SARKAR PART

ADVANCE HEAT TRANSFER ⇒

VECTOR DIFFERENTIATION ⇒

① $\nabla \phi = \text{grad}\,\phi = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

② $\nabla \cdot \phi = \text{div}\,\phi = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

③ $\nabla \times \phi = \text{curl}\,\phi = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi_1 & \phi_2 & \phi_3 \end{vmatrix}$

④ $\vec{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$

⑤ Directional derrivative of scalar function

$$\frac{\partial f}{\partial s} = \bar{N} \cdot \text{grad}(f)$$

VECTOR INTEGRATION ⇒

① $\int_C \bar{f} \cdot d\bar{r}$

② $\int_C \bar{f} \times d\bar{r}$ ⇒ $\bar{f} \times d\bar{r} = \begin{vmatrix} i & j & k \\ f_1 & f_2 & f_3 \\ dx & dy & dz \end{vmatrix}$

③ Green's theorem ⇒

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \cdot dy$$

④ Vector form of Green's theorem ⇒

$$\iint_R \bar{N} \cdot (\nabla \times \bar{f}) \cdot ds = \int_C \bar{f}\, ds \qquad ds = dx \cdot dy \qquad \bar{N} = k$$

⑤ Stokes theorem

$$\iint_S \bar{N} \cdot (\nabla \times \bar{f}) \cdot ds = \int_C \bar{f} \cdot ds \qquad ds = dx \cdot dy$$

⑥ Gauss Divergence theorem

$$\iint \bar{N} \cdot \bar{F} \cdot ds = \iiint_V \nabla \cdot \bar{f} \cdot dV \qquad \bar{N} = \frac{\phi}{|\nabla \phi|}$$

$\frac{\partial f}{\partial s} = \bar{N} \cdot \text{grad}\,\phi \qquad \bar{n} = \frac{\bar{b}}{|b|}$ Directional derrivative of $f(1,1)$ in the direction of b.

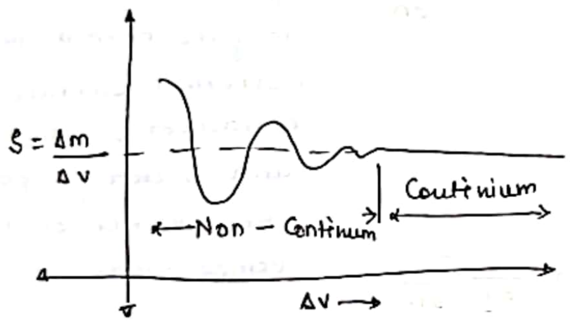
HEAT TRANSFER - CONDUCTION

Solid Conduction ⟶ Lattice vibration + free electron flow

Random movement of particles from both sides of imaginary plane but hotter particles have high energy in terms of translation, rotation & vibration motion, this transfer heat energy to other side.

No net mass transfer but heat transfer

Continum Approach - Ignore molecular details of substance and treat it as continium.



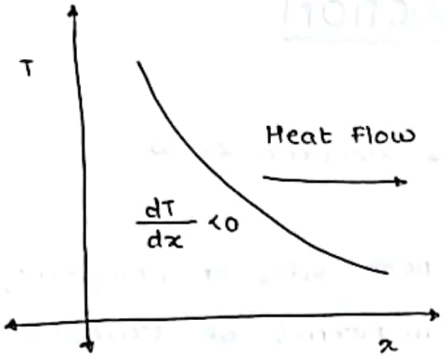
Microscopic / Molecular approach -

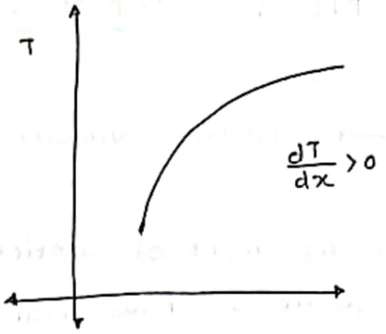
To find system property such as thermal conductivity.

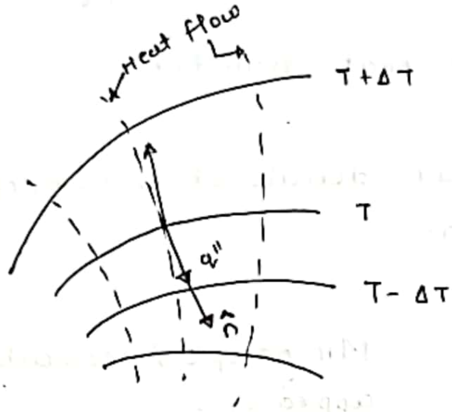
Temperature distribution function - $T(r,t)$

- $T(xi + yj + zk)$

Field effect - field that affect on system volume that result in transport.







$$q'' = -k\frac{\partial T}{\partial n}$$

$\frac{\partial}{\partial n} \rightarrow$ represents the differentiation along the normal to the isothermal surface is charachterized by the unit vector $\hat{n}$ pointing the direction of decreasing temperature.

$$\cancel{q'' = -k\left(\frac{\partial T}{\partial n}\right)}$$

$$q'' = -k\left(\frac{\partial T}{\partial x}\cdot\frac{dx}{dn} + \frac{\partial T}{\partial y}\cdot\frac{\partial y}{\partial n} + \frac{\partial T}{\partial z}\cdot\frac{\partial z}{\partial n}\right)$$

$$q'' = -k\left(\cos\alpha\cdot\frac{\partial T}{\partial x} + \cos\beta\,\frac{\partial T}{\partial y} + \cos\gamma\,\frac{\partial T}{\partial z}\right)$$

$$q'' = -k\left(\hat{i}\cdot\frac{\partial T}{\partial x} + \hat{j}\,\frac{\partial T}{\partial y} + \hat{k}\,\frac{\partial T}{\partial z}\right)\cdot\hat{n}$$

$$\hat{n} = \hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma$$

$$q''_n = -k\nabla T\cdot\hat{n}$$

$$\nabla T = \hat{i}\,\frac{\partial T}{\partial x} + \hat{j}\,\frac{\partial T}{\partial y} + \hat{k}\,\frac{\partial T}{\partial z}$$

Rectangular Co-ordinates (x, y, z) : $\quad \nabla T = \hat{i}\frac{\partial T}{\partial x} + \hat{j}\frac{\partial T}{\partial y} + \hat{k}\frac{\partial T}{\partial z}$

Cylindrical Co-ordinates (r, ϕ, z) : $\quad \nabla T = \hat{e}_r\frac{\partial T}{\partial r} + \hat{e}_\phi\cdot\frac{1}{r}\cdot\frac{\partial T}{\partial \phi} + \hat{e}_z\frac{\partial T}{\partial z}$

$$\hat{e}_r = i\cos\phi + j\sin\phi \qquad \hat{e}_z = \hat{k}$$

$$\hat{e}_\phi = -i\sin\phi + j\cos\phi$$

Spherical Co-ordinates (r, θ, ϕ) :

$$\nabla T = \hat{e}_r \cdot \frac{\partial T}{\partial r} + \hat{e}_\theta \cdot \frac{1}{r} \cdot \frac{\partial T}{\partial \theta} + \hat{e}_\phi \cdot \frac{1}{r\sin\phi} \frac{\partial T}{\partial \phi}$$

$$\hat{e}_r = \hat{i} \sin\theta \cdot \cos\phi + \hat{j} \cos\theta \cdot \sin\phi + \hat{k} \cos\theta$$

$$\hat{e}_\theta = \hat{i} \cos\theta \cdot \cos\phi + j \cos\theta \cdot \sin\phi + \hat{k} \sin\theta$$

$$\hat{e}_\phi = -\hat{i} \sin\phi + j \cos\phi$$

The magnitude of the heat flux across any arbitrary surface passing through P & having the unit direction vector $\hat{s}$ as its normal will be equal to the magnitude of the component of q'' in the s direction

$$q''_s = q'' \cdot \hat{s} = -k \nabla T \cdot \hat{s}$$

$$\nabla T \cdot \hat{s} = \frac{\partial T}{\partial s}$$

$$q''_s = -k \frac{\partial T}{\partial s}$$

$\frac{\partial}{\partial s} \rightarrow$ represents the differentiation in the direction of the normal ie $\hat{s}$.

For anistropic solid, heat flux may not be parallel to temperature gradient.

$$q''_s = q''_s \cdot \hat{n} + q''_s \cdot \hat{t}$$

That is the heat flux due to conduction in a given direction can also be proportional to the temperature gradients in other directions.

$$q''_1 = -k_{11} \frac{\partial T}{\partial x_1} - k_{12} \frac{\partial T}{\partial x_2} - k_{13} \cdot \frac{\partial T}{\partial x_3}$$

$$q''_2 = -k_{21} \cdot \frac{\partial T}{\partial x_1} - k_{22} \cdot \frac{\partial T}{\partial x_2} - k_{23} \frac{\partial T}{\partial x_3}$$

$$q''_3 = -k_{31} \cdot \frac{\partial T}{\partial x_1} - k_{32} \cdot \frac{\partial T}{\partial x_2} - k_{33} \frac{\partial T}{\partial x_3}$$

$$[k_{ij}] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

$$q''_i = -k_{ij} \frac{\partial T}{\partial x_j} \qquad i, j = 1, 2, 3$$

GENERAL HEAT CONDUCTION EQUATION ⇒

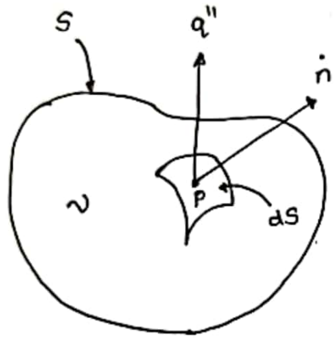
From first law of thermodynamics ⇒

$$\frac{\partial}{\partial t}\int_{cv} e\rho\, dv + \int_{cs} e\rho V\cdot\hat{n}\cdot dA$$

$$= q_{cs} - \int_{cs} pV\cdot\hat{n}\cdot dA - \dot{W}_{shear} - \dot{W}_{shaft} + \int_{cv} \dot{q}_e dv$$

$e \rightarrow$ specific total energy $\quad e = u + \frac{V^2}{2} + gz$

Internal energy (u), bulk kinetic energy ($\frac{V^2}{2}$), bulk potential energy (gz)



1) Stationary ⟶ No convection

2) Opaque ⟶ No Radiation

$$de = du$$

$$\int_v \rho \frac{\partial u}{\partial t}\partial v = q_s + \int_V \dot{q}_e dv$$

For a substance that is homogeneous & invariable in composition.

$$du = \left(\frac{\partial u}{\partial v}\right)_T \cdot dv + C_v dT$$

&

$$dh = \left(\frac{dh}{dP}\right)_T \cdot dP + C_p dT$$

For solids v & P is constant & Incompressible

$$du = dh$$

$$\int_V \rho C \frac{\partial T}{\partial t}\cdot dV = q_s + \int_V \dot{q}\, dv$$

$$q_s = -\int_S q''\cdot\hat{n}\cdot ds$$

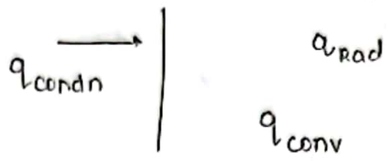
$q_s \rightarrow$ Represents the net rate of heat conducted into the volume V across its bounding surface S

$\hat{n} \rightarrow$ the outward drawn unit vector normal to the surface element ds

Boundary Conditions for Stationary Solids

Need 1 – I.C $\quad T(r,t)\big|_{t\to 0} = T_0(r)$

Need 2 – B.C $\quad$ from energy balance eqn at surface boundary



$$-k\frac{\partial T}{\partial n}\Big|_p = h(T_p - T_\infty) + \varepsilon\sigma(T_p^4 - T_\infty^4)$$

No energy can be accumulated at infinitely thin boundary.

For stationary solid.

i) B C's of 1st Law

Dirchlet – T_p is given / surface temp

$$T_p = T_0 \quad \text{or} \quad T_p = T(r_p, t)$$

ii) BC's of 2nd Law

Neumann – Surface flux is given

$$-k\frac{\partial T}{\partial n}\Big|_p = q_0''$$

$q_0'' = 0$ $\quad$ Insulation / Thermal symmetry

iii) Robin Mixed Boundary Condition

Robin Mixed : Convective at Boundary.

*) Moving B.C

Consider convective flux

for x component $\quad \rho C T u$

$$\text{Net heat flux} = q'' = -k\frac{dT}{dx} + \rho C T u$$

Substituting $\Rightarrow$

$$\int_V \rho c \frac{\partial T}{\partial t} \cdot dV = -\int_S q'' \cdot \hat{n}\, ds + \int_V q''' \cdot dV$$

Gauss - Divergence Theorem $\Rightarrow$ $\int_S q'' \cdot \hat{n}\, ds = \int_V \nabla \cdot q''\, dV$

$$\int_V \rho c \frac{\partial T}{\partial t} \cdot dV = -\int_V \nabla \cdot q''\, dV + \int_V q''' \cdot dV$$

$$\int_V \left(\rho c \frac{\partial T}{\partial t} + \nabla \cdot q'' - q''' \right) dV = 0$$

Integrand vanishes for every volume element V

$$-\nabla \cdot q'' + q''' = \rho c \frac{\partial T}{\partial t}$$

for isotropic solids

$$q'' = -k \nabla T$$

$$+\nabla \cdot (k \nabla T) + q''' = \rho c \frac{\partial T}{\partial t}$$

$$k \nabla^2 T + \nabla k \cdot \nabla T + q''' = \rho c \frac{\partial T}{\partial t}$$

For homogeneous isotropic solid, k is constant & the general heat conduction equation reduces to

$$\nabla^2 T + \frac{\dot{q}'''}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \qquad \alpha = \frac{k}{\rho C_p}$$

Internal heat source is zero

$$\nabla^2 T = \frac{1}{\alpha} \cdot \frac{\partial T}{\partial t} \longrightarrow \text{Heat Diffusion equation}$$

For steady state conditions & in the presence of internal heat sources we get

$$\nabla^2 T + \frac{q'''}{k} = 0 \longrightarrow \text{Poisson Eqn}$$

Steady state & no heat generation

$$\nabla^2 T = 0 \longrightarrow \text{Laplace Eqn}$$

Linear / Non-linear / Quasilinear PDE

Lecture 2

- A PDE is said to be linear if it is linear in the unknown function & all its derrivatives
- An eqn which is not linear is called a non-linear equation
- Linear → $\frac{\partial u}{\partial x} + a(x,y) \cdot \frac{\partial u}{\partial x} = f(x,y)$

  → $\frac{\partial^2 u}{\partial x \cdot \partial y} = 2y - x$

  → $\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x \cdot \partial y} + u(x,y)$

- Non-Linear → $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 0$

  $\frac{\partial u}{\partial x} + a(x,y) \frac{\partial u}{\partial y} = u^2$

- A non-linear equation is said to be quasi-linear if it is linear in all highest ordered derrivatives of the unknown function.

  Quasi-Linear ⇒ $\frac{\partial u}{\partial x} \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} \cdot \frac{\partial^2 u}{\partial y^2} + u^2 = 0$

- The free term in a PDE is the term that contains no unknown functions & its partial derrivatives.
- If the free term is identically zero, a linear equation is called a homogeneous, otherwise it is called a non-homogeneous PDE.
- The most general second-order linear PDE in n-independent variable has the form. $\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_j} + cu = f$
- The above eqn is homogeneous if $f \equiv 0$, otherwise it is non-homogeneous
- Note that the defination of homogenity is only for linear PDE

TEMPERATURE DEPENDENT THERMAL CONDUCTIVITY AND KIRCHOFFS TRANSFORMATION

$$\nabla \cdot [k(T)\, \nabla T] + \dot{q}(r,t) = \rho(T) \cdot c(T) \cdot \frac{\partial T}{\partial t}$$

Because of the dependence of thermal thermophysical properties k, ρ & c on temperature T. Above eqn is a non-linear differential eqn

~~Th~~ And Thermal diffusivity is independent of Temperature

$$\theta(r,t) = \frac{1}{k_R} \int_{T_R}^{T(r,t)} k(T') \cdot dT'$$

$T_R \rightarrow$ Reference temperature

$$\nabla \theta = \frac{k(T)}{k_R} \nabla T$$

$$\frac{\partial \theta}{\partial t} = \frac{k(T)}{k_R} \frac{\partial T}{\partial t}$$

$$\nabla^2 \theta + \frac{\dot{q}(r,t)}{k_R} = \frac{1}{\alpha} \cdot \frac{\partial T}{\partial t}$$

Principle Of Superposition ⇒

Let L be a linear operator $\quad L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c$

The most general homogeneous or non-homogeneous second order linear partial differential equation in n independent variables.

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu = f$$

May respectively be written in the form $Lu = 0$ & $Lu = f$

Property 1

A linear combination of two solution of a homogeneous eqn also a solution the equation

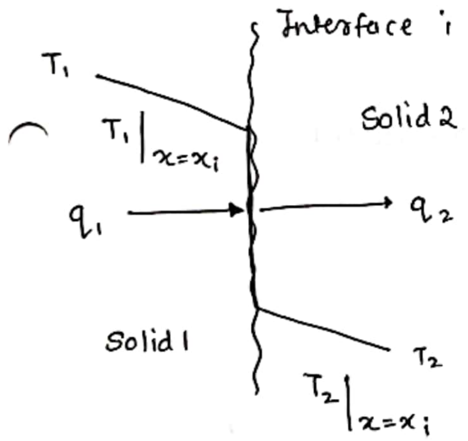
That is $L(C_1u_1 + C_2u_2) = 0 \quad$ if $Lu_1 = 0$ & $Lu_2 = 0$

Here C_1 & C_2 are arbitrary constants.

Boundary Condition At Interface

① Both Surface stationary

Boundary Conditions at interface of two contacting solid surface



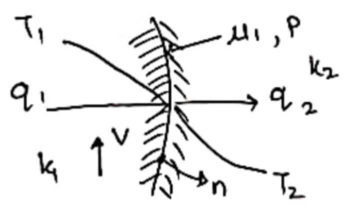
$$q_i'' = -k_1 \frac{\partial T_1}{\partial x}\bigg|_i = h_c (T_1 - T_2)_i = -k_2 \frac{\partial T_2}{\partial x}\bigg|_i$$

Here $h_c \; \frac{W}{m^2 k}$ is called the contact conductance for the interface

$T_1\big|_i = T_2\big|_i$ at the surface interface

$$-k_1 \frac{\partial T_1}{\partial x}\bigg|_i = -k_2 \frac{\partial T_2}{\partial x}\bigg|_i$$

② Solid at Relative ~~Interface~~ Motion



The local pressure on the common boundary is p, the coefficient of dry friction μ & the relative velocity V.

HEAT CONDUCTION EQUATION : NON-LINEARITY

Non-linearity in conduction problem arises when thermo-physical properties are temperature dependent or when boundary conditions are non-linear.

Surface radiation & free convection are typical examples of non-linear boundary conditions.

In phase change problems the interface energy equation is non-linear.

SOURCES OF NON-LINEARITY - Non linear DE

$$\frac{\partial}{\partial x}\left(k \cdot \frac{\partial T}{\partial x}\right) + q''' = \rho C_p \frac{\partial T}{\partial t}$$

If ρ & C_p vary with temperature the transient term is non-linear similarly if $k = k(T)$ the first term becomes non-linear.

$$\Rightarrow \quad k \frac{\partial^2 T}{\partial x^2} + \frac{\partial k}{\partial T} \cdot \left[\frac{\partial T}{\partial x}\right]^2 + q''' = \rho C_p \frac{\partial T}{\partial t}$$

$\Rightarrow$ • The fin exchanging heat by convection & radiation

$$\frac{d^2 T}{dx^2} - \frac{hC}{kA}(T - T_\infty) - \frac{\varepsilon \sigma C}{kA}(T^4 - T^4_{surr}) = 0$$

Non-Linear Boundary Conditions $\Rightarrow$

Free convection Boundary conditions = $-k\dfrac{\partial T}{\partial x} = \beta (T - T_\infty)^{5/4}$

Radiation Boundary Conditions = $-k\dfrac{\partial T}{\partial x} = \varepsilon \sigma (T^4 - T^4_{sur})$

Phase change boundary conditions = $k_s \cdot \dfrac{\partial T}{\partial x} - k_L \dfrac{\partial T}{\partial x} = \rho_s L \dfrac{\partial x_i}{\partial t}$

Orthotropic Bodies

The conductivity matrix depends on the orientation of the co-ordinate system in the body.

If the co-ordinate system is parallel to 3 mutually perpendicular preffered directions of heat conductions then the geometry is said to be orthotropic & the co-ordinate system lies along the principle axes of heat conduction.

An orthotropic body has direction dependent thermal conductivity whose principal values are alligned with the co-ordinate axes. In an orthotropic body the conductivity matrix has a diagonal form

$$\begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{bmatrix}$$

Wood is an example of an orthotropic body.

The energy eqn for orthotropic bodies does not contain any cross derrivatives & it can be transformed into the standard isotropic energy equation by a suitable choice of new spatial Co-ordinates.

Transformation: Orthotropic Solids

The following transformation converts the orthotropic heat conduction equation to the usual conduction equation.

The heat conduction equation in Cartesian co-ordinates for an orthotropic body is given by

$$k_{11}\frac{\partial^2 T}{\partial x^2} + k_{22}\frac{\partial^2 T}{\partial y^2} + k_{33}\frac{\partial^2 T}{\partial z^2} + g(x,y,z,t) = \rho c\frac{\partial T}{\partial x} \quad - \text{Eqn (A)}$$

Define stretched co-ordinate axes of the form -

$$x_1 = x\left(\frac{k}{k_{11}}\right)^{1/2} \qquad y_1 = y\left(\frac{k}{k_{22}}\right)^{1/2} \qquad z_1 = z\left(\frac{k}{k_{33}}\right)^{1/2}$$

where k is reference conductivity is not arbitrary. It must be choosen so that the original differential volume is equal to

The equation of conduction for hetrogeneous isotropic solids (and also for frictionless incompressible fluids):

$$\rho C \frac{dT}{dt} = \nabla \cdot (k \nabla T) + q'''$$

$$\rho C \frac{dT}{dt} = \nabla k \cdot \nabla T + k \nabla^2 T + q'''$$

$$\nabla k = \frac{dk}{dT} \nabla T \longrightarrow k \text{ depends on temperature}$$

$$\rho C \frac{dT}{dt} = \frac{dk}{dT} (\nabla T)^2 + k (\nabla^2 T) + q''' \longrightarrow \text{Non linear eqn}$$

Heat Conduction Equation : Anistropic Solids

Many bodies of engineering interest do not conduct heat equally in all directions & are called anistropic bodies.

Example - Laminates, crystals, composites, graphite, molybdenum Disulphide & woods are among the materials that have preferred direction of heat flow

Thermal Conductivity Matrix in Rectangular Co-ordinates

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

The components of heat flux vector are given by $q_i = \sum_{j=1}^{3} k_{ij} \frac{\partial T}{\partial x_j}$

The energy eqn for anistropic bodies contains cross derivatives.

Scaled differential volume.

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+g(x,y,z,t)=\rho\frac{\partial T}{\partial t}$$

for the 3D cartesian case, the differential volume scales according to

$$dx\,dy\,dz = \frac{\{k_{11}k_{22}k_{33}\}^{1/2}}{k^{3/2}}\cdot dx_1\,dx_2\,dz,$$

and the requirement that $dv = dv$, causes $k = (k_{11}k_{22}k_{33})^{1/3}$

HETROGENEOUS ANISTROPIC SOLIDS

$$\rho C\frac{dT}{dt} = \frac{\partial}{\partial x}\left(k_{11}\frac{\partial T}{\partial x}+k_{12}\frac{\partial T}{\partial y}+k_{13}\frac{\partial T}{\partial z}\right)$$
$$+\frac{\partial}{\partial y}\left(k_{21}\frac{\partial T}{\partial x}+k_{22}\frac{\partial T}{\partial y}+k_{23}\frac{\partial T}{\partial z}\right)$$
$$+\frac{\partial}{\partial z}\left(k_{31}\cdot\frac{\partial T}{\partial x}+k_{32}\frac{\partial T}{\partial y}+k_{33}\frac{\partial T}{\partial z}\right)+q'''$$

VARIABLE THERMAL CONDUCTIVITY K(T) : KIRCHOFF TRANSFORMATION

Consider 1D case : $\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+q''' = \rho C_p\frac{\partial T}{\partial t}$

$$\theta(T) = \frac{1}{k_0}\int_0^T k(T)\cdot dT \Rightarrow \frac{d\theta}{dT} = \frac{k(T)}{k_0}$$

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial \theta}\cdot\frac{\partial \theta}{\partial t} = \frac{k_0}{k(T)}\cdot\frac{\partial\theta}{\partial t}$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial\theta}\cdot\frac{\partial\theta}{\partial x} = \frac{k_0}{k(T)}\cdot\frac{\partial\theta}{\partial x} \Rightarrow \frac{\partial^2 T}{\partial x^2} = \frac{k_0}{k(T)}\cdot\frac{\partial^2\theta}{\partial x^2}$$

$$\frac{\partial^2\theta}{\partial x^2}+\frac{q'''}{k_0} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \qquad \alpha = \alpha(T) = \frac{k(T)}{\rho C_p}$$

VARIABLE THERMAL CONDUCTIVITY $k(T)$: KIRCHOFFS TRANSFORMATION

Consider : $\nabla \cdot [k(T) \nabla T] + \dot{q}(r,t) = \rho(T) \cdot c(T) \dfrac{\partial T}{\partial t}$ — Eqn A

KIRCHHOFF TRANSFORMATION :

$$\theta(r,t) = \frac{1}{k_R} \int_{T_R}^{T(r,t)} k(T') \cdot dT' \quad \text{— Eqn B}$$

where T_R is a reference temperature & $k_R = k(T_R)$

from Eqn (B):

$$\nabla \theta = \frac{k(T)}{k_R} \nabla T \qquad \frac{\partial \theta}{\partial t} = \frac{k(T)}{k_R} \cdot \frac{\partial T}{\partial t}$$

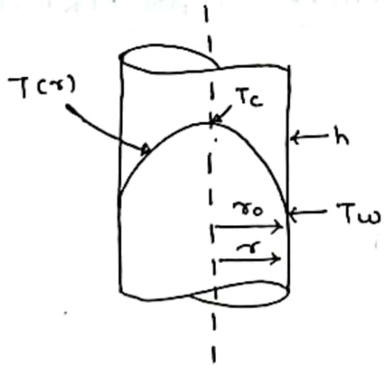
using these with Eqn (A) $\quad \nabla^2 \theta + \dfrac{\dot{q}(r,t)}{k_R} = \dfrac{1}{\alpha} \dfrac{\partial T}{\partial t}$

Example ⇒

find the rate of heat generation per unit volume in a rod that will produce a centerline temperature of 2000°C for the following Conditions

$$r_0 = 1\text{cm}, \quad T_w = 350°C \quad \& \quad k = \frac{3167}{T+273}$$

Also calculate heat flux



Steady State

Solution ⇒

Differential Equation

$$\frac{1}{r} \cdot \frac{d}{dr}\left[r k(T) \cdot \frac{dT}{dr} \right] + \dot{q} = 0$$

Boundary Conditions

$$\left(\frac{dT}{dr}\right)_{r=0} = 0 \quad \& \quad T(r_0) = T_w$$

$$\frac{1}{r}\cdot\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)+\frac{\dot{q}}{k_w}=0 \qquad \left(\frac{d\theta}{dr}\right)_{r=0}=0 \quad \& \quad \theta(r_0)=0$$

$$\theta(r)=\frac{q r_0^2}{4k_w}\left[1-\left(\frac{r}{r_0}\right)^2\right]$$

$$\theta(r)=\frac{1}{k_w}\int_{T_w}^{T(r)} k(T')\,dT' \Rightarrow \int_{T_w}^{T(r)} k(T')\,dT'=\frac{\dot{q} r_0^2}{4}\left[1-\left(\frac{r}{r_0}\right)^2\right]$$

This relation can be written explicitly for $T(r)$ when the relation $k=k(T)$ is given. At $r=0$ this equation yields.

$$\int_{T_w}^{T_c} k(T)\,dT=\frac{q r_0^2}{4}$$

$$\dot{q}=\frac{4}{r_0^2}\int_{350}^{2000}\frac{3167}{T+273}\,dT = 1\cdot 64\times 10^3\ \frac{W}{m^3}$$

Surface Heat flux $\Rightarrow$ $q''_s=\dfrac{\dot{q}\cdot V}{A}$

CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G(x,y)$$

where A, B, C are const. It is said to be

Hyperbolic if $B^2-4AC>0$

Parabolic if $B^2-4AC=0$

Elliptic if $B^2-4AC<0$

unsteady state heat conduction (1D, 2D, 3D) : Parabollic

$$\rho c\frac{\partial T}{\partial t}=k\frac{\partial^2 T}{\partial x^2}$$

Steady state heat conduction (2D, 3D) : Elliptic

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$$

Wave Equation : Hyperbollic

$$\frac{\partial^2 f}{\partial x^2}-\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}=0$$

NON-FOURIER HEAT CONDUCTION :
FINITE SPEED OF HEAT PROPAGATION

for heat conduction in a homogeneous & isotropic medium, the fourier law of heat conduction -

$$\rho c \frac{\partial T}{\partial t} + \nabla \cdot q'' - q''' = 0 \qquad q(r,t) = -k \nabla T(r,t)$$

$$\frac{1}{\alpha} \frac{\partial T}{\partial t} = \nabla^2 T + \frac{q'''}{k}$$

The relation between the heat flux q & the temperature gradient ∇T is called the constitutive relation of heat flux

Parabolic Heat - Conduction Equation -

→ The fourier law of heat conduction is an early emperical law.

→ It assumes that q & ∇T appear at the same time instant t & consequently implies that thermal signals propagate with an infinite speed.

→ If the material is subjected to a thermal disturbance, the effects of the disturbance will be felt instantaneously at distances infinitely far from its source.

→ Although this result is physically unrealistic, it has been confirmed by many experiments that the fourier law of heat conduction holds for many media in the usual range of heat flux q & temperature gradient.

→ what about heat conduction appearing in the range of high heat flux & high unsteadiness?

→ Technology - ultrafast pulse-laser heating on metal films → heat conduction appears in the range of high heat flux and high unsteadiness.

→ Infinite heat propagation speed in the fourier law becomes unacceptable. This has inspired the work of searching for new constitutive relations.

New constitutive relation proposed by Cattaneo (1958) & Vernotte (1958, 1961):

CV Constitutive Relation: $q(r,t) + \tau_0 \dfrac{\partial q(r,t)}{\partial t} = -k \nabla T(r,t)$

Here $\tau_0 > 0$ is a material property and is called the relaxation time

~~Substituting~~ Substitute q in : $\rho c \dfrac{\partial T}{\partial t} + \nabla . q'' - q''' = 0$

The corresponding heat-conduction equation

$$\frac{1}{\alpha} . \frac{\partial T}{\partial t} + \frac{\tau_0}{\alpha} \frac{\partial^2 T}{\partial t^2} = \nabla^2 T + \frac{1}{k}\left(q''' + \tau_0 \frac{\partial q'''}{\partial t}\right)$$

This eqn is of hyperbolic type, charachterizes the combined diffusion & wave-like behaviour of heat conduction, & predicts a finite speed for heat propagation.

$$V_{cv} = \sqrt{\frac{k}{\rho C \tau_0}} = \sqrt{\frac{\alpha}{\tau_0}}$$

Consider no heat generation

$$\frac{\partial^2 q}{\partial t^2} = c^2 \left(\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 q}{\partial z^2}\right)$$

~~Consider the econ~~

SINGLE PHASE LAGGING MODEL

Consider the constitutive relation proposed by Cattaneo & Vernotte.

$$q(r,t) + \tau_0 \frac{\partial q(r,t)}{\partial t} = -k \nabla T(r,t)$$

Note that the CV constitutive relation is actually a first order approximation of more general constitutive relation:

$$q(r, t+\tau_0) = -k \nabla T(r,t)$$

This is called single phase lagging model.

→ This suggests that the temperature gradient established at a point r at time t gives rise to a heat flux vector at r at a later time $t+\tau_0$. There is a finite built-up time τ, for the onset of heat flux at r after a temperature gradient is imposed there.

- Thus the τ represents the time lag needed to establish the heat flux (the results) when a temperature gradient (the cause) is suddenly imposed.

- The higher $\partial q/\partial t$ corresponds to a larger derivation of the CV constitutive relation from the classical fourier law.

- The value of τ_0 is material dependent

- for most solid materials, τ varies from 10^{-10}s to 10^{-14}s

- for gases τ_0 is normally in the range of $10^{-8} \sim 10^{-10}$s

- The value of τ_0 for some biological materials & materials with non homogeneous inner surface structure can be up to 10^{2}s.

- The CV constitutive relation generates a more accurate prediction than the classical fourier law. However some of its predictions do not agree with experimental results.

- The CV constitutive relation has only taken account of the fast-transient effects, but not the micro-structural interactions.

- These two effects can be reasonably represented by the dual-phase-lag between q & ∇T.

$$\boxed{q(r, t+\tau_0) = -k \nabla T(r, t+\tau_T)}$$

Dual-Phase Lagging Model

$$q(r, t+\tau_0) = -k\nabla T(r, t+\tau_T)$$

According to this relation, the temperature gradient at a point r of the material at time $t+\tau_T$ corresponds to the heat flux density vector at r at time $t+\tau_0$.

The delay time τ_T is interpreted as being caused by the micro-structural interactions (small scale heat transport mechanisms occuring in the micro-scale, or small-scale effects of heat transport in space). such as photon-electron interaction or photon scattering and is called the phase lag of the temperature gradient.

$$\boxed{q(r, t+\tau_0) = -k\nabla T(r, t+\tau_T)}$$

Expanding both sides by using the Taylor series & retaining only the first order terms of τ_0 & τ_T we obtain the following constitutive relation that is valid at point r & time t:

$$\boxed{q(r,t) + \tau_0 \cdot \frac{\partial q(r,t)}{\partial t} = -k\left\{ \nabla T(r,t) + \tau_T \frac{\partial}{\partial t}\left[\nabla T(r,t)\right]\right\}}$$

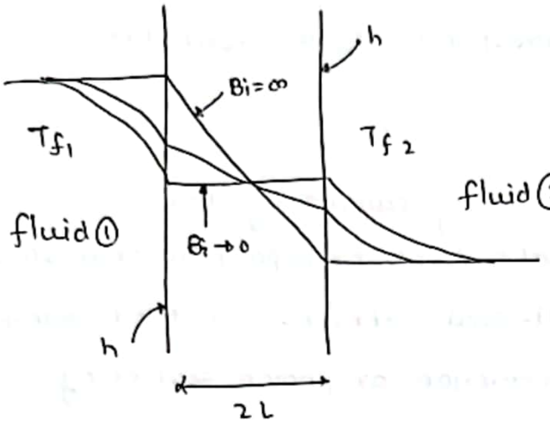
This is known as the jeffreys-type constitutive equation of heat flux (Joseph & Preziosi 1989). In literature this relation is called the dual-phase-lagging constitutive relation.

when $\tau_0 = \tau_T$ this relation reduces to classical fourier law.
when $\tau_T = 0$ it reduces to the CV constitutive relation.

Substituting q in $\boxed{\rho c \frac{\partial T}{\partial t} + \nabla \cdot q'' - q''' = 0}$

Need to check =

Temperature : Distribution in a Plane : Biot Number :
Two limiting Cases



$$q = \frac{T_{f_1} - T_{f_2}}{\Sigma R_f} \qquad \Sigma R_f = \frac{2}{hA} + \frac{2L}{kA}$$

$$\text{Biot Number} = Bi = \frac{\text{Internal Resistance}}{\text{Surface Resistance}}$$

$$= \frac{hL_c}{k}$$

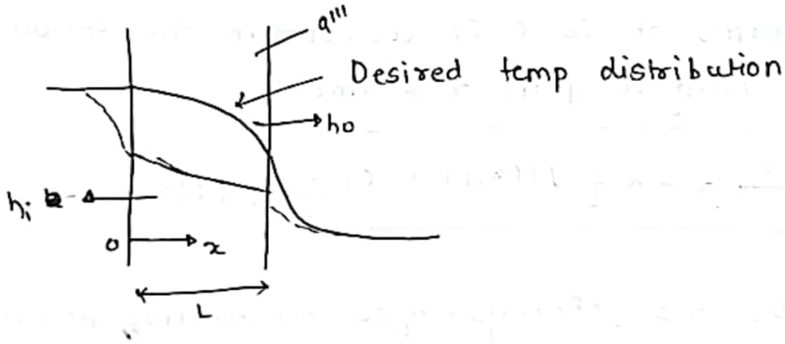
Case 1 $\Rightarrow$ $Bi \to \infty$

$$\Sigma R_t = \frac{2L}{kA}$$

Case 2 $\Rightarrow$ $Bi \to 0$

$$\Sigma R_t = \frac{2}{hA}$$

Heat Loss from One Side



Non-Dimensional Form: 3D Heat Eqn

A stationary, homogeneous, isotropic solid is initially at a const temp T_0 for time $t>0$, heat is generated within the solid and dissipated by convection from the boundary surface into a medium at const temp T_∞. Assume a rectangular geometry & a finite region, R.

BVP of heat conduction eqn $\Rightarrow$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \text{in region } R \quad t>0$$

$$-k_i \frac{\partial T}{\partial n_i} = h_i (T_i - T_\infty) \quad \text{on } S_i \text{ boundary of } R,\ t>0$$

$i = 1,2,\ldots$ no of conditions bounding surface of the solid.

$T = T_0$ in region R $t=0$

Reference length $= L$ = a charachterstic dimension of solid

Define dimensionless length variable

$$\xi = \frac{x}{L}, \quad \eta = \frac{y}{L}, \quad \psi = \frac{z}{L}$$

$\frac{\partial}{\partial N} \equiv$ differentiation ~~only~~ along outward drawn normal in the new dimensionless co-ordinate system (ξ, η, ψ)

choose reference temp $= T_\infty$

Net temp difference $= T_0 - T_\infty$

Define dimensionless excess temperature $\theta = \dfrac{T - T_\infty}{T_0 - T_\infty}$

On substitution:

$$\frac{T_0 - T_\infty}{L}\left(\frac{\partial^2\theta}{\partial\xi^2} + \frac{\partial^2\theta}{\partial\eta^2} + \frac{\partial^2\theta}{\partial\psi^2}\right) + \frac{\dot{q}}{k} = \frac{T_0 - T_\infty}{\alpha}\cdot\frac{\partial\theta}{\partial t} \quad \text{in } R \quad t>0$$

$$-\frac{k_i}{L}\frac{\partial\theta}{\partial N_i} = h_i\theta \quad \text{on } S_i \quad t>0$$

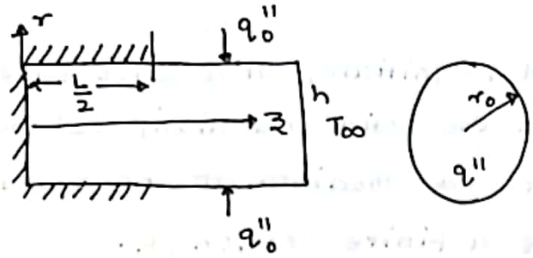
$i = 1,2,\ldots$ = no of continuous bounding surface

$\theta = 1$ in R, $t=0$

$$\frac{\partial^2\theta}{\partial\xi^2} + \frac{\partial^2\theta}{\partial\eta^2} + \frac{\partial^2\theta}{\partial\psi^2} + \frac{\dot{q}L^2}{(T_0 - T_\infty)k} = \frac{L^2}{\alpha}\frac{\partial\theta}{\partial t} = \frac{\partial\theta}{\partial\tau} \quad \text{here } \tau = \frac{\alpha t}{L^2}$$

$$\frac{\partial^2\theta}{\partial\xi^2} + \frac{\partial^2\theta}{\partial\eta^2} + \frac{\partial^2\theta}{\partial\psi^2} + \Phi = \frac{\partial\theta}{\partial\tau} \quad \text{in } R,\ t>0$$

Formulation: 2D



$k \to$ const

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + q''' = 0$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} + \frac{q'''}{k} = 0$$

$$\Rightarrow \frac{\partial T(0,r)}{\partial z} = 0$$

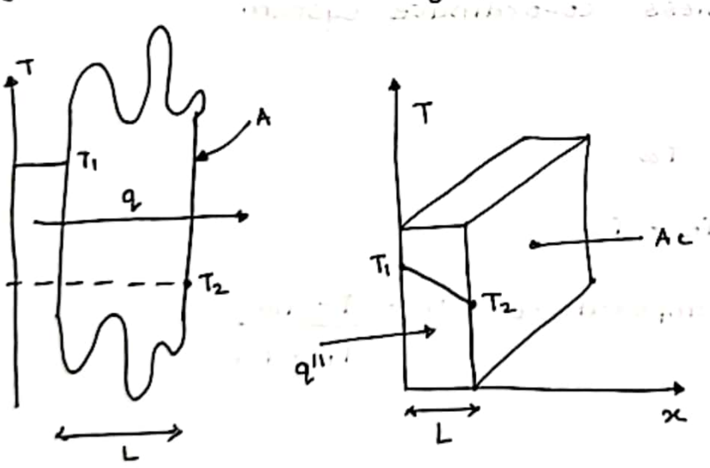
$$\Rightarrow -k\frac{\partial T(L,r)}{\partial z} = h\left[T(r,L) - T_\infty\right]$$

$$\Rightarrow \frac{\partial T(0,z)}{\partial r} = 0 \quad \text{or} \quad T(0,z) = \text{finite}$$

$$\Rightarrow k\frac{\partial T(r_0,z)}{\partial r} = f(z) = \begin{cases} 0 & 0 \leq z \leq L/2 \\ q_0''' & L/2 \leq z \leq L \end{cases}$$

Non-Dimensional form ⇒

why non-dimensional form ?



$$\frac{d^2T}{dx^2} = \frac{-q'''}{k}$$

Define ⇒

① Reference Length
② Reference temperature
③ Reference temp difference

$$\bar{T} = \frac{T - T_1}{T_2 - T_1} \qquad \bar{x} = \frac{x}{L}$$

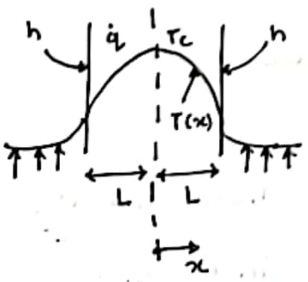
$$\frac{d^2\bar{T}}{d\bar{x}^2} = -\frac{q''' L^2}{k(T_2 - T_1)} = -S$$

$$\bar{T}(\bar{x}=0) = 0 \quad , \quad \bar{T}(\bar{x}=1) = 1$$

$$\bar{T} = -\frac{S\bar{x}^2}{2} + c_1\bar{x} + c_2$$

$$\bar{T} = \bar{x} + S\frac{\bar{x}}{2}(1-\bar{x})$$

Non-Dimensional form : The correct way ⇒



$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0 \qquad \frac{d^2\bar{T}}{d\bar{x}^2} = -1$$

$$\left.\frac{dT}{dx}\right|_{x=0} = 0 \qquad \left.\frac{d\bar{T}}{d\bar{x}}\right|_{\bar{x}=0} = 0 \, , \quad \left.\frac{d\bar{T}}{d\bar{x}}\right|_{\bar{x}=1} = -Bi\,\bar{T}(\bar{x}=1)$$

$$-k\left.\frac{dT}{dx}\right|_{x=L} = h\left(T(x=L) - T_\infty\right)$$

$$\frac{\partial \theta}{\partial N_i} + \frac{h_i L}{K_i}\theta = 0$$

$$\Rightarrow \frac{\partial \theta}{\partial N_i} + B_i \theta = 0$$

$\theta = 1$ in $R, \tau = 0$

$$\tau = \frac{\alpha t}{L^2}$$

$$\tau = \frac{\frac{k}{L} \cdot L^2}{\frac{\rho C_p L^3}{t}} \rightarrow \frac{\text{Rate of heat conduction}}{\text{Rate of heat storage at ref volume.}}$$

$$\Phi = \frac{\dot{q} L^2}{k(T_0 - T_\infty)} = \frac{\dot{q} L^3}{\frac{k}{L} L^2 (T_0 - T_\infty)} \qquad Bi = \frac{h L_c}{k_a k}$$

Uniqueness of Solution for 3D Heat Equation ⇒

Prove that the solution of the following 3D heat problem is unique

$$u_t = \nabla^2 u \qquad x \in D$$

$$u(x,t) = 0 \qquad x \in \partial D \quad \text{on the boundary}$$

$$u(x,0) = f(x) \qquad x \in D$$

Let u_1, u_2 be two solutions. Define $v = u_1 - u_2$. Then v satisfies

$$v_t = \nabla^2 v \qquad x \in D$$

$$v(x,t) = 0 \qquad x \in \partial D$$

$$v(x,0) = 0 \qquad x \in D$$

Define: $V(t) = \iiint_D v^2 dV \geq 0$ $\quad$ $V(t) \geq 0$ since the integrand $v^2(x,t) \geq 0$ for all (x,t)

$$V(t) = \iiint_D v^2 dV \geq 0 \qquad v_t = \nabla^2 v$$

$$\frac{dV(t)}{dt} = \iiint_D 2v \cdot v_t \, dV$$

$$\frac{dV(t)}{dt} = \iiint_D 2v \, \nabla^2 v \cdot dV$$

Now use: $\iint v \nabla v \cdot \hat{n} \cdot ds = \iiint \nabla(v \cdot \nabla v) \, dV$

$$= \iiint (v \nabla^2 v + (\nabla v)^2) \cdot dV$$

$$\iint v \nabla v \cdot \hat{n} \cdot ds - \iiint (\nabla v)^2 dV = \iiint v \nabla^2 v \, dV$$

$$\frac{dV(t)}{dt} = 2\iint_{\partial D} v \nabla v \cdot \hat{n} \, ds - 2\iiint_D |\nabla v|^2 dV$$

But on ∂D, $v = 0$, so that the first integral on r.h.s vanishes. Thus

$$\frac{dV(t)}{dt} = -2\iiint_D |\nabla v|^2 dV \leq 0$$

at $t=0$

$$V(0) = \iiint_D v^2(x,0)\,dV = 0 \qquad \text{Recall IC} - v(x,0)=0$$

Thus $V(0)=0$, $V(t) \geq 0$ & $\frac{dV}{dt} \leq 0$

$V(t)$ is a non-negative, non-increasing functions that starts at zero.

Thus $V(t)$ must be zero for all time t, so that $v(x,t)$ must be identically zero throughout the volume D for all time, implying the two solutions are the same $u_1 = u_2$

Thus the solution to the 3D heat problem is unique.

SOLUTION OF HOMOGENEOUS EQN BY SEPERATION OF VARIABLES ⇒

$$\frac{1}{\alpha}\frac{\partial T(r,t)}{\partial t} = \nabla^2 T(r,t)$$

$$k_i \frac{\partial T}{\partial n_i} + n_i T(r,t) = 0$$

$$T(r,t=0) = f(r)$$

$$T(r,t) = \psi(r)\cdot\Gamma(t)$$

$$\frac{1}{\alpha}\frac{\dot{\Gamma}(t)}{\Gamma(t)} = \frac{\nabla^2\psi(r)}{\psi(r)} = -\lambda^2$$

For time ⇒ $\dot{\Gamma}(t) + \alpha\lambda^2\Gamma(t) = 0$

$$\dot{\Gamma}(t) = e^{-\alpha\lambda^2 t}$$

$$\left.\begin{aligned} &\nabla^2\psi(r) + \lambda^2\psi(r) = 0 \\ &k_i \frac{\partial\psi(r)}{\partial n_i} + h_i\psi(r) = 0 \end{aligned}\right\} \text{Helmholtz Eqn.}$$

The solution for element ⇒ $\lambda m \rightarrow \psi_m(r)$

$$T(r,t) = \sum_{m=1}^{\infty} C_m \cdot \psi_m(r)\cdot e^{-\alpha\lambda_m^2 t}$$

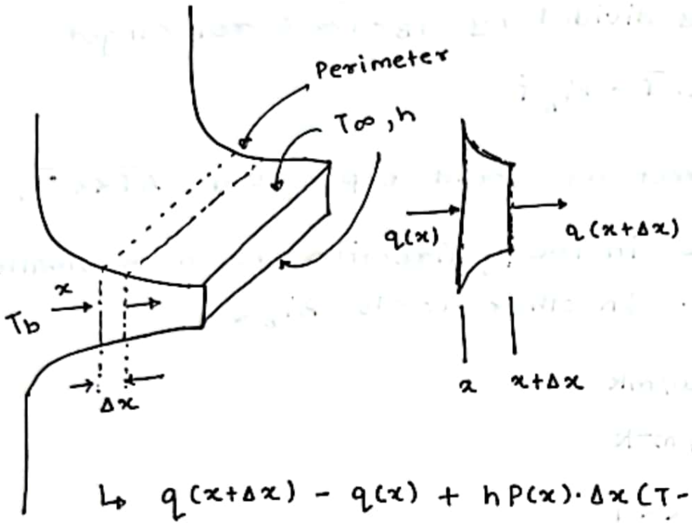
$$\int \psi_m(r)\cdot\psi_n(r)\cdot dr = 0$$

$m \neq n$

$$C_m = \frac{\int_R \psi_m(r)\cdot F(r)\cdot dr}{\int_R \psi_m^2(r)\cdot dr}$$

Normalized Eigen function ⇒ $k = \frac{\psi_m(r)}{\sqrt{N}}$

Extended Surfaces : Fin Equation



$$q(x) = q(x+\Delta x) + q_{conv}$$

since $\Delta x \to 0$

$$q(x+\Delta x) = q(x) + \frac{dq}{dx}\Delta x$$

Also

$$q_{conv} = h\,P(x)\cdot\Delta x\,(T-T_\infty)$$

$$\hookrightarrow q(x+\Delta x) - q(x) + hP(x)\cdot\Delta x\,(T-T_\infty) = 0$$

$$\hookrightarrow \lim_{\Delta x\to 0}\frac{q(x+\Delta x) - q(x)}{\Delta x} + hP(x)(T-T_\infty) = 0 \Rightarrow \frac{dq}{dx} + hP(x)(T-T_\infty) = 0$$

$$q(x) = -kA(x)\frac{dT}{dx} \Rightarrow \frac{d}{dx}\left(-kA(x)\frac{dT}{dx}\right) + hP(x)(T-T_\infty) = 0$$

$$\bar{T} = \frac{T-T_\infty}{T_B - T_\infty} \qquad \bar{x} = \frac{x}{L}$$

$$\frac{d}{d\bar{x}} A_c \frac{d\bar{T}}{d\bar{x}} - \frac{hPL^2}{k}\cdot\bar{T} = 0$$

One BC is at Fin Base : $T = T_b$ $\quad \bar{T}(\bar{x}=0) = 1$

Three types of BC at fin tip :

$\bar{T}(\bar{x}=1) = \bar{T}_t \longrightarrow$ fixed tip T

$\left.\frac{d\bar{T}}{d\bar{x}}\right|_{\bar{x}=1} = 0 \longrightarrow$ Insulated tip

$\left.\frac{d\bar{T}}{d\bar{x}}\right|_{\bar{x}=1} = Bi \,,\ \bar{T}(x=1) \longrightarrow$ tip convection

No each term has unit of area.

Further reduction cannot be made until the specific form of Ac has been set.

- Fin Equation - why 1D?

If y denotes the direction normal to the surface area, the energy balance at the surface would give

$$\left.-k\frac{\partial T}{\partial y}\right|_{y=b} = h(T-T_\infty)$$

where b denotes the thickness of the fin at a particular position x

Thus a temperature gradient must exist in the y direction.

Approximate the derrivative as $\left.\frac{\partial T}{\partial y}\right|_{y=b} = \frac{\Delta T}{b}$ ΔT represents the average temperature difference across the fin in the y direction.

If the surface energy balance is divided by $T_B - T_\infty$ & rearranged

$$\Delta \bar{T} = \frac{\Delta T}{T_B - T_\infty} \approx \frac{hb}{k} \cdot \bar{T} = Bi_b \bar{T}$$

For the 1-D assumption to be correct we would expect that $\Delta \bar{T} << \bar{T}$,

i.e, the variation in temperature in the y direction is much smaller than the variation in the x direction. In other words $Bi_b << 1$.

Consider aluminium fin of thickness 1cm $k \approx 400$ w/mk $h \not> 10$ w/m²k

$$Bi_b = 10 \times 0.01 \times 400 = 0.0025 << 1$$

$$\frac{d}{d\bar{x}} A_c \frac{d\bar{T}}{d\bar{x}} - \frac{hPL^2}{k} \bar{T} = 0$$

If A_c is const: $\bar{T}'' - N^2 \bar{T} = 0$ $\quad N^2 = \frac{hPL^2}{k \cdot A_c}$

Solution: $\bar{T} = Ae^{N\bar{x}} + Be^{-N\bar{x}}$ $\Rightarrow$ $\bar{T} = \frac{e^{N(1-\bar{x})} + e^{-N(1-\bar{x})}}{e^N + e^{-N}} = \cosh N(1-\bar{x})$

$$A = \frac{e^{-N}}{e^N + e^{-N}} \qquad B = 1 - A = \frac{e^N}{e^N + e^{-N}}$$

$$\bar{T} = \frac{\cosh N(1-\bar{x})}{\cosh N}$$

All the heat removed from the fin must be transported into the fin at the base by conduction. This gives

$$q = -kA_{c,B} \left.\frac{dT}{dx}\right|_{x=0} = -\frac{kA_{c,B}(T_B - T_\infty)}{L} \cdot \left.\frac{d\bar{T}}{d\bar{x}}\right|_{\bar{x}=0}$$

$$q = \frac{kA_c(T_B - T_\infty)}{L} \cdot N \cdot \tanh N = \sqrt{hPkA_c} \cdot (T_B - T_\infty) \tanh N$$

$$N^2 = \frac{hPL^2}{k \cdot A_c} \qquad \tanh N \rightarrow 1 \text{ for } N >> 1$$

Longer the fin - higher is the heat removal

How long is long?

Fin Equation use $\theta = T - T_\infty$

$$q(x) = q(x+\Delta x) + q_{conv}$$

$$\frac{d}{dx}\left[A(x)\frac{dT}{dx}\right] - \frac{hP(x)}{k}(T-T_\infty) = 0$$

$$\theta(x) = T(x) - T_\infty$$

$$\frac{d}{dx}\left[A(x)\frac{d\theta}{dx}\right] - \frac{hP(x)}{k}\theta = 0$$

Soln $\Rightarrow$ $\theta(x) = C_1 e^{-mx} + C_2 e^{mx}$ $\qquad$ $\theta(x) = C_3 \sinh mx + C_4 \cosh mx$

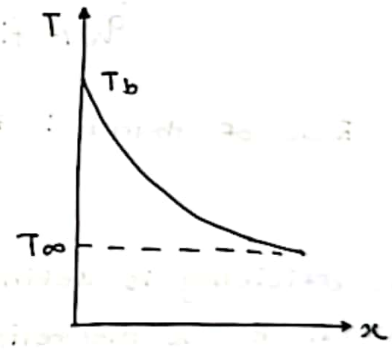
BC at Fin Base $\Rightarrow$ $T(x)\Big|_{x=0} = T_b$ or $\theta(x)\Big|_{x=0} = T_b - T_\infty = \theta_b$

Case I $\Rightarrow$ The extended surface under consideration can be very long

$$T(x)\Big|_{x\to\infty} = T_\infty \quad \text{or} \quad \theta(x)\Big|_{x\to\infty} = 0$$

Solution $\Rightarrow$ $C_1 = \theta_b$ & $C_2 = 0$

$$\theta(x) = \theta_b e^{-mx} \qquad \frac{T-T_\infty}{T_b - T_\infty} = e^{-mx}$$



The rate of heat loss from the extended surface can be determined by integrating the local convective heat transfer over the whole length:

$$q_f = \int_0^\infty hP\,dx\,(T-T_\infty) = hP\int_0^\infty \theta(x)\,dx = hP\theta_b\int_0^\infty e^{-mx}\,dx$$

$$= \frac{hP\theta_b}{m} = \sqrt{hPkA}\cdot\theta_b$$

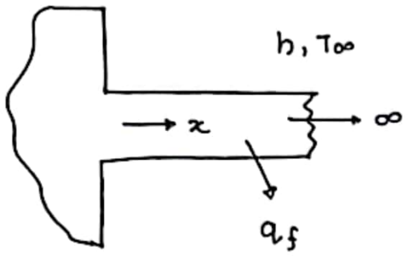
Under steady-state conditions the heat transferred from the extended surface by convection to the surrounding fluid must be equal to the heat conducted to the extended surface at the base

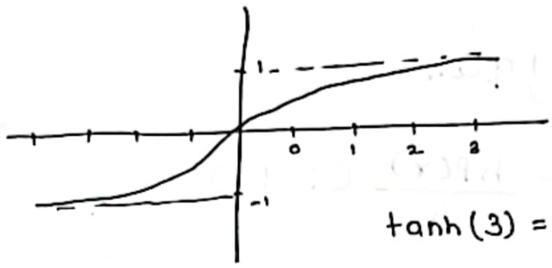
$$q_f = -kA\left(\frac{dT}{dx}\right)_{x=0} = -kA\left(\frac{d\theta}{dx}\right)_{x=0}$$

$$= -kA\theta_b \frac{d\,e^{-mx}}{dx}\Big|_{x=0}$$

$$= kA\theta_b m = \sqrt{hPkA}\;\theta_b$$

Fins : Uniform Cross Section: Heat Removal Long fin





$\tanh(3) = 0.995$

Consequently a fin with $N>3$ is essentially infinite in length. Adding additional length to the fin (& thus increasing N) will not significantly increase the heat transfer from the fin.

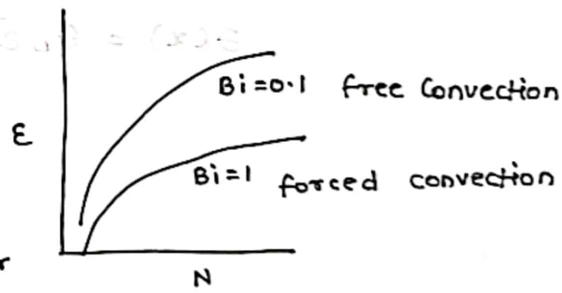
from a design viewpoint Rule of thumb : $N > 2$ to 2.5

PERFORMANCE MEASUREMENT : FIN EFFECTIVENESS

fin effectiveness (Const Ac):

$$\epsilon = \frac{\dot{q}_{fin}}{\dot{q}_{w/o\ fin}} = \frac{\sqrt{hPkA_c}\,(T_B - T_\infty)\tanh N}{hA_c(T_B - T_\infty)} = \frac{N}{B_i}\tanh N$$

Rule of thumb : Fin is justified if $\epsilon > 2$



The efficiency is defined as the ratio of the actual to the theoretical maximum heat transfer from the fin.

for the specific case of the constant ~~area~~ cross section fin, the heat transfer rate:

$$q = \frac{kA_c(T_B - T_\infty)}{L} N\tanh(N) = \sqrt{hPkA_c}\,.\,(T_B - T_\infty)\tanh(N)$$

The maximum heat transfer would occur if the fin was entirely at the temperature of the fin base

$$q_{max} = hA_{fin}(T_B - T_\infty) = hPL(T_B - T_\infty) \quad \text{uniform } A_c$$

for const A_c : $\eta = \frac{\tanh N}{N}$

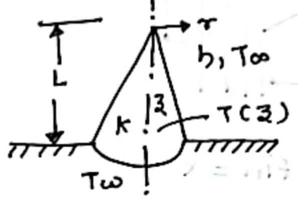
$$N^2 = \frac{hPL^2}{kA_c}$$

$$\frac{d}{dx}[I_0(\alpha x)] = \alpha I_1(\alpha x)$$

$$q_f = 1\sqrt{2hkb}\cdot\theta_b\cdot\frac{I_1(2m\sqrt{L})}{I_0(2m\sqrt{L})}$$

Example ⇒

A spine attached to a wall mainted at a uniform temperature T_w has the shape of a circular cone with base radius r_0 & height L. It is exposed to a fluid at a uniform temperature T_∞.



Assume ⇒

- ↳ Constant thermal conductivity k.
- ↳ Constant heat transfer coefficient.
- ↳ the variation of the temperature in the r direction is negligible.

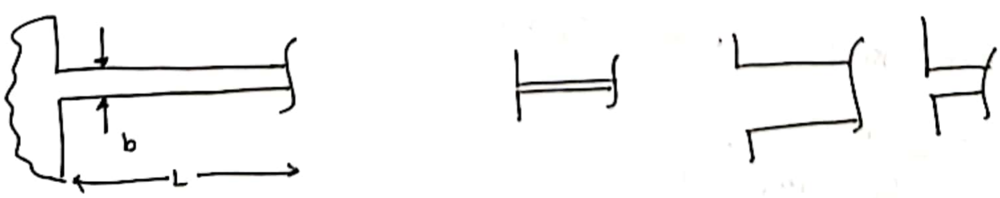
1. Obtain an expression for the steady state temperature distribution $T(z)$ in the spine.
2. Obtain an expression for the rate of heat loss from the spine to the surrounding fluid.

Define a new temperature function by $\theta(z) = T(z) - T_\infty$

$$\frac{d}{dx}\left[A(x)\frac{d\theta}{dx}\right] - \frac{hP(x)}{k}\theta = 0$$

$x \longrightarrow z$

for a given fin shape, fin material, & convection conditions, there exists an optimized design which transfers the maximum amount of heat for a given mass of the fin



Consider adiabatic fin tip

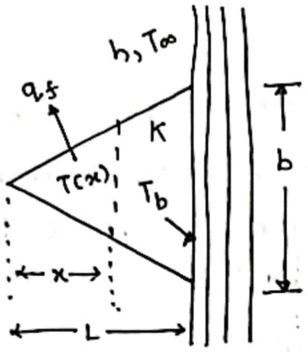
$$q = \sqrt{hPkA_c}\,(T_b - T_\infty)\tanh N \qquad N^2 = \frac{hPL^2}{kA_c}$$

for a long fin $(W \gg b)$ $\quad P = 2W$ & $A_c = bW$ Thus

$$q' = q/W = \sqrt{2bhk}\,(T_b - T_\infty)\tanh N \qquad N^2 = 2hL^2/kb$$

Fins : Non-Uniform Cross Section : Triangular fin

fins of non-uniform cross section can usually transfer more heat for a given mass than those of a const cross-section.



width of fin $= l$

$$\frac{d}{dx}\left[A(x)\frac{dT}{dx}\right] - \frac{hP(x)}{k}(T-T_\infty) = 0$$

Define : $\theta(x) = T(x) - T_\infty$

$$\frac{d}{dx}\left[A(x)\frac{d\theta}{dx}\right] - \frac{hP(x)}{k}\theta = 0$$

$$A(x) = \frac{b}{L}\cdot x \quad \& \quad P(x) = 2\left(\frac{b}{L}x + l\right)$$

If we assume that $b \ll l$,

then $P(x) = 2l$

$$\frac{d}{dx}\left(x\frac{d\theta}{dx}\right) - m^2\theta = 0$$

where $m^2 = \frac{2hL}{kb}$

$$x^2\frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - m^2 x\theta = 0 \qquad \text{Define } \eta = \sqrt{x}$$

$$\eta^2\frac{d^2\theta}{d\eta^2} + \eta\frac{d\theta}{d\eta} - 4m^2\eta^2\theta = 0$$

$$\theta(\eta) = C_1 I_0(2m\eta) + C_2 K_0(2m\eta)$$

$$\theta(x) = C_1 I_0(2m\sqrt{x}) + C_2 K_0(2m\sqrt{x})$$

$$T(0) = \text{finite} \Rightarrow \theta(0) = \text{finite}$$

$$T(L) = T_b \Rightarrow \theta(L) = T_b - T_\infty = \theta_b$$

Since $K_0(0) \longrightarrow \infty$ $\quad C_2 = 0$

$$C_1 = \frac{\theta_b}{I_0(2m\sqrt{L})}$$

$$\frac{\theta(x)}{\theta_b} = \frac{T(x)-T_\infty}{T_b - T_\infty} = \frac{I_0(2m\sqrt{x})}{I_0(2m\sqrt{L})}$$

$$q_f = kA\left(\frac{dT}{dx}\right)_{x=L} = kA\left(\frac{d\theta}{dx}\right)_{x=L}$$

Fin Optimization : Rectangular Fin

The length L can be eliminated using $A_p = bL$. The formula for N, becomes

$$N^2 = \frac{2hA_p^2}{kb^3} \Rightarrow b = \left(\frac{2hA_p^2}{kN^2}\right)^{1/3}$$

$$q' = (4h^2 k A_p)^{1/3} \cdot (T_b - T_\infty) N^{-1/3} \tanh N$$

$$f(N) = N^{-1/3} \cdot \tanh N \qquad \text{set } \frac{df}{dN} = 0 \Rightarrow \cosh N \cdot \sinh N - 3N = 0 \rightarrow \text{solve for } N$$

$$N_{opt} = 1.419 = \left(\frac{2hA_p^2}{k b_{opt}^3}\right)^{1/2}$$

Once A_p is fixed, used the above formula to obtain b_{opt}, & find L from $L = \frac{A_p}{b_{opt}}$.

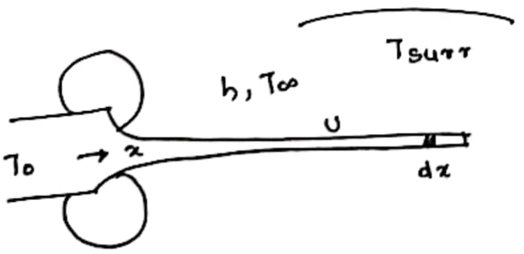
Rate of heat transfer $\Rightarrow$ $q'_{opt} = \left(\frac{4h^2 k A_p}{N_{opt}}\right)^{1/3} (T_b - T_\infty) \tanh N_{opt}$

$$= 1.256\,(h^2 k A_p)^{1/3} (T_b - T_\infty)$$

$$f(N) = N^{-1/3} \cdot \tanh N \qquad N_{opt} = 1.419 = \left(\frac{2hA_p^2}{k b_{opt}^3}\right)^{1/2}$$

For triangular fins $\Rightarrow$ $q'_{opt} = 1.422 \cdot (h^2 k A_p)^{1/3} \cdot (T_b - T_\infty)$

Moving Fin : Example



There are applications where a material exchanges heat with the surrounding while moving through a furnace or a channel.

Examples $\Rightarrow$

↳ The extrusion of plastics

↳ Drawing of wires & sheets

Such problems can be modeled as moving fins as long as the criteron for fin approximation is statisfied.

↳ The Figure shows a sheet being drawn with velocity U through rollers.

↳ The sheet exchangers heat with the surroundings by radiation.

↳ It also exchanges heat with an ambient fluid by convection.

↳ Thus its temperature varies with distance from the rollers. find T(x).

$$\frac{d^2T}{dx^2} - \frac{\rho C_p U}{k} \cdot \frac{dT}{dx} - \frac{hP}{k \cdot A_c}(T - T_\infty) - \frac{\varepsilon \sigma P}{k \cdot A_c}(T^4 - T_{surr}^4) = 0$$

$\varepsilon \rightarrow$ emissivity.

$\sigma \rightarrow$ Stefan-boltzmann const.

$P \rightarrow$ Perimeter.

$U \rightarrow$ velocity.