Linear Momentum Balance (2D) in Cartesian coordinate

For x-component

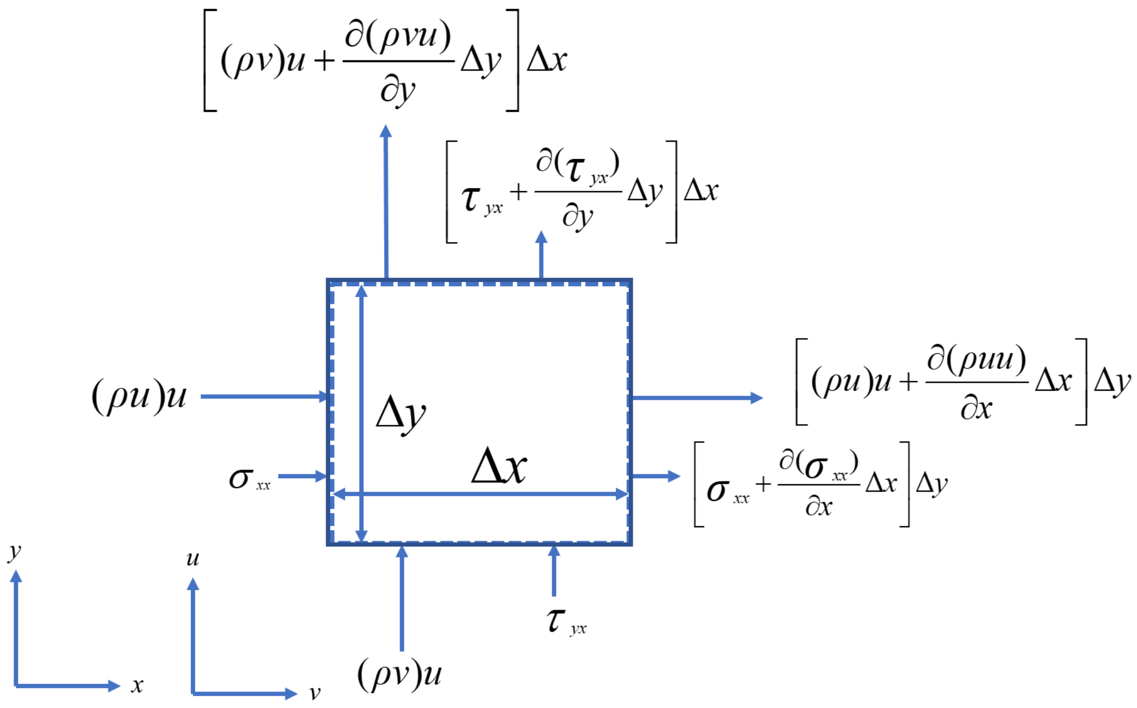

Figure 1. x- Component of the Linear momentum balance in 2D flow field (Cartesian coordinate system)

τ_{yx} = Viscous flux of x momentum in y direction

In general (for special surfaces where the direction of normal to the surface is z, y, or z directions)

τ_{ij} = '*i*' is the direction normal to the surface along which the force is acting and '*j*' is the direction of the applied force.

For any arbitrary surface, applied notation T_i^n means '*n*' is the direction normal to the surface and '*i*' is the direction of applied force.

A general expression for the linear momentum balance:

[Rate of momentum accumulation]1 = [Rate of momentum in]2 – [Rate of momentum out]3 + [Sum of the forces acting on the system]4 (1)

Momentum transfer to or from the system takes place by 2 mechanisms

i) Convection (by the virtue of bulk fluid flow)

ii) Molecular transfer/Diffusion

1) Rate of momentum accumulation (in 'x' direction)

$$\frac{\partial(\rho u \Delta x \Delta y)}{\partial t}$$

2) Rate of momentum in

$$\Delta y[(\rho u)u] + \Delta y[\sigma_{xx}] + \Delta x[(\rho u)v] + \Delta x[\tau_{yx}]$$

3) Rate of momentum out

$$[(\rho u)u + \frac{\partial(\rho uu)}{\partial x}\Delta x]\Delta y + [\sigma_{xx} + \frac{\partial}{\partial x}(\sigma_{xx})\Delta x]\Delta y + [(\rho u)v + \frac{\partial(\rho uv)}{\partial y}\Delta y]\Delta x + [\tau_{yx} + \frac{\partial}{\partial y}(\tau_{yx})\Delta y]\Delta x$$

4) Sum of all forces acting

a) Arising from fluid pressure P

in = $P_x \Delta y$

out = $[P_x + \frac{\partial P_x}{\partial x}\Delta x]\Delta y$

b) Body force

[let 'X' be the B.F per unit volume] = $X \Delta x \Delta y$

Substituting into equation (1)

$$\frac{\partial(\rho u \Delta x \Delta y)}{\partial t} = \Delta y[(\rho u)u] + \Delta y[\sigma_{xx}] + \Delta x[(\rho u)v] + \Delta y[\tau_{yx}] - \Delta y(\rho u)u - \frac{\partial(\rho uu)}{\partial x}\Delta x \Delta y$$

$$-\Delta y \sigma_{xx} - \frac{\partial}{\partial x}(\sigma_{xx})\Delta x \Delta y - \Delta x(\rho u)v - \frac{\partial(\rho uv)}{\partial y}\Delta y \Delta x - \Delta x \tau_{yx} - \frac{\partial}{\partial y}(\tau_{yx})\Delta y \Delta x$$

$$+P_x \Delta y - \Delta y P_x - \frac{\partial P_x}{\partial x}\Delta x \Delta y + X \Delta x \Delta y$$

After simplification,

$$\frac{\partial(\rho u)}{\partial t} = -\frac{\partial(\rho uu)}{\partial x} - \frac{\partial \sigma_{xx}}{\partial x} - \frac{\partial(\rho uv)}{\partial y} - \frac{\partial \tau_{yx}}{\partial y} - \frac{\partial P_x}{\partial x} + X$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} = -\frac{\partial \sigma_{xx}}{\partial x} - \frac{\partial \tau_{yx}}{\partial y} - \frac{\partial P_x}{\partial x} + X \qquad (2)$$

LHS = $\rho \frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} + \rho u \frac{\partial u}{\partial x} + u\frac{\partial(\rho u)}{\partial x} + \rho u \frac{\partial v}{\partial y} + v\frac{\partial(\rho u)}{\partial y}$

$$= \rho \frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} + u\left[u\frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x}\right] + \rho u \frac{\partial u}{\partial x} + \rho u \frac{\partial v}{\partial y} + v\left[u\frac{\partial \rho}{\partial y} + \rho \frac{\partial u}{\partial y}\right]$$

$$= \rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial x}\right]+u\left[\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]$$

$$= \rho\frac{Du}{Dt}+u\left[\frac{D\rho}{Dt}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right] = \rho\frac{Du}{Dt}$$

[As, $\frac{D\rho}{Dt}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$, from continuity equation]

Then,

$$\rho\frac{Du}{Dt}=-\left[\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}\right]-\frac{\partial P_x}{\partial x}+X \tag{3}$$

Equation 3 is the general linear momentum balance (2D) equation in the x-direction of the cartesian coordinate system.

Substitution of the relevant expression of σ_{xx} and τ_{yx} using constitutive relations for Newtonian fluids will lead to the famous Navier Stokes Equations

Energy Transport

A general expression for energy transport

$$[\text{Rate accumulation in the CV}]_1 = [\text{Net transfer of energy by fluid flow}]_2 + [\text{Net heat transfer by conduction}]_3 + [\text{Rate of internal heat generation}]_4 - [\text{Net work transfer from the CV to its environment}]_5 \quad (1)$$

For terms **1** and **2**,

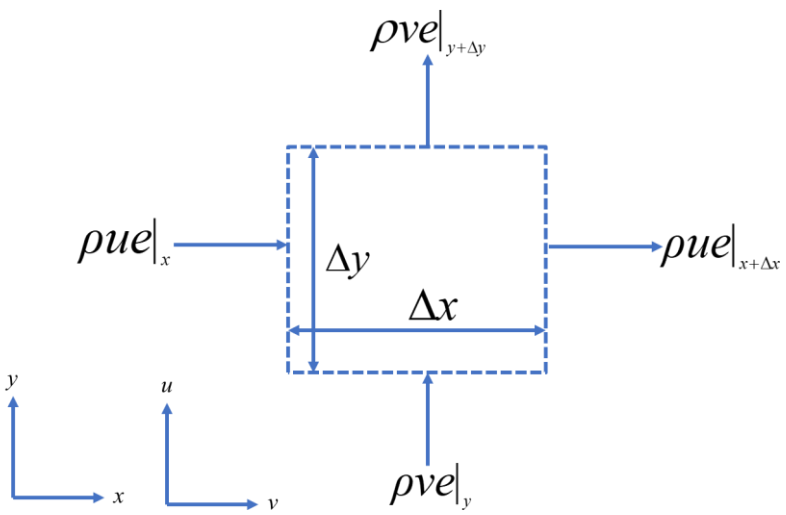

Figure 2. Energy transport by fluid flow in a 2D flow field (Cartesian coordinate system)

Define e as the total specific energy of the system,

$$e = \hat{u} + \frac{1}{2}v^2$$

$\hat{u}$ = specific internal energy = U/m = $\hat{u}$

$\frac{1}{2}v^2$ = specific Kinetic energy = KE/m = $\frac{1}{2}\frac{mv^2}{m} = \frac{1}{2}v^2$

1. Rate of energy accumulation

$$\frac{\partial(\rho e)}{\partial t}\Delta x \Delta y \quad (2)$$

2. Net transfer of energy by fluid flow

i) *Rate of energy in*

$$\Delta y \rho ue + \Delta x \rho ve$$

ii) *Rate of energy out*

$$\Delta y \rho ue\big|_{x+\Delta x} + \Delta x \rho ve\big|_{y+\Delta y} = [\rho ue + \frac{\partial(\rho ue)}{\partial x}\Delta x]\Delta y + [\rho ve + \frac{\partial(\rho ve)}{\partial y}\Delta y]\Delta x$$

Net = (in – out) = (*i*-*ii*)

$$= -[\frac{\partial(\rho ue)}{\partial x} + \frac{\partial(\rho ve)}{\partial y}]\Delta x\Delta y \qquad (3)$$

3. Net heat transfer by conduction

Conduction heat flux is the transport element,

$$q'' = \frac{Conductive\ heat}{Area * time}$$

Similarly, Net conduction heat transfer,

$$-\left[\frac{\partial(q_x'')}{\partial x} + \frac{\partial(q_y'')}{\partial y}\right]\Delta x\Delta y \qquad (4)$$

4. Rate of Internal heat generation

Let the volumetric heat generation rate be q''' (heat/volume*time)

Thus, the total internal heat generation is,

$$q'''\Delta x\Delta y \qquad (5)$$

5. Net work done by the fluid element against its surrounding

It consists of two parts

i) *Work against the body forces (originate from volume)*

ii) *Work against the surface forces*

a) Work against the pressure forces

b) Work against the viscous forces

Recall,

Work = (force) × (distance in the direction of the applied force)

Rate of work = (force) × (velocity in the direction of the applied force)

i) Work against the body forces (originate from volume)

$$-\Delta x \Delta y \rho [v_x g_x + v_y g_y] \qquad 6(i)$$

g_x, g_y are the gravitational accelerations in x and y directions respectively ~ Force/mass.

The negative sign appears in Eqn.5(i) as displacement is against the body force.

ii) Work against the surface forces

a) Pressure

Net force (similar to 2 and 3)

$$-\left[\frac{\partial(pv_x)}{\partial x} + \frac{\partial(pv_y)}{\partial y}\right]\Delta x \Delta y \qquad 6\ [ii(a)]$$

b) **Viscous forces**

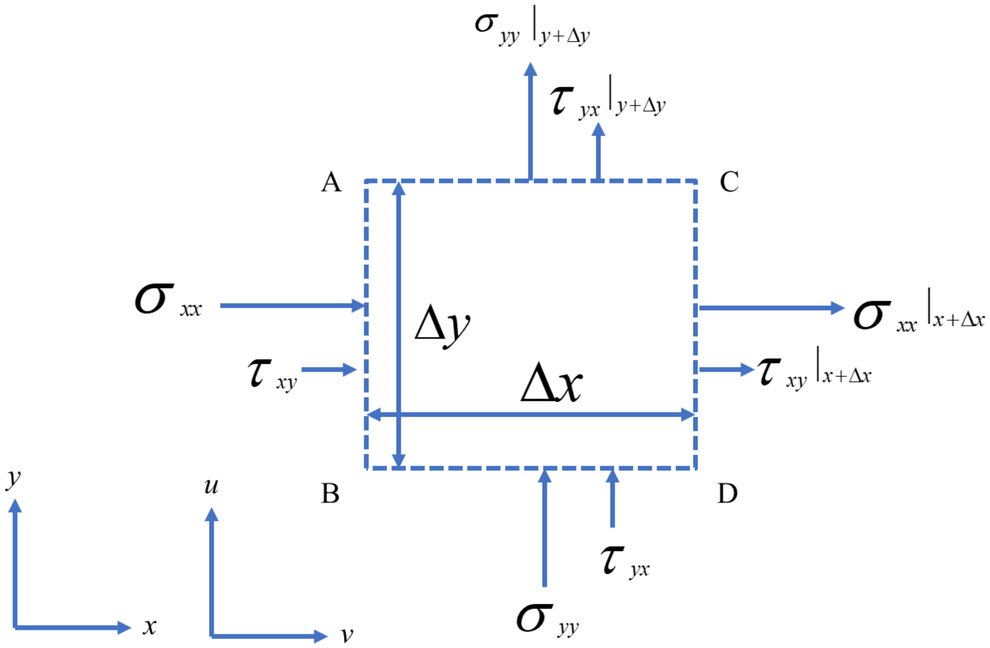

Figure 4. Work done against the viscous force in a 2D flow field (Cartesian coordinate system)

The arrow directions for the τ_{xy} and τ_{yx} are not physical.

The general representation is τ_{ij}, where *i* is the direction normal to the surface along which force is acting, and *j* is the direction of the applied force.

For the AB plane (normal to the *x*-axis at *x=0*), forces/unit area acting are σ_{xx} and τ_{xy}

The velocity in the direction of the force σ_{xx} will be *u* (Since the direction of applied force, the jth index, is *x*)

The velocity in the direction of the force τ_{xy} will be *v* (Since the direction of applied force, the jth index, is *y*).

Hence, the rate of work done by the viscous forces at the AB plane against the surrounding (sign not considered) is

$[\sigma_{xx}u+\tau_{xy}v]\Delta y$

Similarly, at the BD plane (perpendicular to the y-axis at y=0)

Work done by the viscous force

$[\sigma_{yy}v+\tau_{yx}u]\Delta x$.

Thus, net work done by the viscous force-

$$= [\sigma_{xx}u + \tau_{xy}v]\Big|_x \Delta y + [\sigma_{yy}v + \tau_{yx}u]\Big|_y \Delta x - [\sigma_{xx}u + \tau_{xy}v]\Big|_{x+\Delta x} \Delta y - [\sigma_{yy}v + \tau_{yx}u]\Big|_{y+\Delta y} \Delta x$$

$$= -[\frac{\partial \sigma_{xx}u}{\partial x} + \frac{\partial \tau_{xy}v}{\partial x} + \frac{\partial \sigma_{yy}v}{\partial y} + \frac{\partial \tau_{yx}u}{\partial y}]\Delta x \Delta y$$

$$= -[u\frac{\partial \sigma_{xx}}{\partial x} + \sigma_{xx}\frac{\partial u}{\partial x} + \tau_{xy}\frac{\partial v}{\partial x} + v\frac{\partial \tau_{xy}}{\partial x} + v\frac{\partial \sigma_{yy}}{\partial y} + \sigma_{yy}\frac{\partial v}{\partial y} + u\frac{\partial \tau_{yx}}{\partial y} + \tau_{yx}\frac{\partial u}{\partial y}]\Delta x \Delta y$$

Writing the constitutive equations,

$$\tau_{yx} = \tau_{xy} = -\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$$

$$\sigma_{xx} = -2\mu\frac{\partial u}{\partial x} + \frac{2}{3}\mu(\nabla . v)$$

$$\sigma_{yy} = -2\mu\frac{\partial v}{\partial y} + \frac{2}{3}\mu(\nabla . v)$$

For incompressible flow, $\nabla . v = 0$

Substituting τ_{yx}, σ_{xx}, σ_{yy} with $\nabla . v = 0$ in the above equation we get

$$[\mu\phi]\Delta x \Delta y \qquad 6\,[ii(b)]$$

Where,

$$\phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2$$ =Viscous dissipation function

Net work done by fluid on the surroundings,

5(i)+ 5 [ii(a)]+ 5 [ii(b)]

$$= -[\rho(ug_x + vg_y)\phi + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} - \mu\phi]\Delta x \Delta y \qquad (6)$$

Substituting (2), (3), (4),(5) and (6) into equation (1)

$$\frac{\partial(\rho e)}{\partial t} = -[\frac{\partial(\rho ue)}{\partial x} + \frac{\partial(\rho ve)}{\partial y}] - [\frac{\partial(q_x^{''})}{\partial x} + \frac{\partial(q_y^{''})}{\partial y}] + q^{'''} - [\rho(ug_x + vg_y)\phi + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} - \mu\phi]$$

$$\rho\frac{\partial(e)}{\partial t}+e[\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}]+\rho u\frac{\partial e}{\partial x}+\rho v\frac{\partial e}{\partial y}=-(\nabla.q^{''})+q^{'''}+\rho(\vec{v}.g)-\nabla.(\rho\vec{v})+\mu\phi$$

$$\rho\frac{\partial(e)}{\partial t}+e[\frac{\partial\rho}{\partial t}+\nabla.(\rho\vec{v})]=-(\nabla.q^{''})+q^{'''}+\rho(\vec{v}.g)-\nabla.(\rho\vec{v})+\mu\phi$$

Putting $\frac{\partial\rho}{\partial t}+\nabla.(\rho\vec{v})=0$ (continuity),

$$\rho\frac{\partial(e)}{\partial t}=-(\nabla.q^{''})+q^{'''}+\rho(\vec{v}.g)-\nabla.(\rho\vec{v})+\mu\phi \tag{7}$$

Now, gravitational force per unit mass g is a vector and can be expressed as a gradient of a scalar,

$g=-\nabla\phi$ Where, ϕ is the potential energy

Hence,

$$\rho(\vec{v}.g)=-\rho(\vec{v}.\nabla\phi)=-\rho\frac{D\phi}{Dt}+\rho\frac{\partial\phi}{\partial t}$$

Also,

$$\frac{D\phi}{Dt}=\frac{\partial\phi}{\partial t}+\nabla.(v\phi)\approx\frac{\partial\phi}{\partial t}+\vec{v}.\nabla\phi\,;$$

$$(\vec{v}.\nabla\phi)=\frac{D\phi}{Dt}-\frac{\partial\phi}{\partial t}\,;$$

If ϕ is time-independent then, $\frac{\partial\phi}{\partial t}=0$

So, $\rho(\vec{v}.g)=-\rho\frac{D\phi}{Dt}$

Substituting into equation 7,

$$\rho\frac{D(e+\phi)}{Dt}=q'''-\nabla.(P\vec{v})+\mu\phi-\nabla.q''\,'' \tag{8}$$

This is an equation of change for total energy, $e+\phi=\hat{u}+\frac{v^2}{2}+\phi$

Now we have to use the laws of thermodynamics to get the transport equation in terms of Temperature.

According to the First law of thermodynamics

$Q=\Delta u+w$

So, the energy supplied to the system is used to increase the internal energy only. Considering internal energy only, equation 8 can be rewritten as

$$\rho \frac{De}{Dt} = q''' - \nabla.(P\vec{v}) + \mu\phi - \nabla.q'' \tag{9}$$

Where $e = \hat{u}$ only.

Using thermodynamic relation Enthalpy of the system,

$$H = U + PV$$

In terms of specific enthalpy,

$$H / m = h = (U / m + PV / m) = [\hat{u} + P / (m / V)] = [e + P / \rho]$$

Or,

$$\frac{Dh}{Dt} = \frac{De}{Dt} + \frac{1}{\rho}\frac{DP}{Dt} - \frac{P}{\rho^2}\frac{D\rho}{Dt} \tag{10}$$

Using Fourier's law –

$$q'' = -K\nabla T \tag{11}$$

Substituting equation 10 and 11 in equation 9, we get

$$\rho \frac{Dh}{Dt} = q''' - P(\nabla.\vec{v}) + \mu\phi + \frac{DP}{Dt} - \frac{P}{\rho}\frac{D\rho}{Dt} - \nabla.q''$$

$$\rho \frac{Dh}{Dt} = q''' + \mu\phi + \frac{DP}{Dt} - \frac{P}{\rho}\left[\frac{D\rho}{Dt} + \rho(\nabla.\vec{v})\right] + \nabla.(K\nabla T)$$

$$\rho \frac{Dh}{Dt} = q''' + \mu\phi + \frac{DP}{Dt} + \nabla.(K\nabla T) \tag{12}$$

Now, $dh = C_p dT$ is only valid for ideal gases (see **Table 1**)

In general, we know,

$$dH = TdS + VdP$$

[As $H = U + PV$ and, $dH = (dU + PdV) + VdP = dQ + VdP = TdS + VdP$]

Table 1: Summary of thermodynamic relations and models

	Internal Energy $du = T\,ds - P\,dv$	Enthalpy $dh = T\,ds + v\,dP$	Entropy $ds = \frac{1}{T}du + \frac{P}{T}dv$
Pure substance	$du = c_v\,dT + \left[T\left(\frac{\partial P}{\partial T}\right)_v - P\right]dv$	$dh = c_p\,dT + \left[-T\left(\frac{\partial v}{\partial T}\right)_p + v\right]dP$	$ds = \frac{c_p}{T}dT - \left(\frac{\partial v}{\partial T}\right)_P dP = \frac{c_v}{T}dT + \left(\frac{\partial P}{\partial T}\right)_v dv$
Ideal gas	$du = c_v\,dT$	$dh = c_p\,dT$	$ds = c_p\frac{dT}{T} - R\frac{dP}{P} = c_v\frac{dT}{T} + R\frac{dv}{v} = c_v\frac{dP}{P} + c_P\frac{dv}{v}$
Incompressible liquid	$du = c\,dT$	$dh = c\,dT + v\,dP$	$ds = c\frac{dT}{T}$

In terms of specific property

$$dh = Tds + \frac{1}{\rho}dP \tag{13}$$

Now, $s = f(T,P)$

$$ds = \left(\frac{\partial s}{\partial T}\right)_P dT + \left(\frac{\partial s}{\partial P}\right)_T dP \tag{14}$$

From Maxwell's relation [$\left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p$],

we have (for specific properties)

$$\left(\frac{\partial s}{\partial P}\right)_T = -\left(\frac{\partial(1/\rho)}{\partial T}\right)_P = \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_P = \frac{-\beta}{\rho}$$

Where is β the thermal expansion coefficient

$$\beta = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_P = \frac{1}{(1/\rho)}\left(\frac{\partial(1/\rho)}{\partial T}\right)_P = -\frac{\rho}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_P = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P$$

We also have, $\left(\frac{\partial s}{\partial T}\right)_P = \frac{C_P}{T}$

Substituting into equation 14, we have

$$ds = \frac{C_P}{T} dT - \frac{\beta}{\rho} dP$$

Substituting into equation 13, we have

$$dh = T[\frac{C_P}{T} dT - \frac{\beta}{\rho} dP] + \frac{1}{\rho} dP = C_P dT + \frac{1}{\rho}(1 - \beta T) dP$$

Taking substantial derivative and multiplying by ρ we get,

$$\rho \frac{Dh}{Dt} = \rho C_P \frac{DT}{Dt} + (1 - \beta T) \frac{DP}{Dt} \qquad (15)$$

(Relation between *h* and *T*)

Using equation 15 and 12, we get the **general form of the energy transport equation for Newtonian fluids**

$$\rho C_P \frac{DT}{Dt} = q''' + \mu\phi + \beta T \frac{DP}{Dt} + \nabla.(K \nabla T) \qquad (16)$$

(Common form of $\mu\phi$ is $-[\nabla.(\tau.\vec{v})]$)

Special cases,

(A) For ideal gas, $\beta = 1/T$

So,

$$\rho C_P \frac{DT}{Dt} = q''' + \mu\phi + \frac{DP}{Dt} + \nabla.(K \nabla T) \qquad (17)$$

(B) For incompressible liquid, $\beta = 0$

$$\rho C_P \frac{DT}{Dt} = q''' + \mu\phi + \nabla.(K \nabla T) \qquad (18)$$

In most of the convection problems we have

1) Constant K

2) Zero heat generation

3) Negligible viscous dissipation, $\phi = 0$

Using the above three assumptions, the temperature distribution model will be

$$\rho C \frac{DT}{Dt} = K \nabla^2 T \qquad (19)$$

For an incompressible fluid, (specific heat, $C = C_p$)