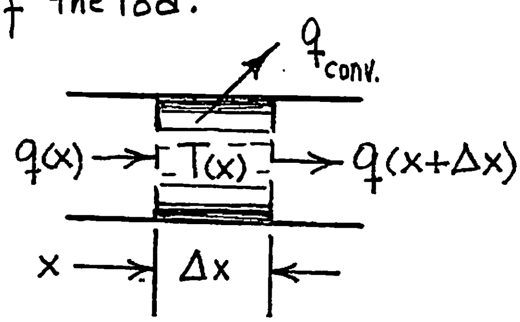
$$T(x) - T_\infty = \frac{\mu P r_0 \omega}{h_1} \; \frac{m_1 \sinh m_1 \frac{L}{2} \cosh m_2 (L-x)}{m_1 \sinh m_1 \frac{L}{2} \cosh m_2 \frac{L}{2} + m_2 \cosh m_1 \frac{L}{2} \sinh m_2 \frac{L}{2}}$$

when $L/2 < x < L$

PROB. 3.28:



Consider a section of the rod:



A thermal energy balance on the section (c.v.) shown gives

$$\cancel{q(x)} + \pi r_o^2 \Delta x\, \dot{q} = q(x+\Delta x) + q_{conv.}$$

$\downarrow$ As $\Delta x \to 0$

$$= \cancel{q(x)} + \frac{dq}{dx}\Delta x + h 2\pi r_o \Delta x\,[T(x)-T_\infty]$$

$$\frac{dq}{dx} + 2\pi r_o h\,[T(x)-T_\infty] = \pi r_o^2 \dot{q}$$

$$q(x) = -k\pi r_o^2 \frac{dT}{dx}$$

Assuming $k = \text{const.}$ $\rightarrow$ $-k\pi r_o^2 \dfrac{d^2T}{dx^2} + 2\pi r_o h\,[T(x)-T_\infty] = \pi r_o^2 \dot{q}$

$$\therefore \quad \boxed{\frac{d^2\theta}{dx^2} - m^2\theta(x) = -\frac{\dot{q}}{k}}\ , \quad m = \frac{2h}{r_o k}$$

$$\theta(x) = T(x) - T_\infty$$

$\underline{\underline{0 < x < L/2}}$: $\dot{q} \neq 0$

$$\theta_1(x) = A \sinh mx + B \cosh mx + \frac{\dot{q}}{m^2 k}$$

$\underline{\underline{L/2 < x < L}}$: $\dot{q} = 0$

$$\theta_2(x) = C \sinh m(L-x) + D \cosh m(L-x)$$

At $x=0$: $\frac{d\theta_1}{dx} = 0 \Rightarrow A = 0$

At $x=L$: $\frac{d\theta_2}{dx} = 0 \Rightarrow C = 0$

At $x = \frac{L}{2}$: $\theta_1 = \theta_2 \Rightarrow B \cosh m\frac{L}{2} + \frac{\dot{q}}{m^2 k} = D \cosh m\frac{L}{2}$

$$\frac{d\theta_1}{dx} = \frac{d\theta_2}{dx} \Rightarrow Bm \sinh m\frac{L}{2} = -Dm \sinh m\frac{L}{2}$$

$$\therefore B = -D$$

$$2B \cosh m\frac{L}{2} = -\frac{\dot{q}}{m^2 k}$$

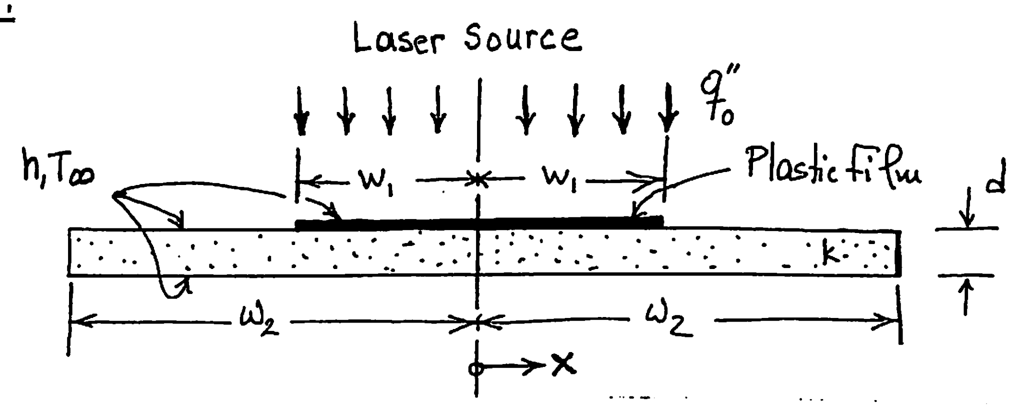
$$\therefore B = -D = -\frac{\dot{q}}{2m^2 k} \frac{1}{\cosh m\frac{L}{2}}$$

Thus,

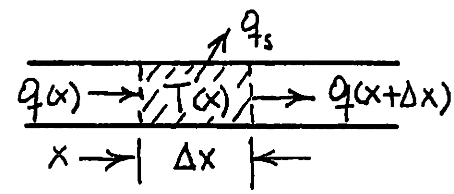
$$T(x) - T_\infty = \frac{\dot{q}}{m^2 k}\left[1 - \frac{1}{2}\frac{\cosh mx}{\cosh m\frac{L}{2}}\right], \quad 0 \leq x \leq \frac{L}{2}$$

$$T(x) - T_\infty = \frac{\dot{q}}{2m^2 k}\frac{\cosh m(L-x)}{\cosh m\frac{L}{2}}, \quad \frac{L}{2} \leq x \leq L$$

PROB. 3.29:



Consider a section of the metal strip:



Energy (thermal) balance on the c.v. shown per unit depth gives

$$\left.\begin{aligned} q(x) &= q(x+\Delta x) + q_s \\ &\downarrow \text{As } \Delta x \to 0 \\ &= q(x) + \frac{dq}{dx}\Delta x + q_s \end{aligned}\right\}$$

where

$$q_s = \begin{cases} 2h(T-T_\infty)\Delta x - q_o''\Delta x, & 0<x<w_1 \\ 2h(T-T_\infty)\Delta x, & w_1<x<w_2 \end{cases}$$

and

$$q(x) = -k(d\cdot 1)\frac{dT}{dx}$$

$$\Downarrow$$

$$\frac{d^2T}{dx^2} - \frac{q_s}{k\cdot d} = 0$$

<u>$0\le x\le w_1$:</u> $T_1(x) \equiv T(x)$.

$$\frac{d^2T}{dx^2} - m^2(T_1 - T_\infty) = \frac{q_o''}{kd}, \quad m^2 = \frac{2h}{kd}$$

$$\therefore\ T_1(x) - T_\infty = C_1\cosh(mx) + C_2\sinh(mx) + \frac{q_o''}{2h}$$

$\omega_1 \leq x \leq \omega_2$: $T_2(x) = T(x)$

$$\frac{d^2T_2}{dx^2} - m^2(T_2 - T_\infty) = 0$$

$$\therefore \quad T_2(x) = C_3 \cosh m(\omega_1 - x) + C_4 \sinh m(\omega_2 - x)$$

At $x=0$: $\dfrac{dT_1}{dx} = 0 \quad \Rightarrow \quad C_2 = 0$

At $x=\omega_2$: $\dfrac{dT_2}{dx} = 0 \quad \Rightarrow \quad C_4 = 0$

At $x=\omega_1$: $T_1 = T_2 \quad \Rightarrow \quad C_1 \cosh(m\omega_1) + \dfrac{q_0''}{2h} = C_3 \cosh m(\omega_2 - \omega_1)$

$$\frac{dT_1}{dx} = \frac{dT_2}{dx} \quad \Rightarrow \quad C_1 m \sinh(m\omega_1) = C_3(-m)\sinh m(\omega_2 - \omega_1)$$

$$\Downarrow$$

$$C_1 = -\frac{q_0''}{2h}\,\frac{\sinh m(\omega_2 - \omega_1)}{\sinh(m\omega_2)}$$

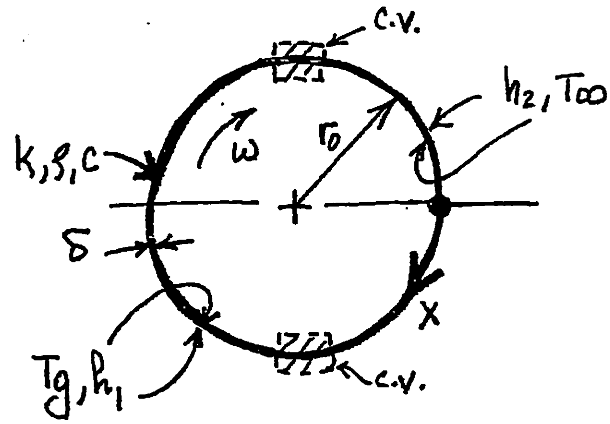
$$C_3 = \frac{q_0''}{2h}\,\frac{\sinh(m\omega_1)}{\sinh(m\omega_2)}$$

Thus,

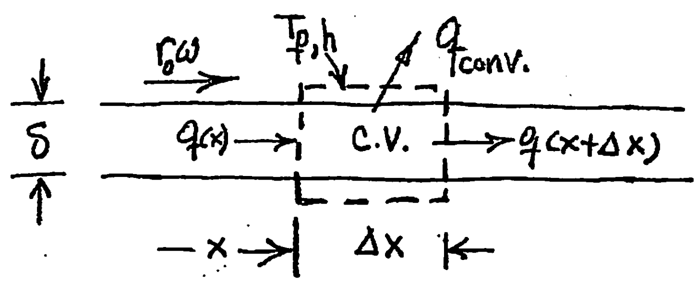
$$T(x) - T_\infty = \frac{q_0''}{2h}\left[1 - \frac{\sinh m(\omega_2 - \omega_1)}{\sinh(m\omega_2)}\cosh(mx)\right], \quad 0 \leq x \leq \omega_1$$

$$T(x) - T_\infty = \frac{q_0''}{2h}\,\frac{\sinh(m\omega_1)}{\sinh(m\omega_2)}\cosh m(\omega_2 - x), \quad \omega_1 \leq x \leq \omega_2$$

PROB. 3.30:



Formulation (by control volume approach):



$$\cancel{q(x)} = q(x+\Delta x) + q_{conv.}$$ ← Thermal energy balance for unit depth.

As $\Delta x \to 0$
$$= \cancel{q(x)} + \frac{dq}{dx}\Delta x + 2\Delta x h\left[T(x) - T_f\right]$$

$$\frac{dq}{dx} + 2h\left[T(x) - T_f\right] = 0$$

where
$$q(x) = c\rho r_0 \delta T(x) + \left(-k\delta\frac{dT}{dx}\right)$$

$$\therefore \quad \frac{d^2T}{dx^2} - 2a\frac{dT}{dx} - m^2\left[T(x) - T_f\right] = 0, \quad m^2 = \frac{2h}{k\delta}, \quad a = \frac{1}{2}\frac{r_0\omega}{\alpha}, \quad \alpha = \frac{k}{\rho c}$$

When $0 < x < L \ (=\pi r_0)$ Let $T(x) = T_1(x)$

$$\frac{d^2T_1}{dx^2} - 2a\frac{dT_1}{dx} - m_1^2\left[T_1(x) - T_g\right] = 0, \quad m_1^2 = \frac{2h_1}{k\delta}$$

$$\therefore \quad T_1(x) - T_g = e^{ax}\left[A\sinh b_1(L-x) + B\cosh b_1(L-x)\right], \quad b_1 = \sqrt{a^2 + m_1^2}$$

When $L(=\pi r_0) < x < 2L$ Let $T(x) = T_2(x)$

$$\frac{d^2T_2}{dx^2} - 2a\frac{dT_2}{dx} - m_2^2\left[T_2(x) - T_\infty\right] = 0, \quad m_2^2 = \frac{2h_2}{k\delta}$$

$$\therefore\ T_2(x) - T_\infty = e^{ax}\left[C\sinh b_2(L-x) + D\cosh b_2(L-x)\right], \quad b_2 = \sqrt{a^2 + m_2^2}$$

Apply B.C.'s:

At $x = L$: $T_1(L) = T_2(L) \quad \Rightarrow D - B = e^{-aL}(T_g - T_\infty)$ — (1)

$$\frac{dT_1}{dx} = \frac{dT_2}{dx} \quad \Rightarrow a(D-B) = Cb_2 - Ab_1 \quad \text{— (2)}$$

At $x = 0$ & $2L$: $T_1(0) = T_2(2L)$

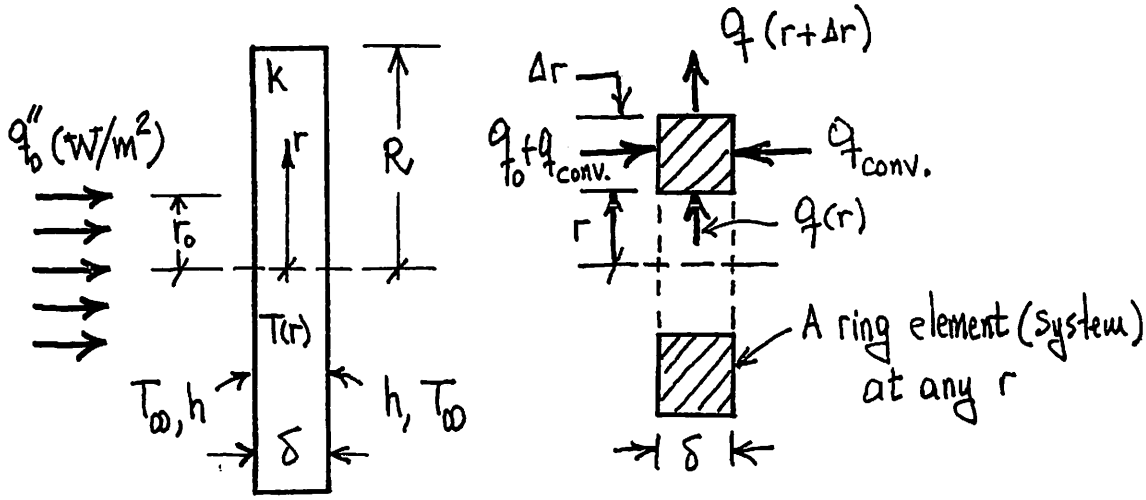
$$\Rightarrow A\sinh b_1L + B\cosh b_1L + (T_g - T_\infty) = e^{2aL}\left[-C\sinh b_2L + D\cosh b_2L\right] \quad \text{— (3)}$$

$$\frac{dT_1(0)}{dx} = \frac{dT_2(L)}{dx}$$

$$\Rightarrow a\left[A\sinh b_1L + B\cosh b_1L\right] + \left[-Ab_1\cosh b_1L - Bb_1\sinh b_1L\right]$$
$$= ae^{2L}\left[-C\sinh b_2L + D\cosh b_2L\right]$$
$$+ a^{2L}\left[-Cb_2\cosh b_2L - b_2D\sinh b_2L\right] \quad \text{— (4)}$$

Solution of the set of four algebraic equations (1), (2), (3) and (4) yields the unknown constants A, B, C, and D in the solutions for $T_1(x)$ & $T_2(x)$.

PROB. 3.31:



Under steady-state conditions, a thermal energy balance on the system shown gives

$$q(r) + q_o + 2\,q_{conv.} = q(r+\Delta r) \qquad (1)$$

As $\Delta r \to 0$,

$$q(r+\Delta r) = q(r) + \frac{dq}{dr}\Delta r \quad \text{with} \quad q(r) = -k\,2\pi r \delta \frac{dT}{dr}$$

$$q_o = A\,2\pi r \Delta r\, q_o'', \qquad A = \begin{cases} 1, & 0<r<r_o \\ 0, & r_o<r<R \end{cases}$$

$$q_{conv.} = 2\pi r\,\Delta r\, h\left[T_\infty - T(r)\right]$$

Thus, assuming $k = \text{const.}$, Eq. (1) reduces to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right) - m^2\,\theta(r) = -\frac{A\,q_o''}{k\delta} \qquad (2)$$

where $\theta(r) = T(r) - T_\infty$ and $m^2 = \dfrac{2h}{k\delta}$

Solution of Eq. (2),

$$\theta(r) = \begin{cases} \theta_1(r) = C_1 I_0(mr_o) + C_2 K_o(mr) + \dfrac{q_o''}{2h}, & 0<r<r_o \\ \theta_2(r) = C_3 I_o(mr_o) + C_4 K_o(mr), & r_o<r<R \end{cases}$$

Boundary conditions:

① $T(0) = \text{finite} \rightarrow \theta_1(0) = \text{finite} \Rightarrow C_2 \equiv 0$

② $\dfrac{dT(R)}{dr} = 0 \rightarrow \dfrac{d\theta(R)}{dr} = 0$ ← Negligible heat loss from edge

$$\Rightarrow C_3\, m\, I_1(mR) - C_4\, m\, K_1(mR) = 0 \Rightarrow C_4 = C_3 \frac{I_1(mR)}{K_1(mR)}$$

Thus, $\theta_2(r) = C_3 \left[I_0(mr) + \dfrac{I_1(mR)}{K_1(mR)} K_0(mr) \right]$

③ $\theta_1(r_0) = \theta_2(r_0)$ ← Continuity of temperature at $r = r_0$.

$$C_1 I_0(mr_0) + \frac{q_0''}{2h} = C_3 \left[I_0(mr_0) + \frac{I_1(mR)}{K_1(mR)} K_0(mr_0) \right] \quad (3)$$

④ $\dfrac{d\theta_1(r_0)}{dr} = \dfrac{d\theta_2(r_0)}{dr}$ ← Continuity of heat flux at $r = r_0$.

$$C_1 \cancel{m} I_1(mr_0) = C_3 \left[\cancel{m} I_1(mr_0) - \frac{I_1(mR)}{K_1(mR)} \cancel{m} K_1(mr_0) \right] \quad (4)$$

Solutions of Eqs. (3) & (4) give

$$C_1 = \frac{\Delta_1}{\Delta_2} \frac{q_0''}{2h} \quad \text{and} \quad C_3 = \frac{K_1(mR)}{\Delta_2} \frac{q_0''}{2h} \Rightarrow C_4 = \frac{I_1(mR)}{\Delta_2} \frac{q_0''}{2h}$$

where

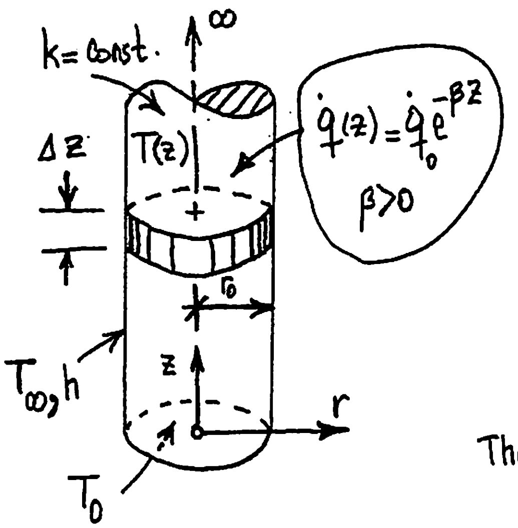
$$\Delta_1 = \frac{1}{I_1(mr_0)} \left[I_1(mr_0) K_1(mR) - I_1(mR) K_1(mr_0) \right]$$

$$\Delta_2 = \frac{I_1(mR)}{I_1(mr_0)} \left[I_1(mr_0) K_0(mr_0) + I_0(mr_0) K_1(mr_0) \right]$$

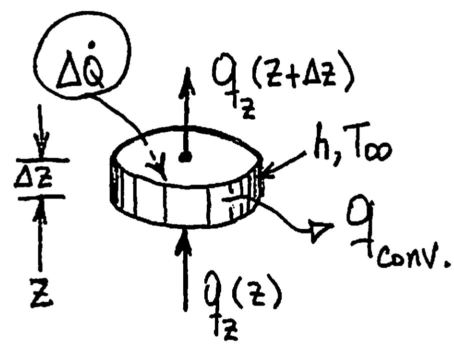
Thus,

$$T(r) - T_\infty = \begin{cases} \theta_1(r) = \dfrac{q_0''}{2h} \left[1 + \dfrac{\Delta_1}{\Delta_2} I_0(mr) \right], & 0 < r < r_0 \\ \theta_2(r) = \dfrac{q_0''}{2h} \dfrac{1}{\Delta_2} \left[K_1(mR) I_0(mr) + I_1(mR) K_0(mr) \right], & r_0 < r < R \end{cases}$$

PROB. 3.32:



Consider a section of the rod as shown:



Thermal energy balance on the section shown:

$$q_z(z) + \Delta\dot{Q} = q_z(z+\Delta z) + q_{conv.}$$

$$\Big\downarrow \text{As } \Delta z \to 0$$

$$\cancel{q_z(z)} + \dot{q}(z)\,\pi r_0^2 \Delta z = \cancel{q_z} + \frac{dq_z}{dz}\Delta z + h\, 2\pi r_0 \Delta z\,[T(z)-T_\infty]$$

$$\left\{ \begin{array}{c} \dfrac{dq_z}{dz} + h\,2\pi r_0\,[T(z)-T_\infty] = \dot{q}(z)\,\pi r_0^2 \\ q_z = -k\,\pi r_0^2 \dfrac{dT}{dz} \end{array} \right\}$$

$$\Downarrow$$

$$\frac{d^2\theta}{dz^2} - m^2\theta(z) = -\frac{\dot{q}_0}{k}e^{-\beta z}, \quad \theta(z)=T(z)-T_\infty$$

$$\theta(0)=T_0-T_\infty, \quad \theta(\infty)=0 \qquad m^2=\frac{2h}{kr_0}$$

$$\theta(z) = Ae^{-mz} + Be^{mz} + \theta_p(z)$$

Particular solution: $$\theta_p(z) = \begin{cases} \dfrac{\dot{q}_0}{k(m^2-\beta^2)}e^{-\beta z}, & \beta \neq m \\ \dfrac{\dot{q}_0}{2k\beta}\, z\, e^{-\beta z}, & \beta = m \end{cases}$$

$$\theta(\infty) = 0 \quad \Rightarrow \quad B = 0$$

$$\theta(0) = T_0 - T_\infty \quad \Rightarrow \quad A = \begin{cases} (T_0 - T_\infty) - \dfrac{\dot{q}_0}{k(m^2-\beta^2)}, & \beta \neq m \\ (T_0 - T_\infty) & , \beta = m \end{cases}$$

$$\therefore \quad \theta(z) = T(z) - T_\infty = \begin{cases} (T_0 - T_\infty)\, e^{-mz} + \dfrac{\dot{q}_0}{k(m^2-\beta^2)}\left[e^{-\beta z} - e^{-mz} \right], & \beta \neq m \\ \left[T_0 - T_\infty + \dfrac{\dot{q}_0}{2k\beta}\, z \right] e^{-\beta z} & , \beta = m \end{cases}$$

CHAPTER 4
THE STURM-LIOUVILLE THEORY AND FOURIER EXPANSIONS

PROB. 4.1: $$y(x) = A\left[B_{12}\, y_1(x) - B_{11}\, y_2(x)\right]$$

$$B_{12} = \alpha_1 y_2(a) + \beta_1 \frac{dy_2(a)}{dx}, \quad B_{11} = \alpha_1 y(a) + \beta_1 \frac{dy_1(a)}{dx}$$

B.C. (4.3b): $$\alpha_1 y(a) + \beta_1 \frac{dy(a)}{dx} = 0$$

$$\alpha_1 \cancel{A}\left[B_{12}\, y_1(a) - B_{11}\, y_2(a)\right] + \beta_1 \cancel{A}\left[B_{12} \frac{dy_1(a)}{dx} - B_{11} \frac{dy_2(a)}{dx}\right] = 0$$

$$\underbrace{\left[\alpha_1 y_1(a) + \beta_1 \frac{dy_1(a)}{dx}\right]}_{B_{11}} B_{12} - \underbrace{\left[\alpha_1 y_2(a) + \beta_1 \frac{dy_2(a)}{dx}\right]}_{B_{12}} B_{11} = 0$$

$$B_{11} B_{12} - B_{12} B_{11} = 0 \quad \checkmark$$

B.C. (4.3c): $$\alpha_2\, y(b) + \beta_2 \frac{dy(b)}{dx} = 0$$

$$\alpha_2 \cancel{A}\left[B_{12}\, y_1(b) - B_{11}\, y_2(b)\right] + \beta_2 \cancel{A}\left[B_{12} \frac{dy_1(b)}{dx} - B_{11} \frac{dy_2(b)}{dx}\right] = 0$$

$$\underbrace{\left[\alpha_2 y_1(b) + \beta_2 \frac{dy_1(b)}{dx}\right]}_{B_{21}} B_{12} - \underbrace{\left[\alpha_2 y_2(b) + \beta_2 \frac{dy_2(b)}{dx}\right]}_{B_{22}} B_{11} = 0$$

$$B_{21} B_{12} - B_{22} B_{11} = 0 \qquad \text{per Eq. (4.7b).}$$

Following the same procedure, it can also be shown that Eq.(4.10) satisfies the B.C.'s (4.3 b,c).

PROB. 4.2:

$$\frac{d^2y}{dx^2} - \lambda^2 y = 0 \quad \longrightarrow \quad y(x) = C_1 \sinh \lambda x + C_2 \cosh \lambda x$$

$$y(0) = 0 \text{ and } y(L) = 0 \qquad y(0)=0 \Rightarrow C_2 = 0$$

$$y(L) = 0 \Rightarrow C_1 \sinh \lambda L = 0$$

$$C_1 \sinh \lambda L = 0 \quad \begin{cases} C_1 = 0 \longrightarrow \text{Trivial solution} \\ \sinh \lambda L = 0 \Rightarrow \lambda L = 0 \Rightarrow \lambda = 0 \end{cases}$$

$\lambda = 0$ leads to trivial solution

Note that $\lambda=0$ is the only <u>real</u> root of $\sinh \lambda L = 0$

PROB. 4.3:

$$A_0; \quad A_1 + A_2 x; \quad A_3 + A_4 x + A_5 x^2$$

$$\int_0^1 A_0 (A_1 + A_2 x)\, dx = 0 \quad \longrightarrow \quad A_1 + \tfrac{1}{2} A_2 = 0 \quad (1)$$

$$\int_0^1 A_0 (A_3 + A_4 x + A_5 x^2)\, dx = 0 \quad \longrightarrow \quad A_3 + \tfrac{1}{2} A_4 + \tfrac{1}{3} A_5 = 0 \quad (2)$$

$$\int_0^1 (A_1 + A_2 x)(A_3 + A_4 x + A_5 x^2)\, dx = 0$$

$$\longrightarrow \quad A_1 A_3 + \tfrac{1}{2}(A_1 A_4 + A_2 A_3) + \tfrac{1}{3}(A_1 A_5 + A_2 A_4) + \tfrac{1}{4} A_2 A_5 = 0 \quad (3)$$

There are 6 constants to be determined, but there are only 3 equations. Hence, 3 of the constants can be taken arbitrarily, say, $A_0 = 1$, $A_1 = 1$ & $A_3 = 1$. Thus, we obtain

$$A_2 = -2, \quad A_4 = -6 \quad \text{and} \quad A_5 = 6$$

Hence, the orthogonal set would be

$$\{1,\ 1-2x,\ 1-6x+6x^2\}$$

PROB. 4.4:

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+[a_2(x)+\lambda a_3(x)]y=0$$

$$\underbrace{\frac{d^2y}{dx^2}+\frac{a_1}{a_0}\frac{dy}{dx}}_{\frac{1}{\bar{p}(x)}\frac{d}{dx}\left[\bar{p}(x)\frac{dy}{dx}\right]}+\left[\frac{a_2}{a_0}+\lambda\frac{a_3}{a_0}\right]y=0$$

$$\therefore \frac{1}{\bar{p}(x)}\frac{d\bar{p}}{dx}=\frac{a_1}{a_0} \to \frac{d\bar{p}}{\bar{p}}=\frac{a_1}{a_0}dx \to \ln\bar{p}(x)=\int\frac{a_1}{a_0}dx+\ln c$$

$$\bar{p}(x)=C_1e^{\int\frac{a_1}{a_0}dx}=c_1p(x);\quad p(x)=e^{\int\frac{a_1}{a_0}dx}$$

$$\frac{1}{\cancel{c_1}p(x)}\frac{d}{dx}\left[\cancel{c_1}p(x)\frac{dy}{dx}\right]=\frac{1}{p(x)}\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]$$

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[\underbrace{\frac{a_2}{a_0}p(x)}_{q(x)}+\lambda\underbrace{\frac{a_3}{a_0}p(x)}_{w(x)}\right]y=0$$

PROB. 4.5:

$$\boxed{\begin{gathered}\frac{d^2y}{dx^2}+\lambda^2y=0\\ \frac{dy(0)}{dx}=0,\quad\frac{dy(L)}{dx}=0\end{gathered}}$$

$$y(x)=A\sin\lambda x+B\cos\lambda x$$

$$\frac{dy}{dx}=A\lambda\cos\lambda x-B\lambda\sin\lambda x$$

$$\frac{dy(0)}{dx}=0\Rightarrow A=0$$

$$\frac{dy(L)}{dx}=0\Rightarrow B\lambda\sin\lambda L=0\Rightarrow\sin\lambda L=0\Rightarrow\lambda_n=\frac{n\pi}{L},\ n=0,1,2\ldots$$

$$\therefore \phi_n(x)=\cos\lambda_nx;\quad\lambda_n=\frac{n\pi}{L},\ n=0,1,2,\ldots$$

PROB. 4.6: (a)

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0 \qquad \longrightarrow \qquad y(x) = C_1 \cos\lambda x + C_2 \sin\lambda x$$

$$\frac{dy(0)}{dx} = 0, \ \alpha y(L) + \beta \frac{dy(L)}{dx} = 0 \qquad \frac{dy(0)}{dx} = 0 \Rightarrow \ C_2 = 0$$

$$\therefore \ \phi(x) = \cos\lambda x$$

$$\alpha y(L) + \beta \frac{dy(L)}{dx} = 0 \Longrightarrow \alpha \cancel{C_1} \cos\lambda L + \beta(-\cancel{C_1}\lambda \sin\lambda L) = 0$$

$$\lambda \tan\lambda L = H, \quad H = \frac{\alpha}{\beta}$$

$$\therefore \ \phi_n(x) = \cos\lambda_n x$$

and λ_n's are the positive roots of $\lambda \tan\lambda L = H$.

(b)

$$f(x) = \sum_{n=1}^{\infty} A_n \cos\lambda_n x, \qquad 0 \le x < L$$

where

$$A_n = \frac{1}{N_n}\int_0^L f(x)\cos\lambda_n x\, dx$$

with

$$N_n = \int_0^L \cos^2\lambda_n x\, dx = \frac{1}{2\lambda_n}\left[\lambda_n L + \sin\lambda_n L \cos\lambda_n L\right]$$

PROB. 4.7: (a)

$$\left\{ \begin{array}{c} \dfrac{d^2y}{dx^2} + \lambda^2 y = 0 \\ y(0) = 0 \\ y(L) - L\dfrac{dy(L)}{dx} = 0 \end{array} \right\}$$

Note that $\lambda = 0$ is an eigenvalue.

$$y_0(x) = a_0 x + b_0$$

$$y_0(0) = 0 \ \rightarrow \ b_0 = 0$$

$$y_0(L) - L\frac{dy_0(L)}{dx} = 0 \ \rightarrow \ a_0 L - L a_0 = 0 \ ✓$$

$$\therefore \ y_0(x) = a_0 \phi_0(x), \quad \phi_0(x) = x$$

When $\lambda \neq 0$,

$$y_n(x) = a_n \sin\lambda_n x + b_n \cos\lambda_n x$$

$$y_n(0) = 0 \implies b_n = 0$$

$$y_n(L) - L\frac{dy_n(L)}{dx} = 0 \implies \sin\lambda_n L - L\lambda_n \cos\lambda_n L = 0$$

$$\underbrace{\tan\lambda_n L = \lambda_n L,}_{\Downarrow}$$

$$\lambda_n, \ n = 1, 2, 3, \ldots$$

Thus, the eigenfunctions are

$$\phi_n(x) = \begin{cases} x, & n = 0 \\ \sin\lambda_n x, & n \neq 0 \end{cases}$$

and the eigenvalues are the zeros of

$$\tan\lambda_n L = \lambda_n L, \quad n = 0, 1, 2, \ldots$$

(b)

$$f(x) = A_0 x + \sum_{n=1}^{\infty} A_n \sin\lambda_n x, \quad 0 < x < L$$

$$A_0 = \frac{1}{N_0}\int_0^L f(x)\, x\, dx, \quad N_0 = \int_0^L x^2 dx = \frac{L}{3}$$

$$A_n = \frac{1}{N_n}\int_0^L f(x) \sin\lambda_n x\, dx, \quad N_n = \int_0^L \sin^2\lambda_n x\, dx$$

$$= \cdots = \frac{L}{2}\sin^2\lambda_n L$$

Thus,

$$A_0 = \frac{3}{L}\int_0^L f(x)\, x\, dx, \quad n = 0$$

$$A_n = \frac{2}{L\sin^2\lambda_n L}\int_0^L f(x)\sin\lambda_n x\, dx, \quad n = 1, 2, 3, \ldots$$

PROB. 4.8:

$$f(x) = \begin{cases} 1, & x < \frac{L}{2} \\ 0, & x > \frac{L}{2} \end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x, \qquad 0<x<L$$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx = \frac{2}{L} \int_0^{L/2} \sin \frac{n\pi}{L} x \, dx$$

$$= \frac{2}{L} \left[-\frac{L}{n\pi} \cos \frac{n\pi}{L} x \right]_{x=0}^{x=L/2} = \frac{2}{n\pi} \left[1 - \cos \frac{n\pi}{2} \right]$$

$$\therefore \; f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[1 - \cos \frac{n\pi}{2} \right] \sin \frac{n\pi}{L} x, \qquad 0<x<L$$

PROB. 4.9:

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{L} x, \qquad 0<x<L$$

$$A_0 = \frac{1}{L} \int_0^L f(x) \, dx = \frac{1}{L} \int_0^{L/2} dx = \frac{1}{2}$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x \, dx$$

$$= \frac{2}{L} \int_0^{L/2} \cos \frac{n\pi}{L} x \, dx = \frac{2}{n\pi} \sin \frac{n\pi}{2}$$

$$\therefore \; f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \cdot \cos \frac{n\pi}{L} x, \qquad 0 \le x < L$$

PROB. 4.10: Since $\{\phi_n(x);\ n=0,1,2,\dots\}$ is the complete set of all the characteristic functions of the Sturm-Liouville problem (4.13), and $f(x)$ is any piecewise differentiable function in the interval (a,b),

$$f(x) = \sum_{n=0}^{\infty} A_n \phi_n(x)$$

is a valid expansion in (a,b), where

$$A_n = \frac{1}{N_n}\int_a^b f(x)\,\phi_n(x)\,w(x)\,dx \quad \text{with} \quad N_n = \int_a^b \phi_n^2(x)\,w(x)\,dx$$

and $w(x)$ is the weight function. Multiply both sides of this expansion by $f(x)w(x)$ and integrate the resultant relation over (a,b):

$$\int_a^b [f(x)]^2 w(x)\,dx = \int_a^b f(x)\left\{\sum_{n=0}^{\infty} A_n \phi_n(x)\right\} w(x)\,dx$$

$$= \sum_{n=0}^{\infty} A_n \underbrace{\int_a^b f(x)\phi_n(x)\,w(x)\,dx}_{N_n A_n} = \underline{\sum_{n=0}^{\infty} N_n A_n^2}$$

PROB. 4.11: (a)

$$1 = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x$$

$$\therefore\quad A_n = \frac{2}{L}\int_0^L 1.\ \sin\frac{n\pi}{L}x\,dx = -\frac{2}{n\pi}\cos\frac{n\pi}{L}x\Big|_0^L = -\frac{2}{n\pi}\left[(-1)^n - 1\right]$$

(b)

$$\underbrace{\int_0^L 1^2.\,dx}_{L} = \underbrace{\sum_{n=1}^{\infty}\frac{L}{2}\left\{-\frac{2}{n\pi}\left[(-1)^n-1\right]\right\}^2}_{\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}} \Longrightarrow \boxed{\frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}}$$

PROB. 4.12:

$$f(x) = B_0 + \sum_{n=1}^{\infty}\left[A_n \sin\frac{n\pi}{L}x + B_n \cos\frac{n\pi}{L}x\right], \quad -L<x<L$$

$$B_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{L/2} dx = \frac{3}{4}$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx, \quad n=1,2,3,\ldots$$

$$= \frac{1}{L}\int_{-L}^{L/2}\sin\frac{n\pi}{L}x\,dx = \frac{1}{n\pi}\Big[\underbrace{\cos(n\pi)}_{(-1)^n} - \cos\left(\frac{n\pi}{2}\right)\Big]$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx, \quad n=1,2,3,\ldots$$

$$= \frac{1}{L}\int_{-L}^{L/2}\cos\frac{n\pi}{L}x\,dx = \frac{1}{n\pi}\sin\frac{n\pi}{2}$$

$$\therefore\ f(x) = \frac{3}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\left[(-1)^n - \cos\frac{n\pi}{2}\right]\sin\frac{n\pi}{L}x + \sin\frac{n\pi}{2}\cos\frac{n\pi}{L}x\right\}$$

PROB. 4.13:

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0 \longrightarrow y(x) = A\sin\lambda_n x + B\cos\lambda x$$

$$\left.\begin{array}{l} y(-L) = y(L) \longrightarrow \\ \dfrac{dy(-L)}{dx} = \dfrac{dy(L)}{dx} \longrightarrow \end{array}\right. \left\{\begin{array}{l} 2A\sin\lambda L = 0 \\ 2B\lambda\sin\lambda L = 0 \end{array}\right\}$$

$$\Downarrow$$

$$\lambda_n = \frac{n\pi}{L}, \quad n=0,1,2,\ldots$$

Note that corresponding to each eigenvalue λ_n there are <u>two</u> eigenfunctions:

$$\phi_n(x) = \sin\lambda_n x \quad \text{and} \quad \psi_n(x) = \cos\lambda_n x, \; n=0,1,2\ldots$$

(b) Since the above eigenvalue problem is a Sturm-Liouville system (with periodic boundary conditions), the set

$$\{ \sin\lambda_n x, \; \cos\lambda_n x \; ; \; n = 0, 1, 2, \ldots \}$$

is a complete orthogonal set over the interval $(-L, L)$ with respect to $w(x) = 1$. Thus, an arbitrary function $f(x)$, piecewise differentiable over $(-L, L)$, can be expanded as

$$f(x) = \sum_{n=0}^{\infty} (a_n \sin\lambda_n x + b_n \cos\lambda_n x)$$

$$= b_0 + \sum_{n=1}^{\infty} (a_n \sin\lambda_n x + b_n \cos\lambda_n x), \quad -L < x < L$$

where expansion coefficients can then be calculated as

$$a_n = \frac{\int_{-L}^{L} f(x) \sin\lambda_n x \, dx}{\int_{-L}^{L} \sin^2\lambda_n x \, dx} = \frac{1}{L} \int_{-L}^{L} f(x) \sin\lambda_n x \, dx$$

$$n = 1, 2, 3, \ldots$$

$$b_0 = \frac{\int_{-L}^{L} f(x) \, dx}{\int_{-L}^{L} dx} = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx$$

$$b_n = \frac{\int_{-L}^{L} f(x) \cos\lambda_n x \, dx}{\int_{-L}^{L} \cos^2\lambda_n x \, dx} = \frac{1}{L} \int_{-L}^{L} f(x) \cos\lambda_n x \, dx$$

$$n = 1, 2, 3, \ldots$$

Here, it should be noted that, due to orthogonality,

$$\int_{-L}^{L} \sin\lambda_n x \cos\lambda_m x \, dx = 0; \; n, m = 0, 1, 2, 3, \cdots$$

$$\int_{-L}^{L} \sin\lambda_n x \sin\lambda_m x \, dx = 0, \quad m \neq n$$

and

$$\int_{-L}^{L} \cos\lambda_n x \cos\lambda_m x \, dx = 0, \quad m \neq n$$

PROB. 4.14:

$$r = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r), \quad 0 < r < r_0$$

where λ_n's are the positive roots of $J_0(\lambda r_0) = 0$.

$$A_n = \frac{1}{N_n} \int_0^{r_0} r . J_0(\lambda_n r) r \, dr, \quad N_n = \frac{r_0^2}{2} J_1^2(\lambda_n r_0)$$

Table 4.2

$$\vdots$$

$$= \frac{1}{N_n} \frac{1}{\lambda_n} \left\{ r_0^2 J_1(\lambda_n r_0) - \frac{1}{\lambda_n} \int_0^{r_0} J_0(\lambda_n r) \, dr \right\}$$

Note that $\int_0^{r_0} J_0(\lambda_n r) \, dr$ cannot be further simplified.

The function,

$$\int_0^{r} J_0(\lambda_n r) \, dr$$

is a tabulated function.

PROB. 4.15:

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-(\lambda^2r^2+\nu^2)R=0 \rightarrow R(r)=AI_\nu(\lambda r)+BK_\nu(\lambda r)$$

$$R(0)=\text{finite} \rightarrow B\equiv 0$$

$$R(r_0)=0 \rightarrow AI_\nu(\lambda r_0)=0$$

If $\nu=0$, $I_0(\lambda r_0)=0$ has no real root.

In this case, $A=0$. But this leads to $R(r)=0$

If $\nu>0$, $I_\nu(\lambda r_0)=0 \Rightarrow \lambda=0$. But this also leads to trivial solution $R(r)=0$.

PROB. 4.16: (a)

$$\left\{\begin{array}{c}\dfrac{d^2R}{dr^2}+\dfrac{1}{r}\dfrac{dR}{dr}+\lambda^2R=0\\[2ex] R(a)=0\ ,\ R(b)=0\end{array}\right\}$$

$$R(r)=C_1J_0(\lambda r)+C_2Y_0(\lambda r)$$

$$C_1J_0(\lambda a)+C_2Y_0(\lambda a)=0$$

$$C_1J_0(\lambda b)+C_2Y_0(\lambda b)=0$$

In order to have non-trivial solution for C_1 and C_2,

$$\begin{vmatrix}J_0(\lambda a) & Y_0(\lambda a)\\ J_0(\lambda b) & Y_0(\lambda b)\end{vmatrix}=0 \quad \text{or} \quad J_0(\lambda a)Y_0(\lambda b)-J_0(\lambda b)Y_0(\lambda a)=0$$

$$\Downarrow$$

$$\lambda_n\ ,\ n=1,2,\ldots$$

Then, the algebraic equations for C_1 and C_2 become identical, yielding

$$C_1 = -C_2 \frac{Y_0(\lambda_n a)}{J_0(\lambda_n a)} = -C_2 \frac{Y_0(\lambda_n b)}{J_0(\lambda_n b)}$$

Thus,

$$\begin{aligned} R_n(r) &= C_1 \left[J_0(\lambda_n r) + \frac{C_2}{C_1} Y_0(\lambda_n r) \right] \\ &= C_1 \left[J_0(\lambda_n r) - \frac{J_0(\lambda_n b)}{Y_0(\lambda_n b)} Y_0(\lambda_n r) \right] \\ &= \underbrace{\frac{C_1}{Y_0(\lambda_n b)}}_{\text{Arbitrary constant}} \left[J_0(\lambda_n r) Y_0(\lambda_n b) - J_0(\lambda_n b) Y_0(\lambda_n r) \right] \end{aligned}$$

$\therefore$ Eigenfunctions : $\boxed{\phi_n(r) = J_0(\lambda_n r) Y_0(\lambda_n b) - J_0(\lambda_n b) Y_0(\lambda_n r)}$

(b)

$$f(r) = \sum_{n=1}^{\infty} A_n \phi_n(r), \quad a < r < b$$

$$\therefore \quad A_n = \frac{1}{N_n} \int_a^b f(r)\, \phi_n(r)\, r\, dr$$

where

$$N_n = \int_a^b \phi_n^2(r)\, r\, dr$$

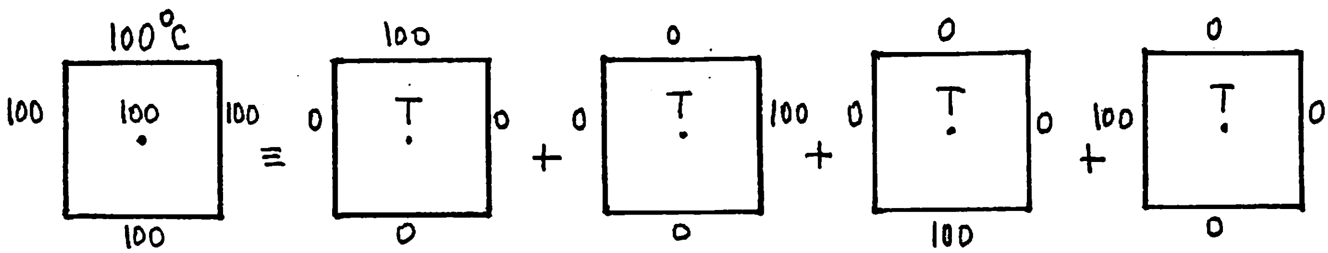
$$\vdots$$

$$= \frac{2}{\pi^2 \lambda_n^2} \left[1 - \frac{J_0^2(\lambda_n b)}{J_0^2(\lambda_n a)} \right]$$

CHAPTER 5

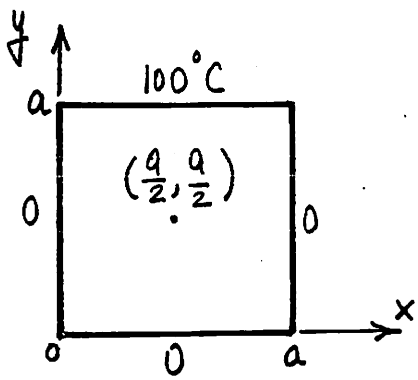
STEADY TWO- AND THREE-DIMENSIONAL HEAT CONDUCTION

PROB. 5.1: Solution by superposition:



$$4T = 100 \Rightarrow \underline{T = 25°C}$$

Solution by using Eq. (5.19):



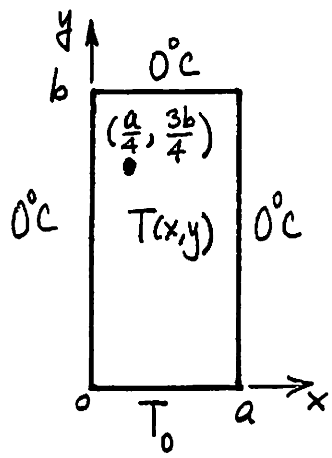
$$T\left(\frac{a}{2},\frac{a}{2}\right) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\,\frac{\sin\frac{n\pi}{a}\frac{a}{2}\sinh\frac{n\pi}{a}\frac{a}{2}}{\sinh\frac{n\pi}{a}a}$$

$$= \frac{200}{\pi}\left\{2\frac{\sinh\frac{\pi}{2}}{\sinh\pi} - \frac{2}{3}\frac{\sinh\frac{3\pi}{2}}{\sinh 3\pi} + \frac{2}{5}\frac{\sinh\frac{5\pi}{2}}{\sinh 5\pi} - \frac{2}{7}\frac{\sinh\frac{7\pi}{2}}{\sinh 7\pi}\cdots\right\}$$

$$T\left(\frac{a}{2},\frac{a}{2}\right) = \frac{400}{\pi}\left(0.199268 - 0.002994 + 0.000078 - 2\times10^{-6} + \cdots\right\}$$

$$\cong 25.00006°C = 25°C$$

PROB. 5.2:



$$\frac{T(x,y)}{T_0} = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{[1-(-1)^n]}{n} \frac{\sin\frac{n\pi}{a}x \sinh\frac{n\pi}{a}(b-y)}{\sinh\frac{n\pi}{a}b}$$

$$\frac{x}{a} = 0.25, \quad \frac{y}{b} = 0.75 \ \& \ b = 2a$$

$$\frac{T(x,y)}{T_0} = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{[1-(-1)^n]}{n} \frac{\sin\frac{n\pi}{4} \sinh\frac{n\pi}{2}}{\sinh 2n\pi}$$

$$\frac{T(x,y)}{T_0} = \frac{2}{\pi}\left\{ 2\,\frac{\sin\frac{\pi}{4}\sinh\frac{\pi}{2}}{\sinh 2\pi} + \frac{2}{3}\,\frac{\sin\frac{3\pi}{4}\sinh\frac{3\pi}{2}}{\sinh 6\pi} + \frac{2}{5}\,\frac{\sin\frac{5\pi}{4}\sinh\frac{5\pi}{2}}{\sinh 10\pi} + \frac{2}{7}\,\frac{\sin\frac{7\pi}{4}\sinh\frac{7\pi}{2}}{\sinh 14\pi} + \cdots \right\}$$

$$= \frac{4}{\pi}\left[6.0777\times 10^{-3} + 1.7086\times 10^{-7} - 8.274\times 10^{-12} - 4.769\times 10^{-16} + \ldots \right]$$

$$\therefore \quad \frac{T(x,y)}{T_0} \cong 7.74\times 10^{-3}$$

PROB. 5.3: Formulation of the problem:

$$\left\{ \begin{array}{c} \dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = 0 \\ T(0,y) = 0; \quad T(a,y) = f(y) \\ T(x,0) = 0; \quad T(x,b) = 0 \end{array} \right\}$$

Let $T(x,y) = X(x)\cdot Y(y)$

Then,

$$\left\{\begin{matrix} \dfrac{d^2Y}{dy^2}+\lambda^2 Y=0 \\ Y(0)=Y(b)=0 \end{matrix}\right\} \Rightarrow \begin{matrix} Y_n(y)=A_n\sin\lambda_n y \\ \lambda_n=\dfrac{n\pi}{b},\ n=1,2,3,\ldots \end{matrix} \Rightarrow \left\{\begin{matrix} \dfrac{d^2X_n}{dx^2}-\lambda_n^2 X_n=0 \\ X_n(0)=0 \end{matrix}\right\}$$

$$\Downarrow$$

Thus,

$$X_n(x)=B_n\sinh\lambda_n x$$

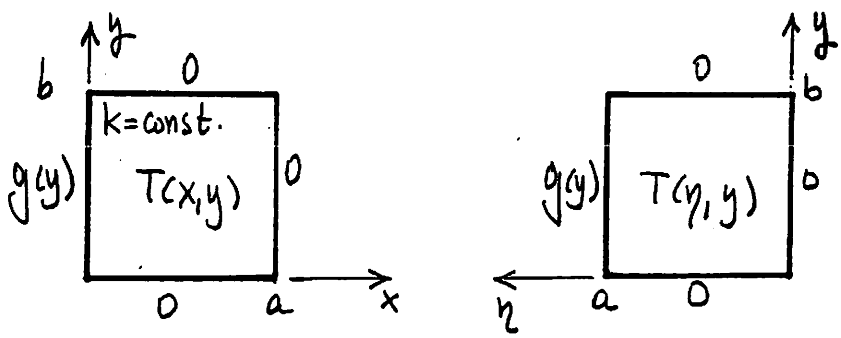
$$T(x,y)=\sum_{n=1}^{\infty} a_n \sinh\lambda_n x\,\sin\lambda_n y$$

$$\Downarrow$$

$$f(y)=\sum_{n=1}^{\infty}\underbrace{a_n\sin\lambda_n a}_{b_n}\,\sin\lambda_n y \Rightarrow b_n=\frac{2}{b}\int_0^b f(y)\sin\lambda_n y\,dy$$

$$\therefore\ \boxed{T(x,y)=\frac{2}{b}\sum_{n=1}^{\infty}\frac{\sinh\frac{n\pi}{b}x\cdot\sin\frac{n\pi}{b}y}{\sinh\frac{n\pi}{b}a}\cdot\int_0^b f(y')\sin\frac{n\pi}{b}y'\,dy'}$$

PROB. 5.4:



$$\boxed{\eta=a-x}$$

From Prob. 5.3:

$$T(\eta,y)=\frac{2}{b}\sum_{n=1}^{\infty}\frac{\sinh\frac{n\pi}{b}\eta\cdot\sin\frac{n\pi}{b}y}{\sinh\frac{n\pi}{b}a}\cdot\int_0^b g(y')\sin\frac{n\pi}{b}y'\,dy'$$

$$\therefore\ \boxed{T(x,y)=\frac{2}{b}\sum_{n=1}^{\infty}\frac{\sinh\frac{n\pi}{b}(a-x)\sin\frac{n\pi}{b}y}{\sinh\frac{n\pi}{b}a}\cdot\int_0^b g(y')\sin\frac{n\pi}{b}y'\,dy'}$$

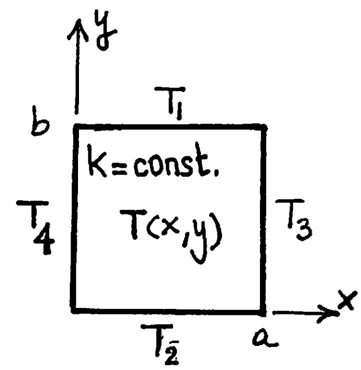
PROB. 5.5:

$$T(x,y) = \frac{2}{\pi}\left\{T_1\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\,\frac{\sin\frac{n\pi}{a}x\,\sinh\frac{n\pi}{a}y}{\sinh\frac{n\pi}{a}b}\right.$$

$$+\,T_2\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\,\frac{\sin\frac{n\pi}{a}x\,\sinh\frac{n\pi}{a}(b-y)}{\sinh\frac{n\pi}{a}b}$$

$$+\,T_3\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\,\frac{\sinh\frac{n\pi}{b}x\,\sin\frac{n\pi}{b}y}{\sinh\frac{n\pi}{b}a}$$

$$\left.+\,T_4\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\,\frac{\sinh\frac{n\pi}{b}(a-x)\,\sin\frac{n\pi}{b}y}{\sinh\frac{n\pi}{b}a}\right\}$$



Assume k = const.

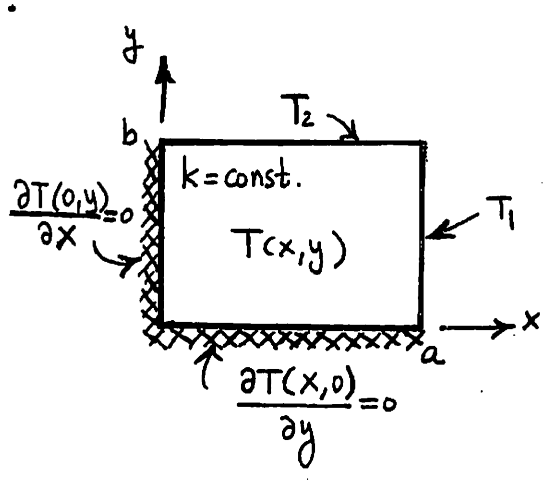
$$T(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\left\{\frac{\sin\frac{n\pi}{a}x}{\sinh\frac{n\pi}{a}b}\left[T_1\sinh\frac{n\pi}{a}y + T_2\sinh\frac{n\pi}{a}(b-y)\right]\right.$$

$$\left.+\,\frac{\sin\frac{n\pi}{b}y}{\sinh\frac{n\pi}{b}a}\left[T_3\sinh\frac{n\pi}{b}x + T_4\sinh\frac{n\pi}{b}(a-x)\right]\right\}$$

PROB. 5.6:

Let $\theta(x,y) = T(x,y) - T_1$.

Then the formulation of the problem is given by



$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

$$\frac{\partial\theta(0,y)}{\partial x} = 0\,; \quad \theta(a,y) = 0$$

$$\frac{\partial\theta(x,0)}{\partial y} = 0\,; \quad \theta(x,b) = T_2 - T_1 \equiv \theta_0$$

Let $\theta(x,y) = X(x)\cdot Y(y)$

Thus,

$$\left\{\begin{array}{c}\dfrac{d^2X}{dx^2}+\lambda^2 X=0\\ \dfrac{dX(0)}{dx}=0\\ X(a)=0\end{array}\right\} \Rightarrow \begin{array}{c} X_n(x)=A_n\cos\lambda_n x\\ \lambda_n=\dfrac{2n+1}{a}\dfrac{\pi}{2},\ n=0,1,2,\ldots\end{array} \Rightarrow \left\{\begin{array}{c}\dfrac{d^2Y_n}{dy^2}-\lambda_n^2 Y_n=0\\ \dfrac{dY_n(0)}{dy}=0\end{array}\right\}$$

$$\Downarrow$$

$$Y_n(y)=B_n\cosh\lambda_n y$$

Then,

$$\theta(x,y)=\sum_{n=0}^{\infty} a_n\cos\lambda_n x\cosh\lambda_n y$$

$$\Downarrow$$

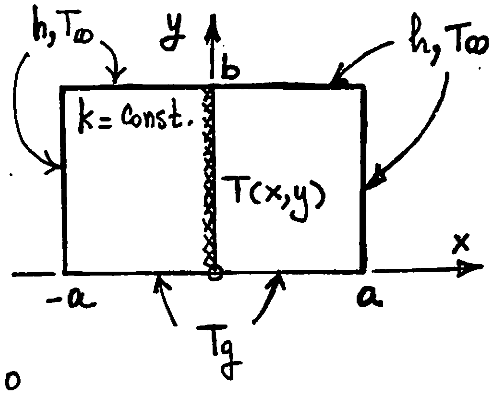
$$\theta_0=\sum_{n=0}^{\infty} a_n\cosh\lambda_n b\cos\lambda_n x$$

$$\Downarrow$$

$$a_n=\frac{\theta_0}{\cosh\lambda_n b}\,\frac{\int_0^a\cos\lambda_n x\,dx}{\int_0^a\cos^2\lambda_n x\,dx}=\cdots=\frac{(-1)^n 4\theta_0}{(2n+1)\pi\cosh\lambda_n b}$$

$$\therefore\ \boxed{\frac{\theta(x,y)}{\theta_0}=\frac{T(x,y)-T_1}{T_2-T_1}=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\cos\frac{2n+1}{a}\frac{\pi}{2}x\cdot\frac{\cosh\frac{2n+1}{a}\frac{\pi}{2}y}{\cosh\frac{2n+1}{a}\frac{\pi}{2}b}}$$

PROB. 5.7: a) Formulation of the problem in terms of $\theta(x,y)=T(x,y)-T_\infty$:



$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0}=0,\quad \left[k\frac{\partial\theta}{\partial x}+h\,\theta(x,y)\right]_{x=a}=0$$

$$\theta(x,0)=T_g-T_\infty,\quad \left[k\frac{\partial\theta}{\partial y}+h\,\theta(x,y)\right]_{y=b}=0$$

Let $\theta(x,y)=X(x)\cdot Y(y)$

Then,

$$\left\{\begin{array}{c} \dfrac{d^2X}{dx^2}+\lambda^2 X=0 \\ \dfrac{dX(0)}{dx}=0 \\ k\dfrac{dX(a)}{dx}+hX(a)=0 \end{array}\right\} \Rightarrow \begin{array}{c} \dfrac{d^2Y_n}{dy^2}-\lambda_n^2 Y_n=0 \\ k\dfrac{dY_n(b)}{dy}+hY(b)=0 \end{array}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$X_n(x)=A_n\cos\lambda_n x \qquad Y_n(y)=B_n\left\{\sinh\lambda_n(b-y)+\frac{\lambda_n a}{Bi}\cosh\lambda_n(b-y)\right\}$$

where λ_n are the roots of

$$(\lambda a)\tan(\lambda a)=Bi\ ,\quad Bi=\frac{ha}{k}$$

$$\Downarrow$$

$$\lambda_n\ ,\quad n=1,2,3,\ldots$$

Thus,

$$\theta(x,y)=\sum_{n=1}^{\infty} a_n\cos\lambda_n x\cdot\left\{\sinh\lambda_n(b-y)+\frac{\lambda_n a}{Bi}\cosh\lambda_n(b-y)\right\}$$

$$\Rightarrow\ T_g-T_\infty=\sum_{n=1}^{\infty} a_n\cos\lambda_n x\cdot\left\{\sinh\lambda_n b+\frac{\lambda_n a}{Bi}\cosh\lambda_n b\right\}$$

$$\therefore\ a_n\left\{\sinh\lambda_n b+\frac{\lambda_n a}{Bi}\cosh\lambda_n b\right\}=\frac{1}{N_n}\int_0^a (T_g-T_\infty)\cos\lambda_n x\,dx=\frac{(T_g-T_\infty)}{N_n}\,\frac{\sin\lambda_n a}{\lambda_n}$$

where $N_n=\dfrac{1}{2\lambda_n}\left[\lambda_n a+\sin\lambda_n a\cos\lambda_n a\right]$ (From Table 4.1)

$$a_n=2(T_g-T_\infty)\frac{\sin\lambda_n a}{\left[\lambda_n a+\sin\lambda_n a\cos\lambda_n a\right]\left[\sinh\lambda_n b+\frac{\lambda_n a}{Bi}\cosh\lambda_n b\right]}$$

$$\therefore \boxed{\frac{T(x,y)-T_\infty}{T_g-T_\infty} = 2\sum_{n=1}^{\infty} \frac{\sin\lambda_n a \cos\lambda_n x}{[\lambda_n a+\sin\lambda_n a\cos\lambda_n a]}\cdot\frac{\sinh\lambda_n(b-y)+\frac{\lambda_n a}{Bi}\cosh\lambda_n(b-y)}{[\sinh\lambda_n b+\frac{\lambda_n a}{Bi}\cosh\lambda_n b]}}$$

b) The rate of heat loss per unit depth can be calculated as

$$q'_L = 2\int_0^a \left\{-k\frac{\partial T}{\partial y}\Big|_{y=0} dx\right\}$$

$$= 4k(T_g-T_\infty)\sum_{n=1}^{\infty}\left\{\frac{\sin^2\lambda_n a}{[\lambda_n a+\sin\lambda_n a\cos\lambda_n a]}\,\frac{\cosh\lambda_n b+\frac{\lambda_n a}{Bi}\sinh\lambda_n b}{\sinh\lambda_n b+\frac{\lambda_n a}{Bi}\cosh\lambda_n b}\right\}$$

PROB. 5.8: Formulation of the problem in terms of $\theta(x,y)=T(x,y)-T_1$ (see Prob. 2.8):

$$2\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0$$

$$\theta(0,y)=0;\quad \theta(a,y)=0$$

$$\theta(x,0)=0;\quad \theta(x,b)=T_2-T_1$$

Let $\eta=\sqrt{2}\,y$. Then, the formulation becomes:

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial \eta^2}=0$$

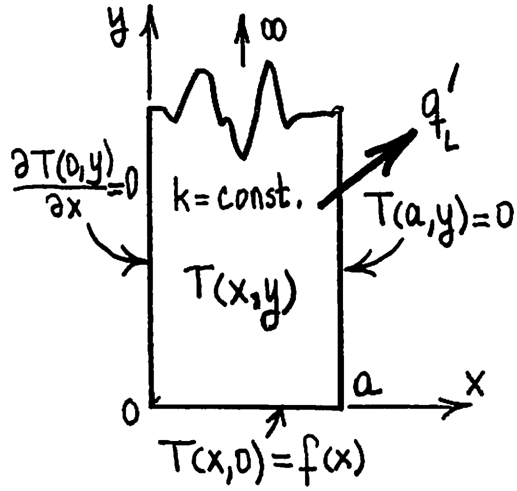
$$\theta(0,\eta)=0;\quad \theta(a,\eta)=0$$

$$\theta(x,0)=0;\quad \theta(x,\sqrt{2}b)=T_2-T_1$$

Now, the solution can be written as [see Eq. (5.19)]:

$$\boxed{\frac{T(x,y)-T_1}{T_2-T_1}=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\,\frac{\sin\frac{n\pi}{a}x\,\sinh\frac{n\pi}{a}\sqrt{2}y}{\sinh\frac{n\pi}{a}\sqrt{2}b}}$$

PROB. 5.9: Formulation:



$$\left\{ \begin{array}{l} \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \\ \frac{\partial T(0,y)}{\partial x} = 0 \; ; \; T(a,y) = 0 \\ T(x,0) = f(x) \; ; \; T(x,\infty) = 0 \end{array} \right\}$$

Let $T(x,y) = X(x) \cdot Y(y)$. Then,

$$\left\{ \begin{array}{c} \frac{d^2 X}{dx^2} + \lambda^2 X = 0 \\ \frac{dX(0)}{dx} = 0 \\ X(a) = 0 \end{array} \right\} \Rightarrow \begin{array}{l} X_n(x) = A_n \cos \lambda_n x \\ \lambda_n = \frac{2n+1}{a}\frac{\pi}{2}, \quad n = 0, 1, 2, \ldots \end{array}$$

and

$$\left\{ \begin{array}{c} \frac{d^2 Y_n}{dy^2} - \lambda_n^2 Y_n = 0 \\ Y_n(\infty) = 0 \end{array} \right\} \Rightarrow Y_n(y) = B_n e^{-\lambda_n y}$$

Thus,

$$T(x,y) = \sum_{n=0}^{\infty} \underbrace{A_n B_n}_{a_n} \cos \lambda_n x \, e^{-\lambda_n y}$$

$$\Downarrow$$

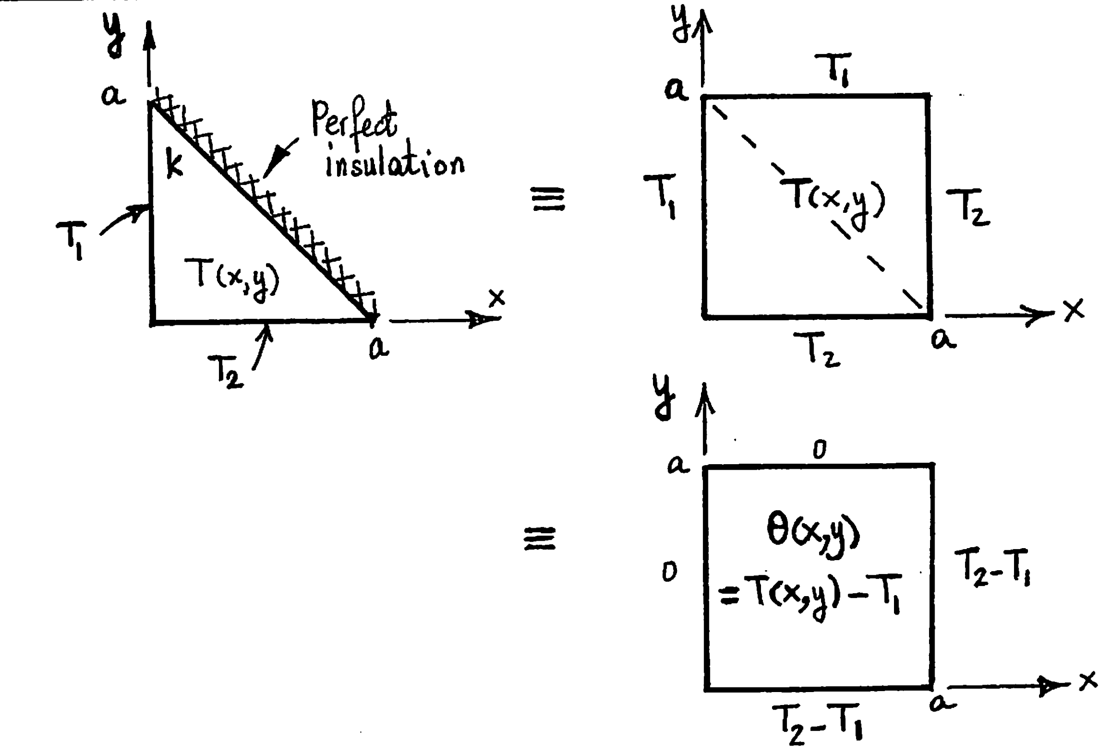
$$f(x) = \sum_{n=0}^{\infty} a_n \cos \lambda_n x \quad \rightarrow \quad a_n = \frac{2}{a} \int_0^a f(x) \cos \lambda_n x \, dx$$

$$\therefore \; T(x,y) = \frac{2}{a} \sum_{n=0}^{\infty} \left\{ \int_0^a f(x') \cos \lambda_n x' \, dx' \right\} \cdot \cos \lambda_n x \, e^{-\lambda_n y}$$

The rate of heat dissipated from the surface at $x=a$ per unit depth can be calculated as

$$q_L' = \int_0^a \left\{ -k \left(\frac{\partial T}{\partial y} \right)_{y=0} \right\} dx = \frac{2k}{a} \sum_{n=0}^{\infty} (-1)^n \int_0^a f(x) \cos \lambda_n x \, dx$$

PROB. 5.10:



Hence, the solution can be written as

$$T(x,y)-T_1=\frac{2}{\pi}(T_2-T_1)\left\{\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\,\frac{\sin\frac{n\pi}{a}x\,\sinh\frac{n\pi}{a}(a-y)}{\sinh n\pi}\right.$$

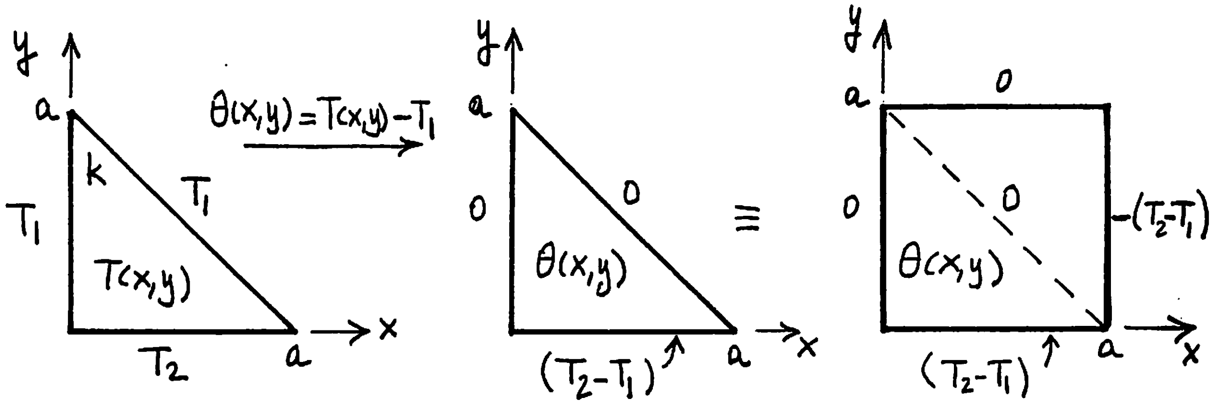
$$\left.+\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\,\frac{\sinh\frac{n\pi}{a}x\,\sin\frac{n\pi}{a}y}{\sinh n\pi}\right\}$$

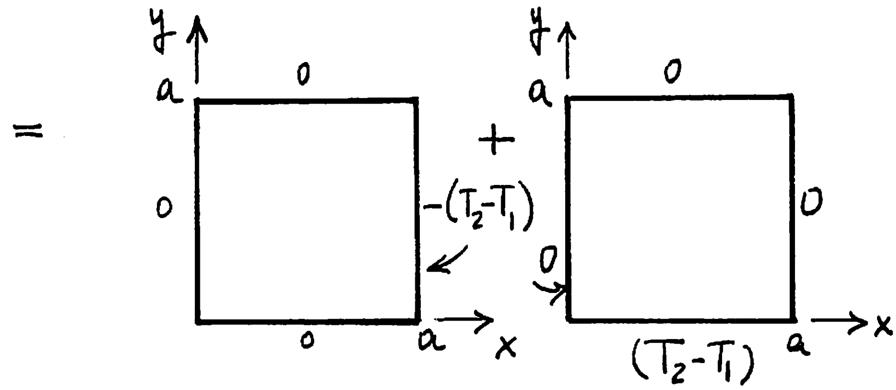
Or,

$$\frac{T(x,y)-T_1}{T_2-T_1}=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n\sinh n\pi}\left\{\sin\frac{n\pi}{a}x\,\sinh\frac{n\pi}{a}(a-y)\right.$$

$$\left.+\sinh\frac{n\pi}{a}x\,\sin\frac{n\pi}{a}y\right\}$$

PROB. 5.11:





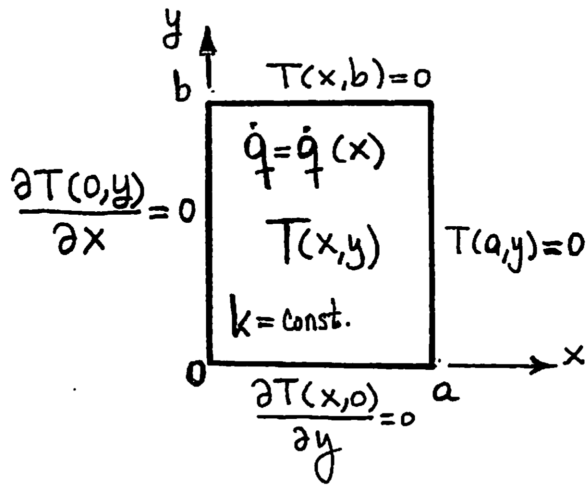
$$T(x,y)-T_1 = \frac{2}{\pi}\left\{-(T_2-T_1)\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\,\frac{\sinh\frac{n\pi}{a}x\,\sin\frac{n\pi}{a}y}{\sinh n\pi}\right.$$

$$\left.+(T_2-T_1)\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\,\frac{\sin\frac{n\pi}{a}x\,\sinh\frac{n\pi}{a}(a-y)}{\sinh n\pi}\right\}$$

$$\boxed{\frac{T(x,y)-T_1}{T_2-T_1} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n\sinh n\pi}\left[\sin\frac{n\pi}{a}x\,\sinh\frac{n\pi}{a}(a-y) - \sinh\frac{n\pi}{a}x\,\sin\frac{n\pi}{a}y\right]}$$

PROB. 5.12: Formulation of the problem:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\dot{q}(x)}{k} = 0$$
$$\frac{\partial T(0,y)}{\partial x} = 0, \quad T(a,y) = 0$$
$$\frac{\partial T(x,0)}{\partial y} = 0, \quad T(x,b) = 0$$



a) Let $T(x,y) = \psi(x,y) + \phi(x)$. Then, the problem separates into two as:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$
$$\frac{\partial \psi(0,y)}{\partial x} = 0, \quad \psi(a,y) = 0$$
$$\frac{\partial \psi(x,0)}{\partial y} = 0, \quad \psi(x,b) = -\phi(x)$$

$$\frac{d^2\phi}{dx^2} + \frac{\dot{q}(x)}{k} = 0$$
$$\frac{d\phi(0)}{dx} = 0, \quad \phi(a) = 0$$

$$\Downarrow$$

$$\phi(x) = \frac{1}{k}\int_x^a \int_0^{x'} \dot{q}(x'')\,dx''\,dx'$$

The solution to $\psi(x,y)$ problem (See Prob. 5.6):

$$\psi(x,y) = \sum_{n=0}^{\infty} A_n \cos\frac{(2n+1)\pi}{2a}x \cosh\frac{(2n+1)\pi}{2a}y$$

$$\Downarrow$$

$$-\phi(x) = \sum_{n=0}^{\infty} A_n \cos\frac{(2n+1)\pi}{2a}x \cosh\frac{(2n+1)\pi}{2a}b$$

$$\therefore \quad A_n \cosh\frac{(2n+1)\pi}{2a}b = -\frac{2}{a}\int_0^a \phi(x)\cos\frac{(2n+1)\pi}{2a}x\,dx$$

Thus, the solution is given by

$$T(x,y) = \frac{1}{k}\int_x^a \int_0^{x'} \dot{q}(x'')\,dx''\,dx'$$

$$-\frac{2}{ak}\sum_{n=0}^{\infty} \frac{\cos\frac{(2n+1)\pi}{2a}x \,\cosh\frac{(2n+1)\pi}{2a}y}{\cosh\frac{(2n+1)}{2a}b} \int_0^a \int_{x'}^a \int_0^{x''} \dot{q}(x''')\,dx'''\,dx''\,dx'$$

b) Let

$$T(x,y) = \sum_{n=0}^{\infty} A_n(y)\cos\frac{(2n+1)\pi}{2a}x, \quad \frac{dA(0)}{dy} = A(b) = 0$$

and

$$\frac{\dot{q}(x)}{k} = \sum_{n=0}^{\infty} B_n \cos\frac{(2n+1)\pi}{2a}x$$

$$\Downarrow$$

Known constants $\rightarrow$ $$B_n = \frac{2}{ak}\int_0^a \dot{q}(x)\cos\frac{(2n+1)\pi}{2a}x\,dx, \quad n=0,1,2,\dots$$

Note that the expansion given for $T(x,y)$ satisfies all the boundary conditions of the problem. Now, substitute both expansions into the differential equation to obtain

$$\sum_{n=0}^{\infty}\left[\frac{d^2A_n}{dy^2} - \lambda_n^2 A_n(y) + B_n\right]\cos\lambda_n x = 0 \quad \text{with } \lambda_n = \frac{(2n+1)\pi}{2a}, \; n=0,1,2,\dots$$

$$\Downarrow$$

$$\left\{\begin{array}{l} \frac{d^2A_n}{dy^2} - \lambda_n^2 A_n(y) = -B_n \\ \frac{dA_n(0)}{dy} = 0;\; A_n(b) = 0 \end{array}\right\} \rightarrow A_n(y) = C_n\sinh\lambda_n y + D_n\cosh\lambda_n y + \frac{B_n}{\lambda_n^2}$$

$$\frac{dA(0)}{dy} = 0 \Rightarrow C_n = 0$$

$$A_n(b) = 0 \Rightarrow D_n = -\frac{B_n}{\lambda_n^2\cosh\lambda_n b}$$

$$\therefore \quad A_n(y) = \frac{B_n}{\lambda_n^2}\left[1 - \frac{\cosh \lambda_n y}{\cosh \lambda_n b}\right]$$

Thus,

$$T(x,y) = \frac{2}{ak}\sum_{n=0}^{\infty} \frac{\cos \lambda_n x}{\lambda_n^2}\left[1 - \frac{\cosh \lambda_n y}{\cosh \lambda_n b}\right] \cdot \int_0^a \dot{q}(x') \cos \lambda_n x' dx'$$

c) Let

$$T(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} \cos \frac{(2m+1)\pi}{2a} x \cdot \cos \frac{(2n+1)\pi}{2b} y \leftarrow$$

satisfies all the boundary conditions

and

$$\frac{\dot{q}(x)}{k} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn} \cos \frac{(2m+1)\pi}{2a} x \cdot \cos \frac{(2n+1)\pi}{2b} y$$

$$\Downarrow$$

$$B_{mn} = \frac{4}{abk}\int_0^b \int_0^a \dot{q}(x) \cos \beta_m x \cos \lambda_n y \, dx\, dy \leftarrow$$

$$\beta_m = \frac{(2m+1)\pi}{2a}$$

$$\lambda_n = \frac{(2n+1)\pi}{2b}$$

$$= \frac{4}{abk}\,\frac{2b}{(2n+1)\pi}(-1)^n \int_0^a \dot{q}(x) \cos \beta_m x \, dx \quad \leftarrow \text{Known Constants}$$

Substituting both expansions into the differential equation, we obtain

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left\{\left[-\beta_m^2 - \lambda_n^2\right] A_{mn} + B_{mn}\right\} \cdot \cos \beta_m x \cdot \cos \lambda_n y = 0$$

$$\Downarrow$$

$$A_{mn} = \frac{B_{mn}}{\beta_m^2 + \lambda_n^2}$$

Thus,

$$T(x,y) = \frac{8}{ak\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)}\,\frac{\cos \beta_m x \cdot \cos \lambda_n y}{\beta_m^2 + \lambda_n^2} \cdot \int_0^a \dot{q}(x') \cos \beta_m x' dx'$$

PROB. 5.13:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\dot{q}(x,y)}{k} = 0$$

$$\frac{\partial T(0,y)}{\partial x} = 0; \quad T(a,y) = 0$$

$$\frac{\partial T(x,0)}{\partial y} = 0; \quad T(x,b) = 0$$

Let

$$T(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} \cos\frac{(2m+1)\pi}{2a}x \cos\frac{(2n+1)\pi}{2b}y$$

and

$$\frac{\dot{q}(x,y)}{k} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn} \cos\frac{(2m+1)\pi}{2a}x \cos\frac{(2n+1)\pi}{2b}y$$

$$\therefore \; B_{mn} = \frac{4}{abk}\int_0^a\int_0^b \dot{q}(x,y)\cos\lambda_m x \cos\beta_n y \, dx\, dy$$

$$\text{with} \quad \lambda_m = \frac{(2m+1)\pi}{2a} \quad \text{and} \quad \beta_n = \frac{(2n+1)\pi}{2b}$$

Substitute both expansions into the DE and obtain

$$\therefore A_{mn} = \frac{B_{mn}}{\lambda_m^2 + \beta_n^2}$$

Thus,

$$\boxed{T(x,y) = \frac{4}{abk}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\cos\lambda_m x \cos\beta_n y}{\lambda_m^2+\beta_n^2}\int_0^a\int_0^b \dot{q}(x',y')\cos\lambda_m x' \cos\beta_n y' \, dx'\, dy'}$$

PROB. 5.14: Obtain the solution from Eq. (5.108) by replacing $F(z)$ by T_0:

$$\frac{T(r,z)}{T_0} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[1-(-1)^n]}{n} \frac{I_0\left(\frac{n\pi}{L}r\right) \sin \frac{n\pi}{L} z}{I_0\left(\frac{n\pi}{L}r_0\right)}$$

PROB. 5.15: Make use of Eq. (5.98) and replace z by $(L-z)$ to obtain

$$T(r,z) = \frac{2}{r_0^2} \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{J_1^2(\lambda_n r_0)} \frac{\sinh \lambda_n z}{\sinh \lambda_n L} \int_0^{r_0} f(r') J_0(\lambda_n r') r' dr'$$

where λ_n's are the positive roots of $J_0(\lambda r_0) = 0$.

PROB. 5.16: a) Obtain the solution by superposition by the use of Eqs (5.98) and (5.108) and the result of Prob. 5.15:

$$T(r,z) = \frac{2}{L} \sum_{n=1}^{\infty} \frac{I_0\left(\frac{n\pi}{L}r\right)}{I_0\left(\frac{n\pi}{L}r_0\right)} \sin \frac{n\pi}{L} z \cdot \int_0^L f_1(z') \sin \frac{n\pi}{L} z' dz'$$

$$+ \frac{2}{r_0^2} \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{J_1^2(\lambda_n r_0) \sinh \lambda_n L} \left\{ \sinh \lambda_n (L-z) \int_0^{r_0} f_2(r') J_0(\lambda_n r') r' dr' + \sinh \lambda_n z \int_0^{r_0} f_3(r') J_0(\lambda_n r') r' dr' \right\}$$

where λ_n's are the positive roots of $J_0(\lambda r_0) = 0$

b) If $f_1(z) = T_0$ and $f_2(r) = f_3(r) = T_1$, then the solution can be written as

$$\frac{T(r,z) - T_1}{T_0 - T_1} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[1-(-1)^n]}{n} \frac{I_0\left(\frac{n\pi}{L}r\right) \sin \frac{n\pi}{L} z}{I_0\left(\frac{n\pi}{L}r_0\right)}$$

or

$$\frac{T(r,z)-T_0}{T_1-T_0}=\frac{2}{r_0}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\frac{J_0(\lambda_n r)}{J_1(\lambda_n r_0)}\;\frac{\sinh\lambda_n(L-z)+\sinh\lambda_n z}{\sinh\lambda_n L}$$

where Eq. (5.99) has been used.

PROB. 5.17: Formulation in terms of $\theta(r,z)=T(r,z)-T_\infty$:

$$\left\{\begin{array}{l}\dfrac{\partial^2\theta}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial\theta}{\partial r}+\dfrac{\partial^2\theta}{\partial z^2}=0\\[2mm] \theta(0,z)=\text{finite};\quad \dfrac{\partial\theta(r_0,z)}{\partial r}+\dfrac{h}{k}\theta(r_0,z)=0\\[2mm] \theta(r,0)=f(r)-T_\infty;\quad \dfrac{\partial\theta(r,L)}{\partial z}+\dfrac{h}{k}\theta(r,L)=0\end{array}\right\}$$

Let $\theta(r,z)=R(r)\cdot Z(z)$. Then,

$$\theta(r,z)=\sum_{n=1}^{\infty}a_n J_0(\lambda_n r)\left[\sinh\lambda_n(L-z)+\frac{\lambda_n k}{h}\cosh\lambda_n L\right]$$

where λ_n are the positive roots of $\dfrac{J_1(\lambda r_0)}{J_0(\lambda r_0)}=\dfrac{h\lambda_n}{k}$. The boundary condition at $z=0$ yields

$$f(r)-T_\infty=\sum_{n=1}^{\infty}a_n J_0(\lambda_n r)\left[\sinh\lambda_n L+\frac{\lambda_n k}{h}\cosh\lambda_n L\right]$$

$$\Rightarrow\quad a_n\left[\sinh\lambda_n L+\frac{\lambda_n k}{h}\cosh\lambda_n L\right]=\frac{1}{N_n}\int_0^{r_0}\left[f(r)-T_\infty\right]J_0(\lambda_n r)r\,dr$$

where $\quad N_n=\dfrac{r_0^2}{2}\left[1+\dfrac{h^2}{k^2\lambda_n^2}\right]J_0^2(\lambda_n r_0)=\dfrac{r_0^2}{2}\left[J_0^2(\lambda_n r_0)+J_1^2(\lambda_n r_0)\right]$

Thus,

$$\theta(r,z)=\frac{2}{r_0^2}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)\left[\sinh\lambda_n(L-z)+\frac{\lambda_n k}{h}\cosh\lambda_n L\right]}{\left[J_0^2(\lambda_n r_0)+J_1^2(\lambda_n r_0)\right]\left[\sinh\lambda_n L+\frac{\lambda_n k}{h}\cosh\lambda_n L\right]}\cdot\int_0^{r_0}\left[f(r')-T_\infty\right]J_0(\lambda_n r')r'\,dr'$$

PROB. 5.18: Formulation of the problem:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0$$

$$T(0,z) = \text{finite}; \quad T(r_0,z) = T_0 + \sin\frac{\pi z}{L}$$

$$T(r,0) = T_0; \quad T(r,L) = T_0$$

$$\text{Let} \quad \theta(r,z) = T(r,z) - T_0 = R(r)\cdot Z(z)$$

Then,

$$\left.\begin{aligned} &\frac{d^2 Z}{dz^2} + \lambda^2 Z = 0 \\ &Z(0) = 0;\ Z(L) = 0 \end{aligned}\right\} \Rightarrow Z_n(z) = A_n \sin\frac{n\pi}{L} z, \ n = 1,2,3,\ldots$$

and

$$\left.\begin{aligned} &\frac{d^2 R_n}{dr^2} + \frac{1}{r}\frac{dR_n}{dr} - \lambda_n^2 R_n = 0 \\ &R_n(0) = \text{finite} \end{aligned}\right\} \Rightarrow R_n(r) = B_n I_0(\lambda_n r)$$

Thus,

$$\theta(r,z) = \sum_{n=1}^{\infty} C_n I_0(\lambda_n r) \sin\frac{n\pi}{L} z$$

$$\Downarrow$$

$$\sin\frac{\pi z}{L} = \sum_{n=1}^{\infty} C_n I_0(\lambda_n r_0) \sin\frac{n\pi}{L} z$$

$$\therefore \quad C_1 = \frac{1}{I_0(\lambda_n r_0)}, \quad \text{and} \quad C_n = 0, \ n = 2, 3, \ldots$$

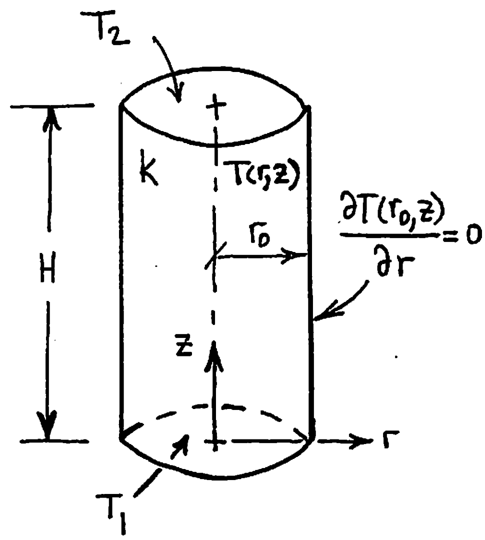
Then, the solution becomes

$$\boxed{\theta(r,z) = T(r,z) - T_0 = \frac{I_0\left(\frac{\pi r}{L}\right)}{I_0\left(\frac{\pi r_0}{L}\right)} \sin\frac{\pi z}{L}}$$

PROB. 5.19: The temperature distribution is, in fact, one-dimensional, that is,

$$T = T(z)$$

because, if $\frac{\partial T}{\partial r}$ is zero at $r = r_0$ then under steady-state conditions it will be zero for all values of r in the rod. Therefore,



$$\boxed{T(z) = T_1 + \frac{T_2 - T_1}{H} z}$$

Alternative Method of Solution with the assumption of $T(r,z)$:
Formulation of the problem in terms of $\theta(r,z) = T(r,z) - T_1$:

$$\boxed{\begin{gathered} \frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} = 0 \\ \theta(0,z) = \text{finite}\ ; \quad \frac{\partial\theta(r_0,z)}{\partial r} = 0 \\ \theta(r,0) = 0\ ; \quad \theta(r,H) = T_2 - T_1 \end{gathered}}$$

Let $\quad \theta(r,z) = R(r) \cdot Z(z)$

$$\left\{ \begin{gathered} r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + \lambda^2 r^2 R = 0 \\ R(0) \neq \infty \\ \frac{dR(r_0)}{dr} = 0 \end{gathered} \right\}$$

$R_n(r) = A_n J_0(\lambda_n r)$, $n = 0, 1, 2, \ldots$ where λ_n's are the positive roots of

$$\frac{d}{dr}\left[J_0(\lambda r)\right] = 0 \quad \text{OR} \quad J_1(\lambda r) = 0$$

Note that $\quad \lambda_0 = 0 \quad$ is a characteristic value.

Corresponding to each λ_n,

$$\left\{ \begin{array}{l} \dfrac{d^2 Z_n}{dz^2} - \lambda_n^2 Z_n = 0 \\ \\ Z_n(0) = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} Z_0(z) = B_0 z \quad \text{for } \lambda_0 = 0 \\ Z_n(z) = B_n \sinh \lambda_n z, \; n = 1, 2, \dots \end{array} \right.$$

Hence, the product solution can be constructed as

$$\theta(r,z) = \underbrace{A_0 B_0}_{C_0} z + \sum_{n=1}^{\infty} \underbrace{A_n B_n}_{C_n} J_0(\lambda_n r) \sinh \lambda_n z$$

Imposing the B.C. at $z = H$:

$$T_2 - T_1 = C_0 H + \sum_{n=1}^{\infty} \underbrace{C_n \sinh \lambda_n H}_{b_n} J_0(\lambda_n r)$$

Fourier-Bessel expansion of $T_2 - T_1$.

$$\therefore \; b_0 = \frac{2}{r_0^2} \int_0^{r_0} (T_2 - T_1) r \, dr = T_2 - T_1 \longrightarrow C_0 = \frac{T_2 - T_1}{H}$$

$$b_n = \frac{1}{N_n} \int_0^{r_0} (T_2 - T_1) J_0(\lambda_n r) r \, dr = \underbrace{\frac{T_2 - T_1}{N_n} \int_0^{r_0} J_0(\lambda_n r) r \, dr = 0}$$

$$\Rightarrow C_n = 0, \; n = 1, 2, 3, \dots$$

Because $\{1, J_0(\lambda_n r); n = 1, 2, 3, \dots\}$ is an orthogonal set (complete).

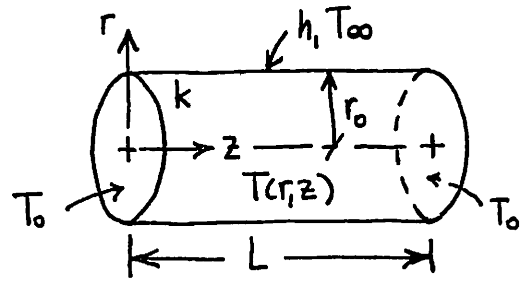
Hence,

$$\theta(r,z) = \frac{T_2 - T_1}{H} z$$

or

$$\boxed{T(r,z) - T_1 = \frac{T_2 - T_1}{H} z}$$

PROB. 5.20: a) Formulation of the problem in terms of $\theta(r,z) = T(r,z) - T_0$:



$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} = 0$$

$$\theta(0,z) = \text{finite}; \quad -k\left.\frac{\partial\theta}{\partial r}\right|_{r=r_0} = h\left[\theta(r_0,z) - \theta_\infty\right], \quad \theta_\infty = T_\infty - T_0$$

$$\theta(r,0) = 0; \quad \theta(r,L) = 0$$

Let $\theta(r,z) = R(r)\cdot Z(z)$ to obtain

$$\left\{\begin{matrix} \dfrac{d^2Z}{dz^2} + \lambda^2 Z = 0 \\ Z(0) = 0 \\ Z(L) = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} Z_n(z) = A_n \sin\lambda_n z \\ \lambda_n = \dfrac{n\pi}{L}, \; n = 1,2,3,\ldots \end{matrix} \quad \text{and} \quad \left\{\begin{matrix} r^2\dfrac{d^2R_n}{dr^2} + r\dfrac{dR_n}{dr} - \lambda_n^2 r^2 R_n = 0 \\ R_n(0) = \text{finite} \end{matrix}\right\}$$

$$\Downarrow$$

$$R_n(r) = B_n I_0(\lambda_n r)$$

$$\therefore \quad \theta(r,z) = \sum_{n=1}^{\infty} a_n I_0(\lambda_n r)\sin\lambda_n z$$

B.C. at $r = r_0$ $\Rightarrow$ $$h\theta_\infty = \sum_{n=1}^{\infty}\left[h I_0(\lambda_n r_0) + k\lambda_n I_1(\lambda_n r_0)\right] a_n \sin\lambda_n z$$

$$\Downarrow$$

$$a_n = \frac{2\theta_\infty}{n\pi}\,\frac{\left[1-(-1)^n\right]}{I_0(\lambda_n r_0) + \frac{n\pi}{Bi} I_1(\lambda_n r_0)}, \quad Bi = \frac{hL}{k}$$

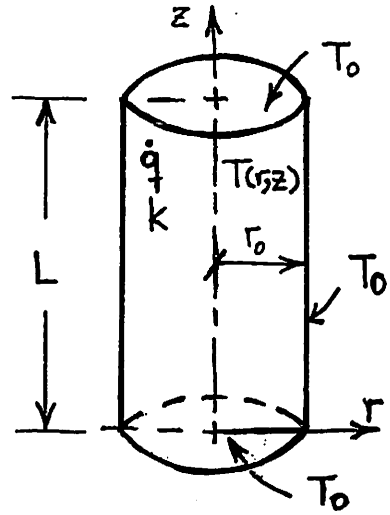
$$\therefore \quad \frac{\theta(r,z)}{\theta_\infty} = \frac{T(r,z) - T_0}{T_\infty - T_0} = 2\frac{Bi}{\pi}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n}\,\frac{I_0(\lambda_n r)\sin\lambda_n z}{Bi\, I_0(\lambda_n r_0) + n\pi I_1(\lambda_n r_0)}$$

b)

$$q_{loss} = 2\times\int_0^{r_0}\left\{-k\,2\pi r\,dr\left.\frac{\partial T}{\partial z}\right|_{z=0}\right\}dr = 8\pi\frac{k}{L}Bi\,\theta_\infty\sum_{n=1}^{\infty}\frac{\left[(-1)^n - 1\right]\int_0^{r_0} I_0(\lambda_n r)\,r\,dr}{Bi\, I_0(\lambda_n r_0) + n\pi I_1(\lambda_n r_0)}$$

$$= 8k\,Bi\,\theta_\infty r_0\sum_{n=1}^{\infty}\frac{\left[(-1)^n - 1\right]}{n}\,\frac{I_1(\lambda_n r_0)}{Bi\, I_0(\lambda_n r_0) + n\pi I_1(\lambda_n r_0)}$$

PROB. 5.21: Formulation in terms of $\theta(r,z) = T(r,z) - T_o$ is:



$$\boxed{\begin{gathered}\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} + \frac{\dot{q}}{k} = 0 \\ \theta(r_o,z) = \theta(r,0) = \theta(r,L) = 0\end{gathered}}$$

Let $\theta(r,z) = \psi(r,z) + \phi(z)$

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = 0$$

$$\psi(r_o,z) = -\phi(z)$$

$$\psi(r,0) = \psi(r,L) = 0$$

$$\frac{d^2\phi}{dz^2} + \frac{\dot{q}}{k} = 0$$

$$\phi(0) = 0$$

$$\phi(L) = 0$$

From Eq. (5.108)

$$\boxed{\phi(z) = \frac{\dot{q}L^2}{2k}\left[\frac{z}{L} - \left(\frac{z}{L}\right)^2\right]}$$

$$\psi(r,z) = \frac{2}{L}\sum_{n=1}^{\infty}\frac{I_o\left(\frac{n\pi r}{L}\right)}{I_o\left(\frac{n\pi r_o}{L}\right)}\sin\frac{n\pi}{L}z \cdot \int_0^L\left\{-\frac{\dot{q}L^2}{2k}\left[\frac{z'}{L} - \left(\frac{z'}{L}\right)^2\right]\right\}\sin\frac{n\pi z'}{L}dz'$$

$$= -\frac{2\dot{q}L^2}{\pi^3 k}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^3}\frac{I_o\left(\frac{n\pi r}{L}\right)}{I_o\left(\frac{n\pi r_o}{L}\right)}\sin\frac{n\pi}{L}z$$

Thus,

$$\boxed{\begin{aligned}T(r,z) - T_o &= \frac{\dot{q}L^2}{2k}\left[\frac{z}{L} - \left(\frac{z}{L}\right)^2\right] \\ &\quad - \frac{2\dot{q}L^2}{\pi^3 k}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^3}\frac{I_o\left(\frac{n\pi r}{L}\right)}{I_o\left(\frac{n\pi r_o}{L}\right)}\sin\frac{n\pi}{L}z\end{aligned}}$$

PROB. 5.22: Formulation in terms of $\theta(r,z) = T(r,z) - T_0$:

$$\left\{ \begin{array}{l} \dfrac{\partial^2\theta}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\theta}{\partial r} + \dfrac{\partial^2\theta}{\partial z^2} + \dfrac{\dot{q}(r,z)}{k} = 0 \\ \theta(r_0,z) = \theta(r,0) = \theta(r,L) = 0 \end{array} \right\}$$

Let

$$\theta(r,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_0(\lambda_n r) \sin\frac{m\pi}{L}z$$

where λ_n are the positive roots of $J_0(\lambda r_0) = 0$.

Also,

$$\frac{\dot{q}(r,z)}{k} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} J_0(\lambda_n r) \sin\frac{m\pi}{L}z$$

where

$$B_{mn} = \frac{2}{kLN_n}\int_0^{r_0}\int_0^{L} \dot{q}(r,z)\sin\frac{m\pi}{L}z\, J_0(\lambda_n r)\, r\,dr\,dz$$

with

$$N_n = \frac{r_0^2}{2} J_1^2(\lambda_n r_0)$$

Note that the expansion for $\theta(r,z)$ satisfies all the boundary conditions. Now, substitute both expansions into the DE and obtain

$$A_{mn} = \frac{B_{mn}}{\lambda_n^2 + \left(\frac{m\pi}{L}\right)^2}$$

Thus,

$$T(r,z) - T_0 = \frac{4}{Lr_0^2 k}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)\sin\frac{m\pi}{L}z}{J_1^2(\lambda_n r_0)\left[\lambda_n^2 + \left(\frac{m\pi}{L}\right)^2\right]}\int_0^{L}\int_0^{r_0} \dot{q}(r',z')\sin\frac{m\pi}{L}z'\, J_0(\lambda_n r')\, r'dr'dz'$$

PROB. 5.23: From Prob. 2.11, the formulation of the problem:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \,, \quad -k \left.\frac{\partial T}{\partial x}\right|_{x=a} = q_1''$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0 \,, \quad k \left.\frac{\partial T}{\partial y}\right|_{y=b} = q_2'' = \frac{b}{a} q_1''$$

Let $\quad T(x,y) = \psi(x,y) + \phi(x) + \Omega(y)$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\left.\frac{\partial \psi}{\partial x}\right|_{x=0} = 0 \,, \quad -k \left.\frac{\partial \psi}{\partial x}\right|_{x=a} = 0$$

$$\left.\frac{\partial \psi}{\partial y}\right|_{y=0} = 0 \,, \quad k \left.\frac{\partial \psi}{\partial y}\right|_{y=b} = 0$$

$\downarrow$ By inspection

$$\psi(x,y) = \text{Const.} = A$$

$$\frac{d^2 \phi}{dx^2} + \frac{d^2 \Omega}{dy^2} = 0$$

$$\downarrow$$

$$\frac{d^2 \phi}{dx^2} = -\frac{d^2 \Omega}{dy^2} = \text{Const.} = C$$

$$\frac{d\phi(0)}{dx} = 0 \,, \quad -k \frac{d\phi(a)}{dx} = q_1''$$

$$\frac{d\Omega(0)}{dy} = 0 \,, \quad k \frac{d\Omega(b)}{dy} = \frac{b}{a} q_1''$$

$$\Downarrow$$

$$\phi(x) = B - \frac{q_1''}{2ak} x^2$$

$$\Omega(y) = D + \frac{q_1''}{2ak} y^2$$

$$\therefore \quad \boxed{T(x,y) = a + \frac{q_1''}{2ak} (y^2 - x^2)}$$

$\uparrow$ Any arbitrary constant $= A + B + D$.

PROB. 5.24: a) Formulation:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\dot{q}}{k}=0$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0}=0; \quad -k\left.\frac{\partial T}{\partial x}\right|_{x=a}=q_1''$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=0}=0; \quad k\left.\frac{\partial T}{\partial y}\right|_{y=b}=q_2''$$

b) Under steady-state conditions,

$$q_2''\times a + \dot{q}\times(a\times b) = q_1''\times b \Rightarrow \boxed{q_2' = b\times\left(\frac{1}{a}q_1''-\dot{q}\right)}$$

c) Note that if $T(x,y)$ is a solution, then $T(x,y)+C$ is also a solution, where C is any constant. This can be proved by direct substitution (see Prob. 2.11).

d) Let $T(x,y)=\psi(x,y)+\phi(x)+\Omega(y)$. This leads to

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$$

$$\left.\frac{\partial\psi}{\partial x}\right|_{x=0}=0; \quad -k\left.\frac{\partial\psi}{\partial x}\right|_{x=a}=0$$

$$\left.\frac{\partial\psi}{\partial y}\right|_{y=0}=0; \quad k\left.\frac{\partial\psi}{\partial y}\right|_{y=b}=0$$

$\Downarrow$ By inspection

$$\psi(x,y)=A=\text{Constant}$$

and

$$\frac{d^2\phi}{dx^2}=-\frac{d^2\Omega}{dy^2}-\frac{\dot{q}}{k}=\text{const.}=C$$

$$\frac{d\phi(0)}{dx}=0; \quad -k\frac{d\phi(a)}{dx}=q_1''$$

$$\frac{d\Omega(0)}{dy}=0; \quad k\frac{d\Omega(b)}{dy}=q_2''$$

$\Downarrow$

$$\phi(x)=B-\frac{q_1''}{2ak}x^2$$

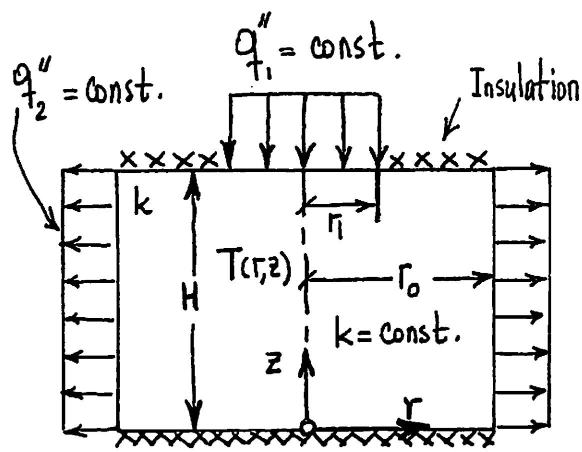
$$\Omega(y)=D+\frac{q_2''}{2bk}y^2$$

Thus,

$$\boxed{T(x,y)=a+\frac{1}{2k}\left(\frac{q_2''}{b}y^2-\frac{q_1''}{a}x^2\right)}$$

a: $A+B+D \leftarrow$ Arbitrary constant

PROB. 5.25: Formulation of the problem:



$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0$$

$$T(0,z) = \text{finite} \; ; \; -k\left.\frac{\partial T}{\partial r}\right|_{r=r_o} = q_2''$$

$$\left.\frac{\partial T}{\partial z}\right|_{z=0} = 0$$

$$-k\left.\frac{\partial T}{\partial z}\right|_{z=H} = \begin{cases} -q_1'', & 0<r<r_1 \\ 0, & r_1<r<r_o \end{cases}$$

Note that $q_1'' \pi r_1^2 = q_2'' 2\pi r_o H \Rightarrow q_1'' = \frac{2 r_o H}{r_1^2} q_2''$ — This condition has to be satisfied to have the steady-state condition.

a) If T is a solution of the problem, then T+C will also be a solution, where C is any constant. This can be proved by direct substitution into the formulation.

b) Let $T(r,z) = \psi(r,z) + \phi(r) + \Omega(z)$

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = 0$$

$$\psi(0,z) = \text{finite} \; ; \; \left.\frac{\partial \psi}{\partial r}\right|_{r=r_o} = 0$$

$$\left.\frac{\partial \psi}{\partial z}\right|_{z=0}$$

$$-k\left.\frac{\partial T}{\partial z}\right|_{z=H} = \begin{cases} -q_2'' + q_3'', & 0<r<r_1 \\ q_3'', & r_1<r<r_o \end{cases}$$

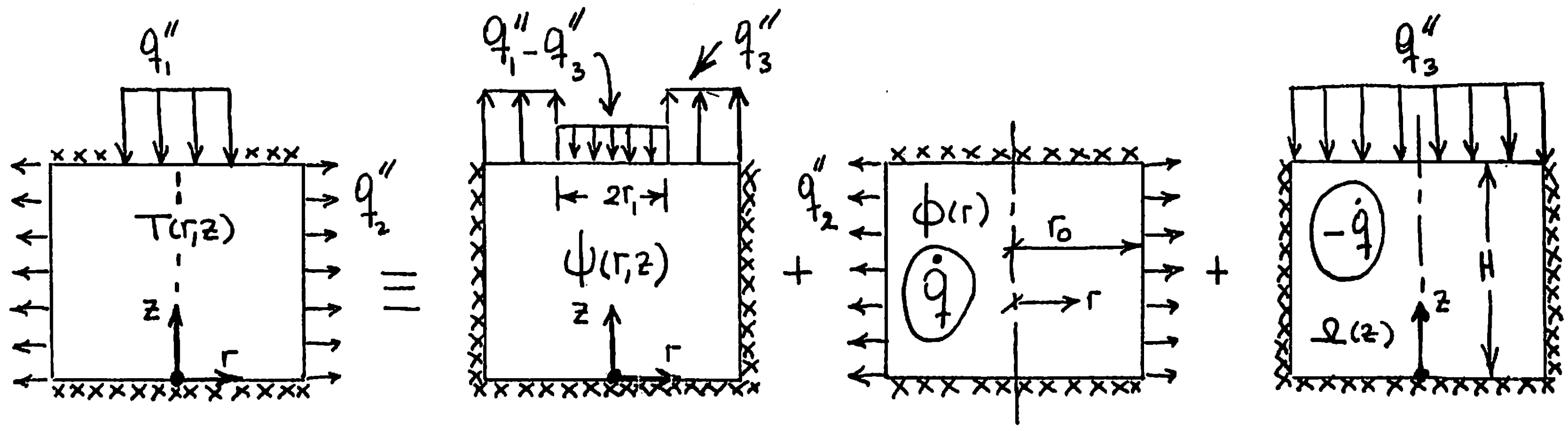
$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} = -\frac{d^2\Omega}{dz^2} = \text{const.} = -\frac{\dot{q}}{k}$$

$$\phi(0) = \text{finite}; \; -k\frac{d\phi(r_o)}{dr} = q_2''$$

$$\frac{d\Omega(0)}{dz} = 0$$

$$-k\frac{d\Omega(H)}{dz} = -q_3''$$

see next page



$$\dot{q}\,\pi r_0^2 H = q''_2\, 2\pi r_0 H$$

$$\boxed{\dot{q} = \frac{2}{r_0} q''_2}$$

$$q''_3\, \pi r_0^2 = \dot{q}\,\pi r_0^2 H$$

$$\boxed{q''_3 = H\dot{q} = \frac{2H}{r_0} q''_2}$$

$$q''_3 = \frac{2H}{r_0}\frac{r_1^2}{2r_0 H} q''_1$$

$$\boxed{q''_3 = \left(\frac{r_1}{r_0}\right)^2 q''_1}$$

check

$$\pi r_1^2 (q''_1 - q''_3) = q''_3\, \pi\, (r_0^2 - r_1^2)$$

$$r_1^2\left[\left(\frac{r_0}{r_1}\right)^2 - 1\right] q''_3 = q''_3\,(r_0^2 - r_1^2)$$

$$\therefore\ r_0^2 - r_1^2 = r_0^2 - r_1^2 \ \checkmark$$

<u>Solution to $\Omega(z)$ problem:</u>

$$\Omega(z) = \frac{q_1''}{2kH}\left(\frac{r_1}{r_0}\right)^2 z^2 + A$$

Any arbitrary constant

<u>Solution to $\phi(r)$ problem:</u>

$$\phi(r) = -\frac{q_1''}{4kH}\left(\frac{r_1}{r_0}\right)^2 r^2 + B$$

Any arbitrary constant

<u>Solution to $\psi(r,z)$ problem:</u>

$$\text{Let } \psi(r,z) = R(r)\cdot Z(z)$$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + \lambda^2 r^2 R = 0$$

$$R(0) = \text{finite}$$

$$\frac{dR(0)}{dr} = 0$$

$$\Downarrow$$

$$R_n(r) = \begin{cases} C_0 \\ C_n J_0(\lambda_n r), \quad n = 1, 2, 3, \ldots \end{cases}$$

$$\frac{d^2 Z_n}{dz^2} - \lambda_n^2 Z_n = 0$$

$$\frac{dZ_n(0)}{dz} = 0$$

$$\Downarrow$$

$$Z_n(z) = \begin{cases} D_0 \\ D_n \cosh\lambda_n z \end{cases}$$

where λ_n are the roots of $J_1(\lambda_n r_0) = 0$

$$\therefore \psi(r,z) = \overset{C_0 D_0}{b_0} + \sum_{n=1}^{\infty} \overset{C_n D_n}{a_n} J_0(\lambda_n r)\cosh\lambda_n z$$

and

$$T(r,z) = \overset{b_0+A+B}{a_0} + \frac{q_1''}{2kH}\left(\frac{r_1}{r_0}\right)^2\left(z^2 - \frac{1}{2}r^2\right) + \sum_{n=1}^{\infty} a_n J_0(\lambda_n r)\cosh\lambda_n z$$

$$k\left.\frac{\partial T}{\partial z}\right|_{z=H}=\underbrace{q_1''\left(\frac{r_1}{r_0}\right)^2}_{C_0}+\sum_{n=1}^{\infty}\underbrace{(k a_n\lambda_n\sinh\lambda_n H)}_{C_n}J_0(\lambda_n r)$$

$$=C_0+\sum_{n=1}^{\infty}C_n J_0(\lambda_n r)\ \leftarrow \text{Valid expansion in } (0,r_0)$$

$$C_0=\frac{2}{r_0^2}\int_0^{r_0}k\left.\frac{\partial T}{\partial z}\right|_{z=H}\cdot r\cdot dr=\frac{2}{r_0^2}\int_0^{r_1}q_1''\,r\,dr=\left(\frac{r_1}{r_0}\right)^2 q_1''$$

$$C_n=\frac{1}{N_n}\int_0^{r_0}k\left.\frac{\partial T}{\partial z}\right|_{z=H}\cdot J_0(\lambda_n r)\cdot r\cdot dr$$

$$\text{where } N_n=\frac{r_0^2}{2}J_0^2(\lambda_n r_0)\ \leftarrow \text{From Table 4.2}$$

$$=\frac{q_1''}{N_n}\int_0^{r_1}J_0(\lambda_n r)\,r\,dr=\frac{q_1''}{N_n}\frac{1}{\lambda_n}\int_0^{r_1}\frac{d}{dr}\left[r\,J_1(\lambda_n r)\right]dr$$

$$=\frac{2}{r_0^2}\,q_1''\,\frac{r_1 J_1(\lambda_n r_1)}{\lambda_n J_0^2(\lambda_n r_0)}$$

$$\therefore\ T(r,z)=a_0+\frac{q_1''}{2kH}\left(\frac{r_1}{r_0}\right)^2\left\{z^2-\frac{1}{2}r^2+4\frac{H}{r_1}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)\,J_1(\lambda_n r_1)\cosh\lambda_n z}{\lambda_n^2 J_0^2(\lambda_n r_0)\sinh\lambda_n H}\right\}$$

Any arbitrary constant.

PROB. 5.26: a) Formulation in terms of $\theta(r,z) = T(r,z) - T_1$:

$$\left\{\begin{array}{l} \dfrac{\partial^2\theta}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\theta}{\partial r} + \dfrac{\partial^2\theta}{\partial z^2} = 0 \\ \theta(0,z) = \text{finite}; \quad \theta(r_0,z) = 0 \\ \theta(r,0) = f(r) - T_1; \quad \theta(r,\infty) = 0 \end{array}\right\}$$

Let $\theta(r,z) = R(r) \cdot Z(z)$. Then,

$$\theta(r,z) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n z} J_0(\lambda_n r), \quad \text{where } \lambda_n \text{ are the positive roots of } J_0(\lambda r_0) = 0$$

$$\Downarrow$$

$$f(r) - T_1 = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r)$$

$$\Downarrow$$

$$c_n = \frac{1}{N_n}\int_0^{r_0} [f(r) - T_1] J_0(\lambda_n r) r\,dr, \quad N_n = \frac{r_0^2}{2} J_1^2(\lambda_n r_0)$$

$$\therefore \quad T(r,z) - T_1 = \frac{2}{r_0^2}\sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{J_1^2(\lambda_n r_0)} e^{-\lambda_n z} \int_0^{r_0} [f(r') - T_1] J_0(\lambda_n r') r'\,dr'$$

b) If $f(r) = T_0 = \text{Const.}$, then

$$\int_0^{r_0} [f(r') - T_1] J_0(\lambda_n r') r'\,dr' = \dots = \frac{T_0 - T_1}{\lambda_n} r_0 J_1(\lambda_n r_0)$$

$$\therefore \quad \frac{T(r,z) - T_1}{T_0 - T_1} = \frac{2}{r_0}\sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n r_0)} e^{-\lambda_n z}$$

c)

$$q_z(0) = \int_0^{r_0} \left\{ -k\, 2\pi r \left(\frac{\partial T}{\partial z}\right)_{z=0} dr \right\} = \dots = 4\pi k (T_0 - T_1) \sum_{n=1}^{\infty} \frac{1}{\lambda_n}$$

PROB. 5.27: a) Formulation in terms of $\theta(r,z) = T(r,z) - T_\infty$:

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} = 0$$

$$\theta(0,z) = \text{finite}; \quad -k\frac{\partial\theta}{\partial r}\bigg|_{r=r_0} = h\,\theta(r_0,z)$$

$$\theta(r,0) = T_0 - T_\infty; \quad \theta(r,\infty) = 0$$

Let $\theta(r,z) = R(r)\cdot Z(z) \Rightarrow \theta(r,z) = \sum_{n=1}^{\infty} a_n J_0(\lambda_n r)\, e^{-\lambda_n z}$

where λ_n are the positive zeros of

$$k\lambda_n J_1(\lambda_n r_0) = h J_0(\lambda_n r_0) \Rightarrow \frac{J_1(\lambda_n r_0)}{J_0(\lambda_n r_0)} = \frac{Bi}{\lambda_n r_0}, \quad Bi = \frac{h r_0}{k}$$

Now,

$$\theta(r,0) = T_0 - T_\infty \Rightarrow T_0 - T_\infty = \sum_{n=1}^{\infty} a_n J_0(\lambda_n r)$$

Thus,

$$a_n = \frac{1}{N_n}\int_0^{r_0} (T_0 - T_\infty) J_0(\lambda_n r)\, r\, dr = \frac{T_0 - T_\infty}{N_n}\frac{r_0}{\lambda_n} J_1(\lambda_n r_0)$$

where (from Table 4.2),

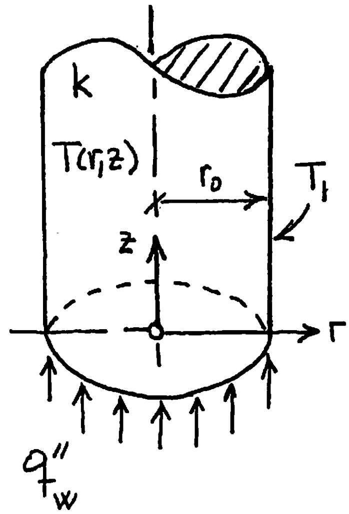
$$N_n = \frac{r_0^2}{2}\left[1 + \frac{1}{\lambda_n^2}\left(\frac{h}{k}\right)^2\right] J_0^2(\lambda_n r_0) = \frac{r_0^2}{2}\left[J_0^2(\lambda_n r_0) + J_1^2(\lambda_n r_0)\right]$$

$$\therefore \theta(r,z) = T(r,z) - T_\infty = \frac{2}{r_0}(T_0 - T_\infty)\sum_{n=1}^{\infty}\frac{J_1(\lambda_n r_0) J_0(\lambda_n r)}{\lambda_n\left[J_0^2(\lambda_n r_0) + J_1^2(\lambda_n r_0)\right]}\, e^{-\lambda_n z}$$

b) The rate of heat loss from the rod,

$$q_{loss} = \int_0^{r_0}\left\{-k\, 2\pi r\, dr\, \frac{\partial T(r,0)}{\partial z}\right\} = \ldots = 4\pi k(T_0 - T_\infty)\sum_{n=1}^{\infty}\frac{J_1^2(\lambda_n r_0)}{\lambda_n\left[J_0^2(\lambda_n r_0) + J_1^2(\lambda_n r_0)\right]}$$

PROB. 5.28: Formulation of the problem in terms of $\theta = T - T_1$:



$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} = 0$$

$$\theta(0,z) \neq \infty \ ; \ \theta(r_0, z) = 0$$

$$-k\left.\frac{\partial\theta}{\partial z}\right|_{z=0} = q''_w \ ; \ \theta(r,\infty) = 0$$

Let $\theta(r,z) = R(r) \cdot Z(z)$

$$\therefore \ \theta(r,z) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n z} J_0(\lambda_n r)$$

where λ_n's are the positive roots of $J_0(\lambda r_0) = 0$.

B.C. at $z = 0$ $\Rightarrow$ $- \dfrac{q''_w}{k} = - \displaystyle\sum_{n=1}^{\infty} C_n \lambda_n J_0(\lambda_n r)$

Thus,

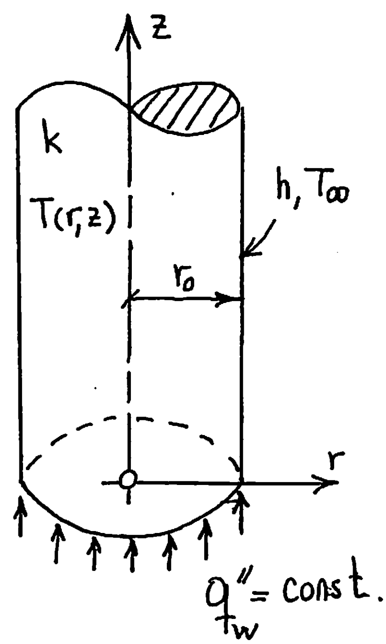
$$C_n \lambda_n = \frac{1}{N_n}\int_0^{r_0} \frac{q''_w}{k} J_0(\lambda_n r)\, r\,dr \ , \ N_n = \frac{r_0^2}{2} J_1^2(\lambda_n r_0)$$

$$= \frac{q''_w}{k} \frac{2}{\lambda_n r_0} \frac{1}{J_1(\lambda_n r_0)}$$

$$\therefore \ T(r,z) - T_1 = \frac{2q''_w}{k r_0} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \frac{J_0(\lambda_n r)}{J_1(\lambda_n r_0)} e^{-\lambda_n z}$$

PROB. 5.29: Formulation in terms of

$$\theta(r,z) = T(r,z) - T_\infty :$$



$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} = 0$$

$$\theta(0,z) \neq \infty \;; \quad -k\left.\frac{\partial\theta}{\partial r}\right|_{r=r_o} = h\,\theta(r_o,z)$$

$$-k\left.\frac{\partial\theta}{\partial z}\right|_{z=0} = q''_w \;; \quad \theta(r,\infty) = 0$$

Let $\theta(r,z) = R(r)\cdot Z(z)$

$$\therefore \quad \theta(r,z) = \sum_{n=1}^{\infty} c_n J_o(\lambda_n r)\, e^{-\lambda_n z}$$

where λ_n are the positive zeros of

$$k\lambda_n J_1(\lambda_n r_o) = h J_o(\lambda_n r_o) \quad \Longrightarrow \quad \frac{J_1(\lambda_n r_o)}{J_o(\lambda_n r_o)} = \frac{Bi}{\lambda_n r_o}, \; Bi = \frac{h r_o}{k}$$

$$\therefore \; -\frac{q''_w}{k} = -\sum_{n=1}^{\infty} c_n \lambda_n J_o(\lambda_n r) \quad \longleftarrow \text{From B.C. at } z=0.$$

Thus,

$$c_n\lambda_n = \frac{1}{N_n}\int_0^{r_o} \frac{q''_w}{k} J_o(\lambda_n r)\, r\,dr,$$

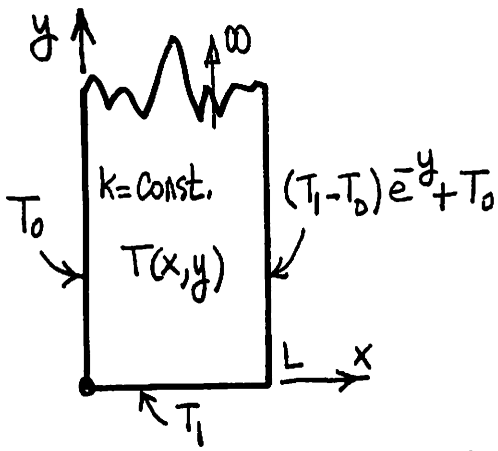
$$= \frac{1}{N_n}\frac{q''_w}{k}\frac{r_o}{\lambda_1} J_1(\lambda_n r_o)$$

where, from Table 4.2,

$$N_n = \frac{r_o^2}{2}\left[1 + \frac{1}{\lambda_n^2}\left(\frac{h}{k}\right)^2\right] J_o^2(\lambda_n r_o) = \frac{r_o^2}{2}\left[J_o^2(\lambda_n r_o) + J_1^2(\lambda_n r_o)\right]$$

$$\therefore \; \theta(r,z) = T(r,z) - T_\infty = \frac{2q''_w}{k r_o}\sum_{n=1}^{\infty}\frac{J_1(\lambda_n r_o)\, J_o(\lambda_n r)}{\lambda_n^2\left[J_o^2(\lambda_n r_o) + J_1^2(\lambda_n r_o)\right]}\, e^{-\lambda_n z}$$

PROB. 5.30:



Formulation in terms of $\theta(x,y)=T(x,y)-T_0$:

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0$$

$$\theta(0,y)=0;\quad \theta(L,y)=(T_1-T_0)e^{-y}$$

$$\theta(x,0)=T_1-T_0;\quad \lim_{y\to\infty}\theta(x,y)=0$$

Let $\theta(x,y)=\psi(x,y)+e^{-y}\phi(x)$

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$$

$$\psi(0,y)=0;\quad \psi(L,y)=0$$

$$\psi(x,0)=(T_1-T_0)-\phi(x)$$

$$\lim_{y\to\infty}\psi(x,y)=0$$

$$\Downarrow\quad \psi(x,y)=X(x)\cdot Y(y)$$

$$\frac{d^2\phi}{dx^2}+\phi(x)=0$$

$$\phi(0)=0$$

$$\phi(L)=T_1-T_0$$

$$\Downarrow$$

$$\boxed{\phi(x)=(T_1-T_0)\frac{\sin x}{\sin L}}$$

$$\left\{\begin{array}{c}\dfrac{d^2X}{dx^2}+\lambda^2X=0\\ X(0)=0\\ X(L)=0\end{array}\right\}\Rightarrow\quad X_n(x)=A_n\sin\lambda_n x,\quad \lambda_n=\frac{n\pi}{L},\ n=1,2,3,\ldots$$

and

$$\left\{\begin{array}{c}\dfrac{d^2Y_n}{dy^2}-\lambda_n^2Y_n=0\\ Y_n(\infty)=0\end{array}\right\}\Rightarrow\quad Y_n(y)=B_ne^{-\lambda_n y}$$

$$\therefore\quad \boxed{\psi(x,y)=\sum_{n=1}^{\infty}a_n\sin\lambda_n x\, e^{-\lambda_n y}}$$

$$\psi(x,0) = (T_1 - T_0)\left[1 - \frac{\sin x}{\sin L}\right] = \sum_{n=1}^{\infty} a_n \sin\lambda_n x$$

$$a_n = \frac{2}{L}(T_1 - T_0)\int_0^L \left[1 - \frac{\sin x}{\sin L}\right]\sin\lambda_n x\,dx$$

$$\vdots$$

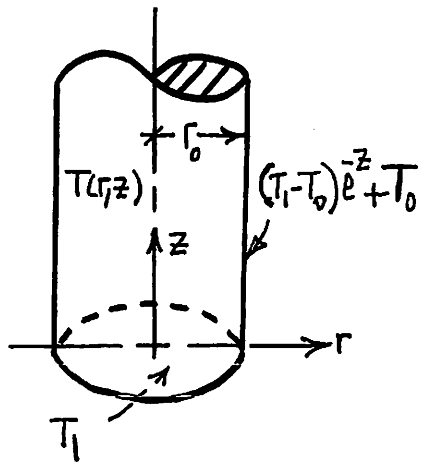
$$= 2(T_1 - T_0)\left\{\frac{1-(-1)^n}{n\pi} - \frac{(-1)^n n\pi}{L^2 - (n\pi)^2}\right\}$$

Thus,

$$\frac{T(x,y) - T_0}{T_1 - T_0} = e^{-y}\frac{\sin x}{\sin L} + 2\sum_{n=1}^{\infty}\left\{\frac{1-(-1)^n}{n\pi} - \frac{(-1)^n n\pi}{L^2 - (n\pi)^2}\right\}\sin\frac{n\pi}{L}x\, e^{-\frac{n\pi}{L}y}$$

PROB. 5.31: Formulation in terms of

$\theta(r,z) = T(r,z) - T_0$:



$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} = 0$$

$\theta(0,z) = \text{finite}; \quad \theta(r_0,z) = (T_1 - T_0)e^{-z}$

$\theta(r,0) = (T_1 - T_0); \quad \lim_{z\to\infty}\theta(r,z) = 0$

Let $\theta(r,z) = \psi(r,z) + e^{-z}\phi(r)$

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = 0$$

$\psi(0,z) = \text{finite}; \quad \psi(r_0,z) = 0$

$\psi(r,0) = (T_1 - T_0) - \phi(r)$

$\lim_{z\to\infty}\psi(r,z) = 0$

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + \phi(r) = 0$$

$\phi(0) = \text{finite}$

$\phi(r_0) = T_1 - T_0$

$\Downarrow$

$$\phi(r) = (T_1 - T_0)\frac{J_0(r)}{J_0(r_0)}$$

If we let $\psi(r,z) = R(r)\cdot Z(z)$, then

$$\left\{\begin{array}{c} \dfrac{d^2R}{dr^2} + \dfrac{1}{r}\dfrac{dR}{dr} + \lambda^2 R = 0 \\ R(0) = \text{finite} \\ R(r_0) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} R_n(r) = A_n J_0(\lambda_n r) \\ \text{where } \lambda_n\text{'s are the positive} \\ \text{zeros of } J_0(\lambda r_0) = 0 \end{array}$$

and

$$\left\{\begin{array}{c} \dfrac{d^2Z_n}{dz^2} - \lambda_n^2 Z_n = 0 \\ Z_n(\infty) = 0 \end{array}\right\} \Rightarrow Z_n(z) = B_n e^{-\lambda_n z}$$

Thus,

$$\psi(r,z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} J_0(\lambda_n r)$$

$$\Downarrow$$

$$\psi(r,0) = (T_1 - T_0)\left[1 - \frac{J_0(r)}{J_0(r_0)}\right] = \sum_{n=1}^{\infty} a_n J_0(\lambda_n r)$$

$$a_n = \frac{1}{N_n}(T_1 - T_0)\int_0^{r_0}\left[1 - \frac{J_0(r)}{J_0(r_0)}\right] J_0(\lambda_n r)\, r\, dr$$

$$\vdots \quad N_n = \frac{r_0^2}{2} J_1^2(\lambda_n r_0)$$

$$= \frac{2}{r_0}\,\frac{T_1 - T_0}{J_1(\lambda_n r_0)}\,\frac{1}{\lambda_n(1-\lambda_n^2)}$$

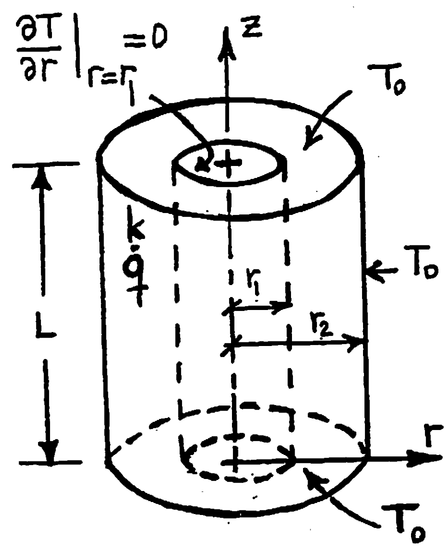
$$\therefore \boxed{\frac{T(r,z) - T_0}{T_1 - T_0} = e^{-z}\frac{J_0(r)}{J_0(r_0)} + \frac{2}{r_0}\sum_{n=1}^{\infty}\frac{1}{\lambda_n(1-\lambda_n^2)}\,\frac{J_0(\lambda_n r)}{J_1(\lambda_n r_0)}\, e^{-\lambda_n z}}$$

PROB. 5.32: Formulation in terms of $\theta(r,z) = T(r,z) - T_0$:

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} + \frac{\dot{q}}{k} = 0$$

$$\left.\frac{\partial\theta}{\partial r}\right|_{r=r_1} = 0 ; \quad \theta(r_2, z) = 0$$

$$\theta(r,0) = 0 ; \quad \theta(r,L) = 0$$



Let

$$\theta(r,z) = \psi(r,z) + \phi(z)$$

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = 0$$

$$\left.\frac{\partial\psi}{\partial r}\right|_{r=r_1} = 0, \quad \psi(r_2, z) = -\phi(z)$$

$$\psi(r,0) = 0, \quad \psi(r,L) = 0$$

$$\Downarrow \quad \psi(r,z) = R(r)\cdot Z(z)$$

$$\frac{d^2\phi}{dz^2} + \frac{\dot{q}}{k} = 0$$

$$\phi(0) = 0$$

$$\phi(L) = 0$$

$$\phi(z) = \frac{\dot{q}L^2}{2k}\left[\frac{z}{L} - \left(\frac{z}{L}\right)^2\right]$$

$$\left\{\begin{array}{c} \frac{d^2Z}{dz^2} + \lambda^2 Z = 0 \\ Z(0) = 0 \\ Z(L) = 0 \end{array}\right\} \Rightarrow \quad Z_n(z) = A_n \sin\lambda_n z, \quad \lambda_n = \frac{n\pi}{L}, \; n = 1, 2, 3, \ldots$$

Corresponding to each λ_n,

$$r^2 \frac{d^2R_n}{dr^2} + r\frac{dR_n}{dr} - \lambda_n^2 r^2 R_n = 0$$

$$\therefore \quad R_n(r) = B_n I_0(\lambda_n r) + C_n K_0(\lambda_n r)$$

$$\left.\frac{\partial\psi}{\partial r}\right|_{r=r_1}=0 \rightarrow \left.\frac{dR_n}{dr}\right|_{r=r_1}=0 \rightarrow \underbrace{B_n\lambda_n I_1(\lambda_n r_1)-C_n\lambda_n K(\lambda_n r_1)=0}$$

$$C_n = B_n\frac{I_1(\lambda_n r_1)}{K_1(\lambda_n r_1)}$$

$$\therefore\ R_n(r) = \frac{B_n}{K_1(\lambda_n r_1)}\left[I_0(\lambda_n r)K_1(\lambda_n r_1)+I_1(\lambda_n r_1)K_0(\lambda_n r)\right]$$

Hence, the solution for $\psi(r,z)$ is

$$\psi(r,z) = \sum_{n=1}^{\infty} a_n\left[I_0(\lambda_n r)K_1(\lambda_n r_1)+I_1(\lambda_n r_1)K_0(\lambda_n r)\right]\sin\lambda_n z$$

$$\therefore\ -\phi(z) = -\frac{\dot{q}L^2}{2k}\left[\frac{z}{L}-\left(\frac{z}{L}\right)^2\right] = \sum_{n=1}^{\infty} b_n\sin\lambda_n z$$

where

$$b_n = a_n\left[I_0(\lambda_n r_2)K_1(\lambda_n r_1)+I_1(\lambda_n r_1)K_0(\lambda_n r_2)\right] = \frac{2}{L}\int_0^L\left[-\phi(z)\right]\sin\lambda_n z\,dz$$

$$= -\frac{\dot{q}L}{k}\int_0^L\left[\frac{z}{L}-\left(\frac{z}{L}\right)^2\right]\sin\lambda_n z\,dz = -\frac{2\dot{q}L^2}{k}\cdot\frac{1-(-1)^n}{(n\pi)^3}$$

$$\therefore\ \psi(r,z) = -\frac{2\dot{q}L^2}{k\pi^3}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^3}\frac{\left[I_0(\lambda_n r)K_1(\lambda_n r_1)+I_1(\lambda_n r_1)K_0(\lambda_n r)\right]}{\left[I_0(\lambda_n r_2)K_1(\lambda_n r_1)+I_1(\lambda_n r_1)K_0(\lambda_n r_2)\right]}\sin\lambda_n z$$

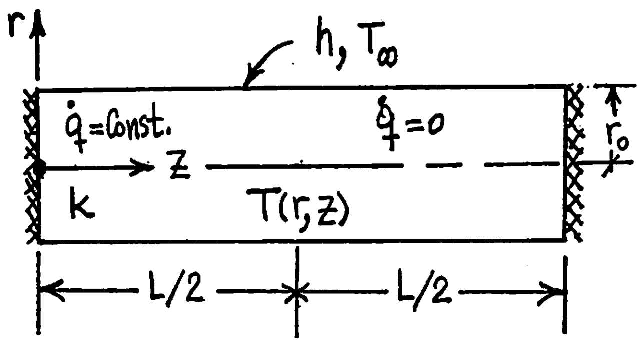
Thus,

$$T(r,z)-T_0 = \frac{\dot{q}L^2}{2k}\left[\frac{z}{L}-\left(\frac{z}{L}\right)^2\right]$$

$$-\frac{2\dot{q}L^2}{k\pi^3}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^3}\frac{\left[I_0(\lambda_n r)K_1(\lambda_n r_1)+I_1(\lambda_n r_1)K_0(\lambda_n r)\right]}{\left[I_0(\lambda_n r_2)K_1(\lambda_n r_1)+I_1(\lambda_n r_1)K_0(\lambda_n r_2)\right]}\sin\lambda_n z$$

PROB. 5.33:

- Because of symmetry, consider only one half of the rod.
- Assume $k = \text{constant}$.



Formulation in terms of $\theta(r,z) = T(r,z) - T_\infty$:

$$\left.\begin{aligned} &\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{\partial z^2} + \frac{\dot{Q}(z)}{k} = 0 \\ &\theta(0,z) = \text{finite} \; ; \; -k\left.\frac{\partial\theta}{\partial r}\right|_{r=r_0} = h\,\theta(r_0,z) \\ &\left.\frac{\partial\theta}{\partial z}\right|_{z=0} = \left.\frac{\partial\theta}{\partial z}\right|_{z=L} = 0 \end{aligned}\right\} \quad \text{where} \quad \dot{Q}(z) = \begin{cases} \dot{q}\,, & 0<z<\frac{L}{2} \\ 0\,, & \frac{L}{2}<z<L \end{cases}$$

Expand both $\theta(r,z)$ and $\dot{Q}(z)/k$ into Fourier series as follows:

$$\theta(r,z) = \sum_{n=0}^{\infty} A_n(r)\cos\lambda_n z = A_0(r) + \sum_{n=1}^{\infty} A_n(r)\cos\lambda_n z\,, \quad 0<z<L$$

and

$$\frac{\dot{Q}(z)}{k} = \sum_{n=0}^{\infty} B_n \cos\lambda_n z = B_0 + \sum_{n=1}^{\infty} B_n \cos\lambda_n z\,, \quad 0<z<L$$

where $\lambda_n = \frac{n\pi}{L}$, $n = 0, 1, 2, \ldots$, and $\cos\lambda_n z$ are the eigenvalues and eigenfunctions of $\left\{\begin{aligned} &\frac{d^2 Z}{dz^2} + \lambda^2 Z = 0 \\ &\frac{dZ(0)}{dz} = \frac{dZ(L)}{dz} = 0 \end{aligned}\right\}$

Therefore,

$$B_0 = \frac{1}{L}\int_0^L \dot{Q}(z)\,dz = \frac{\dot{q}}{Lk}\int_0^{L/2} dz = \frac{\dot{q}}{2k}$$

$$B_n = \frac{2}{L}\int_0^L \dot{Q}(z)\cos\lambda_n z\,dz = \frac{2\dot{q}}{Lk}\int_0^{L} \cos\lambda_n z\,dz = \frac{2\dot{q}}{Lk\lambda_n}\sin\lambda_n\frac{L}{2}$$

Here we note that the expansion for $\theta(r,z)$ satisfies the two boundary conditions in z-direction. Now, substitute both expansions into the heat conduction equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\sum_{n=0}^{\infty}A_n(r)\cos\lambda_n z\right]+\frac{\partial^2}{\partial z^2}\sum_{n=0}^{\infty}A_n(r)\cos\lambda_n z+\sum_{n=0}^{\infty}B_n\cos\lambda_n z=0$$

$$\sum_{n=0}^{\infty}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_n}{dr}\right)-\lambda_n^2A_n(r)+B_n\right]\cos\lambda_n z=0$$

Thus,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_0}{dr}\right)=-B_0$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_n}{dr}\right)-\lambda_n^2A_n(r)=-B_n,\quad n=1,2,3,\ldots$$

Also, the boundary conditions in r-direction yield:

$$\theta(0,z)=\text{finite}\ \Rightarrow\ \sum_{n=0}^{\infty}A_n(0)\cos\lambda_n z=\text{finite}$$

$$\therefore\ A_n(0)=\frac{1}{N_n}\int_0^L(\text{finite})\cos\lambda_n z\,dz=\text{finite}$$

and

$$-k\left(\frac{\partial\theta}{\partial r}\right)_{r=r_0}=h\,\theta(r_0,z)\ \Rightarrow\ \sum_{n=0}^{\infty}\left[k\frac{dA_n(r_0)}{dr}+hA_n(r_0)\right]\cos\lambda_n z=0$$

$$\therefore\ k\frac{dA_n(r_0)}{dr}+hA_n(r_0)=0,\quad n=0,1,2,\ldots$$

Thus, when $n=0$,

$$\left.\begin{array}{c}\dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{dA_0}{dr}\right)=-B_0\\ A_0(0)=\text{finite}\\ k\dfrac{A_0(r_0)}{dr}+hA_0(r_0)=0\end{array}\right\}\Rightarrow A_0(r)=\frac{B_0r_0^2}{4}\left[1+\frac{2}{Bi}-\left(\frac{r}{r_0}\right)^2\right]$$

where $Bi=\dfrac{hr_0}{k}$

When $n = 1, 2, 3, \ldots,$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_n}{dr}\right) - \lambda_n^2 A_n(r) = -B_n$$

$$\therefore \quad A_n(r) = C_n I_0(\lambda_n r) + D_n K_0(\lambda_n r) + \frac{B_n}{\lambda_n^2}$$

Particular solution

$$A_n(0) = \text{finite} \Rightarrow D_n = 0$$

$$k\frac{A_n(r_0)}{dr} + hA_n(r_0) = 0 \Rightarrow k\lambda_n C_n I_1(\lambda_n r_0) + h\left[C_n I_0(\lambda_n r_0) + \frac{B_n}{\lambda_n^2}\right] = 0$$

$$\therefore \quad C_n = -\frac{1}{\lambda_n^2}\frac{hB_n}{k\lambda_n I_1(\lambda_n r_0) + hI_0(\lambda_n r_0)}$$

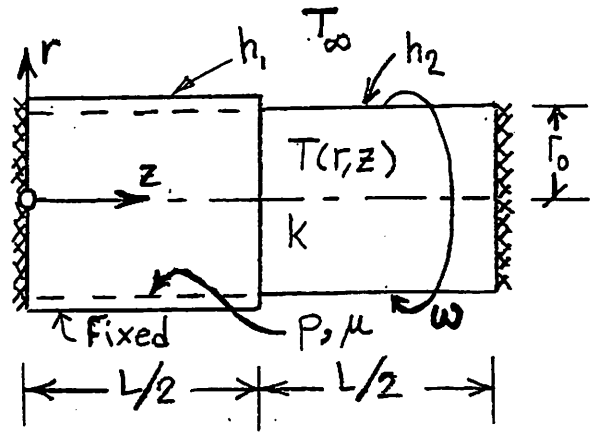
Thus,

$$A_n(r) = \frac{B_n}{\lambda_n^2}\left[1 - \frac{h}{k\lambda_n I_1(\lambda_n r_0) + hI_0(\lambda_n r_0)}\right], \quad n = 1, 2, 3, \ldots$$

and the solution is given by

$$\theta(r,z) = T(r,z) - T_\infty = \frac{\dot{q}r_0^2}{8k}\left[1 + \frac{2k}{hr_0} - \left(\frac{r}{r_0}\right)^2\right]$$
$$+ \frac{2\dot{q}}{Lk}\sum_{n=1}^{\infty}\frac{\sin\lambda_n\frac{L}{k}}{\lambda_n^3}\left[1 - \frac{h}{k\lambda_n I_1(\lambda_n r_0) + hI_0(\lambda_n r_0)}\right]\cos\lambda_n z$$

PROB. 5.34:



- Because of symmetry, consider only one half of the system.
- Since the sleeve is of negligible thickness, its temperature at any z will be the same as the shaft's surface temperature.
- Assume constant thermal conductivity.

Formulation in terms of $\theta(r,z) = T(r,z) - T_\infty$:

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} = 0$$

$$\left.\frac{\partial\theta}{\partial z}\right|_{z=0} = \left.\frac{\partial\theta}{\partial z}\right|_{z=L} = 0$$

$$\theta(0,z) = \text{finite}; \quad k\left.\frac{\partial\theta}{\partial r}\right|_{r=r_o} = q''_w(z) = \begin{cases} \mu P\omega r_o - h_1\theta(r_o,z) \\ \quad 0 < z < L/2 \\ -h_2\theta(r_o,z), \quad L/2 < z < L \end{cases}$$

Let $\theta(r,z) = R(r)\cdot Z(z)$

$$\Downarrow$$

$$\left\{\begin{array}{l} \dfrac{d^2Z}{dz^2} + \lambda^2 Z = 0 \\ \dfrac{dZ(0)}{dz} = \dfrac{dZ(L)}{dz} = 0 \end{array}\right\} \Rightarrow \text{where} \quad \begin{array}{l} Z_n(z) = A_n \cos\lambda_n z \\ \lambda_n = \dfrac{n\pi}{L}, \; n = 0, 1, 2, \ldots \end{array}$$

and

$$\left.\begin{array}{c} \dfrac{d^2R_n}{dr^2} + \dfrac{1}{r}\dfrac{dR_n}{dr} - \lambda_n^2 R_n = 0 \\ R_n(0) = \text{finite} \end{array}\right\} \Rightarrow R_n(r) = B_n I_0(\lambda_n r)$$

Thus,

$$\theta(r,z) = \sum_{n=0}^{\infty} a_n I_0(\lambda_n r)\cos\lambda_n z = a_0 + \sum_{n=1}^{\infty} a_n I_0(\lambda_n r)\cos\lambda_n z \qquad (1)$$

In order to find the expansion coefficients a_n, we now apply the boundary condition at $r=r_0$:

$$k\sum_{n=1}^{\infty} a_n \lambda_n I_1(\lambda_n r_0)\cos\lambda_n z = q_w''(z)$$

Next,

$$\int_0^L \cos\lambda_n z\left\{k\sum_{n=1}^{\infty} a_n\lambda_n I_1(\lambda_n r_0)\cos\lambda_n z = q_w''(z)\right\}dz$$

$$k\,a_n\lambda_n I_1(\lambda_n r_0)\overbrace{\left(\frac{L}{2}\right)}^{=\int_0^L \cos^2\lambda_n z\,dz} = \overbrace{\mathcal{L}_n}^{=\int_0^L q_w''(z)\cos\lambda_n z\,dz}, \quad n=0,1,2,\ldots \qquad (2)$$

where

$$\mathcal{L}_n = \mu p\omega r_0\int_0^{L/2}\cos\lambda_n z\,dz - h_1\int_0^{L/2}\theta(r_0,z)\cos\lambda_n z - h_2\int_{L/2}^{L}\theta(r_0,z)\cos\lambda_n z\,dz$$

$$= Q_n - (h_1 - h_2)\sum_{m=0}^{\infty} a_m I_0(\lambda_m r_0) D_{mn} \qquad (3)$$

Here, $\quad Q_n = \mu p\omega r_0 D_{0n}, \quad n = 0,1,2,\ldots$

and
$$D_{mn} = \int_0^{L/2} \cos\lambda_m z \cos\lambda_n z\,dz = -\int_{L/2}^{L} \cos\lambda_m z \cos\lambda_n z\,dz$$

which yields

$$D_{oo} = \frac{L}{2}, \quad D_{mm} = D_{nn} = \frac{L}{4}, \quad m, n = 1, 2, 3, \ldots$$

$$D_{on} = \frac{1}{\lambda_n} \sin\lambda_n \frac{L}{2}, \quad D_{mo} = \frac{1}{\lambda_m} \sin\lambda_m \frac{L}{2}$$

$$D_{mn} = \frac{\lambda_m^2 D_{mo} \cos\lambda_n \frac{L}{2} - \lambda_n^2 D_{on} \cos\lambda_m \frac{L}{2}}{\lambda_m^2 - \lambda_n^2}, \quad m \neq n$$

Now, substituting ③ into ② gives

$$k\lambda_n I_1(\lambda_n r_o) \frac{L}{2} a_n + (h_1 - h_2) \sum_{m=0}^{\infty} I_o(\lambda_m r_o) D_{mn} a_m = Q_n$$

which can be rewritten as

$$\sum_{m=0}^{\infty} c_{mn} a_m = Q_n, \quad n = 0, 1, 2, \ldots \qquad ④$$

where

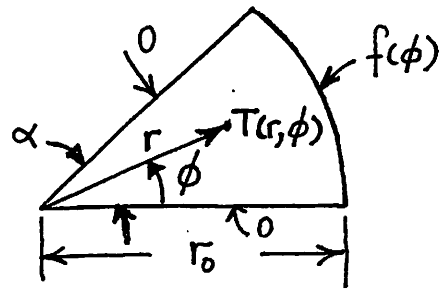
$$c_{mn} = k\lambda_m I_1(\lambda_m r_o) D_{00} \delta_{mn} + (h_1 - h_2) I_o(\lambda_m r_o) D_{mn}, \quad \delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

Equation ④ is a set of (infinite) simultaneous algebraic equations for the coefficients of the solution ①. Of course, the solution of the first N+1 equations $(m, n = 0, 1, 3, \ldots, N)$ gives a_n's of

$$\theta(r, z) \cong \sum_{n=0}^{N} a_n I_o(\lambda_n r) \cos\lambda_n z$$

PROB. 5.35: a) Formulation:

$$\boxed{\begin{array}{l}\dfrac{\partial^2 T}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial T}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 T}{\partial \phi^2}=0\\ T(0,\phi)=0,\ T(r_o,\phi)=f(\phi)\\ T(r,0)=0,\ T(r,\alpha)=0\end{array}}$$



Let $T(r,\phi)=R(r)\cdot\psi(\phi)$

$$\left\{\begin{array}{c}\dfrac{d^2\psi}{d\phi^2}+\lambda^2\psi=0\\ \psi(0)=0\\ \psi(\alpha)=0\end{array}\right\}\Rightarrow \begin{array}{c}\psi_n=A_n\sin\lambda_n\phi\\ \lambda_n=\dfrac{n\pi}{\alpha},\ n=1,2,\dots\end{array}\Rightarrow\left\{\begin{array}{c}r^2\dfrac{d^2R_n}{dr^2}+r\dfrac{dR_n}{dr}-\lambda_n^2R_n=0\\ R_n(0)=0\end{array}\right\}$$

$$\Downarrow$$

$$R_n=B_n r^{\lambda_n}$$

$$\therefore\ T(r,\phi)=\sum_{n=1}^{\infty}a_n r^{\lambda_n}\sin\lambda_n\phi$$

$$\Rightarrow f(\phi)=\sum_{n=1}^{\infty}a_n r_o^{\lambda_n}\sin\lambda_n\phi\ \Rightarrow\ a_n r_o^{\lambda_n}=\frac{2}{\alpha}\int_0^{\alpha}f(\phi')\sin\lambda_n\phi'd\phi'$$

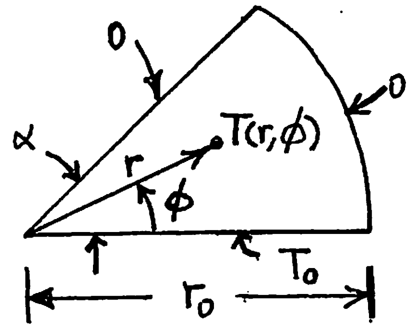
$$T(r,\phi)=\frac{2}{\alpha}\sum_{n=1}^{\infty}\left(\frac{r}{r_o}\right)^{\lambda_n}\sin\lambda_n\phi\cdot\int_0^{\alpha}f(\phi')\sin\lambda_n\phi'd\phi'$$

b) $f(\phi)=T_1=$ constant.

$$T(r,\phi)=\frac{2T_1}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\left(\frac{r}{r_o}\right)^{\lambda_n}\sin\lambda_n\phi$$

PROB. 5.36:

$$\boxed{\begin{gathered}\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}=0\\ T(0,\phi)=\text{finite},\ T(r_0,\phi)=0\\ T(r,0)=T_0,\ T(r,\alpha)=0\end{gathered}}$$



Let $T(r,\phi)=\psi(r,\phi)+\Omega(\phi)$

$$\frac{\partial^2 \psi}{\partial r^2}+\frac{1}{r}\frac{\partial \psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2}=0$$

$$\psi(0,\phi)=\text{finite}-\Omega(\phi)=\underbrace{\text{finite}}_{0}$$

$$\psi(r_0,\phi)=-\Omega(\phi)$$

$$\Omega(r,0)=0,\quad \Omega(r,\alpha)=0$$

$\Downarrow$ See Prob. 5.28.

$$\psi(r,\phi)=\frac{2}{\alpha}\sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^{\frac{n\pi}{\alpha}}\sin\frac{n\pi}{\alpha}\phi\cdot\int_0^{\alpha}\underbrace{\psi(r_0,\phi')}_{-\Omega(\phi)}\sin\frac{n\pi}{\alpha}\phi'\,d\phi'$$

$$=-\frac{2T_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{r_0}\right)^{\frac{n\pi}{\alpha}}\sin\frac{n\pi}{\alpha}\phi$$

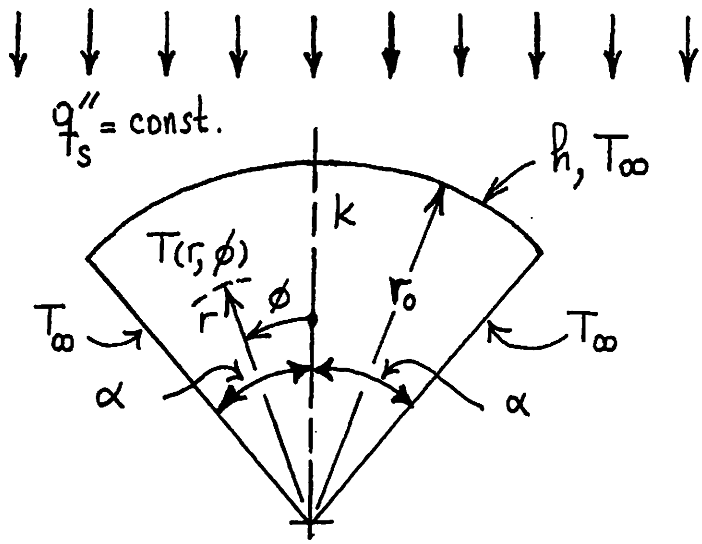
$$\boxed{\frac{d^2\Omega}{d\phi^2}=0 \qquad \Omega(0)=T_0,\ \Omega(\alpha)=0}$$

$\Downarrow$

$$\Omega(\phi)=T_0\left(1-\frac{\phi}{\alpha}\right)$$

$$\therefore\ \boxed{T(r,\phi)=T_0\left(1-\frac{\phi}{\alpha}\right)-\frac{2T_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{r_0}\right)^{\frac{n\pi}{\alpha}}\sin\frac{n\pi}{\alpha}\phi}$$

PROB.5.37:



Formulation in terms of $\theta(r,\phi) = T(r,\phi) - T_\infty$:

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\theta}{\partial\phi^2} = 0$$

$$\theta(0,\phi) = 0; \quad k\left.\frac{\partial\theta}{\partial r}\right|_{r=r_o} = q_s''\cos\phi - h\,\theta(r_o,\phi)$$

$$\left.\frac{\partial\theta}{\partial\phi}\right|_{\phi=0} = 0; \quad \theta(r,\alpha) = 0$$

Let $\theta(r,\phi) = R(r).\psi(\phi)$

$$\left\{\begin{array}{c} \dfrac{d^2\psi}{d\phi^2} + \lambda^2\psi = 0 \\ \dfrac{d\psi(0)}{d\phi} = 0 \\ \psi(\alpha) = 0 \end{array}\right\} \Rightarrow \begin{array}{c} \psi_n = A_n\cos\lambda_n\phi \\ \lambda_n = \dfrac{2n-1}{\alpha}\dfrac{\pi}{2}, \; n=1,2,\ldots \end{array}$$

$$r^2\frac{d^2R_n}{dr^2} + r\frac{dR_n}{dr} - \lambda_n^2 R_n = 0$$

$$R_n(0) = 0$$

$$\Downarrow$$

$$R_n(r) = B_n r^{\lambda_n}$$

$$\therefore \quad \theta(r,\phi) = \sum_{n=1}^{\infty} a_n r^{\lambda_n} \cos\lambda_n\phi$$

Applying the B.C. at $r=r_o$ gives

$$q_s'' \cos\phi = \sum_{n=1}^{\infty} a_n\left[h r_o^{\lambda_n} + k\lambda_n r_o^{\lambda_n - 1}\right] \cos\lambda_n\phi$$

$$= \sum_{n=1}^{\infty} a_n h r_o^{\lambda_n}\left[1+\frac{\lambda_n}{Bi}\right] \cos\lambda_n\phi, \quad Bi = \frac{hr_o}{k}$$

$$\therefore \quad a_n h r_o^{\lambda_n}\left[1+\frac{\lambda_n}{Bi}\right] = \frac{2q_s''}{\alpha}\int_0^{\alpha} \cos\phi \cos\lambda_n\phi \, d\phi$$

$$\vdots$$

$$= q_s'' (-1)^n \cos\alpha \frac{(2n-1)\pi}{1-\lambda_n^2}$$

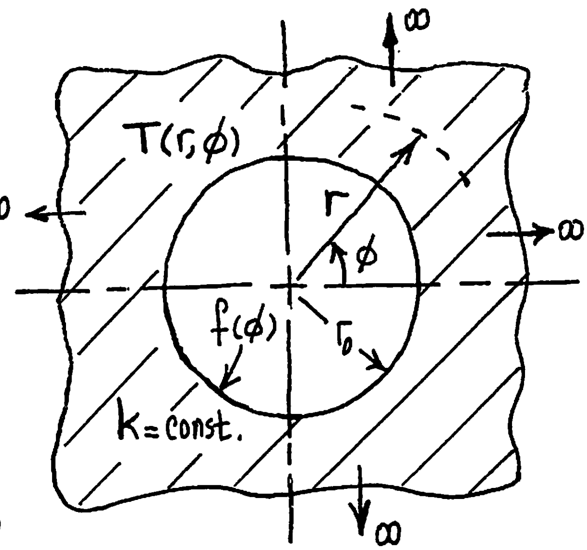
Thus,

$$\theta(r,\phi) = \frac{q_s''\pi\cos\alpha}{h} \sum_{n=1}^{\infty} \left(\frac{r}{r_o}\right)^{\lambda_n} \frac{(-1)^n(2n-1)}{1-\lambda_n^2} \frac{\cos\lambda_n\phi}{\left[1+\frac{\lambda_n}{Bi}\right]}$$

or

$$\boxed{T(r,\phi) - T_\infty = \frac{q_s''\pi r_o\cos\alpha}{k} \sum_{n=1}^{\infty} (-1)^n\left(\frac{r}{r_o}\right)^{\lambda_n} \frac{2n-1}{1-\lambda_n^2} \frac{\cos\lambda_n\phi}{[Bi+\lambda_n]}}$$

PROB. 5.38: Formulation of the problem:



$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} = 0$$

$$T(r_0,\phi) = f(\phi)$$

$$T(r,\phi)\Big|_{r\to\infty} \to \text{finite}$$

$$T(r,\phi) = T(r,\phi+2\pi)$$

$$\frac{\partial T(r,\phi)}{\partial \phi} = \frac{\partial T(r,\phi+2\pi)}{\partial \phi}$$

Seeking a product solution of the form $T(r,\phi) = R(r)\cdot\Phi(\phi)$ yields the general solution

$$T(r,\phi) = (A_1 r^{\lambda} + A_2 r^{-\lambda})(B_1 \sin\lambda\phi + B_2 \cos\lambda\phi)$$

for the differential equation, where the sign of the separation constant is consistent with the fact that the ϕ-direction is the homogeneous direction for $T(r,t)$. On the other hand, imposing the two boundary conditions in the ϕ-direction gives

$$\lambda_n = n\,, \quad n = 0, 1, 2, \ldots$$

Furthermore, $T(\infty,\phi) = $ finite gives $A_1 = 0$. Thus, the solution of the problem can be written as

$$T(r,\phi) = \sum_{n=0}^{\infty} r^{-n}(a_n \sin n\phi + b_n \cos n\phi)$$

The boundary condition $T(r_o, \phi) = f(\phi)$ yields

$$f(\phi) = b_o + \sum_{n=1}^{\infty} r_o^{-n} (a_n \sin n\phi + b_n \cos n\phi)$$

Thus,

$$b_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) d\phi$$

$$a_n = \frac{r_0^n}{\pi} \int_0^{2\pi} f(\phi) \sin n\phi \, d\phi, \quad n = 1, 2, 3, \ldots$$

$$b_n = \frac{r_0^n}{\pi} \int_0^{2\pi} f(\phi) \cos n\phi \, d\phi, \quad n = 1, 2, 3, \ldots$$

Then, the solution becomes

$$T(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) d\phi + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r_0}{r}\right)^n \left\{ \sin n\phi \int_0^{2\pi} f(\phi') \sin n\phi' d\phi' + \cos n\phi \int_0^{2\pi} f(\phi') \cos n\phi' d\phi' \right\}$$

or

$$T(r, \phi) = \frac{1}{\pi} \int_0^{2\pi} \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r_0}{r}\right)^n \cos n(\phi - \phi') \right\} f(\phi') d\phi'$$

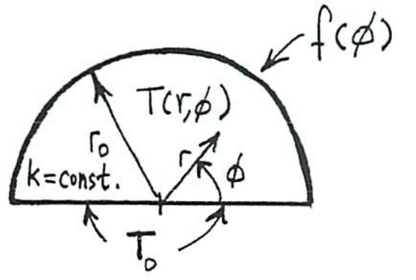
which can also be written as [See Eq. (5.86)]

$$\boxed{T(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - r_o^2}{r^2 - 2 r r_o \cos(\phi - \phi') + r_o^2} f(\phi') d\phi'}$$

Note that, as $r \to \infty$, $T(r, \phi)$ must approach a constant value given by

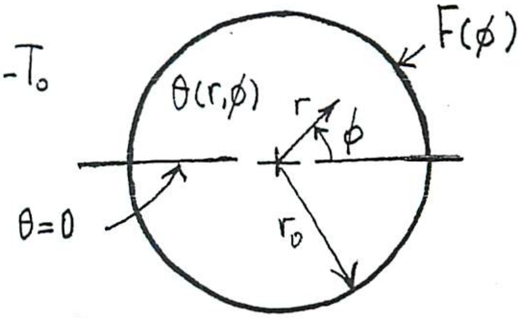
$$T(\infty, \phi) = T_\infty = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) d\phi$$

PROB. 5.39: The problem defined by Eqs. (5.62) and (5.63a-d):



$$\theta(r,\phi) = T(r,\phi) - T_0$$

$$\equiv$$



where

$$F(\phi) = \begin{cases} f(\phi) - T_0 , & 0 < \phi < \pi \\ -[f(2\pi - \phi) - T_0], & \pi < \phi < 2\pi \end{cases}$$

$$\theta(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r_0^2 - r^2}{r_0^2 - 2rr_0\cos(\phi-\phi') + r^2} F(\phi')\, d\phi'$$

$$= \frac{1}{2\pi}\int_0^{\pi} (\quad)[f(\phi') - T_0]\, d\phi' - \underbrace{\frac{1}{2\pi}\int_{\pi}^{2\pi} (\quad)[f(2\pi - \phi') - T_0]\, d\phi'}$$

$$\frac{1}{2\pi}\int_0^{\pi} \frac{r_0^2 - r^2}{r_0^2 - 2rr_0 \underbrace{\cos(\phi + \phi' - 2\pi)}_{\cos(\phi+\phi')} + r^2}[f(\phi') - T_0]\, d\phi'$$

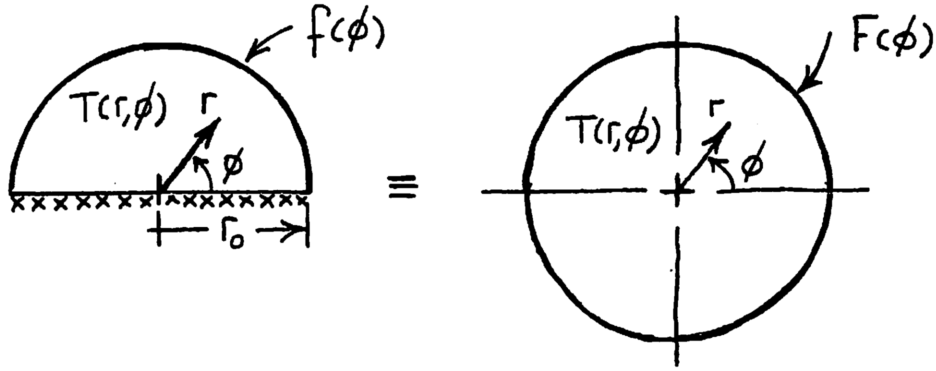
Thus,

$$\boxed{T(r,\phi) - T_0 = \int_0^{\pi} [K(r,\phi-\phi') - K(r,\phi+\phi')][f(\phi') - T_0]\, d\phi'}$$

where

$$K(r, \phi \pm \phi') = \frac{1}{2\pi} \frac{r_0^2 - r^2}{r_0^2 - 2rr_0\cos(\phi \pm \phi') + r^2}$$

PROB. 5.40:



where

$$F(\phi) = \begin{cases} f(\phi) \, , & 0 < \phi < \pi \\ f(2\pi - \phi) \, , & \pi < \phi < 2\pi \end{cases}$$

Thus,

$$T(r,\phi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r_0^2 - r^2}{r_0^2 - 2 r r_0 \cos(\phi - \phi') + r^2} F(\phi')\, d\phi'$$

$$= \frac{1}{2\pi} \int_0^{\pi} \left(\quad \right) f(\phi')\, d\phi' + \underbrace{\frac{1}{2\pi} \int_{\pi}^{2\pi} \left(\quad \right) f(2\pi - \phi')\, d\phi'}$$

$$\frac{1}{2\pi} \int_0^{\pi} \frac{r_0^2 - r^2}{r_0^2 - 2 r r_0 \underbrace{\cos(\phi + \phi' - 2\pi)}_{\cos(\phi + \phi')} + r^2} f(\phi')\, d\phi'$$

$$\therefore \quad \boxed{T(r,\phi) = \int_0^{\pi} \left[K(r, \phi - \phi') + K(r, \phi + \phi') \right] f(\phi')\, d\phi'}$$

where

$$K(r, \phi \pm \phi') = \frac{1}{2\pi} \frac{r_0^2 - r^2}{r_0^2 - 2 r r_0 \cos(\phi \pm \phi') + r^2}$$

PROB. 5.41:

$$f(\theta) = \begin{cases} \cos\theta, & 0 < \theta < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \theta < \pi \end{cases}$$

Let $x = \cos\theta$, then

$$F(x) = f(\cos^{-1}x) = \begin{cases} x, & 0 < x < 1 \\ 0, & -1 < x < 0 \end{cases}$$

$$\therefore \quad F(x) = A_0 P_0(x) + A_1 P_1(x) + A_2 P_2(x) + \cdots, \quad -1 < x < 1$$

From Eq. (5.135),

$$A_n = \frac{2n+1}{2} \int_{-1}^{1} F(x) P_n(x)\, dx$$

$$n=0: \quad A_0 = \frac{1}{2} \cdot \int_0^1 x \cdot 1 \cdot dx = \frac{1}{4}$$

$$n=1: \quad A_1 = \frac{3}{2} \cdot \int_0^1 x \cdot x \cdot dx = \frac{1}{2}$$

$$n=2: \quad A_2 = \frac{5}{2} \cdot \int_0^1 x \cdot \left\{\frac{1}{2}(3x^2 - 1)\right\} \cdot dx = \frac{5}{16}$$

$$\vdots$$

$$F(x) = \frac{1}{4} P_0(x) + \frac{1}{2} P_1(x) + \frac{5}{16} P_2(x) + \cdots, \quad -1 < x < 1$$

Thus,

$$f(\theta) = \frac{1}{4} P_0(\cos\theta) + \frac{1}{2} P_1(\cos\theta) + \frac{5}{16} P_2(\cos\theta) + \cdots, \quad 0 < \theta < \pi$$

where

$$P_0(\cos\theta) = 1, \quad P_1(\cos\theta) = \cos\theta, \quad P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1), \ \ldots$$

PROB. 5.42: The temperature distribution is given by [See Section 5.4.3, Eq. (5.151)]:

$$T(r,\theta) = \sum_{n=0}^{\infty} K_n r^n P_n(\cos\theta)$$

If the surface temperature is specified as

$$T(r_0,\theta) = f(\theta) = \begin{cases} T_0 \cos\theta, & 0<\theta<\frac{\pi}{2} \\ 0, & \frac{\pi}{2}<\theta<\pi \end{cases}$$

then

$$f(\theta) = \sum_{n=0}^{\infty} K_n r_0^n P_n(\cos\theta), \quad 0<\theta<\pi$$

or

$$f(x) = \sum_{n=0}^{\infty} K_n r_0^n P_n(x), \quad -1<x<1$$

where $x = \cos\theta$ and

$$f(x) = \begin{cases} T_0 x, & 0<x<1 \\ 0, & -1<x<0 \end{cases}$$

Now, the expansion coefficients $K_n r_0^n$ can be determined from

$$K_n r_0^n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx$$

$$n=0: \quad K_0 = T_0 \frac{1}{2} \int_0^1 x\, dx = T_0 \frac{1}{4}$$

$$n=1: \quad K_1 r_0 = T_0 \frac{3}{2} \int_0^1 x^2 dx = T_0 \frac{1}{2}$$

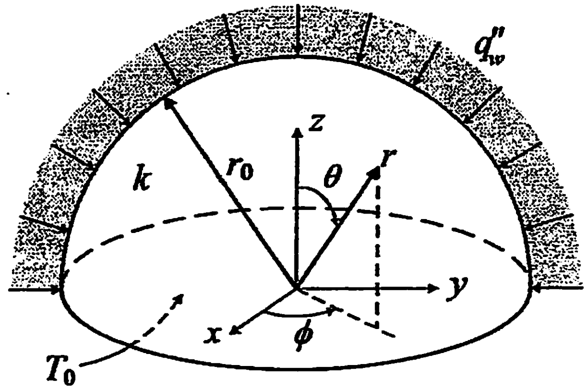
$$n=2: \quad K_2 r_0^2 = T_0 \frac{5}{2} \int_0^1 x \left\{\frac{1}{2}(3x^2-1)\right\} dx = T_0 \frac{5}{16}$$

$$\vdots$$

Thus,

$$\frac{T(r,\theta)}{T_0} = \frac{1}{4} + \frac{1}{2}\left(\frac{r}{r_0}\right) P_1(\cos\theta) + \frac{5}{16}\left(\frac{r}{r_0}\right)^2 P_2(\cos\theta) + \cdots$$

PROB. 5.43:



We note that $T = T(r,\theta)$. The formulation is then given by

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) = 0 \,, \quad \phi(r,\theta) = T(r,\theta) - T_0$$

$$\phi(0,\theta) = 0 \,, \quad \frac{\partial \phi(r_0,\theta)}{\partial r} = \frac{q''_w}{k}$$

$$\phi(r,0) = \text{finite} \,, \quad \phi\left(r, \frac{\pi}{2}\right) = 0$$

Let $\phi(r,\theta) = R(r)\cdot\psi(\theta)$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\psi}{d\theta}\right) + \lambda^2 \psi(\theta) = 0$$

$\downarrow$ $\mu = \cos\theta$

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\psi}{d\mu}\right] + \underbrace{\alpha(\alpha+1)}_{\lambda^2}\psi(\mu) = 0 \quad - (1)$$

$$\psi(0) = 0 \,, \quad \psi(1) = \text{finite}$$

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{\lambda^2}{r^2}R(r) = 0$$

$$R(0) = 0 \quad \rightarrow \quad B_n = 0$$

$$R_n(r) = A_n r^n + \cancel{B_n} r^{-(n+1)}$$

In order to satisfy the finiteness condition at $\mu = 1$, $\alpha = n = 0, 1, 2, \ldots$ and the solutions of Eq. (1) are the Legendre polynomials $P_n(\mu)$.

On the other hand, to satisfy the condition $\psi(0)=0$,

$$P_n(0)=0 \Rightarrow n=1,3,5,\ldots \quad \text{(see Eq. 5.113 in text)}$$

↑ Must be <u>odd</u> integer.

Thus,

$$\psi_n(\mu)=C_nP_n(\mu), \quad n=1,3,5,\ldots$$

Then, the solution for $\phi(r,\mu)$ can be constructed as

$$\phi(r,\mu)=\sum_{n=1,3,5,\ldots}^{\infty} D_n r^n P_n(\mu), \quad D_n \equiv A_nC_n$$

$$\Downarrow$$

$$\frac{q_w''}{k}=\sum_{n=1,3,5,\ldots}^{\infty} D_n n r_o^{(n-1)} P_n(\mu), \quad 0<\mu\leq 1 \quad (\underline{\text{Not}} \; -\leq\mu\leq 1) \quad \text{— ②}$$

Re-visit the orthogonality property of $P_n(\mu)$'s over $-1\leq\mu\leq 1$:

$$\left.\begin{aligned} &\int_{-1}^{1} P_m(\mu)P_n(\mu)\,d\mu=\begin{cases}0, & m\neq n\\ N_n, & m=n\end{cases}\\ &N_n=\int_{-1}^{1}[P_n(\mu)]^2\,d\mu=\frac{2}{2n+1}\end{aligned}\right\} \qquad \int_0^1 P_m(\mu)P_n(\mu)\,d\mu=\begin{cases}0, & \text{if } m \text{ and } n \text{ both odd (or even), and } m\neq n\\ N_n', & m=n\end{cases}$$

$$N_n'=\int_0^1[P_n(\mu)]^2\,d\mu=\frac{1}{2n+1}$$

Now, with the use of the <u>orthogonality property over $0<\mu\leq 1$</u>, Eq. (2) yields,

$$D_n n r_o^{(n-1)}=(2n+1)\frac{q_w''}{k}\int_0^1 P_n(\mu)\,d\mu=\cdots=\frac{q_w''}{k}\left[P_{n+1}(1)-P_{n+1}(0)-P_{n-1}(1)+P_{n-1}(0)\right]$$

Thus,

$$\boxed{\phi(r,\mu)=\frac{q_w''}{k}\sum_{n=1,3,5,\ldots}^{\infty}\frac{r_o}{n}\left(\frac{r}{r_o}\right)^n\left[P_{n+1}(1)-P_{n+1}(0)-P_{n-1}(1)+P_{n-1}(0)\right]P_n(\mu)}$$

PROB. 5.44: The formulation in terms of $\psi(r,\theta)=T(r,\theta)-T_0$:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)=0$$

$$\psi(r_1,\theta)=0$$

$$\psi(r_2,\theta)=f(\theta)-T_0$$

Assuming a product solution in the form $\psi(r,\theta)=R(r)\cdot\phi(\theta)$ yields

$$\psi(r,\theta)=\sum_{n=0}^{\infty}A_n\left[C_n r^n+B_n r^{-(n+1)}\right]P_n(\cos\theta)$$

The B.C. at r_1 gives

$$C_n r_1^n+B_n r_1^{-(n+1)}=0 \quad\Rightarrow\quad B_n=-C_n r_1^{2n+1}$$

Then the solution for $\psi(r,\theta)$ can be rewritten as

$$\psi(r,\theta)=\sum_{n=0}^{\infty}a_n\left[\left(\frac{r}{r_1}\right)^n-\left(\frac{r}{r_1}\right)^{-(n+1)}\right]P_n(\cos\theta)$$

The B.C. at r_2 yields

$$f(\theta)-T_0=\sum_{n=0}^{\infty}a_n\left[\left(\frac{r_2}{r_1}\right)^n-\left(\frac{r_2}{r_1}\right)^{-(n+1)}\right]P_n(\cos\theta)$$

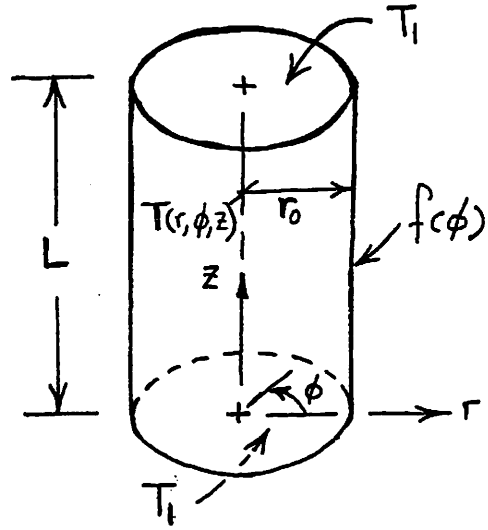
Thus,

$$a_n\left[\left(\frac{r_2}{r_1}\right)^n-\left(\frac{r_2}{r_1}\right)^{-(n+1)}\right]=\frac{2n+1}{2}\int_0^{\pi}\left[f(\theta)-T_0\right]P_n(\cos\theta)\sin\theta\,d\theta$$

Then the solution can be written as

$$\boxed{T(r,\theta)-T_0=\frac{1}{2}\sum_{n=0}^{\infty}(2n+1)\frac{\left[\left(\frac{r}{r_1}\right)^n-\left(\frac{r}{r_1}\right)^{-(n+1)}\right]}{\left[\left(\frac{r_2}{r_1}\right)^n-\left(\frac{r_2}{r_1}\right)^{-(n+1)}\right]}P_n(\cos\theta)\int_0^{\pi}\left[f(\theta')-T_0\right]P_n(\cos\theta')\sin\theta'\,d\theta'}$$

PROB.5.45: a) Formulation in terms of $\theta(r,\phi,z) = T(r,\phi,z) - T_1$:



$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\theta}{\partial\phi^2} + \frac{\partial^2\theta}{\partial z^2} = 0$$

$$\theta(0,\phi,z) = \text{finite}; \quad \theta(r_0,\phi,z) = f(\phi) - T_1$$

$$\theta(r,\phi,0) = 0 \; ; \quad \theta(r,\phi,L) = 0$$

$$\theta(r,\phi,z) = \theta(r,\phi+2\pi,z)$$

$$\frac{\partial\theta(r,\phi,z)}{\partial\phi} = \frac{\partial\theta(r,\phi+2\pi,z)}{\partial\phi}$$

Let $\quad \theta(r,\phi,z) = R(r)\cdot\psi(\phi)\cdot Z(z)$

$$\frac{d^2Z}{dz^2} + \lambda^2 Z = 0$$

$$Z(0) = 0$$

$$Z(L) = 0$$

$$\Downarrow$$

$$Z_n(z) = A_n \sin\lambda_n z$$

$$\lambda_n = \frac{n\pi}{L}, \; n = 1,2,\ldots$$

$$\frac{d^2\psi}{d\phi^2} + \beta^2\psi = 0$$

$$\psi(\phi) = \psi(\phi+2\pi)$$

$$\frac{d\psi(\phi)}{d\phi} = \frac{d\psi(\phi+2\pi)}{d\phi}$$

$$\Downarrow$$

$$\psi_m(\phi) = B_m\cos m\phi + C_m\sin m\phi$$

$$m = 0,1,2,\ldots$$

Thus,

$$\left.\begin{aligned} r^2\frac{d^2R_{mn}}{dr^2} + r\frac{dR_{mn}}{dr} - (m^2 + r^2\lambda_n^2)R_{mn} = 0 \\ R_{mn}(0) = \text{finite} \end{aligned}\right\} \; R_{mn}(r) = D_{mn} I_m(\lambda_n r)$$

$$\therefore\ \theta(r,\phi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} (a_{mn}\cos m\phi + b_{mn}\sin m\phi)\sin\lambda_n z\, I_m(\lambda_n r)$$

$$\Rightarrow\ F(\phi) = f(\phi) - T_1 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} (a_{mn}\cos m\phi + b_{mn}\sin m\phi)\sin\lambda_n z\, I_m(\lambda_n r_0)$$

$$= \sum_{n=1}^{\infty}\left[\sum_{m=0}^{\infty} I_m(\lambda_n r_0)(a_{mn}\cos m\phi + b_{mn}\sin m\phi)\right]\sin\lambda_n z$$

$$\therefore\ \sum_{m=0}^{\infty} I_m(\lambda_n r_0)(a_{mn}\cos m\phi + b_{mn}\sin m\phi) = \frac{2}{L}\int_0^L F(\phi)\sin\lambda_n z\,dz$$

$$= \frac{2}{n\pi}F(\phi)\left[1-(-1)^n\right]$$

$$\frac{2}{n\pi}F(\phi)\left[1-(-1)^n\right] = I_0(\lambda_n r_0)a_{0n} + \sum_{m=1}^{\infty} I_m(\lambda_n r_0)(a_{mn}\cos m\phi + b_{mn}\sin m\phi)$$

$$I_0(\lambda_n r_0)a_{0n} = \frac{1}{2\pi}\int_0^{2\pi}\frac{2}{n\pi}F(\phi)\left[1-(-1)^n\right]d\phi$$

$$= \frac{1}{n\pi^2}\left[1-(-1)^n\right]\int_0^{2\pi}F(\phi)\,d\phi$$

$$I_m(\lambda_n r_0)a_{mn} = \frac{1}{\pi}\int_0^{2\pi}\frac{2}{n\pi}F(\phi)\left[1-(-1)^n\right]\cos m\phi\,d\phi$$

$$= \frac{2}{n\pi^2}\left[1-(-1)^n\right]\int_0^{2\pi}F(\phi)\cos m\phi\,d\phi$$

$$I_m(\lambda_n r_0)b_{mn} = \frac{1}{\pi}\int_0^{2\pi}\frac{2}{n\pi}F(\phi)\left[1-(-1)^n\right]\sin m\phi\,d\phi$$

$$= \frac{2}{n\pi^2}\left[1-(-1)^n\right]\int_0^{2\pi} F(\phi)\sin m\phi\, d\phi$$

Thus,

$$\theta(r,\phi,z) = T(r,\phi,z) - T_1$$

$$= \frac{1}{\pi^2}\int_0^{2\pi} F(\phi)d\phi \cdot \sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\frac{I_0(\frac{n\pi}{L}r)}{I_0(\frac{n\pi}{L}r_0)}\sin\frac{n\pi}{L}z$$

$$+\frac{2}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\frac{I_m(\frac{n\pi}{L}r)}{I_m(\frac{n\pi}{L}r_0)}\sin\frac{n\pi}{L}z$$

$$\cdot\left\{\cos m\phi\int_0^{2\pi}F(\phi')\cos m\phi' d\phi'\right.$$

$$\left.+\sin m\phi\int_0^{2\pi}F(\phi')\sin m\phi' d\phi'\right\}$$

Or,

$$\boxed{\begin{aligned} T(r,\phi,z)-T_1 = \frac{2}{\pi^2}\Bigg\{&\frac{1}{2}\int_0^{2\pi}[f(\phi')-T_1]d\phi' \sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\frac{I_0(\frac{n\pi}{L}r)}{I_0(\frac{n\pi}{L}r_0)}\sin\frac{n\pi}{L}z \\ &+\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\sin\frac{n\pi}{L}z\sum_{m=1}^{\infty}\frac{I_m(\frac{n\pi}{L}r)}{I_m(\frac{n\pi}{L}r_0)}\int_0^{2\pi}[f(\phi')-T_1]\cos m(\phi-\phi')d\phi'\Bigg\}\end{aligned}}$$

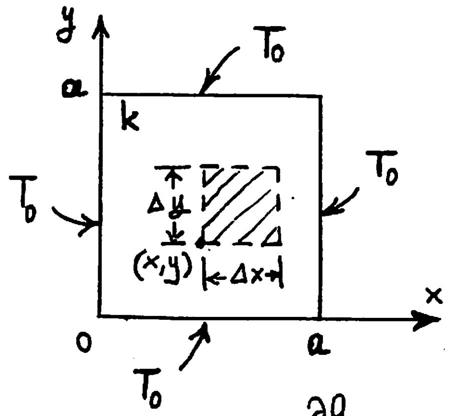
b) Since

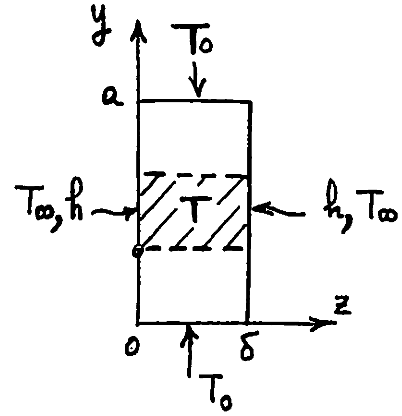
$$\int_0^{2\pi}\cos m(\phi-\phi')d\phi' = 0 \quad , \quad m=1,2,\ldots$$

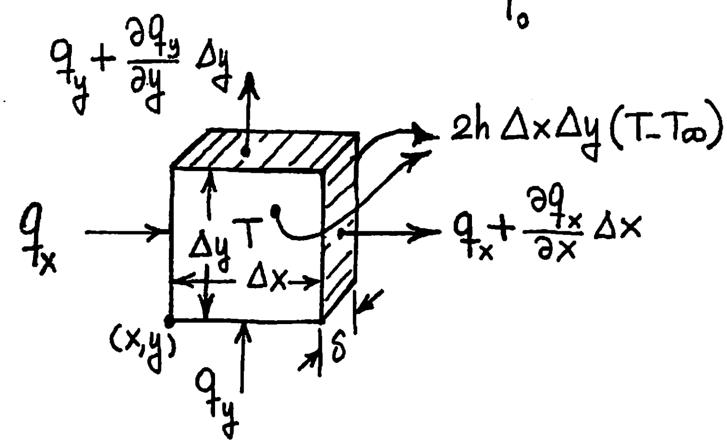
when $f(\phi) = T_0 = $ Const., the solution reduces to

$$T(r,z) - T_1 = (T_0-T_1)\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\frac{I_0(\frac{n\pi}{L}r)}{I_0(\frac{n\pi}{L}r_0)}\sin\frac{n\pi}{L}z$$

PROB. 5.46:







a) Energy balance on the system shown gives:

$$\frac{\partial q_x}{\partial x}\Delta x + \frac{\partial q_y}{\partial y}\Delta y + 2\,h\,\Delta x \Delta y\,(T - T_\infty) = 0$$

$$\frac{\partial}{\partial x}\left(-k\Delta y\delta\frac{\partial T}{\partial x}\right)\Delta x + \frac{\partial}{\partial y}\left(-k\,\Delta x\delta\frac{\partial T}{\partial y}\right)\Delta y + 2h\Delta x\Delta y(T - T_\infty) = 0$$

Assume $k =$ const.

$$\therefore \quad \boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - m^2(T - T_\infty) = 0\,,} \qquad m^2 = \frac{2h}{k\delta}$$

b) Formulation in terms of $\theta(x,y) = T(x,y) - T_\infty$ with $\theta_0 = T_0 - T_\infty$:

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} - m^2\theta = 0$$

$$\theta(0,y) = \theta_0\,; \quad \theta(a,y) = \theta_0$$

$$\theta(x,0) = \theta_0\,; \quad \theta(x,a) = \theta_0$$

Consider the following auxiliary problem:

$$\left\{\begin{array}{l} \dfrac{\partial^2\theta_1}{\partial x^2}+\dfrac{\partial^2\theta_1}{\partial y^2}-m^2\theta_1=0 \\ \theta_1(0,y)=0\ ;\ \ \theta_1(a,y)=0 \\ \theta_1(x,0)=0;\ \ \theta_1(x,a)=\theta_0 \end{array}\right\}$$

Let $\theta_1(x,y)=X(x)\cdot Y(y)$

$$\left.\begin{array}{c} \dfrac{d^2X}{dx^2}+\lambda^2X=0 \\ X(0)=0 \\ X(a)=0 \end{array}\right\} \quad \begin{array}{c} X_n(x)=A_n\sin\lambda_n x \\ \lambda_n=\dfrac{n\pi}{a},\ n=1,2,\dots \end{array}$$

$$\begin{array}{c} \dfrac{d^2Y_n}{dy^2}-(m^2+\lambda_n^2)Y_n=0 \\ Y_n(0)=0 \\ \Downarrow \\ Y_n(y)=B_n\sinh\sqrt{\lambda_n^2+m^2}\,y \end{array}$$

$$\therefore\ \theta_1(x,y)=\sum_{n=1}^{\infty}a_n\sin\lambda_n x\sinh\sqrt{\lambda_n^2+m^2}\,y$$

The B.C. at $y=0$ gives $\Longrightarrow$ $\theta_0=\sum_{n=1}^{\infty}\left[a_n\sinh\sqrt{\lambda_n^2+m^2}\,a\right]\sin\lambda_n x$

$$\therefore\ a_n\sinh\sqrt{\lambda_n^2+m^2}\,a=\frac{2\theta_0}{a}\int_0^a\sin\lambda_n x\,dx=\frac{2\theta_0}{n\pi}\left[1-(-1)^n\right]$$

Thus,

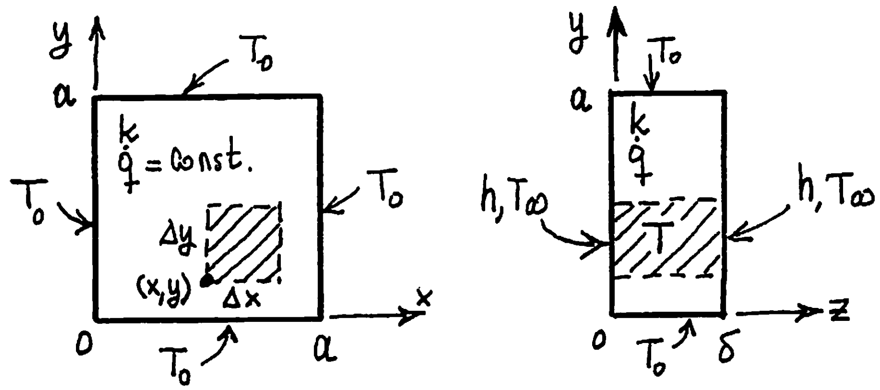
$$\theta_1(x,y)=\frac{2\theta_0}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\,\frac{\sin\lambda_n x\sinh\sqrt{\lambda_n^2+m^2}\,y}{\sinh\sqrt{\lambda_n^2+m^2}\,a}$$

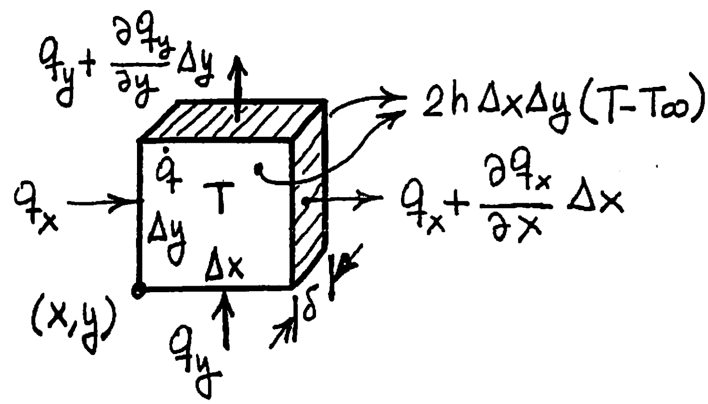
Now, by the method of superposition we obtain

$$\frac{\theta(x,y)}{\theta_0}=\frac{T(x,y)-T_\infty}{T_0-T_\infty}=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n\sinh\sqrt{\lambda_n^2+m^2}\,a}\left\{\sin\lambda_n x\left[\sinh\sqrt{\lambda_n^2+m^2}\,y\right.\right.$$

$$\left.\left.+\sinh\sqrt{\lambda_n^2+m^2}\,(a-y)\right]+\sin\lambda_n y\left[\sinh\sqrt{\lambda_n^2+m^2}\,x+\sinh\sqrt{\lambda_n^2+m^2}\,(a-x)\right]\right\}$$

PROB. 5.47:





a) Energy (thermal) balance on the system shown gives:

$$\cancel{q_x} + \cancel{q_y} + \dot{q}\,\Delta x \Delta y \delta = \cancel{q_x} + \frac{\partial q_x}{\partial x}\Delta x + \cancel{q_y} + \frac{\partial q_y}{\partial y}\Delta y + 2h\,\Delta x \Delta y (T - T_\infty)$$

$$q_x = -k\,\Delta y \delta \frac{\partial T}{\partial x} \quad \text{and} \quad q_y = -k\,\Delta x \delta \frac{\partial T}{\partial y}$$

$$\Downarrow$$

$$\dot{q}\,\Delta x \Delta y \delta = \frac{\partial}{\partial x}\left(-k\,\Delta y \delta \frac{\partial T}{\partial x}\right)\Delta x + \frac{\partial}{\partial y}\left(-k\,\Delta x \delta \frac{\partial T}{\partial y}\right)\Delta y + 2h\,\Delta x \Delta y (T - T_o)$$

$$\Downarrow \text{ Assume } k = \text{const.}$$

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - m^2 (T - T_\infty) + \frac{\dot{q}}{k} = 0,} \quad m^2 = \frac{2h}{k\delta}$$

b) Formulation in terms of

$$\theta(x,y) = T(x,y) - T_\infty - \frac{\dot{q}}{km^2}$$

$$\left\{\begin{array}{l} \dfrac{\partial^2\theta}{\partial x^2} + \dfrac{\partial^2\theta}{\partial y^2} - m^2\theta = 0 \\ \theta(0,y) = \theta_0;\ \theta(a,y) = \theta_0 \\ \theta(x,0) = \theta_0;\ \theta(x,a) = \theta_0 \end{array}\right\}$$

where

$$\theta_0 = T_0 - T_\infty - \frac{\dot{q}}{km^2}$$

Consider the following auxiliary problem:

$$\frac{\partial^2\theta_1}{\partial x^2} + \frac{\partial^2\theta_1}{\partial y^2} - m^2\theta_1 = 0$$

$$\theta_1(0,y) = 0;\ \theta_1(a,y) = 0$$

$$\theta_1(x,0) = 0;\ \theta_1(x,a) = \theta_0$$

The solution to the auxiliary problem can be obtained as (See Prob. 5.46):

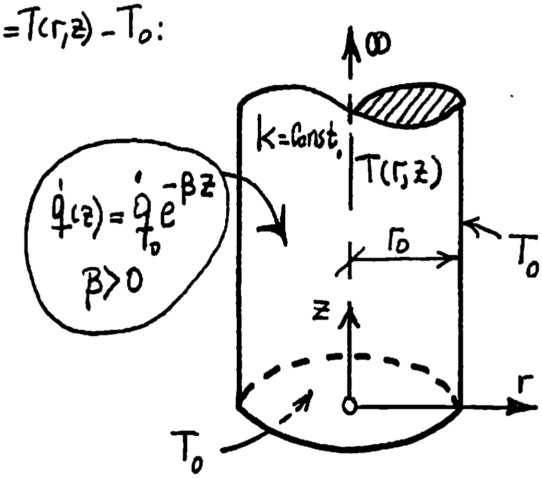
$$\theta_1(x,y) = \frac{2\theta_0}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\,\frac{\sin\lambda_n x \sinh\sqrt{m^2+\lambda_n^2}\,y}{\sinh\sqrt{m^2+\lambda_n^2}\,a}$$

Now, the method of superposition gives

$$\frac{T(x,y) - T_\infty - \dfrac{\dot{q}}{km^2}}{T_0 - T_\infty - \dfrac{\dot{q}}{km^2}} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n\sinh\sqrt{m^2+\lambda_n^2}\,a}\left\{\sin\lambda_n x\left[\sinh\sqrt{m^2+\lambda_n^2}\,y\right.\right.$$

$$\left.\left. + \sinh\sqrt{m^2+\lambda_n^2}\,(a-y)\right] + \sin\lambda_n y\left[\sinh\sqrt{m^2+\lambda_n^2}\,x + \sinh\sqrt{m^2+\lambda_n^2}\,(a-x)\right]\right\}$$

PROB. 5.48: Formulation in terms of $\theta(r,z) = T(r,z) - T_0$:



$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} + \frac{\dot{q}_0}{k} e^{-\beta z} = 0$$

$$\theta(0,z) = \text{finite}; \quad \theta(r_0,z) = 0$$

$$\theta(r,0) = 0; \quad \theta(r,\infty) = 0$$

Let

$$\theta(r,z) = \psi(r,z) + e^{-\beta z}\phi(r)$$

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = 0$$

$$\psi(0,z) = \text{finite}; \quad \psi(r_0,z) = 0$$

$$\psi(r,0) = -\phi(r); \quad \psi(r,\infty) = 0$$

$\Downarrow$ Let $\psi(r,z) = R(r)\cdot Z(z)$

$$\psi(r,z) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n z} J_0(\lambda_n r)$$

where λ_n's are the positive roots of

$$J_0(\lambda_n r_0) = 0, \quad n = 1,2,3,\ldots$$

$$\Downarrow$$

$$A_n = \frac{1}{N_n}\int_0^{r_0}\{-\phi(r)\} J_0(\lambda_n r) r\,dr$$

$$= \cdots = \frac{1}{N_n}\frac{\dot{q}_0}{k}\frac{r_0}{\lambda_n}\frac{J_1(\lambda_n r_0)}{\beta^2 - \lambda_n^2} \quad \text{with } N_n = \frac{r_0^2}{2}J_1^2(\lambda_n r_0)$$

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + \beta^2\phi(r) = -\frac{\dot{q}_0}{k}$$

$$\phi(0) = \text{finite}$$

$$\phi(r_0) = 0$$

$$\phi(r) = B J_0(\beta r) + C Y_0(\beta r) - \frac{\dot{q}_0}{k\beta^2}$$

$$\phi(0) = \text{finite} \Rightarrow C \equiv 0$$

$$\phi(r_0) = 0 \Rightarrow B = \frac{\dot{q}_0}{k\beta^2 J_0(\beta r_0)}$$

$$\Downarrow$$

$$\phi(r) = \frac{\dot{q}_0}{k\beta}\left[\frac{J_0(\beta r)}{J_0(\beta r_0)} - 1\right]$$

$$\therefore\ \theta(r,z) = T(r,z) - T_0 = \frac{\dot{q}_0}{k\beta^2}\left[\frac{J_0(\beta r)}{J_0(\beta r_0)} - 1\right]e^{-\beta z} + \frac{2\dot{q}_0}{r_0 k}\sum_{n=1}^{\infty}\frac{e^{-\lambda_n z}}{\lambda_n(\beta^2-\lambda_n^2)}\frac{J_0(\lambda_n r)}{J_1(\lambda_n r_0)}$$

$$\boxed{T(r,z) - T_0 = \frac{\dot{q}_0}{k}\left\{\left[\frac{J_0(\beta r)}{J_0(\beta r_0)} - 1\right]\frac{e^{-\beta z}}{\beta^2} + \frac{2}{r_0}\sum_{n=1}^{\infty}\frac{e^{-\lambda_n z}}{\lambda_n(\beta^2-\lambda_n^2)}\frac{J_0(\lambda_n r)}{J_1(\lambda_n r_0)}\right\}}$$

Note that the above solution does not converge if $\beta = \lambda_n$, $n = 1, 2, 3, \ldots$

<u>Solution if $\beta = \lambda_n$, $n = 1, 2, 3, \ldots$</u>

First do the following expansion:

$$\phi(r) = \sum_{n=1}^{\infty} B_n J_0(\lambda_n r), \quad 0 \leq r \leq r_0$$

$$\therefore B_n = \frac{1}{N_n}\int_0^{r_0}\phi(r) J_0(\lambda_n r)\, r\, dr = -A_n$$

Thus, the solution can be written as

$$T(r,z) - T_0 = \frac{2\dot{q}_0}{k r_0}\sum_{n=1}^{\infty}\frac{e^{-\lambda_n z} - e^{-\beta z}}{\lambda_n(\beta^2-\lambda_n^2)}\cdot\frac{J_0(\lambda_n r)}{J_1(\lambda_n r_0)}$$

Now, take the following limit,

$$\lim_{\beta\to\lambda_n}\frac{e^{-\lambda_n z} - e^{-\beta z}}{\lambda_n(\beta^2-\lambda_n^2)} = \lim_{\beta\to\lambda_n}\frac{z\, e^{-\beta z}}{2\lambda_n\beta} = \frac{z\, e^{-\lambda_n z}}{2\lambda_n^2}$$

Then, the solution becomes

$$T(r,z) - T_0 = \frac{\dot{q} z}{k r_0}\sum_{n=1}^{\infty}\frac{e^{-\lambda_n z}}{\lambda_n^2}\frac{J_0(\lambda_n r)}{J_1(\lambda_n r_0)}, \quad \beta = \lambda_n,\ n = 1, 2, 3, \ldots$$

PROB. 5.49: Formulation in terms of $\phi(r,\phi) = T(r,\phi) - T_f$:

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\phi^2} = 0$$

$$k\left.\frac{\partial\phi}{\partial r}\right|_{r=r_i} = h\,\phi(r_i,\phi); \quad k\left.\frac{\partial\phi}{\partial r}\right|_{r=r_o} = Q_w''(\phi) = \begin{cases} q_w'', & 0<\phi<\pi \\ 0, & \pi<\phi<2\pi \end{cases}$$

$$\phi(r,\phi) = \phi(r,\phi+2\pi); \quad \frac{\partial\phi(r,\phi)}{\partial\phi} = \frac{\partial\phi(r,\phi+2\pi)}{\partial\phi}$$

Seeking a product solution in the form $\phi(r,\phi) = R(r)\cdot\psi(\phi)$,

$$\lambda_n = n = 0, 1, 2, \ldots$$

$$\psi_n(\phi) = \begin{cases} A_0 + B_0\phi, & n=0 \\ A_n \sin n\phi + B_n \cos n\phi, & n = 1,2,3,\ldots \end{cases}$$

$$R_n(r) = \begin{cases} C_0 \ln r + D_0, & n=0 \\ C_n r^n + D_n r^{-n}, & n = 1,2,3,\ldots \end{cases}$$

The general solution may now be constructed as

$$\phi(r,\phi) = (C_0 \ln r + D_0)(A_0 + B_0\phi) + \sum_{n=1}^{\infty}(C_n r^n + D_n r^{-n})(A_n \sin n\phi + B_n \cos n\phi)$$

Note that $\phi(r,\phi)$ must be single valued. Thus, $B_0 \equiv 0$, and the general solution can be re-arranged as

$$\phi(r,\phi) = (a_0 + b_0 \ln r) + \sum_{n=1}^{\infty}\left[(a_n r^n + b_n r^{-n})\sin n\phi + (c_n r^n + d_n r^{-n})\cos n\phi\right]$$

B.C. at $r=r_i$ gives,

$$\frac{k}{h}\left\{\frac{b_0}{r_i}+\sum_{n=1}^{\infty}\left[na_n r_i^{(n-1)}-nb_n r_i^{-(n+1)}\right]\sin n\phi+\left[nc_n r_i^{(n-1)}-nd_n r_i^{-(n+1)}\right]\cos n\phi\right\}$$

$$=(a_0+b_0\ln r_i)+\sum_{n=1}^{\infty}\left[(a_n r_i^{n}+b_n r_i^{-n})\sin\phi+(c_n r_i^{n}+d_n r_i^{-n})\cos n\phi\right]$$

Thus,

$$\frac{k}{h}\frac{b_0}{r_i}=a_0+b_0\ln r_i \qquad \text{— ①}$$

$$\left(r_i^{n}-n\frac{k}{h}r_i^{(n-1)}\right)a_n+\left(r_i^{-n}+n\frac{k}{h}r_i^{-(n+1)}\right)b_n=0 \qquad \text{— ②}$$

$$\left(r_i^{n}-n\frac{k}{h}r_i^{(n-1)}\right)c_n+\left(r_i^{-n}+n\frac{k}{h}r_i^{-(n+1)}\right)d_n=0 \qquad \text{— ③}$$

B.C. at $r=r_o$ gives,

$$k\left\{\frac{b_0}{r_0}+\sum_{n=1}^{\infty}\left[na_n r_0^{(n-1)}-nb_n r_0^{-(n+1)}\right]\sin n\phi+\left[nc_n r_0^{(n-1)}-nd_n r_0^{-(n+1)}\right]\cos n\phi=Q_w''(\phi)\right.$$

From this we obtain:

$$\frac{kb_0}{r_0}=\frac{1}{2\pi}\int_0^{2\pi}Q_w''(\phi)\,d\phi=\frac{q_w''}{2\pi}\int_0^{\pi}d\phi=\frac{q_w''}{2} \qquad \text{— ④}$$

$$k\left[na_n r_0^{(n-1)}-nb_n r_0^{-(n+1)}\right]=\frac{1}{\pi}\int_0^{2\pi}Q_w''(\phi)\sin n\phi\,d\phi=\ldots=\frac{q_w''}{\pi}\frac{[1-(-1)^n]}{n} \quad ⑤$$

$$k\left[nc_n r_0^{(n-1)}-nd_n r_0^{-(n+1)}\right]=\frac{1}{\pi}\int_0^{2\pi}Q_w''(\phi)\cos n\phi\,d\phi=\ldots=0 \quad ⑥$$

Equations ① & ④ give,

$$b_0=\frac{q_w''r_0}{2k} \quad \text{and} \quad a_0=\frac{q_w''r_0}{2k}\left(\ln r_i-\frac{k}{hr_i}\right)^{-1}$$

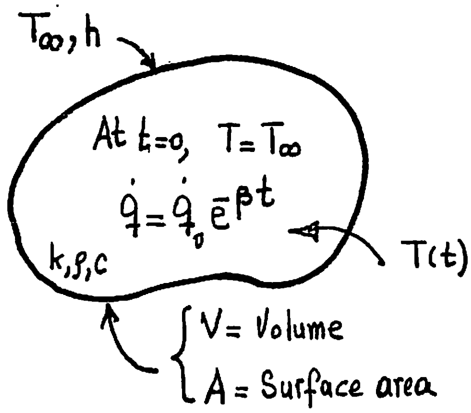
Equations ②, ③, ⑤ & ⑥ yield, for each $n=1,2,3,\ldots$, a set of algebraic set of equation, the solution of which yields a_n, b_n, c_n and d_n, $n=1,2,3,\ldots$

CHAPTER 6

UNSTEADY HEAT CONDUCTION

SOLUTIONS WITH SEPARATION OF VARIABLES

PROB. 6.1: At any instant t:



$$\dot{q}V + hA\left[T_\infty - T(t)\right] = c\rho V \frac{dT}{dt}$$

Let $\theta(t) = T(t) - T_\infty$

$$\frac{d\theta}{dt} + \frac{hA}{\rho V c}\,\theta(t) = \frac{\dot{q}}{\rho c} = \frac{\dot{q}_o}{\rho c} e^{-\beta t}$$

$$T(0) = T_\infty \Rightarrow \theta(0) = 0$$

$$\therefore\ \theta(t) = C_1 e^{-mt} + \theta_p\ , \quad m = \frac{hA}{\rho V c}$$

Particular solution: $\theta_p = C_2 e^{-\beta t} \longrightarrow -C_2\beta e^{-\beta t} + m C_2 e^{-\beta t} = \dfrac{\dot{q}_o}{\rho c} e^{-\beta t}$

$$C_2 = \frac{\dot{q}_o}{\rho c(m-\beta)}$$

$$\theta(t) = C_1 e^{-mt} + \frac{\dot{q}_o}{\rho c\,(m-\beta)} e^{-\beta t}$$

$$\theta(0) = 0 \Rightarrow C_1 + \frac{\dot{q}_o}{\rho c(m-\beta)} = 0 \longrightarrow C_1 = -\frac{\dot{q}_o}{\rho c\,(m-\beta)}$$

$$\theta(t) = T(t) - T_\infty = \frac{\dot{q}_o}{\rho c(m-\beta)}\left[e^{-\beta t} - e^{-mt}\right], \quad \beta \neq m$$

$$\theta(t) = T(t) - T_\infty = \frac{\dot{q}_o}{\rho c}\, t e^{-\beta t}, \quad \beta = m \quad (\text{why?})$$

The maximum temperature will be reached when

$$\frac{dT}{dt} = 0 \quad \Rightarrow \quad t = \begin{cases} \dfrac{1}{m-\beta} \ln \dfrac{m}{\beta}, & \beta \neq m \\ \dfrac{1}{\beta}, & \beta = m \end{cases}$$

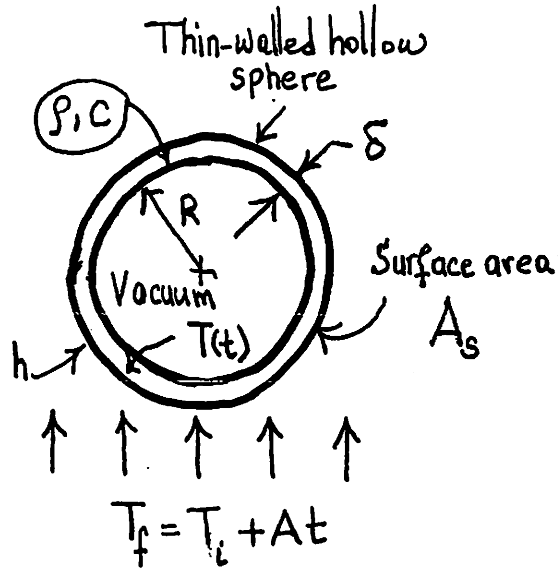
and it will be

$$T_{Max} - T_\infty = \begin{cases} \dfrac{\dot{q}_o}{\rho c\,(m-\beta)}\left[\left(\dfrac{m}{\beta}\right)^{-\frac{\beta}{m-\beta}} - \left(\dfrac{m}{\beta}\right)^{-\frac{m}{m-\beta}}\right], & \beta \neq m \\ \dfrac{\dot{q}_o}{\rho c}\dfrac{1}{\beta e}, & \beta = m \end{cases}$$

PROB. 6.2:

a) Energy balance:

$$h \cancel{A_s}(T_f - T) = \rho c \cancel{A_s} \delta \frac{dT}{dt}$$



$$\left\{ \begin{array}{l} \dfrac{dT}{dt} + \dfrac{h}{\rho c \delta}(T - T_i - At) = 0 \\ T(0) = T_i \end{array} \right\}$$

$$\Downarrow$$

$$\frac{d\theta}{d\tau} + \theta = \tau, \quad \theta = \frac{T - T_i}{mA}, \quad \tau = \frac{t}{m} \quad \& \quad m = \frac{\rho c \delta}{h}$$

$$\theta(0) = 0$$

$$\frac{d}{d\tau}(e^{\tau}\theta) = e^{\tau}\tau \longrightarrow e^{\tau}\theta(\tau) - \cancel{\theta(0)}^{\,0} = \int^{\tau} \tau e^{\tau} d\tau$$

$$= \tau e^{\tau}\Big|_0^{\tau} - \int_0^{\tau} e^{\tau} d\tau$$

$$= \tau e^{\tau} - e^{\tau} + 1$$

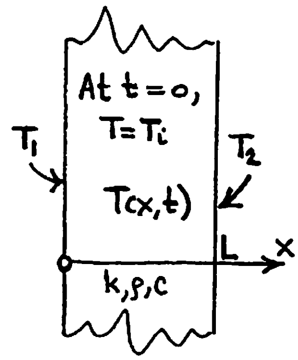
$$\therefore \quad \boxed{\theta(\tau) = \tau - 1 + e^{-\tau}}$$

b) For large times as $t \to \infty$:

$$\theta(\tau) = \tau - 1$$

$$\therefore \quad \frac{T - T_i}{mA} = \frac{t}{m} - 1$$

PROB. 6.3: The solution for $T = T(x,t)$ is given, from Eq. (6.23), by



$$T(x,t) = T_1 - (T_1 - T_2)\frac{x}{L} + \sum_{n=1}^{\infty} A_n\, e^{-\left(\frac{n\pi}{L}\right)^2 \alpha t} \sin \frac{n\pi}{L} x$$

with, from Eq. (6.25),

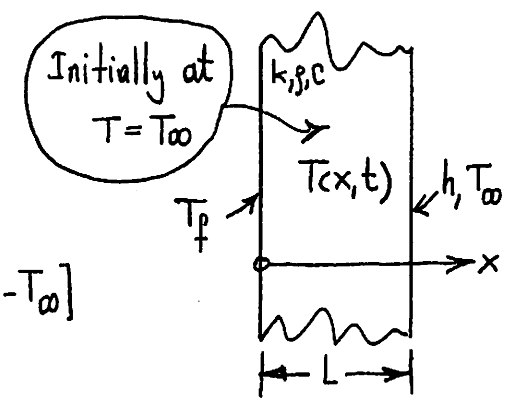
$$A_n = \frac{2}{n\pi}\left[(-1)^n T_2 - T_1\right] + \frac{2}{L}\int_0^L T_i \sin\frac{n\pi}{L} x\, dx$$

$$= \frac{2}{n\pi}\left[(-1)^n T_2 - T_1\right] + \frac{2}{n\pi}\left[1 - (-1)^n\right] T_i$$

Thus,

$$\boxed{T(x,t) = T_1 - (T_1 - T_2)\frac{x}{L} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{[1-(-1)^n]T_i + (-1)^n T_2 - T_1}{n}\, e^{-\left(\frac{n\pi}{L}\right)^2 \alpha t} \sin\frac{n\pi}{L} x}$$

PROB. 6.4: Formulation of the problem:



$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad \alpha = \frac{k}{\rho c}$$

$$T(x,0) = T_\infty$$

$$T(0,t) = T_f\,; \quad -k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h\left[T(L,t) - T_\infty\right]$$

Let $T(x,t) = T_t(x,t) + T_s(x)$

$$\frac{\partial^2 T_t}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T_t}{\partial t} \qquad\qquad \frac{d^2 T_s}{dx^2} = 0$$

$$T_t(x,0) = T_\infty - T_s(x) \qquad\qquad T_s(0) = T_f$$

$$T_t(0,t) = 0\,; \quad -k\left.\frac{\partial T_t}{\partial x}\right|_{x=L} = h\, T_t(L,t) \qquad\qquad -k\frac{dT_s(L)}{dx} = h\left[T_s(L) - T_\infty\right]$$

$$\therefore\ T_s(x) = T_f + Ax\,, \qquad A = -\frac{T_f - T_\infty}{\left(1+\frac{1}{Bi}\right)}\frac{1}{L}\,, \quad Bi = \frac{hL}{k}$$

Now, Let $T_f(x,t) = X(x)\cdot\Gamma(t)$..

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \qquad\qquad \frac{d\Gamma_n}{dt} + \alpha\lambda_n^2\Gamma_n = 0$$

$$X(0) = 0 \qquad\qquad \Downarrow$$

$$h\,X(L) + k\frac{dX(L)}{dx} = 0 \qquad\qquad \Gamma_n(t) = B_n e^{-\alpha\lambda_n^2 t}$$

$$\Downarrow$$

$$X_n(x) = A_n \sin\lambda_n x$$

where λ_n's are the positive roots of $\tan\lambda L = -\dfrac{\lambda L}{Bi}$, $Bi = \dfrac{hL}{k}$.

$$\therefore\ T_t(x,t) = \sum_{n=1}^{\infty} a_n e^{-\alpha\lambda_n^2 t}\sin\lambda_n x$$

$$\Downarrow$$

$$T_\infty - T_s(x) = \sum_{n=1}^{\infty} a_n \sin\lambda_n x$$

From Table 4.1.

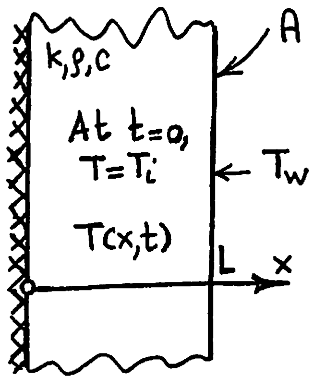
$$\therefore\ a_n = \frac{1}{N_n}\int_0^L \left[T_\infty - T_s(x)\right]\sin\lambda_n x\,dx\,, \qquad N_n = \frac{1}{2\lambda_n}\left[\lambda_n L - \sin\lambda_n L\cos\lambda_n L\right]$$

$$= \cdots = \frac{T_\infty - T_f}{N_n\lambda_n} = \frac{2(T_\infty - T_f)}{[\lambda_n L - \sin\lambda_n L\cos\lambda_n L]}$$

$$\vdots$$

$$\boxed{\frac{T(x,t) - T_f}{T_\infty - T_f} = \frac{Bi}{1+Bi}\frac{x}{L} + 2\sum_{n=1}^{\infty}\frac{e^{-\alpha\lambda_n^2 t}\sin\lambda_n x}{[\lambda_n L - \sin\lambda_n L\cos\lambda_n L]}}$$

PROB. 6.5:



a) Formulation in terms of $\theta(x,t) = T(x,t) - T_w$:

$$\left\{ \begin{array}{c} \dfrac{\partial^2 \theta}{\partial x^2} = \dfrac{1}{\alpha} \dfrac{\partial \theta}{\partial t} \\ \theta(x,0) = T_i - T_w \\ \dfrac{\partial \theta(0,t)}{\partial x} = 0 \; ; \quad \theta(L,t) = 0 \end{array} \right\}$$

Let $\theta(x,t) = X(x) \cdot \Gamma(t)$

$$\left\{ \begin{array}{c} \dfrac{d^2 X}{dx^2} + \lambda^2 X = 0 \\ \dfrac{dX(0)}{dx} = 0 \\ X(L) = 0 \end{array} \right\} \Rightarrow$$

$X_n(x) = A_n \cos \lambda_n x$

where λ_n's are zeros of

$$\cos \lambda L = 0$$

$$\Downarrow$$

$$\lambda_n = \frac{(2n-1)\pi}{2L}, \quad n = 1, 2, 3, \ldots$$

$$\frac{d\Gamma_n}{dt} + \alpha \lambda_n^2 \Gamma_n = 0$$

$$\Downarrow$$

$$\Gamma_n(t) = B_n e^{-\alpha \lambda_n^2 t}$$

$$\therefore \quad \theta(x,t) = \sum_{n=1}^{\infty} a_n e^{-\alpha \lambda_n^2 t} \cos \lambda_n x$$

$$\Rightarrow \quad T_i - T_w = \sum_{n=1}^{\infty} a_n \cos \lambda_n x$$

$$a_n = \frac{2}{L} (T_i - T_w) \int_0^L \cos \lambda_n x \, dx = \frac{2(T_i - T_w)}{L} \left. \frac{\sin \lambda_n x}{\lambda_n} \right|_0^L$$

$$= \frac{4 (T_i - T_w)(-1)^{n+1}}{(2n-1)\pi}$$

$$\boxed{\frac{\theta(x,t)}{T_i - T_w} = \frac{T(x,t) - T_w}{T_i - T_w} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} e^{-\alpha \lambda_n^2 t} \cos \lambda_n x}$$

b)

$$T_m \cancel{\rho}\cancel{c} L \cancel{A} = \int_0^L T \cancel{\rho}\cancel{c}\cancel{A}\,dx \quad \Rightarrow \quad T_m(t) = \frac{1}{L}\int_0^L T(x,t)\,dx$$

$$\Downarrow$$

$$\boxed{\frac{T_m(t)-T_w}{T_i-T_w} = \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{e^{-\alpha\lambda_n^2 t}}{(2n-1)^2}} \quad \Leftarrow \quad \frac{T_m(t)-T_w}{T_i-T_w} = \frac{1}{L}\int_0^L \theta(x,t)\,dx$$

c)

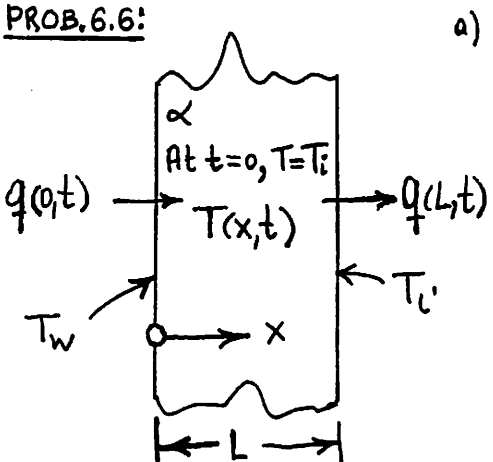
$$q(t) = -kA\left.\frac{\partial T}{\partial x}\right|_{x=L} \quad \Rightarrow \quad \boxed{q(t) = 2kA\,\frac{T_i-T_w}{L}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}}$$

d)

$$Q(t) = \int_0^t q(t)\,dt$$

$$\boxed{\frac{Q(t)}{Q_i} = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1-e^{-\alpha\lambda_n^2 t}}{(2n-1)^2}, \quad Q_i = \rho c A L (T_i - T_w)}$$

PROB. 6.6:



a) Formulation of the problem in terms of $\theta(x,t) = T(x,t) - T_w$:

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta(x,0) = \theta_i \qquad \boxed{\theta_i' = T_i' - T_w}$$

$$\theta(0,t) = 0; \quad \theta(L,t) = \theta_i'$$

Let $\theta(x,t) = \theta_t(x,t) + \theta_s(x)$

$$\frac{\partial^2\theta_t}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta_t}{\partial t} \qquad\qquad \frac{d^2\theta_s}{dx^2} = 0$$

$$\theta_t(x,0) = \theta_i - \theta_s(x) \qquad\qquad \theta_s(0) = 0; \quad \theta_s(L) = \theta_i$$

$$\theta_t(0,t) = 0; \quad \theta_t(L,t) = 0 \qquad\qquad \therefore \ \theta_s(x) = \theta_i\frac{x}{L}$$

$$\therefore \quad \theta_t(x,t) = \sum_{n=1}^{\infty} A_n \, e^{-\alpha \lambda_n^2 t} \sin\lambda_n x \quad ; \quad \lambda_n = \frac{n\pi}{L}, \quad n = 1,2,3,\ldots$$

$$\Downarrow$$

$$\theta_i\left[1-\frac{x}{L}\right] = \sum_{n=1}^{\infty} A_n \sin\lambda_n x \quad \Rightarrow \quad A_n = \frac{2\theta_i}{L}\int_0^L \left[1-\frac{x}{L}\right]\sin\lambda_n x\,dx = \frac{2\theta_i}{n\pi}$$

$$\therefore \quad \boxed{\frac{\theta(x,t)}{\theta_i} = \frac{x}{L} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} e^{-\alpha\lambda_n^2 t} \sin\lambda_n x}$$

b)

$$\cancel{\rho}\cancel{c} L T_m = \int_0^L \cancel{\rho}\cancel{c}\, T\,dx \quad \Rightarrow \quad T_m = \frac{1}{L}\int_0^L T\,dx$$

$$\therefore \quad \boxed{\frac{T_m - T_w}{T_i - T_w} = \frac{1}{2} + \frac{L}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\left[1-(-1)^n\right] e^{-\alpha\lambda_n^2 t}}$$

c)

$$q(0,t) = -k\left.\frac{\partial T}{\partial x}\right|_{x=0} = -\frac{k\theta_i}{L}\left[1 + 2\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\right]$$

$$q(L,t) = -k\left.\frac{\partial T}{\partial x}\right|_{x=L} = -\frac{k\theta_i}{L}\left[1 + 2\sum_{n=1}^{\infty} (-1)^n e^{-\alpha\lambda_n^2 t}\right]$$

<u>PROB. 6.7:</u> Formulation in terms of $\theta(r,t) = T(r,t) - T_w$:

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \quad , \quad \alpha = \frac{k}{\rho c}$$

$$\theta(r,0) = T_i - T_w$$

$$\theta(0,t) = \text{finite} \; ; \quad \theta(r_o,t) = 0$$

Let $\theta(r,t) = R(r)\cdot\Gamma(t)$

$$\Downarrow$$

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+\lambda^2r^2R=0 \qquad\qquad \frac{d\Gamma_n}{dt}+\alpha\lambda_n^2\Gamma_n=0$$

$$R(0)=\text{finite} \qquad\qquad \Downarrow$$

$$R(r_0)=0 \qquad\qquad \Gamma_n(t)=B_n\,e^{-\alpha\lambda_n^2 t}$$

$$\Downarrow$$

$R_n(r)=A_nJ_0(\lambda_n r)$, where λ_n's are zeros of $J_0(\lambda r_0)=0$.

$$\Downarrow$$

$$\theta(r,t)=\sum_{n=1}^{\infty}a_n e^{-\alpha\lambda_n^2t}J_0(\lambda_n r)$$

$$\therefore\ T_i-T_w=\sum_{n=1}^{\infty}a_nJ_0(\lambda_n r)$$

$$\Rightarrow\ a_n=\frac{1}{N_n}\int_0^{r_0}(T_i-T_w)J_0(\lambda_n r)\,r\,dr=\frac{T_i-T_w}{N_n}\frac{r_0}{\lambda_n}J_1(\lambda_n r_0)\ ;\ N_n=\frac{r_0^2}{2}J_1^2(\lambda_n r_0)$$

$$\therefore\ \boxed{\frac{\theta(r,t)}{T_i-T_w}=\frac{T(r,t)-T_w}{T_i-T_w}=\frac{2}{r_0}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{\lambda_nJ_1(\lambda_n r_0)}e^{-\alpha\lambda_n^2t}}$$

b)

$$\pi r_0^2\rho cT_m=\int_0^{r_0}\rho cT\,2\pi r\,dr\ \Rightarrow\ T_m=\frac{2}{r_0^2}\int_0^{r_0}Tr\,dr$$

$$\Rightarrow\ \boxed{\frac{T_m-T_w}{T_i-T_w}=\frac{4}{r_0^2}\sum_{n=1}^{\infty}\frac{e^{-\alpha\lambda_n^2t}}{\lambda_n^2}}$$

c)

$$q'=-k\,2\pi r_0\left(\frac{\partial T}{\partial r}\right)_{r=r_0}=4\pi k(T_i-T_w)\sum_{n=1}^{\infty}e^{-\alpha\lambda_n^2t}$$

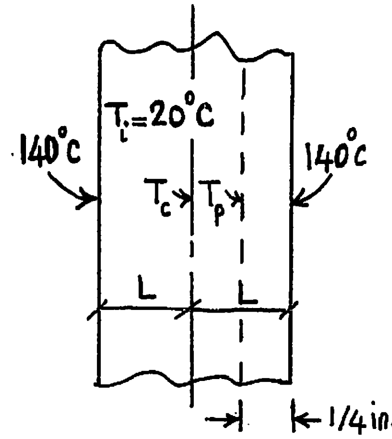
d)

$$Q=\int_0^t q'dt\ \Rightarrow\ \frac{Q}{Q_i}=\frac{4}{r_0^2}\sum_{n=1}^{\infty}\frac{1-e^{-\alpha\lambda_n^2t}}{\lambda_n^2}\ ,\ Q_i=\pi r_0^2\rho c(T_i-T_\infty)$$

PROB. 6.11:

$$2L = 1\text{ in.} \Rightarrow L = 0.5\text{ in}$$

$$\alpha = 8.8\times10^{-8}\ m^2/s$$



Since the surface temperature is specified,

$$Bi \to \infty \quad \text{or} \quad \frac{1}{Bi} = 0$$

a)

$$\frac{T_c - 140}{20-140} = \frac{130-140}{20-140} = 0.083 \Rightarrow \text{From Fig. 6.9} \quad \frac{\alpha t}{L^2} = 1.1$$

$$\therefore\ t = 1.1\frac{L^2}{\alpha} = 1.1\times\frac{(0.5\times2.54\times10^{-2})^2}{8.8\times10^{-8}}$$

$$= 2016\text{ sec} = \underline{33\text{ min } 36\text{ sec}}$$

b)

$$\frac{T_p-140}{T_c-140} \cong 0.72 \longleftarrow \text{From Fig. 6.10 with } \frac{x}{L} = 0.5$$

$$\therefore\ T_p = 140 + 0.72\,(130-140) = 140-7.2$$

$$= \underline{132.8°C}$$

c)

$$\frac{T_p-140}{T_c-140} = \frac{130-140}{T_c-140} = 0.72 \Rightarrow T_c = 140 - \frac{10}{0.72} = 126.1°C$$

$$\frac{T_c-140}{20-140} = \frac{13.9}{120} = 0.116 \Rightarrow \text{From Fig. 6.9} \quad \frac{\alpha t}{L^2} = 1$$

$$\therefore\ t = \frac{L^2}{\alpha} = 1833\text{ Sec} = \underline{30\text{ min } 32\text{ Sec}}$$

PROB. 6.14: Since $T_o = T_s$ at all times, the copper sphere is a lumped-heat capacity system. Thus,

$$\frac{T-T_\infty}{T_i-T_\infty} = \exp\left(-\frac{hA}{\rho c V}t\right) \longrightarrow h = -\frac{\rho c V}{At}\ln\frac{T-T_\infty}{T_i-T_\infty}$$

At $t = 5\,min$, $T = 50°C$.

$$\therefore h = -\frac{(8990)\times(381)\times(\frac{4}{3}\pi r_o^3)}{(4\pi r_o^2)\times(300)}\ln\frac{50-20}{100-20}$$

$$= 112\ W/(m^2.K)$$

At $t = 10\,min$, $T = 31.2°C$.

$$\therefore h = -\frac{(\ )(\ )(\ )}{(\ )\times(600)}\ln\frac{31.2-20}{100-20}$$

$$= 112.25\ W/(m^2.K)$$

$$\underline{h \cong 112.12\ W/(m^2.K)}$$

Now, consider the rubber sphere: The same h applies. (why?)

$$Bi = \frac{hr_o}{k} = \frac{(112.12)\times(0.03)}{0.43} = 7.8 \leftarrow \text{Distributed system}$$

$$Fo = \frac{\alpha t}{r_o^2} = \frac{(0.43)}{(1070)(1630)}\times\frac{(30\times 60)}{(0.03)^2} = 0.493$$

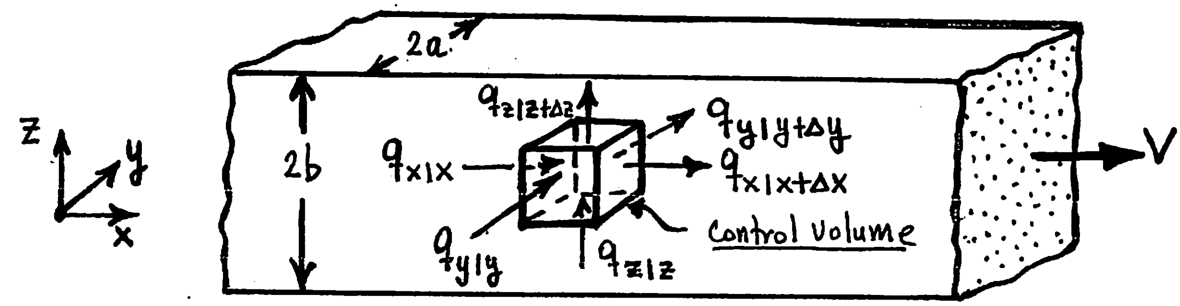
$$\left.\begin{matrix} Fo = 0.493 \\ 1/Bi = 0.128 \end{matrix}\right\} \xrightarrow{\text{Fig. 6.16}} \frac{T_c-T_\infty}{T_i-T_\infty} = 0.045$$

$$\therefore T_c = 0.045\,(100-20)+20 = \underline{23.6°C}$$

$$\text{Fig. 6.17} \Longrightarrow \frac{T_s-T_\infty}{T_c-T_\infty} = 0.25$$

$$\therefore T_s = 0.25\,(23.6-20)+20 = \underline{20.9°C}$$

PROB. 6.16: Consider a control volume of infinitesimal dimensions Δx, Δy and Δz as shown.



where

$$q_{x|x} = -k \underbrace{\cancel{\Delta y \Delta z \frac{\partial T}{\partial x}}}_{\text{Negligible}} + Tc\rho V \Delta y \Delta z \cong Tc\rho V \Delta y \Delta z$$

$$q_{y|y} = -k\, \Delta x \Delta z \frac{\partial T}{\partial y}$$

$$q_{z|z} = -k\, \Delta x \Delta y \frac{\partial T}{\partial z}$$

Energy balance on the c.v. gives:

$$q_{x|x} + q_{y|y} + q_{z|z} = q_{x|x+\Delta x} + q_{y|y+\Delta y} + q_{z|z+\Delta z}$$

$$= q_{x|x} + \frac{\partial q_{x|x}}{\partial x}\Delta x + q_{y|y} + \frac{\partial q_{y|y}}{\partial y}\Delta y + q_{z|z} + \frac{\partial q_{z|z}}{\partial z}\Delta z$$

$$\Downarrow$$

$$\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{V}{\alpha}\frac{\partial T}{\partial x} = \frac{V}{\alpha}\frac{\partial T}{\partial t}\underbrace{\frac{dt}{dx}}_{1/V} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Note that $T(x,y,z) \equiv T(y,z,t)$ with $x = Vt$.

Thus,

$$\underbrace{\left\{\frac{T(y,z,t) - T_\infty}{T_i - T_\infty}\right\}_{\substack{\text{bar}\\ 2a\times 2b}}}_{\phi(y,z,t)} = \underbrace{\left\{\frac{T(y,t) - T_\infty}{T_i - T_\infty}\right\}_{\substack{\text{slab}\\ 2a}}}_{\phi_1(y,t)} \times \underbrace{\left\{\frac{T(z,t) - T_\infty}{T_i - T_\infty}\right\}_{\substack{\text{slab}\\ 2b}}}_{\phi_2(z,t)}$$

$$\underbrace{\frac{150-25}{500-25}}_{\phi(0,0,t)} = 0.263 = \underbrace{\left[\frac{T(0,t)-25}{500-25}\right]_{\substack{slab\\2a}}}_{\phi_1(0,t)} \times \underbrace{\left[\frac{T(0,t)-25}{500-25}\right]_{\substack{slab\\2b}}}_{\phi_2(0,t)}$$

Also

$$\alpha = \frac{k}{\rho c} = \frac{230}{2707 \times 896} = 9.48 \times 10^{-5}\ m^2/s$$

$$\frac{k}{ha} = \frac{230}{5000 \times \underbrace{2.5 \times 10^{-2}}_{a}} = 1.84 \quad \& \quad \frac{k}{h\underbrace{b}_{1.25\times10^{-2} m}} = 3.68$$

Construct the following iteration table using Fig. 6.9:

t (sec)	$\frac{\alpha t}{a^2}$	$\phi_1(0,t)$	$\frac{\alpha t}{b^2}$	$\phi_2(0,t)$	$\phi_1 \cdot \phi_2$
10	1.52	0.52	6.07	0.24	0.1248
5	0.76	0.8	3.04	0.41	0.328
7	1.06	0.7	4.25	0.37	$0.259 \cong 0.263$

$$\therefore t \cong 7 \text{ sec} \implies \underline{\underline{L \cong 0.5 \times 7 = 3.5\ m}}$$

b) The maximum surface temperature that can be reached after the bars emerge from the cooling thank would be the mean temperature at $L = 3.5$ m ($t = 7$ sec). Since the cross-section is rectangular,

$$\bar{T}(t) = \frac{1}{ab}\int_0^a\int_0^b T(y,z,t)\,dy\,dz \longrightarrow \underbrace{\frac{\bar{T}(t)-T_\infty}{T_i - T_\infty}}_{\bar{\phi}(t)} = \frac{1}{ab}\int_0^a\int_0^b \phi\,dy\,dz$$

$$\therefore \bar{\phi}(t) = \underbrace{\frac{1}{a}\int_0^a \phi_1(y,t)\,dy}_{\bar{\phi}_1(t)} \cdot \underbrace{\frac{1}{b}\int_0^b \phi_2(z,t)\,dz}_{\bar{\phi}_2(t)} = \bar{\phi}_1(t) \cdot \bar{\phi}_2(t)$$

Now construct the following "numerical integration" table by making use of Fig. 6.10:

$\frac{y}{a}$ or $\frac{z}{b}$	$\phi_1(y,t)/\phi_1(0,t)$ *	$\phi_2(z,t)/\phi_2(0,t)$ **	$\phi_1(y,t)$	$\phi_2(z,t)$
0.0	1.0	1.0	0.7	0.37
0.2	0.98	0.99	0.686	0.366
0.4	0.96	0.97	0.672	0.359
0.6	0.92	0.95	0.644	0.352
0.8	0.84	0.93	0.588	0.344
1.0	0.78	0.88	0.546	0.326

Means:

$$\bar{\phi}_1 \cong \frac{1}{5}\left[\frac{0.7+0.686}{2}+\frac{0.686+0.672}{2}+\cdots+\frac{0.588+0.546}{2}\right] = 0.643$$

$$\bar{\phi}_2 \cong \frac{1}{5}\left[\frac{0.37+0.366}{2}+\frac{0.366+0.359}{2}+\cdots+\frac{0.344+0.326}{2}\right] = 0.354$$

$$\therefore \bar{\phi} = 0.643 \times 0.354 = 0.228$$

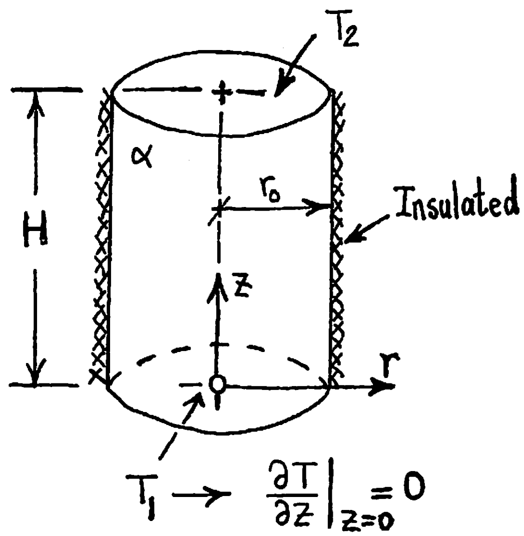
$$\Rightarrow T_m = 0.228\,(T_i - T_\infty) + T_\infty = 0.228\,(500-25)+25 \cong \underline{\underline{133^\circ C}}$$

---

* Read from Fig. 6.10 with $\frac{k}{ha} = 1.84$. Also note that $\phi_1(0,t) = 0.7$ from Part (a).

** Read from Fig. 6.10 with $\frac{k}{hb} = 3.68$. Also note that $\phi_2(0,t) = 0.37$ from Part (a)

PROB. 6.19:



Formulation of the problem:

$$\frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(z,0) = T_1 + \frac{T_2 - T_1}{H} z$$

$$\frac{\partial T(0,t)}{\partial z} = 0 \; ; \; T(H,t) = T_2$$

Note that the temperature distribution will not depend on r. (why?)

$$\therefore \; T(z,t) - T_2 = \sum_{n=1}^{\infty} A_n e^{-\alpha \lambda_n^2 t} \cos \lambda_n x$$

where λ_n's are the positive roots of $\cos \lambda H = 0$.

$$\therefore \; \lambda_n = \frac{2n-1}{H}\frac{\pi}{2}, \quad n = 1, 2, 3, \ldots$$

Applying the initial condition yields

$$(T_1 - T_2)\left(1 - \frac{z}{H}\right) = \sum_{n=1}^{\infty} A_n \cos \lambda_n z$$

$$\Longrightarrow \; A_n = \frac{2}{H}(T_1 - T_2)\int_0^H \left(1 - \frac{z}{H}\right) \cos \lambda_n z \, dz = \frac{2(T_1 - T_2)}{H^2 \lambda_n^2}$$

$$\therefore \; \boxed{\frac{T(z,t) - T_2}{T_1 - T_2} = \frac{2}{H^2} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} e^{-\alpha \lambda_n^2 t} \cos \lambda_n z}$$

PROB. 6.20: Formulation of the problem:

$$\left\{ \begin{array}{c} \dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t} \\ T(x,0) = T_i \\ \left.\dfrac{\partial T}{\partial x}\right|_{x=0} = 0 \; ; \quad k\left.\dfrac{\partial T}{\partial x}\right|_{x=L} = q_w'' \end{array} \right\}$$

Let $T(x,t) = \psi(x,t) + \phi(x) + \Gamma(t)$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \psi}{\partial t}$$

$$\psi(x,0) = -\phi(x)$$

$$\frac{\partial \psi(0,t)}{\partial x} = 0 \; ; \quad \frac{\partial \psi(L,t)}{\partial x} = 0$$

$$\frac{d^2\phi}{dx^2} = \frac{1}{\alpha}\frac{d\Gamma}{dt} = \text{const} = C$$

$$\Gamma(0) = T_i$$

$$\frac{d\phi(0)}{dx} = 0 \; ; \quad \frac{d\phi(L)}{dx} = \frac{q_w''}{k}$$

$$\frac{d\Gamma}{dt} = \alpha C \;\longrightarrow\; \Gamma(t) = \alpha C t + D \quad \& \quad \Gamma(0) = T_i \;\longrightarrow\; D = T_i$$

$$\therefore \; \Gamma(t) = \alpha C t + T_i$$

$$\frac{d^2\phi}{dx^2} = C \;\longrightarrow\; \phi(x) = \frac{Cx^2}{2} + Ax + B$$

$$\frac{d\phi(0)}{dx} = 0 \;\longrightarrow\; A = 0$$

$$\frac{d\phi(L)}{dx} = \frac{q_w''}{k} \;\longrightarrow\; C = \frac{q_w''}{kL}$$

Thus,

$$\phi(x) = \frac{q_w''}{2kL}x^2 + B$$

$$\Gamma(t) = \frac{\alpha q_w''}{kL}t + T_i$$

On the other hand, let $\psi(x,t) = X(x) . \Omega(t)$.

$$\left\{ \begin{array}{l} \dfrac{d^2X}{dx^2} + \lambda^2 X = 0 \\ \dfrac{dX(0)}{dx} = 0 \\ \dfrac{dX(L)}{dx} = 0 \end{array} \right\} \quad X_n(x) = a_n \cos\lambda_n x$$

where

$$\lambda_n = \frac{n\pi}{L}, \; n = 0,1,2,\ldots$$

$$\frac{d\Omega_n}{dt} + \alpha\lambda_n^2 \Omega_n = 0$$

$$\downarrow$$

$$\Omega_n(t) = b_n e^{-\alpha\lambda_n^2 t}$$

$$\therefore \quad \psi(x,t) = C_0 + \sum_{n=1}^{\infty} C_n e^{-\alpha\lambda_n^2 t} \cos\lambda_n x$$

$$\Downarrow$$

$$-\phi(x) = -\frac{q_w''}{2kL} x^2 - B = C_0 + \sum_{n=1}^{\infty} C_n \cos\lambda_n x$$

$$\Downarrow$$

$$C_0 + B = \frac{1}{N_0} \int_0^L \left(-\frac{q_w''}{2kL} x^2 \right) dx, \quad N_0 = L$$

$$= -\frac{q_w'' L}{6k}$$

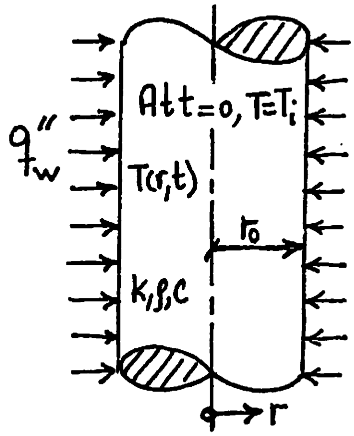
$$C_n = \frac{1}{N_n} \int_0^L \left(-\frac{q_w''}{2kL} x^2 \right) \cos\lambda_n x \, dx, \quad N_n = \frac{L}{2}, n = 1,2,\ldots$$

$$= \ldots = -\frac{2 q_w'' L}{k} \frac{(-1)^n}{(n\pi)^2}$$

$$\vdots$$

$$\boxed{T(x,t) - T_i = \frac{q_w'' L}{k} \left[\frac{\alpha t}{L^2} + \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{6} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} e^{-\alpha\lambda_n^2 t} \cos\lambda_n x \right]}$$

PROB.6.21:



Formulation of the problem:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(r,0) = T_i$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0\ ; \quad k\left(\frac{\partial T}{\partial r}\right)_{r=r_o} = q''_w$$

Based on the physics of the problem, let

$$T(r,t) = \psi(r,t) + \phi(r) + \Omega(t)$$

which yields

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} = \frac{1}{\alpha}\frac{\partial \psi}{\partial t} \qquad \Bigg| \qquad \frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} = \frac{1}{\alpha}\frac{d\Omega}{dt} = \text{const} = C_1$$

$$\psi(r,0) = -\phi(r) \qquad \Bigg| \qquad \Omega(0) = T_i$$

$$\frac{\partial \psi(0,t)}{\partial r} = 0\ ; \ \frac{\partial \psi(r_o,t)}{\partial r} = 0 \qquad \Bigg| \qquad \frac{d\phi(0)}{dr} = 0\ ; \ \frac{d\phi(r_o)}{dr} = \frac{q''_w}{k}$$

$$\frac{d\Omega}{dt} = \alpha C_1 \rightarrow \Omega(t) = \alpha C_1 t + C_2 \quad \& \quad \Omega(0) = T_i \rightarrow C_2 = T_i$$

$$\Omega(t) = \alpha C_1 t + T_i$$

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} = C_1 \longrightarrow \phi(r) = C_1\frac{r^2}{4} + C_3 \ln r + C_4$$

$$\frac{d\phi(0)}{dr} = 0 \rightarrow C_3 = 0$$

$$\frac{d\phi(r_o)}{dr} = \frac{q''_w}{k} \rightarrow C_1 = \frac{2q''_w}{r_o k}$$

Thus,

$$\phi(r) = \frac{q''_w}{2r_o k} r^2 + C_4$$

$$\Omega(t) = \frac{2\alpha q''_w}{r_o k} t + T_i$$

$$\left\{\begin{array}{l} \dfrac{\partial^2\psi}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial\psi}{\partial r}=\dfrac{1}{\alpha}\dfrac{\partial\psi}{\partial t} \\ \psi(r,0)=-\dfrac{q_w''}{2r_0 k}r^2-C_4 \\ \dfrac{\partial\psi(0,t)}{\partial r}=0\,,\quad \dfrac{\partial\psi(r_0,t)}{\partial r}=0 \end{array}\right\} \quad \text{Let } \overline{\psi}(r,t)=\psi(r,t)+C_4=R(r)\cdot\Gamma(t)$$

Thus,

$$\left\{\begin{array}{l} \dfrac{d^2R}{dr^2}+\dfrac{1}{r}\dfrac{dR}{dr}+\lambda^2R=0 \\ \dfrac{dR(0)}{dr}=0,\quad \dfrac{dR(r_0)}{dr}=0 \end{array}\right\} \qquad R_n(r)=A_n J_0(\lambda_n r)$$

Where λ_n's are the positive roots of

$$J_1(\lambda r_0)=0\quad,\quad n=0,1,2,\ldots$$

Note that $\lambda_0=0$ is an eigenvalue.

$$\frac{d\Gamma_n}{dt}+\alpha\lambda_n^2\Gamma_n=0 \longrightarrow \Gamma_n(t)=B_n\exp(-\alpha\lambda_n^2 t)$$

$$\therefore\ \overline{\psi}(r,t)=C_0+\sum_{n=1}^{\infty}C_n e^{-\alpha\lambda_n^2 t}J_0(\lambda_n r)$$

$$\Rightarrow\ -\frac{q_w''}{2r_0K}r^2=C_0+\sum_{n=1}^{\infty}C_n J_0(\lambda_n r)$$

$$C_0=-\frac{2}{r_0^2}\int_0^{r_0}\frac{q_w''}{2r_0 k}r^2\cdot r\,dr=-\frac{q_w'' r_0}{4k}$$

$$C_n=-\frac{1}{N_n}\int_0^{r_0}\frac{q_w''}{2r_0 k}r^2 J_0(\lambda_n r)\,r\,dr=-\frac{2q_w''}{r_0 k\lambda_n^2 J_0(\lambda_n r_0)}\,,\quad N_n=\frac{r_0^2}{2}J_0^2(\lambda_n r_0)$$

$$\therefore\ T(r,t)=\psi(r,t)+\phi(r)+\Omega(t)$$

$$=-\frac{q_w'' r_0}{4k}-\frac{2q_w''}{r_0 k}\sum_{n=1}^{\infty}e^{-\alpha\lambda_n^2 t}\frac{J_0(\lambda_n r)}{\lambda_n^2 J_0(\lambda_n r_0)}-\cancel{C_4}$$

$$+\frac{q_w''}{2r_0 k}r^2+\cancel{C_4}+\frac{2\alpha q_w''}{r_0 k}t+T_i$$

$$\boxed{T(r,t)-T_i=\frac{q_w''}{kr_0}\left[\frac{r^2}{2}+2\alpha t-\frac{r_0^2}{4}-2\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{\lambda_n^2 J_0(\lambda_n r_0)}e^{-\alpha\lambda_n^2 t}\right]}$$

PROB. 6.22: Formulation in terms of $\theta(r,t) = T(r,t) - T_i$:

$$\frac{\partial^2\theta}{\partial r^2} + \frac{2}{r}\frac{\partial\theta}{\partial r} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta(r,0) = 0$$

$$\theta(0,t) = \text{finite}; \quad k\frac{\partial\theta}{\partial r}\bigg|_{r=r_0} = q_w''$$

$$\Big\Vert$$

$$\underline{\overline{U(r,t) = r\cdot\theta(r,t)}}$$

$$\Downarrow$$

$$\frac{\partial^2 U}{\partial r^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t}$$

$$U(r,0) = 0$$

$$U(0,t) = 0; \quad \frac{\partial U}{\partial r}\bigg|_{r=r_0} - \frac{U(r_0,t)}{r_0} = r_0\frac{q_w''}{k}$$

$$\Big\Vert$$

$$\underline{\overline{U(r,t) = \psi(r,t) + \phi(r) + r\,\Omega(t)}}$$

$$\Downarrow$$

$$\frac{\partial^2\psi}{\partial r^2} = \frac{1}{\alpha}\frac{\partial\psi}{\partial t}$$

$$\psi(r,0) = -\phi(r)$$

$$\psi(0,t) = 0; \quad \frac{\partial\psi}{\partial r}\bigg|_{r=r_0} - \frac{\psi(r_0,t)}{r_0} = 0$$

$$\frac{1}{r}\frac{d^2\phi}{dr^2} = \frac{1}{\alpha}\frac{d\Omega}{dt} = \text{const.} = C$$

$$\Omega(0) = 0$$

$$\phi(0) = 0; \quad \frac{d\phi(r_0)}{dr} - \frac{\phi(r_0)}{r_0} = r_0\frac{q_w''}{k}$$

$$\Omega(t) = \alpha C t + \cancel{A}^{\,0} \longrightarrow \Omega(t) = \alpha C t$$

$$\phi(r) = C\frac{r^3}{6} + Br + \cancel{D}^{\,0} \longrightarrow \left(\frac{C}{2}r_0^2 + \cancel{B}\right) - \left(\frac{Cr_0^2}{6} + \cancel{B}\right) = r_0\frac{q_w''}{k}$$

$$C = \frac{3q_w''}{kr_0}$$

$$\Omega(t) = \frac{3\alpha q''_w}{k r_0} t$$

$$\phi(r) = \frac{3 q''_w}{k r_0} \frac{r^3}{6} + Br = \frac{q''_w r^3}{2 k r_0} + Br$$

$$\frac{\partial^2 \psi}{\partial r^2} = \frac{1}{\alpha} \frac{\partial \psi}{\partial t}$$

$$\psi(r,0) = -\frac{q''_w r^3}{2k r_0} - Br$$

$$\psi(0,t) = 0 \; ; \quad \left.\frac{\partial \psi}{\partial r}\right|_{r=r_0} - \frac{\psi(r_0,t)}{r_0} = 0$$

$$\Downarrow \quad \boxed{\bar{\psi}(r,t) = \psi(r,t) + Br}$$

$$\frac{\partial^2 \bar{\psi}}{\partial r^2} = \frac{1}{\alpha} \frac{\partial \bar{\psi}}{\partial t}$$

$$\bar{\psi}(r,0) = -\frac{q''_w r^3}{2k r_0}$$

$$\bar{\psi}(0,t) = 0 \; ; \quad \left.\frac{\partial \bar{\psi}}{\partial r}\right|_{r=r_0} - \frac{\bar{\psi}(r_0,t)}{r_0} = 0$$

$$\Downarrow$$

$$\bar{\psi}(r,t) = R(r)\,\Gamma(t)$$

$$\Downarrow$$

$$\left\{ \begin{array}{c} \dfrac{d^2 R}{dr^2} + \lambda^2 R = 0 \\ R(0) = 0 \\ \dfrac{d R(r_0)}{dr} - \dfrac{R(r_0)}{r_0} = 0 \end{array} \right\}$$

$$\frac{d\Gamma_n}{dt} + \alpha \lambda_n^2 \Gamma_n = 0$$

$$\downarrow$$

$$\Gamma_n(t) = C_n e^{-\alpha \lambda_n^2 t}$$

Note that $\lambda_0 = 0$ is an eigenvalue and the corresponding eigenfunction is (see Prob. 4.7)

$$R_0(r) = a_0 r \, .$$

When $\lambda_n \neq 0$,

$$R_n(r) = a_n \sin\lambda_n r$$

where

$$\tan\lambda_n r_0 = \lambda_n r_0 \, , \quad \lambda_n = 1, 2, 3, \ldots$$

$$\therefore \quad \boxed{\bar{\psi}(r,t) = \underbrace{A_0}_{a_0 c_0} r + \sum_{n=1}^{\infty} \underbrace{A_n}_{a_n c_n} e^{-\alpha\lambda_n^2 t} \sin\lambda_n r}$$

$$\bar{\psi}(r,0) = -\frac{q_w'' r^3}{2k r_0} \Rightarrow -\frac{q_w'' r^3}{2k r_0} = A_0 r + \sum_{n=1}^{\infty} A_n \sin\lambda_n r$$

$$A_0 = \frac{1}{N_0}\int_0^{r_0}\left(-\frac{q_w'' r^3}{2k r_0}\right) r\,dr \, , \quad N_0 = \int_0^{r_0} r^2 dr = \frac{r_0^3}{3}$$

$$= \cdots = -\frac{3}{10}\frac{q_w'' r_0}{k}$$

$$A_n = \frac{1}{N_n}\int_0^{r_0}\left(-\frac{q_w'' r^3}{2k r_0}\right)\sin\lambda_n r\,dr \, , \quad N_n = \int_0^{r_0} \sin^2\lambda_n r\,dr = \ldots = \frac{r_0}{2}\sin^2\lambda_n r_0$$

$$= \cdots = -2\frac{q_w''}{k}\frac{1}{\lambda_n^2 \sin\lambda_n r_0}$$

$$\vdots$$

$$\therefore \quad \boxed{T(r,t) - T_i = \frac{T(r,t)}{r} = \frac{q_w''}{k r_0}\left[\frac{5r^2 - 3r_0^2}{10} + 3\alpha t - 2\frac{r_0}{r}\sum_{n=1}^{\infty}\frac{\sin\lambda_n r}{\lambda_n^2 \sin\lambda_n r_0} e^{-\alpha\lambda_n^2 t}\right]}$$

PROB. 6.23: Assume constant thermophysical properties (k, ρ, c).

Formulation:

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(r,0) = T_i$$

$$T(r_1,t) = 0 \; ; \; -k\left.\frac{\partial T}{\partial r}\right|_{r=r_2} = h\,T(r_2,t)$$

Let $\overline{U(\eta,t) = r\cdot T(r,t) \; ; \; \eta = r - r_1}$

$$\Downarrow$$

$$\frac{\partial^2 U}{\partial \eta^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t}$$

$$U(\eta,0) = (\eta + r_1)T_i$$

$$U(0,t) = 0 \; ; \; \left[k\frac{\partial U}{\partial \eta} + \left(h - \frac{k}{r_2}\right)U(\eta,t)\right]_{\eta = r_2 - r_1} = 0$$

$$\therefore \; U(\eta,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha\lambda_n^2 t}\sin\lambda_n\eta \,,$$

where λ_n's are the positive roots of

$$\cot\lambda(r_2 - r_1) = -\frac{Bi - 1}{\lambda r_2}$$

with $Bi = \frac{hr_2}{k}$

$$\Rightarrow \; A_n = \frac{1}{N_n}\int_0^{r_2 - r_1}(\eta + r_1)T_i\cdot\sin\lambda_n\eta\,d\eta$$

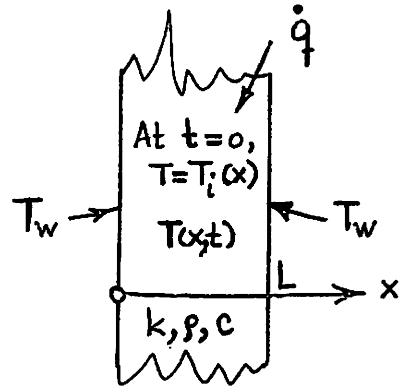
$$= \frac{T_i}{N_n}\left[\frac{Bi}{\lambda_n^2}\sin\lambda_n(r_2 - r_1) + \frac{r_1}{\lambda_n}\right]$$

where

$$N_n = \frac{1}{2\lambda_n}\left[\lambda_n(r_2 - r_1) - \sin\lambda_n(r_2 - r_1)\cos\lambda_n(r_2 - r_1)\right]$$

$$\therefore \; \boxed{T(r,t) = \frac{2T_i}{r}\sum_{n=1}^{\infty}\frac{\left[Bi\sin\lambda_n(r_2 - r_1) + r_1\lambda_n\right]e^{-\alpha\lambda_n^2 t}\sin\lambda_n(r - r_1)}{\lambda_n\left[\lambda_n(r_2 - r_1) - \sin\lambda_n(r_2 - r_1)\cos\lambda_n(r_2 - r_1)\right]}}$$

PROB. 6.24: Formulation of the problem in terms of $\theta(x,t) = T(x,t) - T_w$:



$$\boxed{\begin{array}{l} \dfrac{\partial^2\theta}{\partial x^2} + \dfrac{\dot{q}}{k} = \dfrac{1}{\alpha}\dfrac{\partial \theta}{\partial t} \\ \theta(x,0) = T_i(x) - T_w \triangleq \theta_i(x) \\ \theta(0,t) = \theta(L,t) = 0 \end{array}}$$

Let $\quad \theta(x,t) = \psi(x,t) + \phi(x)$

$$\left\{ \begin{array}{l} \dfrac{\partial^2\psi}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial \psi}{\partial x} \\ \psi(x,0) = \theta_i(x) - \phi(x) \\ \psi(0,t) = \psi(L,t) = 0 \end{array} \right\} \qquad \left\{ \begin{array}{l} \dfrac{d^2\phi}{dx^2} + \dfrac{\dot{q}}{k} = 0 \\ \phi(0) = \phi(L) = 0 \end{array} \right\}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\boxed{\psi(x,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha\lambda_n^2 t}\sin\lambda_n x, \quad \lambda_n = \frac{n\pi}{L}} \qquad \boxed{\phi(x) = \frac{\dot{q}L^2}{2k}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right]}$$

$$\Downarrow$$

$$\theta_i(x) - \phi(x) = \sum_{n=1}^{\infty} A_n \sin\lambda_n x \longrightarrow A_n = \frac{2}{L}\int_0^L [\theta_i(x) - \phi(x)]\sin\lambda_n x\,dx$$

$$\therefore \quad A_n = \frac{2}{L}\left\{\int_0^L T_i(x)\sin\lambda_n x\,dx - \frac{1-(-1)^n}{\lambda_n}\left[T_\infty - \frac{\dot{q}}{k}\frac{1}{\lambda_n^2}\right]\right\}$$

Thus, the solution becomes

$$\boxed{\begin{array}{l} T(x,t) - T_w = \dfrac{\dot{q}L^2}{2k}\left[\dfrac{x}{L} - \left(\dfrac{x}{L}\right)^2\right] + \dfrac{2}{L}\displaystyle\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\sin\lambda_n x \\ \qquad \cdot\left\{\displaystyle\int_0^L T_i(x')\sin\lambda_n x'\,dx' - \dfrac{1-(-1)^n}{\lambda_n}\left[T_\infty + \dfrac{\dot{q}}{k}\dfrac{1}{\lambda_n^2}\right]\right\} \end{array}}$$

PROB. 6.26: Formulation of the problem:

$$\left\{\begin{array}{c} \dfrac{\partial^2 T}{\partial x^2} + \dfrac{\dot{q}(x,t)}{k} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t} \\ T(x,0) = T_i(x) \\ T(0,t) = T_w(t) \,;\; T(L,t) = T_w(t) \end{array}\right\}$$

Let $\theta(x,t) = T(x,t) - T_w(t)$

$$\Downarrow$$

$$\left\{\begin{array}{c} \dfrac{\partial^2 \theta}{\partial x^2} + \dfrac{\dot{q}(x,t)}{k} = \dfrac{1}{\alpha}\left[\dfrac{\partial \theta}{\partial t} + \dfrac{dT_w}{dt}\right] \\ \theta(x,0) = T_i(x) - T_w(0) \\ \theta(0,t) = 0 \,;\; \theta(L,t) = 0 \end{array}\right\}$$

Let

$$\theta(x,t) = \sum_{n=1}^{\infty} A_n(t)\sin\frac{n\pi}{L}x \quad \leftarrow \text{satisfies the B.C.'s.}$$

and

$$\frac{\dot{q}(x,t)}{k} = \sum_{n=1}^{\infty} B_n(t)\sin\frac{n\pi}{L}x$$

$$\therefore\; B_n(t) = \frac{2}{L}\int_0^L \frac{\dot{q}(x,t)}{k}\sin\frac{n\pi}{L}x\,dx$$

$$\frac{dT_\infty}{dt} = \sum_{n=1}^{\infty} C_n(t)\sin\frac{n\pi}{L}x$$

$$\therefore\; C_n(t) = \frac{2}{L}\int_0^L \frac{dT_w}{dt}\sin\frac{n\pi}{L}x\,dx = \frac{2}{n\pi}\left[1-(-1)^n\right]\frac{dT_w}{dt}$$

Substituting the above expansions into the DE yields

$$\sum_{n=1}^{\infty}\left[-\lambda_n^2 A_n(t)+B_n(t)\right]\sin\lambda_n x=\frac{1}{\alpha}\sum_{n=1}^{\infty}\left[\frac{dA_n}{dt}+C_n(t)\right]\sin\lambda_n x$$

$$\Downarrow \quad \text{where } \lambda_n=\frac{n\pi}{L}.$$

$$\frac{dA_n}{dt}+\alpha\lambda_n^2 A_n(t)=\alpha B_n(t)-C_n(t)$$

$$\frac{d}{dt}\left[e^{+\alpha\lambda_n^2 t}A_n(t)\right]=e^{\alpha\lambda_n^2 t}\left[\alpha B_n(t)-C_n(t)\right]$$

$$\therefore\ A_n(t)=e^{-\alpha\lambda_n^2 t}A_n(0)+e^{-\alpha\lambda_n^2 t}\int_0^t e^{\alpha\lambda_n^2 t'}\left[\alpha B_n(t')-C_n(t')\right]dt$$

Now, apply the I.C. to get

$$T_i(x)-T_\infty(0)=\sum_{n=1}^{\infty}A_n(0)\sin\lambda_n x$$

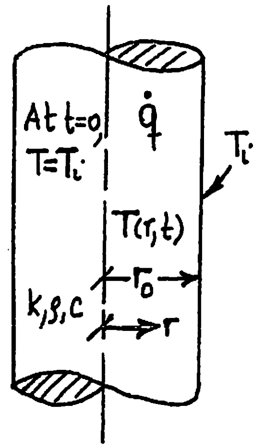
$$\therefore\ A_n(0)=\frac{2}{L}\int_0^L\left[T_i(x)-T_w(0)\right]\sin\lambda_n x$$

$$=\frac{2}{L}\int_0^L T_i(x)\sin\lambda_n x\,dx-\frac{2}{n\pi}\left[1-(-1)^n\right]T_w(0)$$

$$\vdots$$

$$\boxed{\begin{aligned}\theta(x,t)=T(x,t)-T_\infty(t)=\frac{2}{L}\sum_{n=1}^{\infty}e^{-\alpha\lambda_n^2 t}\sin\lambda_n x\Bigg\{&\int_0^L T_i(x')\sin\lambda_n x'\,dx'\\&-\frac{\left[1-(-1)^n\right]}{\lambda_n}\left[T_w(0)+\int_0^t e^{\alpha\lambda_n^2 t'}\frac{dT_w}{dt'}\,dt'\right]\\&+\frac{1}{\rho c}\int_0^t e^{\alpha\lambda_n^2 t'}\int_0^L \dot{q}(x',t)\sin\lambda_n x'\,dx'\,dt'\Bigg\}\end{aligned}}$$

PROB. 6.27: Formulation in terms of $\theta(r,t) = T(r,t) - T_i$:



$$\left\{\begin{aligned} &\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \\ &\theta(r,0) = 0 \\ &\theta(0,t) = \text{finite} \,;\quad \theta(r_0,t) = 0 \end{aligned}\right\}$$

Let $\theta(r,t) = \theta_t(r,t) + \theta_s(r)$

$$\frac{\partial^2\theta_t}{\partial r^2} + \frac{1}{r}\frac{\partial\theta_t}{\partial r} = \frac{1}{\alpha}\frac{\partial\theta_t}{\partial t}$$

$$\theta_t(r,0) = -\theta_s(r)$$

$$\theta_t(0,t) = \text{finite} \,;\quad \theta_t(r_0,t) = 0$$

$$\Downarrow$$

$$\theta_t(r,t) = \sum_{n=1}^{\infty} a_n e^{-\alpha\lambda_n^2 t} J_0(\lambda_n r)$$

$$\frac{d^2\theta_s}{dr^2} + \frac{1}{r}\frac{d\theta_s}{dr} + \frac{\dot{q}}{k} = 0$$

$$\theta_s(0) \neq \infty$$

$$\theta_s(r_0) = 0$$

$$\Downarrow$$

$$\boxed{\theta_s(r) = \frac{\dot{q}r_0^2}{4k}\left[1 - \left(\frac{r}{r_0}\right)^2\right]}$$

where λ_n's are the positive roots of $J_0(\lambda r_0) = 0$.

$$\therefore\ -\theta_s(r) = -\frac{\dot{q}r_0^2}{4k}\left[1 - \left(\frac{r}{r_0}\right)^2\right] = \sum_{n=1}^{\infty} a_n J_0(\lambda_n r)$$

$$\Downarrow$$

$$a_n = -\frac{1}{N_n}\frac{\dot{q}r_0^2}{4k}\int_0^{r_0}\left[1 - \left(\frac{r}{r_0}\right)^2\right] J_0(\lambda_n r)\, r\,dr \,,\quad N_n = \frac{r_0^2}{2} J_1^2(\lambda_n r_0)$$

$$= \cdots = -\frac{2\dot{q}}{k r_0 \lambda_n^3 J_1(\lambda_n r_0)}$$

$$\therefore\ \boxed{T(r,t) - T_i = -\frac{2\dot{q}}{k r_0}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\frac{J_0(\lambda_n r)}{\lambda_n^3 J_1(\lambda_n r_0)} + \frac{\dot{q}r_0^2}{4k}\left[1 - \left(\frac{r}{r_0}\right)^2\right]}$$

PROB. 6.28: Formulation in terms of $\theta(r,t) = T(r,t) - T_i$:

$$\frac{\partial^2\theta}{\partial r^2} + \frac{2}{r}\frac{\partial\theta}{\partial r} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta(r,0) = 0$$

$$\theta(0,t) = \text{finite} \; ; \quad \theta(r_0,t) = 0$$

$$\Downarrow \quad \text{Let } \theta(r,t) = \theta_t(r,t) + \theta_s(r)$$

$$\frac{\partial^2\theta_t}{\partial r^2} + \frac{2}{r}\frac{\partial\theta_t}{\partial r} = \frac{1}{\alpha}\frac{\partial\theta_t}{\partial t}$$

$$\theta_t(r,0) = -\theta_s(r)$$

$$\theta_t(0,t) = \text{finite}; \quad \theta_t(r_0,t) = 0$$

$$\Downarrow$$

$$U(r,t) = r\,\theta_t(r,t)$$

$$\Downarrow$$

$$\frac{\partial^2 U}{\partial r^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t}$$

$$U(r,0) = -r\,\theta_s(r)$$

$$U(0,t) = 0; \quad U(r_0,t) = 0$$

$$\Downarrow$$

$$U(r,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha\lambda_n^2 t}\sin\lambda_n r \quad , \quad \lambda_n = \frac{n\pi}{r_0},\; n=1,2,\ldots$$

$$\frac{d^2\theta_s}{dr^2} + \frac{2}{r}\frac{d\theta_s}{dr} + \frac{\dot{q}}{k} = 0$$

$$\theta_s(0) = \text{finite}$$

$$\theta_s(r_0) = 0$$

$$\Downarrow$$

$$\boxed{\theta_s(r) = \frac{\dot{q}\,r_0^2}{6k}\left[1 - \left(\frac{r}{r_0}\right)^2\right]}$$

$$\therefore \quad -\frac{\dot{q} r_0^2}{6k}\left[r - \frac{r^3}{r_0^2}\right] = \sum_{n=1}^{\infty} A_n \sin\lambda_n r$$

$$\Downarrow$$

$$A_n = \frac{2}{r_0}\int_0^{r_0}\left(-\frac{\dot{q} r_0^2}{6k}\right)\left(r - \frac{r^3}{r_0^2}\right)\sin\lambda_n r\, dr$$

$$= \cdots = (-1)^n \frac{2\dot{q} r_0^3}{k(n\pi)^3}$$

$$\therefore \quad U(r,t) = 2\frac{\dot{q} r_0^3}{k\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}e^{-\alpha\lambda_n^2 t}\sin\lambda_n r$$

$$\Downarrow$$

$$T(r,t) - T_i = \frac{\dot{q} r_0^2}{6k}\left[1 - \left(\frac{r}{r_0}\right)^2\right] + \frac{2\dot{q} r_0^3}{k\pi^3 r}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}e^{-\alpha\lambda_n^2 t}\sin\lambda_n r$$

$$\boxed{\frac{T(r,t) - T_i}{\dot{q} r_0^2/6k} = 1 - \left(\frac{r}{r_0}\right)^2 + \frac{12}{\pi^3}\frac{r_0}{r}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}e^{-\alpha\lambda_n^2 t}\sin\lambda_n r}$$

The temperature distribution as $t \to \infty$:

$$T_s(r,t) - T_i = \frac{\dot{q} r_0^2}{6k}\left[1 - \left(\frac{r}{r_0}\right)^2\right]$$

PROB. 6.29: Formulation in terms of $\theta(x,t) = T(x,t) - T_i$ (See Prob. 2.9):

$$\left\{ \begin{array}{c} \dfrac{\partial^2\theta}{\partial x^2} + \dfrac{\dot{q}_0 e^{-\beta t}}{k} = \dfrac{1}{\alpha}\dfrac{\partial\theta}{\partial t} \\ \theta(x,0) = 0 \\ \left.\dfrac{\partial\theta}{\partial x}\right|_{x=0} = 0 \; ; \; \theta(L,t) = 0 \end{array} \right\}$$

$$\text{Let} \quad \theta(x,t) = e^{-\beta t}\phi(x,t)$$

$$\Downarrow$$

$$\left\{ \begin{array}{c} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\dot{q}_0}{k} = \dfrac{1}{\alpha}\left[-\beta\phi + \dfrac{\partial\phi}{\partial t}\right] \\ \phi(x,0) = 0 \\ \left.\dfrac{\partial\phi}{\partial x}\right|_{x=0} = 0 \; ; \; \phi(L,t) = 0 \end{array} \right\}$$

$$\text{Let} \quad \phi(x,t) = \psi(x,t) + \Omega(x)$$

$$\Downarrow$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{\alpha}\left[\frac{\partial\psi}{\partial t} - \beta\psi\right]$$

$$\psi(x,0) = -\Omega(x)$$

$$\left.\frac{\partial\psi}{\partial x}\right|_{x=0} = 0 \; ; \; \psi(L,t) = 0$$

$$\frac{d^2\Omega}{dx^2} + \frac{\beta}{\alpha}\Omega = -\frac{\dot{q}_0}{k}$$

$$\frac{d\Omega(0)}{dx} = 0$$

$$\Omega(L) = 0$$

$$\Downarrow$$

$$\boxed{\Omega(x) = \frac{\dot{q}_0\alpha}{k\beta}\left[\frac{\cos\sqrt{\frac{\beta}{\alpha}}\,x}{\cos\sqrt{\frac{\beta}{\alpha}}\,L} - 1\right]}$$

$$\text{Let} \quad \psi(x,t) = X(x) \cdot \Gamma(t)$$

$$\left\{ \begin{array}{c} \dfrac{d^2X}{dx^2} + \lambda^2 X = 0 \\ \dfrac{dX(0)}{dx} = 0 \\ X(L) = 0 \end{array} \right\} \qquad X_n(x) = A_n \cos\lambda_n x \qquad \lambda_n = \frac{2n-1}{L}\frac{\pi}{2}, \; n = 1,2,\ldots$$

$$\frac{d\Gamma_n}{dt} + (\alpha\lambda_n^2 - \beta)\Gamma_n = 0 \quad \rightarrow \quad \Gamma_n(t) = B_n e^{-(\alpha\lambda_n^2-\beta)t}$$

$$\therefore \quad \psi(x,t) = \sum_{n=1}^{\infty} a_n e^{-(\alpha\lambda_n^2-\beta)t} \cos\lambda_n x$$

$$\Rightarrow \quad -\Omega(x) = \frac{\dot{q}_0\alpha}{k\beta}\left[1 - \frac{\cos\sqrt{\frac{\beta}{\alpha}}\,x}{\cos\sqrt{\frac{\beta}{\alpha}}\,L}\right] = \sum_{n=1}^{\infty} a_n \cos\lambda_n x$$

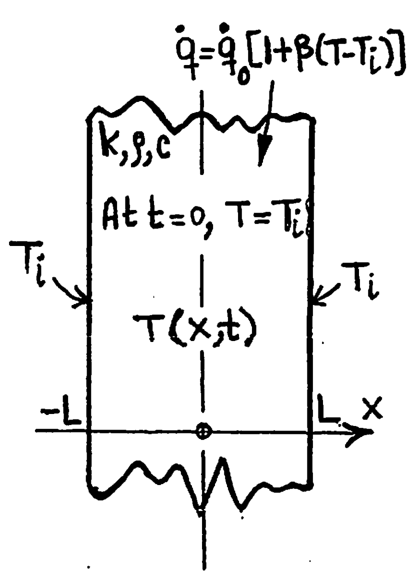
$$a_n = -\frac{2}{L}\int_0^L \Omega(x)\cos\lambda_n x\,dx = -\frac{2}{L}\frac{\dot{q}_0\alpha}{k}\frac{(-1)^n}{\lambda_n(\beta-\alpha\lambda_n^2)}, \; n=1,2,\ldots$$

$$\therefore \quad \phi(x,t) = \Omega(x) - \frac{2}{L}\frac{\dot{q}_0\alpha}{k}\sum_{n=1}^{\infty}\frac{(-1)^n}{\lambda_n(\beta-\alpha\lambda_n^2)} e^{-(\alpha\lambda_n^2-\beta)t}\cos\lambda_n x$$

$$\theta(x,t) = T(x,t) - T_i = e^{-\beta t}\phi(x,t)$$

$$\therefore \quad \boxed{\theta(x,t) = \frac{\dot{q}_0\alpha}{k\beta}\left[\frac{\cos\sqrt{\frac{\beta}{\alpha}}\,x}{\cos\sqrt{\frac{\beta}{\alpha}}\,L} - 1\right]e^{-\beta t} - \frac{2}{L}\frac{\dot{q}_0\alpha}{k}\sum_{n=1}^{\infty}(-1)^n\frac{e^{-\alpha\lambda_n^2 t}\cos\lambda_n x}{\lambda_n(\beta-\alpha\lambda_n^2)}}$$

PROB.6.30: Formulation in terms of $\theta(x,t) = T(x,t) - T_i$:



$$\boxed{\begin{gathered}\frac{\partial^2\theta}{\partial x^2} + \frac{\dot{q}_0}{k}[1+\beta\theta] = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \\ \theta(x,0)=0 \\ \left.\frac{\partial\theta}{\partial x}\right|_{x=0} = 0, \quad \theta(L,t)=0\end{gathered}}$$

Let $\theta(x,t) = \psi(x,t) + \phi(x)$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\dot{q}_0\beta}{k}\psi = \frac{1}{\alpha}\frac{\partial\psi}{\partial t}$$

$$\psi(x,0) = -\phi(x)$$

$$\left.\frac{\partial\psi}{\partial x}\right|_{x=0} = 0, \quad \psi(L,t)=0$$

$$\frac{d^2\phi}{dx^2} + \frac{\dot{q}_0\beta}{k}\phi = -\frac{\dot{q}_0}{k}$$

$$\frac{d\phi(0)}{dx} = 0$$

$$\phi(L) = 0$$

Let $\psi(x,t) = X(x)\cdot\Gamma(t)$

$$\left\{\begin{gathered}\frac{d^2X}{dx^2} + \lambda^2 X = 0 \\ \frac{dX(0)}{dx} = 0 \\ X(L) = 0\end{gathered}\right\} \Rightarrow \quad X_n(x) = A_n \cos\lambda_n x, \quad \lambda_n = \frac{(2n-1)\pi}{2L}, \quad n=1,2,3,\ldots$$

and

$$\frac{d\Gamma_n}{dt} + \alpha\left(\lambda_n^2 - \frac{\dot{q}_0\beta}{k}\right)\Gamma_n = 0 \longrightarrow \Gamma_n(t) = B_n e^{-\alpha\left(\lambda_n^2 - \frac{\dot{q}_0\beta}{k}\right)t}$$

Thus,

$$\psi(x,t) = \sum_{n=1}^{\infty} a_n \cos\lambda_n x \, e^{-\alpha\left(\lambda_n^2 - \frac{\dot{q}_0\beta}{k}\right)t}$$

$$\therefore \quad -\phi(x) = \sum_{n=1}^{\infty} a_n \cos\lambda_n x, \qquad a_n = \frac{2}{L}\int_0^L \{-\phi(x)\}\cos\lambda_n x\, dx$$

Note that $\phi(x) = -\frac{1}{\beta} - \frac{k}{\dot{q}_0\beta}\frac{d^2\phi}{dx^2}$. Then,

$$\int_0^L \phi(x)\cos\lambda_n x\, dx = -\int_0^L \left[\frac{1}{\beta} + \frac{k}{\dot{q}_0\beta}\frac{d^2\phi}{dx^2}\right]\cos\lambda_n x\, dx$$

$$= \frac{(-1)^n}{\beta\lambda_n} + \frac{k\lambda_n^2}{\dot{q}_0\beta}\int_0^L \phi(x)\cos\lambda_n x\, dx$$

$$\therefore \int_0^L \phi(x)\cos\lambda_n x\, dx = \frac{\dot{q}_0}{k\lambda_n}\,\frac{(-1)^n}{[\dot{q}_0\beta/k - \lambda_n^2]}$$

Thus,

$$a_n = \frac{2}{L}\,\frac{\dot{q}_0}{k\lambda_n}\,\frac{(-1)^{n+1}}{[\dot{q}_0\beta/k - \lambda_n^2]}$$

and

$$\boxed{\psi(x,t) = \frac{2\dot{q}_0}{kL}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\lambda_n[\dot{q}_0\beta/k - \lambda_n^2]}\cos\lambda_n x \cdot e^{-\alpha\left(\lambda_n^2 - \frac{\dot{q}_0\beta}{k}\right)t}}$$

This holds for either sign of the product $\dot{q}_0\beta$.

$$\therefore \quad \theta(x,t) = \psi(x,t) + \phi(x)$$

Solution for $\phi(x)$

Case I: $\dot{q}_o \beta < 0$.

Let $r^2 = -\dfrac{\dot{q}_o \beta}{k}$. Then, $\phi(x) = \dfrac{1}{\beta}\left[\dfrac{\cosh rx}{\cosh rL} - 1\right]$.

Thus,

$$\theta(x,t) = T(x,t) - T_i = \frac{1}{\beta}\left[\frac{\cosh rx}{\cosh rL} - 1\right]$$

$$+ \frac{2\dot{q}_o}{kL}\sum_{n=1}^{\infty} \frac{(-1)^n}{\lambda_n[r^2+\lambda_n^2]} \cos\lambda_n x \cdot \exp[-\alpha(\lambda_n^2 + r^2)t]$$

Since the exponential terms in this result all tend to zero as $t \to \infty$, the steady-state solution is given by $\phi(x)$ for all values of $\dot{q}_o \beta < 0$.

Case II: $\dot{q}_o \beta > 0$. Let $p^2 = \dfrac{\dot{q}_o \beta}{k}$. Then,

$$\phi(x) = \frac{1}{\beta}\left[\frac{\cos px}{\cos pL} - 1\right].$$

Thus,

$$\theta(x,t) = T(x,t) - T_i = \frac{1}{\beta}\left[\frac{\cos px}{\cos pL} - 1\right]$$

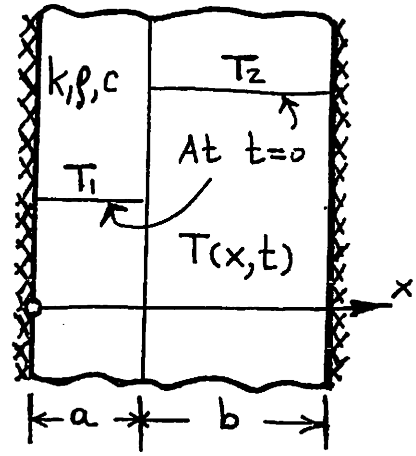
$$- \frac{2\dot{q}_o}{kL}\sum_{n=1}^{\infty} \frac{(-1)^n}{\lambda_n[p^2-\lambda_n^2]} \cos\lambda_n x \cdot \exp[-\alpha(\lambda_n^2 - p^2)t]$$

In this case, the exponential terms vanish as $t \to \infty$ if

$$p^2 < \lambda_1^2 \quad \text{or} \quad \frac{\dot{q}_o \beta}{k} < \left(\frac{\pi}{2L}\right)^2$$

and, if this inequality holds, the steady-state solution is given by $\phi(x)$. If $p^2 > \lambda_1^2$, no steady-state solution exists, because heat would be generated at a rate too great for its removal to be possible.

PROB. 6.33: Formulation of the problem:



$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad \alpha = \frac{k}{\rho c}$$

$$T(x,0) = \begin{cases} T_1, & 0 < x < a \\ T_2, & a < x < a+b \end{cases}$$

$$\frac{\partial T(0,t)}{\partial x} = 0; \quad \frac{\partial T(a+b,t)}{\partial x} = 0$$

Let $T(x,t) = X(x)\cdot\Gamma(t)$

$$\left\{ \begin{aligned} &\frac{d^2X}{dx^2} + \lambda^2 X = 0 \\ &\frac{dX(0)}{dx} = 0 \\ &\frac{dX(a+b)}{dx} = 0 \end{aligned} \right\} \Longrightarrow$$

$X_n(x) = A_n \cos\lambda_n x$

where λ_n's are zeros of

$\sin\lambda(a+b) = 0$

$\therefore \lambda_n = \frac{n\pi}{a+b}, n = 0,1,2,\ldots$

$$\frac{d\Gamma_n}{dt} + \alpha\lambda_n^2\Gamma_n = 0$$

$$\Downarrow$$

$$\Gamma_n(t) = B_n e^{-\alpha\lambda_n^2 t}$$

$$\therefore T(x,t) = \sum_{n=0}^{\infty} a_n e^{-\alpha\lambda_n^2 t}\cos\lambda_n x = a_0 + \sum_{n=1}^{\infty} a_n e^{-\alpha\lambda_n^2 t}\cos\lambda_n x$$

$$\Longrightarrow T(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\lambda_n x$$

Thus,

$$a_0 = \frac{1}{a+b}\int_0^{a+b} T(x,0)\,dx = \frac{1}{a+b}\left\{T_1\int_0^a dx + T_2\int_a^{a+b} dx\right\} = \frac{aT_1 + bT_2}{a+b}$$

$$a_n = \frac{2}{a+b}\int_0^{a+b} T(x,0)\cos\lambda_n x\,dx = \cdots = \frac{2}{a+b}\,\frac{(T_1 - T_2)\sin\lambda_n a}{\lambda_n}$$

$$\therefore \boxed{T(x,t) = \frac{aT_1 + bT_2}{a+b} + \frac{2(T_1 - T_2)}{\pi}\sum_{n=1}^{\infty}\frac{\sin\lambda_n a\,\cos\lambda_n x}{n}\,e^{-\alpha\lambda_n^2 t}}$$

As $t \to \infty$, $T_s = \frac{aT_1 + bT_2}{a+b} \leftarrow$ Steady-state temperature

PROB. 6.34: The formulation in terms of $\theta(x,t) = T(x,t) - (T_i + At)$:

$$\left\{ \begin{array}{c} \dfrac{\partial^2\theta}{\partial x^2} - \dfrac{A}{\alpha} = \dfrac{1}{\alpha}\dfrac{\partial\theta}{\partial t} \\ \theta(x,0) = 0 \\ \left.\dfrac{\partial\theta}{\partial x}\right|_{x=0} = 0; \quad \theta(L,t) = 0 \end{array} \right\}$$

Let $\theta(x,t) = \psi(x,t) + \phi(x)$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\psi}{\partial t}$$

$$\psi(x,0) = -\phi(x)$$

$$\frac{\partial\psi(0,t)}{\partial x} = 0; \quad \psi(L,t) = 0$$

$$\Downarrow$$

$$\psi(x,t) = \sum_{n=1}^{\infty} a_n e^{-\alpha\lambda_n^2 t}\cos\lambda_n x$$

$$\frac{d^2\phi}{dx^2} - \frac{A}{\alpha} = 0$$

$$\frac{d\phi(0)}{dx} = 0$$

$$\phi(L) = 0$$

$$\Downarrow$$

$$\boxed{\phi(x) = \frac{AL^2}{2\alpha}\left[1 - \left(\frac{x}{L}\right)^2\right]}$$

where $\lambda_n = \dfrac{2n-1}{L}\dfrac{\pi}{2}$ (See Table 4.1)

$$\therefore \quad -\phi(x) = \frac{AL^2}{2\alpha}\left[1 - \left(\frac{x}{L}\right)^2\right] = \sum_{n=1}^{\infty} a_n \cos\lambda_n x$$

$$\Downarrow$$

$$a_n = \frac{2}{L}\int_0^L \{-\phi(x)\}\cos\lambda_n x\, dx = \cdots = \frac{2A}{\alpha L}\frac{(-1)^{n+1}}{\lambda^3}$$

$$\therefore \quad \theta(x,t) = -\frac{2A}{\alpha L}\sum_{n=1}^{\infty}\frac{(-1)^n}{\lambda_n^3} e^{-\alpha\lambda_n^2 t}\cos\lambda_n x - \frac{AL^2}{2\alpha}\left[1 - \left(\frac{x}{L}\right)^2\right]$$

or

$$\boxed{T(x,t) - T_i = A\left\{ t - \frac{L^2}{2\alpha}\left[1 - \left(\frac{x}{L}\right)^2\right] - \frac{2}{\alpha L}\sum_{n=1}^{\infty}\frac{(-1)^n}{\lambda_n^3} e^{-\alpha\lambda_n^2 t}\cos\lambda_n x \right\}}$$

PROB. 6.35: Formulation of the problem:

$$\left\{\begin{array}{c} \dfrac{\partial^2 T}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial T}{\partial r} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t} \\ T(r,0) = T_i \\ T(0,t) = \text{finite} \;;\; T(r_o,t) = T_i + At \end{array}\right\}$$

Let $\theta(r,t) = T(r,t) - (T_i + At)$

$$\Downarrow$$

$$\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} = \frac{1}{\alpha}\left[\frac{\partial \theta}{\partial t} + A\right]$$

$$\theta(r,0) = 0$$

$$\theta(0,t) = \text{finite} \;;\; \theta(r_o,t) = 0$$

The solution for $\theta(r,t)$ may now be written as (see Prob. 6.27):

$$\theta(r,t) = \frac{2A}{\alpha r_o}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\frac{J_0(\lambda_n r)}{\lambda_n^3 J_1(\lambda_n r_o)} - \frac{A r_o^2}{4\alpha}\left[1 - \left(\frac{r}{r_o}\right)^2\right]$$

Thus,

$$\boxed{T(r,t) - T_i = \frac{A r_o^2}{4\alpha}\left\{4\frac{\alpha t}{r_o^2} - 1 + \left(\frac{r}{r_o}\right)^2 + \frac{8}{r_o^3}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\frac{J_0(\lambda_n r)}{\lambda_n^3 J_1(\lambda_n r_o)}\right\}}$$

where λ_n's are the positive roots of $J_o(\lambda r_o) = 0$.

PROB. 6.37:

Formulation $\Rightarrow$
$$\left\{\begin{array}{c} \dfrac{\partial^2 T}{\partial s^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t} \\ T(s,0) = T_i(s) \\ T(s,t) = T(s+2L,t) \\ \dfrac{\partial T(s,t)}{\partial s} = \dfrac{\partial T(s+2L,t)}{\partial s} \end{array}\right\}$$

Let $T(s,t) = X(s).\Gamma(t)$

$\Downarrow$

$$\left\{\begin{array}{c} \dfrac{d^2X}{ds^2} + \lambda X = 0 \\ X(s) = X(s+2L) \\ \dfrac{dX(s)}{ds} = \dfrac{dX(s+2L)}{ds} \end{array}\right\} \Rightarrow \begin{array}{l} \lambda_n = \dfrac{n\pi}{L}, \quad n = 0,1,2,\ldots \\ X_n(s) = A_n \sin\lambda_n s + B_n \cos\lambda_n s \end{array}$$

$\Downarrow$

$$\frac{d\Gamma_n}{dt} - \alpha\lambda_n^2 \Gamma_n = 0 \longrightarrow \Gamma_n(t) = C_n e^{-\alpha\lambda_n^2 t}$$

$$\therefore T(s,t) = \underset{\uparrow\, B_0C_0}{b_0} + \sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\left[\underset{\uparrow\, A_nC_n}{a_n} \sin\lambda_n s + \underset{\uparrow\, B_nC_n}{b_n} \cos\lambda_n s\right] \quad (1)$$

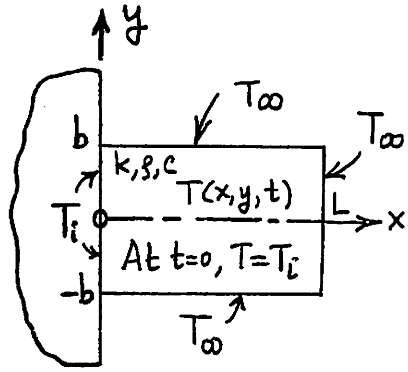
$$T(s,0) = T_i(s) \Rightarrow T_i(s) \underset{\substack{\uparrow \\ \text{Complete Fourier series}}}{=} b_0 + \sum_{n=1}^{\infty}\left[a_n \sin\lambda_n s + b_n \cos\lambda_n s\right]$$

$$\therefore b_0 = \frac{1}{2L}\int_0^{2L} T_i(s)\,ds, \quad a_n = \frac{1}{L}\int_0^{2L} T_i(s)\sin\lambda_n s\,ds, \quad b_n = \frac{1}{L}\int_0^{2L} T_i(s)\cos\lambda_n s\,ds$$

Substitution of b_0, a_n & b_n into Eq (1) gives $T(s,t)$, $t>0$.

Also, note that the <u>steady-state</u> temp. $T(s) = \lim_{t\to\infty} T(s,t) = b_0$.

PROB.6.38: Let $\theta(x,y,t) = T(x,y,t) - T_i$.



Then,

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta(x,y,0) = 0$$

$$\theta(0,y,t) = 0 \,; \quad \theta(L,y,t) = T_\infty - T_i$$

$$\frac{\partial\theta(x,0,t)}{\partial y} = 0 \,; \quad \theta(x,b,t) = T_\infty - T_i$$

Let $\theta(x,y,t) = \theta_s(x,y) + \theta_t(x,y,t)$

$$\frac{\partial^2\theta_s}{\partial x^2} + \frac{\partial^2\theta_s}{\partial y^2} = 0$$

$$\theta_s(0,y) = 0 \,; \quad \theta_s(L,y) = T_\infty - T_i$$

$$\frac{\partial\theta_s(x,0)}{\partial y} = 0 \,; \quad \theta_s(x,b) = T_\infty - T_i$$

$$\frac{\partial^2\theta_t}{\partial x^2} + \frac{\partial^2\theta_t}{\partial y^2} = \frac{1}{\alpha}\frac{\partial\theta_t}{\partial t}$$

$$\theta_t(x,y,0) = -\theta_s(x,y)$$

$$\theta_t(0,y,t) = 0 \,; \quad \theta_t(L,y,t) = 0$$

$$\frac{\partial\theta_t(x,0,t)}{\partial y} = 0 \,; \quad \theta_t(x,b,t) = 0$$

$$\Downarrow$$

$$\theta_s(x,y) - (T_\infty - T_i) = \sum_{n=1}^{\infty} A_n \sinh\frac{2n-1}{b}\frac{\pi}{2}(L-x)\cos\frac{2n-1}{b}\frac{\pi}{2}y$$

$$\Rightarrow -(T_\infty - T_i) = \sum_{n=1}^{\infty} A_n \sinh\frac{2n-1}{b}\frac{\pi}{2}L \cdot \cos\frac{2n-1}{b}\frac{\pi}{2}y$$

$$\therefore\ A_n \sin\frac{2n-1}{b}\frac{\pi}{2}L = -\frac{2}{b}(T_\infty - T_i)\int_0^b \cos\frac{2n-1}{b}\frac{\pi}{2}y\,dy$$

$$= \cdots = \frac{4}{\pi}\frac{(-1)^n}{(2n-1)}(T_\infty - T_i)$$

$$\therefore \boxed{\theta_s(x,y) = (T_\infty - T_i)\left[1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}\frac{\sinh\frac{(2n-1)}{b}\frac{\pi}{2}(L-x)\cos\frac{2n-1}{b}\frac{\pi}{2}y}{\sinh\frac{2n-1}{b}\frac{\pi}{2}L}\right]}$$

On the other hand,

$$\theta_t(x,y,t) = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} B_{mk}\sin\frac{m\pi}{L}x\cdot\cos\frac{2k-1}{b}\frac{\pi}{2}y\cdot\exp\left\{-\left[\left(\frac{m\pi}{L}\right)^2+\left(\frac{2k-1}{b}\frac{\pi}{2}\right)^2\right]t\right\}$$

$$\Rightarrow \quad -\theta_s(x,y) = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} B_{mk}\sin\frac{m\pi}{L}x\cdot\cos\frac{(2k-1)\pi}{2b}y$$

$$\therefore\ B_{mk} = -\frac{4}{Lb}\int_0^L\int_0^b \theta_s(x,y)\sin\frac{m\pi}{L}x\cos\frac{(2k-1)\pi}{2b}y\,dx\,dy$$

$$-\frac{Lb}{4}B_{mk} = (T_\infty - T_i)\int_0^L\int_0^b \sin\frac{m\pi}{L}x\cdot\cos\frac{(2k-1)}{2b}y\,dx\,dy$$

$$+\sum_{n=1}^{\infty}A_n\cdot\int_0^L \sinh\frac{(2n-1)\pi}{2b}(L-x)\sin\frac{m\pi}{L}x\,dx\cdot\underbrace{\int_0^b \cos\frac{(2n-1)\pi}{2b}y\cdot\cos\frac{(2k-1)\pi}{2b}y\,dy}_{\delta_{nk}\frac{b}{2}}$$

$$\vdots$$

$$B_{mk} = -\frac{8(T_\infty - T_i)}{m(2k-1)\pi^2}(-1)^k\left[1-(-1)^m\right] - \frac{8(-1)^k}{\pi L(2k-1)}\frac{\int_0^L \sinh\frac{(2k-1)\pi}{2b}(L-x)\sin\frac{m\pi}{L}x\,dx}{\sinh\frac{2k-1}{b}\frac{\pi}{2}L}$$

Thus,

$$\boxed{\begin{aligned}\frac{T(x,y,t)-T_i}{T_\infty - T_i} &= 1+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}\frac{\sinh\frac{(2n-1)\pi}{2b}(L-x)\cos\frac{(2n-1)\pi}{2b}y}{\sinh\frac{(2n-1)\pi}{2b}L}\\ &-\frac{8}{\pi}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)}\left\{\frac{[1-(-1)^m]}{m\pi}+\frac{1}{L}\frac{\int_0^L \sinh\frac{(2k-1)\pi}{2b}(L-x')\sin\frac{m\pi}{L}x'\,dx'}{\sin\frac{2k-1}{b}\frac{\pi}{2}L}\right\}\\ &\cdot\sin\frac{m\pi}{L}x\cdot\cos\frac{(2k-1)\pi}{2b}y\cdot\exp\left\{-\left[\left(\frac{m\pi}{L}\right)^2+\left(\frac{2k-1}{b}\frac{\pi}{2}\right)^2\right]t\right\}\end{aligned}}$$

PROB. 6.39: Formulation in terms of $\theta(r,\phi,t) = T(r,\phi,t) - T_w$:

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\theta}{\partial\phi^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta(r,\phi,0) = F(r,\phi) - T_w$$

$$\theta(0,\phi,t) = \text{finite}; \quad \theta(r_0,\phi,t) = 0$$

$$\theta(r,0,t) = 0; \quad \theta(r,\pi,t) = 0$$

$$\text{Let} \quad \theta(r,\phi,t) = R(r)\cdot\Phi(\phi)\cdot\Gamma(t)$$

Then,

$$\frac{d^2\Phi}{d\phi^2} + \lambda^2\Phi = 0$$
$$\Phi(0) = 0 \qquad \Rightarrow \qquad \Phi_n(\phi) = A_n \sin n\phi, \quad n = 1,2,3,\ldots$$
$$\Phi(\pi) = 0$$

and

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\beta^2 r^2 - n^2)R = 0$$
$$R(0) = \text{finite} \qquad \Rightarrow R_{mn}(r) = B_{mn} J_n(\beta_{mn} r)$$
$$R(r_0) = 0$$

where β_{mn} are the zeros of

$$J_n(\beta r_0) = 0$$
$$\Downarrow$$
$$\beta_{mn}, \quad m = 1,2,\ldots$$

Finally,

$$\frac{d\Gamma_{mn}}{dt} + \alpha\beta_{mn}^2\Gamma_{mn} = 0$$
$$\Downarrow$$
$$\Gamma_{mn} = C_{mn}\, e^{-\alpha\beta_{mn}^2 t}$$

Thus,

$$\theta(r,\phi,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} J_n(\beta_{mn} r)\sin(n\phi)\, e^{-\alpha\beta_{mn}^2 t}$$

$$\Downarrow$$

$$F(r,\phi) - T_W = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} J_n(\beta_{mn} r)\sin(n\phi)$$

$$\Downarrow$$

$$a_{mn} = \frac{2}{\pi}\frac{1}{N_{nm}}\int_0^{r_0}\int_0^{\pi}\left[F(r,\phi)-T_W\right]\sin(n\phi) J_n(\beta_{mn} r)\, r\,dr\,d\phi$$

where, from Table 4.2,

$$N_{nm} = \frac{r_0^2}{2} J_{n+1}^2(\beta_{mn} r_0)$$

Then the solution can be written as

$$\boxed{\begin{aligned} T(r,\phi,t) - T_W = \frac{4}{\pi r_0^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Bigg\{ & \frac{J_n(\beta_{mn} r)}{J_{n+1}^2(\beta_{mn} r_0)}\sin(n\phi)\, e^{-\alpha\beta_{mn}^2 t} \\ & \cdot \int_0^{r_0}\int_0^{\pi}\left[F(r',\phi')-T_W\right]\sin(n\phi') J_n(\beta_{mn} r')\, r'\,dr'\,d\phi' \Bigg\}\end{aligned}}$$

PROB. 6.40: The formulation in terms of $\theta(r,z,t) = T(r,z,t) - T_i$:

$$\left\{\begin{array}{c} \dfrac{\partial^2\theta}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\theta}{\partial r} + \dfrac{\partial^2\theta}{\partial z^2} = \dfrac{1}{\alpha}\dfrac{\partial\theta}{\partial t} \\ \theta(r,z,0) = 0 \\ \theta(0,z,t) = \text{finite}\,;\quad k\left.\dfrac{\partial\theta}{\partial r}\right|_{r=r_0} = q_w'' \\ \theta(r,0,t) = 0\,;\quad \theta(r,H,t) = 0 \end{array}\right\}$$

$$\theta(r,z,t) = \theta_t(r,z,t) + \theta_s(r,z)$$

$$\frac{\partial^2\theta_t}{\partial r^2} + \frac{1}{r}\frac{\partial\theta_t}{\partial r} + \frac{\partial^2\theta_t}{\partial z^2} = \frac{1}{\alpha}\frac{\partial\theta_t}{\partial t}$$

$$\theta_t(r,z,0) = -\theta_s(r,z)$$

$$\theta_t(0,z,t) = \text{finite}\,;\quad \left.\frac{\partial\theta_t}{\partial r}\right|_{r=r_0} = 0$$

$$\theta_t(r,0,t) = 0\,;\quad \theta_t(r,H,t) = 0$$

$$\frac{\partial^2\theta_s}{\partial r^2} + \frac{1}{r}\frac{\partial\theta_s}{\partial r} + \frac{\partial^2\theta_s}{\partial z^2} = 0$$

$$\theta_s(0,z) = \text{finite}$$

$$\left.\frac{\partial\theta_s}{\partial r}\right|_{r=r_0} = \frac{q_w''}{k}$$

$$\theta_s(r,0) = 0\,;\quad \theta_s(r,H) = 0$$

Solution for $\theta_s(r,z)$:

$$\theta_s(r,z) = \sum_{n=1}^{\infty} A_n I_0\left(\frac{n\pi}{H}r\right)\sin\frac{n\pi}{H}z$$

$$\Downarrow$$

$$\left.\frac{\partial\theta_s}{\partial r}\right|_{r=r_0} = \frac{q_w''}{k} = \frac{\pi}{H}\sum_{n=1}^{\infty} A_n n I_1\left(\frac{n\pi}{H}r_0\right)\sin\frac{n\pi}{H}z$$

Thus,

$$A_n\frac{n\pi}{H}I_1\left(\frac{n\pi}{H}r_0\right) = \frac{2}{H}\left(\frac{q_w''}{k}\right)\int_0^H \sin\frac{n\pi}{H}z\,dz \Rightarrow A_n = \frac{q_w''}{k}\frac{2H}{(n\pi)^2}\frac{[1-(-1)^n]}{I_1\left(\frac{n\pi}{H}r_0\right)}$$

$$\therefore \quad \boxed{\theta_s(r,z) = \frac{q_w''}{k}\,\frac{2H}{\pi^2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^2}\,\frac{I_0(\frac{n\pi}{H}r)}{I_1(\frac{n\pi}{H}r_0)}\sin\frac{n\pi}{H}z}$$

Solution for $\theta_t(r,z,t)$:

$$\theta_t(r,z,t) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} B_{mn}\, e^{-(\lambda_n^2+\beta_m^2)t}\cdot J_0(\lambda_n r)\cdot\sin\beta_m z \ , \quad \beta_m = \frac{m\pi}{H},\ m=1,2,3,\ldots$$

and λ_n's are positive roots of (see Table 4.2)

$$\frac{d J_0(\lambda r_0)}{dr} = 0 \Rightarrow -\lambda J_1(\lambda r_0) = 0 \quad \begin{cases} \lambda_0 = 0 \\ J_1(\lambda_n r_0) = 0 \Rightarrow \lambda_n,\ n=1,2,3,\ldots \end{cases}$$

Thus,

$$-\theta_s(r,z) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} B_{mn}\, J_0(\lambda_n r)\sin\beta_m z$$

Or,

$$-\theta_s(r,z) = \sum_{m=1}^{\infty} B_{m0}\sin\beta_m z + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\, J_0(\lambda_n r)\sin\beta_m z$$

Then,

$$\underbrace{-\int_0^{r_0}\int_0^{H}\theta_s(r,z)\sin\beta_m z\, r\, dr\, dz}_{-\frac{q_w'' r_0}{k}\frac{[1-(-1)^m]}{\beta^3}} = B_{m0}\underbrace{\int_0^{r_0} r\,dr}_{r_0^2/2}\ \underbrace{\int_0^{H}\sin^2\beta_m z\, dz}_{H/2}$$

$$\therefore\ B_{m0} = -\frac{4}{Hr_0}\,\frac{q_w''}{k}\,\frac{[1-(-1)^m]}{\beta^3}$$

and

$$-\underbrace{\int_0^{r_0}\int_0^H \theta_s(r,z)\sin\beta_m z\, J_0(\lambda_n r)\, r\,dr\,dz}_{\vdots} = B_{mn}\underbrace{\int_0^{r_0} J_0^2(\lambda_n r)\, r\,dr}_{\frac{r_0^2}{2}J_0^2(\lambda_n r_0)}\underbrace{\int_0^H \sin^2\beta_m z\,dz}_{\frac{H}{2}}$$

$$-\frac{q_w'' r_0}{k} J_0(\lambda_n r_0)\frac{[1-(-1)^m]}{\beta_m[\lambda_n^2+\beta_m^2]}$$

$$\therefore\ B_{mn} = -\frac{4}{Hr_0}\frac{q_w''}{k}\frac{1}{J_0(\lambda_n r_0)}\frac{[1-(-1)^m]}{\beta_m[\lambda_n^2+\beta_m^2]}$$

Note that $B_{mn}|_{n=0} = B_{m0}$. Thus,

$$\theta_t(r,z,t) = -\frac{4}{Hr_0}\frac{q_w''}{k}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{[1-(-1)^m]}{\beta_m[\lambda_n^2+\beta_m^2]}e^{-(\lambda_n^2+\beta_m^2)t}\frac{J_0(\lambda_n r)}{J_0(\lambda_n r_0)}\sin\beta_m z$$

Hence, the solution is given by

$$\boxed{\begin{aligned} T(r,z,t)-T_i &= \frac{2H}{\pi^2}\frac{q_w''}{k}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^2}\frac{I_0\left(\frac{n\pi}{H}r\right)}{I_1\left(\frac{n\pi}{H}r_0\right)}\sin\frac{n\pi}{H}z \\ &- \frac{4}{Hr_0}\frac{q_w''}{k}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{[1-(-1)^m]}{\beta_m[\lambda_n^2+\beta_m^2]}e^{-(\lambda_n^2+\beta_m^2)t}\frac{J_0(\lambda_n r)}{J_0(\lambda_n r_0)}\sin\beta_m z\end{aligned}}$$

PROB. 6.42: The formulation in terms of $\theta(x,y,t) = T(x,y,t) - T_i$:

$$\left\{ \begin{array}{c} \dfrac{\partial^2\theta}{\partial x^2} + \dfrac{\partial^2\theta}{\partial y^2} + \dfrac{\dot{q}}{k} = \dfrac{1}{\alpha}\dfrac{\partial\theta}{\partial t} \\ \theta(x,y,0) = 0 \\ \theta(0,y,t) = \theta(a,y,t) = \theta(x,0,t) = \theta(x,b,t) = 0 \end{array} \right\}$$

Let $\theta(x,y,t) = \theta_t(x,y,t) + \theta_s(x,y)$

$$\frac{\partial^2\theta_t}{\partial x^2} + \frac{\partial^2\theta_t}{\partial y^2} = \frac{1}{\alpha}\frac{\partial\theta_t}{\partial t}$$

$$\theta_t(x,y,0) = -\theta_s(x,y)$$

$$\theta_t(0,y,t) = \theta_t(a,y,t) = \theta(x,0,t) = \theta(x,b,t) = 0$$

$$\frac{\partial^2\theta_s}{\partial x^2} + \frac{\partial^2\theta_s}{\partial y^2} + \frac{\dot{q}}{k} = 0$$

$$\theta_s(0,y) = \theta_s(a,y) = \theta_s(x,0) = \theta_s(x,b) = 0$$

Solution for $\theta_s(x,y)$: Let $\theta_s(x,y) = \psi(x,y) + \phi(x)$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

$$\psi(0,y) = \psi(a,y) = 0$$

$$\psi(x,0) = \psi(x,b) = -\phi(x)$$

$$\frac{d^2\phi}{dx^2} + \frac{\dot{q}}{k} = 0$$

$$\phi(0) = \phi(a) = 0$$

$$\therefore \boxed{\phi(x) = \frac{\dot{q}a^2}{2k}\left(\frac{x}{a}\right)\left(1 - \frac{x}{a}\right)}$$

Let $\psi(x,y) = \psi_1(x,y) + \psi_2(x,y)$

$$\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial y^2} = 0$$

$$\psi_1(0,y) = \psi_1(a,y) = 0$$

$$\psi_1(x,0) = 0\ ;\ \psi_1(x,b) = -\phi(x)$$

$$\frac{\partial^2\psi_2}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

$$\psi_2(0,y) = \psi_2(a,y) = 0$$

$$\psi_2(x,0) = -\phi(x);\ \psi_2(x,b) = 0$$

<u>Solution for $\psi_1(x,y)$:</u> See Eq. (5.17) in text.

$$\psi_1(x,y) = \frac{2}{a}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{a}x \,\sinh\frac{n\pi}{a}y}{\sinh\frac{n\pi}{a}b}\int_0^a \{-\phi(x')\}\sin\frac{n\pi}{a}x'dx'$$

Here,

$$\int_0^a \phi(x)\sin\frac{n\pi}{a}x\,dx = \cancel{-\phi(x)\frac{\cos\frac{n\pi}{a}x}{(n\pi/a)}\Big|_0^a}^{\,0} + \int_0^a \frac{\cos\frac{n\pi}{a}x}{(n\pi/a)}\frac{d\phi}{dx}dx$$

$$= \cancel{\frac{d\phi}{dx}\frac{\sin\frac{n\pi}{a}x}{(n\pi/a)^2}\Big|_0^a}^{\,0} - \left(\frac{a}{n\pi}\right)^2\int_0^a \sin\frac{n\pi}{a}x\frac{d^2\phi}{dx^2}dx$$

$$= -\left(\frac{a}{n\pi}\right)^2\int_0^a \sin\frac{n\pi}{a}x\left\{-\frac{\dot{q}}{k}\right\}dx$$

$$= -\left(\frac{a}{n\pi}\right)^2\frac{\dot{q}}{k}\frac{\cos\frac{n\pi}{a}x}{n\pi/a}\Big|_0^a = \left(\frac{a}{n\pi}\right)^3\frac{\dot{q}}{k}\left[1-(-1)^n\right]$$

$$\therefore\ \psi_1(x,y) = -\frac{2a^2\dot{q}}{k\pi^3}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^3}\frac{\sin\frac{n\pi}{a}x\,\sinh\frac{n\pi}{a}y}{\sinh\frac{n\pi}{a}b}$$

Then, the solution for $\psi_2(x,y)$ can be written as

$$\psi_2(x,y) = \frac{2a^2\dot{q}}{k\pi^3}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^3}\frac{\sin\frac{n\pi}{a}x\,\sinh\frac{n\pi}{a}(b-y)}{\sinh\frac{n\pi}{a}b}$$

Thus,

$$\theta_s(x,y) = \frac{\dot{q}a^2}{2k}\left(\frac{x}{a}\right)\left(1-\frac{x}{a}\right)$$

$$-\frac{2a^2\dot{q}}{k\pi^3}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^3}\frac{\sin\frac{n\pi}{a}x}{\sinh\frac{n\pi}{a}b}\left[\sinh\frac{n\pi}{a}y + \sinh\frac{n\pi}{a}(b-y)\right]$$

Solution for $\theta_t(x,y,t)$:

$$\theta_t(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi}{a}x \cdot \sin\frac{m\pi}{b}y \cdot e^{-\left[\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right]\alpha t}$$

$$\Downarrow$$

$$-\theta_s(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi}{a}x \cdot \sin\frac{m\pi}{b}y$$

$$\therefore\ -\int_0^a\int_0^b \theta_s(x,y)\sin\frac{n\pi}{a}x \cdot \sin\frac{m\pi}{b}y\,dx\,dy = A_{mn}\underbrace{\int_0^a \sin^2\frac{n\pi}{a}x\,dx}_{\frac{a}{2}} \cdot \underbrace{\int_0^b \sin^2\frac{m\pi}{b}y\,dy}_{\frac{b}{2}}$$

$$A_{mn} = -\frac{4}{ab}\int_0^a\int_0^b \theta_s(x,y)\sin\frac{n\pi}{a}x \cdot \sin\frac{m\pi}{b}y\,dx\,dy$$

$$\vdots$$

$$A_{mn} = -\frac{\dot{q}}{k}\,\frac{1}{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}$$

Thus, the final solution is

$$T(x,y,t) - T_i = \frac{\dot{q}a^2}{2k}\left(\frac{x}{a}\right)\left(1-\frac{x}{a}\right) - \frac{2a^2\dot{q}}{k\pi^3}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^3}\,\frac{\sin\frac{n\pi}{a}x}{\sin\frac{n\pi}{a}b}\left[\sinh\frac{n\pi}{a}y+\sinh\frac{n\pi}{a}(b-a)\right] - \frac{\dot{q}}{k}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin\frac{n\pi}{a}x\,\sin\frac{m\pi}{b}y}{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}\,e^{-\left[\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right]\alpha t}$$

PROB. 6.43: Formulation in terms of $\theta(r,z,t) = T(r,z,t) - T_i$:

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta(r,z,0) = 0$$

$$\theta(0,z,t) = \text{finite}; \quad \theta(r_0,z,t) = 0$$

$$\theta(r,0,t) = 0; \quad \theta(r,H,t) = 0$$

Let $\theta(r,z,t) = \theta_t(r,z,t) + \theta_s(r,z)$

$$\frac{\partial^2\theta_t}{\partial r^2} + \frac{1}{r}\frac{\partial\theta_t}{\partial r} + \frac{\partial^2\theta_t}{\partial z^2} = \frac{1}{\alpha}\frac{\partial\theta_t}{\partial t}$$

$$\theta_t(r,z,0) = -\theta_s(r,z)$$

$$\theta_t(0,z,t) = \text{finite}; \quad \theta_t(r_o,z,t) = 0$$

$$\theta_t(r,0,t) = 0; \quad \theta_t(r,H,t) = 0$$

$$\frac{\partial^2\theta_s}{\partial r^2} + \frac{1}{r}\frac{\partial\theta_s}{\partial r} + \frac{\partial^2\theta_s}{\partial z^2} + \frac{\dot{q}}{k} = 0$$

$$\theta_s(0,z) = \text{finite}; \quad \theta_s(r_o,z) = 0$$

$$\theta_s(r,0) = 0; \quad \theta_s(r,H) = 0$$

From Prob. 5.21:

$$\theta_s(r,z) = \frac{\dot{q}H^2}{2k}\left[\left(\frac{z}{H}\right) - \left(\frac{z}{H}\right)^2\right] - \frac{2\dot{q}H^2}{\pi^3 k}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^3}\frac{I_0\left(\frac{n\pi r}{H}\right)}{I_0\left(\frac{n\pi r_0}{H}\right)}\sin\frac{n\pi}{H}z$$

On the other hand,

$$\theta_t(r,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\, e^{-\alpha[\lambda_n^2+\beta_m^2]t} J_0(\lambda_n r)\sin\beta_m z$$

where

$$\beta_m = \frac{m\pi}{H}, \quad m = 1, 2, 3, \ldots$$

and λ_n's are the positive roots of $J_0(\lambda_n r_0) = 0$.

Thus,

$$-\theta_s(r,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\, J_0(\lambda_n r)\, \sin\beta_m z$$

$$\therefore\ \underbrace{-\int_0^H\int_0^{r_0} \theta_s(r,z)\, \sin\beta_m z\, J_0(\lambda_n r)\, r\,dr\,dz}_{\vdots} = A_{mn} \underbrace{\int_0^{r_0} J_0^2(\lambda_n r)\, r\,dr}_{\frac{r_0^2}{2} J_1^2(\lambda_n r_0)} \cdot \underbrace{\int_0^H \sin^2\beta_m z\, dz}_{\frac{H}{2}}$$

$$\frac{\dot{q}\, r_0}{k} \frac{[1-(-1)^m]\, J_1(\lambda_n r_0)}{\beta_m \lambda_n (\lambda_n^2 - \beta_m^2)}$$

$$\therefore\ A_{mn} = \frac{4}{r_0 H}\, \frac{\dot{q}}{k}\, \frac{[1-(-1)^m]}{\beta_m \lambda_n (\lambda_n^2 - \beta_m^2)\, J_1(\lambda_n r_0)}$$

Now, the solution can be written as

$$T(r,z,t) - T_i = \frac{4}{r_0 H}\frac{\dot{q}}{k} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{[1-(-1)^m]}{\beta_m\lambda_n(\lambda_n^2-\beta_m^2)} \frac{J_0(\lambda_n r)}{J_1(\lambda_n r_0)} \sin\beta_m z\; e^{-\alpha(\lambda_n^2+\beta_m^2)t}$$

$$+\ \frac{\dot{q}H^2}{2k}\left\{ \frac{z}{H} - \left(\frac{z}{H}\right)^2 - \frac{4}{\pi^3}\sum_{n=1}^{\infty} \frac{[1-(-1)^n}{n^3} \frac{I_0(\frac{n\pi}{H} r)}{I_0(\frac{n\pi}{H} r_0)} \sin\frac{n\pi}{H} z \right\}$$

CHAPTER 7

SOLUTIONS WITH INTEGRAL TRANSFORMS

PROB.7.1: From Prob. 4.6:

$$f(x) = \sum_{n=1}^{\infty}\left\{\frac{1}{N_n}\int_0^L f(x')\cos\lambda_n x'\,dx'\right\}\cos\lambda_n x \quad \text{with } N_n = \frac{1}{2\lambda_n}\left[\lambda_n L + \sin\lambda_n L\cos\lambda_n L\right]$$

and λ_n's are the positive roots of $\lambda\tan\lambda L = \frac{\alpha}{\beta}$. Now, rewrite the expansion in two parts:

$$\left.\begin{aligned} \bar{f}_n &= \int_0^L f(x')K_n(x')\,dx' \quad \longleftarrow \underline{\text{Transform}} \\ f(x) &= \sum_{n=1}^{\infty}\bar{f}_n\cdot K_n(x) \quad \longleftarrow \underline{\text{Inversion}} \end{aligned}\right\} \quad \underset{\underline{\text{Kernel}}}{\overset{\uparrow}{K_n(x)}} = \frac{\cos\lambda_n x}{\sqrt{N_n}}$$

PROB.7.2: From Prob. 4.16;

$$f(r) = \sum_{n=1}^{\infty}\left\{\frac{1}{N_n}\int_a^b f(r')\phi_n(r')r'dr'\right\}\phi_n(r)$$

where

$$\phi_n(r) = J_0(\lambda_n r)Y_0(\lambda_n b) - J_0(\lambda_n b)Y_0(\lambda_n r) \ \&\ N_n = \frac{2}{\pi^2\lambda_n^2}\left[1 - \frac{J_0^2(\lambda_n b)}{J_0^2(\lambda_n a)}\right]$$

and λ_n's are the positive roots of

$$J_0(\lambda a)Y_0(\lambda b) - J_0(\lambda b)Y_0(\lambda a) = 0$$

Now, rewrite the expansion in two parts:

$$\left.\begin{aligned} \bar{f}_n &= \int_a^b f(r')K_n(r')r'dr' \quad \longleftarrow \underline{\text{Transform}} \\ f(r) &= \sum_{n=1}^{\infty}\bar{f}_n\cdot K_n(r) \quad \longleftarrow \underline{\text{Inversion}} \end{aligned}\right\} \quad \underset{\underline{\text{Kernel}}}{\overset{\uparrow}{K_n(r)}} = \frac{\phi_n(r)}{\sqrt{N_n}}$$

PROB. 7.3: Re-formulating the problem given by Eqs. (7.16) in terms of $\theta(x,t) = T(x,t) - T_\infty(t)$ we get:

$$\left\{\begin{array}{c} \dfrac{\partial^2\theta}{\partial x^2} + F(x,t) = \dfrac{1}{\alpha}\dfrac{\partial\theta}{\partial x} \\ \theta(x,0) = \theta_i(x) \\ \theta(0,t) = \theta(L,t) = 0 \end{array}\right\} \quad \text{where} \quad \left\{\begin{array}{l} F(x,t) = \dfrac{\dot{q}(x,t)}{k} - \dfrac{1}{\alpha}\dfrac{dT_\infty}{dt} \\ \theta_i(x) = T_i(x) - T_\infty(0) \end{array}\right.$$

Solution to this problem by the integral transform method (see Sec. 7.3) is given by

$$\theta(x,t) = \frac{2}{L}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\sin\lambda_n x\left\{\int_0^L \theta_i(x')\sin\lambda_n x'dx' + \alpha\int_0^t e^{\alpha\lambda_n^2 t'}\int_0^L F(x',t')\sin\lambda_n x'dx'dt'\right\},\quad \lambda_n = \frac{n\pi}{L}$$

Substituting the definitions of $F(x,t)$ and $\theta_i(x)$ into this solution we obtain:

$$\theta(x,t) = T(x,t) - T_\infty(t) = \frac{2}{L}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\sin\lambda_n x\left\{\int_0^L T_i(x')\sin\lambda_n x'dx' - \frac{1}{\lambda_n}\left[1-(-1)^n\right]\left[T_\infty(0) + \int_0^t e^{\alpha\lambda_n^2 t'}\frac{dT_\infty}{dt'}dt'\right] + \frac{1}{\rho c}\int_0^t e^{\alpha\lambda_n^2 t}\int_0^L \dot{q}(x',t')\sin\lambda_n x'dx'dt'\right\}$$

This is the same result as given by Eq. (7.33).

PROB. 7.9:

- Because of symmetry, consider only one half of the rod.
- Assume $k =$ Constant.

Formulation in terms of $\theta(r,z) = T(r,z) - T_\infty$:

$$\left.\begin{aligned} &\underbrace{\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r}}_{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)} + \frac{\partial^2\theta}{\partial z^2} + \frac{\dot{q}(z)}{k} = 0 \\ &\theta(0,z) = \text{finite} \; ; \; -k\left.\frac{\partial\theta}{\partial r}\right|_{r=r_0} = h\,\theta(r_0,z) \\ &\left.\frac{\partial\theta}{\partial z}\right|_{z=0} = \left.\frac{\partial\theta}{\partial z}\right|_{z=L} = 0 \end{aligned}\right\} \text{ where } \dot{q}(z) = \begin{cases} \dot{q}, & 0 < z < \frac{L}{2} \\ 0, & \frac{L}{2} < z < L \end{cases}$$

Define

$$\bar{\theta}_n(r) = \int_0^L \theta(r,z)\,K_n(z)\,dz \quad \longleftarrow \text{Transform}$$

with

$$\theta(r,z) = \sum_{n=0}^{\infty} \bar{\theta}_n(r)\,K_n(z) \quad \longleftarrow \text{Inversion}$$

where $K_n(z)$ are the normalized eigenfunctions of

$$\left.\begin{aligned} &\frac{d^2Z}{dz^2} + \lambda^2 Z = 0 \\ &\frac{dZ(0)}{dz} = 0 \\ &\frac{dZ(L)}{dz} = 0 \end{aligned}\right\}$$

From Table 7.1,

$$K_n(z) = \frac{\cos\lambda_n z}{\sqrt{N_n}}, \quad \lambda_n = \frac{n\pi}{L}, \; n = 0,1,2,\ldots$$

$$N_n = \begin{cases} L, & n = 0 \\ \frac{L}{2}, & n = 1,2,3,\ldots \end{cases}$$

Take the transform of the heat conduction equation:

$$\underbrace{\int_0^L \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)K_n(z)\,dz}_{\frac{1}{r}\frac{d}{dr}\left(r\frac{d\bar{\theta}_n}{dr}\right)} + \underbrace{\int_0^L \frac{\partial^2\theta}{\partial z^2}K_n(z)\,dz}_{I_n} + \underbrace{\int_0^L \frac{\dot{q}(z)}{k}K_n(z)\,dz}_{J_n} = 0$$

$$I_n = \int_0^L \frac{\partial^2\theta}{\partial z^2} K_n(z)\,dz = \cancel{\frac{\partial\theta}{\partial z} K_n(z)}\Big|_{z=0}^{z=L} - \int_0^L \frac{\partial\theta}{\partial z}\frac{dK_n}{dz}\,dz$$

$$= -\,\theta(r,z)\frac{dK_n}{dz}\Big|_{z=0}^{z=L} + \int_0^L \theta \underbrace{\frac{d^2K_n}{dz^2}}_{-\lambda_n^2 K_n(z)}\,dz = -\lambda_n^2 \bar{\theta}_n(r)$$

$$J_n = \int_0^L \frac{\dot{q}(z)}{k} K_n(z)\,dz = \frac{\dot{q}}{k}\int_0^{L/2} K_n(z)\,dz$$

$$\vdots$$

$$= \begin{cases} \dfrac{\dot{q}}{2k}\sqrt{L}\,, & n=0 \\ \dfrac{\dot{q}}{k}\sqrt{\dfrac{2}{L}}\,\dfrac{1}{\lambda_n}\sin\lambda_n\dfrac{L}{2}\,, & n=1,2,3,\ldots \end{cases}$$

Thus,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\bar{\theta}_n}{dr}\right) - \lambda_n^2\bar{\theta}_n(r) = -J_n$$

with

$$\bar{\theta}_n(0) = \text{finite}\;;\quad -k\frac{d\bar{\theta}_n}{dr}\Big|_{r=r_0} = h\bar{\theta}_n(r_0)$$

<u>When $n=0$</u>,

$$\bar{\theta}_0(r) = J_n\frac{r_0^2}{4}\left[1 + \frac{2k}{hr_0} - \left(\frac{r}{r_0}\right)^2\right]$$

<u>When $n=1,2,3,\ldots$</u>,

$$\bar{\theta}_n(r) = \frac{J_n}{\lambda_n^2}\left[1 - \frac{h I_0(\lambda_n r)}{k\lambda_n I_1(\lambda_n r_0) + h I_0(\lambda_n r_0)}\right]$$

Thus, inverting $\bar{\theta}_n(r)$,

$$\theta(r,z) = \frac{J_n r_0^2}{4}\left[1 + \frac{2k}{hr_0} - \left(\frac{r}{r_0}\right)^2\right]\cdot\overbrace{\frac{1}{\sqrt{L}}}^{K_0(z)} + \sum_{n=1}^{\infty}\frac{J_n}{\lambda_n^2}\left[1 - \frac{h I_0(\lambda_n r)}{k\lambda_n I_1(\lambda_n r_0) + h I_0(\lambda_n r_0)}\right]\frac{\cos\lambda_n z}{\sqrt{L/2}}$$

Or

$$\theta(r,z) = T(r,z) - T_\infty = \frac{\dot{q} r_0^2}{8k}\left[1 + \frac{2k}{hr_0} - \left(\frac{r}{r_0}\right)^2\right] + \frac{2\dot{q}}{kL}\sum_{n=1}^{\infty}\frac{\sin\lambda_n\frac{L}{2}}{\lambda_n^3}\left[1 - \frac{h I_0(\lambda_n r)}{k\lambda_n I_1(\lambda_n r_0) + h I_0(\lambda_n r_0)}\right]\cos\lambda_n z$$

Alternative Solution

Define
$$\bar{\bar{\theta}}_{mn} = \int_0^{r_0} \bar{\theta}_n(r)\, K_0(\beta_m, r)\, r\, dr \quad \leftarrow \underline{\text{Transform}}$$

With
$$\bar{\theta}_n(r) = \sum_{m=1}^{\infty} \bar{\bar{\theta}}_{mn}\, K_0(\beta_m, r) \quad \leftarrow \underline{\text{Inversion}}$$

where $K_0(\beta_m, r)$ are normalized eigenfunctions of

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + \beta^2 r^2 R = 0$$

$$R(0) = \text{finite}$$

$$h\,R(r_0) + k\,\frac{R(r_0)}{dr} = 0$$

From Table 7.3,

$$K_0(\beta_m, r) = \frac{\sqrt{2}}{r_0}\, \frac{1}{\left[1 + \left(\frac{h}{k\beta_m}\right)^2\right]^{1/2}}\, \frac{J_0(\beta_m r)}{J_0(\beta_m r_0)}$$

where β_m's are the positive roots of

$$h\,J_0(\beta r_0) + k\,\frac{dJ_0(\beta r_0)}{dr} = 0 \;\Rightarrow\; \frac{J_0(\beta r_0)}{J_1(\beta r_0)} = \frac{\beta k}{h}, \quad m = 1, 2, 3, \ldots$$

Thus, $K_0(\beta_m, r)$ can also be written as

$$K_0(\beta_m, r) = \frac{\sqrt{2}}{r_0}\, \frac{J_0(\beta_m r)}{\left[J_0^2(\beta_m r_0) + J_1^2(\beta_m r_0)\right]^{1/2}}$$

Now, obtain the second transform of $\bar{\theta}_n(r)$ problem:

$$\underbrace{\int_0^{r_0} \frac{1}{\cancel{r}}\frac{d}{dr}\left(r\frac{d\bar{\theta}_n}{dr}\right) K_0(\beta_m, r)\, \cancel{r}\, dr}_{} - \lambda_n^2\, \bar{\bar{\theta}}_{mn} = -J_n \int_0^{r_0} K_0(\beta_m, r)\, r\, dr$$

$\Downarrow$ Integrating twice by parts

$$-\beta_m^2\, \bar{\bar{\theta}}_{mn}$$

Furthermore,

$$\int_0^{r_0} K_o(\beta_m,r)\,r\,dr = \frac{\sqrt{2}}{r_0}\frac{1}{\left[J_0^2(\beta_m r_0)+J_1^2(\beta_m r_0)\right]^{1/2}}\underbrace{\int_0^{r_0} J_0(\beta_m r)\,r\,dr}_{\frac{r_0}{\beta_m}J_1(\beta_m r_0)}$$

$$\vdots$$

$$= \frac{\sqrt{2}}{\beta_m\left[1+\left(\frac{\beta_m k}{h}\right)^2\right]^{1/2}}$$

Thus,

$$\left(\lambda_n^2+\beta_m^2\right)\bar{\bar{\theta}}_{mn} = \frac{\sqrt{2}\; J_n}{\beta_m\left[1+\left(\frac{\beta_m k}{h}\right)^2\right]^{1/2}}$$

$$\bar{\bar{\theta}}_{mn} = \frac{\sqrt{2}\; J_n}{\beta_m(\lambda_n^2+\beta_m^2)\left[1+\left(\frac{\beta_m k}{h}\right)^2\right]^{1/2}}$$

Then,

$$\theta(r,z) = \sum_{n=0}^{\infty}\left\{\sum_{m=1}^{\infty}\bar{\bar{\theta}}_{mn}K_o(\beta_m,r)\right\}K_n(z)$$

$$= \sum_{m=1}^{\infty}\left\{\sum_{n=0}^{\infty}\bar{\bar{\theta}}_{mn}K_n(z)\right\}K_o(\beta_m,r)$$

$$\vdots$$

$$\theta(r,z) = \frac{2}{r_0}\frac{\dot{q}}{k}\sum_{m=1}^{\infty}\frac{1}{\beta_m\left[1+\left(\frac{\beta_m k}{h}\right)^2\right]^{1/2}}\left\{\frac{1}{2\beta_m^2}\right.$$

$$\left.+\frac{2}{L}\sum_{n=1}^{\infty}\frac{\sin\lambda_n\frac{L}{2}\cos\lambda_n z}{\lambda_n(\lambda_n^2+\beta_m^2)}\right\}\frac{J_0(\beta_m r)}{\left[J_0^2(\beta_m r_0)+J_1^2(\beta_m r_0)\right]^{1/2}}$$

PROB. 7.11: Formulation in terms of $\theta(x,y) = T(x,y) - T_0$:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} - m\,(\theta - \theta_\infty) = 0, \qquad \theta_\infty = T_\infty - T_0$$

$$\theta(0,y) = \theta(a,y) = 0$$

$$\theta(x,0) = \theta(x,b) = 0$$

In the interval $(0,a)$, define a <u>finite Fourier transform</u> of $\theta(x,y)$ with respect to x as:

$$\bar\theta(\lambda_n, y) = \int_0^a \theta(x,y)\, K(\lambda_n, x)\, dx$$

with the <u>inversion</u>

$$\theta(x,y) = \sum_{n=1}^{\infty} \bar\theta(\lambda_n, y)\, K(\lambda_n, x)$$

with

(from Table 7.1) $\quad K(\lambda_n, x) = \sqrt{\frac{2}{a}}\,\sin\lambda_n x, \quad \lambda_n = \frac{n\pi}{a}, \quad n = 1, 2, 3, \dots$

Now, obtain the transform of the differential equation:

$$\underbrace{\int_0^a \frac{\partial^2\theta}{\partial x^2} K(\lambda_n, x)\, dx}_{= \dots = -\lambda_n^2\, \bar\theta(\lambda_n, y)} + \frac{d^2\bar\theta}{dy^2} - m^2\, \bar\theta(\lambda_n, y) + m^2\theta_\infty \underbrace{\int_0^a K(\lambda_n, x)\, dx}_{\sqrt{\frac{2}{a}}\,\frac{1-(-1)^n}{\lambda_n}} = 0$$

Thus,

$$\frac{d^2\bar\theta}{dy^2} - (\lambda_n^2 + m^2)\,\bar\theta(\lambda_n, y) = -m^2\theta_\infty \sqrt{\frac{2}{a}}\;\frac{1-(-1)^n}{\lambda_n} \qquad \text{— ①}$$

$$\therefore\ \bar\theta(\lambda_n, y) = A_n \sinh\sqrt{\lambda_n^2 + m^2}\, y + B_n \cosh\sqrt{\lambda_n^2 + m^2} + \underbrace{m^2\theta_\infty \sqrt{\frac{2}{a}}\;\frac{1-(-1)^n}{\lambda_n(\lambda_n^2 + m^2)}}_{\text{Particular solution } \bar\theta_p}$$

$$\theta(x,0)=0 \Rightarrow \bar{\theta}(\lambda_n,0)=0 \Rightarrow B_n=0$$

$$\theta(x,b)=0 \Rightarrow \bar{\theta}(\lambda_n,b)=0 \Rightarrow A_n=-\frac{\bar{\theta}_p}{\sinh\sqrt{\lambda_n^2+m^2}\,b}$$

Thus,

$$\bar{\theta}(\lambda_n,y)=\bar{\theta}_p\left\{1-\frac{\sinh\sqrt{\lambda_n^2+m^2}\,y}{\sinh\sqrt{\lambda_n^2+m^2}\,b}\right\}$$

and

$$\theta(x,y)=\sum_{n=1}^{\infty}\bar{\theta}_p\left\{1-\frac{\sinh\sqrt{\lambda_n^2+m^2}\,y}{\sinh\sqrt{\lambda_n^2+m^2}\,b}\right\}K(\lambda_n,x)$$

$$\Downarrow$$

$$\boxed{\frac{\theta(x,y)}{\theta_\infty}=\frac{T(x,y)-T_0}{T_\infty-T_0}=\frac{2m^2}{a}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{\lambda_n^2(\lambda_n^2+m^2)}\sin\lambda_n x\left\{1-\frac{\sinh\sqrt{\lambda_n^2+m^2}\,y}{\sinh\sqrt{\lambda_n^2+m^2}\,b}\right\}}$$

Alternative approach

Define $$\bar{\bar{\theta}}(\lambda_n,\beta_k)=\int_0^b\bar{\theta}(\lambda_n,y)\,K(\beta_k,y)\,dy$$

with $$\bar{\theta}(\lambda_n,y)=\sum_{n=1}^{\infty}\bar{\bar{\theta}}(\lambda_n,\beta_k)\,K(\beta_k,y)$$

and

$$K(\beta_k,y)=\sqrt{\frac{2}{b}}\sin\beta_k y,\quad \beta_k=\frac{k\pi}{b},\ k=1,2,3,\ldots$$

Now, one more transform of Eq. 1 in the y-direction gives

$$\underbrace{\int_0^b \frac{d^2\bar{\theta}}{dy^2} K(\beta_k, y)\, dy}_{= \dots = -\beta_k^2 \bar{\bar{\theta}}(\lambda_n, \beta_k)} - (\lambda_n^2 + m^2)\, \bar{\bar{\theta}}(\lambda_n, \beta_k) = -m^2 \theta_\infty \sqrt{\frac{2}{a}}\, \frac{1-(-1)^n}{\lambda_n} \underbrace{\int_0^b K(\beta_k, y)\, dy}_{\sqrt{\frac{2}{b}}\, \frac{1-(-1)^k}{\beta_k}}$$

Thus,

$$\left(\lambda_n^2 + \beta_k^2 + m^2\right) \bar{\bar{\theta}}(\lambda_n, \beta_k) = m^2 \theta_\infty \sqrt{\frac{4}{ab}}\, \frac{[1-(-1)^n][1-(-1)^k]}{\lambda_n \beta_k}$$

$$\bar{\bar{\theta}}(\lambda_n, \beta_k) = m^2 \theta_\infty \frac{2}{\sqrt{ab}}\, \frac{[1-(-1)^n][1-(-1)^k]}{\lambda_n \beta_k (\lambda_n^2 + \beta_k^2 + m^2)}$$

$$\Downarrow$$

$$\boxed{\frac{\theta(x,y)}{\theta_\infty} = \frac{T(x,y) - T_0}{T_\infty - T_0} = m^2 \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{[1-(-1)^n][1-(-1)^k]}{\lambda_n \beta_k (\lambda_n^2 + \beta_k^2 + m^2)} \sin\lambda_n x \cdot \sin\beta_k y}$$

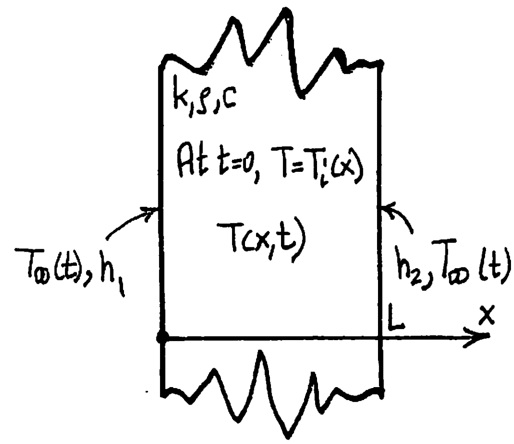
PROB. 7.13: Formulation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}(x,t)}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(x,0) = T_i(x)$$

$$k\left.\frac{\partial T}{\partial x}\right|_{x=0} = h_1\left[T(0,t) - T_\infty(t)\right]$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h_2\left[T(L,t) - T_\infty(t)\right]$$



Define,

$$\bar{T}_n(t) = \int_0^L T(x,t)\, K_n(x)\, dx \leftarrow \text{Transform}$$

with

$$T(x,t) = \sum_{n=1}^{\infty} \bar{T}_n(t)\, K_n(x) \leftarrow \text{Inversion}$$

where, from Table 7.1,

$$K_n(x) = \sqrt{2}\,\frac{\lambda_n \cos\lambda_n x - H_1 \sin\lambda_n x}{\left\{(\lambda_n^2 + H_1^2)\left[L + \frac{H_2}{\lambda_n^2 + H_1^2}\right] - H_1\right\}^{1/2}} \quad \text{with } H_1 = -\frac{h_1}{k} \;\&\; H_2 = \frac{h_2}{k}$$

Where λ_n are the positive roots of

$$\tan\lambda L = \frac{\lambda(H_2 - H_1)}{\lambda^2 + H_1 H_2} \Rightarrow \lambda_n,\; n = 1, 2, 3, \ldots$$

Now, the transform of the D.E.:

$$\int_0^L \frac{\partial^2 T}{\partial x^2} K_n(x)\, dx + \frac{1}{k}\int_0^L \dot{q}(x,t) K_n(x)\, dx = \frac{1}{\alpha}\int_0^L \frac{\partial T}{\partial t} K_n(x)\, dx$$

which yields

$$\bar{T}_n(t) = e^{-\alpha\lambda_n^2 t}\left[\bar{T}_i(\lambda_n) + \alpha\int_0^t F_n(t')\, e^{\alpha\lambda_n^2 t'} dt'\right]$$

where

$$F_n(t) = \left[\frac{h_2}{k} K_n(L) + \frac{h_1}{k} K_n(0)\right] T_\infty(t) + \frac{1}{k}\bar{\dot{q}}_n(t)$$

$$\bar{T}_i(\lambda_n) = \int_0^L T_i(x)\, K_n(x)\, dx \;\&\; \bar{\dot{q}}_n(t) = \int_0^L \dot{q}(x,t)\, K_n(x)\, dx$$

Then, inversion results in

$$\therefore \quad T(x,t) = \sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\left\{\bar{T}_i(\lambda_n) + \alpha\int_0^t F_n(t')\, e^{\alpha\lambda_n^2 t'} dt'\right\}$$

PROB.7.15: Formulation in terms of $\theta(x,t) = T(x,t) - T_i$:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\dot{q}_0 e^{-\beta t}}{k} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta(x,0) = 0$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0} = 0; \quad \theta(L,t) = 0$$

Define,

Transform $\rightarrow$ $\bar{\theta}_n(t) = \int_0^L \theta(x,t)\, K_n(x)\, dx$

Inversion $\rightarrow$ $\theta(x,t) = \sum_{n=1}^{\infty} \bar{\theta}_n(t)\, K_n(x)$

with the kernel $K_n(x) = \sqrt{\frac{2}{L}} \cos\lambda_n x$, $\lambda_n = \frac{2n-1}{L}\frac{\pi}{2}$, $n=1,2,3,\ldots$

From Table 7.1 (Note that B.C. at $x=0$ is 2^{nd} kind and at $x=L$ is 1^{st} kind)

Transform of the D.E.:

$$\underbrace{\int_0^L \frac{\partial^2\theta}{\partial x^2} K_n(x)\,dx}_{-\lambda_n^2\bar{\theta}_n(t)} + \frac{\dot{q}_0 e^{-\beta t}}{k} \underbrace{\int_0^L K_n(x)\,dx}_{\sqrt{\frac{2}{L}}\frac{(-1)^{n+1}}{\lambda_n}} = \frac{1}{\alpha}\underbrace{\int_0^L \frac{\partial\theta}{\partial t} K_n(x)\,dx}_{\frac{d\bar{\theta}_n(t)}{dt}}$$

Integrating by parts twice

$$\Rightarrow \quad \frac{d\bar{\theta}_n}{dt} + \alpha\lambda_n^2\bar{\theta}_n = (-1)^{n+1}\sqrt{\frac{2}{L}}\,\frac{\dot{q}_0\,\alpha\, e^{-\beta t}}{k\lambda_n}$$

with the I.C. : $\bar{\theta}_n(0) = 0$

Thus,

$$\bar{\theta}_n(t) = (-1)^n \sqrt{\frac{2}{L}}\, \frac{\alpha \dot{q}_0}{k\lambda_n}\, \frac{e^{-\beta t} - e^{-\alpha\lambda_n^2 t}}{\beta - \alpha\lambda_n^2}$$

Note that if $\beta = \alpha\lambda_n^2$, then

$$\frac{e^{-\beta t} - e^{-\alpha\lambda_n^2 t}}{\beta - \alpha\lambda_n^2} = -t\, e^{-\beta t}$$

Inverting $\bar{\theta}_n(t)$ we obtain

$$\theta(x,t) = \frac{2}{L}\, \frac{\alpha \dot{q}_0}{k} \sum_{n=1}^{\infty} (-1)^n \frac{e^{-\beta t} - e^{-\alpha\lambda_n^2 t}}{\lambda_n(\beta - \alpha\lambda_n^2)} \cos\lambda_n x$$

It can further be shown that (when $\beta \neq \alpha\lambda_n^2$)

$$\frac{\cos\sqrt{\frac{\beta}{\alpha}}\, x}{\cos\sqrt{\frac{\beta}{\alpha}}\, L} - 1 = \frac{2\beta}{L} \sum_{n=1}^{\infty} (-1)^n \frac{\cos\lambda_n x}{\lambda_n(\beta - \alpha\lambda_n^2)}$$

Therefore, the solution can be rewritten as (when $\beta \neq \alpha\lambda_n^2$)

$$\theta(x,t) = \frac{\alpha \dot{q}_0}{k\beta}\left[\frac{\cos\sqrt{\beta/\alpha}\, x}{\cos\sqrt{\beta/\alpha}\, L} - 1\right] e^{-\beta t} - \frac{2}{L}\, \frac{\alpha \dot{q}_0}{k} \sum_{n=1}^{\infty} (-1)^n \frac{e^{-\alpha\lambda_n^2} \cos\lambda_n x}{\lambda_n(\beta - \alpha\lambda_n^2)}$$

PROB. 7.16: The formulation of the problem in terms of $\theta(x,t) = T(x,t) - T_i$:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\dot{q}_0}{k}\left[1+\beta\theta\right] = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta(x,0) = 0$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0} = 0\,; \quad \theta(L,t) = 0$$

Define

$$\bar{\theta}_n(t) = \int_0^L \theta(x,t)\,K_n(x)\,dx \leftarrow \underline{\text{Transform}}$$

with

$$\theta(x,t) = \sum_{n=1}^{\infty} \bar{\theta}_n(t)\cdot K_n(x) \quad \leftarrow \underline{\text{Inversion}}$$

and (from Table 7.):

$$\underline{\text{Kernel}} \rightarrow K_n(x) = \sqrt{\frac{2}{L}}\cos\lambda_n x\,, \quad \lambda_n = \frac{(2n-1)\pi}{2L},\ n = 1,2,3,\ldots$$

Now, obtain the transform of heat conduction equation:

$$\underbrace{\int_0^L \frac{\partial^2\theta}{\partial x^2}K_n(x)\,dx}_{-\lambda_n^2\,\bar{\theta}_n(t)} + \frac{\dot{q}}{k}\underbrace{\int_0^L K_n(x)\,dx}_{-\frac{(-1)^n}{\lambda_n}\sqrt{\frac{2}{L}}} + \frac{\beta\dot{q}_0}{k}\,\bar{\theta}_n(t) = \frac{1}{\alpha}\frac{d\bar{\theta}_n}{dt}$$

Thus,

$$\frac{1}{\alpha}\frac{d\bar{\theta}_n}{dt} + \left(\lambda_n^2 - \frac{\beta\dot{q}_0}{k}\right)\bar{\theta}_n(t) = -\sqrt{\frac{2}{L}}\,\frac{(-1)^n}{\lambda_n}$$

$$\bar{\theta}_n(t) = A_n\exp\left\{-\alpha\left(\lambda_n^2 - \frac{\beta\dot{q}_0}{k}\right)t\right\} + \bar{\theta}_p$$

where

$$\bar{\theta}_p = \sqrt{\frac{2}{L}}\,\frac{\dot{q}_0}{\lambda_n k}\,\frac{(-1)^n}{(-\lambda_n^2 + \dot{q}_0\beta/k)} \leftarrow \underline{\text{Partial solution}}$$

$$\theta(x,0) = 0 \Rightarrow \bar{\theta}_n(0) = 0 \Rightarrow A_n = -\bar{\theta}_p$$

$$\therefore \quad \bar{\theta}_n(t) = \bar{\theta}_p \left[1 - \exp\left\{ -\alpha \left(\lambda_n^2 - \frac{\beta \dot{q}_0}{k} \right) t \right\} \right.$$

Now, inversion gives

$$\theta(x,t) = \sum_{n=1}^{\infty} \bar{\theta}_p \left[1 - \exp\left\{ -\alpha \left(\lambda_n^2 - \frac{\beta \dot{q}_0}{k} \right) t \right\} \cdot K_n(x) \right.$$

$$\Downarrow$$

$$\theta(x,t) = \frac{2\dot{q}_0}{kL} \sum_{n=1}^{\infty} \frac{(-1)^n}{\lambda_n [\dot{q}_0 \beta / k - \lambda_n^2]} \left\{ 1 - \exp\left[-\alpha \left(\lambda_n^2 - \frac{\beta \dot{q}_0}{k} \right) t \right] \cos \lambda_n x \right.$$

Note that this solution holds for either sign of the product $\beta \dot{q}_0$.

<u>Steady-state temperatures:</u>

<u>Case I:</u> $\beta \dot{q} < 0$

Since the exponential terms in this result all tend to zero as $t \to \infty$, the steady-state solution is given by

$$\theta_s(x) = \frac{2\dot{q}_0}{kL} \sum_{n=1}^{\infty} \frac{(-1)^n}{\lambda_n [\dot{q}_0 \beta / k - \lambda_n^2]} = \cdots = \frac{1}{\beta} \left[\frac{\cosh rx}{\cosh rL} - 1 \right], \quad r = \sqrt{\frac{-\beta \dot{q}_0}{k}}$$

<u>Case II:</u> $\beta \dot{q}_0 > 0$

In this case, the exponential terms vanish as $t \to \infty$, if

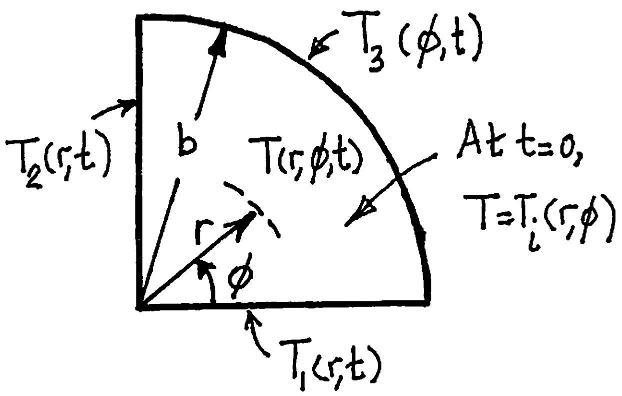
$$p^2 = \frac{\beta \dot{q}_0}{k} < \lambda_1^2 \quad \text{or} \quad \frac{\dot{q}_0 \beta}{k} < \left(\frac{\pi}{2L} \right)^2$$

and

$$\theta_s(x) = \frac{2\dot{q}_0}{kL} \sum_{n=1}^{\infty} \frac{(-1)^n}{\lambda_n [\beta \dot{q}_0 / k - \lambda_n^2]} = \cdots = \frac{1}{\beta} \left[\frac{\cos px}{\cos pL} - 1 \right]$$

If $p^2 > \lambda_1^2$, no steady-state solution exists, because heat would be generated at a rate too great for its removal to be possible.

PROB. 7.23: Assume constant thermophysical properties (k, ρ, c). Then the formulation of the problem is



$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(r,\phi,0)=T_i(r,\phi)$$

$$T(0,\phi,t)=\text{finite};\ T(b,\phi,t)=T_3(\phi,t)$$

$$T(r,0,t)=T_1(r,t);\ T(r,\tfrac{\pi}{2},t)=T_2(r,t)$$

Define, in the interval $(0,\frac{\pi}{2})$, a transform of $T(r,\phi,t)$ with respect to ϕ as

$$\left.\begin{aligned}\bar{T}(r,n,t)&=\int_0^{\pi/2} T(r,\phi,t)\,K(n,\phi)\,d\phi\\ T(r,\phi,t)&=\sum_{n=1}^{\infty}\bar{T}(r,n,t)\cdot K(n,\phi)\end{aligned}\right\}\quad K(n,\phi)=\frac{2}{\sqrt{\pi}}\sin 2n\phi,\quad n=1,2,3,\dots$$

with

Now, obtain the transform of the heat conduction equation w.r.t ϕ:

$$\frac{\partial^2\bar{T}}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{T}}{\partial r}+\frac{1}{r^2}\underbrace{\int_0^{\pi/2}\frac{\partial^2 T}{\partial\phi^2}K(n,\phi)\,d\phi}_{(-1)^n T_2(r,t)\frac{4n}{\sqrt{\pi}}+T_1(r,t)\frac{4n}{\sqrt{\pi}}-4n^2\bar{T}(r,n,\phi)}=\frac{1}{\alpha}\frac{\partial\bar{T}}{\partial t}$$

Thus,

$$\frac{\partial^2\bar{T}}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{T}}{\partial r}+\frac{1}{r^2}\frac{4n}{\sqrt{\pi}}\left[T_1(r,t)+(-1)^n T_2(r,t)\right]-\frac{4n^2}{r^2}\bar{T}(r,n,t)=\frac{1}{\alpha}\frac{\partial\bar{T}}{\partial t}$$

Now, define a Hankel transform as

$$\bar{\bar{T}}(\lambda_m,n,t)=\int_0^b \bar{T}(r,n,t)\,K_n(\lambda_m,r)\,r\,dr$$

$$\bar{T}(r,n,t)=\sum_{m=1}^{\infty}\bar{\bar{T}}(\lambda_m,n,t)\,K_n(\lambda_m,r)$$

where the kernel $K_n(\lambda_m, r)$ are the normalized eigenfunctions of the following eigenvalue problem:

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda^2 r^2 - 4n^2) R(r) = 0$$

$$R(0) = \text{finite}$$

$$R(b) = 0$$

Then, from Table 7.3,

$$K_n(\lambda_m, r) = \frac{\sqrt{2}}{b} \frac{J_{2n}(\lambda_m r)}{J_{2n+1}(\lambda_m b)}$$

and the eigenvalues λ_m are the positive zeros of

$$J_{2n}(\lambda b) = 0$$

Now, a second transform gives

$$\underbrace{\int_0^b \underbrace{\left\{ \frac{\partial^2 \bar{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}}{\partial r} \right\}}_{\frac{1}{\cancel{r}}\frac{\partial}{\partial r}\left(r \frac{\partial \bar{T}}{\partial r}\right)} K_n(\lambda_m, r)\cancel{r}\,dr}_{\vdots} + \frac{4n}{\sqrt{\pi}} \int_0^b \frac{T_1(r,t) + (-1)^n T_2(r,t)}{\cancel{r^2}} K_n(\lambda_m, r)\cancel{r}\,dr$$

$$- 4n^2 \int_0^b \frac{\bar{T}(r,n,t)}{\cancel{r^2}} K_n(\lambda_m, r)\cancel{r}\,dr = \frac{1}{\alpha}\frac{d\bar{\bar{T}}}{dt}$$

$$-b\,\bar{T}_3(n,t) \underbrace{\frac{dK_n(\lambda_m, b)}{dr}}_{-\frac{\sqrt{2}\,\lambda_m}{b}} - \lambda_m^2 \bar{\bar{T}}(\lambda_m, n, t) + 4n^2 \int_0^b \frac{\bar{T}(r,n,t)}{\cancel{r}} K_n(\lambda_m, r)\,dr$$

Thus, we have

$$\frac{1}{\alpha}\frac{d\bar{\bar{T}}}{dt} + \lambda_m^2 \bar{\bar{T}}(\lambda_m, n, t) = F_n^m(t)$$

where

$$F_n^m(t) = \sqrt{2}\,\lambda_m \bar{T}_3(n,t) + \frac{4n}{\pi} \int_0^b \frac{T_1(r,t) + (-1)^n T_2(r,t)}{r} K_n(\lambda_m, r)\,dr$$

Now, the solution of Eq. (1) can be found as

$$\int_0^t \underbrace{e^{\alpha\lambda_m^2 t'}\left[\frac{1}{\alpha}\frac{d\bar{\bar{T}}}{dt}+\lambda_m^2\,\bar{\bar{T}}(\lambda_m,n,t)\right]}_{\frac{1}{\alpha}\frac{d}{dt}\left[e^{\alpha\lambda_m^2 t'}T(\lambda_m,n,t')\right]}dt' = \int_0^t e^{\alpha\lambda_m^2 t'}F_n^m(t')\,dt'$$

Thus,

$$\bar{\bar{T}}(\lambda_m,n,t)=e^{-\alpha\lambda_m^2 t}\Big[\underbrace{\bar{\bar{T}}(\lambda_m,n,0)}_{\bar{\bar{T}}_i(\lambda_m,n)}+\alpha\int_0^t e^{\alpha\lambda_m^2 t'}F_n^m(t')\,dt'\Big]$$

Therefore, the two inversions yield

$$T(r,\phi,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} e^{-\alpha\lambda_m^2 t}\Big[\bar{\bar{T}}_i(\lambda_m,n)+\alpha\int_0^t e^{\alpha\lambda_m^2 t'}F_n^m(t')\,dt'\Big]K_n(\lambda_n r)K(n,\phi)$$

Or,

$$\boxed{\begin{aligned}T(r,\phi,t)=\frac{2}{b\sqrt{\pi}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} e^{-\alpha\lambda_m^2 t}\sin 2n\phi\cdot\frac{J_{2n}(\lambda_m r)}{J_{2n+1}(\lambda_m b)}\\ \times\Big[\bar{\bar{T}}_i(\lambda_m,n)+\alpha\int_0^t e^{\alpha\lambda_m^2 t'}F_n^m(t')\,dt'\Big]\end{aligned}}$$

where

$$\bar{\bar{T}}_i(\lambda_m,n)=\frac{2\sqrt{2}}{b\sqrt{\pi}\,J_{2n+1}(\lambda_m b)}\int_0^b\int_0^{\pi/2} T_i(r,\phi)\sin 2n\phi\, J_{2n}(\lambda_m r)\,r\,dr\,d\phi$$

and

$$F_n^m(t)=\sqrt{2}\,\lambda_m\bar{T}_3(n,t)+\frac{4n}{\pi}\int_0^b \frac{T_1(r,t)+(-1)^n T_2(r,t)}{r}\underbrace{K_n(\lambda_m r)}_{\frac{\sqrt{2}}{b}\frac{J_{2n}(\lambda_m r)}{J_{2n+1}(\lambda_m b)}}dr$$

with

$$\bar{T}_3(n,t)=\frac{2}{\sqrt{\pi}}\int_0^{\pi/2} T_3(\phi,t)\sin 2n\phi\,d\phi$$

PROB. 7.24: If the surface temperatures are independent of time (in Prob. 7.23), then

$$F_n^m = \sqrt{2}\,\lambda_m \bar{T}_3(n) + \frac{4\sqrt{2}\,n}{b\pi}\int_0^b \frac{T_1(r)+(-1)^n T_2(r)}{r\,J_{2n+1}(\lambda_m b)} J_{2n}(\lambda_m r)\,dr$$

where

$$\bar{T}_3(n) = \frac{2}{\sqrt{\pi}}\int_0^{\pi/2} T_3(\phi)\sin 2n\phi\,d\phi$$

Also,

$$\int_0^t e^{\alpha\lambda_m^2 t'} F_n^m\,dt' = F_n^m \int_0^t e^{\alpha\lambda_m^2 t'}\,dt' = F_n^m\,\frac{e^{\alpha\lambda_m^2 t}-1}{\alpha\lambda_m^2}$$

and

$$e^{-\alpha\lambda_m^2 t}\int_0^t e^{\alpha\lambda_m^2 t'} F_n^m\,dt' = F_n^m\,\underbrace{\frac{1-e^{-\alpha\lambda_m^2 t}}{\alpha\lambda_m^2}}_{\text{As } t\to\infty,\ \frac{1}{\alpha\lambda_m^2}}$$

Therefore, from the solution of Prob. 7.23, as $t \to \infty$

$$\boxed{T(r,\phi) = \frac{2}{b\sqrt{\pi}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} F_n^m\,\frac{\sin 2n\phi \cdot J_{2n}(\lambda_m r)}{\lambda_m^2 J_{2n+1}(\lambda_m b)}}$$

CHAPTER 8

SOLUTIONS WITH LAPLACE TRANSFORMS

PROB. 8.1:

$$\mathcal{L}\{\sin\omega t\} = \int_0^\infty e^{-pt}\sin\omega t\,dt = -e^{-pt}\frac{\cos\omega t}{\omega}\Big|_0^\infty - \frac{p}{\omega}\int_0^\infty \cos\omega t\, e^{-pt}dt$$

$$= \frac{1}{\omega} - \frac{p}{\omega}\Big[\cancelto{0}{e^{-pt}\frac{\sin\omega t}{\omega}\Big|_0^\infty} + \frac{p}{\omega}\underbrace{\int_0^\infty \sin\omega t\, e^{-pt}dt}_{\mathcal{L}\{\sin\omega t\}}\Big]$$

$$= \frac{1}{\omega} - \frac{p^2}{\omega^2}\mathcal{L}\{\sin\omega t\}$$

$$\therefore\ \mathcal{L}\{\sin\omega t\} = \frac{\omega}{p^2+\omega^2},\quad p>0$$

Also, from above

$$\mathcal{L}\{\sin\omega t\} = \frac{1}{\omega} - \frac{p}{\omega}\mathcal{L}\{\cos\omega t\}$$

$$\therefore\ \mathcal{L}\{\cos\omega t\} = \frac{\omega}{p}\Big[\frac{1}{\omega} - \mathcal{L}\{\sin\omega t\}\Big] = \frac{\omega}{p}\Big[\frac{1}{\omega} - \frac{\omega}{p^2+\omega^2}\Big] = \frac{p}{p^2+\omega^2},\ p>0$$

PROB. 8.2:

$$\mathcal{L}\Big\{\frac{df}{dt}\Big\} = \int_0^\infty e^{-pt}\frac{df}{dt}dt = e^{-pt}f(t)\Big|_0^\infty + p\underbrace{\int_0^\infty e^{-pt}f(t)\,dt}_{\bar{f}(p)} = -f(0) + p\bar{f}(p),\ p>0$$

PROB. 8.3:

$$\mathcal{L}\Big\{\int_0^t f(t')dt'\Big\} = \mathcal{L}\{F(t)\},\qquad F(t) = \int_0^t f(t')dt'$$

Using the result of Prob. 8.2:

$$\bar{F}(p) = \mathcal{L}\Big\{\int_0^t f(t')dt'\Big\} = \frac{1}{p}\Big[\mathcal{L}\Big\{\underbrace{\frac{dF}{dt}}_{f(t)}\Big\} + \underbrace{F(0)}_{0}\Big] = \frac{1}{p}\bar{f}(p)$$

PROB. 8.4:

$$\bar{g}(p)\cdot\bar{h}(p) = \int_0^\infty e^{-pt} g(t)\,dt \cdot \int_0^\infty e^{-pt} h(t)\,dt$$

$$= \int_0^\infty \int_0^\infty e^{-p(u+v)} g(u)\,h(v)\,du\,dv$$

$$= \int_0^\infty g(u)\left\{\int_0^\infty e^{-p(u+v)} h(v)\,dv\right\} du$$

$u+v=t \rightarrow dv=dt$

$$= \int_0^\infty g(u)\left\{\int_u^\infty e^{-pt} h(t-u)\,dt\right\} du$$

$$= \int_0^\infty \left\{\int_u^\infty e^{-pt} g(u)\,h(t-u)\,dt\right\} du$$

$$= \int_0^\infty h(v)\left\{\int_0^\infty e^{-p(u+v)} g(u)\,du\right\} dv$$

$u+v=t \rightarrow du=dt$

$$= \int_0^\infty h(v)\left\{\int_v^\infty e^{-pt} g(t-v)\,dt\right\} dv$$

$$= \int_0^\infty \left\{\int_v^\infty e^{-pt} h(v)\,g(t-v)\,dt\right\} dv$$

Change the order of integration

$$= \int_0^\infty \left\{\int_0^t e^{-pt} g(u)\,h(t-u)\,du\right\} dt$$

$$= \int_0^\infty e^{-pt}\left\{\int_0^t g(u)\,h(t-u)\,du\right\} dt$$

$$= \mathcal{L}\left\{\int_0^t g(u)\,h(t-u)\,du\right\}$$

$$= \int_0^\infty \left\{\int_0^t e^{-pt} h(v)\,g(t-v)\,dv\right\} dt$$

$$= \int_0^\infty e^{-pt}\left\{\int_0^t h(v)\,g(t-v)\,dv\right\} dt$$

$$= \mathcal{L}\left\{\int_0^t h(v)\,g(t-v)\,dv\right\}$$

$$\therefore\ \mathcal{L}^{-1}\{\bar{g}(p)\cdot\bar{h}(p)\} = \int_0^t g(t-v)\,h(v)\,dv = \int_0^t h(t-u)\,g(u)\,dv$$

PROB. 8.5:

$$\mathcal{L}^{-1}\left\{\frac{\cosh mx}{p\cosh mL}\right\} = \mathcal{L}^{-1}\left\{\frac{P(p)}{Q(p)}\right\} = \sum_{n=1}^{\infty} \frac{P(p_n)}{\frac{dQ(p_n)}{dp}} e^{p_n t}, \quad m=\sqrt{\frac{p}{\alpha}}$$

$$P(p) = \cosh mx = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\left(\frac{p}{\alpha}\right)^n$$

$$Q(p) = p\cosh mL = \sum_{n=0}^{\infty} \frac{L^{2n}}{(2n)!}\frac{p^{n+1}}{\alpha^n}$$

where p_n's are the zeros of

$$p\cosh mL = 0 \begin{cases} \nearrow p_0 = 0 \\ \searrow \cosh mL = 0 \longrightarrow \cos imL = 0 \end{cases}$$

$$\therefore \quad i\sqrt{\frac{p_n}{\alpha}}L = \frac{(2n-1)\pi}{2}$$

$$p_n = -\frac{(2n-1)^2\pi^2\alpha}{4L^2}, \quad n = 1, 2, \ldots$$

$$P(0) = 1, \quad P(p_n) = \cosh\left[-i\frac{(2n-1)\pi}{2L}x\right] = \cos\frac{(2n-1)\pi x}{2L}$$

$$\frac{dQ}{dp} = \cosh mL + \frac{L}{2}\sqrt{\frac{p}{\alpha}}\sinh mL$$

$$\frac{dQ(0)}{dp} = 1, \quad \frac{dQ(p_n)}{dp} = \cosh\left[-i\frac{(2n-1)\pi}{2}\right] - \frac{\cancel{L}}{2}i\frac{(2n-1)}{2\cancel{L}}\sinh\left[-i\frac{(2n-1)\pi}{2}\right]$$

$$= -\frac{(2n-1)\pi}{4}\sin\frac{(2n-1)\pi}{2} = -\frac{(2n-1)\pi}{4}(-1)^{n+1}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{\cosh mx}{p\cosh mL}\right\} = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{-\frac{(2n-1)^2\pi^2\alpha}{4L^2}t}\cdot\cos\frac{(2n-1)\pi x}{2L}$$

PROB. 8.6: Formulation of the problem:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(x,0) = T_i$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = h\left[T_\infty - T(0,t)\right]; \quad \lim_{x\to\infty} T(x,t) = T_i$$

The transformed problem:

$$\frac{d^2\bar{T}}{dx^2} - \frac{p}{\alpha}\bar{T}(x,p) = -\frac{1}{\alpha}T_i$$

$$\therefore \quad \bar{T}(x,p) = Ae^{mx} + Be^{-mx} + \frac{T_i}{p}$$

$$\lim_{x\to\infty}\bar{T}(x,p) = \frac{T_i}{p} \quad \rightarrow \quad A = 0$$

$$-k\left.\frac{d\bar{T}}{dx}\right|_{x=0} = h\left[\frac{1}{p}T_\infty - \bar{T}(0,p)\right] \rightarrow kmB = h\left[\frac{1}{p}T_\infty - B - \frac{T_i}{p}\right]$$

$$B = \frac{h(T_\infty - T_i)}{p(km+h)}$$

$$\therefore \bar{T}(x,p) = \frac{T_i}{p} + \frac{h(T_\infty - T_i)}{p(km+h)}e^{-mx}$$

The inversion gives

$$T(x,t) = T_i + (T_\infty - T_i)\frac{h}{k}\mathcal{L}^{-1}\left\{\frac{e^{-mx}}{p(m+\frac{h}{k})}\right\}$$

$$= T_i + (T_\infty - T_i)\left\{\mathrm{erfc}\left[\frac{x}{2\sqrt{\alpha t}}\right]\right.$$

$$\left. - e^{hx/k + \alpha h^2 t/k^2}\,\mathrm{erfc}\left[\frac{x}{2\sqrt{\alpha t}} + \frac{h}{k}\sqrt{\alpha t}\right]\right\}$$

PROB. 8.7: Formulation:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(x,0) = T_i$$

$$-k\,\frac{\partial T(x,0)}{\partial x} = q_w'' \,; \quad \lim_{x\to\infty} T(x,t) = T_i$$

The transformed problem:

$$\frac{d^2\bar{T}}{dx^2} - \frac{p}{\alpha}\bar{T}(x,p) = -\frac{T_i}{\alpha} \rightarrow \bar{T}(x,p) = Ae^{mx} + Be^{-mx} + \frac{T_i}{p}$$

$$m = \sqrt{\frac{p}{\alpha}}$$

$$\bar{T}(x,p)\Big|_{x\to\infty} = \frac{T_i}{p} \rightarrow A = 0$$

$$-k\,\frac{d\bar{T}(0,p)}{dx} = \frac{q''}{p} \longrightarrow kmB = \frac{q_w''}{p}$$

$$B = \frac{q_w''}{kmp} = \frac{q_w''}{kp}\sqrt{\frac{\alpha}{p}}$$

$$\therefore\ \bar{T}(x,p) = \frac{T_i}{p} + \frac{q_w''}{k}\sqrt{\frac{\alpha}{p}}\,\frac{e^{-mx}}{p}$$

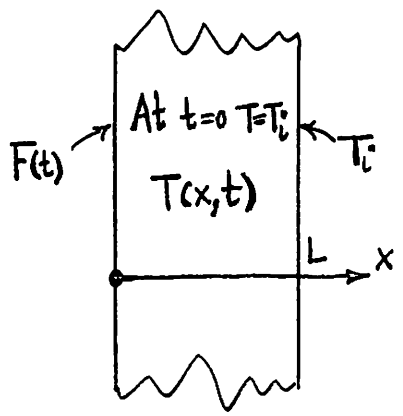
The inversion gives

$$T(x,t) = T_i + \frac{q_w''}{k}\,\mathcal{L}^{-1}\left\{\frac{e^{-\sqrt{\frac{p}{\alpha}}\,x}}{p\sqrt{\alpha/p}}\right\}$$

$$\downarrow \text{Appendix D, Item 28}$$

$$\therefore\quad T(x,t) - T_i = \frac{q_w''}{k}\left[2\left(\frac{\alpha t}{\pi}\right)^{1/2} e^{-x^2/4\alpha t} - x\,\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right]$$

PROB. 8.8:



Formulation:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(x,0) = T_i$$

$$T(0,t) = F(t)\,;\quad T(L,t) = T_i$$

The transformed problem:

$$\frac{d^2\bar{T}}{dx^2} - \frac{p}{\alpha}\bar{T}(x,p) = -\frac{T_i}{\alpha}$$

$$\bar{T}(x,p) = A\sinh mx + B\cosh mx + \frac{T_i}{p}$$

$$m = \sqrt{p/\alpha}$$

$$\bar{T}(0,p) = \bar{F}(p) = B + \frac{T_i}{p} \longrightarrow B = \bar{F}(p) - \frac{T_i}{p}$$

$$\bar{T}(L,p) = \cancel{\frac{T_i}{p}} = A\sinh mL + B\cosh mL + \cancel{\frac{T_i}{p}} \rightarrow A = -B\frac{\cosh mL}{\sinh mL}$$

$$\therefore\ \bar{T}(x,p) = B\left[-\frac{\cosh mL}{\sinh mL}\sinh mx + \cosh mx\right] + \frac{T_i}{p}$$

$$= \left[\bar{F}(p) - \frac{T_i}{p}\right]\frac{\sinh[m(L-x)]}{\sinh mL} + \frac{T_i}{p}$$

$$T(x,t) = \mathcal{L}^{-1}\left\{\bar{F}(p)\frac{\sinh[m(L-x)]}{\sinh mL}\right\} - T_i\,\mathcal{L}^{-1}\left\{\frac{\sinh[m(L-x)]}{p\sinh mL}\right\} + T_i$$

$$T(x,t) - T_i = \int_{t'=0}^{t} F(t-t')\,\mathcal{L}^{-1}\left\{\frac{\sinh[m(L-x)]}{\sinh mL}\right\}dt'$$

$$-T_i\left[\frac{L-x}{L} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-\alpha n^2\pi^2 t/L^2}\sin\frac{n\pi(L-x)}{L}\right]$$

$$T(x,t) - T_i = \frac{2\pi\alpha}{L^2} \int_{t'=0}^{t} F(t-t') \left[\sum_{n=1}^{\infty} (-1)^{n-1} n \, e^{-\alpha n^2 \pi^2 t'/L^2} \sin \frac{n\pi(L-x)}{L} \right] dt'$$

$$- T_i \left[\frac{L-x}{L} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-\alpha n^2 \pi^2 t/L^2} \sin \frac{n\pi(L-x)}{L} \right]$$

Since $\sin \frac{n\pi(L-x)}{L} = -(-1)^n \sin \frac{n\pi}{L} x$

$$\boxed{\begin{aligned} T(x,t) - T_i = {} & \frac{2\pi\alpha}{L^2} \int_{t'=0}^{t} F(t-t') \left[\sum_{n=1}^{\infty} n \, e^{-\alpha n^2 \pi^2 t'/L^2} \sin \frac{n\pi}{L} x \right] dt' \\ & - T_i \left[1 - \frac{x}{L} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-\alpha n^2 \pi^2 t/L^2} \sin \frac{n\pi}{L} x \right] \end{aligned}}$$

In particular, if $F(t) = T_w = \text{const.}$

$$T(x,t) - T_i = \frac{2}{\pi} T_w \sum_{n=1}^{\infty} \frac{1}{n} \left[1 - e^{-\alpha n^2 \pi^2 t/L^2} \right] \sin \frac{n\pi}{L} x$$

$$- T_i \left[1 - \frac{x}{L} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-\alpha n^2 \pi^2 t/L^2} \sin \frac{n\pi}{L} x \right]$$

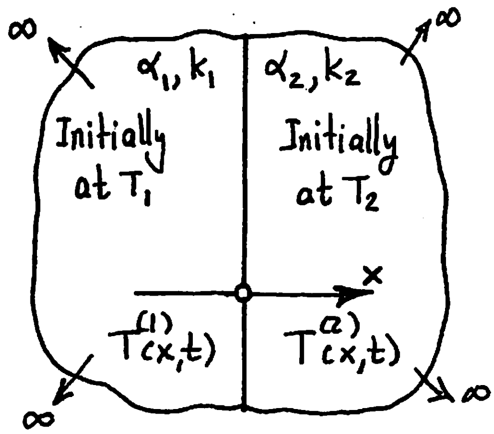
or

$$\boxed{\frac{T(x,t) - T_i}{T_w - T_i} = 1 - \frac{x}{L} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-\alpha n^2 \pi^2 t/L^2} \sin \frac{n\pi}{L} x}$$

where we have used

$$1 - \frac{x}{L} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{L} x$$

PROB. 8.9: (a)



$$\frac{\partial^2 T^{(1)}}{\partial x^2} = \frac{1}{\alpha_1}\frac{\partial T^{(1)}}{\partial t} \quad ; \quad \frac{\partial^2 T^{(2)}}{\partial x^2} = \frac{1}{\alpha_2}\frac{\partial T^{(2)}}{\partial t}$$

$$T^{(1)}(x,0) = T_1 \quad ; \quad T^{(2)}(x,0) = T_2$$

$$\lim_{x\to-\infty} T^{(1)}(x,t) = T_1 \quad ; \quad \lim_{x\to\infty} T^{(2)}(x,t) = T_2$$

$$T^{(1)}(0,t) = T^{(2)}(0,t)$$

$$k_1 \frac{\partial T^{(1)}}{\partial x}\bigg|_{x=0} = k_2 \frac{\partial T^{(2)}}{\partial x}\bigg|_{x=0}$$

Assuming perfect contact at the interface

The formulation of the problem in terms of

$$\theta_1(x,t) = T^{(1)}(x,t) - T_1 \quad \text{and} \quad \theta_2(x,t) = T^{(2)}(x,t) - T_2 :$$

$$\frac{\partial^2 \theta_1}{\partial x^2} = \frac{1}{\alpha_1}\frac{\partial \theta_1}{\partial t} \quad ; \quad \frac{\partial^2 \theta_2}{\partial x^2} = \frac{1}{\alpha_2}\frac{\partial \theta_2}{\partial t}$$

$$\theta_1(x,0) = 0 \quad ; \quad \theta_2(x,0) = 0$$

$$\lim_{x\to-\infty} \theta_1(x,t) = 0 \quad ; \quad \lim_{x\to\infty} \theta_2(x,t) = 0$$

$$\theta_1(0,t) + T_1 = \theta_2(0,t) + T_2$$

$$k_1 \frac{\partial \theta_1}{\partial x}\bigg|_{x=0} = k_2 \frac{\partial \theta_2}{\partial x}\bigg|_{x=0}$$

Let

$$\bar{\theta}_1(x,p) = \int_0^\infty e^{-pt}\theta_1(x,t)\,dt \quad \text{and} \quad \bar{\theta}_2(x,p) = \int_0^\infty e^{-pt}\theta_2(x,t)\,dt$$

Then, the transformed differential equations and boundary conditions become

$$\frac{d^2\bar{\theta}_1}{dx^2} - \frac{p}{\alpha_1}\bar{\theta}_1(x,p) = 0 \quad ; \quad \frac{d^2\bar{\theta}_2}{dx^2} - \frac{p}{\alpha_2}\bar{\theta}_2(x,p) = 0$$

$$\lim_{x\to-\infty}\bar{\theta}_1(x,p) = 0 \quad ; \quad \lim_{x\to\infty}\bar{\theta}_2(x,p) = 0$$

$$\bar{\theta}_1(0,p) + \frac{T_1}{p} = \bar{\theta}_2(0,p) + \frac{T_2}{p}$$

$$k_1 \left.\frac{d\bar{\theta}_1}{dx}\right|_{x=0} = k_2 \left.\frac{d\bar{\theta}_2}{dx}\right|_{x=0}$$

The solutions of the differential equations:

$$\bar{\theta}_1(x,p) = A_1 e^{m_1 x} + A_2 e^{-m_1 x} \quad , \quad m_1^2 = \frac{p}{\alpha_1}$$

$$\bar{\theta}_2(x,p) = B_1 e^{m_2 x} + B_2 e^{-m_2 x} \quad , \quad m_2^2 = \frac{p}{\alpha_2}$$

$$\lim_{x\to-\infty}\bar{\theta}_1 = 0 \implies A_2 = 0 \quad \text{and} \quad \lim_{x\to\infty}\bar{\theta}_2 = 0 \implies B_1 = 0$$

The boundary conditions at $x=0$ give

$$\left.\begin{aligned} A_1 + \frac{T_1}{p} &= B_2 + \frac{T_2}{p} \\ k_1 m_1 A_1 &= -k_2 m_2 B_2 \end{aligned}\right\} \quad \begin{aligned} A_1 &= -\frac{T_1 - T_2}{p}\,\frac{1}{1+m} \\ B_2 &= \frac{T_1 - T_2}{p}\,\frac{m}{1+m} \end{aligned}$$

where

$$m = \frac{k_1}{k_2}\sqrt{\frac{\alpha_2}{\alpha_1}} = \sqrt{\frac{k_1 \rho_1 c_1}{k_2 \rho_2 c_2}}$$

Thus,

$$\bar{\theta}_1(x,p) = -\frac{T_1 - T_2}{p}\,\frac{1}{1+m}\,e^{\sqrt{\frac{p}{\alpha_1}}\,x} \quad \text{and} \quad \bar{\theta}_2(x,p) = \frac{T_1 - T_2}{p}\,\frac{m}{1+m}\,e^{-\sqrt{\frac{p}{\alpha_2}}\,x}$$

The inversions give

$$\theta_1(x,t) = \frac{T_2-T_1}{1+m}\,\mathrm{erfc}\left(-\frac{x}{2\sqrt{\alpha_1 t}}\right), \qquad x<0$$

$$\theta_2(x,t) = m\frac{T_1-T_2}{1+m}\,\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha_2 t}}\right), \qquad x>0$$

Thus,

$$T^{(1)}(x,t) = T_1 + \frac{T_2-T_1}{1+m}\,\mathrm{erfc}\left(\frac{|x|}{2\sqrt{\alpha_1 t}}\right), \qquad x<0$$

$$T^{(2)}(x,t) = T_2 + m\frac{T_1-T_2}{1+m}\,\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha_2 t}}\right), \qquad x>0$$

or

$$T^{(1)}(x,t) = \frac{mT_1+T_2}{1+m}\,\mathrm{erfc}\left(\frac{|x|}{2\sqrt{\alpha_1 t}}\right), \qquad x<0$$

$$T^{(2)}(x,t) = \frac{mT_1+T_2}{1+m}\,\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha_2 t}}\right), \qquad x>0$$

$$\therefore \quad \boxed{T(x,t) = \frac{mT_1+T_2}{1+m}\,\mathrm{erfc}\left(\frac{|x|}{2\sqrt{\alpha t}}\right), \quad -\infty<x<\infty}$$

$$\text{where} \qquad \alpha = \begin{cases} \alpha_1 & \text{when} \quad x<0 \\ \alpha_2 & \text{when} \quad x>0 \end{cases}$$

(b) The interface temperature:

$$T(0,t) = T^{(1)}(0,t) = T^{(2)}(0,t) = \underbrace{\frac{mT_1+T_2}{1+m}}_{\text{constant}}, \qquad t\geq 0$$

PROB. 8.10: Formulation of the problem (see Prob. 6.1):

$$\frac{d\theta}{dt} + m\,\theta(t) = \frac{\dot{q}_0}{\rho c} e^{-\beta t}, \quad m = \frac{hA}{\rho V c}$$

$$\theta(0) = 0$$

The transformed problem:

$$\bar{\theta}(p) = \frac{\dot{q}_0}{\rho c}\frac{1}{p+\beta}\frac{1}{p+m} = \begin{cases} \dfrac{\dot{q}_0}{\rho c}\dfrac{1}{m-\beta}\left[\dfrac{1}{p+\beta} - \dfrac{1}{p+m}\right], & \beta \neq m \\[2ex] \dfrac{\dot{q}_0}{\rho c}\dfrac{1}{(p+\beta)^2}, & \beta = m \end{cases}$$

The inversion gives:

$$\theta(t) = T(t) - T_\infty = \begin{cases} \dfrac{\dot{q}_0}{\rho c}\dfrac{1}{m-\beta}\left[e^{-\beta t} - e^{-mt}\right], & \beta \neq m \\[2ex] \dfrac{\dot{q}_0}{\rho c}\, t\, e^{-\beta t}, & \beta = m \end{cases}$$

The maximum temperature will be reached when

$$\frac{dT}{dt} = 0 \quad \Rightarrow \quad t = \begin{cases} \dfrac{1}{m-\beta}\ln\dfrac{m}{\beta}, & \beta \neq m \\[2ex] \dfrac{1}{\beta}, & \beta = m \end{cases}$$

and it will be

$$T_{max} - T_\infty = \begin{cases} \dfrac{\dot{q}_0}{\rho c}\dfrac{1}{m-\beta}\left[\left(\dfrac{m}{\beta}\right)^{-\frac{\beta}{m-\beta}} - \left(\dfrac{m}{\beta}\right)^{-\frac{m}{m-\beta}}\right], & \beta \neq m \\[2ex] \dfrac{\dot{q}_0}{\rho c}\dfrac{1}{\beta e}, & \beta = m \end{cases}$$

PROB. 8.12: Formulation of the problem in terms of $\theta(x,t) = T(x,t) - T_i$:

$$\left\{ \begin{array}{c} \dfrac{\partial^2\theta}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial\theta}{\partial t} \\ \theta(x,0) = 0 \\ \left.\dfrac{\partial\theta}{\partial x}\right|_{x=0} = 0 \,; \quad k\left.\dfrac{\partial\theta}{\partial x}\right|_{x=L} = q''_w \end{array} \right\}$$

Let $\quad \bar{\theta}(x,p) = \mathcal{L}\{\theta(x,t)\} = \int_0^\infty e^{-pt}\theta(x,t)\,dt$

Then, the transformed differential equation and the boundary conditions become

$$\frac{d^2\bar{\theta}}{dx^2} - m^2\,\bar{\theta}(x,p) = 0 \quad , \quad m^2 = \frac{p}{\alpha}$$

$$\left.\frac{d\bar{\theta}}{dx}\right|_{x=0} = 0 \quad ; \quad \left.\frac{d\bar{\theta}}{dx}\right|_{x=L} = \frac{q''_w}{kp}$$

$$\Downarrow$$

$$\bar{\theta}(x,p) = \frac{q''_w}{kpm}\,\frac{\cosh mx}{\sinh mL}$$

$$\frac{\cosh mx}{\sinh mL} = \frac{e^{mx} + e^{-mx}}{e^{mL} - e^{-mL}} = e^{-mL}\,\frac{e^{mx} + e^{-mx}}{1 - e^{-2mL}} = \left[e^{-m(L-x)} + e^{-m(L+x)} \right]\frac{1}{1 - e^{-2mL}}$$

$$= \left[e^{-m(L-x)} + e^{-m(L+x)} \right] \sum_{k=0}^{\infty} e^{-2kmL}$$

$$= \sum_{k=0}^{\infty} \left\{ \exp[-m(nL-x)] + \exp[-m(nL+x)] \right\}$$

where $\qquad n = 2k+1$

$$\therefore \quad \bar{\theta}(x,p) = \frac{q_w''}{kpm} \sum_{k=0}^{\infty} \left\{ \exp\left[-m(nL-x)\right] + \exp\left[-m(nL+x)\right] \right\}$$

From Appendix D,

$$\frac{e^{-x\sqrt{p/\alpha}}}{p\sqrt{p/\alpha}} = 2\left(\frac{\alpha t}{\pi}\right)^{1/2} e^{-x^2/4\alpha t} - x \, \mathrm{erfc}\left[\frac{x}{2(\alpha t)^{1/2}}\right]$$

Thus, the inversion gives

$$\theta(x,t) = \frac{q_w''}{k} \sum_{k=0}^{\infty} \left\{ 2\left(\frac{\alpha t}{\pi}\right)^{1/2} e^{-(nL-x)^2/4\alpha t} - (nL-x)\,\mathrm{erfc}\left[\frac{nL-x}{2(\alpha t)^{1/2}}\right] \right.$$

$$\left. + \, 2\left(\frac{\alpha t}{\pi}\right)^{1/2} e^{-(nL+x)^2/4\alpha t} - (nL+x)\,\mathrm{erfc}\left[\frac{nL+x}{2(\alpha t)^{1/2}}\right] \right\}$$

$$\Downarrow$$

$$\boxed{\theta(x,t) = \frac{2q_w''\sqrt{\alpha t}}{k} \sum_{k=0}^{\infty} \left\{ \mathrm{ierfc}\, \frac{(2k+1)L-x}{2\sqrt{\alpha t}} + \mathrm{ierfc}\, \frac{(2k+1)L+x}{2\sqrt{\alpha t}} \right\}}$$

where

$$\mathrm{ierfc}(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} - x\,\mathrm{erfc}(x)$$

PROB.8.14: Formulation in terms of $\theta(r,t) = T(r,t) - T_i$:

$$\frac{\partial^2\theta}{\partial r^2} + \frac{2}{r}\frac{\partial\theta}{\partial r} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta(r,0) = 0$$

$$\theta(0,t) = \text{finite}; \quad k\left.\frac{\partial\theta}{\partial r}\right|_{r=r_0} = q_w''$$

The transformed problem:

$$\frac{d^2\bar\theta}{dr^2} + \frac{2}{r}\frac{d\bar\theta}{dr} = \frac{1}{\alpha}\left[p\,\bar\theta(r,p) - \cancelto{0}{\theta(r,0)}\right] \quad \text{where} \quad \bar\theta(r,p) = \int_0^\infty e^{-pt}\theta(r,t)\,dt$$

$$\Rightarrow \frac{d^2(r\bar\theta)}{dr^2} - m^2(r\bar\theta) = 0, \qquad m^2 = \frac{p}{\alpha}$$

$$\therefore \quad \bar\theta(r,p) = \frac{1}{r}\left[C_1 \sinh mr + C_2 \cosh mr\right]$$

$$\bar\theta(0,p) = \text{finite} \quad \Rightarrow C_2 \equiv 0$$

$$\left.\frac{d\bar\theta}{dr}\right|_{r=r_0} = \frac{1}{p}\frac{q_w''}{k} \quad \Rightarrow C_1 = \frac{q_w'' r_0^2}{k}\,\frac{1}{p}\,\frac{1}{mr_0\cosh mr_0 - \sinh mr_0}$$

$$\therefore \quad \bar\theta(r,p) = \frac{q_w'' r_0}{k}\left(\frac{r_0}{r}\right)\frac{1}{p}\,\frac{\sinh mr}{mr_0\cosh mr_0 - \sinh mr_0}$$

Thus,

$$\theta(r,t) = \frac{q_w'' r_0}{k}\left(\frac{r_0}{r}\right)\mathcal{L}^{-1}\left\{\frac{1}{p}\,\frac{\sinh mr}{mr_0\cosh mr_0 - \sinh mr_0}\right\}$$

Note that

$$\mathcal{L}^{-1}\left\{\frac{1}{p}\,\frac{\sinh mr}{mr_0\cosh mr_0 - \sinh mr_0}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{\sinh mr}{mr_0\cosh mr_0 - \sinh mr_0}\right\}dt$$

On the other hand,

$$\mathcal{L}^{-1}\left\{\frac{\sinh mr}{mr_0\cosh mr_0-\sinh mr_0}\right\}=\mathcal{L}^{-1}\left\{\frac{P(p)}{Q(p)}\right\}=\sum_{n=0}^{\infty}\frac{P(p_n)}{\frac{dQ(p_n)}{dp}}e^{p_n t}$$

Heaviside's expansion theorem

where p_n's are the zeros of

$$mr_0\cosh mr_0-\sinh mr_0=0 \quad \text{or} \quad mr_0\cos' imr_0+i\sin' imr_0=0,\ i=\sqrt{-1}$$

or

$$\tan imr_0=imr_0 \quad \text{or} \quad \tan\lambda r_0=\lambda r_0 \quad \text{where } \lambda^2=-m^2=-\frac{p}{\alpha}$$

$$\Downarrow$$

$$\lambda_n^2\left(=-\frac{p_n}{\alpha}\right),\quad n=0,1,2,\ldots$$

<u>For $n=0$</u>,

$$\lambda_0=0 \Rightarrow \underbrace{p_0=0}_{m=0} \Rightarrow P(0)=0$$

$$\frac{dQ}{dp}=\frac{r_0}{2\sqrt{\alpha p}}\left\{\cosh mr_0+mr_0\sinh mr_0-\cosh mr_0\right\}=\frac{r_0^2}{2\alpha}\sinh mr_0$$

$$\frac{dQ(0)}{dp}=0$$

that the first term in Heaviside's expansion is $\frac{0}{0}$. Thus, this term can be calculated as

$$\lim_{p_n\to 0}\frac{\sinh mr}{\frac{r_0^2}{2\alpha}\sinh mr_0}=\frac{2\alpha}{r_0^2}\lim_{p\to 0}\frac{r\cosh mr\cdot\cancel{\frac{dm}{dp}}}{r_0\cosh mr_0\cdot\cancel{\frac{dm}{dp}}}=\frac{2\alpha}{r_0^2}\left(\frac{r}{r_0}\right)$$

<u>For $n\geq 1$</u>, $\quad p_n=-\alpha\lambda_n^2 \longrightarrow m_n=-i\lambda_n$

where λ_n's are non-zero zeros of $\tan\lambda r_0=\lambda r_0$

$$P(p_n)=\sinh m_n r=\sinh(-i\lambda_n)r=-i\sin\lambda_n r$$

$$\frac{dQ}{dp} = \frac{r_0^2}{2\alpha}\sinh m_n r_0 = \frac{r_0}{2\alpha}\sinh(-i\lambda_n)r_0 = -\frac{r_0}{2\alpha} i \sin\lambda_n r_0$$

Thefore,

$$\mathcal{L}^{-1}\left\{\frac{\sinh mr}{mr_0\cosh mr_0 - \sinh mr_0}\right\} = \underbrace{\frac{2\alpha}{r_0^2}\left(\frac{r}{r_0}\right) + \sum_{n=1}^{\infty}\frac{2\alpha}{r_0^2}\frac{\sin\lambda_n r}{\sin\lambda_n r_0}e^{-\alpha\lambda_n^2 t}}$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{p}\frac{\sinh mr}{mr_0\cosh mr_0 - \sinh mr_0}\right\} = \int_0^t \{\cdots\}\,dt$$

$$= \frac{2\alpha r}{r_0^3}t + \frac{2}{r_0^2}\sum_{n=1}^{\infty}\frac{\sin\lambda_n r}{\lambda_n^2\sin\lambda_n r_0}\left(1-e^{-\alpha\lambda_n^2 t}\right)$$

Now, the solution can be written as

$$\theta(r,t) = T(r,t) - T_i = \frac{q_w''}{kr_0}\left\{2\alpha t + 2\frac{r_0}{r}\sum_{n=1}^{\infty}\frac{\sin\lambda_n r}{\lambda_n^2\sin\lambda_n r_0}\left(1-e^{-\alpha\lambda_n^2 t}\right)\right\}$$

It can be shown that

$$2\frac{r_0}{r}\sum_{n=1}^{\infty}\frac{\sin\lambda_n r}{\lambda_n^2\sin\lambda_n r_0} = \frac{r_0^2}{2}\left[\left(\frac{r}{r_0}\right)^2 - \frac{3}{5}\right] = \frac{5r^2 - 3r_0^2}{10}$$

Then, the solution can be rewritten as

$$\boxed{T(r,t) - T_i = \frac{q_w''}{kr_0}\left\{\frac{5r^2 - 3r_0^2}{10} + 2\alpha t - 2\frac{r_0}{r}\sum_{n=1}^{\infty}\frac{\sin\lambda_n r}{\lambda_n^2\sin\lambda_n r_0}e^{-\alpha\lambda_n^2 t}\right\}}$$

PROB. 8.16: Formulation in terms of $\theta(r,t) = T(r,t) - T_i$:

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta(r,0) = 0$$

$$\theta(0,t) = \text{finite}; \quad \theta(r_0,t) = 0$$

The transformed problem:

$$\frac{d^2\bar{\theta}}{dr^2} + \frac{1}{r}\frac{d\bar{\theta}}{dr} + \frac{\dot{q}}{kp} = \frac{1}{\alpha}\left[p\bar{\theta}(r,p) - \cancelto{0}{\theta(r,0)}\right]$$

$$\bar{\theta}(0,p) = \text{finite} \quad ; \quad \bar{\theta}(r_0,p) = 0$$

$$\Downarrow$$

$$\bar{\theta}(r,p) = \frac{\dot{q}\alpha}{kp^2}\left[1 - \frac{I_0(mr)}{I_0(mr_0)}\right], \quad m = \sqrt{\frac{p}{\alpha}}$$

$$\therefore \quad \theta(r,t) = \frac{\dot{q}\alpha}{k}\left[t - \int_0^t \left\{1 - 2\sum_{k=1}^{\infty} \frac{J_0(\lambda_k \frac{r}{r_0})}{\lambda_k J_1(\lambda_k)} e^{-\frac{\alpha\lambda_k^2}{r_0^2}t'}\right\}dt'\right]$$

where λ_k's are the zeros of $J_0(z)$.

Thus,

$$\theta(r,t) = -2\frac{\dot{q}r_0^2}{k}\left\{\sum_{n=1}^{\infty} \frac{J_0(\lambda_n \frac{r}{r_0})}{\lambda_n^3 J_1(\lambda_n)} e^{-\frac{\alpha\lambda_n^2}{r_0^2}t} - \underbrace{\sum_{n=1}^{\infty} \frac{J_0(\lambda_n \frac{r}{r_0})}{\lambda_n^3 J_1(\lambda_n)}}_{\frac{1}{8}\left[1-\left(\frac{r}{r_0}\right)^2\right]}\right\}$$

or

$$\boxed{T(r,t) - T_i = -2\frac{\dot{q}r_0^2}{k}\sum_{n=1}^{\infty} \frac{J_0(\lambda_n \frac{r}{r_0})}{\lambda_n^3 J_1(\lambda_n)} e^{-\frac{\alpha\lambda_n^2}{r_0^2}t} + \frac{\dot{q}r_0^2}{4k}\left[1-\left(\frac{r}{r_0}\right)^2\right]}$$

PROB. 8.17: The formulation in terms of $\theta(r,t) = T(r,t) - T_i$:

$$\frac{\partial^2\theta}{\partial r^2} + \frac{2}{r}\frac{\partial\theta}{\partial r} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta(r,0) = 0$$

$$\theta(0,t) = \text{finite}; \quad \theta(r_0,t) = 0$$

The transformed problem:

$$\frac{d^2\bar{\theta}}{dr^2} + \frac{2}{r}\frac{d\bar{\theta}}{dr} + \frac{\dot{q}}{kp} = \frac{1}{\alpha}\left[p\,\bar{\theta}(r,p) - \cancelto{0}{\theta(r,0)}\right]$$

where $$\bar{\theta}(r,p) = \int_0^\infty e^{-pt}\,\theta(r,t)\,dt$$

$$\frac{d^2\bar{\theta}}{dr^2} + \frac{2}{r}\frac{d\bar{\theta}}{dr} - m^2\,\bar{\theta}(r,p) = -\frac{\dot{q}}{kp}, \quad m^2 = \frac{p}{\alpha}$$

or

$$\frac{d^2}{dr^2}(r\bar{\theta}) - m^2(r\bar{\theta}) = -\frac{\dot{q}}{kp}r$$

$$\therefore \quad \bar{\theta}(r,p) = \frac{1}{r}\left\{C_1 \sinh mr + C_2 \cosh mr\right\} + \frac{\dot{q}\alpha}{kp^2}$$

$$\bar{\theta}(0,p) = \text{finite} \quad \Rightarrow \quad C_2 = 0$$

$$\bar{\theta}(r_0,p) = 0 \quad \Rightarrow \quad C_1 = -\frac{\dot{q}\alpha r_0}{kp^2 \sinh mr_0}$$

$$\bar{\theta}(r,p) = \frac{\dot{q}\alpha}{kp^2}\left[1 - \frac{r_0}{r}\,\frac{\sinh mr}{\sinh mr_0}\right]$$

$$\therefore \quad \theta(r,t) = \frac{\dot{q}\alpha}{k}\left[t - \frac{r_0}{r}\,\mathcal{L}^{-1}\left\{\frac{\sinh mr}{p^2 \sinh mr_0}\right\}\right]$$

$$\frac{\sinh mr}{\sinh mr_0} = \frac{e^{mr} - e^{-mr}}{e^{mr_0} - e^{-mr_0}} = \left[e^{-m(r_0-r)} - e^{-m(r_0+r)} \right] \frac{1}{1-e^{-2mr_0}}$$

$$= \left[e^{-m(r_0-r)} - e^{-m(r_0+r)} \right] \sum_{k=0}^{\infty} e^{-2mkr_0}$$

$$= \sum_{k=0}^{\infty} \left[e^{-m(nr_0-r)} - e^{-m(nr_0+r)} \right], \quad n = 2k+1$$

Thus,

$$\mathcal{L}^{-1}\left\{ \frac{\sinh mr}{p^2 \sinh mr_0} \right\} = \sum_{k=0}^{\infty} \left[\mathcal{L}^{-1}\left\{ \frac{e^{-m(nr_0-r)}}{p^2} \right\} - \mathcal{L}^{-1}\left\{ \frac{e^{-m(nr_0+r)}}{p^2} \right\} \right]$$

From Appendix D, item 29:

$$\mathcal{L}^{-1}\left\{ \frac{e^{-x\sqrt{p/a}}}{p^2} \right\} = t + \frac{x^2}{2a} \operatorname{erfc}\left[\frac{x}{2\sqrt{at}} \right] - x \left(\frac{t}{\pi a} \right)^{1/2} e^{-x^2/4at}$$

$$= t - x\sqrt{\frac{t}{a}}\, i\operatorname{erfc}\left[\frac{x}{2\sqrt{at}} \right]$$

where $\quad i\operatorname{erfc}(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} - x \operatorname{erfc}(x)$

Thus,

$$\mathcal{L}^{-1}\left\{ \frac{\sinh mr}{p^2 \sinh mr_0} \right\} = \sqrt{\frac{t}{\alpha}} \sum_{k=0}^{\infty} \left\{ (nr_0+r)\, i\operatorname{erfc}\left[\frac{nr_0+r}{2\sqrt{\alpha t}} \right] - (nr_0-r)\, i\operatorname{erfc}\left[\frac{nr_0-r}{2\sqrt{\alpha t}} \right] \right\}$$

Then,

$$\boxed{\begin{aligned} \frac{T(r,t)-T_i}{\dot{q}\alpha/k} = t - \frac{r_0}{r}\sqrt{\frac{t}{\alpha}} \sum_{k=0}^{\infty} \Bigg\{ &[(2k+1)r_0+r]\, i\operatorname{erfc}\left[\frac{(2k+1)r_0+r}{2\sqrt{\alpha t}} \right] \\ &- [(2k+1)r_0-r]\, i\operatorname{erfc}\left[\frac{(2k+1)r_0-r}{2\sqrt{\alpha t}} \right] \Bigg\} \end{aligned}}$$

PROB. 8.19: Refer to Section 8.10 in text.

$$\frac{\bar{\theta}}{\theta_w} = \frac{r_0}{r}\,\frac{\sinh mr}{p \sinh mr_0} = \frac{r_0}{rp}\,\frac{e^{mr} - e^{-mr}}{e^{mr_0} - e^{-mr_0}}$$

$$= \frac{r_0}{rp}\,\frac{e^{-m(r_0-r)} - e^{-m(r_0+r)}}{1 - e^{-2mr_0}}$$

$$= \frac{r_0}{r}\frac{1}{p}\left[e^{-m(r_0-r)} - e^{-m(r_0+r)}\right]\sum_{k=0}^{\infty} e^{-2mkr_0}$$

$$= \frac{r_0}{r}\frac{1}{p}\sum_{k=0}^{\infty}\left\{e^{-m(nr_0-r)} - e^{-m(nr_0+r)}\right\}$$

$$n = 2k+1$$

$$\therefore\ \frac{\theta(r,t)}{\theta_w} = \frac{r_0}{r}\sum_{k=0}^{\infty}\left[\mathcal{L}^{-1}\left\{\frac{e^{-m(nr_0-r)}}{p}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-m(nr_0+r)}}{p}\right\}\right]$$

$$\boxed{\frac{T(r,t)-T_i}{T_w-T_i} = \frac{r_0}{r}\sum_{k=0}^{\infty}\left\{\operatorname{erfc}\left[\frac{(2k+1)r_0-r}{2\sqrt{\alpha t}}\right] - \operatorname{erfc}\left[\frac{(2k+1)r_0+r}{2\sqrt{\alpha t}}\right]\right\}}$$

PROB. 8.20:

$$\left\{\begin{array}{c} \dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t} \\ T(x,0)=T_i \\ T(0,t) = T_o + (\Delta T)_o \sin\omega t \\ T(x,t)\Big|_{x\to\infty} = T_i \end{array}\right\} \xrightarrow{\theta(x,t) = \frac{T(x,t)-T_o}{(\Delta T)_o}} \left\{\begin{array}{c} \dfrac{\partial^2 \theta}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial \theta}{\partial t} \\ \theta(x,0) = \dfrac{T_i - T_o}{(\Delta T)_o} = \theta_i \\ \theta(0,t) = \sin\omega t \\ \theta(x,t)\Big|_{x\to\infty} = \theta_i \end{array}\right\}$$

$$\frac{d^2\bar\theta}{dx^2} = \frac{1}{\alpha}\left\{ p\,\bar\theta(x,p) - \theta_i \right\} \quad , \quad \bar\theta(x,p) = \int_0^\infty e^{-pt}\theta(x,t)\,dt$$

$$\bar\theta(x,p) = A\,e^{mx} + B\,e^{-mx} + \frac{\theta_i}{p} \quad , \quad m = \sqrt{\frac{p}{\alpha}}$$

$$\bar\theta(x,p)\Big|_{x\to\infty} \rightarrow \frac{\theta_i}{p} \Rightarrow A \equiv 0$$

$$\bar\theta(0,p) = \frac{\omega}{p^2+\omega^2} = B + \frac{\theta_i}{p} \Rightarrow B = \frac{\omega}{p^2+\omega^2} - \frac{\theta_i}{p}$$

$$\therefore \quad \bar\theta(x,p) = \frac{\theta_i}{p}\left(1 - e^{-mx}\right) + \frac{\omega}{p^2+\omega^2}\,e^{-mx}$$

$$\Downarrow$$

$$\theta(x,t) = \theta_i\left[1 - \mathcal{L}^{-1}\left\{\frac{e^{-x\sqrt{p/\alpha}}}{p}\right\}\right] + \omega\,\mathcal{L}^{-1}\left\{\frac{e^{-x\sqrt{p/\alpha}}}{p^2+\omega^2}\right\}$$

$$= \theta_i \underbrace{\left[1 - \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right]}_{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)} + \int_0^t \sin\omega t' \, \frac{x\, e^{-\frac{x^2}{4\alpha(t-t')}}}{2\sqrt{\pi\alpha\,(t-t')^3}}\,dt'$$

Let $\dfrac{x}{2\sqrt{\alpha(t-t')}} = \mu \Rightarrow t-t' = \dfrac{x^2}{4\alpha\mu^2}$ & $dt' = \dfrac{4\sqrt{\alpha}(t-t')^{3/2}}{x} d\mu$

Thus,

$$\int_0^t \sin\omega t' \, \frac{x\, e^{-\frac{x^2}{4\alpha}(t-t')}}{2\sqrt{\pi\alpha(t-t')^3}}\, dt' = \frac{2}{\pi} \int_{\frac{x}{2\sqrt{\alpha t}}}^{\infty} \sin\left\{\omega\left[t - \frac{x^2}{4\alpha\mu^2}\right]\right\} e^{-\mu^2} d\mu$$

Also, by a known definite integral,*

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} \sin\left\{\omega\left[t - \frac{x^2}{4\alpha\mu^2}\right]\right\} e^{-\mu^2} d\mu = e^{-x\sqrt{\frac{\omega}{2\alpha}}} \sin\left\{\omega t - x\left(\frac{\omega}{2\alpha}\right)^{1/2}\right\}$$

it follows that

$$\frac{2}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{\alpha t}}}^{\infty} \sin\left\{\omega\left[t - \frac{x^2}{4\alpha\mu^2}\right]\right\} e^{-\mu^2} d\mu = e^{-x\sqrt{\frac{\omega}{2\alpha}}} \sin\left\{\omega t - x\left(\frac{\omega}{2\alpha}\right)^{1/2}\right\} - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{\alpha t}}} \sin\left\{\omega\left[t - \frac{x^2}{4\alpha\mu^2}\right]\right\} e^{-\mu^2} d\mu$$

Then, the solution can be written as

$$\boxed{\theta(x,t) = \frac{T(x,t) - T_0}{(\Delta T)_0} = e^{-x\sqrt{\frac{\omega}{2\alpha}}} \sin\left\{\omega t - x\left(\frac{\omega}{2\alpha}\right)^{1/2}\right\} + \frac{T_i - T_0}{(\Delta T)_0} \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{\alpha t}}} \sin\left\{\omega\left[t - \frac{x^2}{4\alpha\mu^2}\right]\right\} e^{-\mu^2} d\mu}$$

Note that, as $t \to \infty$, the last two terms die away, leaving the first term which is a steady oscillation of $\frac{2\pi}{\omega}$.

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* See Ref. [3], Carslaw and Jaeger, p. 65.

PROB. 8.25: The formulation in terms of $\theta(x,t) = T(x,t) - T_i$:

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta(x,0) = 0$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0} = 0; \quad \theta(L,t) = At$$

The transformed problem:

$$\frac{d^2\bar\theta}{dx^2} - \frac{p}{\alpha}\bar\theta(x,p) = 0, \qquad \bar\theta(x,p) = \int_0^\infty e^{-pt}\theta(x,t)\,dt$$

$$\left.\frac{d\bar\theta}{dx}\right|_{x=0} = 0; \quad \theta(L,p) = \frac{A}{p^2}$$

$$\Downarrow$$

$$\bar\theta(x,p) = \frac{A}{p^2}\frac{\cosh mx}{\cosh mL}, \quad m^2 = \frac{p}{\alpha}$$

$$\therefore \quad \theta(x,t) = A\int_0^t \mathcal{L}^{-1}\left\{\frac{\cosh mx}{p\cosh mL}\right\}dt$$

Transform No. 40, Appendix D

$$= A\int_0^t\left\{1 + \frac{2}{L}\sum_{n=1}^\infty \frac{(-1)^n}{\lambda_n} e^{-\alpha\lambda_n^2 t}\cos\lambda_n x\right\}dt, \quad \lambda_n = \frac{(2n-1)\pi}{2L}$$

$$\therefore \quad \theta(x,t) = T(x,t) - T_i = A\left\{t + \frac{2}{\alpha L}\sum_{n=1}^\infty \frac{(-1)^n}{\lambda_n^3}\left(1 - e^{-\alpha\lambda_n^2 t}\right)\cos\lambda_n x\right\}$$

or

$$T(x,t) - T_i = A\left\{t - \frac{L^2}{2\alpha}\left[1 - \left(\frac{x}{L}\right)^2\right] - \frac{2}{\alpha L}\sum_{n=1}^\infty \frac{(-1)^n}{\lambda_n^3} e^{-\alpha\lambda_n^2 t}\cos\lambda_n x\right\}$$

Here, we used the relation,

$$1 - \left(\frac{x}{L}\right)^2 = \frac{4}{L^3}\sum_{n=1}^\infty \frac{(-1)^{n+1}}{\lambda_n^3}\cos\lambda_n x, \quad 0 \le x \le L$$

Note that the above solution does not converge rapidly for small values of t. An alternative solution that converges rapidly for small values of t can be obtained as follows:

$$\frac{\cosh mx}{\cosh mL} = \sum_{k=0}^{\infty} (-1)^k \left\{ e^{-m(nL-x)} + e^{-m(nL+x)} \right\}, \quad n = 2k+1$$

Thus,

$$\theta(x,t) = \mathcal{L}^{-1}\left\{ \frac{A}{p^2} \frac{\cosh mx}{\cosh mL} \right\}$$

$$= A \sum_{k=0}^{\infty} (-1)^k \mathcal{L}^{-1}\left\{ \frac{e^{-m(nL-x)}}{p^2} + \frac{e^{-m(nL+x)}}{p^2} \right\}$$

Transform No. 29, Appendix D:

$$\mathcal{L}^{-1}\left\{ \frac{e^{-x\sqrt{p/a}}}{p^2} \right\} = \left(t + \frac{x^2}{2a}\right) \operatorname{erfc}\left[\frac{x}{2\sqrt{\alpha t}}\right] - x\left(\frac{t}{\pi a}\right)^{1/2} e^{-x^2/4at}$$

Then, it follows that

$$\begin{aligned} \theta(x,t) = T(x,t) - T_i = A \sum_{k=0}^{\infty} (-1)^k \Big\{ & \left[t + \frac{(nL-x)^2}{2\alpha}\right] \operatorname{erf}\left[\frac{nL-x}{2\sqrt{\alpha t}}\right] \\ & + \left[t + \frac{(nL+x)^2}{2\alpha}\right] \operatorname{erfc}\left[\frac{nL+x}{2\sqrt{\alpha t}}\right] \\ & - (nL-x)\left(\frac{t}{\pi\alpha}\right)^{1/2} \exp\left[-\frac{(nL-x)^2}{4\alpha t}\right] \\ & - (nL+x)\left(\frac{t}{\pi\alpha}\right)^{1/2} \exp\left[-\frac{(nL+x)^2}{4\alpha t}\right] \Big\} \end{aligned}$$

PROB. 8.26: The formulation in terms of $\theta(r,t) = T(r,t) - T_i$:

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta(r,0) = 0$$

$$\theta(0,t) = \text{finite}; \quad \theta(r_0,t) = At$$

The transformed problem:

$$\frac{d^2\bar{\theta}}{dr^2} + \frac{1}{r}\frac{d\bar{\theta}}{dr} - \frac{p}{\alpha}\bar{\theta}(r,p) = 0, \quad \bar{\theta}(r,p) = \int_0^\infty e^{-pt}\theta(r,t)\,dt$$

$$\bar{\theta}(0,p) = \text{finite}; \quad \bar{\theta}(r_0,p) = \frac{A}{p^2}$$

$$\Downarrow$$

$$\bar{\theta}(r,p) = \frac{A}{p^2}\frac{I_0(mr)}{I_0(mr_0)}, \quad m^2 = \frac{p}{\alpha}$$

$$\therefore \theta(r,t) = A\int_0^t \left\{1 - 2\sum_{n=1}^{\infty}\frac{J_0(\lambda_n \frac{r}{r_0})}{\lambda_n J_1(\lambda_n)} e^{-\alpha\frac{\lambda_n^2}{r_0^2}t'}\right\} dt'$$

where λ_n are the zeros of $J_0(z)$. (See Prob. 8.16)

Thus,

$$\theta(r,t) = A\left\{ t + \frac{2r_0^2}{\alpha}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n\frac{r}{r_0})}{\lambda_n^3 J_1(\lambda_n)} e^{-\frac{\alpha\lambda_n^2}{r_0^2}t} - \frac{2r_0^2}{\alpha}\underbrace{\sum_{n=1}^{\infty}\frac{J_0(\lambda_n\frac{r}{r_0})}{\lambda_n J_1(\lambda_n)}}_{\frac{1}{8}\left[1-\left(\frac{r}{r_0}\right)^2\right]}\right\}$$

or

$$\boxed{\theta(r,t) = \frac{Ar_0^2}{4\alpha}\left\{4\frac{\alpha t}{r_0^2} - \left[1-\left(\frac{r}{r_0}\right)^2\right] + 8\sum_{n=1}^{\infty}\frac{J_0(\lambda_n\frac{r}{r_0})}{\lambda_n^3 J_1(\lambda_n)}\exp\left(-\frac{\alpha\lambda_n^2}{r_0^2}t\right)\right\}}$$

PROB. 8.27: The formulation in terms of $\theta(r,t) = T(r,t) - T_i$:

$$\frac{\partial^2\theta}{\partial r^2} + \frac{2}{r}\frac{\partial\theta}{\partial r} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta(r,0) = 0$$

$$\theta(0,t) = \text{finite}; \quad \theta(r_0,t) = At$$

The transformed problem:

$$\frac{d^2\bar\theta}{dr^2} + \frac{2}{r}\frac{d\bar\theta}{dr} - \frac{p}{\alpha}\bar\theta(r,p) = 0, \qquad \bar\theta(r,p) = \int_0^\infty e^{-pt}\,\theta(r,t)\,dt$$

$$\bar\theta(0,p) = \text{finite}; \quad \bar\theta(r_0,p) = \frac{A}{p^2}$$

$$\frac{d^2}{dr^2}(r\bar\theta) - m^2(r\bar\theta) = 0, \qquad m^2 = \frac{p}{\alpha}$$

$$\therefore \quad \bar\theta(r,p) = \frac{1}{r}\left\{C_1\cosh mr + C_2\sinh mr\right\}$$

$$\bar\theta(0,p) = \text{finite} \Rightarrow C_1 = 0$$

$$\bar\theta(r_0,p) = \frac{A}{p^2} \Rightarrow \frac{1}{r_0}C_2\sinh mr_0 = \frac{A}{p^2} \Rightarrow C_2 = \frac{Ar_0}{p^2\sinh mr_0}$$

$$\bar\theta(r,p) = A\frac{r_0}{r}\frac{\sinh mr}{p^2\sinh mr_0} = A\frac{r_0}{r}\frac{1}{p^2}\frac{e^{mr}-e^{-mr}}{e^{mr_0}-e^{-mr_0}}$$

$$= A\frac{r_0}{r}\frac{1}{p^2}\left[e^{-m(r_0-r)} - e^{-m(r_0+r)}\right]\sum_{k=0}^{\infty} e^{-2mkr_0}$$

$$= A\frac{r_0}{r}\frac{1}{p^2}\sum_{k=0}^{\infty}\left\{e^{-m(nr_0-r)} - e^{-m(nr_0+r)}\right\}, \quad n = 2k+1$$

$$\theta(r,t) = A\frac{r_0}{r}\sum_{k=0}^{\infty}\left[\mathcal{L}^{-1}\left\{\frac{e^{-m(nr_0-r)}}{p^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-m(nr_0+r)}}{p^2}\right\}\right]$$

From Appendix D, Item 29:

$$\mathcal{L}^{-1}\left\{\frac{e^{-x\sqrt{p/a}}}{p^2}\right\} = \left(t+\frac{x^2}{2a}\right)\operatorname{erfc}\left[\frac{x}{2\sqrt{at}}\right] - x\left(\frac{t}{\pi a}\right)^{1/2} e^{-x^2/4at}$$

$$= t\operatorname{erfc}\left[\frac{x}{2\sqrt{at}}\right] - x\sqrt{\frac{t}{a}}\ \operatorname{ierfc}\left[\frac{x}{2\sqrt{at}}\right]$$

where

$$\operatorname{ierfc}(x) = \frac{1}{\sqrt{\pi}}\, e^{-x^2} - x\operatorname{erfc}(x)$$

Then,

$$\mathcal{L}^{-1}\left\{\frac{e^{-m(nr_0\pm r)}}{p^2}\right\} = t\operatorname{erfc}\left[\frac{(nr_0\pm r)}{2\sqrt{\alpha t}}\right] - (nr_0\pm r)\sqrt{\frac{t}{\alpha}}\ \operatorname{ierfc}\left[\frac{nr_0\pm r}{2\sqrt{\alpha t}}\right]$$

$$n = 2k+1$$

Thus,

$$\theta(r,t) = A\frac{r_0}{r}t\sum_{k=0}^{\infty}\left\{\operatorname{erfc}\left[\frac{(2k+1)r_0-r}{2\sqrt{\alpha t}}\right] + \operatorname{erfc}\left[\frac{(2k+1)r_0+r}{2\sqrt{\alpha t}}\right]\right\}$$

$$+A\frac{r_0}{r}\sqrt{\frac{t}{\alpha}}\sum_{k=0}^{\infty}\left\{[(2k+1)r_0+r]\operatorname{ierfc}\left[\frac{(2k+1)r_0+r}{2\sqrt{\alpha t}}\right]\right.$$

$$\left.-\left[(2k+1)r_0-r\right]\operatorname{ierfc}\left[\frac{(2k+1)r_0-r}{2\sqrt{\alpha t}}\right]\right\}$$

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