# ANS

## (Practice Problem)

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① $\frac{dC_b}{dt} = (\mu - F_d)C_b \rightarrow f_1$

$\frac{dC_s}{dt} = F_d(C_{sf} - C_s) - \frac{\mu C_b}{P} - f_2$

$$\mu = \frac{\mu_m C_s}{(k_m + C_s + k_1 C_s^2)}$$

Sol$^n$

(a) Linear state space model

$x_1 = C_b - C_b^s \qquad u = F_d - F_d^s$

$x_2 = C_s - C_s^s$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} C_b - C_b^s \\ C_s - C_s^s \end{bmatrix} \quad \& \quad y = [F_d - F_d^s]$$

$\frac{-u\,dv + v\,du}{v^2}$

$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$

$$\gamma = \left.\frac{\partial \mu}{\partial C_s}\right|_s = \frac{-2\mu_m C_s k_1 + \mu_m(k_m + C_s + k_1 C_s^2)}{(k_m + C_s + k_1 C_s^2)^2}$$

$$\gamma = \frac{+\mu_m k_m + \mu_m C_s - \mu_m C_s^2 k_1}{(k_m + C_s + k_1 C_s^2)^2}$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \mu^s - F_d^s & C_b^s \gamma \\ \frac{-\mu^s}{P} & -F_d^s - \frac{\gamma C_b^s}{P} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} \\ \frac{\partial f_2}{\partial u_1} \end{bmatrix} = \begin{bmatrix} -C_b^s \\ C_{sf} - C_s^s \end{bmatrix}$$

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At. S.S $\quad \frac{dC_p}{dt} = 0 \Rightarrow \mu^s = F_d^s$

$\frac{dC_s}{dt} = 0 \Rightarrow \cancel{d}(C_{sf} - C_s) - \frac{C_b}{P} =$

$C_{sf} - C_s = \frac{C_b}{P}$

$$A = \begin{bmatrix} 0 & C_b^s \gamma \\ -\frac{\mu^s}{P} & -\mu^s - \frac{\gamma C_b^s}{P} \end{bmatrix}$$

$$B = \begin{bmatrix} -C_b^s \\ \frac{C_b}{P} \end{bmatrix} = C_b^s \begin{bmatrix} -1 \\ +\frac{1}{P} \end{bmatrix}$$

(b) not fully State Controllable.

$$C_B = [B \quad AB]$$

$$= \begin{bmatrix} -C_b^s & +C_b^s \gamma \frac{C_b^s}{P} \\ \frac{C_b^s}{P} & \cancel{+ \frac{C_b^s \mu^s}{P} - \frac{\mu^s C_b^s}{P}} - \frac{\gamma (C_b^s)^2}{P^2} \end{bmatrix}$$

$$= \det \begin{vmatrix} -C_b^s & +\frac{(C_b^s)^2 \gamma}{P} \\ \frac{C_b^s}{P} & -\frac{\gamma (C_b^s)^2}{P^2} \end{vmatrix}$$

$$= \left| +\frac{(C_b^s)^3 \gamma}{P^2} - \frac{(C_b^s)^3 \gamma}{P^2} \right| = 0$$

$\because \det(C_B) = 0$

$\therefore$ Rank $(C_B) < 2$, not Controllable

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(2) $\frac{dx_1}{dt} = -x_1 + x_2 + u_1$ (test Q2)

$\frac{dx_2}{dt} = 2x_2 + u_2$

$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$y = x_1$

Sol^n (a) system open loop Stable.

$A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$ $|SI - A| = \begin{vmatrix} S+1 & -1 \\ 0 & S-2 \end{vmatrix} = 0$

$(S+1)(S-2) = 0$

$S = (-1, 2)$

for Stable, both $\lambda$ must be -ve

$\Rightarrow$ Thus not stable.

(b)

Choosing $u_1$ $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$C_B = [B \;\; AB] = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, Rank $(C_B) = 1$

$\therefore$ syst^m is not controllable

choosing $u_2$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$C_B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$, Rank $(C_B) = 2$

$\Rightarrow$ Controllable

(c) full order observer Eq$^n$

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \& \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

A full order Obs. Eq$^n$:

Cheat Data

$$\frac{d\bar{x}}{dt} = (A - LC)\bar{x} + BU + Ly$$

$$\frac{d\bar{x}}{dt} = \left( \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \bar{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U_2 + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} y$$

$$\frac{d\bar{x}}{dt} = \begin{bmatrix} (-L_1 - 1) & 1 \\ -L_2 & 2 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U_2 + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} y$$

Eq$^n$

$$\frac{d\bar{x}_1}{dt} = -(L_1 + 1)\bar{x}_1 + \bar{x}_2 + L_1 y$$

$$\frac{d\bar{x}_2}{dt} = -L_2 \bar{x}_1 + 2\bar{x}_2 + U_2 + L_2 y$$

To find observer gain,

$$|sI - A + LC| = (s+2)(s+2)$$

$$\left| \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} + \underset{2\times 1}{\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}} \underset{1\times 2}{\begin{pmatrix} 1, 0 \end{pmatrix}} \right| = s^2 + 4s + 4$$

$$\begin{matrix} L_1 & 0 \\ L_2 & 0 \end{matrix}$$

$$\begin{vmatrix} s + L_1 + 1 & -1 \\ L_2 & s - 2 \end{vmatrix} = s^2 + 4s + 4$$

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$(s+l_1+1)(s-2)+l_2 = s^2+4s+4$

$s^2+(l_1-1)s-2(l_1+1)+l_2 = s^2+4s+4$

Comparing

$l_1-1=4$ $\qquad$ $l_2-2(l_1+1)=4$

$l_1=5$ $\qquad$ $l_2-12=4$

$l_2=16$

Thus, full order Obs. Eqn.

$$\frac{d\bar{x}_1}{dt} = -6\bar{x}_1+\bar{x}_2+5y$$

$$\frac{d\bar{x}_2}{dt} = -16\bar{x}_1+2\bar{x}_2+u_2+16y$$

(3)

$\dot{x} = AX+BU$

$y = CX$

Show that $O^TA(O^T)^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$

Soln WKT

$O^T = [C^T \; A^TC^T \; \cdots \; (A^{n-1})^TC^T]$

Assuming state Eqn is in Control Canonical

$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$

$(O^T)^{-1} = \frac{Adj(O^T)}{|O^T|}$

Post mult $(O^T)^{-1}$ both side

$$O^T A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} O^T$$

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LHS

$$O^T A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} O^T \quad \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

=

LHS

$$O^T A = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}^T_{1\times 3} A_{3\times 3} = \begin{bmatrix} CA \\ CA^2 \\ CA^3 \end{bmatrix}^T$$

RHS

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}_{3\times 3} \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}^T_{3\times 1} = \begin{bmatrix} CA \\ CA^2 \\ -a_3 C - a_2 CA - a_1 CA^2 \end{bmatrix}^T$$

Now by caley Hamitton them.

$$A^3 + a_1 A^2 + a_2 A + a_3 = 0$$

$$CA^3 + a_1 CA^2 + a_2 CA + a_3 C = 0$$

$$CA^3 = -a_3 C - a_2 CA - a_1 CA^2 \quad \text{--- (1)}$$

Thus RHS will be come.

$$= \begin{bmatrix} CA \\ CA^2 \\ CA^3 \end{bmatrix} = \text{LHS}$$

(5) $\frac{dC_A}{dt} = \frac{F}{V}(C_{Af} - C_A) - k_1 C_A - k_3 C_A^2$

$\frac{dC_B}{dt} = -\frac{F}{V} C_B + k_1 C_A - k_2 C_B$

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(4) $k_1, k_2, k_3$ & $C_{Af}$, $V$

Sol^n (a) SS val of $C_A$ & $C_B$ .. for $F = 60$.

$\frac{dC_A}{dt} = 0 = 60(10 - C_A) - 50 C_A - 10 C_A^2 = 0$

$600 - 110 C_A - 10 C_A^2 = 0$

$\Rightarrow \boxed{C_A^2 + 11 C_A - 60 = 0}$

$(C_A + 15)(C_A - 4) = 0$

$C_A = 4$

||ly $C_B = 1.25$

(b) Linear State Space Model (SSM),

$\begin{pmatrix} -190 & 0 \\ 50 & -160 \end{pmatrix}$

$A = \begin{bmatrix} \frac{\partial f_1}{\partial C_A} & \frac{\partial f_1}{\partial C_B} \\ \frac{\partial f_2}{\partial C_A} & \frac{\partial f_2}{\partial C_B} \end{bmatrix} = \begin{bmatrix} -\frac{F}{V} - k_1 - 2k_3 C_A^s & 0 \\ k_1 & -\frac{F}{V} - k_2 \end{bmatrix}$

$B = \begin{bmatrix} \frac{\partial f_1}{\partial F} \\ \frac{\partial f_2}{\partial F} \end{bmatrix} = \begin{bmatrix} \frac{C_{Af}^s - C_A^s}{V} \\ -\frac{C_B^s}{V} \end{bmatrix}$

$y = cx$

$C = [0 \;\; 1]$ $C_B$ is controlled

(c) $C_B = [B \;\; AB]$

Rank (2) → controllable.

$|sI - A| = \begin{vmatrix} s+190 & 0 \\ -50 & s+160 \end{vmatrix} = s^2 + \underbrace{350}_{a_1} s + \underbrace{30400}_{a_2}$

Poles

$(s+190)(s+170) = s^2 + \underbrace{360}_{\alpha_1} s + \underbrace{32300}_{\alpha_2}$ ($\mu_1$, $\mu_2$)

$K = [1900 \quad 10]$

$\alpha_n - a_n$

$\alpha_2 - a_2 \qquad \alpha_1 - a_1$

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(4) $\frac{dx_1}{dt} = a_{11} x_1 + a_{12} x_2$

$\frac{dx_2}{dt} = a_{21} x_1 + a_{22} x_2 + bU$

$y = x_2$

Sol^n

(a) reduced order State observer & Dynamic Eq

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ b \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$

To derive

RO observer Eqn : $\dot{\hat{x}} = (A_{bb} - L A_{ab}) \hat{x} + [(A_{bb} - L A_{ab}) L + A_{ba} - L A_{ab}] y + (B - LB_a) V$

$= \hat{X}_b - Ly$

error dynamic

$\dot{x}_b - \dot{\hat{x}}_b = E = (A_{bb} - L A_{ab}) E$

$\dot{x} = AX + BU$

$y = CX$

$\begin{bmatrix} \dot{x}_a \\ \dot{x}_b \end{bmatrix} = \begin{bmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{bmatrix}$ state Eq^n.

$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_a \\ x_b \end{bmatrix}$

Eq^n for measured portion.

$\dot{x}_a = A_{aa} x_a + A_{ab} x_b + B_a U$

$A_{ab} X_b = \underbrace{\dot{X}_a - A_{aa} x_a - B_a U}_{(a)}$ output Eq^n for RO

Eqⁿ for unmeasured portion.

$$\dot{x}_b = A_{ba}x_a + A_{bb}x_b + B_bU$$

$A_{ba}x_a$ & $B_bU \rightarrow$ are known

Eqⁿ for full order obser.

$$\dot{\bar{x}} = (A - K_eC)\bar{x} + BU + K_ey$$

making proper substitution from table

$$\dot{\bar{x}}_b = (A_{bb} - K_eA_{ab})\bar{x}_b + A_{ba}x_a + B_bV + K_e(\dot{x}_a - A_{aa}x_a - B_aU)$$

we. $(y = x_a)$

$$\dot{\bar{x}}_b - K_e\dot{x}_a = (A_{bb} - K_eA_{ab})\bar{x}_b + (A_{ba} - K_eA_{aa})y + (B_b - K_eB_a)U$$

$$= (A_{bb} - K_eA_{ab})\bar{x}_b - (A_{bb} - K_eA_{ab})K_ey + (A_{bb} - K_eA_{ab})K_e + (A_{ba} - K_eA_{aa})y + (B_b - K_eB_a)U$$

$$= (A_{bb} - K_eA_{ab})(\bar{x}_b - K_ey) + [(A_{bb} - K_eA_{ab})K_e + A_{ba} - K_eA_{aa}]y + (B_b - K_eB_a)U$$

Now take $(x_b - K_ey) = \eta = (x_b - K_ex_a)$

$$(\bar{x}_b - K_ey) = \hat{\eta} = (\bar{x}_b - K_e\bar{x}_a)$$

Thus,

reduced order observer.

$$\dot{\hat{\eta}} = (A_{bb} - K_eA_{ab})\hat{\eta} + [(A_{bb} - K_eA_{ab})K_e + A_{ba} - K_eA_{aa}]y + (B_b - K_eB_a)U$$

Now

$$\dot{x}_b = A_{bb}x_b + (A_{ba}x_a + B_aU)$$

$$\dot{\bar{x}}_b = (A_{bb} - K_eA_{ab})\bar{x}_b + (A_{ba}x_a + B_bU) + K_eA_{ab}x_b$$

$$\dot{x}_b - \dot{\bar{x}}_b = (A_{bb} - K_eA_{ab})\underbrace{(x_b - \bar{x}_b)}_{E}$$

$$\Rightarrow \boxed{\dot{E} = (A_{bb} - K_eA_{ab})E}$$ dynamic error