Practice Sheet

Prob 2: $\frac{dx_1}{dt} = -x_1 + x_2 + u_1$

$\frac{dx_2}{dt} = 2x_2 + u_2$

$y = x_1$

Sol^n: Linear state space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{U}$$

(a) To check for stability of the open loop system, we need to look into the eigen value of 'A'

$|\lambda I - A| = 0$

$\begin{vmatrix} \lambda+1 & -1 \\ 0 & \lambda-2 \end{vmatrix} = 0$

$(\lambda+1)(\lambda-2) = 0$

$\lambda_1 = -1, \lambda_2 = 2$

Since, one of the ~~roots~~ eigen value is positive.

$\therefore$ open loop system is unstable

(b) In order to choose the control variable, the Controllable Matrix must be a fully controllable for the given unstable open loop system.

Case-I $u_1 \equiv$ Controll variable

$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$

$C_B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$\therefore \det(C_B) = 0 \rightarrow \text{Rank}(C_B) \neq 2$

Case-2 $u_2 \equiv$ control variable

$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ & $A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$

$C_B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

$\det(C_B) \neq 0 \quad \therefore \text{Rank}(C_B) = 2$

$\therefore$ $u_2 \equiv$ Control variable for our given system

$$\therefore \begin{bmatrix} \dot{x}_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2$$

Prob 3 :

$$\dot{X} = AX + BU$$

$$y = CX$$

The linear state space model is in completely observable system →

$$A = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \quad \& \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_1 \\ 1 & -a_1 & -a_2 + a_1^2 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -a_3 & a_1 a_3 \\ 0 & -a_2 & -a_3 + \\ 1 & -a_1 & -a_2 + \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 1 & -a_1 & -a_2 + a_1^2 \end{bmatrix}$$

$$O^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_1 \\ 1 & -a_1 & -a_2 + a_1^2 \end{bmatrix} \qquad (O^T)^{-1} = \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$O^T A (O^T)^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_1 \\ 1 & -a_1 & -a_2 + a_1^2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_1 \\ 1 & -a_1 & -a_2 + a_1^2 \end{bmatrix} \begin{bmatrix} -a_3 & 0 & 0 \\ 0 & a_1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

Prob 4 :

$$\frac{dx_1}{dt} = a_{11} x_1 + a_{12} x_2$$

$$\frac{dx_2}{dt} = a_{21} x_1 + a_{22} x_2 + bu \qquad y = x_2$$

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 \qquad y = x_2$$
$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + bu$$

$\boxed{\dot{x}_2 - a_{22}x_2 - bu = a_{21}x_1}$ & $\boxed{\dot{x}_1 = a_{11}x_1 + a_{12}x_2}$

$\boxed{A \equiv a_{11}, \quad BU \equiv a_{12}x_2 \quad X \equiv x_1}$ $\boxed{y \equiv \dot{x}_2 - a_{22}x_2 - bu}$

$\boxed{C \equiv a_{21}}$ Generally $\rightarrow \dot{\hat{x}} = (A - LC)\hat{x} + BU + LY$

$$\dot{\hat{x}}_1 = (a_{11} - la_{21})\hat{x}_1 + a_{12}x_2 + l(\dot{x}_2 - a_{22}x_2 - bu)$$

$$\dot{\hat{x}}_1 - l\dot{x}_2 = (a_{11} - la_{21})(\hat{x}_1 - lx_2) + (a_{11} - la_{21})lx_2 + a_{12}x_2 - la_{22}x_2 - lbu$$

Let $\hat{x}_1 - lx_2 = \eta$

$$\therefore \dot{\hat{\eta}} = (a_{11} - la_{21})\hat{\eta} + [(a_{11} - la_{21})l + a_{12} - la_{22}]x_2 - lbu$$

$$\boxed{\dot{\hat{\eta}} = (a_{11} - la_{21})\hat{\eta} + [(a_{11} - la_{21})l + a_{12} - la_{22}]y - lbu}$$

Reduced order observer eq^n

$e = \eta - \hat{\eta}$ when $\eta = x_1 - Lx_2 = x_1 - Ly$
$\hat{\eta} = \hat{x}_1 - Lx_2 = \hat{x}_1 - Ly \rightarrow$ $l\ a_{21}\ x_1$ ↑ ↑ ↑ LCX $\boxed{LY}$

$\therefore e = x_1 - \hat{x}_1$

$\dot{x}_1 = a_{11}x_1 + a_{12}x_2$ | $\dot{\hat{x}}_1 = (a_{11} - la_{21})\hat{x}_1 + a_{12}x_2 + \overline{la_{21}x_1}$

$$\dot{e} = \dot{x}_1 - \dot{\hat{x}}_1 = a_{11}x_1 + \cancel{a_{12}}x_1 - (a_{11} - la_{21})\hat{x} - \cancel{a_{12}}x_2 - la_{21}x_1$$

$$\dot{e} = (a_{11}x_1 - a_{11}\hat{x} + la_{21}\hat{x} - la_{21}x$$

$$\boxed{\dot{e} = (a_{11} - la_{21})e}$$ error dynamic eq^n (i)

Prob 5 :

$$\frac{dC_A}{dt} = \frac{F}{V}(C_{Af} - C_A) - K_1 C_A - K_3 C_A^2$$

$$\frac{dC_B}{dt} = -\frac{F}{V} C_B + K_1 C_A - K_2 C_B$$

Data: $K_1 = 50$, $K_2 = 100$, $K_3 = 10$, $C_{Af} = 10$, $V = 1$

(a) steady state values of C_A & C_B from $F_s = 60$

$$0 = \frac{F_s}{V}(C_{Af} - C_{As}) - K_1 C_{As} - K_3 C_{As}^2$$

$$1\cancel{0} C_{As}^2 + 5\cancel{0} C_{As} - 6\cancel{0}(10 - C_{As}) = 0$$

$$C_{As}^2 + 11 C_{As} - 60 = 0$$

$$(C_{As} + 15)(C_{As} - 4) = 0$$

$$C_{As} = +4 \text{ or } -15$$

* C_{As} must be positive, since it's concentration & conc^n cannot be negative quantity.

$$\therefore \boxed{C_{As} = 4}$$

$$0 = -\frac{F_s}{V} C_{Bs} + K_1 C_{As} - K_2 C_{Bs}$$

$$0 = -60 C_{Bs} + 50 \times 4 - 100 C_{Bs}$$

$$C_{Bs} = \frac{20\cancel{0}}{16\cancel{0}} = \frac{5}{4} = 1.25$$

(b) Let, $x_1 = C_A - C_{As}$ | $x_2 = C_B - C_{Bs}$ | $u = F - F_s$

$$\frac{dC_A}{dt} = \frac{F}{V}(C_{Af} - C_A) - K_1 C_A - K_3 C_A^2 = f_1(C_A, F)$$

$$\frac{dC_B}{dt} = -\frac{F}{V} C_B + K_1 C_A - K_2 C_B = f_2(C_A, C_B, F)$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial C_A} & \frac{\partial f_1}{\partial C_B} \\ \frac{\partial f_2}{\partial C_A} & \frac{\partial f_2}{\partial C_B} \end{bmatrix} = \begin{bmatrix} -\frac{F_s}{V} - K_1 - 2K_3 C_{As} & 0 \\ K_1 & -\frac{F_s}{V} - K_2 \end{bmatrix}$$

$$A = \begin{bmatrix} -190 & 0 \\ 50 & -160 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial F} \\ \frac{\partial f_2}{\partial F} \end{bmatrix} = \begin{bmatrix} \frac{C_{Af}-C_A}{V} \\ -\frac{C_{Bs}}{V} \end{bmatrix} = \begin{bmatrix} 6 \\ -1.25 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\therefore \dot{X} = AX + BU \quad \& \quad Y = CX$

(c) $\dot{X} = AX + BU$

$U = -KX$

for finding the controller gain 'k' for state feedback controller using Bass-Gura-method $\longrightarrow$ $C_B = [B \;\; AB] = \begin{bmatrix} 6 & -1140 \\ -1.25 & 500 \end{bmatrix}$ $\det(C_B) \neq 0$ $\therefore \text{Rank}(C_B) = 2$

$|SI - A| = (S+190)(S+160) = S^2 + 350S + 30400$ $\quad a_1 = 350$, $a_2 = 30400$

Desire pole location: $[\mu_1 = -190, \; \mu_2 = -170$

$(S-\mu_1)(S-\mu_2) = (S+190)(S+170) = S^2 + 360S + 32300$ $\quad \alpha_1 = 360$, $\alpha_2 = 32300$

$$W = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 350 & 1 \\ 1 & 0 \end{bmatrix} \qquad C_B = \begin{bmatrix} 6 & -1140 \\ -1.25 & 500 \end{bmatrix}$$

$$T = C_B W = \begin{bmatrix} 960 & 6 \\ 62.5 & -1.25 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} \frac{1}{1260} & \frac{2}{525} \\ \frac{5}{126} & -\frac{64}{105} \end{bmatrix}$$

$$K = [\alpha_2 - a_2 \quad \alpha_1 - a_1] T^{-1} = [1900 \quad 10] \begin{bmatrix} \frac{1}{1260} & \frac{2}{525} \\ \frac{5}{126} & -\frac{64}{105} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{40}{21} & \frac{8}{7} \end{bmatrix} = [1.9047 \quad 1.1428]$$

$\therefore$ state feed back controller –

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -190 & 0 \\ 50 & -160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ -1.25 \end{bmatrix} U$$

$$U = -\begin{bmatrix} 1.9047 & 1.1428 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(d) The full order observer eq^n for the given system →

$$\dot{\hat{X}} = (A - LC)\hat{X} + BU + LY$$

When L → observer Gain,

Now for calculating the observer Gain 'L', using Ackermann's method, we get →

$$L = \phi(A) \begin{bmatrix} C \\ CA \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For $\phi(A)$ →

$$\phi(s) = (s - \mu_1)(s - \mu_2) \quad | \quad \mu_1 = -180 \ \& \ \mu_2 = -175$$
$$= (s+180)(s+175)$$
$$= s^2 + 355s + 31500$$

$$\phi(A) = A^2 + 355A + 31500I \rightarrow \begin{bmatrix} 36100 & 0 \\ -17500 & 25600 \end{bmatrix} + \begin{bmatrix} -67450 & 0 \\ 17750 & -56 \end{bmatrix} + \begin{bmatrix} 31500 & 0 \\ 0 & 315 \end{bmatrix}$$

Putting the value of A we get,

$$\phi(A) = \begin{bmatrix} 150 & 0 \\ 250 & 300 \end{bmatrix}$$

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 50 & -160 \end{bmatrix},$$

$$\therefore L = \begin{bmatrix} 150 & 0 \\ 250 & 300 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 50 & -160 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$\therefore$ Observer gain $(L) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

(e) Let for $\rightarrow$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$

$a_{11} = -190 \mid a_{12} = 0$

$y = x_2$ $\qquad a_{21} = 50 \quad a_{22} = -160$

$O_B = \begin{bmatrix} 0 & 1 \\ 50 & -160 \end{bmatrix}$

$\boxed{\text{Rank}(O_B) = 2}$

$\therefore \quad \dot{x}_1 = a_{11} x_1 + a_{12} x_2 + b_1 u$

$\dot{x}_2 = a_{21} x_1 + a_{22} x_2 + b_2 u$

$\therefore \quad \dot{x}_1 = a_{11} x_1 + a_{12} x_2 + b_1 u \quad \text{---} (1)$

$\dot{x}_2 - a_{22} x_2 - b_2 u = a_{21} x_1 \quad \text{---} (2)$

Simply comparing above eq^n with General full order sy

$X = x_1 \qquad Y \equiv \dot{x}_2 - a_{22} x_2 - b_2 u \qquad A \equiv a_{11}$

$C \equiv a_{21} \qquad BU \equiv a_{12} x_2 + b_1 u$

$\therefore$ General full order observer eq^n

$$\dot{\hat{X}} = (A - LC)\hat{X} + BU + LY$$

$\therefore$ Reduced observer eq^n becomes.

$$\dot{\hat{x}}_1 = (a_{11} - l a_{21})\hat{x}_1 + a_{12} x_2 + b_1 u + l(\dot{x}_2 - a_{22} x_2 - b_2 u)$$

$$\dot{\hat{x}}_1 - l\dot{x}_2 = (a_{11} - l a_{21})(\hat{x}_1 - l x_2) + (a_{11} - l a_{21}) l x_2 + a_{12} x_2 + b_1 u - l a_{22} x_2 - l b_2 u$$

$$\dot{\hat{x}}_1 - l\dot{x}_2 = (a_{11} - l a_{21})(\hat{x}_1 - l x_2) + \left((a_{11} - l a_{21}) l - l a_{22} + a_{12}\right) \underset{Y}{x_2} + (b_1 - l b_2) u$$

$$\hat{\eta} = \hat{x}_1 - l x_2 = \hat{x}_1 - l y$$

$$\therefore \boxed{\dot{\hat{\eta}} = (a_{11} - l a_{21})\hat{\eta} + [(a_{11} - l a_{21}) l + a_{12} - l a_{22}] y + (b_1 - l b_2) u}$$

putting the value, we get,

$$\boxed{\dot{\hat{\eta}} = (-190 - 50l)\hat{\eta} + [(-190 - 50l) l + 160 l] y + (8 + 1.25l) u}$$

For finding the value of l,

$$\dot{\hat{x}}_1 = \underbrace{(a_{11} - l a_{21})\hat{x}_1}_{(A - LC)\hat{x}_1} + \overbrace{a_{12} x_2 + b_1 u}^{BU} + \overbrace{l a_{21} x_1}^{LY}$$

$$\dot{x}_1 = a_{11} x_1 + a_{12} x_2 + b_1 u$$

$$e = \hat{\eta} - \hat{\eta} = x_1 - \cancel{l x_2} - \hat{x}_1 + \cancel{l x_2}$$

$$= x_1 - \hat{x}_1$$

$$\dot{e} = a_{11} x_1 - l a_{21} x_1 - (a_{11} - l a_{21})\hat{x}_1$$

$$= (a_{11} - l a_{21})(x_1 - \hat{x}_1) = (a_{11} - l a_{21}) e$$

$$(s - a_{11} + l a_{21}) e(s) = 0$$

$$(s - a_{11} + l a_{21}) = (s - \mu_1)$$

$$\boxed{l = \frac{a_{11} - \mu_1}{a_{21}}} = \frac{-190 - (-180)}{50}$$

$$l = -\frac{1}{5} = -0.2$$

$$(s - a_{11} + l a_{21})$$

$$= s + 180$$

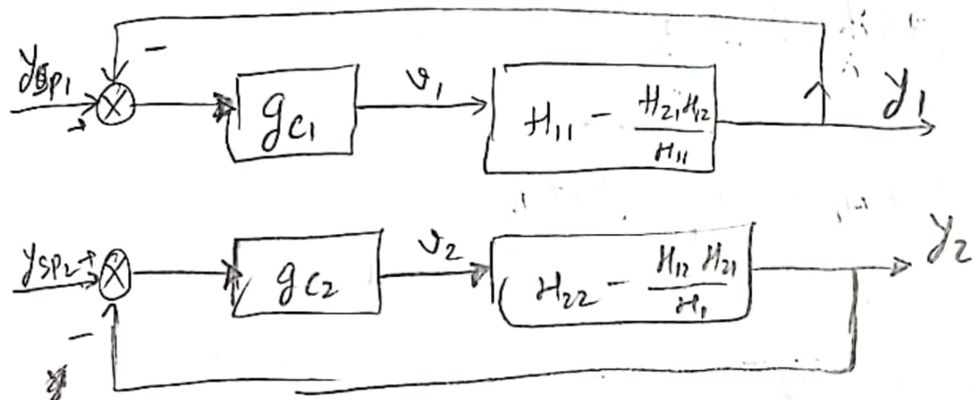
$$180 = -a_{11} + l a_{21}$$

$$180 + a_{11} = l a_{21}$$

$$l = \frac{-10}{50} = -\frac{1}{5}$$

$$y_1 = \left(H_{11} - \frac{H_{21} H_{12}}{H_{22}}\right) m_1 + \cancel{\left(-H_{12} + H_{12}\right) m_2}$$

$$y_2 = \left(H_{22} - \frac{H_{12} H_{21}}{H_{11}}\right) m_2$$



Practice sheet

Problem-6 : Regulator System —

$$\frac{dx_1}{dt} = x_2 \quad ; \quad \frac{dx_2}{dt} = 20.6\, x_1 + u \quad ; \quad y = x_1$$

State-space model →

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad \rightarrow \quad \dot{X} = AX + BU$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \rightarrow \quad y = CX$$

let,

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

State feedback controller gain using Bass gura method:

$$\therefore |sI - A| = \begin{vmatrix} s & -1 \\ -20.6 & s \end{vmatrix} = s^2 - 20.6$$

$$a_1 = 0, \quad a_2 = -20.6$$

$$(s-\mu_1)(s-\mu_2) = s^2 - (\mu_1+\mu_2)s + \mu_1\mu_2 \quad \alpha_1 = -(\mu_1+\mu_2)$$

$$\alpha_2 = \mu_1\mu_2$$

$$W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad ; \quad C_B = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \text{rank}(C_B) = 2$$

$\det(C_B) \neq 0$

$$T = C_B W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$T^{-1} = I$$

$$\therefore \; K = \begin{bmatrix} \alpha_2 - a_2 & \alpha_1 - a_1 \end{bmatrix} T^{-1}$$

$$= \begin{bmatrix} \mu_1 \mu_2 + 20.6 & -(\mu_1 + \mu_2) \end{bmatrix}_{1\times 2} \underset{0}{I}$$

$$K_1 = \begin{bmatrix} \mu_1 \mu_2 + 20.6 & -(\mu_1 + \mu_2) \end{bmatrix}$$

∴ For optimal state feed back gain, we need to minimize the performance Index, $J = \int_0^{\infty} (x^T x + u^2)\,dt$

From general ⇒ $J = \int_0^{t_f} (x^T Q x + u^T R u)\,dt$

For our case: $Q = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ & $R = [1]$

$t_f \to \infty$

∴ we can use Algebraic Riccati eqⁿ —

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \quad \& \; K = R^{-1}B^T P$$

where $P = \begin{bmatrix} \overset{a}{p_{11}} & \overset{b}{p_{12}} \\ \overset{c}{p_{21}} & \overset{d}{p_{22}} \end{bmatrix}$

$$\therefore \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$- \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 20.6\,p_{12} & p_{11} \\ 20.6\,p_{22} & p_{21} \end{bmatrix} + \begin{bmatrix} 20.6\,p_{21} & 20.6\,p_{22} \\ p_{11} & p_{12} \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$+\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} 20.6(p_{12}+p_{21})+1 & 20.6p_{22}+p_{11} \\ 20.6p_{22}+p_{11} & p_{21}+p_{12}+1 \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ p_{21} & p_{22} \end{bmatrix} = 0$$

$$\begin{bmatrix} 20.6(p_{12}+p_{21})+1 & 20.6p_{22}+p_{11} \\ 20.6p_{22}+p_{11} & p_{21}+p_{12}+1 \end{bmatrix} - \begin{bmatrix} p_{21}p_{12} & p_{12}p_{22} \\ p_{22}p_{21} & p_{22}^2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 20.6(p_{12}+p_{21})+1-p_{21}p_{12} & 20.6p_{22}+p_{11}-p_{12}p_{22} \\ 20.6p_{22}+p_{11}-p_{22}p_{21} & p_{21}+p_{12}+1-p_{22}^2 \end{bmatrix} = 0$$

$$20.6p_{12} + 20.6p_{21} - p_{21}p_{12} + 1 = 0 \quad --- \; (1)$$

$$20.6p_{22} + p_{11} - p_{12}p_{22} = 0 \quad --- \; (2)$$

$$20.6p_{22} + p_{11} - p_{22}p_{21} = 0 \quad -- \; (3)$$

$$p_{21} + p_{12} - p_{22}^2 + 1 = 0 \quad -- \; (4)$$

from (2) & (3) $\rightarrow$ $p_{22} \neq 0$ & $p_{21} = p_{12} = x$

from eqn (1) $\rightarrow$ $41.2x - x^2 + 1 = 0$

$$x = -0.024,\ 41.224$$

Because of positive definitness of matrix 'P'. $\leftarrow$ $(x \neq -0.024)$, $x = 41.224 \Rightarrow p_{21} = p_{12} = 41.224$

from eqn (4) $\quad p_{22} = \sqrt{2 \times 41.2 + 1} = 9.135$

from eqn (3) $\quad p_{11} = 188.4$

$$P = \begin{bmatrix} 188.4 & 41.224 \\ 41.224 & 9.135 \end{bmatrix}$$

$$K = 1 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 188.4 & 41.224 \\ 41.224 & 9.135 \end{bmatrix}$$

$$K = \begin{bmatrix} 41.224 & 9.135 \end{bmatrix}$$

(c) from (a) we get,

$$K = \begin{bmatrix} \mu_1\mu_2 + 20.6 & (\mu_1 + \mu_2) \end{bmatrix}$$

from (b) we get,

$$K = \begin{bmatrix} 41.224 & 9.135 \end{bmatrix}$$

$$\mu_1\mu_2 = 20.624$$

$$\mu_1 + \mu_2 = 9.135$$

$$(9.135 - \mu_2)\mu_2 = 20.624$$

$$\mu_2 = 5.055, \ 4.08$$

$$\mu_1 = 4.08, \ 5.055$$

$\therefore$ pole location. $[\mu_1 \quad \mu_2] = [4.08 \quad 5.055]$

or

$[5.055 \quad 4.08]$

7 (a) $G(s) = \dfrac{4}{s(s+2)}$

$$\frac{y(s)}{z(s)} \cdot \frac{z(s)}{U(s)} = 4 \cdot \frac{1}{s(s+2)}$$

$$s^2 z(s) + 2s\, z(s) = U(s)$$

$$\frac{d^2 z}{dt^2} + 2\frac{dz}{dt} = u$$

$$z = x_1, \quad \frac{dz}{dt} = x_2$$

$$\therefore \dot{x}_2 = -2x_2 + u$$

$$\dot{x}_1 = x_2$$

$y = 4x_1$

state space model →

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\dot{X} = AX + BU$$

$$y = \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow y = CX$$

(b) For state feed back controller →

$$\dot{X} = AX + BU$$

$$\& \quad U = -KX$$

Check C_B:

$$C_B = \begin{bmatrix} B \mid AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

$\det(C_B) \neq 0$

$\therefore \text{rank}(C_B) = 2$

For calculating 'K' using Ackermann formula –

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} B & AB \end{bmatrix}^{-1} \phi(A)$$

when, $\phi(s) = (s-\mu_1)(s-\mu_2) = s^2 - (\mu_1+\mu_2)s + \mu_1\mu_2$

$$\mu_{1,2} = 2 \pm j\sqrt{2}$$

$$\phi(s) = s^2 - 4s + 6$$

$$\phi(A) = A^2 - 4A + 6I = \begin{bmatrix} 6 & 0 \\ 0 & 18 \end{bmatrix}$$

$$P = \begin{bmatrix} 188.4 & 41.224 \\ 41.224 & 9.135 \end{bmatrix}$$

$$K = 1 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 188.4 & 41.224 \\ 41.224 & 9.135 \end{bmatrix}$$

$$K = \begin{bmatrix} 41.224 & 9.135 \end{bmatrix}$$

(c) from (a) we got,

$$K = \begin{bmatrix} \mu_1 \mu_2 + 20.6 & (\mu_1 + \mu_2) \end{bmatrix}$$

from (b) we get,

$$K = \begin{bmatrix} 41.224 & 9.135 \end{bmatrix}$$

$$\mu_1 \mu_2 = 20.624$$

$$\mu_1 + \mu_2 = 9.135$$

$$(9.135 - \mu_2)\mu_2 = 20.624$$

$$\mu_2 = 5.055 \, , \, 4.08$$

$$\mu_1 = 4.08 \, , \, 5.055$$

$\therefore$ pole location. $\begin{bmatrix} \mu_1 & \mu_2 \end{bmatrix} = \begin{bmatrix} 4.08 & 5.055 \end{bmatrix}$

or

$\begin{bmatrix} 5.055 & 4.08 \end{bmatrix}$

(7) (a) $G(s) = \frac{4}{s(s+2)}$

$$\frac{y(s)}{z(s)} \cdot \frac{z(s)}{U(s)} = 4 \cdot \frac{1}{s(s+2)}$$

$$s^2 z(s) + 2s\, z(s) = U(s)$$

$$\frac{d^2 z}{dt^2} + 2\frac{dz}{dt} = u$$

$$z = x_1 \quad , \frac{dz}{dt} = x_2$$

$$\therefore \dot{x}_2 = -2x_2 + u$$

$$\dot{x}_1 = x_2$$

$$y = 4x_1$$

state space model →

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\dot{X} = AX + BU$$

$$y = \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow y = CX$$

(b) for state feed back controller →

$$\dot{X} = AX + BU$$

$$\& \quad U = -KX$$

Check C_B:

$$C_B = [B \mid AB] = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

$\det(C_B) \neq 0$

$\therefore \text{rank}(C_B) = 2$

For calculating 'K' using Ackermann formula –

$$K = [0 \quad 1][B \quad AB]^{-1} \phi(A)$$

where, $\phi(s) = (s-\mu_1)(s-\mu_2) = s^2 - (\mu_1+\mu_2)s + \mu_1\mu_2$

$$\mu_{1,2} = 2 \pm j\sqrt{2}$$

$$\phi(s) = s^2 - 4s + 6$$

$$\phi(A) = A^2 - 4A + 6I = \begin{bmatrix} 6 & 0 \\ 0 & 18 \end{bmatrix}$$

$$[B \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\therefore \quad K = [0 \quad 1] \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 6 & 0 \\ 0 & 18 \end{bmatrix}$$

$$K = [6 \quad 0]$$

$\therefore$ Controller Eq^n :— $\quad U = -KX$

$$\therefore \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$u = -[6 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(c) The full order observer eq^n $\longrightarrow$

$$\dot{\hat{X}} = (A - LC)\hat{X} + BU + LY$$

where 'L' observer gain.

Check for O_B:

$$O_B = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4I$$

$$\text{rank}(O_B) = 2$$

By using Ackermann's formula →

$$L = \phi(A) \begin{bmatrix} C \\ CA \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

for $\phi(s) = (s - \mu_1')(s - \mu_2') = s^2 - (\mu_1' + \mu_2') + \mu_1'\mu_2'$

$$\mu_1' = -8 \; ; \; \mu_2' = -8$$

$$\therefore \phi(s) = s^2 + 16s + 64 \quad \Big| \quad \phi(A) = A^2 + 16A + 64I$$

$$\phi(A) = \begin{bmatrix} 64 & 14 \\ 0 & 36 \end{bmatrix}$$

$$\therefore L = \begin{bmatrix} 64 & 14 \\ 0 & 36 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 9 \end{bmatrix}$$

$\therefore$ Observer eq^n Becoms $\rightarrow$

$$\dot{\hat{x}} = (A - LC)\hat{x} + BU + LY$$

when $\quad U = -K\hat{x}$

$$K = \begin{bmatrix} 6 & 0 \end{bmatrix} \quad \& \quad L = \begin{bmatrix} 3.5 \\ 9 \end{bmatrix}$$

(d) Now, for observer controller transfer function $\rightarrow$

$$\dot{\hat{x}} = (A - LC)\hat{x} + BU + LY \quad \& \quad U = -K\hat{x}$$

$$\dot{\hat{x}} = (A - LC)\hat{x} - BK\hat{x} + LY$$

$$\dot{\hat{x}} = (A - LC - BK)\hat{x} + LY$$

$$s\hat{x}(s) = (A - LC + BK)\hat{x} + LY$$

$$\hat{x}(s) = (sI - A + LC + BK)^{-1} L\, Y(s)$$

we ke $\quad U(s) = -K\hat{x}(s)$

$$\therefore \frac{U(s)}{-Y(s)} = K(sI - A + LC + BK)^{-1} L$$

$$\therefore \quad sI - A + LC + BK =$$

$$= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 3.5 \\ 9 \end{bmatrix} \begin{bmatrix} 4 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix} + \begin{bmatrix} 14 & 0 \\ 36 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} s+14 & -1 \\ 42 & s+8 \end{bmatrix}$$

$$(sI - A + LC + BK)^{-1} = \frac{1}{s^2 + 22s + 154} \begin{bmatrix} s+8 & -42 \\ +1 & s+14 \end{bmatrix}$$

$$\therefore \quad G\ \frac{U(s)}{-Y(s)} = \frac{\begin{bmatrix} 6 & 0 \end{bmatrix} \begin{bmatrix} s+8 & -42 \\ 1 & s+14 \end{bmatrix} \begin{bmatrix} 3.5 \\ 9 \end{bmatrix}}{s^2 + 22s + 154}$$

$$= \frac{\begin{bmatrix} 6s+48 & -252 \end{bmatrix} \begin{bmatrix} 3.5 \\ 9 \end{bmatrix}}{s^2 + 22s + 154}$$

$$= \frac{21s + 168 - 2268}{s^2 + 22s + 154} = \frac{21(s-100)}{s^2 + 22s + 154}$$

$$\therefore \quad G_{oc}(s) = \frac{U(s)}{-Y(s)} = \frac{21(s-100)}{s^2 + 22s + 154}$$