Q2. Consider the following system **[5]**

$$\frac{dx_1}{dt} = -x_1 + x_2 + u_1$$
$$\frac{dx_2}{dt} = 2x_2 + u_2$$
$$y = x_1$$

a) Is the system open loop stable?

Soln

$A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$ $\quad calculate\ |sI - A|$ to get poles or calculate $|\lambda I - A|$ to get eigenvalues of the system. The Pole / eigenvalue locations are [-1, 2]. So the system is unstable.

b) In order to control y using a state feedback controller, we have to choose 1 control variable out of 2 control variables u_1 and u_2. Which control variable should be chosen and why? Give your answer with mathematical justification.

Soln

Choosing u_1 as control variable, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; Controllability matrix $C_B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ Rank (C_B) =1. So, the system is not controllable with u_1 as control variable

Choosing u_2 as control variable, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; Controllability matrix $C_B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ Rank (C_B) =2. So, the system is controllable with u_2 as control variable

c) Based on your choice of input variable, write full order observer equation for the system and calculate observer gain for the desired observer pole location at [-2,-2].

Soln

So, $A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $C = [1 \quad 0]$

Full order observer equation is

$$\frac{d\hat{x}}{dt} = (A - LC)\hat{x} + Bu + Ly$$

$$\frac{d\hat{x}}{dt} = \left(\begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}[1 \quad 0]\right)\hat{X} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u_2 + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}y$$

$$\frac{d\hat{x}}{dt} = \begin{bmatrix} -L_1 - 1 & 1 \\ -L_2 & 2 \end{bmatrix}\hat{X} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u_2 + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}y$$

The full order observer equation is

$$\frac{d\hat{x}_1}{dt} = -(L_1 + 1)\,\hat{x}_1 + \hat{x}_2 + L_1 y$$

$$\frac{d\hat{x}_2}{dt} = -L_2\,\hat{x}_1 + 2\hat{x}_2 + u_2 + L_2 y$$

To find observer gain, $\quad |sI - A + LC| = (s+2)(s+2)$

$$\begin{vmatrix} s + L_1 + 1 & -1 \\ L_2 & s - 2 \end{vmatrix} = s^2 + 4s + 4$$

$$(s + L_1 + 1)(s - 2) + L_2 = s^2 + 4s + 4$$

$s^2 + (L_1 - 1)s + L_2 - 2(L_1 + 1) = s^2 + 4s + 4$

Or $L_1 - 1 = 4$ $i.e$ $L_1 = 5$
and $L_2 - 2(L_1 + 1) = 4$ i.e, $L_2 = 16$

The full order observer equation is

$$\frac{d\hat{x}_1}{dt} = -6\,\hat{x}_1 + \hat{x}_2 + 5y$$

$$\frac{d\hat{x}_2}{dt} = -16\,\hat{x}_1 + 2\hat{x}_2 + u_2 + 16y$$