Prof. Debasis Sarkar

Advanced Heat Transfer

Steady & Unsteady State conduction → Prof. D.S.

Forced and Free convection

Thermal Radiation

Books: Heat Conduction by M. Necati Ozisik

" " by Sadik Kadak, Yaman Yener, Carolina P.

" " by Latif Jiji

Evaluation via -

- Class tests
- Assignments
- Term Paper
- Viva

Course Outline

- Introduction
- Mechanisms of HT
- Fourier law
- Thermal conductivity tensor, measurement
- Heat conduction eq, Moving solids, Boundary conditions
- Kirchoff transformation
- Fins - uniform and non-uniform cross sections, Fin optimisation
- Orthogonal functions, Strum-Liouville problems
- Fourier expansions: Fourier sine and cosine series
- Separation of variables: Steady state and unsteady state conduction in rectangular/spherical & cylindrical co-ordinate systems.

- Solutions with Integral transforms : Finite / Semi-infinite / Infinite fourier transform, Hankel transform, Laplace transform
- Heat conduction with phase change : Stefan condition, P.C. probs.
- Numerical methods for solution of heat conduction problems.

What are the basic questions study of HT aims to answer -

Mode of HT

Rate of HT

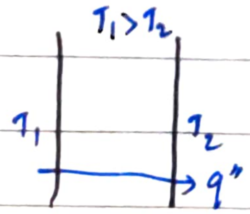
Temperature distribution in the body.

Modes of H.T. - Conduction, Convection & Radiation

- First mechanism - Molecular interaction (fluids)
  - ↳ Greater motion of molecules at higher temp.
- Second mechanism - Free electrons (solids)

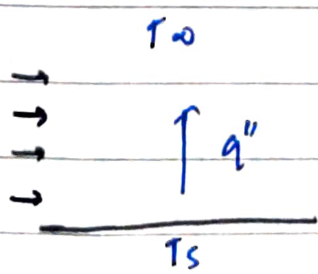
In general temp distribution in a body is controlled by a combination of all the three modes.

Conduction



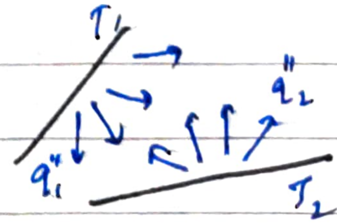
$$q'' = -k\frac{dT}{dx} = k\frac{T_1 - T_2}{L}$$

Convection



$$q'' = h(T_s - T_\infty)$$

Radiation



$$q'' = \varepsilon\sigma T_s^4$$

3 scientific approaches for problem solving -

1. Experimental approach - Realistic but expensive with technical hurdles, not extrapolatable.
2. Analytical approach - Elegant but difficult in application of adv. physics, complex geometries and non-linearity
3. Computational approach - Getting cheaper by the day, accurate, needs validation by experiments.

Concept of continuum

To determine macroscopic properties of a solid / fluid; it is assumed to be a continuous medium. It is an idealisation even though on a microscopic scale, it is composed of individual molecules.

But in study of thermophysics or transport then system has to be studied on a microscopic level (mol. structure) where continuum concept is invalid.

→ Valid when size and mean free path are much smaller than the other dimensions in the medium.

Temperature Distribution Function - Instantaneous values of temperatures at all points in the medium of interest. It can be

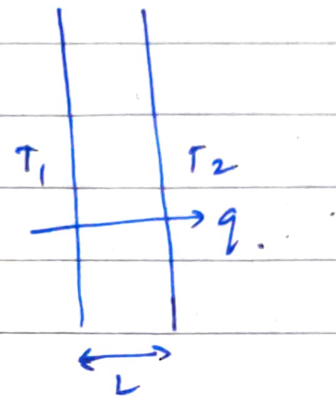
Steady	vs.	unsteady (transient)
$T = f(r)$ (position)		$T = f(r, t)$ (position, time)
$\frac{\partial T}{\partial t} = 0$		Can be 1D or 2D or 3D.

Fourier's law of Heat Conduction.

It is an observed phenomenon

$$q \propto A \cdot \frac{T_1 - T_2}{L}$$

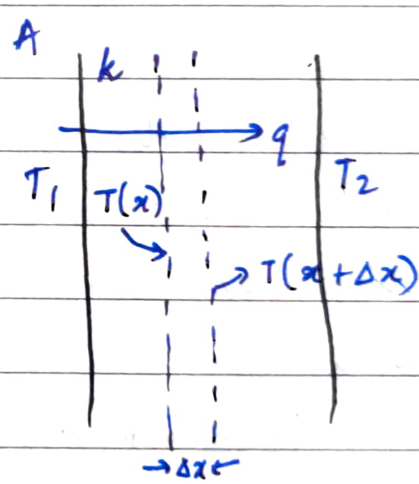
$$q = kA\left(\frac{T_1 - T_2}{L}\right)$$



where $k = \dfrac{(q/A) \rightarrow \text{heat flux}}{(T_1 - T_2)/L \rightarrow \text{temp. gradient.}}$

$k \downarrow$ Thermal conductivity

For a 1D Steady system;



$$q = kA\left(\frac{T(x) - T(x+\Delta x)}{\Delta x}\right)$$

$$\Rightarrow q = -kA \lim_{\Delta x \to 0} \frac{T(x+\Delta x) - T(x)}{\Delta x}$$

$$\Rightarrow \boxed{q = -kA\frac{dT}{dx}} \Rightarrow \boxed{\frac{q}{A} = q'' = -k\frac{dT}{dx}}$$

$k \rightarrow$ Units $W/(m.K)$

Diff. types of medium based on k –

- Homogeneous
- Heterogeneous

} Depending on whether k varies with position

- Isotropic
- Anisotropic

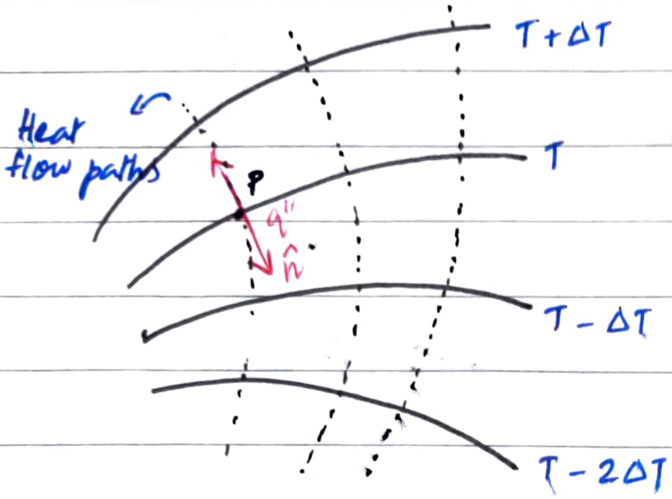
} Whether k is same/different in any direction

Porous media – heterogeneous, fibrous materials – anisotropic

classmate

Fourier's law of heat conduction : Vector Form

Consider isothermal surfaces -



Heat flux due to conduction at P -

$$q''_n = -k\frac{\partial T}{\partial n}$$

where $\frac{\partial}{\partial n}$ represents differentiation along the normal $\hat{n}$.

In rectangular system,

$$q''_n = -k\left[\frac{\partial T}{\partial x}\frac{dx}{dn} + \frac{\partial T}{\partial y}\frac{dy}{dn} + \frac{\partial T}{\partial z}\frac{dz}{dn}\right]$$

or, $\hat{n} = \hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma$ where $(\alpha, \beta, \gamma) \rightarrow$ direction cosines of $\hat{n}$.

Hence, $q''_n = -k\left(\frac{\partial T}{\partial x}\cos\alpha + \frac{\partial T}{\partial y}\cos\beta + \frac{\partial T}{\partial z}\cos\gamma\right)$

Using vector calculus,

$$q''_n = -k\left[\hat{i}\frac{\partial T}{\partial x} + \hat{j}\frac{\partial T}{\partial y} + \hat{k}\frac{\partial T}{\partial z}\right]\cdot\hat{n}$$

$\Rightarrow q''_n = -k\nabla T\cdot\hat{n}$

$\nabla T = \hat{i}\frac{\partial T}{\partial x} + \hat{j}\frac{\partial T}{\partial y} + \hat{k}\frac{\partial T}{\partial z}$ in rectangular coordinates.

$\longrightarrow q'' = \hat{n}q''_n$

Use $q''_n = -k\nabla T\cdot\hat{n}$

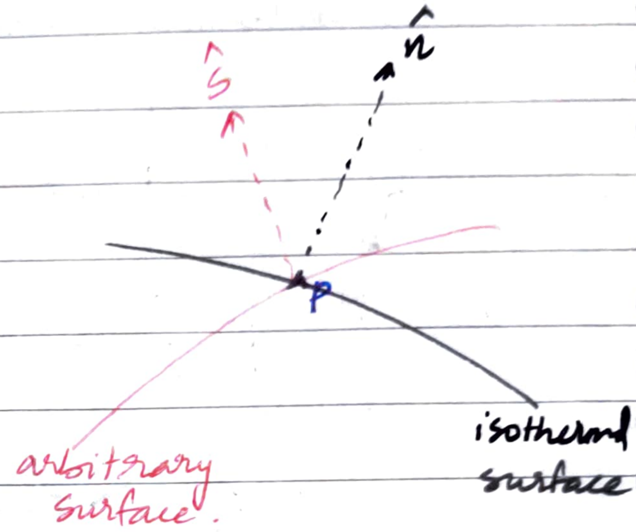
$\Rightarrow \boxed{q'' = -k\nabla T} \longrightarrow$ vector form of Fouriers law.

$\downarrow$

diff. forms in diff. coordinate systems

Extension of Fouriers' law to non-isothermal surface

Magnitude of heat flux across any arbitrary surface passing through P and having $\hat{s}$ as its normal unit vector will be equal to the component of q'' in s direction; i.e.;



$$q''_s = q''.\hat{s} = k\nabla T.\hat{s}$$

Since $\nabla T.\hat{s} = \frac{\partial T}{\partial s}$; we have $q''_s = -k\frac{\partial T}{\partial s}$

In rectangular coordinate system;

$$q''_x = -k\frac{\partial T}{\partial x} \quad ; \quad q''_y = -k\frac{\partial T}{\partial y} \quad ; \quad q''_z = -k\frac{\partial T}{\partial z}$$

Thermal Conductivity : Anisotropic solids.

Heat flux may not necessarily be parallel to temp. gradient. In anisotropic solids, heat flux due to conduction in a given direction can also be proportional to temp. gradients in other directions.

$$q''_1 = -k_{11}\frac{\partial T}{\partial x_1} - k_{12}\frac{\partial T}{\partial x_2} - k_{13}\frac{\partial T}{\partial x_3}$$

$$q''_2 = -k_{21}\frac{\partial T}{\partial x_1} - k_{22}\frac{\partial T}{\partial x_2} - k_{23}\frac{\partial T}{\partial x_3}$$

$$q''_3 = -k_{31}\frac{\partial T}{\partial x_1} - k_{32}\frac{\partial T}{\partial x_2} - k_{33}\frac{\partial T}{\partial x_3}$$

where x_1, x_2, x_3 are three orthogonal directions

Thermal conductivity tensor $[k_{ij}] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$

$$\therefore \quad q_i'' = \sum_{j=1}^{3} \left(-k_{ij} \frac{\partial T}{\partial x_j} \right) \text{ for } i = 1, 2, 3.$$

For isotropic materials, $k_{11} = k_{22} = k_{33} = k$; other terms ($k_{12}$, k_{13} ...) are zero.

Orthotropic meterials.

18.01.21

$$q_i'' = -k \cdot \frac{1}{a_i} \frac{\partial T}{\partial u_i} \quad \text{for } i = 1, 2, 3.$$

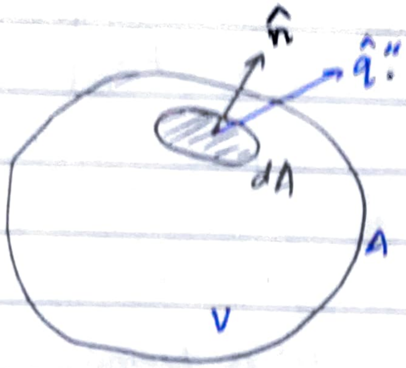
a_i → scale factor

$q_r'' = -k \dfrac{\partial T}{\partial r}$; $q_\phi'' = -\dfrac{k}{r} \dfrac{\partial T}{\partial \phi}$; $q_z'' = -k \dfrac{\partial T}{\partial z}$ in cylindrical coordinates.

$q_r'' = -k \dfrac{\partial T}{\partial r}$; $q_\phi'' = -\dfrac{k}{r \sin\theta} \dfrac{\partial T}{\partial \phi}$; $q_\theta'' = -\dfrac{k}{r} \dfrac{\partial T}{\partial \theta}$ in spherical coordinates.

These scale factors also correct the unit of flux as temp. gradient has no length term in ϕ and θ directions.

Total heat flow by Conduction



To find the total heat flow by conduction, it needs to first be found for surface element dA.

The rate of heat flow out of the solid through surface element dA per unit time $= q'' \cdot \hat{n}\, dA$

Rate of heat flowing out of the entire bounding surface $= \int_A q'' \cdot \hat{n}\, dA$

$q''_n = -k \frac{\partial T}{\partial n}$

Heat flowing out of the bounding surface in time interval t_1 to t_2 $= \int_{t_1}^{t_2} \int_A q'' \cdot \hat{n}\, dA$

$\hookrightarrow -k \frac{\partial T}{\partial n}$

Need temperature distribution in the solid that can be obtained from the energy balance eq as done below

<u>Divergence theorem</u>:

$$\iint_S F \cdot dS = \iiint_E \operatorname{div} F\, dV$$

Converts surface integral to volume integral.

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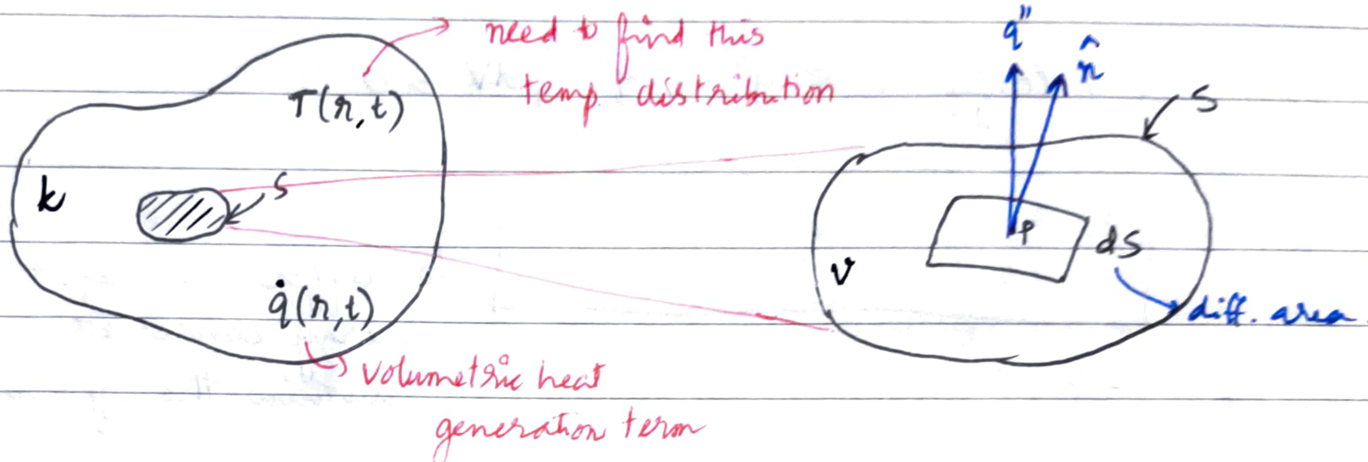
q''' and $\dot{q}$ are the same thing
dS and dA are the same thing

Consider stationary (only conduction), homogeneous, isotropic, opaque (no radiation) solid.



Energy balance:

Rate of heat entering through the bounding surfaces of V + rate of heat generation =

$$\frac{dE}{dt} = \dot{Q}$$

E can be considered as the sum of many small volume elements.

$$\frac{dE}{dt} = \int_V \rho \frac{\partial e}{\partial t}\, dV = \int_V \rho c \frac{\partial T}{\partial t}\, dV$$

specific heat

$\dot{Q}$ = net rate of heat conduction into V + heat generation in volume V.

$$= -\int_A q''\cdot n\, dA + \int_V q''\, dV.$$

$$\therefore \int_V \rho c \frac{\partial T}{\partial t}\, dV = -\int_A q''\cdot n\, dA + \int_V \dot{q}\, dV.$$

surface integral – converting it to volume integral

From divergence theorem,

$$\int_A q''.n\,dA = \int_V \nabla . q''\,dV.$$

$$\therefore \int_V \left(\rho c \frac{\partial T}{\partial t} + \nabla . q'' - q''' \right) dV = 0$$

$$\therefore \boxed{\rho c \frac{\partial T}{\partial t} + \nabla . q'' - q''' = 0.}$$

↳ differential eq. for energy conservation within the system.

$$q'' = -k \nabla T$$

$$\therefore \nabla q'' = -\nabla . (k \nabla T).$$

Upon substitution,

$$\nabla . (k \nabla T) + \dot{q} = \rho c \frac{\partial T}{\partial t}$$

$$\Rightarrow \boxed{k \nabla^2 T + \nabla k . \nabla T + \dot{q} = \rho c \frac{\partial T}{\partial t}} \longrightarrow \text{A}$$

If k, ρ, c are functions of space only, then eq A is a linear PDE. If k is constant, we have,

$$\boxed{k \nabla^2 T + \dot{q} = \rho c \frac{\partial T}{\partial t}} \longrightarrow \text{Fourier - Biot eq.}$$

$$\Rightarrow \Delta^2 T + \frac{\dot{q}}{k} = \frac{1}{(k/\rho c)} \frac{\partial T}{\partial t}$$

$$\Rightarrow \Delta^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

↳ thermal diffusivity (unit: m^2/s)

Considering constant properties + No heat gen. $\nabla^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t}$ Heat diffusion eq.

Steady state, constant k $\nabla^2 T + \frac{\dot{q}}{k} = 0$ Poisson eq.

Steady state, no heat gen. $\nabla^2 T = 0$ Laplace eq.

depends on shape of solid and B.Cs only.

In rectangular coordinates, $\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$

In cylindrical coordinates, $\nabla^2 T = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$

In spherical coordinates,

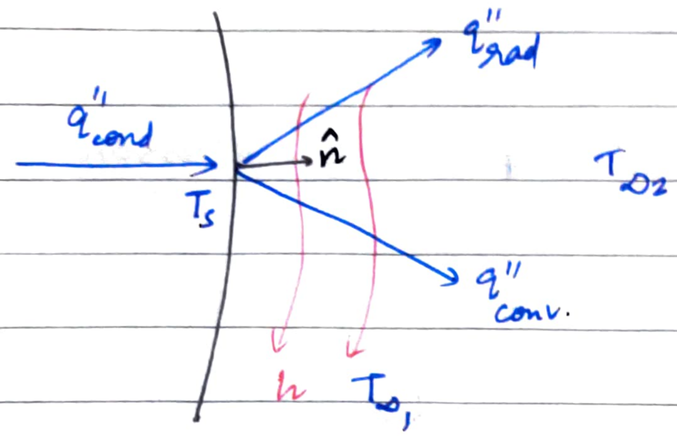
$$\nabla^2 T = \frac{1}{r^2}\frac{\partial T}{\partial r} + \left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}$$

<u>Boundary Conditions:</u>

Solution will require 2 B.Cs. in each spatial direction and one initial condition for the non-steady state problem.

Initial condition : $T(r,t)\big|_{t\to 0} = T_0(r)$

General B.C.



Assume stationary, thin surface (no accumulation).

$\therefore$ in = out

$$q''_{cond} = q''_{rad} + q''_{conv.}$$

$$\Rightarrow \underbrace{-k \left.\frac{\partial T}{\partial n}\right|_{surf}}_{\text{Fourier's Law}} = \underbrace{h(T|_{surf} - T_{\infty,})}_{\text{Newton's Law}} + \underbrace{\varepsilon\sigma(T^4|_{surf} - T_{\infty_2}^4)}_{\text{Stefan-Boltzmann law}}$$

Fourier's Law (+ve flux in +ve co-ordinate direction)

Newton's Law (+ve flux in direction of surface normal)

Stefan-Boltzmann law (+ve flux away from surface when $T_s > T_\infty$)

B.C. of 1st kind (Dirichlet B.C.) — Surface temp is known.

$$T_s = T_0 \text{ or } f(\hat{r}, t)$$

B.C. of 2nd kind (Neumann B.C.) — Heat flux is given.

$$-k \left.\frac{\partial T}{\partial n}\right|_{surface} = q''_0 \text{ or } f(\hat{r}, t).$$

B.C. of 3rd kind (Robbin's B.C.) — T_∞ given — convection

$$-k\left(\frac{\partial T}{\partial n}\right) = h\left[T(r_s, t) - T_\infty\right]$$

(or) $$h T_\infty = \left[k \frac{\partial T}{\partial n} + hT\right]$$

Heat conduction eq: Moving solids.

Motion of solid adds to convective flux ($\rho c T u$, $\rho c T v$, $\rho c T w$)

$\therefore q''_x = -k\frac{\partial T}{\partial x} + \rho c T u$; $q''_y = -k\frac{\partial T}{\partial y} + \rho c T v$; $q''_z = -k\frac{\partial T}{\partial z} + \rho c T w$

Substituting in $\rho c \frac{\partial T}{\partial t} + \nabla \cdot q'' - q''' = 0$

$$\Rightarrow \nabla\cdot(k\nabla T) + q''' = \rho c\left[\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right]$$

$$\Rightarrow \boxed{\nabla\cdot(k\nabla T) + q''' = \rho c\frac{DT}{Dt}}$$

Variable Thermal Conductivity : Linear Variation.

Consider $k = k_0(1+b\theta) \Rightarrow \boxed{q = -k_0(1+b\theta)A\cdot\frac{d\theta}{dx}} \rightarrow$ eq(i)

$$\Rightarrow \frac{q}{A}\int_{x_1}^{x_2} dx = -k_0\int_{\theta_1}^{\theta_2}(1+b\theta)\,d\theta$$

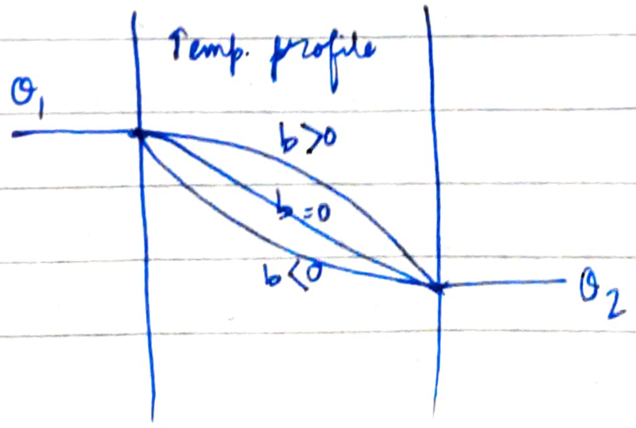
$$\Rightarrow \frac{q}{A}(x_2-x_1) = k_0\left[\theta_1-\theta_2+\frac{b}{2}(\theta_1^2-\theta_2^2)\right]$$

$$= k_0(\theta_1-\theta_2)\left(1+b\left(\frac{\theta_1+\theta_2}{2}\right)\right)$$

$$= k_0(\theta_1-\theta_2)(1+b\theta_m)$$ where $k_m = k_0(1+b\theta_m)$ and $\theta_m = \frac{\theta_1+\theta_2}{2}$

$$= k_m(\theta_1-\theta_2)$$

$$= -k_m(\theta_2-\theta_1)$$



From eq(i); upon diff. wrt x

$$\frac{d^2\theta}{dx^2} = -\left(\frac{b}{1+b\theta}\right)\left(\frac{d\theta}{dx}\right)^2$$

If $b>0$, $\left(\frac{-b}{1+b\theta}\right) < 0$ and $T(x)$ is concave.

classmate

where $k(T)$ is any function of temperature.

Variable Thermal conductivity $k(T)$: Kirchoff Transformation

Consider, $\nabla \cdot (k(T) \cdot \nabla T) + \dot{q}(r, T) = \rho(T) c(T) \dfrac{\partial T}{\partial t}$ from eq (A)

Kirchoff transformation :

$$\theta(r, t) = \frac{1}{k_R} \int_{T_R}^{T(r,t)} k(T')\, dT'$$

where, T_R is a reference temp. and $k_R = k(T_R)$ $\longrightarrow$ eq (B)

From eq. B,
$$\nabla \theta = \frac{k(T)}{k_R} \nabla T$$

$$\frac{\partial \theta}{\partial t} = \frac{k(T)}{k_R} \cdot \frac{\partial T}{\partial t}$$

Using these with eq A;

$$\boxed{\nabla^2 \theta + \frac{\dot{q}(r, t)}{k_R} = \frac{1}{\alpha} \frac{\partial T}{\partial t}}$$ where $\alpha(T) = \dfrac{k(T)}{\rho(T) \cdot c(T)}$

For many solids, variation of $\alpha(T)$ with temp. is not that significant. If it can be approximated to be a constant, then the above transformed eq. becomes linear.

Also, in steady state; $\nabla^2 \theta + \dfrac{\dot{q}(r, t)}{k_R} = 0$ $\Rightarrow$ Linear PDE irrespective of α

It is imp. that even B.C.s are transformed. Kirchoff's transformation poses no problem in transformation of B.C.s of 1st kind and 2nd kind. However, BCs of 3rd kind may not be possible.

classmate

Question:

Find the rate of heat gen. per unit vol in a rod that will produce a centerline temp. of 2000°C for the following conditions:

$$r_o = 1\text{ cm}$$

$$T_w = 350°C$$

$$k = \frac{3167}{T + 273}$$

Also, calculate surface heat flux.

B.C.s:

$$\left.\frac{dT}{dr}\right|_{r=0} = 0$$

$$T(r_o) = T_w$$

Solution: Steady state system.

In cylindrical co-ordinate system,

$$\frac{1}{r}\frac{d}{dr}\left[r\,k(T)\cdot\frac{dT}{dr}\right] + \dot{q} = 0$$

Defining θ (from kirchoff transformation) with $T_R = T_w$

$$\theta(r) = \frac{1}{k_w}\int_{T_w}^{T(r)} k(T')\,dT'$$

$$\therefore \frac{1}{r}\frac{d}{dr}\left[r\frac{d\theta}{dr}\right] + \frac{\dot{q}}{k_w} = 0 \quad \text{where,} \quad \left(\frac{d\theta}{dr}\right)_{r=0} = 0 \text{ and } \theta(r_o) = 0.$$

Upon integration,

$$\theta(r) = \frac{\dot{q}r_o^2}{4k_w}\left[1 - \left(\frac{r}{r_o}\right)^2\right]$$

Now, we need to get it back in terms of T.

$$\frac{\dot{q} r_o^2}{4 \cancel{k_w}}\left[1-\left(\frac{r}{r_o}\right)^2\right] = \frac{1}{\cancel{k_w}}\int_{T_w}^{T(r)} k(T')\,dT'$$

At $r=0$, $T = T_c$

$$\frac{\dot{q} r_o^2}{4} = \int_{T_w}^{T_c} k(T)\,dT$$

$$\dot{q} = \frac{4}{r_o^2}\int_{T_w}^{T_c} \frac{3167}{T+273}\,dT = \frac{4 \times 3167}{(0.01)^2} \ln\left(\frac{2273}{623}\right)$$

$$\dot{q} = 1.64 \times 10^8 \ W/m^3$$

Surface Heat flux :

$$q_s'' = \frac{\dot{q} V}{A} = \frac{\dot{q}\pi r_o^2 L}{2\pi r_o L} = \frac{\dot{q} r_o}{2} = 8.2 \times 10^6 \ W/m^3$$

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Heat Conduction Eq: Review

For heterogeneous isotropic solids (and frictionless incompressible fluids):

$$\rho c \frac{dT}{dt} = \nabla\cdot(k\nabla T) + q'''$$

$$\rho c \frac{dT}{dt} = \nabla k \cdot \nabla T + k\nabla^2 T + q'''$$

When $k = f(T)$; $\nabla k = \left(\frac{dk}{dT}\right)(\nabla T)$

$$\therefore \rho c \frac{dT}{dt} = \frac{dk}{dT}(\nabla T)^2 + k(\nabla^2 T) + q'''$$

Non-linear.

For heterogeneous anisotropic solids, k becomes a tensor.

$$\rho c \frac{dT}{dt} = \frac{\partial}{\partial x}\left(k_{11}\frac{\partial T}{\partial x} + k_{12}\frac{\partial T}{\partial y} + k_{13}\frac{\partial T}{\partial z}\right) + \frac{\partial}{\partial y}\left(k_{21}\frac{\partial T}{\partial x} + k_{22}\frac{\partial T}{\partial y} + k_{23}\frac{\partial T}{\partial z}\right)$$

$$+ \frac{\partial}{\partial z}\left(k_{31}\frac{\partial T}{\partial x} + k_{32}\frac{\partial T}{\partial y} + k_{33}\frac{\partial T}{\partial z}\right) + q'''$$

For homogeneous anisotropic solids (k independent of space) -

$$\rho c \frac{dT}{dt} = k_{11}\frac{\partial^2 T}{\partial x^2} + k_{22}\frac{\partial^2 T}{\partial y^2} + k_{33}\frac{\partial^2 T}{\partial z^2} + (k_{12}+k_{21})\frac{\partial^2 T}{\partial x \partial y} +$$

$$(k_{23}+k_{32})\frac{\partial^2 T}{\partial y \partial z} + (k_{13}+k_{31})\frac{\partial^2 T}{\partial x \partial z} + q'''$$

Heat conduction eq is a PDE. They are classified as :

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G(x,y)$$

Hyperbolic if $B^2 - 4AC > 0$

Parabolic if $B^2 = 4AC$

Elliptic if $B^2 - 4AC < 0$

Unsteady state Heat conduction (1D, 2D, 3D) → are parabolic

Steady state " " (2D, 3D) → are elliptic

What about hyperbolic?

classmate

Non-Fourier heat conduction : Finite speed of heat propagation.

We know that for homogeneous isotropic medium, the Fourier law of heat conduction is $q(r,t) = -k\nabla T(r,t)$.

Also,

$$\rho c \frac{\partial T}{\partial t} + \nabla \cdot q'' - q''' = 0 \quad \text{from energy balance.}$$

$$\therefore \left(\frac{\rho c}{k}\right)\frac{\partial T}{\partial t} = \nabla^2 T + \frac{q'''}{k} \longrightarrow \text{Unsteady state. Parabolic H.C. equation.}$$

But Fourier's law doesn't ~~for~~ consider delayed propagation. It assumes infinite speed of propagation. It assumes q and ∇T appear at the same time instant t. Fourier's law is an early emporical law and its unacceptable in some cases.

Cattaneo (1958) and vernotte (1958, 1961) proposed new constitutive equations to tackle this –

$$\boxed{\text{CV equation: } q(r,t) + \tau_0 \frac{\partial q(r,t)}{\partial t} = -k\nabla T(r,t)}$$

known as CV constitutive relation

$\tau_0 > 0$ is a material property called relaxation time. It takes care of delay in heat propagation.

Now, let's derive heat conduction eq. for Non-Fourier heat conduction.

Substituting $q(r,t) + \tau_0 \frac{\partial q}{\partial t} = -k\nabla T$;

$$\rho c \frac{\partial T}{\partial t} + \nabla \cdot q'' = q''' \quad \text{becomes}$$

Try yourself!

$$\boxed{\frac{1}{\alpha}\frac{\partial T}{\partial t} + \frac{\tau_0}{\alpha}\frac{\partial^2 T}{\partial t^2} = \nabla^2 T + \frac{1}{k}\left(q''' + \tau_0 \frac{\partial q'''}{\partial t}\right)} \longrightarrow (1)$$

This equation is of hyperbolic type. This predicts a finite speed for heat propagation $V_{cv} = \sqrt{\frac{k}{\rho c \tau_0}} = \sqrt{\frac{\alpha}{\tau_0}}$

Note: To derive this speed; put $q''' = 0$ in eq (1).

Std. wave eq. is given by $\frac{\partial^2 q}{\partial t^2} = c^2\left(\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 q}{\partial z^2}\right)$

c → speed of wave.

CV constitutive relation is actually a 1st order approximation of a more general constitutive relation called 'Single-Phase-lagging model' given by-

$$\boxed{q(r, t + \tau_0) = -k\nabla T(r,t).}$$

There is also something called a 'double-phase-lagging model'.

So, Fourier's law considers both time and space as macroscopic.

Single-phase-lagging model considers only time as microscopic

Dual-phase-lagging model considers both time & space as microscopic.

Also, effects of micro-structural interactions (in microscopic 'space') are not considered in single-phase-lagging model.

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1st order approximation

$$q(r, t+\tau_0) \simeq q(r,t) + \tau_0 \frac{\partial q(r,t)}{\partial t} = -k \nabla T(r,t)$$

This suggests that the temp. gradient established at (r,t) gives rise to a heat flux vector at r at a later time $t+\tau_0$. So, there is a finite lag τ_0 for heat flux to build up as a result of temp gradient.

Higher the τ_0; larger the deviation from Fourier's law.
For most solids, τ_0 ranges from 10^{-10} to 10^{-14} s.
gases, τ_0 ranges from 10^{-8} to 10^{-10} s.
For some biological materials and non-homogeneous structures, it can be upto 10^{2} s which is very significant. In such cases, thermal relaxation effects can't be ignored.

Dual phase lagging model: DPLM

$$\boxed{q(r, t+\tau_0) = -k \nabla T(r, t+\tau_T)}$$

Temperature gradient at 'r' at time '$t+\tau_T$' leads to a heat flux at 'r' at time '$t+\tau_q$'.

τ_T is caused by the micro-structural interactions (small scale heat transport mechanisms occuring in micro-scale) and is called the phase lag of heat flux.

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→ Reduces to Fouriers law when $\tau_0 = \tau_T$

Taylor series and 1st order approx. of D-P-L-M

$$\boxed{q(r,t) + \tau_0 \frac{\partial q(r,t)}{\partial t} = -k\left(\nabla T(r,t) + \tau_T \frac{\partial}{\partial t}\left(\nabla T(r,t)\right)\right)}$$

Called Dual-phase-lagging-constitutive relation.

Substituting dual-phase lagging model in heat conduction eq:

$$\rho c \frac{\partial T}{\partial t} + \nabla . q'' - q''' = 0$$

(Try deriving yourself!)

$$\Rightarrow \boxed{\frac{1}{\alpha}\frac{\partial T}{\partial t} + \frac{\tau_0}{\alpha}\frac{\partial^2 T}{\partial t^2} = \nabla^2 T + \tau_T \frac{\partial}{\partial t}(\nabla^2 T) + \frac{1}{k}\left(q''' + \tau_0 \frac{\partial q'''}{\partial t}\right)}$$

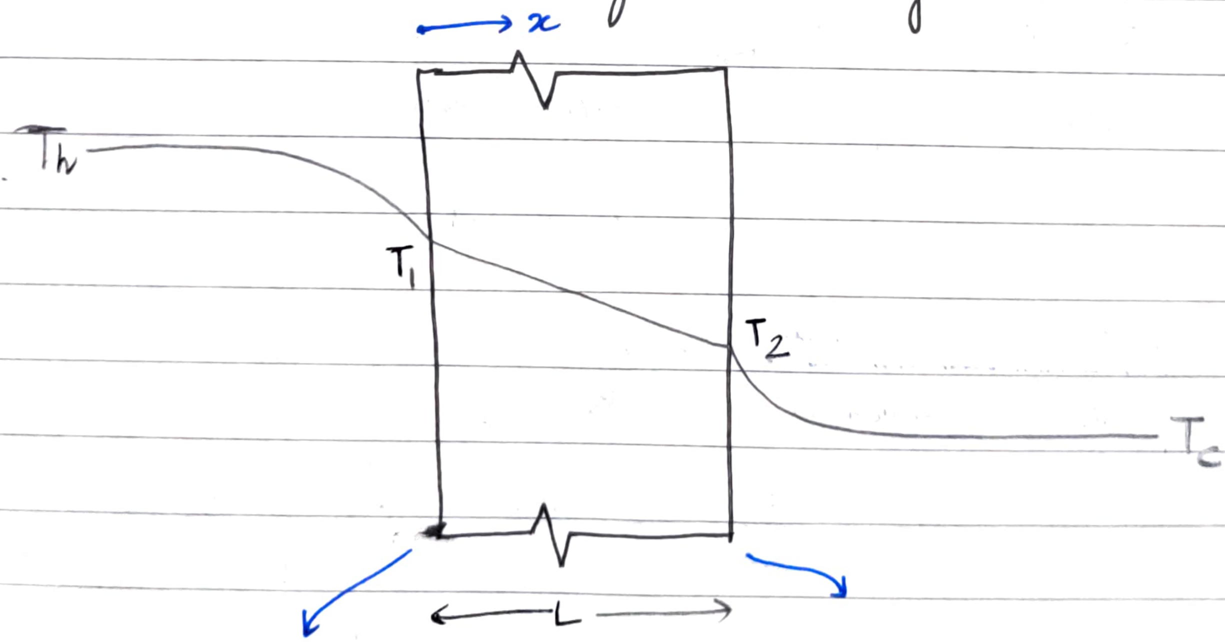
→ This eq. is parabolic when $\tau_0 < \tau_T$

This eq is hyperbolic when $\tau_0 > \tau_T$

This eq. can be approximated to represent wave-like thermal signals much like in CV constitutive relations.

predicts a non-wave like heat conduction

Conduction and convection at system boundary.

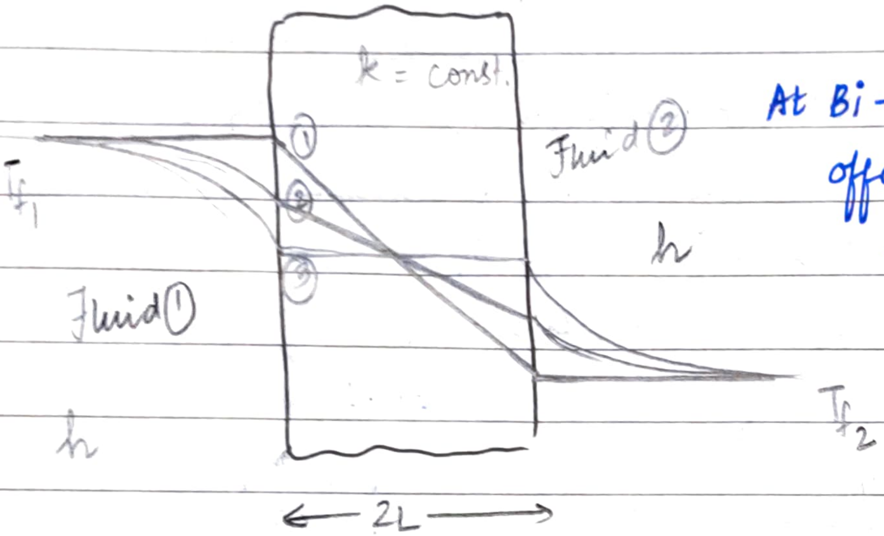


$$h_h(T_h - T|_{x=0}) = -k \frac{\partial T}{\partial x}\bigg|_{x=0}$$

$$h_c(T|_{x=L} - T_c) = -k \frac{\partial T}{\partial x}\bigg|_{x=L}$$

Temperature distribution in a plane : Biot Number → 2 Limiting cases.

Consider both fluids have same 'h' for simplicity.



At $Bi \to \infty$, the surface doesn't offer any resistance and there is a temp profile within the solid → gives case ①.

Consider the other case, $Bi \to 0$, there is no temp drop within the body. Leads to case ③. <u>How to explain these cases</u>?

$$q = \frac{T_{f_1} - T_{f_2}}{\sum R_t} \quad ; \quad \sum R_t = \frac{2}{hA} + \frac{2L}{kA}$$

$\frac{2L}{kA}$ → within the body

$\frac{2}{hA}$ → $1/hA$ at each of the 2 surfaces.

$$Bi = \frac{\text{internal resistance}}{\text{surface resistance}} = \frac{hL}{k}.$$

① At $Bi \to \infty$, $\sum R_t \approx \frac{2L}{kA}$ as $hL >> k$ → No resistance at surface

② At $Bi \to 0$; $\sum R_t \approx \frac{2}{hA}$ as $k >> hL$ → No resistance inside the body.

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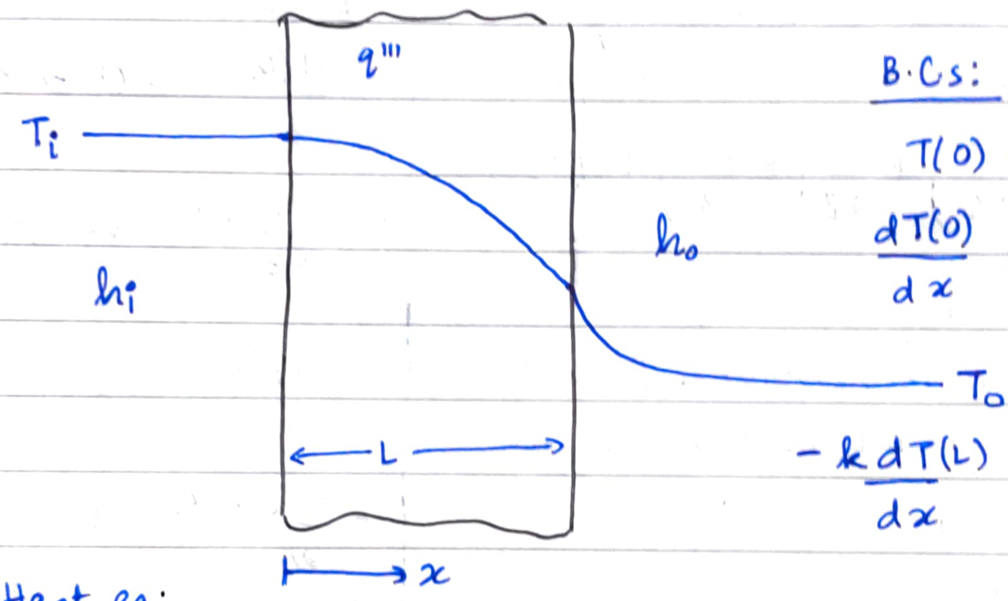
Special Case: Eliminate heat loss from one side ($x=0$)

In a slab, to eliminate loss of heat from one side ($\left.\frac{dT}{dx}\right|_{x=0}=0$), we used to insulate that surface. Another way to achieve the same result is by placing a heat source in the slab at $x=0$ such that it counter balances any heat transfer (loss/gain) occuring from the surface at $x=0$. Let's say that the heat generated by that source is q'''. Now the temp. profile must look like -



B.Cs:

$T(0) = T_i$

$\frac{dT(0)}{dx} = 0.$

$-k\frac{dT(L)}{dx} = h_o(T(L) - T_o)$

Heat eq:

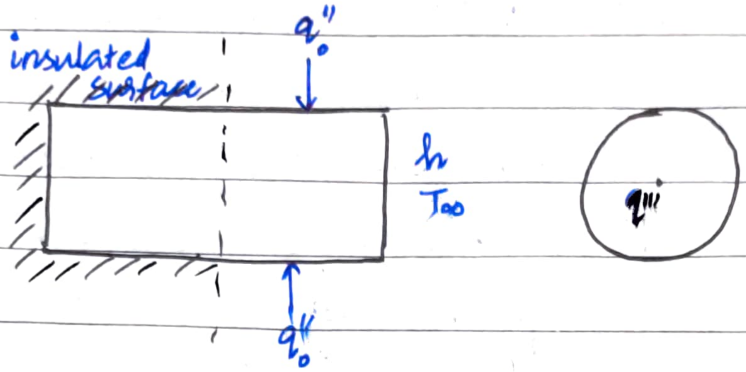
$$\frac{d^2T}{dx^2} + \frac{q'''}{k} = 0.$$

By solving this heat eq. with the three B.C.s given above, q''' can be found, that is required to maintain zero heat transfer at $x=0$.

Do it yourself!

Problem Formulation : 2D

Consider a cylinder of length L and radius r_0.



In cylindrical coordinates, heat eq:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + q''' = 0$$

$\downarrow$

Convert to non-dimensional form to simplify calculations & terms.

$\downarrow$

Define: Reference length(s)
Reference temperature
Reference temperature difference
Time scale (for unsteady state).

B.C.s:

$$\frac{\partial T(0,r)}{\partial z} = 0 \rightarrow \text{insulated}$$

$$-k\frac{\partial T(L,r)}{\partial z} = h\left[T(r,L) - T_\infty\right]$$

↳ convection

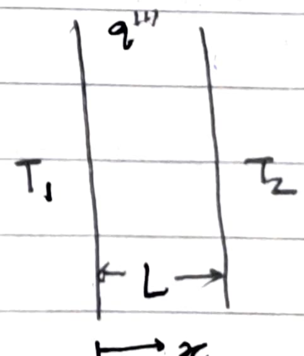
$$\frac{\partial T(0,z)}{\partial r} = 0 \longrightarrow \text{symmetry}$$

$$k\frac{\partial T(r_0,z)}{\partial r} = f(z)\begin{cases} 0 & z \le \frac{L}{2} \\ q_0'' & z > \frac{L}{2} \end{cases}$$

constant flux.

Example of non-dimensionalisation:

(1) Rectangular slab: $\dfrac{\partial^2 T}{\partial x^2} = -\dfrac{q'''}{k}$



$$\bar{T} = \frac{T - T_1}{T_2 - T_1} \quad ; \quad \bar{x} = \frac{x}{L}$$

$$\therefore \frac{d^2\bar{T}}{d\bar{x}^2} = \frac{-q'''L^2}{k(T_2 - T_1)} = -S$$

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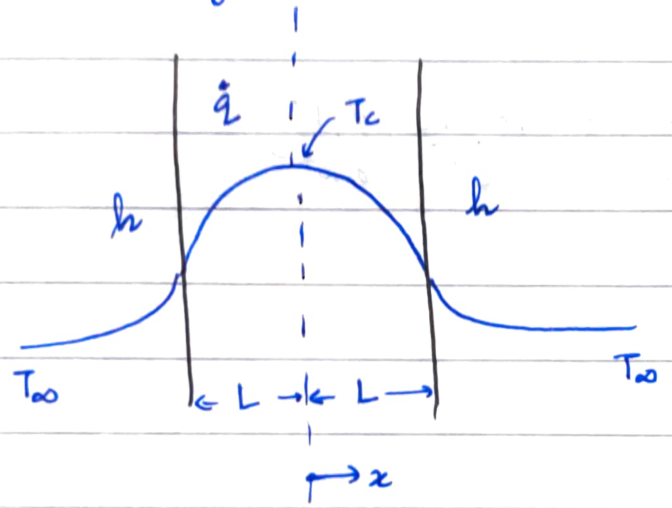
$$\therefore \quad \bar{T} = -\frac{S\bar{x}^2}{2} + C_1\bar{x} + C_2$$

~~At~~ B.C.s; $\bar{T}(\bar{x}=0) = 0, \quad \bar{T}(\bar{x}=1) = 1$

$$\therefore \quad \bar{T} = \bar{x} + \frac{S\bar{x}}{2}(1-\bar{x}).$$

② Rectangular slab with different B.C.s –



Ref. length → L

Ref. temp. → T_∞

Ref. temp. difference → ?

$\bar{x} = \frac{x}{L}$, $\bar{T} = \frac{(T-T_\infty)k}{q''' L^2}$ S.T.

$$\frac{d^2\bar{T}}{d\bar{x}^2} = -1$$

B.C.s: $\left.\frac{d\bar{T}}{d\bar{x}}\right|_{\bar{x}=0} = 0 \; ; \quad \left.\frac{d\bar{T}}{d\bar{x}}\right|_{x=1} = -Bi\,\bar{T}(\bar{x}=0)$

Now it can be easily solved.

③ 1D unsteady state conduction. –

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

B.Cs: $T(x, t=0) = T_0$

$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$ and $-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h\left(T|_{x=L} - T_\infty\right)$

Define: $x^* = \frac{x}{L}$; $T^* = \frac{T - T_\infty}{T_0 - T_\infty}$

$t^* = \frac{\alpha t}{L^2}$ → from governing eq.

Use chain rule:

$$\frac{\partial T}{\partial x} = \frac{(T_0 - T_\infty)}{L}\left(\frac{\partial T^*}{\partial x^*}\right)$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_0 - T_\infty}{L} \cdot \frac{\partial}{\partial x}\left(\frac{\partial T^*}{\partial x^*}\right) = \frac{T_0 - T_\infty}{L^2} \cdot \frac{\partial^2 T^*}{\partial x^{*2}}$$

$$\frac{\partial T}{\partial t} = \frac{\alpha (T_0 - T_\infty)}{L^2} \frac{\partial T^*}{\partial t^*}$$

B.C.s :- $T^*(x^*_0, t^* = 0) = 1$

$$\left.\frac{\partial T^*}{\partial x^*}\right|_{x^* = 0} = 0$$

where $Bi = \frac{hL}{k}$,

$$-\left.\frac{\partial T^*}{\partial x^*}\right|_{x^* = 1} = Bi \left. T^* \right|_{x^* = 1}$$

(4) Non-dimensional form, 3D Heat eq.

Medium of temp T_∞, body initial temp T_0, rectangular geometry

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

B.C.s : $-k_i \frac{\partial T}{\partial h_i} = h_i (T_i - T_\infty)$ where $i = 1, 2, \ldots$ are no. of bounding surfaces. (S_i)

$T = T_0$ at $t = 0$.

Length (dimensionless):

$$\xi = \frac{x}{L} \; ; \quad \eta = \frac{y}{L} \; ; \quad \psi = \frac{z}{L}$$

$\frac{\partial}{\partial N} \rightarrow$ diff. along outward normal in dimensionless co-ordinate system.

Temperature:

$$\theta = \frac{T - T_\infty}{T_0 - T_\infty}$$

$$\therefore \left(\frac{T_0 - T_\infty}{L^2}\right)\left(\frac{\partial^2 \theta}{\partial \xi^2} + \frac{\partial^2 \theta}{\partial \eta^2} + \frac{\partial^2 \theta}{\partial \psi^2}\right) + \frac{\dot{q}}{k} = \frac{(T_0 - T_\infty)}{\alpha} \frac{\partial \theta}{\partial t}$$

$$\Rightarrow \frac{\partial^2 \theta}{\partial \xi^2} + \frac{\partial^2 \theta}{\partial \eta^2} + \frac{\partial^2 \theta}{\partial \psi^2} + \frac{\dot{q} L^2}{(T_0 - T_\infty) k} = \frac{L^2}{\alpha} \frac{\partial \theta}{\partial t}$$

ϕ : dimensionless heat gen.

τ : dimensionless time

$$\Rightarrow \frac{\partial^2 \theta}{\partial \xi^2} + \frac{\partial^2 \theta}{\partial \eta^2} + \frac{\partial^2 \theta}{\partial \psi^2} + \phi = \frac{\partial \theta}{\partial \tau}$$

where $\tau = \frac{\alpha t}{L^2}$

$\phi = \frac{\dot{q} L^2}{(T_0 - T_\infty) k}$.

B.C.s: $-\frac{k_i}{L} \frac{\partial \theta}{\partial N_i} = h_i \theta$ on S_i at $t > 0$

$\theta = 1$ at $t = 0$

$$\frac{\partial \theta}{\partial N_i} + Bi \, \theta = 0 \text{ on } S_i, \tau > 0$$

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