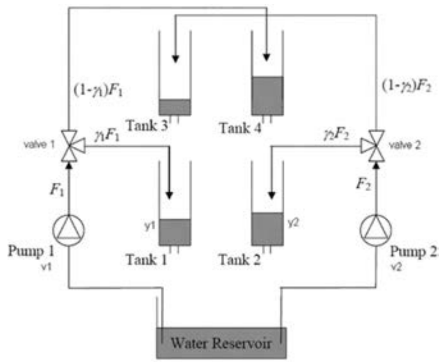
Tutorial problem

Prob 1. Consider the quadruple tank system where levels of tank1 and tank2 are manipulated by voltages supplied to the pumps.

1. Derive Dynamic model
2. Derive Nonlinear state space model in vector-matrix form
3. Derive Linear state space model in vector-matrix form
4. Compute state transition matrix using the following data



Data: A_1, A_3 = 28 cm^2 , A_2, A_4 = 32 cm^2, a_1, a_3 = 0.071 cm^2, a_2, a_4 = 0.057 cm^2 , k_1, k_2 = 3.33, 3.35, v_1, v_2 = 3.0, 3.0, γ_1, γ_2 = 0.7, 0.6

Solution

Dynamic model:

Overall mass balance for Tank3 during Δt

$$A_3 h_3 \rho|_{t+\Delta t} - A_3 h_3 \rho|_t = k_2 v_2 (1-\gamma_2)\rho \Delta t - a_3\sqrt{2gh_3}\,\rho\Delta t$$

Cancelling ρ from both sides and dividing by Δt

$$A_3\frac{dh_3}{dt} = k_2 v_2 (1-\gamma_2) - a_3\sqrt{2gh_3}$$

$$\frac{dh_3}{dt} = \frac{k_2 v_2 (1-\gamma_2)}{A_3} - \frac{a_3}{A_3}\sqrt{2gh_3} = f_3(h_1, h_2, h_3, h_4, v_1, v_2)$$

Similarly, for tank4,

$$\frac{dh_4}{dt} = \frac{k_1 v_1 (1-\gamma_1)}{A_4} - \frac{a_4}{A_4}\sqrt{2gh_4} = f_4(h_1, h_2, h_3, h_4, v_1, v_2)$$

For tank 2:

$$\frac{dh_2}{dt} = \frac{k_2 v_2 \gamma_2}{A_2} + \frac{a_4}{A_2}\sqrt{2gh_4} - \frac{a_2}{A_2}\sqrt{2gh_2} = f_2(h_1, h_2, h_3, h_4, v_1, v_2)$$

For tank 1:

$$\frac{dh_1}{dt} = \frac{k_1 v_1 \gamma_1}{A_1} + \frac{a_3}{A_1}\sqrt{2gh_3} - \frac{a_1}{A_1}\sqrt{2gh_1} = f_1(h_1, h_2, h_3, h_4, v_1, v_2)$$

Nonlinear state space model:

$$\frac{dh}{dt}=\begin{bmatrix}\frac{dh_1}{dt}\\ \frac{dh_2}{dt}\\ \frac{dh_3}{dt}\\ \frac{dh_4}{dt}\end{bmatrix}=\begin{bmatrix}-\frac{a_1}{A_1}\sqrt{2gh_1}+\frac{a_3}{A_1}\sqrt{2gh_3}\\ -\frac{a_2}{A_2}\sqrt{2gh_2}+\frac{a_4}{A_2}\sqrt{2gh_4}\\ -\frac{a_3}{A_3}\sqrt{2gh_3}\\ -\frac{a_4}{A_4}\sqrt{2gh_4}\end{bmatrix}\begin{bmatrix}h_1\\ h_2\\ h_3\\ h_4\end{bmatrix}+\begin{bmatrix}\frac{k_1\gamma_1}{A_1} & 0\\ 0 & \frac{k_2\gamma_2}{A_2}\\ 0 & \frac{k_2(1-\gamma_2)}{A_3}\\ \frac{k_1(1-\gamma_1)}{A_4} & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}$$

i.e, $\frac{dh}{dt}=f(h)+g(h)u$

and $y=\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix}\begin{bmatrix}h_1\\ h_2\\ h_3\\ h_4\end{bmatrix}$

Linear State Space model

$$\frac{dX}{dt}=AX+BU$$
$$Y=CX+DU$$

Here, $X=\begin{bmatrix}X_1\\ X_2\\ X_3\\ X_4\end{bmatrix}=\begin{bmatrix}h_1-h_1^s\\ h_2-h_2^s\\ h_3-h_3^s\\ h_4-h_4^s\end{bmatrix}$ $\quad Y=\begin{bmatrix}Y_1\\ Y_2\end{bmatrix}=\begin{bmatrix}X_1\\ X_2\end{bmatrix}$ $\quad U=\begin{bmatrix}U_1\\ U_2\end{bmatrix}=\begin{bmatrix}v_1-v_1^s\\ v_2-v_2^s\end{bmatrix}$

$$A=\begin{bmatrix}\left.\frac{\partial f_1}{\partial h_1}\right|_s & \left.\frac{\partial f_1}{\partial h_2}\right|_s & \left.\frac{\partial f_1}{\partial h_3}\right|_s & \left.\frac{\partial f_1}{\partial h_4}\right|_s\\ \left.\frac{\partial f_2}{\partial h_1}\right|_s & \left.\frac{\partial f_2}{\partial h_2}\right|_s & \left.\frac{\partial f_2}{\partial h_3}\right|_s & \left.\frac{\partial f_2}{\partial h_4}\right|_s\\ \left.\frac{\partial f_3}{\partial h_1}\right|_s & \left.\frac{\partial f_3}{\partial h_2}\right|_s & \left.\frac{\partial f_3}{\partial h_3}\right|_s & \left.\frac{\partial f_3}{\partial h_4}\right|_s\\ \left.\frac{\partial f_4}{\partial h_1}\right|_s & \left.\frac{\partial f_4}{\partial h_2}\right|_s & \left.\frac{\partial f_4}{\partial h_3}\right|_s & \left.\frac{\partial f_4}{\partial h_4}\right|_s\end{bmatrix}\quad B=\begin{bmatrix}\left.\frac{\partial f_1}{\partial v_1}\right|_s & \left.\frac{\partial f_1}{\partial v_2}\right|_s\\ \left.\frac{\partial f_2}{\partial v_1}\right|_s & \left.\frac{\partial f_2}{\partial v_2}\right|_s\\ \left.\frac{\partial f_3}{\partial v_1}\right|_s & \left.\frac{\partial f_3}{\partial v_2}\right|_s\\ \left.\frac{\partial f_4}{\partial v_1}\right|_s & \left.\frac{\partial f_4}{\partial v_2}\right|_s\end{bmatrix}\quad C=\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix}$$

Or, $A=\begin{bmatrix}-\frac{1}{T_1} & 0 & \frac{A_3}{A_1T_3} & 0\\ 0 & -1/T_2 & 0 & \frac{A_4}{A_2T_4}\\ 0 & 0 & -1/T_3 & 0\\ 0 & 0 & 0 & -1/T_4\end{bmatrix}$ where $T_i=\frac{A_i}{a_i}\sqrt{\frac{2h_i^s}{g}}$ for i=1..4

and

$$B = \begin{bmatrix} \frac{k_1\gamma_1}{A_1} & 0 \\ 0 & \frac{k_2\gamma_2}{A_2} \\ 0 & \frac{k_2(1-\gamma_2)}{A_3} \\ \frac{k_1(1-\gamma_1)}{A_4} & 0 \end{bmatrix}$$

Calculation of steady state values:

$$a_4\sqrt{2gh_4} = (1-\gamma_1)k_1v_1 \quad i.e, h_4 = 1.41$$

Similarly, $h_1 = 12.26, h_2 = 12.78, h_3 = 1.63$

Based on steady state values, the state-space model in vector matrix form

$$\frac{dX}{dt} = \begin{bmatrix} -0.016 & 0 & 0.044 & 0 \\ 0 & -0.011 & 0 & 0.033 \\ 0 & 0 & -0.044 & 0 \\ 0 & 0 & 0 & -0.033 \end{bmatrix} X + \begin{bmatrix} 0.083 & 0 \\ 0 & 0.063 \\ 0 & 0.048 \\ 0.031 & 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} X + [0]U$$

Calculation of state transition matrix

$$\phi(t) = e^{At} = L^{-1}[(SI-A)^{-1}]$$

$$SI - A = \begin{bmatrix} s+0.016 & 0 & -0.044 & 0 \\ 0 & s+0.011 & 0 & -0.033 \\ 0 & 0 & s+0.044 & 0 \\ 0 & 0 & 0 & s+0.033 \end{bmatrix}$$

Computing $(SI-A)^{-1}$ by gaussian elimination

$$\left\langle \begin{array}{cccc|cccc} s+0.016 & 0 & -0.044 & 0 & 1 & 0 & 0 & 0 \\ 0 & s+0.011 & 0 & -0.033 & 0 & 1 & 0 & 0 \\ 0 & 0 & s+0.044 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s+0.033 & 0 & 0 & 0 & 1 \end{array} \right\rangle$$

Perform the following row manipulation

a. Divide ith row by its ith element.
b. Multiply 3rd row by $\frac{0.044}{s+0.016}$ and add with 1st row.
c. Multiply 4th row by $\frac{0.033}{s+0.011}$ and add with 2nd row.
d. Divide 3rd row by $\frac{0.044}{s+0.016}$
e. Divide 4th row by $\frac{0.033}{s+0.011}$

We get,

$$\left\langle \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \middle| \begin{matrix} \frac{1}{s+0.016} & 0 & \frac{0.044}{(s+0.016)(s+0.044)} & 0 \\ 0 & \frac{1}{s+0.011} & 0 & \frac{0.033}{(s+0.011)(s+0.033)} \\ 0 & 0 & \frac{1}{s+0.044} & 0 \\ 0 & 0 & 0 & \frac{1}{s+0.033} \end{matrix} \right\rangle$$

Therefore,

$$(SI-A)^{-1} = \begin{bmatrix} \frac{1}{s+0.016} & 0 & \frac{0.044}{(s+0.016)(s+0.044)} & 0 \\ 0 & \frac{1}{s+0.011} & 0 & \frac{0.033}{(s+0.011)(s+0.033)} \\ 0 & 0 & \frac{1}{s+0.044} & 0 \\ 0 & 0 & 0 & \frac{1}{s+0.033} \end{bmatrix}$$

$$\phi(t) = e^{At} = L^{-1}[(SI-A)^{-1}]$$

$$= \begin{bmatrix} e^{-0.016t} & 0 & 1.57(e^{-0.016t} - e^{-0.044t}) & 0 \\ 0 & e^{-0.011t} & 0 & 1.5(e^{-0.011t} - e^{-0.033t}) \\ 0 & 0 & e^{-0.044t} & 0 \\ 0 & 0 & 0 & e^{-0.033t} \end{bmatrix}$$

Prob 2. Consider the following dynamic model of a reactor

$$\frac{dC_A}{dt} = 10 - C_A - 3.5 \times 10^7 \exp\left(-\frac{6000}{T}\right) C_A$$
$$\frac{dT}{dt} = 298 - 1.3T + 4.2 \times 10^8 \exp\left(-\frac{6000}{T}\right) C_A + 0.3T_c$$

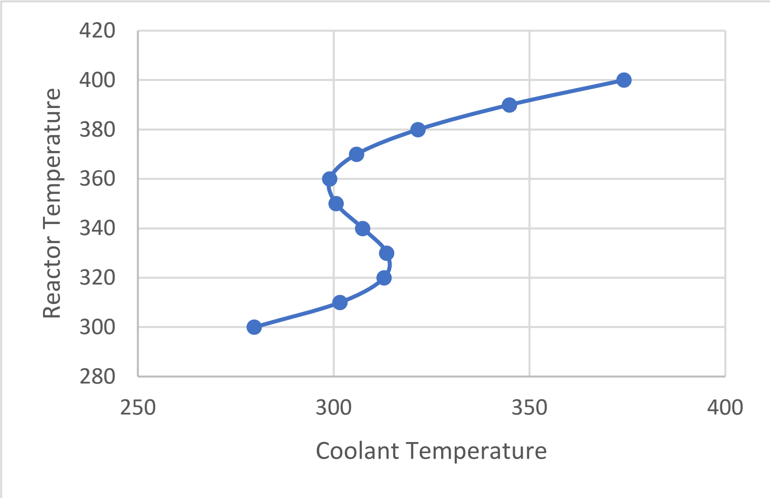
The control objective is to control T by manipulating T_c.

1. Plot steady state input(T_c)-output(T) curve for T ranging 300 to 400 K.
2. Derive linear state space model for steady state T = 320 K
3. Compute state transition matrix
4. Derive the expression for dynamic response of T for unit step change in T_c.

Solution

$$\frac{dC_A}{dt} = 10 - C_A - 3.5 \times 10^7 \exp\left(-\frac{6000}{T}\right) C_A = 0$$
$$C_A = \frac{10}{1 + 3.5 \times 10^7 \exp\left(-\frac{6000}{T}\right)}$$
$$\frac{dT}{dt} = 298 - 1.3T + 4.2 \times 10^8 \exp\left(-\frac{6000}{T}\right) C_A + 0.3T_c = 0$$
$$T_c = \frac{1}{0.3}\left(1.3T - \frac{4.2 \times 10^9 \exp\left(-\frac{6000}{T}\right)}{1 + 3.5 \times 10^7 \exp\left(-\frac{6000}{T}\right)} - 298\right)$$

T_C	T	C_A
279.75	300	9.33
301.64	310	8.79
312.87	320	7.99
313.59	330	6.92
307.45	340	5.69
300.63	350	4.43
299.02	360	3.31
305.86	370	2.4
321.57	380	1.71
344.93	390	1.21
374.17	400	0.85



Linear State Space model

$$C_A^s = 8.0, T^s = 320, T_c^s = 312.9$$

$$\frac{dC_A}{dt} = 10 - C_A - 3.5 \times 10^7 \exp\left(-\frac{6000}{T}\right) C_A = f_1(C_A, T, T_c)$$

$$\frac{dT}{dt} = 298 - 1.3T + 4.2 \times 10^8 \exp\left(-\frac{6000}{T}\right) C_A + 0.3T_c$$

$$= f_2(C_A, T, T_c)$$

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} C_A - C_A^s \\ T - T^s \end{bmatrix}; \quad U = T_c - T_c^s \quad Y = X_2$$

$$\left.\frac{\partial f_1}{\partial C_A}\right|_s = -1 - 3.5 \times 10^7 \exp\left(-\frac{6000}{T^s}\right) = -1.25$$

$$\left.\frac{\partial f_1}{\partial T}\right|_s = -3.5 \times 10^7 \times \frac{6000}{\mathrm{T}^{s2}} \times \exp\left(-\frac{6000}{T^s}\right) \times C_A^s = -0.12$$

$$\left.\frac{\partial f_2}{\partial C_A}\right|_s = 4.2 \times 10^8 \exp\left(-\frac{6000}{T^s}\right) = 3.02$$

$$\left.\frac{\partial f_2}{\partial T}\right|_s = -1.3 + 4.2 \times 10^8 \times \frac{6000}{\mathrm{T}^{s2}} \times \exp\left(-\frac{6000}{T^s}\right) \times C_A^s = 0.114$$

$$\left.\frac{\partial f_1}{\partial T_c}\right|_s = 0 \quad and \; \left.\frac{\partial f_2}{\partial T_c}\right|_s = 0.3$$

$$\frac{dX}{dt} = \begin{bmatrix} \left.\frac{\partial f_1}{\partial C_A}\right|_s & \left.\frac{\partial f_1}{\partial T}\right|_s \\ \left.\frac{\partial f_2}{\partial C_A}\right|_s & \left.\frac{\partial f_2}{\partial T}\right|_s \end{bmatrix} X + \begin{bmatrix} \left.\frac{\partial f_1}{\partial T_c}\right|_s \\ \left.\frac{\partial f_2}{\partial T_c}\right|_s \end{bmatrix} U = \begin{bmatrix} -1.25 & -0.12 \\ 3.02 & 0.114 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0.3 \end{bmatrix} U$$

$$Y = [0 \quad 1]X + [0]U$$

State transition matrix

$$\phi(t) = e^{At}$$

Eigenvalues of A: $|\lambda I - A| = 0$ or, $\begin{bmatrix} \lambda + 1.25 & 0.12 \\ -3.02 & \lambda - 0.114 \end{bmatrix} = 0$

$$(\lambda + 1.25)(\lambda - 0.114) + 3.02 * 0.12 = \lambda^2 + 1.136\lambda + 0.22 = 0$$

So, $\lambda_1 = 0.89 \;\; and \; \lambda_2 = 0.25$

So, $$e^{0.89t} = a_0 + 0.89\, a_1$$
$$e^{0.25t} = a_0 + 0.25 a_1$$
Solving, $a_1 = 1.56e^{0.89t} - 1.56e^{0.25t}$ $\quad a_0 = 1.39e^{0.25t} - 0.39e^{0.89t}$
$$e^{At} = a_0 I + a_1 A$$

$$= \begin{bmatrix} 1.39e^{0.25t} - 0.39e^{0.89t} & 0 \\ 0 & 1.39e^{0.25t} - 0.39e^{0.89t} \end{bmatrix} + (1.56e^{0.89t} - 1.56e^{0.25t}) \begin{bmatrix} -1.25 & -0.12 \\ 3.02 & 0.114 \end{bmatrix}$$
$$= \begin{bmatrix} 1.39e^{0.25t} - 0.39e^{0.89t} & 0 \\ 0 & 1.39e^{0.25t} - 0.39e^{0.89t} \end{bmatrix} + \begin{bmatrix} 1.95e^{0.25t} - 1.95e^{0.89t} & 0.19e^{0.25t} - 0.19e^{0.89t} \\ 4.71e^{0.89t} - 4.71e^{0.25t} & 0.18e^{0.89t} - 0.18e^{0.25t} \end{bmatrix}$$

$$\phi(t) = e^{At} = \begin{bmatrix} 3.34e^{0.25t} - 2.34e^{0.89t} & 0.19e^{0.25t} - 0.19e^{0.89t} \\ 4.71e^{0.89t} - 4.71e^{0.25t} & 1.21e^{0.25t} - 0.21e^{0.89t} \end{bmatrix}$$

dynamic response of T for unit step change in T_c.

$$X(t) = X(0) + \int_0^t \phi(t-\tau) d\tau\, B$$
$$Y(t) = CX(t) = CX(0) + \int_0^t C\, \phi(t-\tau)\, B\, dt$$
$$Y(t) = Y(0) + \int_0^t (1.21e^{0.25(t-\tau)} - 0.21e^{0.89(t-\tau)})\, d\tau = 4.84e^{0.25t} - 0.236e^{0.89t} - 4.6$$

Prob 3. Consider the following dynamic model of a bioreactor

$$\frac{dc_1}{dt} = \frac{0.5c_1c_2}{0.1 + c_2} - uc_1$$
$$\frac{dc_2}{dt} = 4u - uc_2 - \frac{1.25c_1c_2}{0.1 + c_2}$$

1. Find the optimum value of u to maximize rate of cell production per unit reactor volume, uc_1.
2. Derive linear state space model using the optimum operating condition obtained in (1).
3. Compute state transition matrix using results of (2).

Solution

$$\frac{dc_1}{dt} = \frac{0.5c_1c_2}{0.1 + c_2} - uc_1 = 0 \;\; gives \;\; c_2 = \frac{0.1u}{0.5 - u}$$

$$\frac{dc_2}{dt} = 4u - uc_2 - \frac{1.25c_1c_2}{0.1 + c_2} = 0 \;\; gives \;\; c_1 = \frac{4}{2.5} - \frac{\frac{0.1}{2.5}}{0.5 - u}$$
$$2.5\, uc_1 = 4u - \frac{0.1u}{0.5 - u}$$
$$\frac{d(2.5\, uc_1)}{du} = 4 - \frac{0.05}{(0.5 - u)^2} = 0 \;\; gives \;\; u = 0.5 \pm \frac{\sqrt{0.05}}{2} = 0.5 \pm 0.11$$

For maximizing uc_1,

$$\frac{d^2(2.5uc_1)}{du^2} = \frac{0.1}{(u-0.5)^3} < 0\ for\ u < 0.5\ \ i.e, u = 0.39$$

So optimum value of u=0.39 and optimum c_1=1.2364 ≈ 1.24 c_2 = 0.3545 $\approx$ 0.35

linear state space model

$$\frac{dc_1}{dt} = \frac{0.5c_1c_2}{0.1+c_2} - uc_1 = f_1(c_1, c_2, u)$$

$$\frac{dc_2}{dt} = 4u - uc_2 - \frac{1.25c_1c_2}{0.1+c_2} = f_2(c_1, c_2, u)$$

Consider, $X_1 = c_1 - c_1^s \quad X_2 = c_2 - c_2^s \quad Y = X_1\ and\ U = u - u^s$

$$\frac{\partial f_1}{\partial c_1} = \frac{0.5c_2^s}{0.1+c_2^s} - u^s = 0.0 \qquad \frac{\partial f_1}{\partial c_2} = \frac{0.05c_1^s}{(0.1+c_2^s)^2} = 0.3062\ \approx 0.31$$

$$\frac{\partial f_2}{\partial c_1} = \frac{-1.25c_2^s}{0.1+c_2^s} = -2.5 * u^s \approx -0.97 \qquad \frac{\partial f_2}{\partial c_2} = -u^s - \frac{0.125c_1^s}{(0.1+c_2)^2} = -1.1554 \approx -1.15$$

$$\frac{\partial f_1}{\partial u} = -c_1^s = -1.24 \qquad \frac{\partial f_2}{\partial u} = 4 - c_2^s = 3.65$$

So, state space model can be written as

$$\frac{dX}{dt} = \begin{bmatrix} 0 & 0.31 \\ -0.97 & -1.15 \end{bmatrix} X + \begin{bmatrix} -1.24 \\ 3.65 \end{bmatrix} U$$

$$Y = [1 \quad 0]X + [0]U$$

State transition matrix

Eigenvalue calculation:

$$|\lambda I - A| = 0 \quad or, \quad \begin{vmatrix} \lambda & -0.31 \\ 0.97 & \lambda + 1.15 \end{vmatrix} = 0 \quad or, \lambda^2 + 1.15\lambda + 0.3 = 0$$

So, $\lambda_1 = -0.75\ \ and\ \lambda_2 = -0.4$

$$e^{-0.75t} = a_0 - 0.75a_1$$
$$e^{-0.4t} = a_0 - 0.4a_1$$

Solving we get, $a_1 = \frac{1}{0.35}(e^{-0.4t} - e^{-0.75t})$ and $a_0 = \frac{0.75}{0.35}e^{-0.4t} - \frac{0.4}{0.35}e^{-0.75t}$

$$e^{At} = a_0 I + a_1 A$$

$$= \begin{bmatrix} \frac{0.75}{0.35}e^{-0.4t} - \frac{0.4}{0.35}e^{-0.75t} & 0 \\ 0 & \frac{0.75}{0.35}e^{-0.4t} - \frac{0.4}{0.35}e^{-0.75t} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \frac{0.31}{0.35}(e^{-0.4t} - e^{-0.75t}) \\ \frac{-0.97}{0.35}(e^{-0.4t} - e^{-0.75t}) & \frac{-1.15}{0.35}(e^{-0.4t} - e^{-0.75t}) \end{bmatrix}$$

$$e^{At}=\begin{bmatrix}\frac{0.75}{0.35}e^{-0.4t}-\frac{0.4}{0.35}e^{-0.75t} & \frac{0.31}{0.35}\left(e^{-0.4t}-e^{-0.75t}\right)\\ \frac{-0.97}{0.35}\left(e^{-0.4t}-e^{-0.75t}\right) & \frac{-0.4}{0.35}e^{-0.4t}+\frac{0.75}{0.35}e^{-0.75t}\end{bmatrix}$$

Prob 4. Consider the following dynamic model of van-de-vusse reactor

$$\frac{dC_A}{dt}=10u-\left(u+\frac{5}{6}\right)C_A-\frac{1}{6}C_A^2$$

$$\frac{dC_B}{dt}=\frac{5}{6}C_A-\left(\frac{5}{3}+u\right)C_B$$

1. Find the optimum value of u to maximize production of B (i.e, C_B).
2. Derive linear state space model using steady state value of u=2.
3. Compute state transition matrix using results of (2).

Solution

Optimum value of u to maximize production of B (i.e, C_B)

$$\frac{dC_A}{dt}=10u-\left(u+\frac{5}{6}\right)C_A-\frac{1}{6}C_A^2=0$$

$$\frac{dC_B}{dt}=\frac{5}{6}C_A-\left(\frac{5}{3}+u\right)C_B=0$$

Solving above equations,

$$C_A=-\frac{6u+5}{2}+\frac{\sqrt{36u^2+300u+25}}{2}$$

$$C_B=\frac{\frac{5}{6}C_A}{\frac{5}{3}+u}=\frac{-\frac{30u+25}{12}+\frac{5\sqrt{36u^2+300u+25}}{12}}{\frac{5}{3}+u}$$

$$for\ C_B\ to\ be\ maximum,\frac{dC_B}{du}=0$$

$$\frac{dC_B}{du}=\frac{\left(\frac{d}{du}\left(\frac{-30u+25}{12}+\frac{5\sqrt{36u^2+300u+25}}{12}\right)\left(\frac{5}{3}+u\right)-\left(-\frac{30u+25}{12}+\frac{5\sqrt{36u^2+300u+25}}{12}\right)\right)}{\left(\frac{5}{3}+u\right)^2}$$

$$=\frac{\left(-\frac{5}{2}+\frac{5(72u+300)}{24\sqrt{36u^2+300u+25}}\right)\left(\frac{5}{3}+u\right)-\left(-\frac{30u+25}{12}+\frac{5\sqrt{36u^2+300u+25}}{12}\right)}{\left(\frac{5}{3}+u\right)^2}=0$$

After simplification ***(you must do these steps)*** $u=1.29$

linear state space model using steady state value of u=2.

$$C_A^s = 1.22, \qquad C_B^s = 5.37$$

$$A = \begin{bmatrix} -u^s - \frac{5}{6} - \frac{2}{6}C_A^s & 0 \\ \frac{5}{6} & -\frac{5}{3} - u^s \end{bmatrix}$$

$$B = \begin{bmatrix} 10 - C_A^s \\ -C_B^s \end{bmatrix} \qquad C = [0 \quad 1]$$

$$SI - A = \begin{bmatrix} s + u^s + \frac{5}{6} + \frac{2}{6}C_A^s & 0 \\ -\frac{5}{6} & s + \frac{5}{3} + u^s \end{bmatrix}$$

$$(SI - A)^{-1} = \left(\frac{1}{det}\right)\begin{bmatrix} s + \frac{5}{3} + u^s & 0 \\ \frac{5}{6} & s + u^s + \frac{5}{6} + \frac{2}{6}C_A^s \end{bmatrix} \;; \; det = \left(s + \frac{5}{3} + u^s\right)\left(s + u^s + \frac{5}{6} + \frac{2}{6}C_A^s\right)$$

Rest of the problem should be done by you

Prob 5. The model of a chemical process

s gives the following Process Transfer function.

$$G(s) = \frac{180}{(s+2)(s+3)(s+4)(s+5)}$$

a) Find an equivalent first order with dead time (FODT) model using moment method.
b) Find an equivalent second order with dead time model having equal time constants $\frac{K\,e^{-\theta s}}{(\tau s+1)^2}$ using moment method.

Solution

a) G(s)= $\frac{K\,e^{-\theta s}}{(\tau s+1)}$ **or** $logG(s) = \log K - \theta s - log(\tau s + 1)$

$$\frac{dlogG(s)}{ds} = \frac{G'(s)}{G(s)} = -\theta - \frac{\tau}{\tau s + 1}$$

$$\frac{G'(0)}{G(0)} = -\theta - \tau = -(\theta + \tau) = -\tau_{ar}$$

$$\frac{d^2logG(s)}{ds^2} = \frac{G''(s)}{G(s)} - \left(\frac{G'(s)}{G(s)}\right)^2 = \frac{\tau^2}{(\tau s + 1)^2}$$

$$\frac{G''(0)}{G(0)} - \tau_{ar}^2 = \tau^2$$

$$G(s) = \frac{180}{(s+2)(s+3)(s+4)(s+5)} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+4} + \frac{D}{s+5}$$

$$A = \left.\frac{180}{(s+3)(s+4)(s+5)}\right|_{s=-2} = \frac{180}{6} = 30$$

$$B = \left.\frac{180}{(s+2)(s+4)(s+5)}\right|_{s=-3} = -\frac{180}{2} = -90$$

$$C = \left.\frac{180}{(s+2)(s+3)(s+5)}\right|_{s=-4} = \frac{180}{2} = 90$$

$$A = \left.\frac{180}{(s+2)(s+3)(s+4)}\right|_{s=-5} = -\frac{180}{6} = -30$$

$$G(s) = \frac{180}{(s+2)(s+3)(s+4)(s+5)} = \frac{30}{s+2} - \frac{90}{s+3} + \frac{90}{s+4} - \frac{30}{s+5}$$

$$G(0) = \frac{30}{2} - \frac{90}{3} + \frac{90}{4} - \frac{30}{5} = 1.5$$

$$\frac{dG(s)}{ds} = G'(s) = \frac{-30}{(s+2)^2} + \frac{90}{(s+3)^2} - \frac{90}{(s+4)^2} + \frac{30}{(s+5)^2}$$

$$G'(0) = -\frac{30}{4} + \frac{90}{9} - \frac{90}{16} + \frac{30}{25} = -1.925$$

$$\frac{d^2G(s)}{ds^2} = G''(s) = \frac{60}{(s+2)^3} - \frac{180}{(s+3)^3} + \frac{180}{(s+4)^3} - \frac{60}{(s+5)^3}$$

$$G''(0) = 60/8 - 180/27 + 180/64 - 60/125 = 3.166$$

$$\tau_{ar} = -\frac{G'(0)}{G(0)} = \frac{1.925}{1.5} = 1.283$$

$$\tau^2 = \frac{G''(0)}{G(0)} - \tau_{ar}^2 = \frac{3.166}{1.5} - (1.283)^2 = 0.464 \;\; so, \tau = 0.681 \; and \; \theta = 0.6$$

b) G(s)= $\frac{K\,e^{-\theta s}}{(\tau s+1)^2}$ **or** $logG(s) = \log K - \theta s - 2log(\tau s + 1)$

$$\frac{dlogG(s)}{ds} = \frac{G'(s)}{G(s)} = -\theta - \frac{2\tau}{\tau s + 1}$$

$$\frac{G'(0)}{G(0)} = -\theta - 2\tau = -(\theta + 2\tau) = -\tau_{ar}$$

$$\frac{d^2logG(s)}{ds^2} = \frac{G''(s)}{G(s)} - \left(\frac{G'(s)}{G(s)}\right)^2 = \frac{2\tau^2}{(\tau s + 1)^2}$$

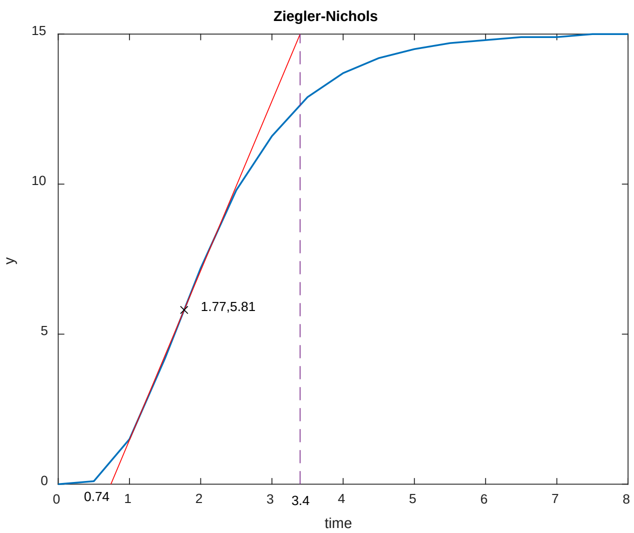
$$\frac{G''(0)}{G(0)} - \tau_{ar}^2 = 2\tau^2$$

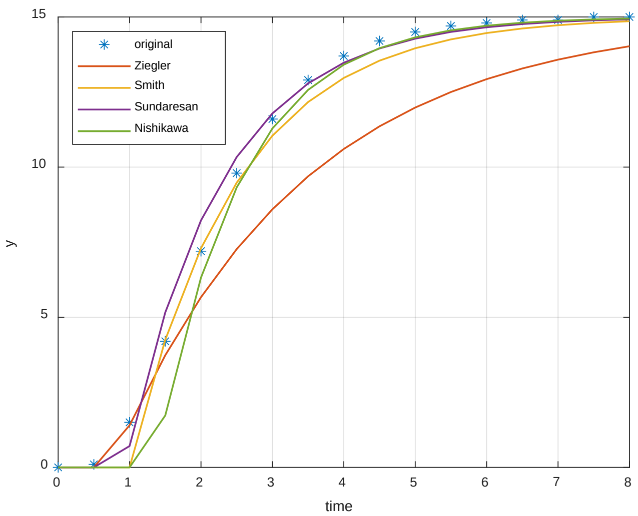
$$So, 2\tau^2 = 0.464 \; or \; \tau = 0.482 \; and \; \theta = \tau_{ar} - 2 * \tau = 1.283 - 2 * 0.482 = 0.32$$

Prob 6. The following data is generated from a process unit by giving unit step change in the input at time t=0. The first row is time (t) in min and the second row is change in output.

0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0
0	0.1	1.5	4.2	7.2	9.8	11.6	12.9	13.7	14.2	14.5	14.7	14.8	14.9	14.9	15.0	15.0

1. Find first order with dead time (FODT) model using a) Ziegler-Nichols b) Smith's method c) Sundaresan method d) Nishikawa method
2. Compare the Mean Absolute Prediction Error(MAPE)





Inflection point calculated at $\frac{d^2y}{dt^2} = 0$

Method	Time constant	Time delay	MAPE
Ziegler-Nichols	2.66	0.74	21.07
Smith	1.5	1	15.04
Sundaresan	1.34	0.935	12.97
Nishikawa	1.18	1.35	18.09

Prob 7. Consider the transfer function $G(s) = \frac{120(s+6)(s+7)}{(s+2)(s+3)(s+4)(s+5)}$

Derive state space model in

a) Controllable canonical form
b) Jordon canonical form
c) Observable canonical form

Solution Taught in the class. Do yourself