MONOJEET SIR PART

*) TRANSPORT EQUATION ⇒

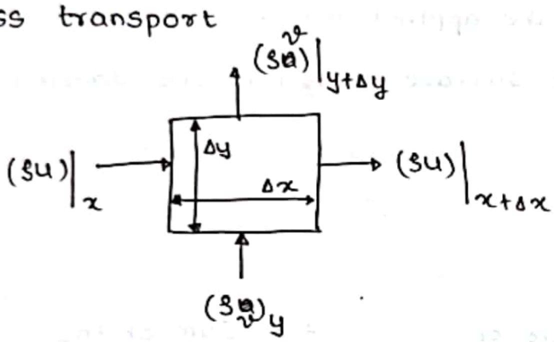
$$\begin{bmatrix}\text{change in} \\ \text{property in } \Delta t\end{bmatrix}_1 = \text{Rate of property}\left[\text{In} - \text{Out}\right]_2 + \left[\text{generation}\right]_3$$

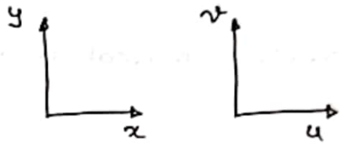
Properties ⇒ Density $\rho = \frac{M}{\Delta V} \Rightarrow \lim_{\Delta V \to 0} \frac{M}{\Delta V} = \rho$

Momentum (ρu)

Energy $e\left[\frac{\text{Energy}}{\text{mass}}\right] \rightarrow \rho \times e\left[\frac{\text{Energy}}{\text{Volm.}}\right]$

*) Mass transport





Mass In − Mass Out = Rate of accumulation

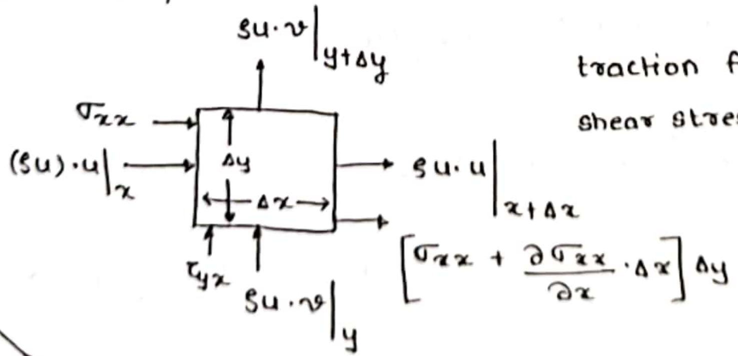
$$\rho u \cdot \Delta y + \rho v \cdot \Delta x - \left\{\rho u + \frac{\partial}{\partial x}\rho u \cdot \Delta x\right\}\Delta y - \left\{\rho v + \frac{\partial}{\partial y}\rho v \cdot \Delta y\right\}\Delta x = \frac{\partial \rho}{\partial t}\cdot \Delta x \cdot \Delta y$$

$$\frac{\partial \rho}{\partial t} = -\left\{\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y}\right\}$$

$$\frac{\partial \rho}{\partial t} = -\left\{\rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial y} + v\frac{\partial \rho}{\partial y}\right\} \Rightarrow \frac{\partial \rho}{\partial t} + \rho\left\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right\} = -\left\{u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y}\right\}$$

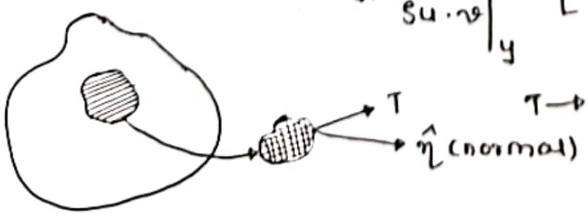
$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\left\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right\} = 0 \Rightarrow \frac{D\rho}{Dt} + \rho\, \nabla\cdot\vec{V} = 0$$

*) Mommentum transport ⇒



traction force τ_{ij}

shear stress $= \frac{\text{Tangential force}}{\text{Area}}$



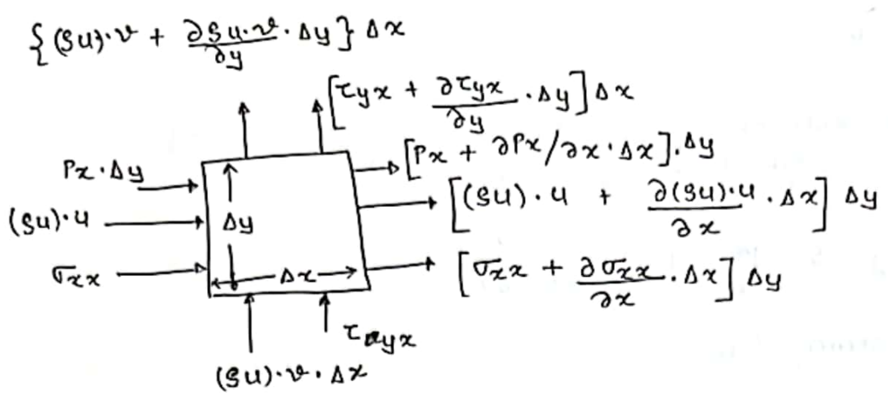
T → force

T_i^{η} → Traction force

$\approx T_{\eta i} \approx \tau_{ji}$

Doubt
what is
traction
force
f all?

Linear Mommentum Balance (2D) in Cartesian Co-ordinate



$\tau_{yx} \rightarrow$ viscous Flux of momentum in y direction

$\tau_{ij} \rightarrow$ i is the direction normal to the surface along which the force is acting & j is the direction of the applied force.

$T_i^n \rightarrow$ n is the direction normal to the surface & i is the direction of applied force.

General Expression for linear mommentum balance $\Rightarrow$

Rate of momentum accumulation = Rate of momentum In − Rate of momentum Out + Sum of the Forces acting on the system

Momentum transfer to or from the system takes place by 2 mechanism

i) Convection

ii) Molecular transfer / Diffusion

$$\frac{\partial \rho u}{\partial t}\cdot \Delta x \cdot \Delta y = (\rho u)\cdot u \cdot \Delta y + \sigma_{xx}\cdot \Delta y - \left[(\rho u)\cdot u + \frac{\partial (\rho u)\cdot u}{\partial x}\cdot \Delta x\right]\cdot \Delta y - \left[\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\cdot \Delta x\right]\Delta y$$

$$(\rho u)\cdot v \cdot \Delta x + \tau_{yx}\cdot \Delta x - \left[(\rho u)\cdot v + \frac{\partial (\rho u) v}{\partial y}\cdot \Delta y\right]\Delta x - \left[\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\cdot \Delta y\right]\Delta x$$

$$+ P_x \cdot \Delta y - \left[P_x + \frac{\partial P_x}{\partial x}\cdot \Delta x\right]\Delta y$$

$$X \cdot \Delta x \cdot \Delta y$$

where X be the Body force per unit volume.

$$\frac{\partial \rho u}{\partial t} = -\left[\frac{\partial (\rho u)\cdot u}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial (\rho u)\cdot v}{\partial y} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial P_x}{\partial x}\right] + X$$

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial \rho u \cdot u}{\partial x} + \frac{\partial \rho u \cdot v}{\partial y} = -\left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial P_x}{\partial x}\right]$$

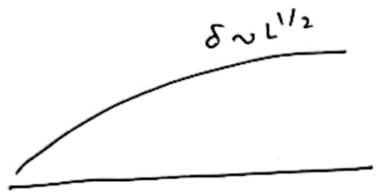
$$\rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial t} + u\left[u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x}\right] + \rho u\frac{\partial u}{\partial x} + \rho u\frac{\partial v}{\partial y} + v\left[u\frac{\partial \rho}{\partial y} + \rho\frac{\partial u}{\partial y}\right]$$

In Boundary Layer we got, $\frac{\delta}{L} \sim Re^{-1/2}$

$$Re = \frac{\text{Inertia force}}{\text{viscous force}}$$

Inside Boundary Layer change in Re will not change the Inertial & viscous force

Inside BL Inertial force = viscous force always. also inside BL Re is just a geometric quantity



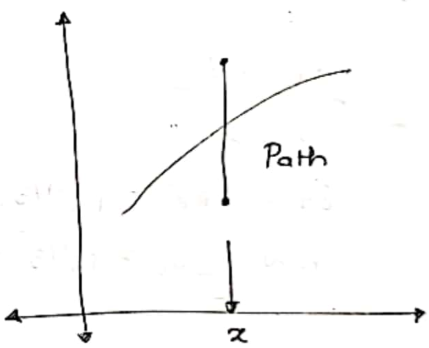
This is limitation of BL theory as we cannot comment at $x \to 0$

* Integral solution to Boundary Layer — In Exam – practice it

why? – Scaling analysis give approximate soln over same length, we cannot get particular value at some point

Momentum Balance Eqn ⇒

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho}\frac{\partial P_\infty}{\partial x}$$



Take along path v at some const 'x'

$$\frac{d}{dx}\int_0^Y u(u_\infty - u)\,dy = \frac{1}{\rho} Y \frac{dP_\infty}{dx} + \frac{du_\infty}{dx}\int_0^Y u\,dy + \nu\left(\frac{\partial u}{\partial y}\right)_0 \quad \text{—— (1)}$$

$$\frac{d}{dx}\int_0^Y u(T_\infty - T)\,dy = \frac{dT_\infty}{dx}\int_0^Y u\,dY + \alpha\left(\frac{\partial T}{\partial y}\right)_0 \quad \text{—— (2)}$$

$$\int \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho}\frac{dP_\infty}{dx}$$

Energy Balance Eqn $\Rightarrow$

$$\Rightarrow \quad u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$

$$\Rightarrow \quad u \cdot \frac{\Delta T}{L} \sim \alpha \frac{\Delta T}{\delta_T^2} \Rightarrow \frac{u}{L} \sim \frac{\alpha}{\delta_T^2}$$

$$\Rightarrow \quad \frac{u_\infty}{L}\frac{\delta_T}{\delta} \sim \frac{\alpha}{\delta_T^2}$$

$$\Rightarrow \quad \frac{\delta_T^3}{L^3} \sim \frac{\delta}{L} \times \frac{\alpha}{u_\infty L}$$

$$\Rightarrow \quad \frac{\delta_T^3}{L^3} \sim Re^{-1/2}\; Pe^{-1} \qquad Pe = Re \times Pr$$

$$\Rightarrow \quad \frac{\delta_T^3}{L^3} \sim Re^{-3/2}\; Pr^{-1} \Rightarrow \frac{\delta_T}{L} \sim Re^{-1/2}\; Pr^{-1/3}$$

$$\Rightarrow \quad h \sim \frac{k}{\delta_T} \Rightarrow h \sim \frac{k}{L} Re^{1/2}\; Pr^{1/3}$$

$$\Rightarrow \quad \frac{\delta_T}{\delta} << 1 \longrightarrow Pr^{-1/3} << 1 \longrightarrow Pr >> 1$$

$\delta_T << \delta$	$\delta_T >> \delta$
$Pr >> 1$	$Pr << 1$
$\delta_T \sim L Re^{-1/2} . Pr^{-1/3}$	$\delta_T \sim L Re^{-1/2}\; Pr^{-1/2}$
$h \sim \frac{k}{L} Re^{1/2}\; Pr^{1/3}$	$h \sim \frac{k}{L} Re^{1/2}\; Pr^{1/2}$

$$\frac{\partial u}{\partial t} \sim \nu \left(\frac{\partial^2 u}{\partial y^2}\right) \qquad \frac{u_\infty^2}{L} \sim \frac{u_\infty}{t_L}$$

$$\frac{u_\infty}{t_\delta} \sim \nu \frac{u_\infty}{\delta^2} \qquad t_L \sim \frac{L}{u_\infty}$$

$$t_\infty = t_\delta \sim \frac{\delta^2}{\nu}$$

$$t_\delta \sim t_L$$

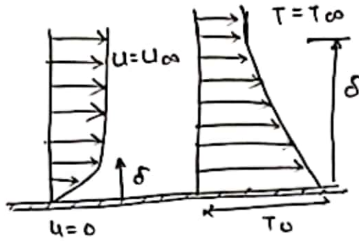
$$\frac{\delta^2}{\nu} \sim \frac{L}{u_\infty} \Rightarrow \delta^2 \sim \frac{L\nu}{u_\infty} \Rightarrow \delta \sim \sqrt{\frac{\nu L}{u_\infty}}$$

$$\frac{\delta}{L} \sim \frac{\nu}{L u_\infty} \Rightarrow \frac{\delta}{L} \sim Re^{-1/2}$$

↳ This is same result we get previously.

↳ For heat transfer time scale, compare convection & heat generation

Case - I ⇒ $\delta << \delta_T$



$u \sim u_\infty$

From continuity eqn ⇒ $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ ⇒ $\frac{u_\infty}{L} + \frac{v}{\delta_T} = 0$

$\Rightarrow v \sim u_\infty \frac{\delta_T}{L}$

$\Rightarrow u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$

$\Rightarrow u_\infty \frac{\Delta T}{L} \sim v\frac{\Delta T}{\delta_T} \sim \alpha \frac{\Delta T}{\delta_T^2}$

$\Rightarrow u_\infty \frac{\Delta T}{\delta_T} \cdot \frac{\delta_T}{L} \sim \alpha \frac{\Delta T}{\delta_T^2}$

$\Rightarrow \frac{u_\infty}{L} \sim \alpha \frac{\Delta T}{\delta_T^2}$

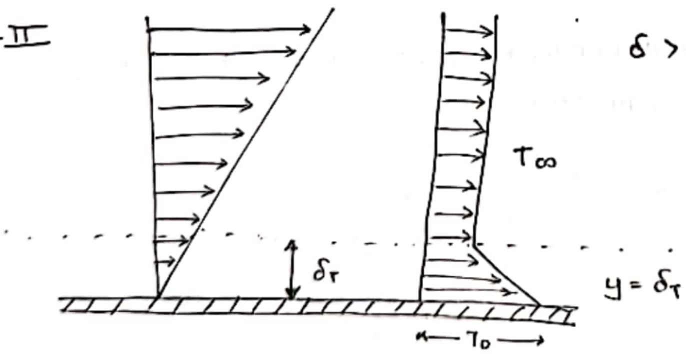
$\Rightarrow \frac{\delta_T^2}{L^2} \sim \frac{\alpha}{u_\infty L} \sim \frac{1}{Pe}$ $\qquad Pe = Re \times Pr$

$\Rightarrow \frac{\delta_T}{L} \sim Re^{-1/2} \times Pr^{-1/2}$ $\qquad \frac{\delta}{L} \sim Re^{-1/2}$

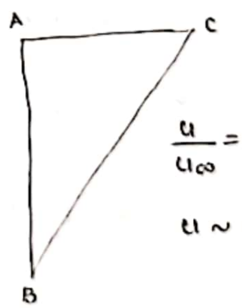
$\Rightarrow \frac{\delta}{\delta_T} \sim Pr^{+1/2} << 1$ ⟶ It hold true for liquid metals

$\Rightarrow h \sim \frac{k}{\delta_T} \sim \frac{k}{L} \times Re^{1/2} Pr^{1/2}$ $\qquad \boxed{\therefore Nu \sim Re^{1/2} Pr^{1/2}}$

Case-II



$\delta >> \delta_T$



$\frac{u}{u_\infty} = \frac{\delta_T}{\delta}$

$u \sim u_\infty \frac{\delta_T}{\delta}$

$\delta \longrightarrow u_\infty$

$\delta_T \longrightarrow u$

$u \sim u_\infty \frac{\delta_T}{\delta}$

Continuity Eqn ⇒ at $y = \delta_T$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Scaling Analysis ⇒

$\frac{u}{L} \sim \frac{v}{\delta_T}$ ⇒ $v \sim u \cdot \frac{\delta_T}{L}$

CamScanner

From x-momentum balance scaling ⇒

$$\frac{1}{\rho}\frac{\partial P}{\partial x} \sim \nu \frac{u_\infty}{\delta^2}$$

From y-momentum balance scaling ⇒

$$\frac{1}{\rho}\frac{\partial P}{\partial y} \sim$$

From scaling of $\frac{dy}{dx} \sim \frac{\delta}{L}$

On substituting, we get ⇒ $$\frac{\frac{dP}{dx}}{\frac{\partial P}{\partial x}} = 1 + \frac{\nu \cdot \frac{v}{\delta^2} \cdot \frac{\delta}{L}}{\nu \cdot \frac{u_\infty}{\delta^2}} = 1 + \frac{v}{u_\infty} \cdot \frac{\delta}{L} = 1 + \left(\frac{\delta}{L}\right)^2$$

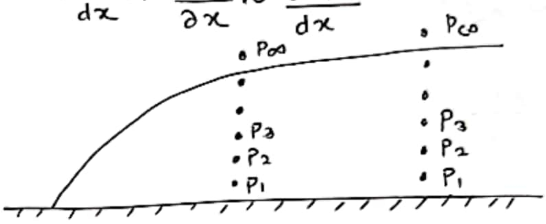
$$\left(\frac{\delta}{L}\right)^2 \approx 0$$

From continuity eqn ⇒ $\frac{v}{u_\infty} \sim \frac{\delta}{L}$

$$\Rightarrow \frac{dP}{dx} \Big/ \frac{\partial P}{\partial x} = 1$$

⇒ P does not vary with 'y' ⇒ ∴ $\frac{dP}{dx} \sim \frac{\partial P}{\partial x} \sim \frac{dP_\infty}{dx}$

⇒ Inside Boundary Layer ⇒



⇒ Final set of eqns → $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\cdot\frac{\partial P_\infty}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$

Scaling analysis of interested variable

$$\tau_0 \sim \mu \cdot \frac{u_\infty}{\delta} \qquad q'' = h\Delta T \Rightarrow k\frac{\Delta T}{\Delta x} \sim h\Delta T$$

$$h \sim \frac{k}{\Delta x} \sim \frac{k}{\delta_T}$$

⇒ We assumed const temperature independent properties → Scale = ?

Using reduced x-momentum equation

⇒ Convection ~ diffusion ⇒ $$\frac{u_\infty^2}{L} \sim \nu\frac{u_\infty}{\delta^2} \to \delta^2 \sim \frac{\nu L}{u_\infty} \to \frac{\delta^2}{L^2} \sim \frac{\nu}{u_\infty L}$$

$$\frac{\delta^2}{L^2} \sim \frac{1}{\frac{u_\infty L}{\nu}} \sim \frac{1}{Re^{1/2}} \sim \boxed{Re^{-1/2} = \frac{\delta}{L}}$$

$$\tau \sim \frac{\mu u_\infty}{\delta} \sim \frac{\mu u_\infty}{L\,Re^{-1/2}} \sim \frac{\mu u_\infty}{L} Re^{1/2}$$

$$\Rightarrow \tau \sim \frac{u_\infty^2 \rho}{\frac{L u_\infty \rho}{\mu}} Re^{1/2} \sim \frac{\rho u_\infty^2 Re^{1/2}}{Re} \sim \rho u_\infty^2 Re^{-1/2}$$

⇒ friction factor ⇒ $$f_f = \frac{\tau}{\frac{\rho u_\infty^2}{2}} \sim Re^{-1/2}$$

Scale

$x \sim L \qquad y \sim \delta \qquad u \sim u_\infty$

$\delta_t \rightarrow$ Thermal Boundary Layer upto which the presence of heat source is significant

$\delta \rightarrow$ Hydrodynamic B.L upto which presence of hydrodynamic plate is significant

Applying scaling analysis to x-momentum eqn

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u_\infty \cdot \frac{u_\infty}{L} \sim v \cdot \frac{u_\infty}{L} \sim -\frac{1}{\rho}\frac{P_\infty}{L} \sim \nu\frac{U_\infty}{L^2} \; q \sim \nu\frac{u_\infty}{\delta^2}$$

$\delta << L$

$$\Rightarrow \frac{\nu u_\infty}{L^2} << \frac{\nu u_\infty}{\delta^2}$$

Applying scaling analysis to continiuity eqn

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{u_\infty}{L} \sim \frac{v}{\delta} \Rightarrow v \sim u_\infty\frac{\delta}{L}$$

$$v \cdot \frac{u_\infty}{\delta} \approx \delta \cdot \frac{u_\infty^2}{L}$$

So our eqn can be reduced to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

Similarly energy equation is reduced to

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

The pressure is given by

$$dP = \frac{\partial P}{\partial x}\cdot dx + \frac{\partial P}{\partial y}\cdot dy$$

$$\frac{dP}{dx} = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y}\cdot\frac{dy}{dx} \Rightarrow \frac{\frac{dP}{dx}}{\frac{\partial P}{\partial x}} = 1 + \frac{\frac{\partial P}{\partial y}\cdot\frac{dy}{dx}}{\frac{\partial P}{\partial x}} \approx 1$$

Br - Brinkmann No $= \dfrac{\mu v_b^2}{k \cdot \Delta T} = \dfrac{\mu \cdot v_b \cdot \dfrac{v_b}{\Delta y}}{k \cdot \dfrac{\Delta T}{\Delta y}}$

$= \dfrac{\text{Transport of heat by viscous dissipation}}{\text{Transport of heat by conduction}}$

Boundary Layer ⇒ Equation

$$\rho C_p \frac{\partial T}{\partial t} = k \cdot \frac{\partial^2 T}{\partial y^2}$$

$c = a + b$ $\quad O(c) \sim O(a) \sim O(b)$

$O(a) >> O(b)$

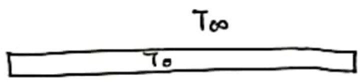
$O(c) \sim O(a)$

$c = a \times b$ $\quad O(c) \sim O(a) \times O(b)$

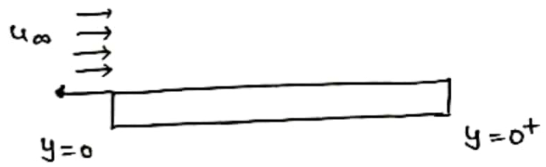
$c = \dfrac{a}{b}$ $\quad \dfrac{O(a)}{O(b)}$

FUNDAMENTAL PROBLEM IN CONVECTIVE HEAT TRANSFER

$F = \int \tau_0 \cdot dA \qquad \tau_0 = -\mu \cdot \left(\dfrac{\partial u}{\partial y}\right)_{y=0} \qquad u = f(y)$



$\dot{q} = \int_A q'' \cdot dA \qquad q'' = h \cdot \Delta T$



$q'' = -k\left(\dfrac{\partial T}{\partial y}\right)_{y=0} \qquad \dfrac{-k\left(\dfrac{\partial T}{\partial y}\right)_{y=0}}{\Delta T} = h \qquad \dfrac{\partial T}{\partial y}$

$T = f(y)$

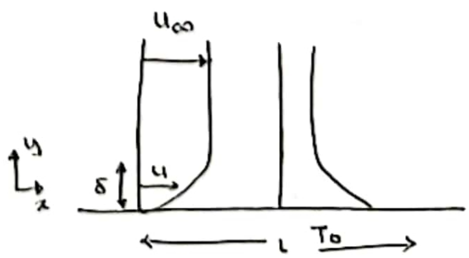
$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial u}{\partial y} = -\frac{1}{\rho} \cdot \frac{\partial P}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2}\right)$$

$$u \frac{\partial v}{\partial x} + v \cdot \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$u \frac{\partial T}{\partial x} + v \cdot \frac{\partial T}{\partial y} = \cancel{\rho c_p \partial k} \; \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

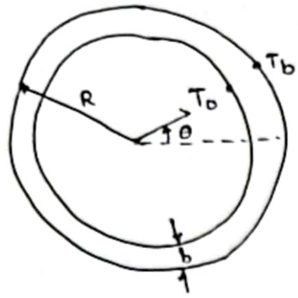
Boundary Condition ⇒ $y = 0$ $\quad u = 0$, $v = 0$, $T = T_0$

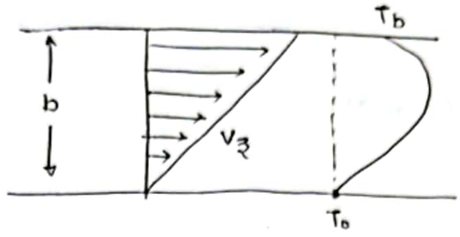


$y \to \infty$ $\quad u = u_\infty$, $v = 0$, $T = T_\infty$

Soln of entire domain is different from the soln close to the wall
— Nusselt

Heat Conduction With A Viscous Heat Source {Bird's Derrivation}





$$WL \cdot e\Big|_x - W \cdot L\, e\Big|_{x+\Delta x} = 0$$

$$\lim_{\Delta x \to 0} \frac{e\big|_{x+\Delta x} - e\big|_x}{\Delta x} = 0$$

$$\frac{de_x}{dx} = 0 \Rightarrow e_x = C_1$$

$$e_x = q_x + [\tau \cdot v]$$

$$= q_x + \tau_{xx} \cdot v_x + \tau_{xy} v_y + \tau_{xz} \cdot v_z$$

$$C_1 = -k\frac{dT}{dx} - \mu v_z \cdot \frac{\partial v_z}{\partial x}$$

$$v_z = v_b\left(\frac{x}{b}\right) \Rightarrow \frac{dv_z}{dx} = \frac{v_b}{b}$$

$$C_1 = k\frac{dT}{dx} - \mu\left(\frac{v_b}{b}\right)^2 \cdot x$$

$$C_1 \cdot x + C_2 = k \cdot T - \mu\left(\frac{v_b}{b}\right)^2 \cdot \frac{x^2}{2}$$

$$T = -\frac{\mu}{k} \cdot \left(\frac{v_b}{b}\right)^2 \cdot \frac{x^2}{2} - \frac{C_1}{k} x + C_2$$

BC-I at $x=0$ $T = T_0$

$$C_2 = T_0$$

BC-II at $x=b$ $T = T_b$

$$T_b = -\frac{\mu}{k} \cdot \left(\frac{v_b}{b}\right)^2 \cdot \frac{b^2}{2} - \frac{C_1}{k} b + T_0$$

$$T_b - T_0 + \frac{\mu}{k}\frac{v_b^2}{2} = -\frac{C_1}{k} \cdot b$$

$$\frac{k}{b}(T_0 - T_b) - \frac{\mu}{\cancel{k}} \cdot \frac{\cancel{k}}{b}\frac{v_b^2}{2} = C_1$$

$$\rho \frac{Dh}{Dt} = q''' + \mu\phi + \frac{DP}{Dt} + \nabla\cdot(k\nabla T)$$

$$dh = C_p \cdot dT$$

$$dH = TdS + VdP$$

In terms of specific property

$$dh = Tds + \frac{dP}{\rho}$$

$$s = f(T,P)$$

$$ds = \left(\frac{\partial s}{\partial T}\right)_P \cdot dT + \left(\frac{\partial s}{\partial P}\right)_T \cdot dP$$

Maxwell Relation $\Rightarrow \left(\frac{\partial s}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P$

we have (for specific properties)

$$\left(\frac{\partial s}{\partial P}\right)_T = -\left(\frac{\partial\, 1/\rho}{\partial T}\right)_P = \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_P = \frac{-\beta}{\rho}$$

$\beta \rightarrow$ Thermal Expansion coefficient $\beta = \frac{1}{v}\cdot\left(\frac{\partial v}{\partial T}\right)_P = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P$

$$\left(\frac{\partial s}{\partial T}\right)_P = \frac{C_p}{T}$$

$$ds = \frac{C_p}{T}\cdot dT - \frac{\beta}{\rho}\cdot dP$$

$$dh = T\left\{\frac{C_p}{T}\, dT - \frac{\beta}{\rho}\, dP\right\} + \frac{1}{\rho}\cdot dP$$

$$dh = C_p \cdot dT + \frac{1}{\rho}(1-\beta T)\, dP$$

$$\frac{\rho Dh}{Dt} = \rho C_p \frac{DT}{Dt} + (1-\beta T)\frac{DP}{Dt}$$

$$\rho C_p \frac{DT}{Dt} = q''' + \mu\phi + \beta T\frac{DP}{Dt} + \nabla\cdot(k\nabla T)$$

A) for ideal gas $\beta = \frac{1}{T}$

$$\rho C_p \cdot \frac{DT}{Dt} = q''' + \mu\phi + \frac{DP}{Dt} + \nabla\cdot(k\nabla T)$$

B) For incompressible liquid $\beta = 0$

$$\rho C_p \cdot \frac{DT}{Dt} = q''' + \mu\phi + \nabla\cdot(k\nabla T)$$

In most of the convection problems we have

1. Constant k
2. Zero heat generation
3. Negligiable viscous dissipation.

$$\rho C_p \frac{DT}{Dt} = k\cdot\nabla^2 T$$

$g = -\nabla\phi$ where ϕ is the potential energy

Hence,

$$\rho(\vec{V}\cdot g) = (-\rho\cdot\nabla\phi) = -\rho\frac{D\phi}{Dt} + \rho\frac{\partial\phi}{\partial t}$$

Also,

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \nabla\cdot(v\phi) \approx \frac{\partial\phi}{\partial t} + \vec{V}\cdot\nabla\phi$$

$$\vec{V}\cdot\nabla\phi = \frac{D\phi}{Dt} - \frac{\partial\phi}{\partial t}$$

If ϕ is time-independent $\frac{\partial\phi}{\partial t} = 0$

$$\rho(\vec{V}\cdot g) = -\rho\frac{D\phi}{Dt}$$

$$\rho\frac{D(e+\phi)}{Dt} = q''' - \nabla(P\vec{V}) + \mu\phi - \nabla\cdot q''$$

$e+\phi \longrightarrow$ Is the equation of total energy

$Q = \Delta u + \omega$

So, the energy supplied to the system is used to increase the internal energy only.

Considering internal energy only -

$$\rho\cdot\frac{D(e)}{Dt} = q''' - \nabla(P\vec{V}) + \mu\phi - \nabla\cdot q''$$

$e = \hat{u}$

$H = U + PV$

In terms of specific enthalpy $\Rightarrow$ $\frac{H}{m} = \frac{U}{m} + \frac{PV}{m}$

$h = e + \frac{P}{\rho}$

OR

$$\frac{Dh}{Dt} = \frac{De}{Dt} + \frac{1}{\rho}\cdot\frac{DP}{Dt} - \frac{P}{\rho^2}\cdot\frac{D\rho}{Dt} \Rightarrow \frac{De}{Dt} = \frac{Dh}{Dt} - \frac{1}{\rho}\frac{DP}{Dt} + \frac{P}{\rho^2}\cdot\frac{D\rho}{Dt}$$

Using fourier's law -

$q'' = -k\nabla T$

$$\rho\cdot\frac{Dh}{Dt} = \frac{DP}{Dt} - \frac{P}{\rho}\frac{D\rho}{Dt} + q''' - \nabla(P\vec{V}) + \mu\phi + \nabla k\cdot\nabla T$$

$$= q''' + \mu\phi + \frac{DP}{Dt} - \frac{P}{\rho}\left[\frac{D\rho}{Dt} + \rho(\nabla\vec{V})\right] + \nabla\cdot(k\cdot\nabla T)$$

Thus, net work done by the viscous force -

$$= \left[\sigma_{xx}\cdot u + \tau_{xy}\cdot v\right]\Big|_x \cdot \Delta y + \left[\sigma_{yy}\cdot v + \tau_{yx}\cdot u\right]\Big|_y \cdot \Delta x$$

$$- \left[\sigma_{xx}\cdot u + \tau_{xy}\cdot v\right]\Big|_{x+\Delta x} \cdot \Delta y + \left[\sigma_{yy}\cdot v + \tau_{yx}\cdot u\right]\Big|_{y+\Delta y} \cdot \Delta x$$

$$= \left\{ - \left[\frac{\partial \sigma_{xx}\cdot u}{\partial x} + \frac{\partial \tau_{xy}\cdot v}{\partial x} \right] - \left[\frac{\partial \sigma_{yy}\cdot v}{\partial y} + \frac{\partial \tau_{yx}\cdot u}{\partial y} \right] \right\} \Delta x \cdot \Delta y$$

$$= - \left[\frac{u\cdot \partial\sigma_{xx}}{\partial x} + \sigma_{xx}\cdot\frac{\partial u}{\partial x} + \tau_{xy}\frac{\partial v}{\partial x} + v\frac{\partial \tau_{xy}}{\partial x} + \frac{v\cdot\partial\sigma_{yy}}{\partial y} + \sigma_{yy}\cdot\frac{\partial v}{\partial y} + u\frac{\partial \tau_{yx}}{\partial y} + \tau_{yx}\frac{\partial u}{\partial y} \right] \Delta x\cdot\Delta y$$

writing the constitutive equations,

$$\tau_{yx} = \tau_{xy} = -\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\sigma_{xx} = -2\mu\frac{\partial u}{\partial x} + \frac{2}{3}\mu\,\nabla\cdot u$$

$$\sigma_{yy} = -2\mu\frac{\partial u}{\partial y} + \frac{2}{3}\mu\,\nabla\cdot v$$

For incompressible flow $\nabla V = 0$

$$\phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2 = \text{viscous dissipation function}$$

$$= -\left[\rho(g_x\cdot u + g_y\cdot v)\cdot\phi + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} - \mu\phi\right]\Delta x\cdot\Delta y$$

$$\frac{\partial(\rho e)}{\partial t} = -\left[\frac{\partial(\rho u e)}{\partial x} + \frac{\partial(\rho v e)}{\partial y}\right] - \left[\frac{\partial(q''_x)}{\partial x} + \frac{\partial q''_y}{\partial y}\right] + q'''$$

$$- \left[\rho(u\cdot g_x + v g_y)\cdot\phi + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} - \mu\phi\right]$$

$$\rho\frac{\partial e}{\partial t} + e\left[\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y}\right] + \rho u\frac{\partial e}{\partial x} + \rho v\cdot\frac{\partial e}{\partial y}$$

$$= -(\nabla\cdot q'') + q''' + \rho(\vec{v}\cdot g) - \nabla\cdot(\rho\vec{v}) + \mu\phi$$

$$\rho\frac{\partial e}{\partial t} + e\left[\frac{\partial\rho}{\partial t} + \nabla(\rho\vec{v})\right] = -(\nabla\cdot q'') + q''' + \rho(\vec{v}\cdot g) - \nabla(\rho\cdot\vec{v}) + \mu\phi$$

Putting $\frac{\partial\rho}{\partial t} + \nabla(\rho\vec{V}) = 0$

$$\rho\frac{\partial e}{\partial t} = -(\nabla q'') + q''' + \rho(\vec{V}\cdot g) - \nabla\cdot(\rho\vec{V}) + \mu\phi$$

→ Rate of internal heat generation ⇒

$$q'''\left[\frac{\text{heat}}{\text{volm} * \text{time}}\right] \Rightarrow q'''\cdot\Delta x\cdot\Delta y$$

↳ Net work done by the fluid element against its surrounding

It consist of two parts

i) work against the body forces (originate from volume)

ii) work against the surface forces.

a) work against pressure forces

b) work against viscous forces

Recall,

work = force × distance in the direction of the applied force

Rate of work = force × velocity in the direction of the applied force

i) Work against the body forces (originate from volume).

$$-\Delta x\,\Delta y\,\rho\,[v_x g_x + v_y g_y]$$

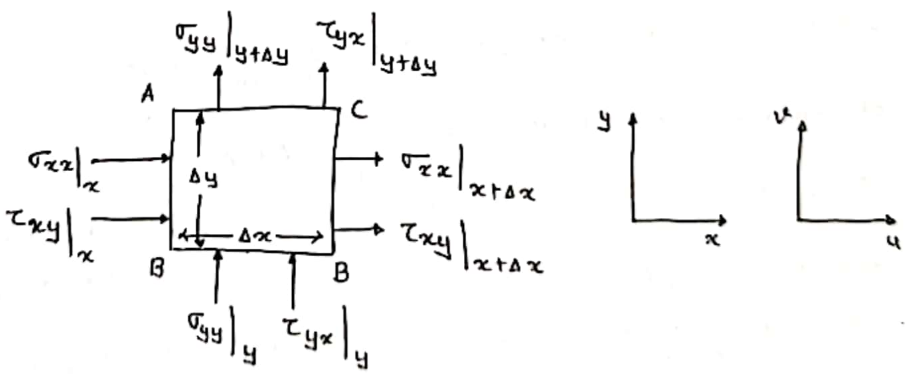
-ve Sign indicates displacement against body force.

ii) work against the surface forces

a) Pressure - Net forces

$$-\left[\frac{\partial\, P v_x}{\partial x} + \frac{\partial\, P v_y}{\partial y}\right]\Delta x\,\Delta y$$

b) Viscous force



↳ The arrow direction of τ_{yx} & τ_{xy} are not physical

↳ The velocity in the direction of the Force σ_{xx} will be u

↳ The velocity in the direction of the Force τ_{xy} will be v

↳ Hence the work done by the viscous force at the AB plane against the Surrounding is

$[\sigma_{xx}\cdot u + \tau_{xy}\cdot v]\cdot\Delta y$

↳ Similarly ⇒ $[\sigma_{yy}\cdot v + \tau_{yx}\cdot u]\cdot\Delta x$

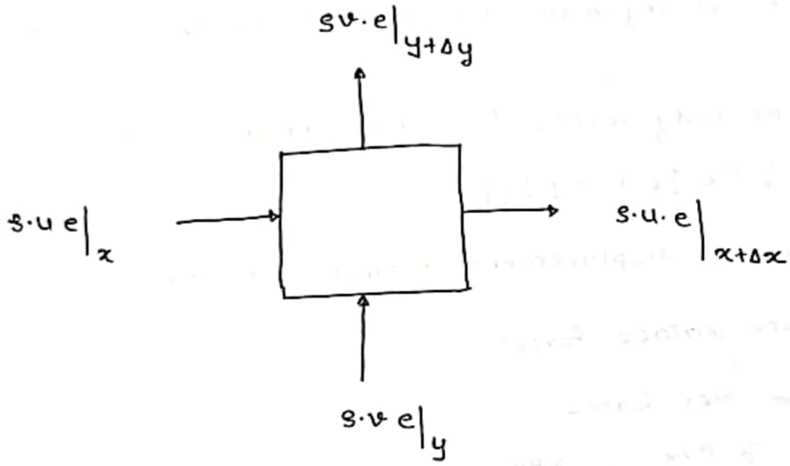
$$= \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial x}\right] + u\left[\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right]$$

$$= \rho\frac{Du}{Dt} + u\left[\frac{D\rho}{Dt} + \rho\nabla\cdot V\right] = -\left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}\right] - \frac{\partial P_x}{\partial x} + X$$

From Continiuit Equation $\Rightarrow \frac{D\rho}{Dt} + \rho\nabla\cdot V = 0 \longrightarrow$ {why this is zero need to ask -?}

Note – Substitution of the relevant expression of σ_{xx} & τ_{yx} using consecutive relations for Newtonian fluids will lead to the famous Naviers – Stokes Eqn.

ENERGY TRANSPORT $\Rightarrow$ A general Expression for energy transport



$e \longrightarrow$ Total specific energy of the system.

$e = \hat{u} + \frac{1}{2}v^2$

$\hat{u}$ = specific internal Energy $= \frac{U}{m} = \hat{u}$

$\frac{1}{2}v^2$ = specific kinetic Energy $= \frac{K.E}{m} = \frac{1}{2}\frac{mv^2}{m} = \frac{1}{2}v^2$

↳ Rate of Energy accumulation $\longrightarrow \frac{\partial \rho e}{\partial t}\cdot\Delta x\cdot\Delta y$

↳ Net Energy transport by fluid flow $\longrightarrow$ i) Rate of energy in

$\rho ue\cdot\Delta x + \rho v\cdot e\cdot\Delta y$

ii) Rate of energy out

$$\Delta y\cdot\rho ue|_{x+\Delta x} + \Delta x\cdot\rho ve|_{y+\Delta y} = \left[\rho ue + \frac{\partial(\rho ue)}{\partial x}\Delta x\right]\Delta y + \left[\rho ve + \frac{\partial \rho ve}{\partial y}\cdot\Delta y\right]\Delta x$$

$$\text{In} - \text{Out} = -\left\{\frac{\partial \rho ue}{\partial x} + \frac{\partial \rho ve}{\partial y}\right\}\Delta x\Delta y$$

↳ Net heat transfer by Conduction

Conduction heat flux is the transport element,

$$q'' = \frac{\text{Conductive heat}}{\text{Area} * \text{time}}$$

$$\text{In} - \text{Out} = \left(-\partial q''/\partial x - \partial q''/\partial y\right)\Delta x\,\Delta y$$