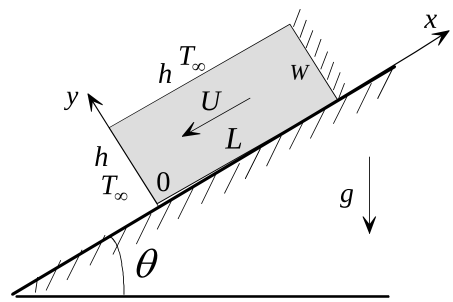
PROBLEM 1.11

A rectangular plate of length L and height H slides down an inclined surface with a velocity U. Sliding friction results in surface heat flux q''_o. The front and top sides of the plate exchange heat by convection. The heat transfer coefficient is h and the ambient temperature is T_∞. Neglect heat loss from the back side and assume that no frictional heat is conducted through the inclined surface. Write the two-dimensional steady state heat equation and boundary conditions.



(1) Observations. (i) This is a two-dimensional steady state problem. (ii) Four boundary conditions are needed. (iii) Rectangular geometry. (iv) Specified flux at surface (x,0) points inwards. (v) Plate velocity has no effect other than generating frictional heat at the inclined plane. (vi) Use Section 1.6 as a guide to writing boundary conditions.

(2) Origin and Coordinates. The origin and Cartesian coordinates x,y are as shown. The coordinates move with the plate.

(3) Formulation.

(i) Assumptions. (1) Steady state, (2) two-dimensional, (3) no energy generation, (4) all frictional heat is added to plate (inclined surface is insulated) and (5) stationary material (plate does not move relative to coordinates).

(ii) Governing Equations. The heat equation in Cartesian coordinates is given by

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+q'''=\rho c_p\left(\frac{\partial T}{\partial t}+U\frac{\partial T}{\partial x}+V\frac{\partial T}{\partial y}+W\frac{\partial T}{\partial z}\right) \quad (1.7)$$

where c_p is specific heat and ρ is density. The above assumptions give

$$U=V=W=\frac{\partial}{\partial z}=q'''=\frac{\partial}{\partial t}=0$$

Eq. (1.7) simplifies to

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)=0 \quad (a)$$

If we further assume constant conductivity, we obtain

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \quad (b)$$

(iii) Boundary Conditions. Four boundary conditions are needed. They are:

(1) Specified flux at boundary (x,0)

(2)

$$-k\frac{\partial T(x,0)}{\partial y} = q_o''$$

(2) Convection at surface (x, W). Pretending that heat flows in the positive *y*-direction, conservation of energy, Fourier's law of conduction and Newton's law of cooling give

$$-k\frac{\partial T(x,W)}{\partial y} = h\left[T(x,W) - T_\infty\right]$$

(3) Convection at surface (0,*y*). Pretending that heat flows in the positive *x*-direction, conservation of energy, Fourier's law of conduction and Newton's law of cooling give

$$-k\frac{\partial T(0,y)}{\partial x} = h\left[T_\infty - T(0,y)\right]$$

(4) Insulated boundary at surface (L, y). Fourier's law gives

$$\frac{\partial T(L,y)}{\partial x} = 0$$

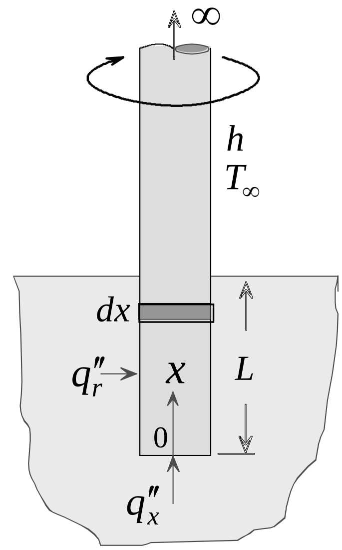
(4) Checking. *Dimensional check*: Each term in boundary conditions (1), (2) and (3) has units of flux.

Limiting check: Boundary conditions (2) and (3) should reduce to an insulated condition if *h* = 0. Similarly, boundary condition (1) should describe an insulated surface if $q_o'' = 0$. Setting *h* = 0 in conditions (2) and (3) and $q_o'' = 0$ in (1) gives the description for an insulated surface.

(5) Comments. (i) The four boundary conditions are valid for constant and variable conductivity *k*. (ii) Formulating boundary conditions requires an understanding of what takes place physically at the boundaries. (iii) Plate motion in this example should not be confused with convection velocity appearing in the heat equation (1.7).

PROBLEM 2.17

A section of a long rotating shaft of radius r_o is buried in a material of very low thermal conductivity. The length of the buried section is L. Frictional heat generated in the buried section can be modeled as surface heat flux. Along the buried surface the radial heat flux is q_r'' and the axial heat flux is q_x''. The exposed surface exchanges heat by convection with the ambient. The heat transfer coefficient is h and the ambient temperature is T_∞. Model the shaft as semi-infinite fin and assume that all frictional heat is conducted through the shaft. Determine the temperature distribution.



(1) Observations. (i) This is a constant area fin. (ii) Temperature distribution can be assumed one dimensional. (iii) The shaft has two sections. Surface conditions are different for each of sections. Thus two equations are needed. (iv) Shaft rotation generates surface flux in the buried section and does not affect the fin equation.

(2) Origin and Coordinates. The origin is selected at one end of the fin and the coordinate x is directed as shown.

(3) Formulation.

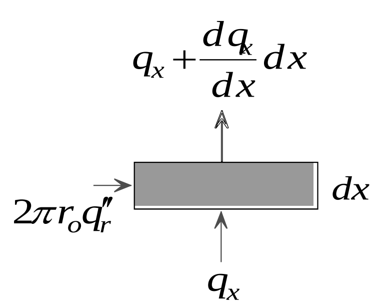
(i) Assumptions. (1) One-dimensional, (2) steady state, (3) constant cross-sectional area, (4) uniform surface flux in the buried section, (5) $Bi << 1$, (6) constant conductivity, (7) no radiation (8) no energy generation and (9) uniform h and T_∞.

(ii) Governing Equations. Since surface conditions are different for the two sections, two fin equations are needed. Let

$T_1(x)$ = temperature distribution in the buried section, $0 \le x \le L$
$T_2(x)$ = temperature distribution in the exposed section, $L \le x \le 2L$

The heat equation for the buried section must be formulated using fin approximation. Conservation of energy for the fin element dx gives

$$q_x + q_r'' 2\pi r_o dx = q_x + \frac{dq_x}{dx} dx \qquad \text{(a)}$$



where q_x is the rate of heat conducted in the x-direction, given by Fourier's law

$$q_x = -k \frac{dT_1}{dx} \qquad \text{(b)}$$

Substituting (b) into (a) and rearranging

$$\frac{d^2T_1}{dx^2} + \frac{2q_r''}{kr_o} = 0, \qquad 0 \le x \le L \qquad \text{(c)}$$

The heat equation for constant area fin with surface convection is given by equation (2.9)

PROBLEM 2.17 (continued)

$$\frac{d^2T_2}{dx^2} - \frac{hC}{kA_c}(T_2 - T_\infty) = 0, \qquad x \ge L \tag{d}$$

where C is the circumferance and A_c is the cross section area given by

$$C = 2\pi r_o \tag{e}$$

$$A\pi r_o^2 \tag{f}$$

Using (c) and (d) to define m

$$m = \sqrt{\frac{hC}{kA_c}} = \sqrt{\frac{2h}{kr_o}} \tag{g}$$

Subatituting (e) into (d)

$$\frac{d^2T_2}{dx^2} - m^2(T_2 - T_\infty) = 0 \tag{h}$$

(iii) Boundary Conditions. Four boundary conditions are needed.

(1) Specified flux at $x = 0$

$$-k\frac{dT_1(0)}{dx} = q''_x \tag{i}$$

(2) Equality of temperature at $x = L$

$$T_1(L) = T_2(L) \tag{j}$$

(3) Equality of flux at $x = L$

$$\frac{dT_1(L)}{dx} = \frac{dT_2(L)}{dx} \tag{k}$$

(4) Finite temperature at $x = \infty$

$$T_2(\infty) = \text{ finite} \tag{l}$$

(4) Solution. The solutions to (c) is

$$T_1(x) = -\frac{q''_r}{kr_o}x^2 + A_1 x + B_1 \tag{m}$$

where A_1 and B_1 are constants of integration. The solution to (h) is given in Appendix A, equation (A-2b)

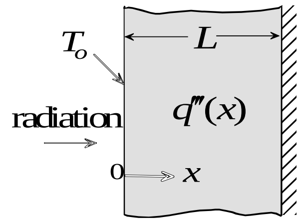
$$T_2(x) = T_\infty + A_2 e^{mx} + B_2 e^{-mx} \tag{n}$$

where A_2 and B_2 are constants of integration. Application of the four boundary conditions gives the four constants

$$A_1 = -\frac{q''_x}{k} \tag{o}$$

$$B_1 = T_\infty + \frac{q''_r}{kr_o}L^2 + \frac{q''_x}{k}L + \frac{1}{m}\left[2\frac{q''_r}{kr_o}L + \frac{q''_x}{k}\right] \tag{p}$$

PROBLEM 2.1



Radiation is used to heat a plate of thickness L and thermal conductivity k. The radiation has the effect of volumetric energy generation at a variable rate given by

$$q'''(x) = q_o''' \, e^{-bx}$$

where q_o''' *and b are constant and x is the distance along the plate. The heated surface at x = 0 is maintained at uniform temperature* T_o *while the opposite surface is insulated. Determine the temperature of the insulated surface.*

(1) Observations. (i) This is a steady state one-dimensional problem. (ii) Use a rectangular coordinate system. (iii) Energy generation is non-uniform. (iv) Two boundary conditions are needed. The temperature is specified at one boundary. The second boundary is insulated. (v) To determine the temperature of the insulated surface it is necessary to determine the temperature distribution in the plate.

(2) Origin and Coordinates. A rectangular coordinate system is used with the origin at one of the two surfaces. The coordinate *x* is oriented as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) stationary and (4) constant properties.

(ii) Governing Equations. Introducing the above assumptions into eq. (1.8) gives

$$\frac{d^2T}{dx^2} + \frac{q'''}{k} = 0 \tag{a}$$

where

k = thermal conductivity
q''' = rate of energy generation per unit volume
T = temperature
x = independent variable

Energy generation q''' varies with location according to

$$q''' = q_o''' e^{-bx} \tag{b}$$

where q_o''' and b are constant. Substituting (b) into (a) gives

$$\frac{d^2T}{dx^2} + \frac{q_o'''}{k} e^{-bx} = 0 \tag{c}$$

(iii) Boundary Conditions. Since (c) is second order with a single independent variable x, two boundary conditions are needed.

(1) Specified temperature at $x = 0$

$$T(0) = T_o \tag{d}$$

(2) The surface at $x = L$ is insulated

PROBLEM 2.1 (continued)

$$\frac{dT(L)}{d\,x} = 0 \tag{e}$$

(4) Solution. Separating variables and integrating (c) twice

$$T(x) = -\frac{q_o'''}{b^2 k}\, e^{-bx} + C_1 x + C_2 \tag{f}$$

where C_1 and C_2 are constants of integration. Application of boundary conditions (d) and (e) gives C_1 and C_2

$$C_1 = -\frac{q_o'''}{bk}\, e^{-bL}, \quad C_2 = T_o + \frac{q_o'''}{b^2 k} \tag{g}$$

Substituting (g) into (f)

$$T(x) = \frac{q_o'''}{b^2 k}\left[1 - e^{-bx} - bxe^{-bL}\right] + T_o \tag{h}$$

The temperature of the insulated surface is obtained by setting $x = L$ in (h)

$$T(L) = \frac{q_o'''}{b^2 k}\left[1 - (1 + bL)e^{-bL}\right] + T_o \tag{i}$$

Expressing (h) and (i) in dimensionless form gives

$$\frac{T(x) - T_o}{\dfrac{q_o'''}{b^2 k}} = 1 - e^{-bL(x/L)} - (bL)(x/L)e^{-bL} \tag{j}$$

and

$$\frac{T(L) - T_o}{\dfrac{q_o'''}{b^2 k}} = 1 - (1 + bL)\, e^{-bL} \tag{k}$$

(5) Checking. *Dimensional check*: Each term in solution (h) has units of temperature.

Boundary conditions check: Setting x = 0 in (h) gives $T(0) = T_o$. Thus boundary condition (d) is satisfied. To check condition (e) solution (h) is differentiated to obtain

$$\frac{dT}{d\,x} = \frac{q_o'''}{b^2 k}\left[be^{-bx} - be^{-bL}\right] \tag{m}$$

Evaluation (m) at $x = L$ shows that condition (e) is satisfied.

Differential equation check: Direct substitution of (h) into equation (c) shows that the solution satisfies the governing equation.

Global energy conservation check: The total energy generated in the plate, $\dot{E}_g$, must be equal to the energy conducted from the plate at x = 0. Energy generated is given by

$$\dot{E}_g = \int_0^L q'''(x)dx = q_o'''\int_0^L e^{-bx}dx = \frac{q_o'''}{b}(1 - e^{-bL}) \tag{k}$$

Energy conducted at x = 0, $\dot{E}(0)$, is determined by substituting (m) into Fourier's law

$$\dot{E}(0) = -k\frac{dT(0)}{d\,x} = -\frac{q_o'''}{b}(1 - e^{-bL}) \tag{l}$$

The minus sign indicates that heat is conducted in the negative *x*-direction. Thus energy is conserved.

Limiting check: For the special case of $q''' = 0,$ temperature distribution should be uniform given by $T(x) = T_o.$ Setting $q_o''' = 0$ in (h) gives the anticipated result.

(6) Comments. (i) The solution to the special case of constant energy generation corresponds to $b = 0.$ However, setting $b = 0$ in solution (h) gives 0/0. To obtain the solution to this case it is necessary to set $b = 0$ in differential equation (c) and solve the resulting equation. (ii) The dimensionless solution is characterized by a single dimensionless parameter *bL*.

PROBLEM 2.2

Repeat Example 2.3 with the outer plates generating energy at a rate q''' and no energy is generated in the inner plate.

(1) Observations. (i) The three plates form a composite wall. (ii) The geometry can be described by a rectangular coordinate system. (iii) Due to symmetry, no heat is conducted through the center plate. Thus the center plate is at a uniform temperature. (iv) Heat conduction is assumed to be in the direction normal to the three plates. (v) This problem reduces to a single heat generating plate in which one side is maintained at a specified temperature and the other side is insulated.

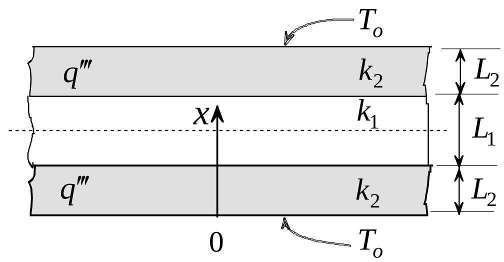
(2) Origin and Coordinates. Since only one plate needs to be analyzed, the origin and coordinate x are selected as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state and (3) constant conductivity of the heat generating plate.

(ii) Governing Equations. Let the subscript 2 refer to the heat generating plate. Based on the above assumptions, eq.(1.8) gives

$$\frac{d^2T_2}{dx^2} + \frac{q'''}{k_2} = 0 \tag{a}$$



where

k_2 = thermal conductivity
q''' = rate of energy generation per unit volume
T = temperature
x = coordinate

(iii) Boundary Conditions. Since (a) is second order with a single independent variable x, two boundary conditions are needed.

(1) Specified temperature at $x = 0$

$$T_2(0) = T_o \tag{b}$$

(2) Insulated surface at $x = L_2$

$$\frac{dT_2(L_2)}{dx} = 0 \tag{c}$$

where L_2 is the thickness of the heat generating plate.

(4) Solution. Integrating (a) twice gives

$$T_2(x) = -\frac{q'''}{2k_2}x^2 + C_1 x + C_2 \tag{d}$$

where C_1 and C_2 are constants of integration. Application of boundary conditions (b) and (c) gives C_1 and C_2

$$C_1 = \frac{q''' L_2}{k_2}, \quad C_2 = T_o \tag{e}$$

Substituting (e) into (d) gives the temperature distribution in the heat generating plate

$$T_2(x) = \frac{q''' x^2}{2k_2}\left[(2L_2/x) - 1\right] + T_o \tag{f}$$

To determine the temperature distribution in the center plate we apply continuity of temperature at $x = L_2$

$$T_1(L_2) = T_2(L_2) \tag{g}$$

Noting that the center plate must be at a uniform temperature, evaluating (f) at $x = L_2$ and substituting into (g) gives

$$T_1(x) = T_1(L_2) = T_2(L_2) = \frac{q''' L_2^2}{2k_2} + T_o \tag{h}$$

Expressing (f) and (h) in dimensionless form gives

$$\frac{T_2(x) - T_o}{\dfrac{q''' L_2^2}{k_2}} = \frac{1}{2}\frac{x}{L_2}\left[2 - (x/L_2)\right] \tag{i}$$

and

$$\frac{T_2(x) - T_o}{\dfrac{q''' L_2^2}{k_2}} = \frac{1}{2} \tag{j}$$

(5) Checking. *Dimensional check*: Each term in (f) has units of temperature.

Boundary conditions check: Substitution of solutions (f) into (b) and (c) shows that the two boundary conditions are satisfied.

Differential equation check: Direct substitution of (f) into equation (a) shows that the solution satisfies the governing equation.

Global energy conservation check: The total energy generated in the plate, $\dot{E}_g$, must be equal to the energy conducted from the plate at $x = 0$. Energy generated is given by

$$\dot{E}_g = q''' A L_2 \tag{k}$$

where A is surface area. Energy conducted at $x = 0$, $\dot{E}(0)$, is determined by differentiation (f) and substituting into Fourier's law

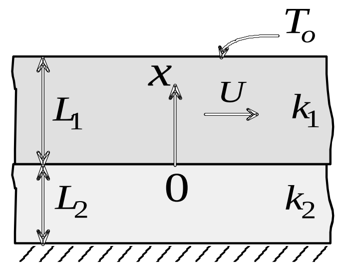
$$\dot{E}(0) = -k_2 A \frac{dT_2(0)}{d\,x} = -q''' A L_2 \tag{m}$$

The minus sign indicates that heat is conducted in the negative x-direction. Equations (k) and (m) show that energy is conserved.

Limiting check: If energy generation vanishes, the composite wall should be at a uniform temperature T_o. Setting $q''' = 0$ in (f) and (h) gives $T_1(x) = T_2(x) = T_o$.

(6) Comments. (i) An alternate approach to solving this problem is to consider half the center plate and the heat generating plate as a composite wall, note symmetry, write two differential equations and four boundary conditions and solve for $T_1(x)$ and $T_2(x)$. Clearly this is a longer approach to solving the problem. (ii) No parameter characterizes the solution.

PROBLEM 2.3



A plate of thickness L_1 and conductivity k_1 moves with a velocity U over a stationary plate of thickness L_2 and conductivity k_2. The pressure between the two plates is P and the coefficient of friction is μ. The surface of the stationary plate is insulated while that of the moving plate is maintained at constant temperature T_o. Determine the steady state temperature distribution in the two plates.

(1) Observations. (i) The two plates form a composite wall. (ii) The geometry can be described by a rectangular coordinate system. (iii) Heat conduction is assumed to be in the direction normal to the two plates. (iv) Since the stationary plate is insulated no heat is conducted through it. Thus it is at a uniform temperature. (v) All energy dissipated due to friction at the interface is conducted through the moving plate. (vi) This problem reduces to one-dimensional steady state conduction in a single plate with specified flux at one surface and a specified temperature at the opposite surface. (vii) Plate motion plays no role in the governing equation since no heat is conducted in the direction of motion.

(2) Origin and Coordinates. Since only one plate needs to be analyzed, the origin is selected at the interface and the coordinate x is directed as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) constant conductivity of the moving plate, (4) no energy generation and (5) uniform energy dissipation at the interface.

(ii) Governing Equations. Let the subscript 1 refer to the moving plate. Based on the above assumptions, eq.(1.8) applied to the moving plate gives

$$\frac{d^2T_1}{dx^2} = 0 \tag{a}$$

(iii) Boundary Conditions. Since (a) is second order with a single independent variable x, two boundary conditions are needed.

(1) Specified heat flux at $x = 0$. Fourier's law gives

$$-k_1 \frac{dT_1(0)}{dx} = \mu PU \tag{b}$$

where

k_1 = thermal conductivity
P = interface pressure
U = plate velocity
μ = coefficient of friction

(2) Specified temperature at $x = L_1$

$$T_1(L_1) = T_o \tag{c}$$

where L_1 is the thickness of the moving plate.

(4) Solution. Integrating (a) twice gives

$$T_1(x) = C_1 x + C_2 \tag{d}$$

where C_1 and C_2 are constants of integration. Application of boundary conditions (b) and (c) gives C_1 and C_2

$$C_1 = -\frac{\mu PU}{k_1}, \quad C_2 = T_o + \frac{\mu PU}{k_1} L_1 \tag{e}$$

Substituting (e) into (d) gives the temperature distribution in the moving plate

$$T_1(x) = \frac{\mu PUL_1}{k_1}\left[1 - (x/L_1)\right] + T_o \tag{f}$$

To determine the temperature distribution in the stationary plate we apply continuity of temperature at $x = 0$

$$T_2(0) = T_1(0) \tag{g}$$

Noting that the stationary plate must be at a uniform temperature, evaluating (f) at $x = 0$ and substituting into (g) gives

$$T_2(x) = T_2(0) = T_1(0) = \frac{\mu PUL_1}{k_1} + T_o \tag{h}$$

Expressing (f) and (h) in dimensionless form gives

$$\frac{T_1(x) - T_o}{\dfrac{\mu PUL_1}{k_1}} = 1 - \frac{x}{L_1} \tag{i}$$

$$\frac{T_1(x) - T_o}{\dfrac{\mu PUL_1}{k_1}} = 1 \tag{j}$$

(5) Checking. *Dimensional check*: Each term in (f) has units of temperature

Boundary conditions check: Substitution of solutions (f) into (b) and (c) shows that the two boundary conditions are satisfied.

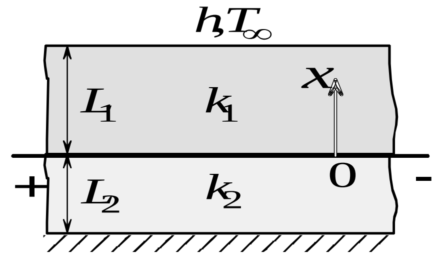
Differential equation check: Direct substitution of (f) into equation (a) shows that the solution satisfies the governing equation.

Limiting check: If no energy is added at the interface ($\mu = 0$ or U = 0 or P = 0), the two plates should be at a uniform temperature T_o. Setting any of the quantities μ, U or P equal to zero in (f) and (h) gives $T_1(x) = T_2(x) = T_o$.

(6) Comments. (i) An alternate approach to solving this problem is to consider the two plates as a composite wall, write two differential equations and four boundary conditions and solve for $T_1(x)$ and $T_2(x)$. Clearly this is a longer approach to solving the problem. (ii) No parameter characterizes the solution.

PROBLEM 2.4

A thin electric element is wedged between two plates of conductivities k_1 and k_2. The element dissipates uniform heat flux q_o''. The thickness of one plate is L_1 and that of the other is L_2. One plate is insulated while the other exchanges heat by convection. The ambient temperature is T_∞ and the heat transfer coefficient is h. Determine the temperature of the insulated surface for one-dimensional steady state conduction.



(1) Observations. (i) The two plates form a composite wall. (ii) The geometry can be described by a rectangular coordinate system. (iii) Heat conduction is assumed to be in the direction normal to the two plates. (iv) Since no heat is conducted through the insulated plate, its temperature is uniform equal to the interface temperature. (v) All energy dissipated by the element at the interface is conducted through the upper plate. (vi) This problem reduces to one-dimensional steady state conduction in a single plate with specified flux at one surface and convection at the opposite surface. (vii) To determine the temperature of the insulated surface it is necessary to determine the temperature distribution in the upper plate.

(2) Origin and Coordinates. Since only one plate needs to be analyzed, the origin is selected at the interface and coordinate *x* is directed as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) constant conductivity of the upper plate, (4) no energy generation and (5) uniform energy dissipation at the interface.

(ii) Governing Equations. Let the subscript 1 refer to the upper plate. Based on the above assumptions, eq.(1.8) applied to the upper plate gives

$$\frac{d^2T_1}{dx^2} = 0 \tag{a}$$

(iii) Boundary Conditions. Since (a) is second order with a single independent variable *x*, two boundary conditions are needed.

(1) Specified heat flux at $x = 0$

$$-k_1 \frac{dT_1(0)}{dx} = q_o'' \tag{b}$$

(2) Convection at $x = L_1$. Fourier's law and Newton's law of cooling give

$$-k_1 \frac{dT_1(L_1)}{dx} = h\left[T_1(L_1) - T_\infty\right] \tag{c}$$

where L_1 is the thickness of the upper plate.

(4) Solution. Integrating (a) twice gives

$$T_1(x) = C_1 x + C_2 \tag{d}$$

where C_1 and C_2 are constants of integration. Application of boundary conditions (b) and (c) gives C_1 and C_2

$$C_1 = -\frac{q''_o}{k_1}, \quad C_2 = \frac{q''_o}{h}\left[(hL_1 / k_1) + 1\right] + T_\infty \tag{e}$$

Substituting (e) into (d) and rearranging gives the temperature distribution in the upper plate

$$T_1(x) = \frac{q''_o}{h}\left[\frac{hL_1}{k_1}\left(1 - \frac{x}{L_1}\right) + 1\right] + T_\infty \tag{f}$$

To determine the temperature distribution in the insulated plate we apply continuity of temperature at $x = 0$

$$T_2(0) = T_1(0) \tag{g}$$

Noting that the stationary lower plate must be at a uniform temperature, evaluating (f) at $x = 0$ and substituting into (g) gives

$$T_2(x) = T_2(0) = T_1(0) = \frac{q''_o}{h}\left[\frac{hL_1}{k_1} + 1\right] + T_\infty \tag{h}$$

Equation (h) gives the temperature of the insulated surface.

Expressing (f) and (h) in dimensionless form gives

$$\frac{T_1(x) - T_\infty}{\dfrac{q''_o}{h}} = Bi\left(1 - \frac{x}{L_1}\right) + 1 \tag{i}$$

and

$$\frac{T_1(x) - T_\infty}{\dfrac{q''_o}{h}} = 1 + Bi \tag{j}$$

where Bi is the Biot number defined as

$$Bi = \frac{hL_1}{k_1} \tag{k}$$

(5) Checking. *Dimensional check*: Each term in (f) has units of temperature.

Boundary conditions check: Substitution of solution (f) into (b) and (c) shows that the two boundary conditions are satisfied.

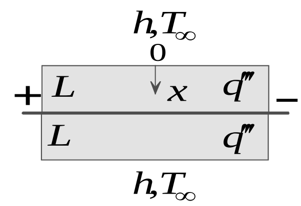
Differential equation check: Direct substitution of (f) into equation (a) shows that the solution satisfies the governing equation.

Limiting check: (i) If no energy is added at the interface, the two plates should be at a uniform temperature T_∞. Setting $q''_o = 0$ in (f) and (h) gives $T_1(x) = T_2(x) = T_\infty$. (ii) If $h = 0$, no heat can leave the plates and thus their temperature will be infinite. Setting h = 0 in (f) and (h) gives $T_1(x) = T_2(x) = \infty$.

*Qualitative check***:** Increasing interface energy input increases the temperature level in the two plates. Equations (f) and (h) exhibit this behavior.

(6) Comments. (i) An alternate approach to solving this problem is to consider the two plates as a composite wall, write two differential equations and four boundary conditions and solve for $T_1(x)$ and $T_2(x)$. Clearly this is a longer approach to solving the problem. (ii) The problem is characterized by a single dimensionless parameter hL_1 / k_1 which is the Biot number. This parameter appears in problems involving surface convection.

PROBLEM 2.5

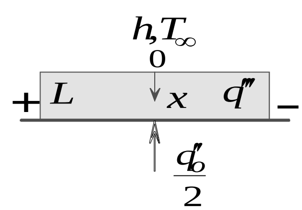


A thin electric element is sandwiched between two plates of conductivity k and thickness L each. The element dissipates a flux q_o''. Each plate generates energy at a volumetric rate of q''' and exchanges heat by convection with an ambient fluid at T_∞. The heat transfer coefficient is h. Determine the temperature of the electric element.

(1) Observations. (i) The two plates form a composite wall. (ii) The geometry can be described by a rectangular coordinate system. (iii) Heat conduction is assumed to be in the direction normal to the two plates. (iv) Symmetry requires that half the energy dissipated by the element at the interface is conducted through the upper plate and the other half is conducted through the lower plate. Thus only one plate needs to be analyzed.(v) The problem reduces to one-dimensional steady state conduction in a single plate with energy generation having specified flux at one surface and convection at the opposite surface. (vi) To determine the temperature of the element it is necessary to determine the temperature distribution in one of the two plates.

(2) Origin and Coordinates. The upper plate is considered for analysis. The origin is selected at the upper surface and coordinate x is directed as shown.

(3) Formulation.



(i) Assumptions. (1) One-dimensional, (2) steady state, (3) constant conductivity of the upper plate, (4) uniform energy generation and (5) uniform energy dissipation at the interface.

(ii) Governing Equations. Let the subscript 1 refer to the upper plate. Based on the above assumptions, eq.(1.8) applied to the upper plate gives

$$\frac{d^2T_1}{dx^2} + \frac{q'''}{k} = 0 \tag{a}$$

(iii) Boundary Conditions. Since (a) is second order with a single independent variable x, two boundary conditions are needed.

(2) Specified heat flux at $x = L$

$$-k\frac{dT_1(L)}{dx} = -\frac{q_o''}{2} \tag{b}$$

(2) Convection at $x = 0$. Fourier's law and Newton's law of cooling give

$$-k\frac{dT_1(0)}{dx} = h\left[T_\infty - T_1(0)\right] \tag{c}$$

where L is the thickness of the upper plate.

(4) Solution. Integrating (a) twice gives

$$T_1(x) = -\frac{q'''}{2k}x^2 + C_1 x + C_2 \tag{d}$$

where C_1 and C_2 are constants of integration. Application of boundary conditions (b) and (c) gives C_1 and C_2

$$C_1 = \frac{q''_o}{2k} + \frac{q'''L}{k}, \quad C_2 = T_\infty + \frac{q''_o}{2h} + \frac{q'''L}{h} \tag{e}$$

Substituting (e) into (d) and rearranging gives the temperature distribution in the upper plate

$$T_1(x) = T_\infty + \frac{q''_o}{2h} + \frac{q'''L}{h} + \left[\frac{q''_o}{2k} + \frac{q'''L}{k}\right] x - \frac{q'''}{2k} x^2 \tag{f}$$

Expressing (f) in dimensionless form gives

$$\frac{T_1(x) - T_\infty}{q'''L/h} = \left[1 + \frac{q''_o}{2q'''L}\right]\left[1 + Bi\,\frac{x}{L}\right] - \frac{Bi}{2}\left[\frac{x}{L}\right]^2 \tag{g}$$

where Bi is the Biot number defined as

$$Bi = \frac{hL}{k} \tag{h}$$

To determine the temperature of the electric element set $x = L$ in (g)

$$\frac{T_1(L) - T_\infty}{q'''L/h} = \frac{q''_o}{2q'''L}\left[1 + Bi\right] + \frac{Bi}{2} + 1 \tag{i}$$

(5) Checking. *Dimensional check*: Each term in (g) is dimensionless.

Boundary conditions check: Substitution of solution (f) into (b) and (c) shows that the two boundary conditions are satisfied.

Differential equation check: Direct substitution of (f) into equation (a) shows that the solution satisfies the governing equation.

Limiting check: (i) If no energy is added at the interface and there is no energy generation the two plates should be at a uniform temperature T_∞. Setting $q''_o = q''' = 0$ in (f) gives $T_1(x) = T_\infty$. (ii) If $h = 0$, no heat can leave the plate and thus the temperature will be infinite. Setting $h = 0$ in (f) gives $T_1(x) = \infty$.

Qualitative check: Increasing interface energy input and/or energy generation increases the temperature level in the plate. Equation (f) exhibit this behavior.

(6) Comments. (i) An alternate approach to solving this problem is to consider the two plates as a composite wall, write two differential equations and four boundary conditions and solve for $T_1(x)$ and $T_2(x)$. Clearly this is a longer approach to solving the problem. (ii) The solution is characterized by two parameters: the convection parameter Bi and the ratio of energy dissipated in the element to energy generation, $q''_o / q'''L$.

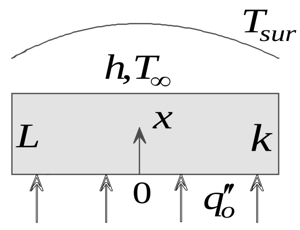
PROBLEM 2.6

One side of a plate is heated with uniform flux q''_o while the other side exchanges heat by convection and radiation. Surface emissivity is ε, heat transfer coefficient h, ambient temperature T_∞, surroundings temperature T_{sur}, plate thickness L and the conductivity is k. Assume one-dimensional steady state conduction and use a simplified radiation model. Determine:

[*a*] *The temperature distribution in the plate.*

[*b*] *The two surface temperatures for:*

$h = 27\,\text{W/m}^2\text{–}^\circ\text{C}$, $k = 15\,\text{W/m–}^\circ\text{C}$, $L = 8$ cm, $\varepsilon = 0.95$, $q''_o = 19{,}500\,\text{W/m}^2$, $T_{sur} = 18^\circ\text{C}$, $T_\infty = 22\,^\circ\text{C}$.



(1) Observations. (i) This is a one-dimensional steady state problem. (ii) Two boundary conditions are needed. (iii) Cartesian geometry. (iv) Specified flux at one surface and convection and radiation at the opposite surface. (v) Since radiation is involved in this problem, temperature units should be expressed in kelvin

(2) Origin and Coordinates. The origin and coordinate x are shown.

(3) Formulation.

(i) Assumptions. (1) Steady state, (2) one-dimensional, (3) constant properties, (4) no energy generation, (5) small gray surface which is enclosed by a much larger surface, (6) uniform conditions at the two surfaces and (7) stationary material.

(ii) Governing Equations. Based on the above assumptions, the heat equation in Cartesian coordinates is given by

$$\frac{d^2T}{dx^2} = 0 \tag{a}$$

(iii) Boundary Conditions. Two boundary conditions are needed. They are:

(1) Specified flux at $x = 0$

$$-k\frac{dT(0)}{dx} = q''_o \tag{b}$$

where

$k =$ thermal conductivity = $15\,\text{W/m–}^\circ\text{C}$

$q''_o =$ surface flux = $19{,}500\,\text{W/m}^2$

(2) Convection and radiation at $x = L$. Conservation of energy, Fourier's law of conduction, Newton's law of cooling and Stefan-Boltzmann radiation law give

$$-k\frac{dT(L)}{dx} = h\left[T(L) - T_\infty\right] + \varepsilon\,\sigma\left[T^4(L) - T_{sur}^4\right] \tag{c}$$

where

$h =$ heat transfer coefficient = $27\,\text{W/m}^2\text{–}^\circ\text{C} = 27\,\text{W/m}^2 - \text{K}$

$L =$ plate thickness = 8 cm

T_∞ = ambient temperature = 22 $^{\text{o}}$C = (22 + 273.15) K = 295.15 K

T_{sur} = surroundings temperature = 18 $^{\text{o}}$C = (18 + 273.15) K = 291.15 K

$\varepsilon =$ emissivity = 0.95

$\sigma =$ Stefan-Boltzmann constant = 5.67×10^{-8} W/m^2 – K^4

(4) Solution. Integrating (a) twice gives

$$T(x) = C_1\, x + C_2 \tag{d}$$

where C_1 and C_2 are constants of integration. Application of boundary conditions (b) and (c) gives

$$C_1 = -\frac{q_o''}{k} \tag{e}$$

$$q_o'' = h\left[-\frac{q_o''}{k}L + C_2 - T_\infty\right] + \varepsilon\,\sigma\left[(-\frac{q_o''}{k}L + C_2)^4 - T_{sur}^4\right] \tag{f}$$

Equation (f) gives the constant C_2. However, this equation must be solved for C_2 by trial and error.

(5) Checking. *Dimensional Check*: Each term in equations (b), (c) and (f) has units of flux.

Differential equation check: Solution (d) satisfies the governing equation (a).

(6) Computations. Equation (e) gives

$$\mathrm{C}_1 = -\frac{19{,}500\ (\mathrm{W/m^2})}{15\,(\mathrm{W/m\text{-}^oC})} = -1300\frac{^{\mathrm{o}}\mathrm{C}}{\mathrm{m}} = -1300\frac{\mathrm{K}}{\mathrm{m}}$$

Equation (f) gives

$$19{,}500(\mathrm{W/m^2}) = 27\,(\mathrm{W/m^2\text{ - }K})\left[\frac{\text{-}19{,}500\ (\mathrm{W/m^2})}{15\,(\mathrm{W/m\text{ - }K})}0.08(\mathrm{m}) + C_2 - 295.15(\mathrm{K})\right]$$

$$+ (0.95)5.67\times 10^{-8}\ (\mathrm{W/m^2\text{ - }K^4})\left[\left\{\frac{\text{-}19{,}500\ (\mathrm{W/m^2})}{15\,(\mathrm{W/m\text{ - }K})}0.08(\mathrm{m}) + C_2\right\}^4 - (291.15)^4(\mathrm{K}^4)\right]$$

Solving this equation for C_2 by trial and error gives

$$\mathrm{C}_2 = 761.9\,\mathrm{K}$$

Substituting C_1 and C_2 into (d) gives

$$T(x) = 761.9\,(\mathrm{K}) - 1300(\mathrm{K/m})\,x \tag{g}$$

Surface temperatures at x = 0 and $x = L$ are determined using (g)

$$T(0) = 761.9\,\text{K}$$

$$T(L) = 657.9\,\text{K}$$

(7) Comments. (i) A trial and error procedure was required to solve this problem. (ii) Care should be exercised in selecting the correct units for temperature in radiation problems. (iii) With the two temperatures determined a numerical check based on conservation of energy can be made.

$$\text{Energy flux entering plate} = \text{Energy flux conducted through the plate} = \text{Energy flux leaving} \qquad \text{(h)}$$

$$\text{Energy flux entering plate} = q_o'' = 19{,}500\ \text{W/m}^2$$

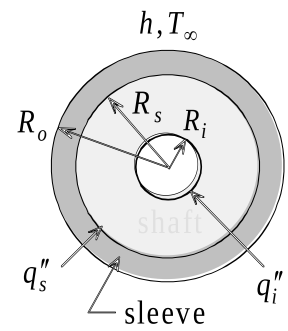
$$\text{Energy flux conducted through the plate} = \frac{T(0) - T(L)}{L/k}$$

$$\text{Energy flux leaving} = h\left[-\frac{q_o''}{k}L + C_2 - T_\infty\right] + \varepsilon\sigma\left[\left(-\frac{q_o''}{k}L + C_2\right)^4 - T_{sur}^4\right]$$

Numerical computations show that each term in (h) is equal to 19,500 W/m^2.

PROBLEM 2.7

A hollow shaft of outer radius R_s and inner radius R_i rotates inside a sleeve of inner radius R_s and outer radius R_o. Frictional heat is generated at the interface at a flux q_s''. At the inner shaft surface heat is added at a flux q_i''. The sleeve is cooled by convection with a heat transfer coefficient h. The ambient temperature is T_∞. Determine the steady state one-dimensional temperature distribution in the sleeve.



(1) Observations. (i) The shaft and sleeve form a composite cylinder. (ii) Use a cylindrical coordinate system. (iii) Heat conduction is assumed to be in the radial direction only. (iv) At steady state, energy entering the shaft at R_i must be equal to energy leaving at R_o. (v) Frictional energy dissipated at the interface and energy added at R_i must be conducted through the sleeve. (vi) This problem reduces to one-dimensional steady state conduction in a hollow cylinder (sleeve) with specified flux at the inside surface and convection at the outside surface.

(2) Origin and Coordinates. The origin is selected at the center and the coordinate r is directed as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) constant conductivity of the sleeve, (4) no energy generation, and (5) uniform frictional energy at the interface.

(ii) Governing Equations. Based on the above assumptions, eq.(1.11) applied to the sleeve gives

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \tag{a}$$

(iii) Boundary Conditions. Since (a) is second order with a single independent variable r, two boundary conditions are needed.

(1) Specified heat flux at $r = R_s$. Fourier's law gives

$$-k\frac{dT(R_s)}{dr} = q_{sl}'' \tag{b}$$

where q_{sl}'' is the heat flux added to the sleeve at $r = R_s$. This flux is the sum of frictional energy generated at the interface and energy added at R_i. That is

$$q_{sl}'' = q_s'' + q_o'' \tag{c}$$

where

q_o'' = heat flux leaving the shaft at R_s
q_s'' = heat flux generated by friction at the interface R_s
q_{sl}'' = total heat flux entering the sleeve at R_s

At steady state, conservation of energy applied to the shaft requires that energy entering the shaft at R_i be equal to energy leaving it at R_s. Thus

$$2\pi R_i q_i'' = 2\pi R_s q_o''$$

Or

$$q_o'' = \frac{R_i}{R_s} q_i'' \tag{d}$$

Substituting (c) and (d) into (b) gives the boundary condition at R_s

$$-k\frac{dT(R_s)}{dr} = q_s'' + \frac{R_i}{R_s} q_i'' \tag{e}$$

(2) Convection at $r = R_o$. Fourier's law and Newton's law of cooling give

$$-k\frac{dT(R_o)}{dr} = h\left[T(R_o) - T_\infty\right] \tag{f}$$

(4) Solution. Integrating (a) twice gives

$$T(r) = C_1 \ln r + C_2 \tag{g}$$

where C_1 and C_2 are constants of integration. Application of boundary conditions (e) and (f) gives C_1 and C_2

$$C_1 = -\frac{R_s}{k}\left[q_s'' + (R_i/R_s)q_i''\right], \quad C_2 = T_\infty + \frac{R_s}{k}\left[q_s'' + (R_i/R_s)q_i''\right]\left[(k/hR_o) + \ln R_o\right] \tag{h}$$

Substituting (h) into (g) and rearranging gives the temperature distribution in the sleeve

$$T(r) = \frac{R_s}{k}\left[q_s'' + (R_i/R_s)q_i''\right]\left[\ln(R_o/r) + (k/hR_o)\right] + T_\infty \tag{i}$$

Expressing (i) in dimensionless form gives

$$\frac{T(r) - T_\infty}{\dfrac{q_s'' R_s}{k}} = \left[1 + \frac{q_i'' R_i}{q'' R_s}\right]\left[\ln(R_o/r) + (1/Bi)\right] \tag{j}$$

where the Biot number Bi is defined as

$$Bi = \frac{hR_o}{k} \tag{k}$$

(5) Checking. *Dimensional check*: Each term in (i) has units of temperature. Each term in (j) is dimensionless.

Boundary conditions check: Substitution of solutions (i) into (e) and (f) shows that the two boundary conditions are satisfied.

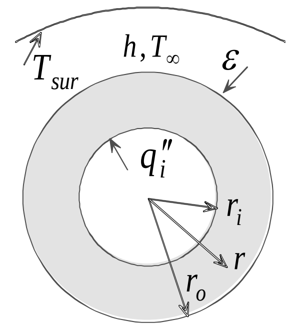
Differential equation check: Direct substitution of (i) into (a) shows that the solution satisfies the governing equation.

Limiting check: (i) If no energy is added to the shaft at R_i and no frictional heat is generated at the interface R_s, the sleeve should be at a uniform temperature T_∞. Setting $q_i'' = q_s'' = 0$ in

(i) gives $T(r) = T_\infty$. (ii) If h = 0, no heat can leave the sleeve and the temperature will be infinite. Setting h = 0 in (i) gives $T(r) = \infty$.

(6) Comments. (i) An alternate approach to solving this problem is to consider the shaft and sleeve as a composite cylinder, write two differential equations and four boundary conditions and solve for the temperature distribution in both shaft and cylinder. Clearly this is a longer approach to solving the problem. (ii) Two parameters characterize the solution: The Biot number and heat ratio $(q_i'' R_i / q_s'' R_s)$.

PROBLEM 2.8



A hollow cylinder of inner radius r_i *and outer radius* r_o *is heated with flux* q_i'' *at its inner surface. The outside surface exchanges heat with the ambient and surroundings by convection and radiation. The heat transfer coefficient is h, ambient temperature* T_∞*, surroundings temperature* T_{sur} *and surface emissivity is* ε*. Assume steady state one-dimensional conduction and use a simplified radiation model. Determine:*

[a] The temperature distribution in the cylinder.

[b] The inner and outer surface temperatures for:

$q_i'' = 35{,}500\ \text{W/m}^2$, $T_\infty = 24\ ^\circ\text{C}$, $T_{sur} = 14\,^\circ\text{C}$, $k = 3.8\ \text{W/m–}^\circ\text{C}$, $r_o = 12$ *cm,*

$r_i = 5.5$ *cm,* $h = 31.4\ \text{W/m}^2\text{–}^\circ\text{C}$, $\varepsilon = 0.92$

(1) Observations. (i) This is a one-dimensional steady state problem. (ii) Two boundary conditions are needed. (iii) Cylindrical geometry. (iv) Specified flux at the inner surface and convection and radiation at the outer surface. (v) Since radiation is involved in this problem, temperature units should be expressed in kelvin

(2) Origin and Coordinates. The origin and coordinate *r* are shown.

(3) Formulation.

(i) Assumptions. (1) Steady state, (2) one-dimensional, (3) constant properties, (4) no energy generation, (5) small gray surface which is enclosed by a much larger surface, (6) uniform conditions at the two surfaces and (7) stationary material.

(ii) Governing Equations. Based on the above assumptions, the heat equation in Cartesian coordinates is given by

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \tag{a}$$

(iii) Boundary Conditions. Two boundary conditions are needed. They are:

(2) Specified flux at $r = r_i$

$$-k\frac{dT(r_i)}{dr} = q_i'' \tag{b}$$

where

$k =$ thermal conductivity = 3.8 $\text{W/m–}^\circ\text{C}$

$q_o'' =$ surface flux = 35,500 W/m^2

$r_i =$ inside radius = 5.5 cm

(2) Convection and radiation at $r = r_o$. Conservation of energy, Fourier's law of conduction, Newton's law of cooling and Stefan-Boltzmann radiation law give

$$-k\frac{dT(r_o)}{dr} = h\left[T(r_o) - T_\infty\right] + \varepsilon\,\sigma\left[T^4(r_o) - T_{sur}^4\right] \tag{c}$$

where

$h =$ heat transfer coefficient = 31.4 $\text{W/m}^2{-}^{\text{o}}\text{C}$ = 27 $\text{W/m}^2-\text{K}$
r_o = outside radius = 12 cm
T_∞ = ambient temperature = 24 $^{\text{o}}$C = (24 + 273.15) K = 297.15 K
T_{sur} = surroundings temperature = 14 $^{\text{o}}$C = (14 + 273.15) K = 287.15 K
ε = emissivity = 0.92
σ = Stefan-Boltzmann constant = 5.67×10^{-8} $\text{W/m}^2-\text{K}^4$

(4) Solution. Integrating (a) twice gives

$$T(r) = C_1 \ln r + C_2 \tag{d}$$

where C_1 and C_2 are constants of integration. Application of boundary conditions (b) and (c) gives

$$C_1 = -\frac{q_i''}{k} r_i \tag{e}$$

$$\frac{r_i}{r_o} q_i'' = h\left[-\frac{q_i''}{k} r_i \ln r_o + C_2 - T_\infty\right] + \varepsilon\,\sigma\left[(-\frac{q_i''}{k} r_i \ln r_o + C_2)^4 - T_{sur}^4\right] \tag{f}$$

Equation (f) gives the constant C_2. However, this equation must be solved for C_2 by trial and error.

(5) Checking. *Dimensional Check*: Each term in equations (b), (c) and (f) has units of flux.

Differential equation check: Solution (d) satisfies the governing equation (a).

(6) Computations. Equation (e) gives

$$\text{C}_1 = -\frac{35{,}500\ (\text{W/m}^2)}{3.8\ (\text{W/m-}^{\text{o}}\text{C})} 0.055(\text{m}) = -513.82 \frac{^{\text{o}}\text{C}}{\text{m}} = -513.82 \frac{\text{K}}{\text{m}}$$

Equation (f) gives

$$\frac{0.055(\text{m})}{0.12(\text{m})} 35{,}500\ (\text{W/m}^2) = 31.4\,(\text{W/m}^2\ \text{-K})\left[\frac{-0.055\,(\text{m})}{3.8\,(\text{W/m - K})} 35{,}500(\text{W/(W/m}^2)\ln 0.12(\text{m}). + C_2 - 297.15\,(\text{K})\right]$$
$$+ (0.92)5.67\times10^{-8}\ (\text{W/m}^2\ \text{-K}^4)\left[\left\{\frac{-0.055(\text{m})}{3.8\,(\text{W/m - K})} 35{,}500\,(\text{W/m}^2)\ln 0.12(\text{m}) + C_2\right\}^4 - (287.15)^4\,(\text{K}^4)\right]$$

Solving this equation for C_2 by trial and error gives

$$\text{C}_2 = -484.8\,\text{K}$$

Substituting (e) and C_2 into (d) gives

$$T(r) = -484.8\,(\text{K}) - \frac{q_i}{k} r_i \ln r \tag{g}$$

To determine the inside surface temperature set $r = r_i$ in (g)

$$T(r_i) = -484.8\,(\mathrm{K}) - \frac{35{,}500(\mathrm{W/m^2{-}^oC})}{3.8(\mathrm{W/m{-}^oC})} 0.055(\mathrm{m}) \ln 0.055(\mathrm{m}) = 1005.48\,\mathrm{K}$$

Similarly, the outside surface temperature is

$$T(r_o) = -484.8\,(\mathrm{K}) - \frac{35{,}500(\mathrm{W/m^2{-}^oC})}{3.8(\mathrm{W/m{-}^oC})} 0.055(\mathrm{m}) \ln 0.055(\mathrm{m}) = 604.625\,\mathrm{K}$$

(7) Comments. (i) A trial and error procedure was required to solve this problem. (ii) Care should be exercised in selecting the correct units for temperature in radiation problems. (iii) With the two temperatures determined a numerical check based on conservation of energy for a tube of length *L* can be made:

Energy entering tube at r_i = Energy conducted through tube = Energy leaving tube at r_o (h)

Energy entering tube = $\dfrac{q}{L} = 2\pi\, r_i\, q_i''$ W/m

Energy conducted through tube = $\dfrac{q}{L} = \dfrac{T(r_i) - T(r_o)}{\dfrac{1}{2\pi k} \ln \dfrac{r_o}{r_i}}$

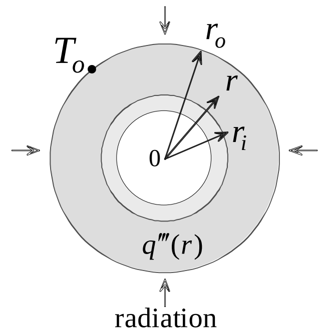
Energy flux leaving at r_o = $\dfrac{q}{L} = 2\pi\, r_o h\left[T(r_o) - T_\infty\right] + 2\pi\, r_o \varepsilon\, \sigma\left[T^4(r_o) - T_{sur}^4\right]$

Numerical computations show that each term in (h) is equal to 12267.9 W/m.

PROBLEM 2.9

Radiation is used to heat a hollow sphere of inner radius r_i*, outer radius* r_o *and conductivity k. Due to the thermal absorption characteristics of the material the radiation results in a variable energy generation rate given by*

$$q'''(r) = q_o''' \frac{r^2}{r_o^2}$$



where q_o''' *is constant and r is the radial coordinate. The inside surface is insulated and the outside surface is maintained at temperature* T_o*. Assume steady state one-dimensional conduction, determine the temperature distribution.*

(1) Observations. (i) Radiation results in a non-uniform energy generation. (ii) Use a spherical coordinate system. (iii) At steady state the total energy generated in the sphere must be equal to the energy leaving the surface by convection and radiation. (iv) This is a one-dimensional steady state conduction problem in a hollow sphere with insulated inner surface and convection and radiation at the outer surface.

(2) Origin and Coordinates. The origin is selected at the center and the coordinate *r* is directed as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) constant conductivity, (4) uniform surface and ambient conditions, and (5) the sphere is modeled as a small gray surface which is enclosed by a much larger surface.

(ii) Governing Equations. Based on the above assumptions, eq.(1.13) gives

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) + \frac{q'''}{k} = 0 \tag{a}$$

where q''' depends on *r* according to

$$q'''(r) = q_o''' \frac{r^2}{r_o^2} \tag{b}$$

Substituting (b) into (a)

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) + \frac{q_o'''}{k}\frac{r^2}{r_o^2} = 0 \tag{c}$$

(iii) Boundary Conditions. Since (a) is second order with a single independent variable *r*, two boundary conditions are needed.

(1) Insulated boundary at r_i

$$\frac{dT(r_i)}{dr} = 0 \tag{d}$$

PROBLEM 2.9 (continued)

(2) Specified temperature at r_o

$$T(r_o) = T_o \tag{e}$$

(4) Solution. Separating variables and integrating (a) twice gives

$$T = -\frac{q_o''}{20kr_o^2} r^4 - \frac{C_1}{r} + C_2 \tag{f}$$

where C_1 and C_2 are constants of integration. Application of boundary condition (d) gives

$$C_1 = \frac{q_o''}{5k} \frac{r_i^5}{r_o^2} \tag{g}$$

Boundary condition (2) gives

$$C_2 = T_o + \frac{q_o'' r_o^2}{5k} \left[\frac{1}{4} + \frac{r_i^5}{r_o^5} \right] \tag{h}$$

Substituting (g) and (h) into (f)

$$T(r) = T_o - \frac{q_o''}{20kr_o^2} \left[r^4 + 4\frac{r_i^5}{r} - r_o^4 - 4\frac{r_i^5}{r_o} \right] \tag{i}$$

Expressing (i) in dimensionless form gives

$$\frac{T(r) - T_o}{\dfrac{q'' r_o^2}{k}} = \frac{1}{20} + \frac{1}{5}\left[\frac{r_i}{r_o}\right]^5 - \frac{1}{5}\left[\frac{r_i}{r_o}\right]^5 \frac{r_o}{r} - \frac{1}{20}\left[\frac{r}{r_o}\right]^4 \tag{j}$$

(5) Checking. *Dimensional check*: Each term in (i) has units of temperature and each term in (j) is dimensionless.

Boundary conditions check: Equation (i) satisfies boundary conditions (d) and (e).

Differential equation check: Direct substitution of (i) into (c) shows that the solution satisfies the governing equation.

Limiting check: If no energy is generated in the sphere the temperature will be uniform equal to T_o. Setting $q_o''' = 0$ in (i) gives $T(r) = T_o$.

Conservation of energy check: The total energy generated in the sphere must leave the outer surface by conduction. Total energy generate is obtained by integrating (b) over the volume of the hollow sphere. Energy conducted the outer surface is obtained by applying Fourier's law at $r = r_o$ using solution (i). Both are found to be equal to

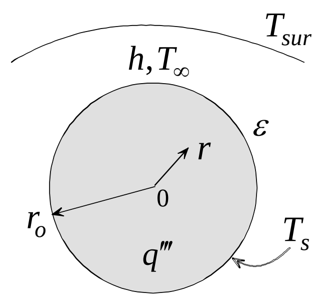
$$\text{Total energy generated} = \frac{4\pi}{5} \frac{q_o'''}{r_o^2} (r_o^5 - r_i^5)$$

(7) Comments. According to (j), the solution is characterized by a single dimensionless geometric parameter r_i / r_o.

PROBLEM 2.10

An electric wire of radius 2 mm and conductivity 398 $\text{W/m–}^{\text{o}}\text{C}$ *generates energy at a rate of* $1.25\times10^5\ \text{W/m}^3$. *The surroundings and ambient temperatures are* 78^{o}C *and* 82^{o}C, *respectively. The heat transfer coefficient is 6.5* $\text{W/m}^2\text{–}^{\text{o}}\text{C}$ *and surface emissivity is 0.9. Neglecting axial conduction and assuming steady state, determine surface and centerline temperatures.*

(1) Observations. (i) The wire generates energy. (ii) Use a cylindrical coordinate system. (iii) Heat conduction is assumed to be in the radial direction only. (iv) At steady state the total energy generated in the wire must be equal to the energy leaving the surface by convection and radiation. (v) To determine center and surface temperatures it is necessary to solve for the temperature distribution in the wire. (vi) This is a one-dimensional steady state conduction problem in a solid cylinder with energy generation.



(2) Origin and Coordinates. The origin is selected at the center and the coordinate r is directed as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) constant conductivity, (4) uniform energy generation, (5) uniform surface and ambient conditions, and (6) the wire is modeled as a small gray surface which is enclosed by a much larger surface.

(ii) Governing Equations. Based on the above assumptions, eq.(1.11) gives

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{q'''}{k}=0 \tag{a}$$

where

k = thermal conductivity = 398 $\text{W/m–}^{\text{o}}\text{C}$
q''' = energy generation rate = $1.25\times10^5\ \text{W/m}^3$

(iii) Boundary Conditions. Since (a) is second order with a single independent variable r, two boundary conditions are needed.

(2) Symmetry gives

$$\frac{dT(0)}{dr}=0 \tag{b}$$

(2) Convection and radiation at the surface $r = r_o$. Conservation of energy at the surface and using Fourier's law, Newton's law of cooling and Stefan-Boltzmann radiation law give

$$-k\frac{dT(r_o)}{dr}=h\left[T(r_o)-T_\infty\right]+\varepsilon\sigma\left[T^4(r_o)-T_{sur}^4\right] \tag{c}$$

where

h = heat transfer coefficient = 6.5 $\text{W/m}^2\text{–}^{\text{o}}\text{C}$
r_o = wire radius = 2 mm = 0.002 m
T_{sur} = surroundings temperature = 82^{o}C = 355 K

T_∞ = ambient temperature = $78^\circ\text{C} = 351\text{ K}$
ε = emissivity = 0.9
σ = Stefan-Boltzmann constant = $5.67\times10^{-8}\ \text{W/m}^2-\text{K}^4$

(4) Solution. Separating variables and integrating (a) twice gives

$$T(r) = -\frac{q'''}{4k}r^2 + C_1 \ln r + C_2 \tag{d}$$

where C_1 and C_2 are constants of integration. Application of boundary condition (b) gives

$$C_1 = 0 \tag{e}$$

Thus (d) becomes

$$T(r) = C_2 - \frac{q'''}{4k}r^2 \tag{f}$$

Boundary condition (2) gives

$$\frac{q'''r_o}{2} = h\left[C_2 - (q'''r_o^2/4k) - T_\infty\right] + \varepsilon\sigma\left[C_2 - (q'''r_o^2/4k)\right]^4 - \varepsilon\sigma T_{sur}^4 \tag{g}$$

This equation gives C_2. However, it can not be solved explicitly for C_2. Solution must be obtained by trial and error. Once C_2 is determined, equation (f) gives center and surface temperatures by setting $r = 0$ and $r = r_o$, respectively. Thus

$$T(0) = C_2 \tag{h}$$

and

$$T(r_o) = C_2 - \frac{q'''}{4k}r_o^2 \tag{i}$$

(5) Checking. *Dimensional check*: Each term in (g) has units of flux and each term in (i) has units of temperature.

Boundary conditions check: Equation (f) satisfies boundary condition (b). Condition (c) can not be checked since the constant C_2 is not known explicitly.

Differential equation check: Direct substitution of (f) into (a) shows that the solution satisfies the governing equation.

Limiting check: If no energy is generated in the wire and $T_\infty = T_{sur}$, the wire temperature will be uniform equal to the ambient temperature. Setting $q''' = 0$ and $T_\infty = T_{sur}$ in (f) gives $T(r) = C_2$. Equation (g) gives $C_2 = T_\infty$.

(6) Computations. Substituting numerical values in (g) and solving for C_2 by trial and error gives $C_2 = 361.2\,\text{K}$. Equations (h) and (i) give center and surface temperatures as

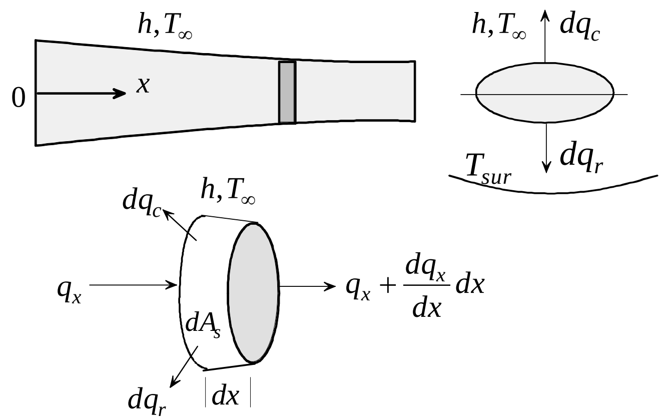
$T(0) = 361.2\,\text{K}$ and $T(r_o) \cong 361.2\,\text{K}$

(7) Comments. The wire is essentially at uniform temperature. This is due to the fact that k is high and r_o and h are small.

PROBLEM 2.11

The cross-sectional area of a fin is $A_c(x)$ and its circumference is $C(x)$. Along its upper half surface heat is exchanged by convection with an ambient fluid at T_∞. The heat transfer coefficient is h. Along its lower half surface heat is exchanged by radiation with a surroundings at T_{sur}. Using a simplified radiation model formulate the steady state heat equation for this fin.

(1) Observations. (i) This is a fin problem with surface convection and radiation. (ii) To formulate the fin equation apply conservation of energy to an element of the fin using fin approximation and Fourier, Newton and Stefan-Boltzmann laws.



(2) Origin and Coordinates. The origin is selected at one end of the fin and the coordinate x is directed as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) no energy generation, (4) $Bi << 1$, and (5) the fin is modeled as a small gray surface which is enclosed by a much larger surface.

(ii) Governing Equations. To formulate the heat equation for this fin we select an element dx at location x as shown. This element is enlarged showing its geometric characteristics and energy interchange at its surfaces. Conservation of energy for the element under steady state conditions requires that

$$\dot{E}_{in} = \dot{E}_{out} \tag{a}$$

where

$\dot{E}_{in}$ = rate of energy added to element
$\dot{E}_{out}$ = rate of energy removed from element

Energy enters the element by conduction at a rate q_x and leaves at a rate $q_x + (dq_x/dx)\,dx$. Energy also leaves by convection at a rate dq_c and by radiation at a rate dq_r. We now formulate each term in (a)

$$\dot{E}_{in} = q_x \tag{b}$$

and

$$\dot{E}_{out} = q_x + \frac{dq_x}{dx}dx + dq_c + dq_r \tag{c}$$

Substituting (b) and (c) into (a)

$$\frac{dq_x}{dx}dx + dq_c + dq_r = 0 \tag{d}$$

We introduce Fourier, Newton and Stefan-Boltzmann laws to eliminate q_x, dq_c and dq_r, respectively. Thus,

$$q_x = -k\,A_c\,\frac{dT}{dx} \tag{e}$$

$$dq_c = h(T - T_\infty)dA_s\,/\,2 \tag{f}$$

and

$$dq_r = \varepsilon\,\sigma\left[T^4 - T_{sur}^4\right]dA_s\,/\,2 \tag{g}$$

where $A_c = A_c(x)$ is the cross-sectional area through which heat is conducted. The infinitesimal area $dA_s\,/\,2$ is half the surface area of the element where heat is exchanged by convection at the top half and by radiation at the bottom half.

Substituting (e), (f) and (g) into (d) we obtain

$$\frac{d}{dx}\left(kA_c(x)\frac{dT}{dx}\right) - h(T - T_\infty)\frac{1}{2}\frac{dA_s}{dx} - \varepsilon\,\sigma(T^4 - T_{sur}^4)\frac{1}{2}\frac{dA_s}{dx} = 0 \tag{h}$$

The gradient of surface area $dA_s\,/\,dx$ is given by eq. (2.6a)

$$\frac{dA_s}{dx} = C(x)\left[1 + (dy_s\,/\,dx)^2\right]^{1/2} \tag{i}$$

where $y_s(x)$ is variable which describes the fin profile and $C(x)$ is the circumference. If k is assumed constant, (h) becomes

$$\frac{d^2T}{dx^2} + \frac{1}{A_c}\frac{dA_c}{dx}\frac{dT}{dx} - \frac{h}{2kA_c}\frac{dA_s}{dx}(T - T_\infty) - \frac{\varepsilon\,\sigma}{2kA_c}\frac{dA_s}{dx}(T^4 - T_{sur}^4) = 0 \tag{j}$$

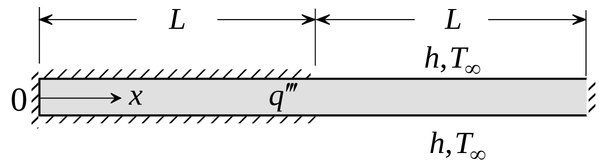
(4) Checking. *Dimensional check*: Each term in (j) has the same units.

Limiting check: If there is no radiation, (j) should reduce to eq. (2.5b). Setting $\varepsilon = 0$ in (j) gives eq. (2.9).

(5) Comments. (i) Radiation introduces non-linearity in the fin equation. (ii) To obtain the corresponding equation for a fin with no surface convection set h = 0 in (j) and remove the ½ factor if radiation takes place over the entire surface.

PROBLEM 2.12

A constant area fin of length $2L$ and cross-sectional area A_c generates heat at a volumetric rate q'''. Half the fin is insulated while the other half exchanges heat by convection. The heat transfer coefficient is h and the ambient temperature is T_∞. The base and tip are insulated. Determine the steady state temperature at the mid-section. At what location is the temperature highest?



(1) Observations. (i) This is a constant area stationary fin. (ii) Temperature distribution can be assumed one dimensional. (iii) Surface conditions are different for each half. Thus two equations are needed. (iv) Both ends are insulated. (v) The highest temperature is at the insulated end $x = 0$.

(2) Origin and Coordinates. The origin is selected at one end of the fin and the coordinate x is directed as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) constant cross-sectional area, (4) uniform energy generation, (5) $Bi << 1$, (6) constant conductivity, (7) no radiation and (8) uniform h and T_∞.

(ii) Governing Equations. Since half the fin is insulated while the other half exchanges heat by convection, two fin equations are needed. Let

$T_1(x)$ = temperature distribution in the first half, $0 \le x \le L$
$T_2(x)$ = temperature distribution in the second half, $L \le x \le 2L$

The fin equation for each section is obtained from eq. (2.5b)

$$\frac{d^2T}{dx^2} + \frac{1}{A_c(x)}\frac{dA_c}{dx}\frac{dT}{dx} - \frac{h}{kA_c(x)}(T - T_\infty)\frac{dA_s}{dx} + \frac{q'''}{k} = 0 \tag{2.5b}$$

Note that A_c is constant and $dA_s / dx = C$, where C is the circumference. Eq. (2.5b) becomes

$$\frac{d^2T}{dx^2} - \frac{hC}{kA_c}(T - T_\infty) + \frac{q'''}{k} = 0 \tag{a}$$

In the first half the fin is insulated along its surface and thus the convection term vanishes. Equation (a) gives

$$\frac{d^2T_1}{dx^2} + \frac{q'''}{k} = 0, \qquad 0 \le x \le L \tag{b}$$

For the second half (a) gives

$$\frac{d^2T_2}{dx^2} - m^2(T_2 - T_\infty) + \frac{q'''}{k} = 0, \qquad L \le x \le 2L \tag{c}$$

where

$$m^2 = \frac{hC}{kA_c} \tag{d}$$

(iii) Boundary Conditions. Four boundary conditions are needed.

(1) Insulated surface at $x = 0$

$$\frac{dT_1(0)}{dx} = 0 \tag{e}$$

(2) Equality of temperature at $x = L$

$$T_1(L) = T_2(L) \tag{f}$$

(3) Equality of flux at $x = L$

$$\frac{dT_1(L)}{dx} = \frac{dT_2(L)}{dx} \tag{g}$$

(4) Insulated surface at $x = 2L$

$$\frac{dT_2(2L)}{dx} = 0 \tag{h}$$

(4) Solution. Separating variables and integrating (b) twice

$$T_1(x) = -\frac{q'''}{2k}x^2 + Ax + B \tag{i}$$

where A and B are constants of integration. The solution to (c) is given in Appendix A, equation (A-2b)

$$T_2(x) = T_\infty + \frac{q'''}{km^2} + C\sinh mx + D\cosh mx \tag{j}$$

where C and D are constants of integration. Application of the four boundary conditions gives the four constants

$$A = 0 \tag{k}$$

$$B = T_\infty + \frac{q'''L^2}{2k} + \frac{q'''}{km^2} - \frac{q'''L}{km}\,\frac{\tanh 2mL \sinh mL - \cosh mL}{\tanh 2mL \cosh mL - \sinh mL} \tag{l}$$

$$C = \frac{q'''L}{km}\,\frac{\tanh 2ml}{\sinh mL - \tanh 2mL \cosh mL} \tag{m}$$

$$D = \frac{q'''L}{km}\,\frac{1}{\tanh 2mL \cosh mL - \sinh mL} \tag{n}$$

Substituting into (i) and (j) gives the solutions as

$$T_1(x) = T_\infty + \frac{q'''L^2}{2k} + \frac{q'''}{km^2} - \frac{q'''L}{km}\,\frac{\tanh 2mL \sinh mL - \cosh mL}{\tanh 2mL \cosh mL - \sinh mL} - \frac{q'''}{2k}x^2 \tag{o}$$

and

$$T_2(x) = T_\infty + \frac{q'''}{km^2} + \frac{q'''L}{km}\,\frac{\cosh mx - \tanh 2mL \sinh mx}{\tanh 2mL \cosh mL - \sinh mL} \tag{p}$$

PROBLEM 2.12 (continued)

Expressing (o) and (p) in dimensionless form gives

$$\frac{T_1(x)-T_\infty}{\dfrac{q'''L^2}{k}} = \frac{1}{2} + \frac{1}{(mL)^2} - \frac{1}{(mL)}\frac{\tanh 2mL \sinh mL - \cosh mL}{\tanh 2mL \cosh mL - \sinh mL} - \frac{1}{2}(x/L)^2 \qquad \text{(q)}$$

and

$$\frac{T_2(x)-T_\infty}{\dfrac{q'''L^2}{k}} = \frac{1}{(mL)^2} + \frac{1}{(mL)}\frac{\cosh(mL)(x/L) - \tanh 2mL \sinh(mL)(x/L)}{\tanh 2mL \cosh mL - \sinh mL} \qquad \text{(r)}$$

Examination of (o) shows that the temperature of the insulated half decreases as the distance from the origin is increased. Thus the highest temperature occurs at $x = 0$.

(5) Checking. *Dimensional check*: Each term in (o) and (p) has units of temperature and each term in (q) and (r) is dimensionless.

Boundary conditions check: Substitution of solutions (o) and (p) into equations (e)-(h) shows that the four boundary conditions are satisfied.

Differential equation check: Direct substitution of (o) into (b) and (p) into (c) shows that the governing equations are satisfied.

Limiting check: (i) If no energy is generated, the entire fin should be at the ambient temperature. Setting $q''' = 0$ in (o) and (p) gives

$$T_1(x) = T_2(x) = T_\infty$$

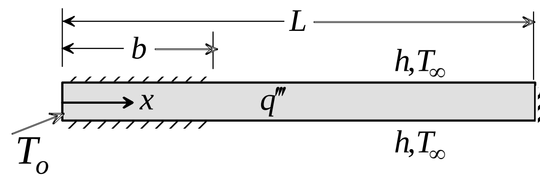
(ii) If $h = 0$, no heat can leave the fin and thus the temperature should be infinite. According to (d) if $h = 0$, then $m = 0$. Setting $m = 0$ in (o) and (p) gives

$$T_1(x) = T_2(x) = \infty$$

(6) Comments. (i) When conditions along a fin are not uniform it is necessary to use more than one fin equation. (ii) An alternate approach to solving this problem is to recognize that all the energy generated in the insulated half must enter the second half at $x = L$. Thus one can consider the second half as a constant area fin with specified flux at the base $x = L$ and an insulated tip at $x = 2L$. The solution to the temperature distribution in this half gives the temperature at $x = L$. The insulated section can be treated as a fin with insulated base at $x = 0$ and a specified temperature at the tip $x = L$. (iii) The solution is characterized by a single parameter mL.

PROBLEM 2.13

A constant area fin of length L and cross-sectional area A_c is maintained at T_o at the base and is insulated at the tip. The fin is insulated along a distance b from the base end while heat is exchanged by convection along its remaining surface. The ambient temperature is T_∞ and the heat transfer coefficient is h. Determine the steady state heat transfer rate from the fin.



(1) Observations. (i) This is a constant area stationary fin. (ii) Temperature distribution can be assumed one-dimensional. (iii) The fin has two sections. Surface conditions are different for each section. Thus two equations are needed. (iv) One end is maintained at a specified temperature and the other end is insulated. (v) To determine fin heat transfer rate the temperature distribution in the two-section fin must be determined.

(2) Origin and Coordinates. The origin is selected at one end of the fin and the coordinate x is directed as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) constant cross-sectional area, (4) uniform energy generation, (5) $Bi << 1$, (6) constant conductivity, (7) uniform h and T_∞, and (8) no radiation.

(ii) Governing Equations. Since one section is insulated while the other exchanges heat by convection, two fin equations are needed. Let

$T_1(x)$ = temperature distribution in the first section, $0 \le x \le b$
$T_2(x)$ = temperature distribution in the second half, $b \le x \le L$

The fin equation for each section is obtained from eq. (2.5b)

$$\frac{d^2T}{dx^2} + \frac{1}{A_c(x)}\frac{dA_c}{dx}\frac{dT}{dx} - \frac{h}{kA_c(x)}(T - T_\infty)\frac{dA_s}{dx} + \frac{q'''}{k} = 0 \tag{2.5b}$$

Note that A_c is constant and $dA_s/dx = C$, where C is the circumference. Eq. (2.5b) becomes

$$\frac{d^2T}{dx^2} - \frac{hC}{kA_c}(T - T_\infty) + \frac{q'''}{k} = 0 \tag{a}$$

In the first section the fin is insulated along its surface and thus the convection term vanishes. Equation (a) gives

$$\frac{d^2T_1}{dx^2} + \frac{q'''}{k} = 0, \qquad 0 \le x \le b \tag{b}$$

For the second section equation (a) gives

$$\frac{d^2T_2}{dx^2} - m^2(T_2 - T_\infty) + \frac{q'''}{k} = 0, \qquad b \le x \le L \tag{c}$$

where

$$m^2 = \frac{hC}{kA_c} \tag{d}$$

(iii) Boundary Conditions. Four boundary conditions are needed.

(1) Specified temperature at $x = 0$

$$T_1(0) = T_o \tag{e}$$

(2) Equality of temperature at $x = b$

$$T_1(b) = T_2(b) \tag{f}$$

(3) Equality of flux at $x = b$

$$\frac{dT_1(b)}{dx} = \frac{dT_2(b)}{dx} \tag{g}$$

(4) Insulated surface at $x = L$

$$\frac{dT_2(L)}{dx} = 0 \tag{h}$$

(4) Solution. Separating variables and integrating (b) twice

$$T_1(x) = -\frac{q'''}{2k}x^2 + A_1 x + B_1 \tag{i}$$

where A_1 and B_1 are constants of integration. The solution to (c) is given in Appendix A, equation (A-2b)

$$T_2(x) = T_\infty + \frac{q'''}{km^2} + A_2 \sinh mx + B_2 \cosh mx \tag{j}$$

where A_2 and B_2 are constants of integration. Application of the four boundary conditions give the four constants

$$A_1 = \frac{q'''b}{k} + \frac{\left(T_o - T_\infty + \dfrac{q'''b^2}{2k} - \dfrac{q'''}{km^2}\right)\left(m \sinh mb - m \tanh mL \cosh mb\right)}{\cosh mb - \tanh mL \sinh mb - mb \sinh mb + mb \tanh mL \cosh mb} \tag{k}$$

$$B_1 = T_o \tag{l}$$

$$A_2 = -\frac{\left(T_o - T_\infty + \dfrac{q'''b^2}{2k} - \dfrac{q'''}{km^2}\right)\tanh mL}{\cosh mb - \tanh mL \sinh mb - mb \sinh mb + mb \tanh mL \cosh mb} \tag{m}$$

$$B_2 = \frac{T_o - T_\infty + \dfrac{q'''b^2}{2k} - \dfrac{q'''}{km^2}}{\cosh mb - \tanh mL \sinh mb - mb \sinh mb + mb \tanh mL \cosh mb} \tag{n}$$

Fin heat transfer rate q_f is equal to the energy leaving surface area $C(L\text{-}b)$ by convection. Using Newton's law of cooling

$$q_f = hC\int_b^L (T_2 - T_\infty)dx \tag{o}$$

where $T_2(x)$ is given by (j). An alternate and simpler approach is based on conservation of energy for the two-section fin. Thus

$$q_f = q(0) + \dot{E}_g \tag{p}$$

where

$\dot{E}_g$ = rate of energy generated in the fin

$q(0)$ = rate of energy conducted at the base

$\dot{E}_g$ is given by

$$\dot{E}_g = CLq''' \tag{q}$$

Fourier's law gives

$$q(0) = -kA_c \frac{dT_1(0)}{dx} \tag{r}$$

Using (i)

$$q(0) = -kA_c A_1 \tag{s}$$

Substituting (q) and (s) into (p) gives the fin heat transfer rate

$$q_f = -kA_c A_1 + CLq''' \tag{t}$$

(5) Checking. *Dimensional check*: Each term in (i) and (j) has units of temperature.

Boundary conditions check: Substitution of solutions (i) and (j) into equations (e)-(h) shows that the four boundary conditions are satisfied.

Differential equation check: Direct substitution of (i) into (b) and (j) into (c) confirms that the governing equations are satisfied.

Limiting check: (i) If $q''' = 0$ and $T_o = T_\infty$, the temperature distribution in the fin should be uniform equal to T_∞. For this case equations (k), (m) and (n) give $A_1 = A_2 = B_2 = 0$. Substituting into equations (i) and (j) gives $T_1(x) = T_2(x) = T_\infty$. (ii) If $q''' = b = 0$, the problem reduces to a constant area fin with specified temperature at the base, insulated tip and convection at its surface. The solution to this case is given eq. (2.14) of Section 2.2.9. Setting $q''' = b = 0$ in (j) and rearranging the result gives

$$T_2(x) = T_\infty + (T_o - T_\infty)\frac{\cosh m(L-x)}{\cosh mL}$$

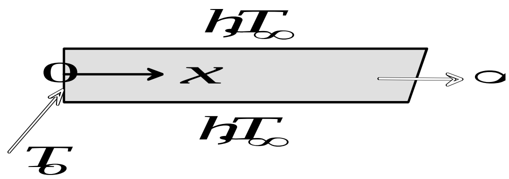
This is identical to eq. (2.14).

(6) Comments. When conditions along a fin are not uniform it is necessary to use more than one fin equation.

PROBLEM 2.14

Heat is removed by convection from an electronic package using a single fin of circular cross section. Of interest is increasing the heat transfer rate without increasing the weight or changing the material of the fin. It is recommended to use two identical fins of circular cross-section with a total weight equal to that of the single fin. Evaluate this recommendation using a semi-infinite fin model.

(1) Observations. (i) This is a constant area semi-infinite fin problem. (ii) Determine the heat transfer rate from a circular fin and examine the effect of radius on the heat transfer rate. (iii) Assume that the base is at a specified temperature and that heat transfer from the surface is by convection.



(2) Origin and Coordinates. The origin is selected at the base and the coordinate x is directed as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) constant cross-sectional area, (4) no energy generation, (5) $Bi \ll 1$, (6) constant conductivity, (7) convection at the surface, (8) uniform heat transfer coefficient and ambient temperature, (9) specified base temperature and (10) no radiation.

(ii) Governing Equations. The fin equation for this case is given by eq. (2.10)

$$\frac{d^2\theta}{dx^2} - m^2\,\theta = 0 \tag{2.10}$$

where

$$\theta = T(x) - T_\infty \tag{a}$$

and

$$m^2 = \frac{hC}{kA_c} \tag{b}$$

For a circular rod of radius r the cross-sectional area A_c and the circumference C are given by

$$A_c = \pi r^2 \tag{c}$$

and

$$C = 2\pi r \tag{d}$$

Substituting (c) and (d) into (b)

$$m^2 = \frac{2h}{kr} \tag{e}$$

(iii) Boundary Conditions. The two boundary conditions are

(1) Specified temperature at the base $x = 0$

$$T(0) = T_o \text{ or } T(0) - T_o = \theta(0) = 0 \tag{f}$$

(3) Finite temperature at $x = \infty$

$$T(\infty) = \text{finite or } \theta(\infty) = \text{ finite} \tag{g}$$

(4) Solution. The solution to eq. (2.10) is

$$\theta(x) = C_1 \exp(mx) + C_2 \exp(-mx) \tag{h}$$

where C_1 and C_2 are constants of integration. Conditions (f) and (g) give C_1 and C_2

$$C_1 = 0 \text{ and } C_2 = T_o - T_\infty \tag{i}$$

Substituting into (h) and using (a) gives the temperature distribution in the fin

$$T(x) = T_\infty + (T_o - T_\infty)e^{-mx} \tag{j}$$

Fourier's law gives the heat transfer rate from the fin

$$q_f = -kA_c \frac{dT(0)}{dx} = k A_c m(T_o - T_\infty) \tag{k}$$

Substituting (c) and (e) into (k)

$$q_f = \pi (T_o - T_\infty)\sqrt{2hk}\, r^{3/2} \tag{l}$$

We now compare the heat transfer rate from a single fin of radius *r* with that from two fins having the same weight as the single fin. Let r_2 be the radius of each of the two fins. Equality of mass gives

$$2\pi r_2^2 = \pi r^2$$

Thus

$$r_2 = \frac{r}{\sqrt{2}} \tag{m}$$

The total heat transfer rate from two fins of radius r_2 each, q_{2f}, is obtained by substituting (m) into (l)

$$q_{2f} = 2\pi (T_o - T_\infty)\sqrt{2hk}\, (r/\sqrt{2})^{3/2} = 2^{1/4} \pi (T_o - T_\infty)\sqrt{2hk}\, r^{3/2} \tag{n}$$

Taking the ratio of (n) and (l)

$$\frac{q_{2f}}{q_f} = 2^{1/4} = 1.19$$

(5) Checking. *Dimensional check*: Each term in (j) has units of temperature. The heat transfer rate q_f in equations (l) and (n) is expressed in units of watts.

Boundary conditions check: Substitution of solutions (j) into equations (f) and (g) shows that the boundary conditions are satisfied.

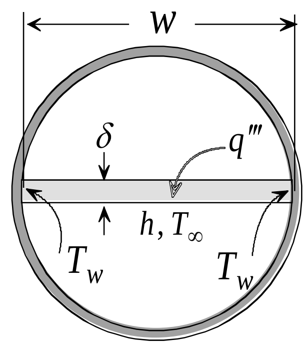
Differential equation check: Direct substitution of (j) into eq. (2.10) confirms that the governing equation is satisfied.

Limiting check: (i) If $T_o = T_\infty$, the temperature distribution in the fin should be uniform equal to T_∞. Setting $T_o = T_\infty$ in (j) gives $T(x) = T_\infty$. (ii) If k = 0 or h = 0, the heat transfer from the fin should vanish. Setting k = 0 or h = 0 in equation (l) gives $q_f = 0$.

(6) Comments. Two fins having the same weight as a single fin transfer 19% more heat than a single fin.

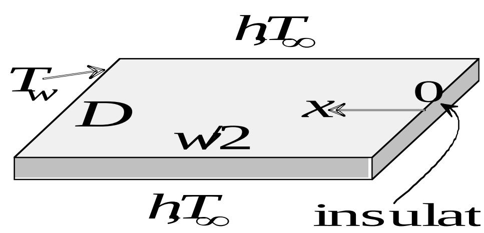
PROBLEM 2.15

A plate which generates heat at a volumetric rate q''' is placed inside a tube and cooled by convection. The width of the plate is w and its thickness is δ. The heat transfer coefficient is h and coolant temperature is T_∞. The temperature at the interface between the tube and the plate is T_w. Because of concern that the temperature at the center may exceed design limit, you are asked to estimate the steady state center temperature using a simplified fin model. The following data are given:



$h = 62\ \text{W/m}^2\text{–}^\circ\text{C}$, $\quad w = 3\,\text{cm}$, $\quad \delta = 2.5\ \text{mm}$,
$q''' = 2.5\times10^7\ \text{W/m}^3$, $T_w = 110^\circ\text{C}$, $T_\infty = 94\,^\circ\text{C}$,
$k = 18\ \text{W/m–}^\circ\text{C}$.

(1) Observations. (i) This is a constant area fin. (ii) Temperature distribution can be assumed one dimensional. (iii) Temperature distribution is symmetrical about the center. Thus the temperature gradient at the center is zero. (iv) Consider half the plate as a fin which extends from the center to the tube surface. One end is maintained at a specified temperature and the other end is insulated.



(2) Origin and Coordinates. The origin is selected at the center of tube and the coordinate x is directed as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) constant area, (4) uniform energy generation, (5) $Bi \ll 1$, (6) constant conductivity, (7) uniform h and T_∞, (8) negligible radiation, and (9) negligible temperature variation along the tube.

(ii) Governing Equations. The fin equation is obtained from eq. (2.5b)

$$\frac{d^2T}{dx^2} + \frac{1}{A_c(x)}\frac{dA_c}{dx}\frac{dT}{dx} - \frac{h}{kA_c(x)}(T - T_\infty)\frac{dA_s}{dx} + \frac{q'''}{k} = 0 \tag{2.5b}$$

Note that A_c is constant and $dA_s/dx = C$, where C is the circumference. Eq. (2.5b) becomes

$$\frac{d^2T}{dx^2} - m^2(T - T_\infty) + \frac{q'''}{k} = 0 \tag{c}$$

where

$$m^2 = \frac{hC}{kA_c} \tag{d}$$

where A_c is the cross-sectional given by

$$A_c = \delta D \tag{e}$$

where D is the length of plate along the tube and δ is plate thickness. Neglecting heat loss from the end surface $\delta \times w/2$, the circumference is given by

$$C = 2D \tag{f}$$

Substituting (e) and (f) into (d)

$$m^2 = \frac{2h}{k\delta} \tag{g}$$

(iii) Boundary Conditions. The two conditions are

(1) Insulated surface at $x = 0$

$$\frac{dT(0)}{dx} = 0 \tag{h}$$

(2) Specified temperature at $x = w/2$

$$T(w/2) = T_w \tag{i}$$

(4) Solution. The solution to (c) is given in Appendix A, equation (A-2b)

$$T(x) = T_\infty + \frac{q'''}{km^2} + C_1 \sinh mx + C_2 \cosh mx \tag{j}$$

where C_1 and C_2 are constants of integration. Application of boundary conditions (h) and (i) gives the two constants

$$C_1 = 0 \tag{k}$$

and

$$C_2 = \left(T_w - T_\infty - \frac{q'''}{km^2}\right)\frac{1}{\cosh mw/2} \tag{l}$$

Substituting into (j)

$$T(x) = T_\infty + \frac{q'''}{km^2} + \left(T_w - T_\infty - \frac{q'''}{km^2}\right)\frac{\cosh mx}{\cosh mw/2} \tag{m}$$

The maximum temperature may occur at $x = 0$. Setting $x = 0$ in equation (m) gives

$$T(0) = T_\infty + \frac{q'''}{km^2} + \left(T_w - T_\infty - \frac{q'''}{km^2}\right)\frac{1}{\cosh mw/2} \tag{n}$$

(5) Checking. *Dimensional check*: Each term in solution (m) has units of temperature.

Boundary conditions check: Substitution of solution (m) into equations (h) and (i) shows that the boundary conditions are satisfied.

Differential equation check: Direct substitution of (m) into (e) confirms that the governing equation is satisfied.

Limiting check: If $T_w = T_\infty$ and $q''' = 0$, the temperature distribution in the plate should be uniform equal to T_∞. Setting $T_w = T_\infty$ and $q''' = 0$ in (m) gives $T(x) = T_\infty$.

(6) Computations. Equation (g) is used to compute m

$$m^2 = \frac{2(62)(\text{W/m}^2-{}^\text{o}\text{C})}{18(\text{W/m}-{}^\text{o}\text{C})(0.0025)(\text{m})} = 2755.6\ (1/\text{m}^2)$$

m = 52.493 (1/m)

Substituting the given data into (n)

$$T(0) = 94({}^\text{o}\text{C}) + \frac{2.5\times10^7\,(\text{W/m}^3)}{18(\text{W/m}-{}^\text{o}\text{C})2755.6(1/\text{m}^2)} +$$

$$\left[110^\text{o}\text{C} - 94^\text{o}\text{C} - \frac{2.5\times10^7\,(\text{W/m}^3)}{18(\text{W/m}-{}^\text{o}\text{C})2755.6(1/\text{m}^2)}\right]\frac{1}{\cosh 52.493(1/\text{m})0.015(\text{m})}$$

$T(0)$ = 230.1 $^\text{o}$C

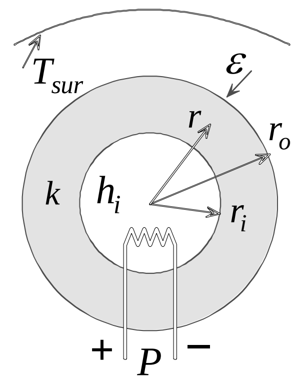
(7) Comments. (i) Examination of (m) shows that the maximum temperature may not always be at the center. For example, in the limiting case of $q''' = 0$ the highest temperature will be at the tube surface while the center will be at the lowest temperature. However, for the data given in this problem the maximum temperature occurs at the center. (ii) The solution is expressed in dimensionless form to determine the governing parameters. Equation (m) is rewritten as

$$\frac{T(x)-T_\infty}{T_w - T_\infty} = \frac{q'''}{km^2(T_w - T_\infty)} + \left[1 - \frac{q'''}{k\,m^2(T_w - T_\infty)}\right]\frac{\cosh(mw)(x/w)}{\cosh(mw)/2} \tag{o}$$

This form shows that two parameters characterizes the solution: mw and $\dfrac{q'''}{km^2(T_w - T_\infty)}$.

PROBLEM 2.16

An electric heater with a capacity P is used to heat air in a spherical chamber. The inside radius is r_i, *outside radius* r_o *and the conductivity is* k. *At the inside surface heat is exchanged by convection. The inside heat transfer coefficient is* h_i. *Heat loss from the outside surface is by radiation. The surroundings temperature is* T_{sur} *and surface emissivity is* ε. *Assuming one-dimensional steady state conduction, use a simplified radiation model to determine:*



[*a*] *The temperature distribution in the spherical wall.*

[*b*] *The inside air temperatures for the following conditions:*

$h = 6.5\ \text{W/m}^2\text{–}^\circ\text{C}$, $P = 1,500$ *W*, $\varepsilon = 0.81$,
$k = 2.4\ \text{W/m–}^\circ\text{C}$, $T_{sur} = 18^\circ\text{C}$, $r_i = 10$ *cm*, $r_o = 14$ *cm.*

(1) Observations. (i) This is a one-dimensional steady state problem. (ii) Two boundary conditions are needed. (iii) Spherical geometry. (iv) Specified heat transfer rate at the inner surface and radiation at the outer surface. (v) Since radiation is involved in this problem, temperature units should be expressed in kelvin

(2) Origin and Coordinates. The origin and coordinate r are shown.

(3) Formulation.

(i) Assumptions. (1) Steady state, (2) one-dimensional, (3) constant properties, (4) no energy generation, (5) small gray surface which is enclosed by a much larger surface, (6) uniform conditions at the two surfaces and (7) stationary material.

(ii) Governing Equations. Based on the above assumptions, the heat equation in Cartesian coordinates is given by

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0 \tag{a}$$

(iii) Boundary Conditions. Two boundary conditions are needed. They are:

(3) Specified heat transfer rate at $r = r_i$

$$-k4\pi r_i^2\frac{dT(r_i)}{dr} = P \tag{b}$$

where

$k =$ thermal conductivity = $2.4\ \text{W/m–}^\circ\text{C} = 2.4\ \text{W/m–}^\circ\text{K}$
$P =$ electric heater capacity = heat transfer rate = 1,500 W
$r_i =$ inside radius = 10 cm

(2) Radiation at $r = r_o$. Conservation of energy, Fourier's law of conduction and Stefan-Boltzmann radiation law give

$$-k\frac{dT(r_o)}{dr}=\varepsilon\,\sigma\left[T^4(r_o)-T_{sur}^4\right] \tag{c}$$

where

$r_o=$ outside radius = 14 cm

T_{sur} = surroundings temperature = 18 ^{o}C = (18 + 273.15) K = 291.15 K

$\varepsilon=$ emissivity = 0.81

$\sigma=$ Stefan-Boltzmann constant = 5.67×10^{-8} W/m$^2-$K^4

(4) Solution. Integrating (a) twice gives

$$T(r)=-\frac{C_1}{r}+C_2 \tag{d}$$

where C_1 and C_2 are constants of integration. Application of boundary conditions (b) and (c) gives C_1 and C_2

$$C_1=-\frac{P}{4\pi k} \tag{e}$$

$$C_2=-\frac{P}{4\pi k r_o}+\left[T_{sur}^4+\frac{P}{4\pi\varepsilon\sigma\,r_o^2}\right]^{\frac{1}{4}} \tag{f}$$

Substituting (e) and (f) into (d)

$$T(r)=\frac{P}{4\pi kr}-\frac{P}{4\pi kr_o}+\left[T_{sur}^4+\frac{P}{4\pi\varepsilon\sigma\,r_o^2}\right]^{\frac{1}{4}} \tag{g}$$

Expressing (g) in dimensionless form gives

$$\frac{T(r)}{\dfrac{P}{kr_o}}=\frac{1}{4\pi}\left[\frac{r_o}{r}-1\right]+\frac{kr_oT_{sur}}{P}\left[1+\frac{P}{4\pi\varepsilon\sigma r_o^2T_{sur}^4}\right]^{\frac{1}{4}} \tag{h}$$

To determine the inside air temperature T_i apply Newton's law of cooling at the inside surface

$$P=4\pi r_i^2h[T_i-T(r_i)) \tag{i}$$

where

$h=6.5(\text{W/m}^2-{}^{\text{o}}\text{C})=6.5(\text{W/m}^2-{}^{\text{o}}\text{K})$

Solving (i) for T_i

$$T_i=T(r_i)+\frac{P}{4\pi\,r_i^2h} \tag{j}$$

where $T(r_i)$ is obtained from (h) by setting $r=r_i$.

(5) Checking. *Dimensional Check*: Each term in equations (g) and (j) has units of temperature. Each term in (h) is dimensionless.

Differential equation check: Solution (g) satisfies the governing equation (a).

Boundary conditions check: Solution (g) satisfies boundary conditions (b) and (c).

Limiting check: If the heater is turned off (P = 0) the sphere and inside air will be at the same temperature as the surroundings. Setting P = 0 in (g) and (j) gives $T(r) = T_i = T_{sur}$.

(6) Computations. Equation (g) is used to determine $T(r_i)$

Equation (f) gives

$$T(r_i) = \frac{1500(\text{W})}{4\pi 2.4(\text{W/m-}^\circ\text{K})0.1(\text{m})} - \frac{1500(\text{W})}{4\pi 2.4(\text{W/m-}^\circ\text{K})0.14(\text{m})} +$$

$$\left[(291.15)^4(\text{K}^4) + \frac{1500(\text{W})\times 10^8}{4\pi(0.81)5.67(\text{W/m}^2\text{-K}^4)(0.14)^2(\text{m}^2)}\right]^{\frac{1}{4}} = 753.74\text{K}$$

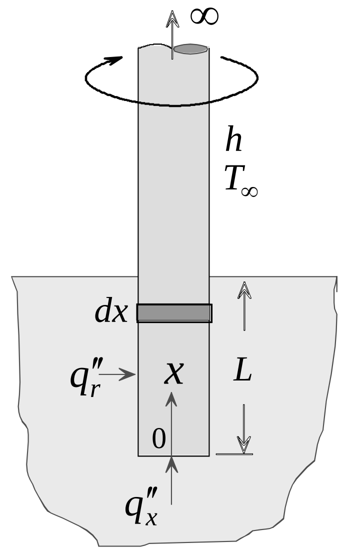
Substituting into (j)

$$T_i = 753.74(\text{K}) + \frac{1{,}500(\text{W})}{4\pi(0.1)^2(\text{m}^2)6.5(\text{W/m}^2-\text{K})} = 2{,}590\text{K}$$

(7) Comments. (i) The inside air temperature is high. This is due to the small inside surface area and small heat transfer coefficient. (ii) Care should be exercised in selecting the correct units for temperature in radiation problems.

PROBLEM 2.17

A section of a long rotating shaft of radius r_o is buried in a material of very low thermal conductivity. The length of the buried section is L. Frictional heat generated in the buried section can be modeled as surface heat flux. Along the buried surface the radial heat flux is q_r'' and the axial heat flux is q_x''. The exposed surface exchanges heat by convection with the ambient. The heat transfer coefficient is h and the ambient temperature is T_∞. Model the shaft as semi-infinite fin and assume that all frictional heat is conducted through the shaft. Determine the temperature distribution.



(1) Observations. (i) This is a constant area fin. (ii) Temperature distribution can be assumed one dimensional. (iii) The shaft has two sections. Surface conditions are different for each of sections. Thus two equations are needed. (iv) Shaft rotation generates surface flux in the buried section and does not affect the fin equation.

(2) Origin and Coordinates. The origin is selected at one end of the fin and the coordinate x is directed as shown.

(3) Formulation.

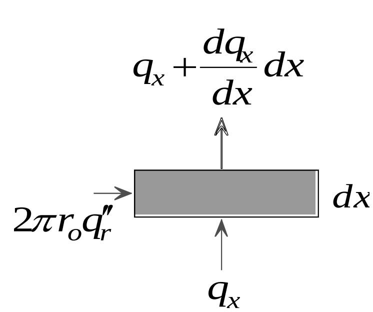
(i) Assumptions. (1) One-dimensional, (2) steady state, (3) constant cross-sectional area, (4) uniform surface flux in the buried section, (5) $Bi \ll 1$, (6) constant conductivity, (7) no radiation (8) no energy generation and (9) uniform h and T_∞.

(ii) Governing Equations. Since surface conditions are different for the two sections, two fin equations are needed. Let

$T_1(x) =$ temperature distribution in the buried section, $0 \le x \le L$
$T_2(x) =$ temperature distribution in the exposed section, $L \le x \le 2L$

The heat equation for the buried section must be formulated using fin approximation. Conservation of energy for the fin element dx gives

$$q_x + q_r'' 2\pi r_o dx = q_x + \frac{dq_x}{dx} dx \qquad \text{(a)}$$



where q_x is the rate of heat conducted in the x-direction, given by Fourier's law

$$q_x = -k \frac{dT_1}{dx} \qquad \text{(b)}$$

Substituting (b) into (a) and rearranging

$$\frac{d^2 T_1}{dx^2} + \frac{2q_r''}{k r_o} = 0, \qquad 0 \le x \le L \qquad \text{(c)}$$

The heat equation for constant area fin with surface convection is given by equation (2.9)

$$\frac{d^2T_2}{dx^2} - \frac{hC}{kA_c}(T_2 - T_\infty) = 0\,, \qquad x \ge L \tag{d}$$

where C is the circumferance and A_c is the cross section area given by

$$C = 2\pi\, r_o \tag{e}$$

$$A\pi\, r_o^2 \tag{f}$$

Using (c) and (d) to define m

$$m = \sqrt{\frac{hC}{kA_c}} = \sqrt{\frac{2h}{kr_o}} \tag{g}$$

Subatituting (e) into (d)

$$\frac{d^2T_2}{dx^2} - m^2(T_2 - T_\infty) = 0 \tag{h}$$

(iii) Boundary Conditions. Four boundary conditions are needed.

(1) Specified flux at $x = 0$

$$-k\frac{dT_1(0)}{dx} = q''_x \tag{i}$$

(2) Equality of temperature at $x = L$

$$T_1(L) = T_2(L) \tag{j}$$

(3) Equality of flux at $x = L$

$$\frac{dT_1(L)}{dx} = \frac{dT_2(L)}{dx} \tag{k}$$

(4) Finite temperature at $x = \infty$

$$T_2(\infty) = \text{ finite} \tag{l}$$

(4) Solution. The solutions to (c) is

$$T_1(x) = -\frac{q''_r}{kr_o}x^2 + A_1 x + B_1 \tag{m}$$

where A_1 and B_1 are constants of integration. The solution to (h) is given in Appendix A, equation (A-2b)

$$T_2(x) = T_\infty + A_2\, e^{mx} + B_2\, e^{-mx} \tag{n}$$

where A_2 and B_2 are constants of integration. Application of the four boundary conditions gives the four constants

$$A_1 = -\frac{q''_x}{k} \tag{o}$$

$$B_1 = T_\infty + \frac{q''_r}{kr_o}L^2 + \frac{q''_x}{k}L + \frac{1}{m}\left[2\frac{q''_r}{kr_o}L + \frac{q''_x}{k}\right] \tag{p}$$

PROBLEM 2.17 (continued)

$$A_2 = 0 \tag{q}$$

$$B_2 = \frac{1}{m}\left[2\frac{q_r''}{kr_o}L + \frac{q_x''}{k}\right]e^{mL} \tag{r}$$

Substituting into (m) and (n) gives the solutions as

$$T_1(x) = T_\infty + \frac{q_r''}{kr_o}L^2 + \frac{q_x''}{k}L + \frac{1}{m}\left[2\frac{q_r''}{kr_o}L + \frac{q_x''}{k}\right] - \frac{q_r''}{kr_o}x^2 - \frac{q_x''}{k}x \tag{s}$$

and

$$T_2(x) = T_\infty + \frac{1}{m}\left[2\frac{q_r''}{kr_o}L + \frac{q_x''}{k}\right]e^{m(L-x)} \tag{t}$$

Expressing (s) and (t) in dimensionless form gives

$$\frac{T_1(x) - T_\infty}{\dfrac{q_r''L^2}{kr_o}} = 1 - \left[\frac{x}{L}\right]^2 + \frac{q_x''r_o}{q_r''L}\left[1 - \frac{x}{L}\right] + \frac{1}{mL}\left[2 + \frac{q_x''r_o}{q_r''L}\right] \tag{u}$$

and

$$\frac{T_2(x) - T_\infty}{\dfrac{q_r''L^2}{kr_o}} = \frac{1}{mL}\left[2 + \frac{q_x''r_o}{q_r''L}\right]e^{mL(1-x/L)} \tag{v}$$

(5) Checking. *Dimensional check*: Each term in (s) and (t) has units of temperature and each term in (u) and (v) is dimensionless.

Boundary conditions check: Substitution of solutions (s) and (t) into equations (i)-(l) shows that the four boundary conditions are satisfied.

Differential equation check: Direct substitution of (s) into (c) and (t) into (d) shows that the governing equations are satisfied.

Limiting check: (i) If no energy is added to the buried section, the entire fin should be at the ambient temperature T_∞.. Setting $q_x'' = q_r'' = 0$ in (s) and (t) gives

$$T_1(x) = T_2(x) = T_\infty$$

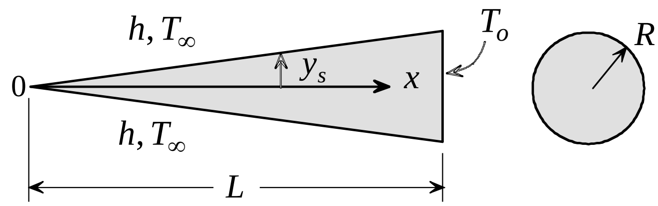
(ii) If $h = 0$, no heat can leave the fin and thus the temperature should be infinite. According to (g) if $h = 0$, then $m = 0$. Setting $m = 0$ in (s) and (t) gives

$$T_1(x) = T_2(x) = \infty$$

(6) Comments. (i) When conditions along a fin are not uniform it is necessary to use more than one fin equation. (ii) The solution is characterized by two dimensionless parameters: mL and $\dfrac{q_x''r_o}{q_r''L}$.

PROBLEM 2.18

A conical spine with a base radius R and length L exchanges heat with the ambient by convection. The heat transfer coefficient is h and the ambient temperature is T_∞. The base is maintained at T_o. Using a fin model, determine the steady state heat transfer rate.



(1) Observations. (i) Temperature distribution in the fin and Fourier's law of conduction give the heat transfer rate. (ii) This is a variable area fin with specified temperature at the base and convection at the surface. (iii) Temperature distribution can be assumed one-dimensional. (iv) Eq. (2.5) should be used to formulate the heat equation for this fin.

(2) Origin and Coordinates. The origin is selected at the tip and the coordinate x is directed as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) no energy generation, (4) $Bi << 1$, (5) constant properties, (6) uniform h and T_∞, and (7) negligible radiation.

(ii) Governing Equations. Applying Fourier's law at $x = L$ gives the fin heat transfer rate

$$q_f = -kA_c(L)\frac{dT(L)}{dx} \tag{a}$$

where $A_c(L)$ is the cross-sectional area at $x = L$, given by

$$A_c(L) = \pi R^2 \tag{b}$$

Substituting (b) into (a)

$$q_f = -\pi k R^2 \frac{dT(L)}{dx} \tag{c}$$

Temperature distribution in the fin is obtained from the solution to the heat equation for the fin. Based on the above assumptions, eq. (2.5b) gives

$$\frac{d^2T}{dx^2} + \frac{1}{A_c(x)}\frac{dA_c}{dx}\frac{dT}{dx} - \frac{h}{kA_c(x)}(T - T_\infty)\frac{dA_s}{dx} = 0 \tag{d}$$

The cross-sectional area $A_c(x)$ for a conical fin is given by

$$A_c = \pi y_s^2 \tag{e}$$

where y_s is the local radius given by

$$y_s = (R/L)x \tag{f}$$

Substituting (f) into (e)

$$A_c(x) = \pi (R/L)^2 x^2 \tag{g}$$

Eq. (2.6a) is used to formulate dA_s / dx

$$\frac{dA_s}{dx} = C(x)\left[1 + (dy_s / dx)^2\right]^{1/2}$$

Using (f) into the above

$$\frac{dA_s}{dx} = 2\pi(R/L)\left[1 + (R/L)^2\right]^{1/2} x \tag{h}$$

Substituting (g) and (h) into (d) and rearranging

$$x^2 \frac{d^2\theta}{dx^2} + 2x\frac{d\theta}{dx} - \beta^2 x\theta = 0 \tag{i}$$

where

$$\theta = T - T_\infty \tag{j}$$

and

$$\beta^2 = \frac{2hL}{kR}\sqrt{1 + (R/L)^2} \tag{k}$$

(iii) Boundary Conditions. The two boundary conditions are

(1) Finite temperature at $x = 0$

$$T(0) = \text{ finite, or } \theta(0) = T(0) - T_\infty = \text{finite} \tag{l}$$

(2) Specified temperature at $x = L$

$$T(L) = T_o, \text{ or } \theta(L) = T(L) - T_\infty = T_o - T_\infty \tag{m}$$

(4) Solution. Equation (i) is a second order differential equation with variable coefficient. It may be a Bessel equation. Comparing (i) with eq. (2.26) gives

$$A = -1/2$$
$$B = 0$$
$$C = 1/2$$
$$D = 2\beta i, \quad p = D/i = 2\beta$$
$$n = 1$$

Since n is an integer and D is imaginary the solution to (i) is given by eq. (2.29)

$$\theta(x) = T(x) - T_\infty = x^{-1/2}\left[C_1 I_1(2\beta x^{1/2}) + C_2 K_1(2\beta x^{1/2})\right] \tag{n}$$

where C_1 and C_2 are constants of integration. Since $K_1(0) = \infty$ (Table 2.1), condition (l) requires that

$$C_2 = 0 \tag{o}$$

Boundary condition (m) gives

$$C_1 = \frac{(T_o - T_\infty)L^{1/2}}{I_1(2\beta L^{1/2})} \tag{p}$$

Substituting (o) and (p) into (n)

$$\frac{T(x)-T_\infty}{T_o-T_\infty}=(L/x)^{1/2}\,\frac{I_1(2\beta x^{1/2})}{I_1(2\beta L^{1/2})} \tag{q}$$

With the temperature distribution known, equation (c) gives the heat transfer rate. Using eq. (2.42) to determine the derivative of $I_1(2\beta x^{1/2})$ and substituting into (c) gives

$$q_f=-\pi R^{3/2}\sqrt{2hk}\left[1+(R/L)^2\right]^{1/4}(T_o-T_\infty)\,\frac{I_2(2\beta L^{1/2})}{I_1(2\beta L^{1/2})} \tag{r}$$

Expressing the above in dimensionless form gives

$$\frac{q_f}{\pi\sqrt{2}R^{3/2}\sqrt{hk}\,\left[1+(R/L)^2\right]^{1/4}(T_o-T_\infty)}=-\frac{I_2(2\beta L^{1/2})}{I_1(2\beta L^{1/2})} \tag{s}$$

(5) Checking. *Dimensional check*: (i) The argument of the Bessel function, $(2\beta x^{1/2})$ is dimensionless. (ii) Equation (r) gives the correct units for q_f.

Boundary conditions check: Equation (q) satisfies boundary conditions (l) and (m).

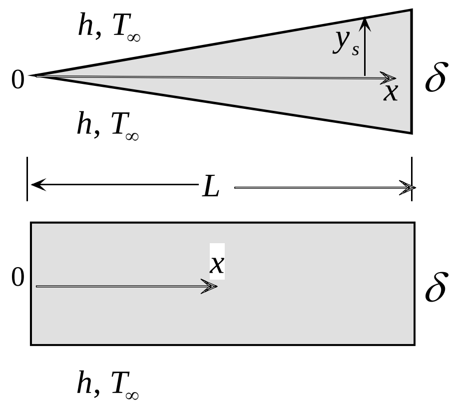
Limiting check: If $T_o=T_\infty$, the temperature distribution in the fin should be uniform equal to T_∞ and the heat transfer should vanish. Setting $T_o=T_\infty$ in (q) gives $T(x)=T_\infty$. Similarly, setting $T_o=T_\infty$ in (r) gives $q_f=0$.

(6) Comments. (i) Selecting the origin at the tip is motivated by the knowledge that $K_n(0)=\infty$. This leads to $C_2=0$. (ii) The minus sign in equation (r) indicates that for $T_o>T_\infty$ heat flows in the negative *x*-direction. This is consistent with the second law of thermodynamics. (iii) The solution is characterized by a single dimensionless parameter $\beta L^{1/2}$

PROBLEM 2.19

In many applications it is desirable to reduce the weight of a fin without significantly reducing its heat transfer capacity. Compare the heat transfer rate from two straight fins of identical material. The profile of one fin is rectangular while that of the other is triangular. Both fins have the same length L and base thickness δ. Surface heat transfer is by convection. Base temperature is T_o and ambient temperature is T_∞. The following data are given:

$$h = 28\ \text{W/m}^2\text{–}^\circ\text{C}, \quad k = 186\ \text{W/m–}^\circ\text{C}, \quad L = 18\ \text{cm}, \quad \delta = 1\,\text{cm}$$



(1) Observations. (i) The heat transfer rate from a straight rectangular fin should be compared with that of a straight triangular fin. (ii) To determine the temperature distribution and heat transfer rate for the two fins the heat equation for each geometry should be formulated and solved. Alternatively, the efficiency graph of Fig. 2.16 can be used to determine the heat transfer rate from each fin. (iii) The heat transfer rate for insulated rectangular fins is given by eq. (2.15).

(2) Origin and Coordinates. The origin is selected at the tip of the triangular fin and the coordinate x is directed as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) no energy generation, (4) $Bi \ll 1$, (5) constant properties, (6) uniform h and T_∞, (7) the tip of the rectangular fin is insulated, (8) the side surfaces are insulated and (10) no radiation.

(ii) Governing Equations. The heat transfer rate for the straight rectangular fin is given by eq. (2.15)

$$q_{fr} = \left[hC_r k A_{cr}\right]^{1/2}(T_o - T_\infty)\tanh m_r L \tag{a}$$

where the subscript r refers to the rectangular fin and

$$m_r = \sqrt{hC_r / kA_{cr}} \tag{b}$$

A_{cr} is the cross-sectional area and C_r is the circumference. Let W be the depth of each plate. Thus

$$A_{cr} = W\delta \tag{c}$$

and

$$C_r = 2W \tag{d}$$

Substituting (b), (c) and (d) into (a) and rearranging yields

$$\frac{q_{fr}}{(T_o - T_\infty)W} = \sqrt{2hk\delta}\,\tanh\sqrt{2h/k\delta}\,L \tag{e}$$

We now turn our attention to the triangular fin. Applying Fourier's law at $x = L$ gives the heat transfer rate

$$q_{ft} = -kA_{ct}(L)\frac{dT(L)}{dx} \tag{f}$$

where the subscript t refers to the straight triangular fin and $A_{ct}(L)$ is the cross-sectional area at $x = L$, given by

$$A_{ct}(L) = W\delta \tag{g}$$

Substituting (g) into (f)

$$q_{ft} = -kW\delta\frac{dT(L)}{dx} \tag{h}$$

Temperature distribution in the fin is obtained from the solution to the heat equation for the fin. Based on the above assumptions, eq. (2.5b) gives

$$\frac{d^2T}{dx^2} + \frac{1}{A_{ct}(x)}\frac{dA_{ct}}{dx}\frac{dT}{dx} - \frac{h}{kA_{ct}(x)}(T - T_\infty)\frac{dA_{st}}{dx} = 0 \tag{i}$$

The cross-sectional area $A_{ct}(x)$ for a straight triangular fin is given by

$$A_{ct}(x) = 2W\,y_s \tag{j}$$

where y_s is the half local width

$$y_s = (\delta/2L)x \tag{k}$$

Substituting (k) into (j)

$$A_{ct}(x) = (W\delta/L)x \tag{l}$$

Eq. (2.6a) is used to formulate dA_s/dx

$$\frac{dA_{st}}{dx} = C_t(x)\left[1 + (dy_s/dx)^2\right]^{1/2} \tag{m}$$

where the circumference $C_t(x)$ is given by

$$C_t = 2W \tag{n}$$

Using (k) and (o) into (m)

$$\frac{dA_{st}}{dx} = 2W\left[1 + (\delta/2L)^2\right]^{1/2}x \tag{o}$$

Substituting (l) and (o) into (i) and rearranging

$$x^2\frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - \beta^2 x\theta = 0 \tag{p}$$

where

$$\theta = T - T_\infty \tag{q}$$

and

$$\beta^2 = \frac{2hL}{k\delta}\sqrt{1+(\delta/2L)^2} \tag{r}$$

(iii) Boundary Conditions. The two boundary conditions are

(1) Finite temperature at $x = 0$

$$T(0) = \text{ finite, or } \theta(0) = T(0) - T_\infty = \text{finite} \tag{s}$$

(2) Specified temperature at $x = L$

$$T(L) = T_o, \text{ or } \theta(L) = T(L) - T_\infty = T_o - T_\infty \tag{t}$$

(4) Solution. Equation (p) is a second order differential equation with variable coefficient. It may be a Bessel equation. Comparing (p) with eq. (2.26) gives

$$A = 0$$
$$B = 0$$
$$C = 1/2$$
$$D = 2\beta i, \quad p = D/i = 2\beta$$
$$n = 0$$

Since n is zero and D is imaginary the solution to (p) is given by eq. (2.29)

$$\theta(x) = T(x) - T_\infty = C_1 I_0(2\beta x^{1/2}) + C_2 K_0(2\beta x^{1/2}) \tag{u}$$

where C_1 and C_2 are constants of integration. Since $K_1(0) = \infty$ (Table 2.1), condition (s) requires that

$$C_2 = 0 \tag{v}$$

Boundary condition (t) gives

$$C_1 = \frac{(T_o - T_\infty)}{I_0(2\beta L^{1/2})} \tag{w}$$

Substituting (v) and (w) into (u)

$$\frac{T - T_\infty}{T_o - T_\infty} = \frac{I_0(2\beta x^{1/2})}{I_0(2\beta L^{1/2})} \tag{x}$$

With the temperature distribution known, equation (h) gives the heat transfer rate. Using eq. (2.46) to determine the derivative of $I_0(2\beta x^{1/2})$ and substituting into (h) gives

$$\frac{q_{ft}}{(T_o - T_\infty)W} = -\sqrt{2hk\delta}\left[1+(\delta/2L)^2\right]^{1/4}\frac{I_1(2\beta L^{1/2})}{I_0(2\beta L^{1/2})} \tag{y}$$

(5) Checking. *Dimensional check*: (i) The argument of the Bessel function, $(2\beta x^{1/2})$, is dimensionless. (ii) Each of equation (e) and (y) is dimensionally consistent.

Boundary conditions check: Equation (x) satisfies boundary conditions (s) and (t).

Limiting check: If $T_o = T_\infty$, the temperature distribution in the fin should be uniform equal to T_∞ and the heat transfer should vanish. Setting $T_o = T_\infty$ in (x) gives $T(x) = T_\infty$. Similarly, setting $T_o = T_\infty$ in (y) gives $q_{ft} = 0$.

(6) Computations. Substituting numerical values in (e) gives the heat transfer rate for the rectangular fin

$$\frac{q_{fr}}{(T_o - T_\infty)W} = \sqrt{2(28)\left(\frac{\text{W}}{\text{m}^2-^\circ\text{C}}\right)186\left(\frac{\text{W}}{\text{m}-^\circ\text{C}}\right)(0.01)(\text{m})}\,\tanh\sqrt{\frac{2(28)(\text{W/m}^2-^\circ\text{C})}{186\text{W/m}-^\circ\text{C})0.01(\text{m})}}\,0.18(\text{m}) = 7.72\frac{\text{W}}{\text{m}-^\circ\text{C}}$$

To determine the heat transfer rate for the triangular fin we first compute the argument $2\beta L^{1/2}$ of the Bessel functions $I_o(2\beta L^{1/2})$ and $I_1(2\beta L^{1/2})$

$$2\beta L^{1/2} = 2\sqrt{\frac{2(28)(\text{W/m}^2-^\circ\text{C})0.18(\text{m})}{186(\text{W/m}-^\circ\text{C})0.01(\text{m})}\sqrt{1+\left[\frac{0.01(\text{m})}{0.36(\text{m})}\right]^2}}\,[0.18(\text{m})]^{1/2} = 1.9757$$

Tables of Bessel functions give

$$I_o(1.9757) = 2.2427$$
$$I_1(1.9757) = 1.556$$

Substituting into (y) gives the heat transfer rate for the triangular fin

$$\frac{q_{ft}}{(T_o - T_\infty)W} = -\sqrt{2(28)\left(\frac{\text{W}}{\text{m}^2-^\circ\text{C}}\right)186\left(\frac{\text{W}}{\text{m}-^\circ\text{C}}\right)0.01(\text{m})}\left[1+(0.01(\text{m})/0.36(\text{m}))^2\right]^{1/4}\frac{1.556}{2.2427} = -7.08\frac{\text{W}}{\text{m}-^\circ\text{C}}$$

The minus sign indicates that heat flows in the negative x-direction if $T_o > T_\infty$.

(7) Comments. (i) Selecting the origin at the tip is motivated by the knowledge that $K_n(0) = \infty$. This leads to $C_2 = 0$. (ii) The minus sign in equation (y) indicates that for $T_o > T_\infty$ heat flows in the negative x-direction. This is consistent with the second law of thermodynamics. (iii) By using a triangular fin the weight is reduced by 50% while the heat transfer rate is decreased by 8.3%. (iv) The efficiency graph of Fig. 2.16 can be used to solve this problem. However, this approach introduces small errors due to reading the efficiency graph. Fin efficiency is defined as

$$\eta_f = \frac{q_f}{hA_s(T_o - T_\infty)} \tag{A}$$

or

$$\frac{q_f}{(T_o - T_\infty)W} = \eta_f \frac{hA_s}{W} \tag{B}$$

where η_f is fin efficiency and A_s is surface area. Surface area for the two fins is given by

$$A_{sr} = 2WL \tag{C}$$

$$A_{st} = 2W\sqrt{L^2 + (\delta/2)^2} \tag{D}$$

Applying (B) to the two fins and using (C) and (D)

PROBLEM 2.19 (continued)

$$\frac{q_{fr}}{(T_o - T_\infty)W} = 2\eta_{fr} hL \tag{E}$$

and

$$\frac{q_{ft}}{(T_o - T_\infty)W} = 2\eta_{ft} h\sqrt{L^2 + (\delta/2)^2} \tag{F}$$

To determine η_f from Fig. 2.16 the parameter $L_c\sqrt{2h/k\delta}$ is computed

$$L_c\sqrt{2h/k\delta} = 0.18(\text{m})\sqrt{2(28)(\text{W/m}^2-{}^\text{o}\text{C})/186(\text{W/m}-{}^\text{o}\text{C})0.01(\text{m})} = 0.99$$

Fig. 2.16 gives

$$\eta_{fr} \cong 0.78 \text{ and } \eta_{ft} \cong 0.7$$

Substituting into (E) and (F) gives

$$\frac{q_{fr}}{(T_o - T_\infty)W} = 7.86\ \frac{\text{W}}{\text{m}-{}^\text{o}\text{C}}$$

and

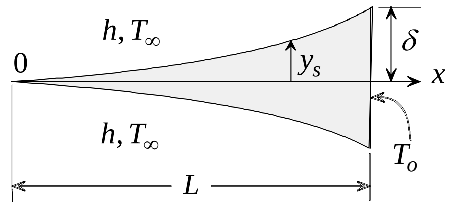
$$\frac{q_{ft}}{(T_o - T_\infty)W} = 7.06\ \frac{\text{W}}{\text{m}-{}^\text{o}\text{C}}$$

This result shows that the error in using the efficiency graphs is small.

PROBLEM 2.20

The profile of a straight fin is given by

$$y_s = \delta (x/L)^2$$



where L is the fin length and δ is half the base thickness. The fin exchanges heat with the ambient by convection. The ambient temperature is T_∞ and the heat transfer coefficient is h. The base temperature is T_o. Assume that $(\delta / L) << 1$, determine the steady state fin heat transfer rate.

(1) Observations. (i) Temperature distribution in the fin and Fourier's law of conduction give the heat transfer rate. (ii) This is a variable area straight fin with specified temperature at the base. (iii) Temperature distribution can be assumed one-dimensional. (iv) Eq. (2.5) should be used to formulate the heat equation for this fin.

(2) Origin and Coordinates. The origin is selected at the tip and the coordinate x is directed as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) no energy generation, (4) $Bi << 1$, (5) constant properties, (6) uniform h and T_∞, and (7) negligible radiation.

(ii) Governing Equations. Applying Fourier's law at $x = L$ gives the fin heat transfer rate

$$q_f = -kA_c(L)\frac{dT(L)}{dx} \tag{a}$$

where $A_c(L)$ is the cross-sectional area at $x = L$, given by

$$A_c(L) = 2\delta W \tag{b}$$

where W is the fin depth. Substituting (b) into (a)

$$q_f = -2k\delta W \frac{dT(L)}{dx} \tag{c}$$

Temperature distribution in the fin is obtained from the solution to the heat equation for the fin. Based on the above assumptions eq. (2.5b) gives

$$\frac{d^2T}{dx^2} + \frac{1}{A_c(x)}\frac{dA_c}{dx}\frac{dT}{dx} - \frac{h}{kA_c(x)}(T - T_\infty)\frac{dA_s}{dx} = 0 \tag{d}$$

The cross-sectional area $A_c(x)$ for a straight fin is

$$A_c = 2y_s^2 W \tag{e}$$

where y_s is the local width given by

$$y_s = \delta (x/L)^2 \tag{f}$$

Substituting (f) into (e)

$$A_c(x) = 2\delta W(x/L)^2 \tag{g}$$

Eq. (2.6a) is used to formulate dA_s/dx

$$\frac{dA_s}{dx} = C(x)\left[1 + (dy_s/dx)^2\right]^{1/2}$$

Using (f) into the above and noting that $C = 2W$

$$\frac{dA_s}{dx} = 2W\left[1 + (2\delta x/L^2)^2\right]^{1/2} \approx 2W \tag{h}$$

Substituting (g) and (h) into (d) and rearranging

$$x^2\frac{d^2\theta}{dx^2} + 2x\frac{d\theta}{dx} - \beta\theta = 0 \tag{i}$$

where

$$\theta = T - T_\infty \tag{j}$$

and

$$\beta = \frac{hL^2}{k\delta} \tag{k}$$

(iii) Boundary Conditions. The two boundary conditions are

(1) Finite temperature at $x = 0$

$$T(0) = \text{ finite, or } \theta(0) = T(0) - T_\infty = \text{ finite} \tag{l}$$

(2) Specified temperature at $x = L$

$$T(L) = T_o, \text{ or } \theta(L) = T(L) - T_\infty = T_o - T_\infty \tag{m}$$

(4) Solution. Equation (i) is a second order differential equation with variable coefficient. Examination of the coefficients shows that it is an Euler equation. Comparing (i) with eq. (2.49) gives

$$a_0 = -\beta \text{ and } a_1 = 2 \tag{n}$$

The solution to (i) depends on the roots $r_{1,2}$ of eq. (2.50)

$$r_{1,2} = \frac{-(a_1 - 1) \pm \sqrt{(a_1 - 1)^2 - 4a_0}}{2} \tag{o}$$

Substituting (n) into (o) and using (k)

$$r_{1,2} = -(1/2) \pm \sqrt{(1/4) + hL^2/k\delta} \tag{p}$$

Thus the two roots are

$$r_1 = -(1/2) + \sqrt{(1/4) + hL^2/k\delta} \tag{q}$$

and

$$r_2 = -(1/2) - \sqrt{(1/4) + hL^2 / k\delta} \tag{r}$$

The solution to (i) is given by eq. (2.51)

$$\theta(x) = T(x) - T_\infty = C_1 x^{r_1} + C_2 x^{r_2} \tag{s}$$

where C_1 and C_2 are constants of integration. Application of boundary condition (l) gives

$$C_2 = 0 \tag{t}$$

Boundary condition (m) gives

$$C_1 = (T_o - T_\infty) / L^{r_1} \tag{u}$$

Substituting (t) and (u) into (s) gives the temperature solution

$$T(x) = T_\infty + (T_o - T_\infty)(x / L)^{r_1} \tag{v}$$

Expressing the above in dimensionless form gives

$$\frac{T(x) - T_\infty}{(T_o - T_\infty)} = (x / L)^{r_1} \tag{w}$$

Fin heat transfer rate is obtained by substituting (v) into (c) and using (q)

$$\frac{q_f}{Wk(\delta / L)(T_o - T_\infty)} = -2\left[\sqrt{(1/4) + hL^2 / k\delta} - (1/2)\right] \tag{x}$$

(5) Checking. *Dimensional check*: (i) Each term in (v) has units of temperature. (ii) Each term in (w) and (x) is dimensionless.

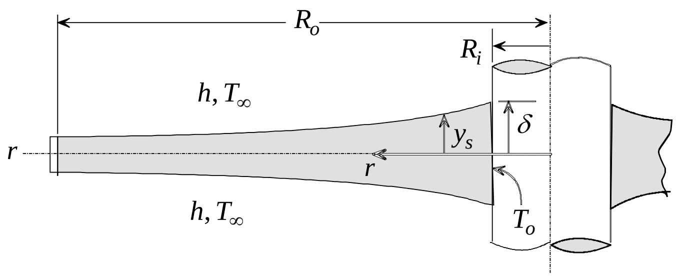
Boundary conditions check: Equation (v) satisfies boundary conditions (l) and (m).

Differential equation check: Direct substitution of (v) into (i) confirms that the governing equation is satisfied.

Limiting check: If $T_o = T_\infty$, the temperature distribution in the fin should be uniform equal to T_∞ and the heat transfer should vanish. Setting $T_o = T_\infty$ in (v) gives $T(x) = T_\infty$. Similarly, setting $T_o = T_\infty$ in (x) gives $q_f = 0$.

(6) Comments. (i) Not every second order differential equation with variable coefficient is a Bessel equation. In this example the equation is Euler. (ii) The minus sign in equation (x) indicates that for $T_o > T_\infty$ heat flows in the negative *x*-direction. This is consistent with the second law of thermodynamics. (iii) The solution is characterized by a single dimensionless parameter $hL^2 / k\delta$. This parameter appears in (q) and (x).

PROBLEM 2.21



A circular disk of radius R_o is mounted on a tube of radius R_i. The profile of the disk is described by

$$y_s = \delta R_i^2 / r^2$$

where δ is half the thickness at the tube. The disk exchanges heat with the surroundings by convection. The heat transfer coefficient is h and the ambient temperature is T_∞. The base is maintained at T_o and the tip is insulated. Assume that $4\delta / R_i << 1$, use a fin model to determine the steady state heat transfer rate from the disk.

(1) Observations. (i) Temperature distribution in the fin and Fourier's law of conduction give the heat transfer rate. (ii) This is a variable area annular fin with specified temperature at the base, insulated tip and convection at the surface . (iii) Temperature distribution can be assumed one-dimensional. (iv) Use eq. (2.5) to formulate the heat equation for the disk.

(2) Origin and Coordinates. The origin is selected at the tube center and the coordinate r is directed as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) no energy generation, (4) $Bi << 1$, (5) constant properties, (6) uniform h and T_∞, and (7) negligible radiation.

(ii) Governing Equations. Applying Fourier's law at $r = R_i$ gives the heat transfer rate

$$q_f = -kA_c(R_i)\frac{dT(R_i)}{dr} \tag{a}$$

where $A_c(R_i)$ is the cross-sectional area at $r = R_i$, given by

$$A_c(R_i) = 4\pi\delta R_i \tag{b}$$

Substituting (b) into (a)

$$q_f = -4\pi\delta R_i k\frac{dT(R_i)}{dr} \tag{c}$$

Temperature distribution in the fin is obtained from the solution to the heat equation for the fin. Based on the above assumptions eq. (2.5b) gives

$$\frac{d^2T}{dr^2} + \frac{1}{A_c(r)}\frac{dA_c}{dr}\frac{dT}{dr} - \frac{h}{kA_c(r)}(T - T_\infty)\frac{dA_s}{dr} = 0 \tag{d}$$

The cross-sectional area $A_c(r)$ for an annular fin is

$$A_c(r) = 4\pi r y_s \tag{e}$$

where y_s is the local fin thickness given by

$$y_s = \delta R_i^2 / r^2 \tag{f}$$

Substituting (f) into (e)

$$A_c(r) = 4\pi\delta R_i^2 / r \tag{g}$$

Eq. (2.6a) is used to formulate dA_s / dr

$$\frac{dA_s}{dr} = C(r)\left[1 + (dy_s / dr)^2\right]^{1/2}$$

Using (f) into the above and noting that $C(r) = 4\pi r$

$$\frac{dA_s}{dr} = 4\pi r\left[1 + (4\delta^2 R_i^4 / r^6)\right]^{1/2}$$

However, since $4\delta / R_i << 1,$ it follows that $(4\delta^2 R_i^4 / r^6) << 1$. Thus the above simplifies to

$$\frac{dA_s}{dr} \approx 4\pi r \tag{h}$$

Substituting (g) and (h) into (d) and rearranging

$$r^2 \frac{d^2\theta}{dr^2} - r\frac{d\theta}{dr} - \beta^2 r^4 \theta = 0 \tag{i}$$

where

$$\theta = T - T_\infty \tag{j}$$

and

$$\beta^2 = \frac{h}{k\delta R_i^2} \tag{k}$$

(iii) Boundary Conditions. The two boundary conditions are

(1) Specified temperature at $r = R_i$

$$\theta(R_i) = T_o - T_\infty \tag{l}$$

(2) Insulated tip at $r = R_o$

$$\frac{d\theta(R_o)}{dr} = 0 \tag{m}$$

(4) Solution. Equation (i) is a second order differential equation with variable coefficient. It may be a Bessel equation. Comparing (i) with eq. (2.26) gives

$A = 1$
$B = 0$
$C = 2$
$D = \beta i / 2, \quad p = D / i = \beta / 2$
$n = 1/2$

Since n is not an integer and D is imaginary the solution to (i) is given by eq. (2.30)

$$\theta(r) = T(r) - T_\infty = r\left[C_1 I_{1/2}(\beta r^2 / 2) + C_2 I_{-1/2}(\beta r^2 / 2)\right] \tag{n}$$

where C_1 and C_2 are constants of integration. However using equations (2.35) and (2.36), the Bessel functions in (n) can be expressed in terms of hyperbolic functions as

$$I_{1/2}(\beta r^2/2) = (2/r)\sqrt{1/\pi\beta}\,\sinh(\beta r^2/2) \tag{o}$$

and

$$I_{-1/2}(\beta r^2/2) = (2/r)\sqrt{1/\pi\beta}\,\cosh(\beta r^2/2) \tag{p}$$

Substituting (o) and (p) into (n)

$$T(r) = T_\infty + \frac{2C_1}{\sqrt{\pi\beta}}\sinh(\beta r^2/2) + \frac{2C_2}{\sqrt{\pi\beta}}\cosh(\beta r^2/2) \tag{q}$$

Boundary conditions (l) and (m) give C_1 and C_2

$$C_1 = -\frac{(\sqrt{\pi\beta}/2)(T_o - T_\infty)\tanh(\beta R_o^2/2)}{\cosh(\beta R_i^2/2) - \tanh(\beta R_o^2/2)\sinh(\beta R_i^2/2)} \tag{r}$$

$$C_2 = \frac{(\sqrt{\pi\beta}/2)(T_o - T_\infty)}{\cosh(\beta R_i^2/2) - \tanh(\beta R_o^2/2)\sinh(\beta R_i^2/2)} \tag{s}$$

Substituting equations (r) and (s) into(q)

$$\frac{T - T_\infty}{T_o - T_\infty} = \frac{\cosh(\beta r^2/2) - \tanh(\beta R_o^2/2)\ \sinh(\beta r^2/2)}{\cosh(\beta R_i^2/2) - \tanh(\beta R_o^2/2)\ \sinh(\beta R_i^2/2)} \tag{t}$$

Differentiating (t) and substituting into (c) gives the fin heat transfer rate

$$\frac{q_f}{(T_o - T_\infty)\sqrt{hk\delta R_i^2}} = -4\pi\frac{\sinh(\beta R_i^2/2) - \tanh(\beta R_o^2/2)\ \ \cosh(\beta R_i^2/2)}{\cosh(\beta R_i^2/2) - \tanh(\beta R_o^2/2)\ \sinh(\beta R_i^2/2)} \tag{u}$$

(5) Checking. *Dimensional check*: (i) The arguments of the sinh and cosh are dimensionless. (ii) Each term in (t) and (u) is dimensionless.

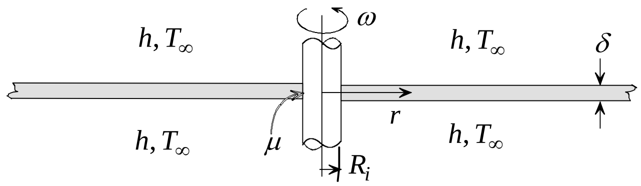
Boundary conditions check: Equation (t) satisfies boundary conditions (l) and (m).

Differential equation check: Direct substitution of (t) into (i) confirms that the governing equation is satisfied.

Limiting check: If $T_o = T_\infty$ the temperature distribution in the fin should be uniform equal to T_∞ and the heat transfer should vanish. Setting $T_o = T_\infty$ in (t) gives $T(r) = T_\infty$. Similarly, setting $T_o = T_\infty$ in (u) gives $q_f = 0$.

(6) Comments. (i) The minus sign in equation (u) indicates that for $T_o > T_\infty$ heat flows in the negative *x*-direction. (ii) Neglecting the term $(4\delta^2 R_i^4/r^6)$ reduces the fin equation to a Bessel differential equation and facilitates obtaining a solution. (iii) The problem is characterized by two dimensionless parameters: βR_i^2 and R_o/R_i.

PROBLEM 2.22



A very large disk of thickness δ is mounted on a shaft of radius R_i. The shaft rotates with angular velocity ω. The pressure at the interface between the shaft and the disk is P and the coefficient of friction is μ. The disk exchanges heat with the ambient by convection. The heat transfer coefficient is h and the ambient temperature is T_∞. Neglecting heat transfer to the shaft at the interface, use a fin model to determine the interface temperature.

(1) Observations. (i) Temperature distribution in the fin gives the temperature at the interface. (ii) This is a variable area annular fin with uniform thickness. (iii) Heat is generated at the shaft-disk interface due to friction and is conducted through the disk. Thus the heat flux is specified at the base $r = R_o$. Disk temperature far away from the interface approaches ambient level. (iv) Heat exchange at the surface is by convection. (v) Temperature distribution can be assumed one-dimensional. (vi) Eq. (2.5) should be used to formulate the heat equation for the disk.

(2) Origin and Coordinates. The origin is selected at the shaft center and the coordinate r is directed as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) no energy generation, (4) $Bi << 1$, (5) constant properties, (6) uniform h and T_∞, (7) all frictional energy is conducted through the disk and (8) negligible radiation.

(ii) Governing Equations. Temperature distribution in the disk is obtained from the solution to the fin heat equation. Based on the above assumptions eq. (2.5b) gives

$$\frac{d^2T}{dr^2} + \frac{1}{A_c(r)}\frac{dA_c}{dr}\frac{dT}{dr} - \frac{h}{kA_c(r)}(T - T_\infty)\frac{dA_s}{dr} = 0 \qquad \text{(a)}$$

The cross-sectional area $A_c(r)$ for an annular fin of constant thickness δ is

$$A_c(r) = 4\pi\delta r \qquad \text{(b)}$$

Eq. (2.6a) is used to formulate dA_s / dr

$$\frac{dA_s}{dr} = C(r)\left[1 + (dy_s / dr)^2\right]^{1/2} \qquad \text{(c)}$$

where $C(r)$ is the circumference and y_s is the disk half thickness given by

$$C(r) = 4\pi r \qquad \text{(d)}$$

$$y_s = \delta \qquad \text{(e)}$$

Substituting (d) and (e) into (c)

$$\frac{dA_s}{dr} = 4\pi r \qquad \text{(f)}$$

Substituting (b) and (f) into (a) and rearranging

$$r^2 \frac{d^2\theta}{dr^2} + r\frac{d\theta}{dr} - \beta^2 r^2 \theta = 0 \tag{g}$$

where

$$\theta = T - T_\infty \tag{h}$$

and

$$\beta^2 = \frac{2h}{k\delta} \tag{i}$$

(iii) Boundary Conditions. The two boundary conditions are:

(1) Specified flux at $r = R_i$. Frictional heat generated at the interface is proportional to the tangential force and tangential velocity. Assuming that frictional energy is conducted through the disk, Fourier's law gives

$$-k\frac{d\theta(R_i)}{dr} = \mu P\omega R_i \tag{j}$$

(2) Finite temperature at $r = \infty$

$$\theta(\infty) = \text{finite} \tag{k}$$

(4) Solution. Equation (g) is a second order differential equation with variable coefficient. It may be a Bessel equation. Comparing (g) with eq. (2.26) gives

$A = 0$
$B = 0$
$C = 1$
$D = \beta i, \quad p = D/i = \beta$
$n = 0$

Since *n* is zero and *D* is imaginary the solution to (g) is given by eq. (2.29)

$$\theta(r) = T(r) - T_\infty = C_1 I_0(\beta r) + C_2 K_0(\beta r) \tag{l}$$

where C_1 and C_2 are constants of integration. Since $I_0(\infty) = \infty$ (Table 2.1) boundary condition (k) gives

$$C_1 = 0 \tag{m}$$

Thus solution (l) becomes

$$T(r) = T_\infty + C_2 K_0(\beta r) \tag{n}$$

Using eq. (2.46) to determine the derivative of $K_0(\beta r)$, boundary condition (j) gives C_2

$$C_2 = \frac{\mu P\omega R_i}{k\beta K_1(\beta R_i)} \tag{o}$$

Substituting (o) into (n) and using (i) gives

$$T(r) = T_\infty + \frac{\mu P\omega R_i}{\sqrt{2hk/\delta}} \frac{K_0(\beta r)}{K_1(\beta R_i)} \tag{p}$$

To determine interface temperature we set $r = R_i$ in (p)

$$T(R_o) = T_\infty + \frac{\mu P \omega R_i}{\sqrt{2hk/\delta}} \frac{K_0(\beta R_i)}{K_1(\beta R_i)} \tag{q}$$

Expressing (p) in dimensionless form gives

$$\frac{T(r) - T_\infty}{\dfrac{\mu P \omega R_i}{\sqrt{2hk/\delta}}} = \frac{K_0(\beta r)}{K_1(\beta R_i)} \tag{r}$$

(5) Checking. *Dimensional checks*: (i) The argument of the Bessel function, (βr) is dimensionless. (ii) Each term in (p) has units of temperature.

Boundary conditions check: Equation (p) satisfies boundary conditions (j) and (k).

Differential equation check: Direct substitution of (p) into (g) confirms that the governing equation is satisfied.

Limiting check: If no frictional energy is generated, the disk temperature distribution should be uniform equal to T_∞. Setting any of the quantities μ, P or ω equal to zero in (p) gives $T(r) = T_\infty$.

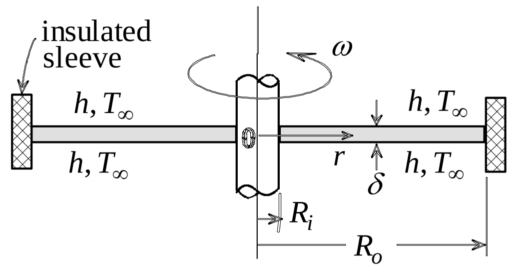
(6) Comments. (i) Since $K_o(\beta r)$ decreases monotonically with r (see Fig. 2.15), equation (p) shows that the maximum disk temperature occurs at the interface. (ii) The solution is characterized by a single dimensionless parameter βR_i.

PROBLEM 2.23

A circular disk of thickness δ and outer radius R_o is mounted on a shaft of radius R_i. The shaft and disk rotate inside a stationary sleeve with angular velocity ω. Because of friction between the disk and sleeve, heat is generated at a flux q''_o. The disk exchanges heat with the surroundings by convection. The ambient temperature is T_∞. Due to radial variation in the tangential velocity, the heat transfer coefficient varies with radius according to

$$h = h_i (r / R_i)^2$$

Assume that no heat is conducted to the sleeve and shaft, use a fin model to determine the steady state temperature distribution.



(1) Observations. (i) This is a variable area annular fin with uniform thickness. (ii) Frictional heat generated at the sleeve is conducted through the disk. (iii) Heat exchange at the surface is by convection. (iv) Temperature distribution can be assumed one-dimensional. (v) The heat transfer coefficient varies over the disk. (vi) Eq. (2.5) should be used to formulate the heat equation for the disk.

(2) Origin and Coordinates. The origin is selected at the center of shaft and the coordinate r is directed as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) no energy generation, (4) $Bi << 1$, (5) constant properties, (6) uniform ambient temperature T_∞, (7) all frictional energy is conducted inwardly through the disk, (8) the shaft is insulated and (9) negligible radiation.

(ii) Governing Equations. Temperature distribution in the disk is obtained from the solution to the fin heat equation. Based on the above assumptions eq. (2.5b) gives

$$\frac{d^2T}{dr^2} + \frac{1}{A_c(r)}\frac{dA_c}{dr}\frac{dT}{dr} - \frac{h}{kA_c(r)}(T - T_\infty)\frac{dA_s}{dr} = 0 \tag{a}$$

The cross-sectional area $A_c(r)$ for an annular fin of constant thickness δ is

$$A_c(r) = 2\pi\delta r \tag{b}$$

Eq. (2.6a) is used to formulate dA_s / dr

$$\frac{dA_s}{dr} = C(r)\left[1 + (dy_s / dr)^2\right]^{1/2} \tag{c}$$

where $C(r)$ is the circumference and y_s is the disk half thickness given by

$$C(r) = 4\pi r \tag{d}$$

$$y_s = \delta / 2 \tag{e}$$

The heat transfer coefficient varies with radius according to

$$h = h_i (r / R_i)^2 \tag{f}$$

Substituting (d), (e) and (f) into (a)

$$r^2 \frac{d^2\theta}{dr^2} + r\frac{d\theta}{dr} - 4\beta^2 r^4 \theta = 0 \tag{g}$$

where

$$\theta = T - T_\infty \tag{h}$$

and

$$\beta^2 = \frac{h_i}{2k\delta R_i^2} \tag{i}$$

(iii) Boundary Conditions. The two boundary conditions are

(1) Insulated surface at $r = R_i$

$$\frac{d\theta(R_i)}{dr} = 0 \tag{j}$$

(2) Specified flux at $r = R_o$

$$-k\frac{d\theta(R_o)}{dr} = -q_o'' \tag{k}$$

(4) Solution. Equation (g) is a second order differential equation with variable coefficient. It may be a Bessel equation. Comparing (g) with eq. (2.26) gives

$A = 0$
$B = 0$
$C = 2$
$D = \beta i, \quad p = D / i = \beta$
$n = 0$

Since n is zero and D is imaginary the solution to (g) is given by eq. (2.29)

$$\theta(r) = T(r) - T_\infty = C_1 I_0(\beta r^2) + C_2 K_0(\beta r^2) \tag{l}$$

where C_1 and C_2 are constants of integration. Application of boundary conditions (j) and (k) and using eqs. (2.45) and (2.46) to determine the derivatives of the Bessel functions in (l) give the constants C_1 and C_2

$$C_1 = \frac{q_o''}{\sqrt{2h_i k R_o^2 / \delta R_i^2}} \left[I_1(\beta R_o^2) - \frac{I_1(\beta R_i^2)}{K_1(\beta R_i^2)} K_1(\beta R_o^2) \right]^{-1} \tag{m}$$

$$C_2 = \frac{q_o''}{\sqrt{2h_i k R_o^2 / \delta R_i^2}} \left[I_1(\beta R_o^2) - \frac{I_1(\beta R_i^2)}{K_1(\beta R_i^2)} K_1(\beta R_o^2) \right]^{-1} \frac{I_1(\beta R_i^2)}{K_1(\beta R_i^2)} \tag{n}$$

Substituting (m) and (n) into (l) gives the temperature distribution in dimensionless form

$$\frac{T(r)-T_\infty}{\dfrac{q_o''}{\sqrt{2h_i k R_o^2/\delta R_i^2}}} = \left[I_1(\beta R_o^2) - \frac{I_1(\beta R_i^2)}{K_1(\beta R_i^2)} K_1(\beta R_o^2) \right]^{-1} \left[I_o(\beta r^2) + \frac{I_1(\beta R_i^2)}{K_1(\beta R_i^2)} K_o(\beta r^2) \right] \quad \text{(o)}$$

(5) Checking. *Dimensional checks*: (i) The argument of the Bessel function, (βr), is dimensionless. (ii) The quantity $q_o'' / \sqrt{2h_i k R_o^2 / \delta R_i^2}$ in (o) has units of temperature. Thus each term in (o) is dimensionless.

Boundary conditions check: Equation (o) satisfies boundary conditions (j) and (k).

Differential equation check: Direct substitution of (o) into (g) confirms that the governing equation is satisfied.

Limiting check: If no frictional energy is generated, the disk temperature distribution should be uniform equal to T_∞. Setting q_o'' equal to zero in (o) gives $T(r) = T_\infty$.

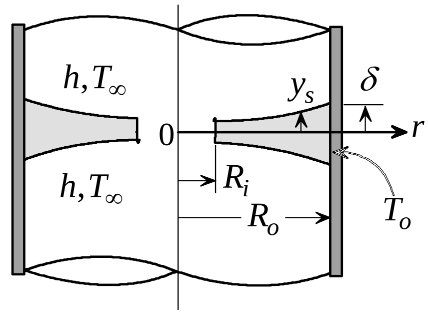
(6) Comments. (i) Since $K_o(\beta r^2)$ decreases with *r* and $I_o(\beta r^2)$ increases with *r* (see Fig. 2.15), equation (o) shows that the maximum disk temperature occurs at the interface $r = R_o$. (ii) The solution is characterized by two dimensionless parameter: βR_i^2 and R_o / R_i.

PROBLEM 2.24

A specially designed heat exchanger consists of a tube with fins mounted on its inside surface. The fins are circular disks of inner radius R_i and variable thickness given by

$$y_s = \delta(r/R_o)^{3/2}$$

where R_o is the tube radius. The disks exchange heat with the ambient fluid by convection. The heat transfer coefficient is h and the fluid temperature is T_∞. The tube surface is maintained at T_o. Modeling each disk as a fin, determine the steady state heat transfer rate. Neglect heat loss from the disk surface at R_i and assume that $(\delta/R_o) << 1$.



(1) Observations. (i) This is a variable area annular fin with variable thickness. (ii) The base is maintained at a specified temperature and the tip is insulated. (iii) Heat exchange at the surface is by convection. (iv) Temperature distribution can be assumed one-dimensional. (v) Eq. (2.5) should be used to formulate the heat equation for the fin.

(2) Origin and Coordinates. The origin is selected at the center of tube and the coordinate r is directed as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) no energy generation, (4) $Bi << 1$, (5) constant properties, (6) uniform heat transfer coefficient h and ambient temperature T_∞, and (7) no radiation.

(ii) Governing Equations. Temperature distribution in the fin is obtained from the solution to the fin heat equation. Based on the above assumptions eq. (2.5b) gives

$$\frac{d^2T}{dr^2} + \frac{1}{A_c(r)}\frac{dA_c}{dr}\frac{dT}{dr} - \frac{h}{kA_c(r)}(T - T_\infty)\frac{dA_s}{dr} = 0 \tag{a}$$

The cross-sectional area $A_c(r)$ for an annular fin of is

$$A_c(r) = 4\pi r y_s \tag{b}$$

where y_s is the local fin thickness given by

$$y_s = \delta(r/R_o)^{3/2} \tag{c}$$

Substituting (c) into (b)

$$A_c(r) = 4\pi\delta r^{5/2}/R_o^{3/2} \tag{d}$$

Eq. (2.6a) is used to formulate dA_s/dr

$$\frac{dA_s}{dr} = C(r)\left[1 + (dy_s/dr)^2\right]^{1/2} \tag{e}$$

Using (c) into (e) and noting that $C(r) = 4\pi r$

$$\frac{dA_s}{dr} = 4\pi r\left[1 + (9\delta^2 r / 4R_o^3)\right]^{1/2} \tag{f}$$

However, since $\delta / R_i << 1$, it follows that $9\delta^2 r / 4R_o^3 << 1$. Thus the above simplifies to

$$\frac{dA_s}{dr} \approx 4\pi r \tag{g}$$

Substituting (d) and (g) into (a) and rearranging

$$r^2 \frac{d^2\theta}{dr^2} + (5/2) r \frac{d\theta}{dr} - \beta^2 r^{1/2} \theta = 0 \tag{h}$$

where

$$\theta = T - T_\infty \tag{i}$$

and

$$\beta^2 = \frac{hR_o^{3/2}}{k\delta} \tag{j}$$

(iii) Boundary Conditions. The two boundary conditions are

(1) Insulated surface at $r = R_i$

$$\frac{d\theta(R_i)}{dr} = 0 \tag{k}$$

(2) Specified temperature at $r = R_o$

$$\theta(R_o) = T_o - T_\infty \tag{l}$$

(4) Solution. Equation (h) is a second order differential equation with variable coefficient. It may be a Bessel equation. Comparing (h) with eq. (2.26) gives

$A = -3/4$
$B = 0$
$C = 1/4$
$D = 4\beta i, \quad p = D/i = 4\beta$
$n = 3$

Since *n* is and integer and *D* is imaginary the solution to (h) is given by eq. (2.29)

$$\theta(r) = T(r) - T_\infty = r^{-3/4}\left[C_1 I_3(4\beta r^{1/4}) + C_2 K_3(4\beta r^{1/4})\right] \tag{m}$$

where C_1 and C_2 are constants of integration. Application of boundary conditions (k) and (l) and using eqs. (2.45) and (2.46) to determine the derivatives of the Bessel functions in (k) give the constants C_1 and C_2:

$$C_1 = \frac{(T_o - T_\infty) R_o^{3/4}}{K_3(4\beta R_o^{1/4})}\left[\frac{(2/3)\beta R_i^{1/4} I_2(4\beta R_i^{1/4}) - I_3(4\beta R_i^{1/4})}{(2/3)\beta R_i^{1/4} K_2(4\beta R_i^{1/4}) - K_3(4\beta R_i^{1/4})} + \frac{I_3(4\beta R_o^{1/4})}{K_3(4\beta R_o^{1/4})}\right]^{-1} \tag{n}$$

and

$$C_2 = \frac{(T_o - T_\infty)R_o^{3/4}}{K_3(4\beta R_o^{1/4})}\left[\frac{(2/3)\beta R_i^{1/4} I_2(4\beta R_i^{1/4}) - I_3(4\beta R_i^{1/4})}{(2/3)\beta R_i^{1/4} K_2(4\beta R_i^{1/4}) - K_3(4\beta R_i^{1/4})} + \frac{I_3(4\beta R_o^{1/4})}{K_3(4\beta R_o^{1/4})}\right]^{-1} \times \frac{(2/3)\beta R_i^{1/4} I_2(4\beta R_i^{1/4}) - I_3(4\beta R_i^{1/4})}{(2/3)\beta R_i^{1/4} K_2(4\beta R_i^{1/4}) + K_3(4\beta R_i^{1/4})} \tag{n}$$

Fin heat transfer rate q is determined by applying Fourier's law at $r = R_o$

$$q = -4\pi k R_o \delta \frac{dT(R_o)}{dr} \tag{o}$$

Using (m) into (o)

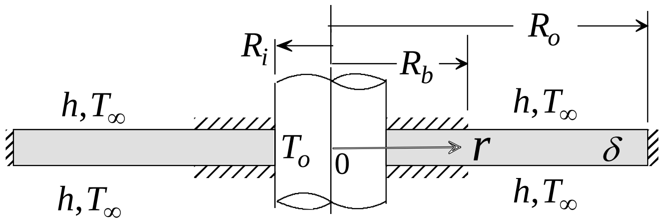
$$\begin{aligned} q = -4\pi k R_o \delta \Big\{ &-\frac{3}{4} R_o^{-7/4}\left[C_1 I_3(4\beta R_o^{1/4}) + C_2 K_3(4\beta R_o^{1/4})\right] \\ &+ \frac{1}{4} R_o^{-3/2}\Big[4\beta C_1 I_2(4\beta R_o^{1/4}) - 3R_o^{-1/4} C_1 I_3(4\beta R_o^{1/4}) \\ &- 4\beta C_2 K_2(4\beta R_o^{1/4}) - \frac{3}{1} R_o^{-1/4} C_2 K_3(4\beta R_o^{1/4})\Big\} \end{aligned} \tag{p}$$

(5) Checking. *Dimensional checks*: (i) The argument of the Bessel function, $(\beta r^{1/4})$, is dimensionless. (ii) C_1 and C_2 have units of $(^{o}\mathrm{C} - \mathrm{m}^{3/4})$. Thus each term in (m) has units of temperature.

(6) Comments. (i) Neglecting the term $(9\delta^2 r / 4R_o^3)$ reduces the fin equation to a Bessel differential equation and facilitates obtaining a solution. (ii) The solution is characterized by two dimensionless parameter: βR_i^2 and R_o / R_i.

PROBLEM 2.25

A disk of radius R_o and thickness δ is mounted on a steam pipe of outer radius R_i. The disk's surface is insulated from $r = R_i$ to $r = R_b$. The remaining surface exchanges heat with the surroundings by convection. The heat transfer coefficient is h and the ambient temperature is T_∞. The temperature of the pipe is T_o and the surface at $r = R_o$ is insulated. Formulate the governing equations and boundary conditions and obtain a solution for the temperature distribution in terms of constants of integration.



(1) Observations. (i) This is an annular disk with uniform thickness. (ii) One section of the disk's surface is insulated while the other exchanges heat by convection. Thus, two governing equations are needed for the temperature distribution. (iii) Assuming that $Bi << 1$ the disk can be modeled as a fin with inner surface at a specified temperature and the outer surface insulated. (iv) Temperature distribution can be assumed one-dimensional. (v) Use eq. (2.5) to formulate the heat equation for each section of the disk.

(2) Origin and Coordinates. The origin is selected at the center of shaft and the coordinate r is directed as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) no energy generation, (4) $Bi << 1$, (5) constant properties, (6) uniform heat transfer coefficient h and ambient temperature T_∞ and (7) negligible radiation.

(ii) Governing Equations. Since one section is insulated while the other exchanges heat by convection, two fin equations are needed. Let

$T_1(r)$ = temperature distribution in the first section, $R_i < r < R_b$
$T_2(r)$ = temperature distribution in the second half, $R_b < r < R_o$

The fin equation is obtained from eq. (2.5b)

$$\frac{d^2T}{dr^2} + \frac{1}{A_c(r)}\frac{dA_c}{dr}\frac{dT}{dr} - \frac{h}{kA_c(r)}(T - T_\infty)\frac{dA_s}{dr} = 0 \tag{a}$$

The cross-sectional area $A_c(r)$ for an annular fin of constant thickness δ is

$$A_c(r) = 2\pi\delta r \tag{b}$$

Eq. (2.6a) is used to formulate dA_s / dr

$$\frac{dA_s}{dr} = C(r)\left[1 + (dy_s / dr)^2\right]^{1/2} \tag{c}$$

where $y_s = \delta$ and $C(r)$ is the circumference given by

$$C(r) = 4\pi r \tag{d}$$

In the first section the fin is insulated along its surface and thus the convection term vanishes. Substituting (b) into (a) and setting $h = 0$ gives

$$\frac{d^2T_1}{dr^2} + \frac{1}{r}\frac{dT_1}{dr} = 0, \qquad R_i \le r \le R_b \tag{e}$$

Similarly, substituting (b), (c) and (d) into (a) gives the fin equation for the second section

$$r^2\frac{d^2\theta}{dr^2} + r\frac{d\theta}{dr} - \beta^2 r^2\theta = 0, \qquad R_b \le r \le R_0 \tag{f}$$

where

$$\beta^2 = \frac{2h}{k\delta} \tag{g}$$

and

$$\theta(r) = T_2(r) - T_\infty \tag{h}$$

(iii) Boundary Conditions. Four boundary conditions are needed.

(1) Specified temperature at $r = R_i$

$$T_1(0) = T_o \tag{i}$$

(2) Equality of temperature at $r = R_b$

$$T_1(R_b) = T_2(R_b) = \theta(R_b) + T_\infty \tag{j}$$

(3) Equality of flux at $r = R_b$

$$\frac{dT_1(R_b)}{dr} = \frac{d\theta(R_b)}{dr} \tag{k}$$

(4) Insulated surface at $r = R_o$

$$\frac{d\theta(R_o)}{dr} = 0 \tag{l}$$

(4) Solution. Separating variables and integrating (e) twice

$$T_1(r) = C_1 \ln r + C_2 \tag{m}$$

where C_1 and C_2 are constants of integration. Equation (f) is a second order differential equation with variable coefficient. It may be a Bessel equation. Comparing (f) with eq. (2.26) gives

$A = 0$
$B = 0$
$C = 1$
$D = \beta i, \quad p = D/i = \beta$
$n = 0$

Since n is zero and D is imaginary the solution to (f) is given by eq. (2.29)

$$\theta(r) = T_2(r) - T_\infty = C_3 I_0(\beta r) + C_4 K_0(\beta r) \tag{n}$$

where C_3 and C_4 are constants of integration.

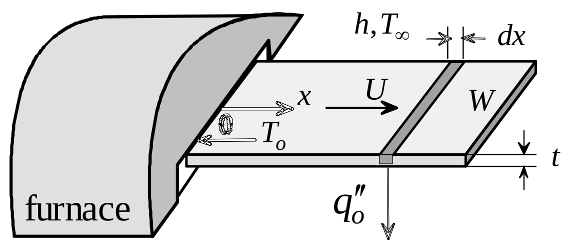
(5) Checking. *Dimensional check*: The argument of the Bessel function is dimensionless.

Differential equation check: Direct substitution of (i) into (e) and (j) into (f) confirms that the governing equations are satisfied.

(6) Comments. When conditions along a fin are not uniform, it is necessary to use more than one fin equation.

PROBLEM 2.26

Consider the moving plastic sheet of Example 2.4. In addition to cooling by convection at the top surface, the sheet is cooled at the bottom at a constant flux q''_o. Determine the temperature distribution in the sheet.



(1) Observations. (i) This is a constant area moving fin problem. (ii) Temperature distribution can be assumed one-dimensional. (iii) Along the upper surface heat is exchanged with the surroundings by convection. At the lower surface heat is removed at uniform flux. (iv) The temperature is specified at the outlet of the furnace. (v) The sheet is semi-infinite.

(2) Origin and Coordinates. A rectangular coordinate system is used with the origin at the exit of furnace. The coordinate x is directed as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) constant pressure, (4) constant properties, (5) constant conveyor speed, (6) uniform heat transfer coefficient h and ambient temperature T_∞, (7) uniform flux q''_o, (8) no radiation, (9) insulated sides and (10) $Bi << 1$.

(ii) Governing Equations. Formulation of the heat equation for this moving fin must take into consideration energy transport due to material motion, surface convection and heat loss at uniform flux from the lower surface. Consider a constant area fin moving with constant velocity U. For steady state, conservation of energy for the element dx gives

$$q_x + \dot{m}\hat{h} = q_x + \frac{dq_x}{dx}dx + \dot{m}\hat{h} + \dot{m}\frac{d\hat{h}}{dx}dx + dq_c + dq_o \tag{a}$$

where $\hat{h}$ is enthalpy per unit mass, $\dot{m}$ is mass flow rate and q_x and q_c are heat transfer rates by conduction and convection, respectively and q_o is heat loss from the bottom surface. Mass flow rate is given by

$$\dot{m} = \rho U A_c \tag{b}$$

where A_c is the cross-sectional area and ρ is density. For a constant pressure process the change in enthalpy is

$$d\hat{h} = c_p dT \tag{c}$$

where c_p is specific heat. Using Fourier's and Newton's laws, we obtain

$$q_x = -kA_c \frac{dT}{dx} \tag{d}$$

and

$$dq_c = h(T - T_\infty)dA_s \tag{e}$$

where A_c is the heat conduction area, dA_s is the surface area of the element through which heat is convected and T_∞ is the ambient temperature. The conduction area is given by

$$A_c = W\,t \tag{f}$$

where W is the sheet width and t is its thickness. Since the sides are assumed to be insulated the convection area dA_s is

$$dA_s = W dx \tag{g}$$

Heat removed from the lower surface of the element, dq_o, is given by

$$dq_o = q_o'' W dx \tag{h}$$

Substituting (b)-(h) into (a)

$$\frac{d^2T}{dx^2} - \frac{\rho c_p U}{k}\frac{dT}{dx} - \frac{h}{kt}(T - T_\infty) - \frac{q_o''}{kt} = 0 \tag{i}$$

This is the fin equation for the moving sheet.

(iii) Boundary Conditions. Since equation (i) is second order with a single independent variable x, two boundary conditions are needed.

(1) The temperature is specified at $x = 0$

$$T(0) = T_o \tag{j}$$

(2) Far away from the furnace the temperature approaches T_∞. Thus

$$T(\infty) = \text{finite} \tag{k}$$

(4) Solution. Equation (i) is a linear, second order differential equation with constant coefficients. Its solution is presented in Appendix A. Equation (i) is rewritten as

$$\frac{d^2T}{dx^2} + 2b\frac{dT}{dx} + m^2 T = c \tag{l}$$

where b, c and m are defined as

$$b = -\frac{\rho c_p U}{2k}, \quad c = \frac{q_o''}{kt} - \frac{h}{kt}T_\infty, \quad m^2 = -\frac{h}{kt} \tag{m}$$

Noting that $b^2 > m^2$, the solution is given by equation (A-6c)

$$T = C_1 \exp(-bx + \sqrt{b^2 - m^2}\, x) + C_2 \exp(-bx - \sqrt{b^2 - m^2}\, x) + \frac{c}{m^2} \tag{n}$$

where C_1 and C_2 are constants of integration. Since $m^2 < 0$, equation (k) requires that

$$C_1 = 0 \tag{o}$$

Equation (j) gives C_2

$$C_2 = T_o - \frac{c}{m^2} \tag{p}$$

Substituting (m), (o) and (p) into (n) gives the temperature distribution in the sheet

$$\frac{T(x) - T_\infty}{T_o - T_\infty} = -\frac{(q_o'')}{h(T_o - T_\infty)} + \left[1 + \frac{(q_o'')}{h(T_o - T_\infty)}\right] \exp\left[\frac{\rho c_p U t}{2k} - \sqrt{\left(\frac{\rho c_p U t}{2k}\right)^2 + \frac{ht}{k}}\right]\frac{x}{t} \tag{q}$$

(5) Checking. *Dimensional check*: (i) The exponent in (q) must be dimensionless

$$\frac{\rho(\text{kg/m}^3)c_p(\text{J/kg–}^\circ\text{C})U(\text{m/s})x(\text{m})}{k(\text{W/m–}^\circ\text{C})} = \text{dimensionless}$$

and

$$\left[\frac{h(\text{W/m}^2\text{–}^\circ\text{C})}{k(\text{W/m–}^\circ\text{C})t(\text{m})}\right]^{1/2} x(\text{m}) = \text{dimensionless}$$

(ii) The term $q_o''/h(T_o - T_\infty)$ must be dimensionless

$$\frac{q_o''(\text{W/m}^2)}{h(\text{W/m}^2\text{–}^\circ\text{C})(T_o - T_\infty)(^\circ\text{C})} = \text{dimensionless}$$

Boundary conditions check: Substitution of (q) into (j) and (k) shows that the two boundary conditions are satisfied.

Differential equation check: Direct substitution of (q) into (i) confirms that the governing equation is satisfied.

Limiting checks: (i) If h = 0 and $q_o'' = 0$ no energy can be exchanged with the sheet after leaving the furnace and therefore its temperature must remain constant. Setting h = 0 and $q_o'' = 0$ in (q) gives $T(x) = T_o$.

(ii) If the velocity U is infinite, the time it takes the fin to reach a distance x vanishes and therefore energy added or removed also vanishes. Thus for $U = \infty$ the temperature remains constant. Setting $U = \infty$ in (q) gives $T(x) = T_o$.

(iii) For the special case of $q_o'' = 0$ the solution should agree with the result of Example 2.4. Setting $q_o'' = 0$ in (q) gives the same result as eq. (2.22) for the case of insulated sides.

(iv) Far away from the exit of the furnace $(x = \infty)$ the temperature becomes uniform and thus energy removed from the bottom must be equal to energy added by convection at the top. Setting $x = \infty$ in (q) gives

$$T(\infty) - T_\infty + (q_o''/h) = 0$$

or

$$h\left[T_\infty - T(\infty)\right] = q_o''$$

This is recognized as Newton's law of cooling applied to the upper surface.

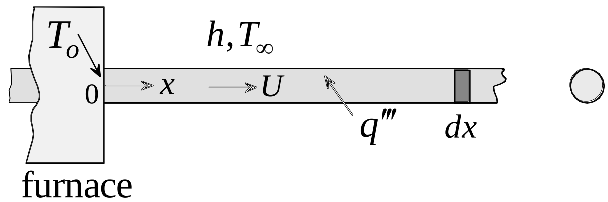
(6) Comments. (i) The temperature decays exponentially with distance x. (ii) Sheet motion has the effect of slowing down the decay. In the limit, $U = \infty$, no decay takes place and the temperature remains constant. (iii) The solution is characterized by three dimensionless parameter:

$$\frac{ht}{k},\ \frac{q_o''}{h(T_o - T_\infty)},\ \frac{\rho c_p U t}{k}$$

Note that the first parameter is the Biot number. The second is the ratio of the heat removed at the lower surface to the heat convected at the upper surface at x = 0. The third parameter is the Peclet number (Reynolds times Prandtl) represents the effect of motion (axial convection).

PROBLEM 2.27

A cable of radius r_o, conductivity k, density ρ and specific heat c_p moves through a furnace with velocity U and leaves at temperature T_o. Electric energy is dissipated in the cable resulting in a volumetric energy generation rate of q'''. Outside the furnace the cable is cooled by convection. The heat transfer coefficient is h and the ambient temperature is T_∞. Model the cable as a fin and assume that it is infinitely long. Determine the steady state temperature distribution in the cable.

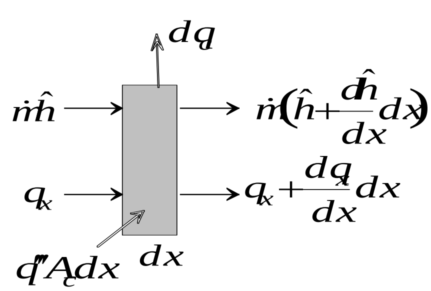


(1) Observations. (i) This is a constant area moving fin problem. Temperature distribution can be assumed one-dimensional. (ii) Heat is exchanged with the surroundings by convection. (iii) The temperature is specified at the outlet of the furnace. (iv) The fin is semi-infinite. (v) The cable generates volumetric energy.

(2) Origin and Coordinates. A rectangular coordinate system is used with the origin at the exit of furnace, as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) constant pressure, (4) constant properties, (5) constant speed, (6) constant *h*, (7) no radiation, (8) uniform energy generation, and (9) $Bi << 1$.



(ii) Governing Equations. Consider an element dx of the cable. Energy balance for the element gives

$$q_x + \dot{m}\hat{h} + q'''A_c dx = q_x + \frac{dq_x}{dx}dx + \dot{m}\hat{h} + \dot{m}\frac{d\hat{h}}{dx}dx + dq_c \tag{a}$$

where $\hat{h}$ is enthalpy per unit mass, $\dot{m}$ is mass flow rate, A_c is conduction area, and q_x, q_c and q''' are heat transfer rates by conduction, convection and energy generation, respectively. Mass flow rate $\dot{m}$ is given by

$$\dot{m} = \rho U A_c \tag{b}$$

where ρ is density. For constant pressure process the change in enthalpy is

$$d\hat{h} = c_p dT \tag{c}$$

where c_p is specific heat. Using Fourier's and Newton's laws, we obtain

$$q_x = -kA_c \frac{dT}{dx} \tag{d}$$

and

$$dq_c = h(T - T_\infty)dA_s \tag{e}$$

where dA_s is the surface area of the element and T_∞ is the ambient temperature. Surface area is expressed in terms of fin circumference C as

$$dA_s = C\,dx \tag{f}$$

Substituting (b)-(f) into (a)

$$\frac{d^2T}{dx^2} - \frac{\rho c_p U}{k}\frac{dT}{dx} - \frac{hC}{kA_c}(T - T_\infty) + \frac{q'''}{k} = 0 \tag{g}$$

The cross-sectional area and circumference are given by

$$A_c = \pi r_o^2 \tag{h}$$

and

$$C = 2\pi r_o \tag{i}$$

Define

$$\theta = T - T_\infty \tag{j}$$

Substituting (h)-(j) into(g)

$$\frac{d^2\theta}{dx^2} + 2b\frac{d\theta}{dx} + m^2\theta = c \tag{k}$$

where the constants b and m^2 are defined as

$$b = -\frac{\rho c_p U}{2k}, \quad m^2 = -\frac{2h}{kr_o}, \quad c = -\frac{q'''}{k} \tag{l}$$

(iii) Boundary Conditions. Since equation (c) is second order with a single independent variable x, two boundary conditions are needed. The temperature is specified at $x = 0$. Thus

$$\theta(0) = T_o - T_\infty \tag{m}$$

Far away from the furnace the temperature is T_∞. Thus, the second boundary condition is

$$\theta(\infty) = \text{finite} \tag{n}$$

(4) Solution. Equation (c) is a linear, second order differential equation with constant coefficients. Its solution is presented in Appendix A. Noting that $b^2 > m^2$, the solution is given by equation (A-6c)

$$\theta = C_1 \exp(-bx + \sqrt{b^2 - m^2}\,x) + C_2 \exp(-bx - \sqrt{b^2 - m^2}\,x) + \frac{c}{m^2} \tag{o}$$

where C_1 and C_2 are constants of integration. Since $m^2 < 0$, boundary condition (f) requires that

$$C_1 = 0 \tag{p}$$

Equation (m) gives C_2

$$C_2 = T_o - T_\infty - \frac{c}{m^2} \tag{q}$$

Substituting (p) and (q) into (o) gives the temperature distribution in the cable

$$\theta = \left(T_o - T_\infty - \frac{c}{m^2}\right)\exp(-bx - \sqrt{b^2 - m^2}\, x) + \frac{c}{m^2} \tag{r}$$

Substituting the definitions of *c*, *b* and *m*, given in equation (l), solution (r) is rewritten in dimensionless form

$$\frac{T(x) - T_\infty}{T_o - T_\infty} = \left[1 - \frac{q'''r_o}{2h(T_o - T_\infty)}\right]\exp\left[\frac{\rho c_p U r_o}{2k} - \sqrt{\left(\frac{\rho c_p U r_o}{2k}\right)^2 + \frac{2hr_o}{k}}\right]\frac{x}{r_o} + \frac{q'''r_o}{2h(T_o - T_\infty)} \tag{s}$$

(5) Checking. *Dimensional check*: (1) The exponent in (r) must be dimensionless:

$$\frac{\rho(\text{kg/m}^3)\, c_p(\text{J/kg–}^\circ\text{C})\, U(\text{m/s})\, r_o(\text{m})}{k(\text{W/m–}^\circ\text{C})} = \text{dimensionless}$$

and

$$\frac{h(\text{W/m}^2\text{–}^\circ\text{C})\, r_o(\text{m})}{k(\text{W/m–}^\circ\text{C})} = \text{dimensionless}$$

(2) The coefficient in (r) must be dimensionless:

$$\frac{q'''(\text{W/m}^3)\, r_o(\text{m})}{2h(\text{W/m}^2\text{–}^\circ\text{C})(T_o - T_\infty)(^\circ\text{C})} = \text{dimensionless}$$

Boundary conditions check: Substitution (s) into (m) and (n) shows that the two boundary conditions are satisfied.

Differential equation check: Direct substitution of (s) into (g) confirms that the governing equation is satisfied.

Limiting checks: (i) If $h = 0$, no energy can be removed from the cable after leaving the furnace and therefore its temperature must continue to increase due to energy generation until it becomes infinite. Setting $h = 0$ in (s) gives $T(x) = \infty$. (ii) At $x = \infty$ the cable reaches a uniform temperature $T(\infty)$ representing a balance between energy generation and surface convection. Note that motion plays not role. Setting $x = \infty$ in (s) gives

$$T(\infty) = T_\infty + \frac{q'''r_o}{2h} \tag{t}$$

Equation energy generated to surface energy convected for an cable element Δx at $x = \infty$ gives

$$2\pi r_o \Delta x h[T(\infty) - T_\infty] = \pi r_o^2 \Delta x q''' \tag{u}$$

Solving (u) for $T(\infty)$ gives the result shown in (t).

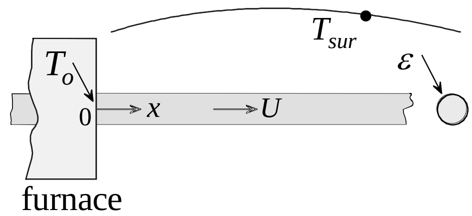
(6) Comments. (i) The temperature decays exponentially with distance *x*. (ii) Cable motion has the effect of slowing down the decay. In the limit, $U = \infty$, no decay takes place and the temperature remains constant. (iii) The solution is characterized by three dimensionless parameter:

$$\frac{hr_o}{k}, \ \frac{q'''r_o}{h(T_o - T_\infty)}, \ \frac{\rho c_p U r_o}{k}$$

Note that the first parameter is the Biot number. The second is the ratio of the heat generated at the lower surface to the heat convected at the upper surface at $x = 0$. The third parameter represents the effect of motion.

PROBLEM 2.28

A cable of radius r_o*, conductivity k, density* ρ *and specific heat* c_p *moves with velocity U through a furnace and leaves at temperature* T_o*. Outside the furnace the cable is cooled by radiation. The surroundings temperature is* T_{sur} *and surface emissivity is* ε*. In certain cases axial conduction can be neglected. Introduce this simplification and assume that the cable is infinitely long. Use a fin model and determine the steady state temperature distribution in the cable.*



(1) Observations. (i) This is a constant area moving fin with surface radiation problem. Temperature distribution can be assumed one-dimensional. (ii) Heat is exchanged with the surroundings by radiation only. (iii) The temperature is specified at the outlet of the furnace. (iv) The fin is semi-infinite. (v) Axial conduction can be neglected.

(2) Origin and Coordinates. A rectangular coordinate system is used with the origin at the exit of furnace, as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) constant pressure, (4) constant properties (5) constant speed, (6) no surface convection, (7) negligible axial conduction, and (8) $Bi << 1$.

(ii) Governing Equations. The heat equation for a moving fin with surface radiation is given by eq. (2.19)

$$\frac{d^2T}{dx^2} - \frac{\rho c_p U}{k}\frac{dT}{dx} - \frac{hC}{kA_c}(T - T_\infty) - \frac{\varepsilon\sigma C}{kA_c}(T^4 - T_{sur}^4) = 0 \tag{a}$$

where

$$A_c = \pi r_o^2 \tag{b}$$

$$C = 2\pi r_o \tag{c}$$

Neglecting axial conduction and surface convection and using (b) and (c), equation (c) simplifies to

$$\frac{\rho c_p U}{k}\frac{dT}{dx} + \frac{2\varepsilon\sigma}{kr_o}(T^4 - T_{sur}^4) = 0 \tag{d}$$

(iii) Boundary Conditions. Since equation (d) is first order equation with a single independent variable *x*, one boundary condition is needed. The temperature is specified at *x* = 0. Thus

$$T(0) = T_o \tag{e}$$

(4) Solution. Separating variables, (d) becomes

$$\frac{dT}{T^4 - T_{sur}^4} = \frac{2\varepsilon\sigma}{\rho c_p U r_o}dx \tag{f}$$

Direct integration of (f) gives the solution $T(x)$

$$\frac{2\varepsilon\sigma}{\rho c_p U r_o} x = \frac{1}{4T_{sur}^3} \ln\left|\frac{T_{sur}+T}{T_{sur}-T}\right| + \frac{2}{T_{sur}^3}\tan^{-1}\frac{T}{T_{sur}} + C \tag{g}$$

where boundary condition (e) gives the constant of integration C

$$C = -\frac{1}{4T_{sur}^3} \ln\left|\frac{T_{sur}+T_o}{T_{sur}-T_o}\right| + \frac{2}{T_{sur}^3}\tan^{-1}\frac{T_o}{T_{sur}} \tag{h}$$

Substituting (h) into (g) and rearranging the resulting equation in dimensionless form gives

$$\frac{x}{r_o} = \frac{\rho c_p U}{8\varepsilon\sigma T_{sur}^3}\left\{\ln\left|\frac{1+\dfrac{T}{T_{sur}}}{1-\dfrac{T}{T_{sur}}}\right| - \ln\left|\frac{1+\dfrac{T_o}{T_{sur}}}{1-\dfrac{T_o}{T_{sur}}}\right| + 8\left[\tan^{-1}\frac{T}{T_{sur}} - \tan^{-1}\frac{T_o}{T_{sur}}\right]\right\} \tag{i}$$

(5) Checking. *Dimensional check*: (1) Each term in (i) is dimensionless:

Boundary conditions check: Solution (i) satisfies boundary condition (e).

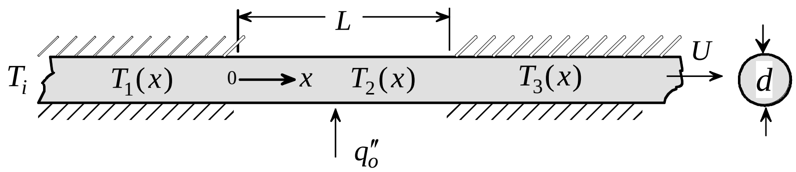
Limiting checks: at $x = \infty$ cable temperature must be equal to T_{sur}. Setting $T = T_{sur}$ in (i) gives $x = \infty$.

(6) Comments. The solution is characterized by two dimensionless parameter:

$$\frac{T_o}{T_{sur}} \text{ and } \frac{\rho c_p U}{\varepsilon\sigma T_{sur}^3}$$

PROBLEM 2.29

A thin wire of diameter d, specific heat c_p, *conductivity k and density* ρ *moves with velocity U through a furnace of length L. Heat is added to the wire in the furnace at a uniform surface flux* q''_o. *Far away from the inlet of the furnace the wire is at temperature* T_i. *Assume that no heat is exchanged with the wire before it enters and after it leaves the furnace. Determine the temperature of the wire leaving the furnace.*



(1) Observations. (i) This is a constant area moving fin. (ii) Temperature distribution can be assumed one dimensional. (iii) The fin has three sections. Surface and boundary conditions are different for each section. Thus three equations are needed. (iv) The surface of two sections is insulated while heat is added at uniform flux at the third (middle section).

(2) Origin and Coordinates. The origin is selected at the start of the middle section and the coordinate *x* points in the direction of motion.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) constant cross-sectional area, (4) $Bi << 1$, (5) constant properties, (6) uniform surface flux, (7) no energy generation and (8) no radiation.

(ii) Governing Equations. Three fin equations are needed. Let

$T_1(x)$ = temperature distribution in the first section, $-\infty < x < 0$
$T_2(x)$ = temperature distribution in the second (middle) section, $0 < x < L$
$T_3(x)$ = temperature distribution in the third section, $L < x < \infty$

We begin by formulating the fin equation for the second section. Consider a constant area fin moving with constant velocity *U*. For steady state, conservation of energy for an element *dx* gives

$$q_x + \dot{m}\hat{h} = q_x + \frac{dq_x}{dx}dx + \dot{m}\hat{h} + \dot{m}\frac{d\hat{h}}{dx}dx + dq_o \tag{a}$$

where $\hat{h}$ is enthalpy per unit mass, $\dot{m}$ is mass flow rate, q_x is conduction heat transfer rate conduction and q_o is heat added at the surface. Mass flow rate is given by

$$\dot{m} = \rho U A_c \tag{b}$$

where A_c is the cross-sectional area and ρ is density. For a constant pressure process the change in enthalpy is

$$d\hat{h} = c_p dT_2 \tag{c}$$

where c_p is specific heat. Using Fourier's law we obtain

$$q_x = -kA_c \frac{dT_2}{dx} \tag{d}$$

where A_c is the cross-sectional area given by

$$A_c = \pi d^2 / 4 \tag{e}$$

where d is the wire diameter. Heat added at the surface of the element, dq_o, is given by

$$dq_o = (\pi d) q_o'' dx \tag{f}$$

Substituting (b)-(f) into (a)

$$\frac{d^2T_2}{dx^2} - \frac{\rho c_p U}{k}\frac{dT_2}{dx} + \frac{4q_o''}{kd} = 0, \qquad 0 \le x \le L \tag{g}$$

Equation (g) is the fin equation for the middle section. To obtain the fin equations for the first and third sections, set $q_o'' = 0$ in equation (g)

$$\frac{d^2T_1}{dx^2} - \frac{\rho c_p U}{k}\frac{dT_1}{dx} = 0, \qquad -\infty \le x \le 0 \tag{h}$$

and

$$\frac{d^2T_3}{dx^2} - \frac{\rho c_p U}{k}\frac{dT_3}{dx} = 0, \qquad x \ge L \tag{i}$$

(iii) Boundary Conditions. Six boundary conditions are needed.

(1) Specified temperature at $x = -\infty$

$$T_1(-\infty) = T_i \tag{j}$$

(2) Equality of temperature at $x = 0$

$$T_1(0) = T_2(0) \tag{k}$$

(3) Equality of flux at $x = 0$

$$\frac{dT_1(0)}{dx} = \frac{dT_2(0)}{dx} \tag{l}$$

(4) Equality of temperature at $x = L$

$$T_2(L) = T_3(L) \tag{m}$$

(5) Equality of flux at $x = L$

$$\frac{dT_2(L)}{dx} = \frac{dT_3(L)}{dx} \tag{n}$$

(6) Finite temperature at $x = \infty$

$$T_3(\infty) = \text{finite} \tag{o}$$

(4) Solution. The solutions to (g), (h) and (i) are

$$T_1 = A_1 \exp(\rho c_p U x / k) + B_1 \tag{p}$$

$$T_2 = A_2 \exp(\rho c_p Ux/k) + \frac{4q''_o}{\rho c_p U d} x + B_2 \tag{q}$$

$$T_3 = A_3 \exp(\rho c_p Ux/k) + B_3 \tag{r}$$

where A_1, A_2, A_3, B_1, B_2 and B_3 are constants of integration. Application of boundary conditions (j) to (o) give six equations for the six constants

$$B_1 = T_i$$

$$A_1 + T_i = A_2 + B_2$$

$$\frac{\rho c_p U}{k} A_1 = \frac{\rho c_p U}{k} A_2 + \frac{4q''_o}{\rho c_p U d}$$

$$A_2 \exp(\rho c_p UL/k) + \frac{4q''_o}{\rho c_p U d} L + B_2 = A_3 \exp(\rho c_p UL/k) + B_3$$

$$A_2 \frac{\rho c_p U}{k} \exp(\rho c_p UL/k) + \frac{4q''_o}{\rho c_p U d} = A_3 \frac{\rho c_p U}{k} \exp(\rho c_p UL/)$$

$$A_3 = 0$$

The above six equations are solved for the six constants. Substitution into equations (p), (q) and (r) gives

$$T_1(x) = (k/\rho c_p U)^2 \frac{4q''_o}{kd} \left[1 - \exp(-\rho c_p UL/k)\right] \exp(\rho c_p Ux/k) + T_i \tag{s}$$

$$T_2(x) = -(k/\rho c_p U)^2 \frac{4q''_o}{kd} \exp\left[(\rho c_p U/k)(x - L)\right] + \frac{4q''_o}{\rho c_p U d} x + (k/\rho c_p U)^2 \frac{4q''_o}{kd} + T_i \tag{t}$$

$$T_3(x) = T_i + \frac{4q''_o}{\rho c_p U d} L \tag{u}$$

Expressing (s)-(u) in dimensionless form

$$\frac{T_1(x) - T_i}{\dfrac{4kq''_o}{(\rho c_p U)^2 d}} = \exp \frac{\rho c_p UL(x/L)}{k} \left[1 - \exp\left(-\frac{\rho c_p UL}{k}\right)\right] \tag{v}$$

$$\frac{T_2(x) - T_i}{\dfrac{4kq''_o}{(\rho c_p U)^2 d}} = 1 + \frac{\rho c_p UL}{k} \frac{x}{L} - \exp \frac{\rho c_p UL}{k} \left[(x/L) - 1\right] \tag{w}$$

$$\frac{T_3(x) - T_i}{\dfrac{4q''_o L}{\rho c_p U d}} = 1 \tag{x}$$

(5) Checking. *Dimensional checks*: (i) The exponent of the exponential, $\rho c_p Ux/k$, is dimensionless. (ii) Each term in equations (s), (t) and (u) has units of temperature and each term in (v), (w) and (x) is dimensionless.

Boundary conditions check: Substitution of solutions (s), (t) and (u) into equations (j)-(o) shows that the six boundary conditions are satisfied.

Differential equations check: Direct substitution of (s), (t) and (u) into (g), (h) and (i), respectively, shows that the governing equations are satisfied.

Global energy conservation: energy added to the wire at the middle section must be axially convected by the wire. Thus conservation of energy between $x = -\infty$ and $x = \infty$ gives

$$\dot{m} c_p \left[T_3(\infty) - T_i \right] = q_o'' \pi L d$$

Substituting (b), (e) and (u) into the above gives

$$\rho U \pi (d^2/4) c_p \left[T_i + \frac{4 q_o''}{\rho c_p U d} L - T_i \right] = q_o'' \pi L d$$

or

$$q_o'' \pi L d = q_o'' \pi L d$$

Thus energy is conserved.

Limiting check: (i) If wire velocity vanishes, the temperature of the wire will continue to rise. Thus for $U = 0$ the wire temperature should be infinite. Setting $U = 0$ in (s), (t) and (u) gives

$$T_1(x) = T_2(x) = T_3(x) = \infty$$

(ii) If the wire is not heated at the middle section, its temperature should be uniform equal to T_i. Setting $q_o'' = 0$ in (s), (t) and (u) gives

$$T_1(x) = T_2(x) = T_3(x) = T_i$$

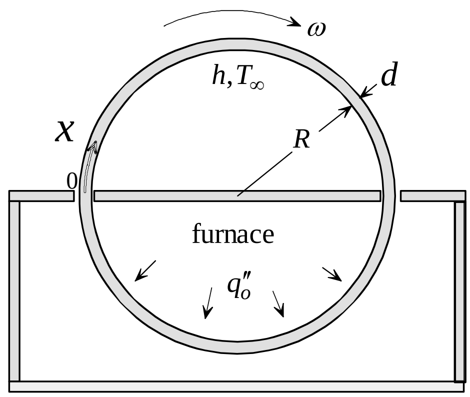
(iii) If the wire moves with infinite velocity, no heating will take place at the middle section and consequently its temperature should be uniform equal to T_i. Setting $U = \infty$ in (s), (t) and (u) gives

$$T_1(x) = T_2(x) = T_3(x) = \infty$$

(6) Comments. (i) Wire temperature increases with distance along the first and second sections. However, in the third section the temperature is uniform. (ii) a simpler approach to solving this problem is to recognize that the temperature in the third section must be uniform, use global conservation of energy to determine it and solve the temperature distribution in the first and second sections. (iii) The solution is characterized by a single parameter $\rho c_p UL/k$.

PROBLEM 2.30

A wire of diameter d is formed into a circular loop of radius R. The loop rotates with constant angular velocity ω. One half of the loop passes through a furnace where it is heated at a uniform surface flux q_o''. *In the remaining half the wire is cooled by convection. The heat transfer coefficient is h and the ambient temperature is* T_∞. *The specific heat of the wire is* c_p, *conductivity k and density ρ. Use fin approximation to determine the steady state temperature distribution in the wire.*



(1) Observations. (i) This is a constant area moving fin problem. (ii) Temperature distribution can be assumed one dimensional. (iii) The fin has two sections. Since surface conditions are different for each section, two fin equations are needed. (iv) In the furnace section heat is added to the wire at uniform flux. Outside the furnace heat is removed by convection.

(2) Origin and Coordinates. The origin is selected at the outlet end of the furnace. The coordinate x points in the direction of motion.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) constant cross-sectional area, (4) $Bi << 1$, (5) constant properties, (6) uniform surface flux, (7) uniform heat transfer coefficient and ambient temperature, (8) negligible curvature effect ($d/R << 1$), (9) no energy generation and (10) no radiation.

(ii) Governing Equations. Two fin equations are needed. Let

$T_1(x)$ = wire temperature distribution outside furnace, $0 < x < \pi R_c$
$T_2(x)$ = wire temperature distribution inside furnace, $\pi R_c < x < 2\pi R_c$

where R_c is the distance from the center of loop to the wire center, given by

$$R_c = R + d/2 \tag{a}$$

We begin by formulating the fin equation outside the furnace. Consider a constant area fin moving with constant velocity. For steady state, conservation of energy for an element dx gives

$$q_x + \dot{m}\hat{h} = q_x + \frac{dq_x}{dx}dx + \dot{m}\hat{h} + \dot{m}\frac{d\hat{h}}{dx}dx + dq_c \tag{b}$$

where $\hat{h}$ is enthalpy per unit mass, $\dot{m}$ is mass flow rate, q_x is conduction heat transfer rate and q_c is heat removed by convection. Mass flow rate is

$$\dot{m} = \rho U A_c \tag{c}$$

where A_c is the cross-sectional area and ρ is density. For a constant pressure process the change in enthalpy is

$$d\hat{h} = c_p dT_1 \tag{d}$$

where c_p is specific heat. Using Fourier's law we obtain

$$q_x = -kA_c \frac{dT_1}{dx} \tag{e}$$

The cross-sectional area A_c is given by

$$A_c = \pi d^2 / 4 \tag{f}$$

where d is the wire diameter. Heat removed from the element by convection, dq_c, is

$$dq_c = (\pi d)h(T_1 - T_\infty)dx \tag{g}$$

Substituting (c)-(g) into (b)

$$\frac{d^2 T_1}{dx^2} - \frac{\rho c_p U}{k}\frac{dT_1}{dx} - \frac{4h}{kd}(T_1 - T_\infty) = 0 \qquad 0 < x < \pi R_c \tag{h}$$

where U is the tangential velocity given by

$$U = \omega R_c$$

Equation (h) is the fin equation for the wire outside the furnace. To obtain the fin equation inside the furnace consider an element of the wire dx and apply conservation of energy

$$q_x + \dot{m}\hat{h} + dq_o = q_x + \frac{dq_x}{dx}dx + \dot{m}\hat{h} + \dot{m}\frac{d\hat{h}}{dx}dx \tag{i}$$

where dq_o is the heat added at the surface given by

$$dq_o = (\pi d)q_o'' dx \tag{j}$$

Substituting (c)-(f) and (j) into (i)

$$\frac{d^2 T_2}{dx^2} - \frac{\rho c_p U}{k}\frac{dT_2}{dx} + \frac{4q_o''}{kd} = 0 \qquad \pi R_c < x < 2\pi R_c \tag{k}$$

This is the fin equation for the wire inside the furnace.

(iii) Boundary Conditions. Four boundary conditions are needed.

(1) Equality of temperature at $x = 0$

$$T_1(0) = T_2(0) \tag{l}$$

(2) Equality of flux at $x = 0$

$$\frac{dT_1(0)}{dx} = \frac{dT_2(0)}{dx} \tag{m}$$

(3) Equality of temperature at $x = \pi R_c$

$$T_1(\pi R_c) = T_2(\pi R_c) \tag{n}$$

(4) Equality of flux at $x = \pi R_c$

$$\frac{dT_1(\pi R_c)}{dx} = \frac{dT_2(\pi R_c)}{dx} \tag{o}$$

(4) Solution. The solutions to (h) and (k) are

$$T_1(x) = A_1 \exp\left[-bx + \sqrt{b^2 - m^2}\right] + B_1 \exp\left[-bx - \sqrt{b^2 - m^2}\right] + \frac{c}{m^2} \tag{p}$$

and

$$T_2 = A_2 + B_2 \exp(\rho c_p U x / k) + Q_o x \tag{q}$$

where A_1, A_2, B_1 and B_2 are constants of integration and b, c, m^2 and Q_o are constant defined as

$$b = -\frac{\rho c_p U}{2k}, \quad m^2 = -\frac{4h}{kd}, \quad c = -\frac{4h}{kd} T_\infty, \quad Q_o = \frac{4 q_o''}{\rho c_p U d} \tag{r}$$

Application of the four boundary conditions give

$$A_1 + B_1 + (c / m^2) = A_2 + B_2 \tag{s}$$

$$\left[-b + \sqrt{b^2 - m^2}\right] A_1 + \left[-b - \sqrt{b^2 - m^2}\right] B_1 = (\rho c_p U U / k) B_2 + Q_o \tag{t}$$

$$\begin{aligned} A_1 \exp\left[(-b + \sqrt{b^2 - m^2}) \pi R_c\right] + B_1 \exp\left[(-b - \sqrt{b^2 - m^2}) \pi R_c\right] + (c / m^2) = \\ A_2 + B_2 \exp(\pi R_c \rho c_p U / k) + \pi R_c Q_o \end{aligned} \tag{u}$$

$$\begin{aligned} A_1(-b + \sqrt{b^2 - m^2}) \exp\left[(-b + \sqrt{b^2 - m^2}) \pi R_c\right] + \\ B_1(-b - \sqrt{b^2 - m^2}) \exp\left[(-b - \sqrt{b^2 - m^2}) \pi R_c\right] = B_2 (U / \alpha) \exp(\pi R_c \rho c_p U / k) + Q_o \end{aligned} \tag{v}$$

Equations (s)-(v) are solved for the four constants.

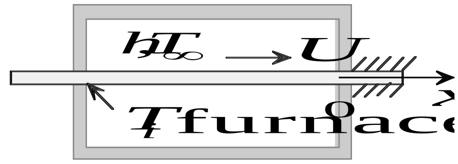
(5) Checking. *Dimensional check*: (i) Each term in (h) and (k) has units of ($^{\circ}$C/m^2). The exponents of the exponentials in (p) and (q) are dimensionless. (ii) Each term in (p) and (q) has units of temperature.

(6) Comments. The rotating wire can be thought of as a heat exchanger removing heat from the furnace and adding it to the ambient fluid outside the furnace. The net heat transfer rate can be determined by making an energy balance for the wire. For steady state energy added to the surface inside the furnace is equal to energy removed by convection outside the furnace. Considering the energy added in the furnace, we obtain

$$q = \pi d (\pi R_c) q_o''$$

This shows that the rate of heat transfer is independent of angular velocity!

PROBLEM 2.31



A wire of radius r_o moves with a velocity U through a furnace of length L. It enters the furnace at T_i where it is heated by convection. The furnace temperature is T_∞ and the heat transfer coefficient is h. Assume that no heat is removed from the wire after it leaves the furnace. Using fin approximation determine the wire temperature leaving the furnace.

(1) Observations. (i) This is a constant area moving fin. (ii) Temperature distribution can be assumed one dimensional. (iii) The wire has two sections. Surface and boundary conditions are different for each section. Thus two equations are needed. (iv) In the furnace section convection takes place at the surface. The section outside the furnace is insulated. (v) To determine the wire temperature at the furnace outlet one must determine the temperature distribution in the wire.

(2) Origin and Coordinates. The origin is selected at the exit of the furnace section (start of the insulated section) as shown.

(3) Formulation.

(i) Assumptions. (1) One-dimensional, (2) steady state, (3) constant cross-sectional area, (4) $Bi << 1$, (5) constant properties, (6) uniform heat transfer coefficient and furnace temperature, (7) no energy generation and (8) no radiation.

(ii) Governing Equations. Two fin equations are needed. Let

$T_1(x)$ = temperature distribution in the first section, $-L \le x \le 0$
$T_2(x)$ = temperature distribution in the second (middle) section, $0 \le x \le \infty$

The heat equation for both sections is obtained from eq. (2.19)

$$\frac{d^2T}{dx^2} - \frac{\rho c_p U}{k}\frac{dT}{dx} - \frac{hC}{kA_c}(T - T_\infty) - \frac{\varepsilon \sigma C}{kA_c}(T^4 - T_{sur}^4) = 0 \tag{2.19}$$

Neglecting radiation in eq. (2.19) gives the heat equation in the furnace section

$$\frac{d^2T_1}{dx^2} - \frac{\rho c_p U}{k}\frac{dT_1}{dx} - \frac{hC}{kA_c}(T_1 - T_\infty) = 0 \tag{a}$$

Similarly, neglecting radiation and convection in eq. (2.19) gives the heat equation in the insulated section

$$\frac{d^2T_2}{dx^2} - \frac{\rho c_p U}{k}\frac{dT_2}{dx} = 0 \tag{b}$$

(iii) Boundary Conditions. Four boundary conditions are needed.

(1) Specified temperature at $x = -L$

$$T_1(-L) = T_i \tag{c}$$

(2) Equality of temperature at $x = 0$

$$T_1(0) = T_2(0) \tag{d}$$

(3) Equality of flux at $x = 0$

$$\frac{dT_1(0)}{dx} = \frac{dT_2(0)}{dx} \tag{e}$$

(6) Finite temperature at $x = \infty$

$$T_2(\infty) = \text{finite} \tag{f}$$

(4) Solution. The solutions to (a) is given by eq. (A-6c) of Appendix A. Rewriting (a) as

$$\frac{d^2T_1}{dx^2} + 2b\frac{dT_1}{dx} + m^2T_1 = c \tag{g}$$

where

$$b = -\frac{\rho c_p U}{2k} \tag{h}$$

$$m^2 = -\frac{2h}{kr_o} \tag{i}$$

$$c = -\frac{2h}{kr_o}T_\infty \tag{j}$$

The solution is

$$T_1 = A_1 \exp(-bx + \sqrt{b^2 - m^2}\,x) + B_1 \exp(-bx - \sqrt{b^2 - m^2}\,x) + \frac{c}{m^2} \tag{k}$$

The solution to (b) is

$$T_2 = A_2 \exp(-2bx) + B_2 \tag{l}$$

where A_1, B_1, A_2 and B_2 are constants of integration. The four boundary conditions give

$$A_1 = \frac{T_i - T_\infty}{\exp(bL - \sqrt{b^2 - m^2}\,L) + \dfrac{-b + \sqrt{b^2 - m^2}}{b + \sqrt{b^2 - m^2}}\exp(bL + \sqrt{b^2 - m^2}\,L)} \tag{m}$$

$$B_1 = \frac{\dfrac{-b + \sqrt{b^2 - m^2}}{b + \sqrt{b^2 - m^2}}(T_i - T_\infty)}{\exp(bL - \sqrt{b^2 - m^2}\,L) + \dfrac{-b + \sqrt{b^2 - m^2}}{b + \sqrt{b^2 - m^2}}\exp(bL + \sqrt{b^2 - m^2}\,L)} \tag{n}$$

$$A_2 = 0 \tag{o}$$

$$B_2 = T_\infty + \frac{(T_i - T_\infty)}{\exp(bL - \sqrt{b^2 - m^2}\, L) + \dfrac{-b + \sqrt{b^2 - m^2}}{b + \sqrt{b^2 - m^2}} \exp(bL + \sqrt{b^2 - m^2}\, L)} \left[1 + \frac{-b + \sqrt{b^2 - m^2}}{b + \sqrt{b^2 - m^2}} \right] \tag{p}$$

The wire temperature outlet temperature is obtained by evaluating solution (k) or (l) at $x = 0$

$$T_1(0) = T_2(0) = B_2 \tag{q}$$

(5) Checking. *Dimensional check*: (i) The argument of the exponentials in equations (k)-(n) is dimensionless. (ii) Each term in solutions (k) and (l) has units of temperature.

Boundary conditions check: Solutions (k) and (l) satisfy boundary conditions (c)-(f).

Differential equations check: Direct substitution shows that (k) satisfies (a) and (l) satisfies (b).

Limiting check: If furnace temperature is the same as inlet temperature the entire wire will be at the inlet temperature. Setting $T_\infty = T_i$ into (m), (n) and (p) gives

$$A_1 = B_1 = 0$$

and

$$B_2 = T_\infty$$

Equations (i) and (j) give

$$\frac{c}{m^2} = T_\infty$$

Substituting these results into (k) and (l) gives

$$T_1(x) = T_2(x) = T_\infty$$

(6) Comments. (i) Because the wire has two sections having different surface conditions it was necessary to use two heat equations. (ii) The wire remains at uniform temperature once it exits the furnace. (iii) Rewriting solutions (k) and (l) in dimensionless form shows that the problem is characterized by two dimensionless parameters:

$$bL = -\frac{\rho c_p UL}{2k} \quad \text{and} \quad m^2 L^2 = \frac{2hL^2}{kr_o}$$

PROBLEM 2.32

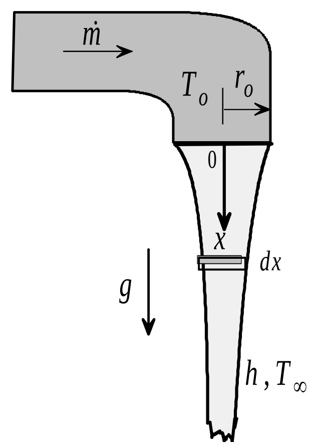
Hot water at T_o flows downward from a faucet of radius r_o at a rate $\dot{m}$. The water is cooled by convection as it flows. The ambient temperature is T_∞ and the heat transfer coefficient is h. Suggest a model for analyzing the temperature distribution in the water. Formulate the governing equations and boundary conditions.

(1) Observations. (i) If the Biot number, hr_o / k, is small compared to unity a fin model can be used to analyze the temperature distribution in the water. (ii) Since water velocity increases with distance *x*, conservation of mass requires that the cross-sectional area decreases with distance. Thus this is a variable area fin which is moving with variable velocity. (iii) Temperature distribution can be assumed one dimensional.

(2) Origin and Coordinates. The origin is selected at the outlet of the faucet and the coordinate *x* points in the direction of motion and gravity.

(3) Formulation.

(i) Assumptions. (1) One-dimensional temperature distribution, (2) steady state, (3) $Bi << 1$, (4) constant conductivity, (5) constant density, (6) uniform heat transfer coefficient and ambient temperature, (7) uniform velocity at any given section, (8) circular cross section, (9) no energy generation, (10) negligible air resistance and (11) no radiation.



(ii) Governing Equations. Consider an element *dx* of the moving fluid. Conservation of energy for steady state gives

$$q_x + \dot{m}\hat{h} = q_x + \frac{dq_x}{dx}dx + \dot{m}\hat{h} + \dot{m}\frac{d\hat{h}}{dx}dx + dq_c \qquad \text{(a)}$$

where $\hat{h}$ is enthalpy per unit mass, $\dot{m}$ is mass flow rate, q_x is conduction heat transfer rate and q_c is heat exchange by convection. For a constant pressure process the change in enthalpy is

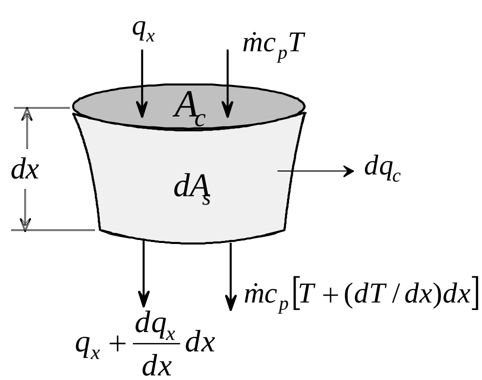
$$d\hat{h} = c_p dT \qquad \text{(b)}$$

where c_p is specific heat. Using Fourier's law we obtain

$$q_x = -kA_c \frac{dT}{dx} \qquad \text{(c)}$$

where $A_c(x)$ is the cross-sectional area. Heat exchange by convection is given by

$$dq_c = (T - T_\infty)dA_s \qquad \text{(d)}$$



where dA_s is surface area of the element. Substituting (b)-(d) into (a)

$$\frac{d}{dx}\left[A_c(x)\frac{dT}{dx}\right]-\frac{\dot{m}c_p}{k}\frac{dT}{dx}-\frac{h}{k}\frac{dA_s}{dx}(T-T_\infty)=0 \tag{e}$$

It remains to determine $A_c(x)$ and dA_s/dx. The conduction area is given by

$$A_c(x)=\pi r^2 \tag{f}$$

where $r=r(x)$ is the local radius. To determine $r(x)$ we apply conservation of mass

$$\dot{m}=\rho U_o \pi r_o^2=\rho U(x)\pi r^2 \tag{g}$$

where U_o is the discharge velocity and $U(x)$ is the local velocity. Solving (g) for U_o gives

$$U_o=\dot{m}/\rho\pi r_o^2 \tag{h}$$

where ρ is density. Neglecting air resistance Bernoulli's equation gives the local velocity $U(x)$

$$U(x)=\sqrt{U_o^2+2gx}=\sqrt{(\dot{m}/\rho\pi r_o^2)^2+2gx} \tag{i}$$

where g is the gravitational acceleration. Substituting (i) into (g) and solving for $r(x)$

$$r(x)=r_o\left[1+(\rho\pi r_o^2/\dot{m})^2\,2gx\right]^{-1/4} \tag{j}$$

Using (j) into (f)

$$A_c(x)=\pi r_o^2\left[1+(\rho\pi r_o^2/\dot{m})^2\,2gx\right]^{-1/2} \tag{k}$$

Next we determine dA_s/dx using eq. (2.6a)

$$\frac{dA_s}{dx}=C(x)\sqrt{1+(dr/dx)^2} \tag{l}$$

where

$$C(r)=2\pi r \tag{m}$$

and

$$\frac{dr}{dx}=-\frac{1}{2}gr_o(\rho\pi r_o^2/\dot{m})^2\left[1+(\rho\pi r_o^2/\dot{m})^2\,2gx\right]^{-5/4} \tag{n}$$

Substituting (j), (m) and (n) into (l)

$$\frac{dA_s}{dx}=2\pi r_o\left[1+(\rho\pi r_o^2/\dot{m})^2\,2gx\right]^{-1/4}\left\{1+(gr_o/2)^2(\rho\pi r_o^2/\dot{m})^4\left[1+(\rho\pi r_o^2/\dot{m})^2\,2gx\right]^{-5/2}\right\}^{1/2} \tag{o}$$

Substituting (k) and (o) into (e) gives the heat equation for the flowing water.

(5) Checking. *Dimensional check*: (i) Each term in (e) has units of temperature. (ii) Each term in (i) has units of velocity. (iii) The term $(\rho\pi r_o^2/\dot{m})^2\,2gx$ is dimensionless. (iv) The slope in (n) is dimensionless. (v) Equation (o) has units of length.

Limiting check: For the special case of zero gravity the velocity and cross sectional area remain constant. The fin equation for this case is given by eq. (2.19). Setting g equal to zero in (i), (k) and (o) and substituting the result into (e) and using (g) gives

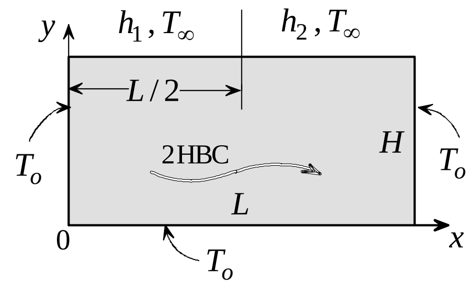
$$\frac{d^2T}{dx^2} - \frac{\rho c_p U_o}{k}\frac{dT}{dx} - \frac{2h}{kr_o}(T - T_\infty) = 0$$

This is identical to eq. (2.19) with the radiation term set equal to zero.

(6) Comments. The problem can be simplified by neglecting the dr/dx term in (l) and using an average value for the circumference C.

PROBLEM 3.1

Three sides of a rectangular plate $L \times H$ are maintained at uniform temperature T_o while the fourth side exchanges heat by convection. The heat transfer coefficient is non-uniform, being h_1 over half the surface and h_2 over the remaining half. The ambient temperature is T_∞. Determine the two-dimensional steady state temperature distribution.



(1) Observations. (i) This is a steady state two-dimensional conduction problem. (ii) Four boundary conditions are needed. (iii) Three conditions can be made homogeneous by defining a new temperature variable $\theta(x, y) = T(x, y) - T_o$. (iv) The heat transfer coefficient along boundary (x,H) is not uniform. This requires special consideration.

(2) Origin and Coordinates. The origin and coordinate axes are selected as shown.

(3) Formulation.

(i) Assumptions. (1) Two-dimensional, (2) steady, (3) no energy generation and (4) constant thermal conductivity.

(ii) Governing Equations. Introducing the above assumptions into eq. (1.8) gives the heat equation

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 \tag{a}$$

where

$$\theta(x, y) = T(x, y) - T_o \tag{b}$$

(iii) Independent Variable with Two Homogeneous Boundary Conditions. The two boundaries at $x = 0$ and $x = L$ have homogeneous conditions. Thus, the x-variable has two homogeneous boundary conditions.

(iv) Boundary Conditions. The boundary conditions are listed in the following order: starting with the two boundaries in the variable having two homogeneous conditions and ending with the non-homogeneous condition. Thus, the first three conditions in this order are homogeneous and the fourth is non-homogeneous. Therefore, we write

(1) $\theta(0, y) = 0$, homogeneous

(2) $\theta(L, y) = 0$, homogeneous

(3) $\theta(x, 0) = 0$, homogeneous

The fourth boundary condition is first written in terms of the variable T and then transformed in terms of the variable θ. Using Fourier's law and Newton's law

$$-k\frac{\partial T(x,H)}{\partial y} = \begin{cases} h_1\left[T(x,H) - T_\infty\right], & 0 < x < L/2 \\ h_2\left[T(x,H) - T_\infty\right], & L/2 < x < L \end{cases}$$

Expressed in terms of θ, the above gives the fourth boundary condition

$$(4)\ -k\frac{\partial\theta(x,H)}{\partial y} = f(x) = \begin{cases} h_1\left[\theta(x,H)-(T_\infty-T_o)\right], & 0<x<L/2 \\ h_2\left[\theta(x,H)-(T_\infty-T_o)\right], & L/2<x<L \end{cases} \quad \text{non-homogeneous}$$

This condition represents non-uniform convection.

(4) Solution.

(i) Assumed Product Solution. Assume a solution in the form

$$\theta(x,y) = X(x)\,Y(y) \tag{c}$$

Substituting into (a), separating variables and setting the resulting equation equal to constant

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \pm\lambda_n^2$$

Assuming that λ_n is multi-valued, the above gives

$$\frac{d^2X_n}{dx^2} \mp \lambda_n^2 X_n = 0 \tag{d}$$

and

$$\frac{d^2Y_n}{dy^2} \pm \lambda_n^2 Y_n = 0 \tag{e}$$

(ii) Selecting the Sign of the λ_n^2 Terms. Since the x-variable has two homogeneous boundary conditions, the $\lambda_n^2 X_n$ term in (d) takes the positive sign. Therefore, (d) and (e) become

$$\frac{d^2X_n}{dx^2} + \lambda_n^2 X_n = 0 \tag{f}$$

and

$$\frac{d^2Y_n}{dy^2} - \lambda_n^2 Y_n = 0 \tag{g}$$

For the important case of $\lambda_n = 0,$ equations (f) and (g) become

$$\frac{d^2X_0}{dx^2} = 0 \tag{h}$$

and

$$\frac{d^2Y_0}{dy^2} = 0 \tag{i}$$

(iii) Solutions to the Ordinary Differential Equations. The solutions to eqs. (f)-(i) are

$$X_n(x) = A_n \sin\lambda_n x + B_n \cos\lambda_n x \tag{j}$$

$$Y_n(y) = C_n \sinh\lambda_n y + D_n \cosh\lambda_n y \tag{k}$$

$$X_0(x) = A_0 x + B_0 \tag{l}$$

$$Y_0(y) = C_0 y + D_0 \tag{m}$$

Corresponding to each value of λ_n there is a temperature solution $\theta_n(x, y)$. Thus

$$\theta_n(x, y) = X_n(x) Y_n(y) \tag{n}$$

and

$$\theta_0(x, y) = X_0(x) Y_0(y) \tag{o}$$

The complete solution becomes

$$\theta(x, y) = X_0(x) Y_0(y) + \sum_{n=1}^{\infty} X_n(x) Y_n(y) \tag{p}$$

(iv) Application of Boundary Conditions. Boundary condition (1) applied to solutions (j) and (l) gives

$$B_n = B_0 = 0 \tag{q}$$

Boundary condition (2) applied to (j) gives the characteristic equation for λ_n

$$\sin\lambda_n L = 0 \tag{r}$$

Thus λ_n

$$\lambda_n = \frac{n\pi}{L} \qquad (n = 1,2,3,\ldots.) \tag{s}$$

Similarly, boundary condition (2) applied to (l) gives

$$A_0 = 0$$

With $A_0 = B_0 = 0$, the solution corresponding to $\lambda_n = 0$ vanishes. Application of boundary condition (3) to (k) gives

$$D_n = 0$$

Solution (p) becomes

$$\theta(x, y) = \sum_{n=1}^{\infty} a_n (\sin\lambda_n x)(\sinh\lambda_n y) \tag{t}$$

where $a_n = A_n C_n$. The only remaining unknown is a_n. Application of condition (4) gives

$$f(x) = -k \sum_{n=1}^{\infty} (a_n \lambda_n \cosh\lambda_n H) \sin\lambda_n x \tag{u}$$

where $f(x)$ is defined in boundary condition (4).

(v) Orthogonality. To determine a_n in equation (u) we apply orthogonality. Note that the characteristic functions $\phi_n(x) = \sin\lambda_n x$ in (u) are solutions to equation (f). Comparing (f) with eq. (3.5a) shows that it is a Sturm-Liouville equation with

PROBLEM 3.1 (continued)

$$a_1(x) = a_2(x) = 0 \text{ and } a_3(x) = 1$$

Thus eq. (3.6) gives

$$p(x) = w(x) = 1 \text{ and } q(x) = 0$$

Since the boundary conditions at x = 0 and x = L are homogeneous, it follows that the characteristic functions $\phi_n(x) = \sin\lambda_n x$ are orthogonal with respect to the weighting function $w(x) = 1$. Multiplying both sides of (u) by $w(x)\sin\lambda_m x\,dx$ and integrating from x = 0 to $x = L$

$$\int_0^L f(x)w(x)\sin\lambda_m x\,dx = -k\int_0^L \left\{\sum_{n=1}^{\infty}(a_n \cosh\lambda_n H)\sin\lambda_n x\right\} w(x)(\sin\lambda_m x)\,dx$$

Using the definition of $f(x)$ in condition (4) and noting that $w(x) = 1$, the above gives

$$h_1\int_0^{L/2}\left[\theta(x,H) - (T_\infty - T_o)\right]\sin\lambda_m x\,dx + h_2\int_{L/2}^{L}\left[\theta(x,H) - (T_\infty - T_o)\right]\sin\lambda_m x\,dx = -k\int_0^L\left\{\sum_{n=1}^{\infty}(a_n\lambda_n \cosh\lambda_n H)\sin\lambda_n x\right\}\sin\lambda_m x\,dx \quad \text{(v)}$$

Evaluating (t) at $y = H$ and substituting into (v)

$$h_1\int_0^{L/2}\left\{\sum_{n=1}^{\infty}(a_n \sinh\lambda_n H)\sin\lambda_n x\right\}\sin\lambda_m x\,dx + h_2\int_{L/2}^{L}\left\{\sum_{n=1}^{\infty}(a_n \sinh\lambda_n H)\sin\lambda_n x\right\}\sin\lambda_m x\,dx -$$
$$h_1(T_\infty - T_o)\int_0^{L/2}\sin\lambda_m x\,dx - h_2(T_\infty - T_o)\int_{L/2}^{L}\sin\lambda_m x\,dx = -k\int_0^L\left\{\sum_{n=1}^{\infty}(a_n\lambda_n \cosh\lambda_n H)\sin\lambda_n x\right\}\sin\lambda_m x\,dx$$

Interchanging the integration and summation signs and applying orthogonality, eq. (3.7),

$$h_1 a_n \sinh\lambda_n H\int_0^{L/2}\sin^2\lambda_n x\,dx + h_2 a_n \sinh\lambda_n H\int_{L/2}^{L}\sin^2\lambda_n x\,dx -$$
$$h_1(T_\infty - T_o)\int_0^{L/2}\sin\lambda_n x\,dx - h_2(T_\infty - T_o)\int_{L/2}^{L}\sin\lambda_n x\,dx = -k a_n\lambda_n \cosh\lambda_n H\int_0^L \sin^2\lambda_n x\,dx$$

Evaluating the integrals and solving for a_n we obtain

$$a_n = \frac{2(T_\infty - T_o)\left\{(h_1 L/k)\left[3 - (-1)^{n+1}\right] - (h_2 L/k)\left[1 - (-1)^{n+1} + 2(-1)^n\right]\right\}}{\lambda_n L\left\{(h_1 + h_2)(L/k)\sinh\lambda_n H + 2(\lambda_n L)\cosh\lambda_n H\right\}} \quad \text{(w)}$$

(5) Checking. *Dimensional check*: (i) The arguments of sin, sinh and cosh are dimensionless. (ii) The coefficient a_n in (t) must have units of temperature. Equation (w) shows that a_n has units of temperature.

Limiting check: (i) If $h_1 = h_2 = 0$, no heat can be exchanged at the (*x*,*H*) boundary and consequently the entire plate should be at a uniform temperature T_o. Setting $h_1 = h_2 = 0$ in equation (w) gives $a_n = 0$. When this is substituted into (t) we obtain

PROBLEM 3.1 (continued)

$$\theta(x,y) = T(x,y) - T_o = 0$$

which gives $T(x,y) = T_o$.

(ii) If $T_\infty = T_o$**,** no heat can be exchanged at the (x,H) boundary and thus the entire plate should be at a uniform temperature T_o. Setting $T_\infty = T_o$ in equation (w) gives $a_n = 0$. When this is substituted into (t) we obtain $T(x,y) = T_o$.

Boundary conditions check: Direct substitution into solution (t) shows that boundary conditions (1), (2) and (3) are satisfied.

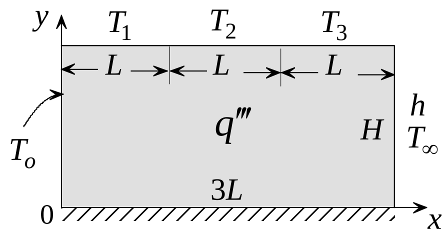
Special case: For the special case of $h_1 = h_2 = h$ equation (w) gives

$$a_n = \frac{2(Lh/h)(T_\infty - T_o)\left[1-(-1)^n\right]}{(\lambda_n L)\left[(hL/k)\sinh\lambda_n L + (\lambda_n L)\cosh\lambda_n L\right]}$$

Formal determination of a_n for this case gives the same result.

(6) Comments. (i) A non-uniform non-homogeneous boundary condition along a boundary does not introduce added complications. (ii) The solution is expressed in term of hL/k which appears in a_n. This dimensionless parameter is the Biot number. Equation (s) gives the dimensionless roots $\lambda_n L$. However, the argument $\lambda_k H$ is determined by multiplying and dividing by L according to $\lambda_k H = (\lambda_k L)(H/L)$. This introduces an additional dimensionless parameters H/L.

PROBLEM 3.2



Heat is generated in a rectangular plate $3L \times H$ at a uniform volumetric rate of q'''. The surface at (0,y) is maintained at T_o. Along surface (3L,y) the plate exchanges heat by convection. The heat transfer coefficient is h and the ambient temperature is T_∞. Surface (x,H) is divided into three equal segments which are maintained at uniform temperatures T_1, T_2, and T_3, respectively. Surface (x,0) is insulated. Determine the two-dimensional steady state temperature distribution.

(1) Observations. (i) This is a steady state two-dimensional conduction problem. (ii) Four boundary conditions are needed. (iii) Only one boundary condition is homogeneous. (iv) The energy generation term makes the differential equation non-homogeneous.

(2) Origin and Coordinates. The origin and coordinate axes are selected as shown.

(3) Formulation.

(i) Assumptions. (1) Two-dimensional, (2) steady, (3) uniform energy generation and (4) constant thermal conductivity.

(ii) Governing Equations. Introducing the above assumptions into eq. (1.8) gives the heat equation for this problem

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{q'''}{k} = 0 \qquad \text{(a)}$$

(iii) Independent Variable with Two Homogeneous Boundary Conditions. Neither variable has two homogeneous conditions. Only the *x*-variable can have two homogeneous conditions.

(iv) Boundary Conditions.

(1) $T(0,y) = T_o$, non-homogeneous

(2) $-k\dfrac{\partial T(3L,y)}{\partial x} = h\left[T(3L,y) - T_\infty\right]$, non-homogeneous

(3) $\dfrac{\partial T(x,0)}{\partial y} = 0$, homogeneous

(4) $T(x,H) = f(x) = \begin{cases} T_1, & 0 < x < L \\ T_2, & L < x < 2L, \\ T_3, & 2L < x < 3L \end{cases}$ non-homogenous

This condition represents a boundary with non-uniform temperature given by $f(x)$.

(4) Solution. Equation (a) is non-homogeneous due to the heat generation term. Thus we can not proceed directly with the application of the method of separation of variables. Instead, we assume a solution of the form

$$T(x,y)=\psi(x,y)+\phi(x) \tag{b}$$

Note that $\psi(x,y)$ depends on two variables while $\phi(x)$ depends on a single variable. Substituting (b) into (a)

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{d^2\phi}{dx^2}+\frac{q'''}{k}=0 \tag{c}$$

The next step is to split (c) into two equations, one for $\psi(x,y)$ and the other for $\phi(x)$. We select

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0 \tag{d}$$

Thus, the second part must be

$$\frac{d^2\phi}{dx^2}+\frac{q'''}{k}=0 \tag{e}$$

Boundary conditions on $\psi(x,y)$ and $\phi(x)$ are obtained by substituting (b) into the four boundary conditions on T. In this process the function $\psi(x,y)$ is assigned homogeneous conditions whenever possible. Using (b) and boundary condition (1)

$$\psi(0,y)+\phi(0)=T_o$$

Let

$$\psi(0,y)=0 \tag{d-1}$$

Thus

$$\phi(0)=T_o \tag{e-1}$$

Boundary condition (2) and (b) give

$$-k\frac{\partial\psi(3L,y)}{\partial x}-k\frac{d\phi(3L)}{dx}=h\psi(3L,y)+h\left[\phi(3L)-T_\infty\right]$$

Let

$$-k\frac{\partial\psi(3L,y)}{\partial x}=h\psi(3L,y) \tag{d-2}$$

Thus

$$-k\frac{d\phi(3L)}{dx}=h\left[\phi(3L)-T_\infty\right] \tag{e-2}$$

Boundary condition (3) and (b) give

$$\frac{\partial\psi(x,0)}{\partial y}=0 \tag{d-3}$$

Finally, boundary condition (4) gives

$$\psi(x,H)+\phi(x)=f(x)$$

Or

$$\psi(x,H)=f(x)-\phi(x)\;=F(x)$$

Using the definition of $f(x)$ in boundary condition (4) the above gives

$$\psi(x,H) = f(x) - \phi(x) = F(x) = \begin{cases} T_1 - \phi(x), & 0 < x < L \\ T_2 - \phi(x), & L < x < 2L \\ T_3 - \phi(x), & 2L < x < 3L \end{cases}, \text{ non-homogeneous} \tag{d-4}$$

Thus non-homogeneous equation (a) is replaced by two equations: (d), which is a homogeneous partial differential equation and (e) which is a non-homogeneous ordinary differential equation. Equation (d) has three homogeneous conditions, (d-1), (d-2) and (d-3) and one non-homogeneous condition, (d-4). The solution to (e) is

$$\phi(x) = -\frac{q'''}{2k}x^2 + E\,x + F \tag{f}$$

where E and F are constants of integration. Application of boundary conditions (e-1) and (e-2) gives the two constants

$$E = \frac{(3q'''L/k)\left[1 + (3hL/2k)\right] - (h/k)(T_o - T_\infty)}{1 + (3hL/k)}, \qquad F = T_o \tag{g}$$

(i) Assumed Product Solution. To solve partial differential equation (d) we assume a product solution. Let

$$\psi(x,y) = X(x)Y(y) \tag{h}$$

Substituting (h) into (d), separating variables and setting the resulting two equations equal to a constant, $\pm\lambda_n^2$, we obtain

$$\frac{d^2X_n}{dx^2} \pm \lambda_n^2 X_n = 0 \tag{i}$$

and

$$\frac{d^2Y_n}{dy^2} \mp \lambda_n^2 Y_n = 0 \tag{j}$$

(ii) Selecting the Sign of the λ_n^2 Terms. Since the *x*-variable has two homogeneous boundary conditions, the plus sign is selected in (i). Thus (i) and (j) become

$$\frac{d^2X_n}{dx^2} + \lambda_n^2 X_n = 0 \tag{k}$$

and

$$\frac{d^2Y_n}{dy^2} - \lambda_n^2 Y_n = 0 \tag{l}$$

For the special case of $\lambda_n = 0$ the above equations take the form

$$\frac{d^2X_0}{dx^2} = 0 \tag{m}$$

and

PROBLEM 3.2 (continued)

$$\frac{d^2 Y_0}{d y^2} = 0 \tag{n}$$

(iii) Solutions to the Ordinary Differential Equations. The solutions to equations (k)-(n) are

$$X_n(x) = A_n \sin\lambda_n x + B_n \cos\lambda_n x \tag{o}$$

$$Y_n(y) = C_n \sinh\lambda_n y + D_n \cosh\lambda_n y \tag{p}$$

$$X_0(x) = A_0 x + B_0 \tag{q}$$

and

$$Y_0(y) = C_0 y + D_0 \tag{r}$$

Corresponding to each value of λ_n there is a solution $\psi_n(x, y)$. Thus

$$\psi_n(x, y) = X_n(x) Y_n(y) \tag{s}$$

and

$$\psi_0(x, y) = X_0(x) Y_0(y) \tag{t}$$

The complete solution to $T(x, y)$ becomes

$$T(x, y) = \phi(x) + X_0(x) Y_0(y) + \sum_{n=1}^{\infty} X_n(x) Y_n(y) \tag{u}$$

(iv) Application of Boundary Conditions. Applying boundary condition (d-1) to equation (s) and (t) gives

$$B_n = B_0 = 0 \tag{v}$$

Condition (d-2) applied to (s) gives the characteristic equation for λ_n

$$-3\lambda_n L = (3hL/k)\tan 3\lambda_n L \tag{w}$$

Condition (d-2) applied to (t)

$$A_0 = 0 \tag{x}$$

Boundary conditions (d-3) and (s) give

$$C_n = 0 \tag{y}$$

With $A_0 = B_0 = 0$, the solution $X_0 Y_0$ vanishes. Substituting into (s) and using (v) and (y), the solution to $\psi(x, y)$ becomes

$$\psi(x, y) = \sum_{n=1}^{\infty} a_n (\cosh\lambda_n y) \sin\lambda_n x \tag{z}$$

where $a_n = A_n D_n$. Thus the only remaining unknown is a_n. Application of condition (d-4) gives

$$F(x) = \sum_{n=1}^{\infty} a_n (\cosh\lambda_n H) \sin\lambda_n x \tag{z-1}$$

where $F(x)$ is defined in (d-4). To determine a_n from this result we apply orthogonality.

(v) Orthogonality. The characteristic functions, $\sin\lambda_n x$, in (z-1) are solutions to equation (k). Comparing (k) with eq. (3.5a) shows that it is a Sturm-Liouville equation with

$$a_1(x) = a_2(x) = 0 \text{ and } a_3(x) = 1$$

Thus eq. (3.6) gives

$$p(x) = w(x) = 1 \text{ and } q(x) = 0$$

Since boundary conditions (d-1) and (d-2) at $x = 0$ and $x = 3L$ are homogeneous, it follows that the characteristic functions $\sin\lambda_n x$ are orthogonal with respect to $w = 1$. Multiplying both sides of (aa) by $\sin\lambda_m x\, dx$, integrating from $x = 0$ to $x = 3L$

$$\int_0^{3L} F(x)w(x)\sin\lambda_m x\, dx = \int_0^{3L}\left\{\sum_{n=1}^{\infty}(a_n \cosh\lambda_n L)\sin\lambda_n x\right\}w(x)(\sin\lambda_m x)\, dx$$

Interchanging the integration and summation signs, noting that $w(x) = 1$ and applying orthogonality, the above gives

$$\int_0^{3L} F(x)\sin\lambda_n x\, dx = a_n \cosh\lambda_n H \int_0^{3L}\sin^2\lambda_n x\, dx$$

Evaluating the integral on the right-hand-side and solving the above for a_n

$$a_n = \frac{\int_0^{3L} F(x)\sin\lambda_n x dx}{(1/2\lambda_n)\cosh\lambda_n H\left[3\lambda_n L - (1/2)\sin 6\lambda_n L\right]} \tag{z-2}$$

The integral in (z-2) is evaluated using the definition of $F(x)$ in (d-4) and the solution to $\phi(x)$ given in (f). Thus

$$\begin{aligned}\int_0^{3L} F(x)\sin\lambda_n x dx = &\int_0^{L}\left[T_1 + (q'''/2k)x^2 - Ex - F\right]\sin\lambda_n x dx + \\ &\int_L^{2L}\left[T_2 + (q'''/2k)x^2 - Ex - F\right]\sin\lambda_n x dx + \\ &\int_{2L}^{3L}\left[T_3 + (q'''/2k)x^2 - Ex - F\right]\sin\lambda_n x dx +\end{aligned}$$

where E and F are given in equation (g). Evaluating each of the three integrals in the above

$$\begin{aligned}\int_0^{L}\left[T_1 + (q'''/2k)x^2 - Ex - F\right]\sin\lambda_n x dx = (F - T_1)(1/\lambda_n)(\cos\lambda_n L - 1) + \\ (q'''/2k\lambda_n^3)\left[2\lambda_n L\sin\lambda_n L - (\lambda_n^2 L^2 - 2)(\cos\lambda_n L) - 2\right] - (E/\lambda_n^2)\left[\sin\lambda_n L - \lambda_n L\cos\lambda_n L\right]\end{aligned} \tag{z-3}$$

$$\int_L^{2L}\left[T_2+(q'''/2k)x^2-Ex-F\right]\sin\lambda_n x dx=(F-T_2)(1/\lambda_n)(\cos 2\lambda_n L-\cos\lambda_n L)+$$
$$(q'''/2k\lambda_n^3)\left[4\lambda_n L\sin 2\lambda_n L-(4\lambda_n^2L^2-2)(\cos 2\lambda_n L)-2\lambda_n L\sin\lambda_n L+(\lambda_n^2L^2-2)(\cos\lambda_n L)\right]-$$
$$(E/\lambda_n^2)\left[\sin 2\lambda_n L-2\lambda_n L\cos 2\lambda_n L-\sin\lambda_n L+\lambda_n L\cos\lambda_n L\right] \qquad \text{(z-4)}$$

and

$$\int_{2L}^{3L}\left[T_3+(q'''/2k)x^2-Ex-F\right]\sin\lambda_n x dx=(F-T_3)(1/\lambda_n)(\cos 3\lambda_n L-\cos 2\lambda_n L)+$$
$$(q'''/2k\lambda_n^3)\left[6\lambda_n L\sin 3\lambda_n L-(9\lambda_n^2L^2-2)(\cos 3\lambda_n L)-4\lambda_n L\sin 2\lambda_n L+(4\lambda_n^2L^2-2)(\cos 2\lambda_n L)\right]-$$
$$(E/\lambda_n^2)\left[\sin 3\lambda_n L-3\lambda_n L\cos 3\lambda_n L-\sin 2\lambda_n L+2\lambda_n L\cos 2\lambda_n L\right] \qquad \text{(z-5)}$$

(5) Checking. *Dimensional checks*: (i) The coefficient a_n in (z-2) should have units of temperature. Since $F(x)$ has units of temperature and λ_n is measured in (1/m), (z-2) has units of temperature. (ii) Each term in (z-3), (z-4) and (z-5) has units of $(^\circ\text{C}-\text{m})$.

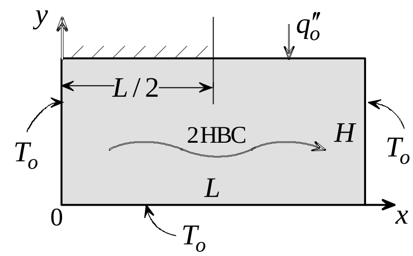
Limiting check: For the special case of $q'''=0$ and $T_1=T_2=T_3=T_\infty=T_o$, physical consideration requires that the plate be at a uniform temperature T_o. For this case E = 0 and $\phi(x)=T_o$. Substitution into (z-2) gives $a_n=0$. Equation (u) gives $T(x,y)=T_o$.

Boundary conditions check: Boundary conditions (1) and (3) are readily shown to be satisfied.

(6) Comments. (i) The same approach can be used to solve the more general problem where the heat generation rate is variable, $q'''=q'''(x)$. This will affect the solution of $\phi(x)$ and the integral in (z-2). (ii) The solution is expressed in terms of the Biot number, hL/k, which appears in equation (w).

PROBLEM 3.3

A rectangular plate $L \times H$ is maintained at uniform temperature T_o along three sides. Half the fourth side is insulated while the other half is heated at uniform flux q''_o. Determine the steady state heat transfer rate through surface (0,y).



(1) Observations. (i) To determine the heat transfer through surface (0,y) it is necessary to first determine the temperature distribution in the plate. (ii) This is a steady state two-dimensional conduction problem. (iii) Four boundary conditions are needed. (iv) Three conditions can be made homogeneous by defining a new temperature variable $\theta(x,y) = T(x,y) - T_o$. (v) The heat flux along boundary (x,H) is non-uniform. This requires special consideration.

(2) Origin and Coordinates. The origin and coordinate axes are selected as shown.

(3) Formulation.

(i) Assumptions. (1) Two-dimensional, (2) steady, (3) no energy generation and (4) constant thermal conductivity.

(ii) Governing Equations. Introducing the above assumptions into eq. (1.8) gives the heat equation for this case as

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \tag{a}$$

where

$$\theta(x,y) = T(x,y) - T_o \tag{b}$$

Once the temperature distribution is determined, Fourier's law give the heat transfer rate through surface (0,y). Thus

$$q(0) = -k\int_0^H \frac{\partial T(0,y)}{\partial x} dy \tag{c}$$

where $q(0)$ is the heat transfer rate through surface (0,y)

(iii) Independent Variable with Two Homogeneous Boundary Conditions. The two boundaries at x = 0 and x = L have homogeneous conditions. Thus, the x-variable has two homogeneous boundary conditions.

(iv) Boundary Conditions. The required four boundary conditions are

(1) $\theta(0,y) = 0$, homogeneous
(2) $\theta(L,y) = 0$, homogeneous
(3) $\theta(x,0) = 0$, homogeneous

The fourth boundary condition is first written in terms of the variable T and then transformed in terms of the variable θ. Using Fourier's law

$$-k\frac{\partial T(x,H)}{\partial y} = \begin{cases} 0, & 0 < x < L/2 \\ -q''_o, & L/2 < x < L \end{cases}$$

Expressed in terms of θ, the above gives the fourth boundary condition

$$(4)\ \ -k\frac{\partial \theta(x,H)}{\partial y} = \ f(x) \ = \begin{cases} 0, & 0 < x < L/2 \\ -q''_o, & L/2 < x < L \end{cases} \quad \text{non-homogeneous}$$

This condition represents a heat flux which varies along the boundary according to $f(x)$.

(4) Solution.

(i) Assumed Product Solution. Assume a solution in the form

$$\theta(x,y) \ = \ X(x)\ Y(y) \tag{d}$$

Substituting into (a), separating variables and setting the resulting equation equal to constant

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \pm\lambda_n^2$$

Assuming that λ_n is multi-valued, the above gives

$$\frac{d^2X_n}{dx^2} \mp \lambda_n^2 X_n = 0 \tag{e}$$

and

$$\frac{d^2Y_n}{dy^2} \pm \lambda_n^2 Y_n = 0 \tag{f}$$

(ii) Selecting the Sign of the λ_n^2 Terms. Since the x-variable has two homogeneous boundary conditions the $\lambda_n^2 X_n$ term in (e) takes the positive sign. Therefore, (e) and (f) become

$$\frac{d^2X_n}{dx^2} + \lambda_n^2 X_n = 0 \tag{g}$$

and

$$\frac{d^2Y_n}{dy^2} - \lambda_n^2 Y_n = 0 \tag{h}$$

For the important case of $\lambda_n = 0,$ equations (g) and (h) become

$$\frac{d^2X_0}{dx^2} = 0 \tag{i}$$

and

$$\frac{d^2Y_0}{dy^2} = 0 \tag{j}$$

PROBLEM 3.3 (continued)

(iii) Solutions to the Ordinary Differential Equations. The solutions to eqs. (g)-(j) are

$$X_n(x) = A_n \sin\lambda_n x + B_n \cos\lambda_n x \tag{k}$$

$$Y_n(y) = C_n \sinh\lambda_n y + D_n \cosh\lambda_n y \tag{l}$$

$$X_0(x) = A_0 x + B_0 \tag{m}$$

$$Y_0(y) = C_0 y + D_0 \tag{n}$$

Corresponding to each value of λ_n there is a temperature solution $\theta_n(x, y)$. Thus

$$\theta_n(x, y) = X_n(x)Y_n(y) \tag{o}$$

and

$$\theta_0(x, y) = X_0(x)Y_0(y) \tag{p}$$

The complete solution becomes

$$\theta(x, y) = X_0(x)Y_0(y) + \sum_{n=1}^{\infty} X_n(x)Y_n(y) \tag{q}$$

(iv) Application of Boundary Conditions. Boundary condition (1) applied to solutions (k) and (m) gives

$$B_n = B_o = 0 \tag{r}$$

Boundary condition (2) applied to (k) gives the characteristic equation for λ_n

$$\sin\lambda_n L = 0 \tag{s}$$

Or

$$\lambda_n = \frac{n\pi}{L} \qquad n = 1,2,3\ldots. \tag{t}$$

Similarly, boundary condition (2) applied to (m) gives

$$A_0 = 0$$

With $A_0 = B_0 = 0$, the solution corresponding to $\lambda_n = 0$ vanishes. Application of boundary condition (3) to (l) gives

$$D_n = 0$$

Solution (q) becomes

$$\theta(x, y) = \sum_{n=1}^{\infty} a_n (\sin\lambda_n x)(\sinh\lambda_n y) \tag{u}$$

where $a_n = A_n C_n$. The only remaining unknown is the set of constants a_n. Application of condition (4) gives

$$f(x) = -k\sum_{n=1}^{\infty} (a_n\lambda_n \cosh\lambda_n H)\sin\lambda_n x \tag{v}$$

where $f(x)$ is defined in boundary condition (4).

(v) Orthogonality. To determine a_n in equation (v) we apply orthogonality. Note that the characteristic functions $\phi_n(x) = \sin\lambda_n x$ in (v) are solutions to equation (g). Comparing (g) with eq. (3.5a) shows that it is a Sturm-Liouville equation with

$$a_1(x) = a_2(x) = 0 \text{ and } a_3(x) = 1$$

Thus eq. (3.6) gives

$$p(x) = w(x) = 1 \text{ and } q(x) = 0$$

Since the boundary conditions at $x = 0$ and $x = L$ are homogeneous, it follows that the characteristic functions $\phi_n(x) = \sin\lambda_n x$ are orthogonal with respect to the weighting function $w(x) = 1$. Multiplying both sides of (v) by $w(x)\sin\lambda_m x\,dx$ and integrating from $x = 0$ to $x = L$

$$\int_0^L f(x)w(x)\sin\lambda_m x\,dx = -k\int_0^L \left\{\sum_{n=1}^{\infty}(a_n\lambda_n\cosh\lambda_n H)\sin\lambda_n x\right\}w(x)\sin\lambda_m x\,dx$$

Interchanging the summation with integration and invoking orthogonality gives

$$\int_0^L f(x)w(x)\sin\lambda_n x\,dx = -k\,a_n\lambda_n\cosh\lambda_n H\int_0^L w(x)\sin^2\lambda_n x$$

Solving for a_n and recalling that $w(x) = 1$ the above gives

$$a_n = \frac{\int_0^L f(x)\sin\lambda_n x\,dx}{-k\lambda_n\cosh\lambda_n H\int_0^L \sin^2\lambda_n x\,dx} = \frac{-2}{k\lambda_n L\cosh\lambda_n H}\int_0^L f(x)\sin\lambda_n x\,dx \tag{w}$$

Using the definition of $f(x)$ in condition (4) the integral in (w) is evaluated

$$\int_0^L f(x)\sin\lambda_n x\,dx = \int_0^{L/2}(0)\sin\lambda_n x\,dx - \int_{L/2}^L q_o''\sin\lambda_n x\,dx = (q''/\lambda_n)(\cos\lambda_n L - \cos\lambda_n L/2)$$

Substituting into (w) and using (t) to eliminate $\lambda_n L$ and $(\cos\lambda_n L - \cos\lambda_n L/2)$, we obtain

$$a_n = \frac{q_o''L/k}{(n\pi)^2(\cosh\lambda_n H)}\left[(-1)^{n+1} - 2(-1)^n - 1\right] \tag{x}$$

Substituting (x) into (u) and rearranging the result gives the dimensionless temperature distribution in the plate

$$\frac{T(x,y) - T_o}{q_o''L/k} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1} - 2(-1)^n - 1}{(n\pi)^2(\cosh\lambda_n H)}(\sin\lambda_n x)\sinh\lambda_n y \tag{y}$$

To determine the heat transfer rate through surface (0,*y*) we substitute (y) into (c)

PROBLEM 3.3 (continued)

$$q(0) = -k(q_o''L/k)\int_0^H \sum_{n=1}^{\infty} \frac{(-1)^{n+1} - 2(-1)^n - 1}{(n\pi)^2\,(\cosh\lambda_n H)} \lambda_n (\cos\lambda_n 0)\sinh\lambda_n y dy =$$

$$-q_o''L\int_0^H \sum_{n=1}^{\infty} \frac{(-1)^{n+1} - 2(-1)^n - 1}{(n\pi)^2\,(\cosh\lambda_n H)} \int_0^H \sinh\lambda_n y d(\lambda_n y)$$

Evaluating the integral

$$q(0) = -q_o''L\int_0^H \sum_{n=1}^{\infty} \frac{(-1)^{n+1} - 2(-1)^n - 1}{(n\pi)^2\,(\cosh\lambda_n H)} \left[\cosh\lambda_n H - 1\right] \tag{z}$$

(5) Checking. *Dimensional check*: (i) The arguments of the sin, sinh and cosh are dimensionless. (ii) Each term in solution (y) is dimensionless.(iii) Equation (z) has the correct units of heat transfer rate per unit length.

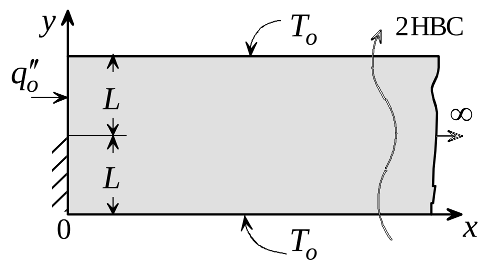
Limiting check: If $q_o'' = 0$ the entire plate should be at a uniform temperature T_o. Setting $q_o'' = 0$ in (y) gives $T(x, y) = T_o$.

Boundary conditions check: Conditions (1), (2) and (3) can be shown to be satisfied by direct substitution.

(6) Comments. (i) The same approach can be used to solve the more general problem where the heat flux at the boundary is variable, $q_o'' = q_o''(x)$. This will affect the integral in (w). (ii) Equation (t) gives the dimensionless roots $\lambda_n L$. However, the argument $\lambda_k H$ is determined by multiplying and dividing by L according to $\lambda_k H = (\lambda_k L)(H/L)$. Thus the solution is characterized by a single dimensionless parameters H/L.

PROBLEM 3.4

A semi-infinite plate of width 2L is maintained at uniform temperature T_o along its two semi-infinite sides. Half the surface (0,y) is insulated while the remaining half is heated at a flux q_o''. Determine the two-dimensional steady state temperature distribution.



(1) Observations. (i) This is a steady state two-dimensional conduction problem. (ii) Four boundary conditions are needed. (iii) Far away from the heated end the temperature should be uniform equal to T_o. (iv) Three conditions can be made homogeneous by defining a new temperature variable $\theta(x,y) = T(x,y) - T_o$. (v) The heat flux along boundary (0,*y*) is non-uniform. This requires special consideration.

(2) Origin and Coordinates. The origin and coordinate axes are selected as shown.

(3) Formulation.

(i) Assumptions. (1) Two-dimensional, (2) steady, (3) no energy generation and (4) constant thermal conductivity.

(ii) Governing Equations. Introducing the above assumptions into eq. (1.8) gives the heat equation for this as

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \tag{a}$$

where

$$\theta(x,y) = T(x,y) - T_o \tag{b}$$

(iii) Independent Variable with Two Homogeneous Boundary Conditions. The two boundaries at $y = 0$ and $y = 2L$ have homogeneous conditions. Thus, the *y*-variable has two homogeneous boundary conditions.

(iv) Boundary Conditions. The required four boundary conditions are

(1) $\theta(x,0) = 0$, homogeneous

(2) $\theta(x,2L) = 0$, homogeneous

(3) $\theta(\infty, y) = 0 =$ finite, homogeneous

(4) $-k\dfrac{\partial\theta(0,y)}{\partial x} = f(y) = \begin{cases} 0, & 0 < x < L \\ q_o'', & L < x < 2L \end{cases}$ non-homogeneous

This condition represents a heat flux which varies along the boundary according to $f(y)$.

(4) Solution.

(i) Assumed Product Solution. Assume a solution in the form

$$\theta(x,y) = X(x)\,Y(y) \tag{c}$$

Substituting into (a), separating variables and setting the resulting equation equal to constant

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \pm\lambda_n^2$$

Assuming that λ_n is multi-valued, the above gives

$$\frac{d^2X_n}{dx^2} \mp \lambda_n^2 X_n = 0 \tag{d}$$

and

$$\frac{d^2Y_n}{dy^2} \pm \lambda_n^2 Y_n = 0 \tag{e}$$

(ii) Selecting the Sign of the λ_n^2 Terms. Since the *y*-variable has two homogeneous boundary conditions the $\lambda_n^2 Y_n$ term in (e) takes the positive sign. Therefore, (d) and (e) become

$$\frac{d^2X_n}{dx^2} - \lambda_n^2 X_n = 0 \tag{f}$$

and

$$\frac{d^2Y_n}{dy^2} + \lambda_n^2 Y_n = 0 \tag{g}$$

For the important case of $\lambda_n = 0,$ equations (f) and (g) become

$$\frac{d^2X_0}{dx^2} = 0 \tag{h}$$

and

$$\frac{d^2Y_0}{dy^2} = 0 \tag{i}$$

(iii) Solutions to the Ordinary Differential Equations. The solutions to eqs. (f)-(i) are

$$X_n(x) = A_n \exp(-\lambda_n x) + B_n \exp(\lambda_n x) \tag{j}$$

$$Y_n(y) = C_n \sin\lambda_n y + D_n \cos\lambda_n y \tag{k}$$

$$X_0(x) = A_0 x + B_0 \tag{l}$$

$$Y_0(y) = C_0 y + D_0 \tag{m}$$

Corresponding to each value of λ_n there is a temperature solution $\theta_n(x, y).$ Thus

$$\theta_n(x, y) = X_n(x)Y_n(y) \tag{n}$$

and

$$\theta_0(x, y) = X_0(x)Y_0(y) \tag{o}$$

The complete solution becomes

$$\theta(x,y) = X_0(x)Y_0(y) + \sum_{n=1}^{\infty} X_n(x)Y_n(y) \tag{p}$$

(iv) Application of Boundary Conditions. Boundary condition (1) applied to solutions (k) and (m) gives

$$D_n = D_o = 0 \tag{q}$$

Boundary condition (2) applied to (j) gives the characteristic equation for λ_n

$$\sin 2\lambda_n L = 0 \tag{r}$$

Thus λ_n

$$\lambda_n = \frac{n\pi}{2L} \qquad (n = 1,2,3\ldots.) \tag{s}$$

Similarly, boundary condition (2) applied to (m) gives

$$C_0 = 0$$

With $C_0 = D_0 = 0,$ the solution corresponding to $\lambda_n = 0$ vanishes. Application of boundary condition (3) to (j) gives

$$B_n = 0$$

Solution (p) becomes

$$\theta(x,y) = \sum_{n=1}^{\infty} a_n \left[\exp(-\lambda_n x)\right] \sin\lambda_n y \tag{t}$$

where $a_n = A_n C_n$. The only remaining unknown is the set of constants a_n. Application of condition (4) gives

$$f(y) = k\sum_{n=1}^{\infty} a_n \lambda_n \sin\lambda_n y \tag{u}$$

where $f(y)$ is defined in boundary condition (4).

(v) Orthogonality. To determine a_n in equation (u) we apply orthogonality. Note that the characteristic functions $\phi_n(y) = \sin\lambda_n y$ in (u) are solutions to equation (g). Comparing (g) with eq. (3.5a) shows that it is a Sturm-Liouville equation with

$$a_1(y) = a_2(y) = 0 \text{ and } a_3(y) = 1$$

Thus eq. (3.6) gives

$$p(y) = w(y) = 1 \text{ and } q(y) = 0$$

Since the boundary conditions at y = 0 and y = 2L are homogeneous, it follows that the characteristic functions $\phi_n(y) = \sin\lambda_n y$ are orthogonal with respect to the weighting function $w(y) = 1$. Multiplying both sides of (u) by $w(y)\sin\lambda_m y\,dy$ and integrating from y = 0 to $y = L$

$$\int_0^{2L} f(y)w(y)\sin\lambda_m y\,dy = k\int_0^{2L}\left\{\sum_{n=1}^{\infty} a_n\lambda_n \sin\lambda_n y\right\}w(y)\sin\lambda_m y\,dy$$

Interchanging the summation with integration and invoking orthogonality gives

$$\int_0^{2L} f(y)w(y)\sin\lambda_n y\,dy = k\,a_n\lambda_n\int_0^{2L} w(y)\sin^2\lambda_n y$$

Solving for a_n and recalling that $w(y) = 1$ the above gives

$$a_n = \frac{\int_0^{2L} f(y)\sin\lambda_n y\,dy}{k\lambda_n\int_0^{2L}\sin^2\lambda_n y\,dy} = \frac{1}{k\lambda_n L}\int_0^{2L} f(y)\sin\lambda_n y\,dy$$

Evaluating the integral in the denominator, the above gives

$$a_n = \frac{1}{k\lambda_n L}\int_0^{2L} f(y)\sin\lambda_n y\,dy \qquad \text{(v)}$$

Using the definition of $f(y)$ in condition (4) the integral in (v) is evaluated

$$\int_0^{2L} f(y)\sin\lambda_n y\,dy = \int_0^{L}(0)\sin\lambda_n y\,dy + \int_L^{2L} q_o''\sin\lambda_n y\,dy = -(q''/\lambda_n)(\cos 2\lambda_n L - \cos\lambda_n L)$$

Substituting into (v) and using (s) to eliminate $\lambda_n L$ and $(\cos\lambda_n L - \cos\lambda_n L/2)$, we obtain

$$a_n = \frac{2q_o''L/k}{(n\pi)^2}\left[(-1)^{n+1} - 2(-1)^n - 1\right] \qquad \text{(w)}$$

Substituting (w) into (t) and rearranging the result gives

$$\frac{T(x,y)-T_o}{q_o''L/k} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}-2(-1)^n-1}{(n\pi)^2}\,e^{-\lambda_n x}\sin\lambda_n y$$

(5) Checking. *Dimensional check*: According to (t) the coefficient a_n should have units of temperature. The term $q_o''L/k$ in (w) has units of temperature.

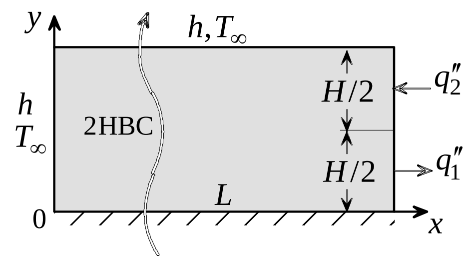
Limiting check: If $q_o'' = 0$, the entire plate should be at a uniform temperature T_o. Setting $q_o'' = 0$ in (w) gives $a_n = 0$. When this is substituted into (t) gives $\theta = 0$, or $T(x,y) = T_o$.

Boundary conditions check: Conditions (1), (2) and (3) can be shown to be satisfied by direct substitution.

(6) Comments. The same approach can be used to solve the more general problem where the heat flux at the boundary is variable, $q_o'' = q_o''(y)$. This will affect the integral in (v).

PROBLEM 3.5

Consider two-dimensional steady state conduction in a rectangular plate $L \times H$. The sides (0,y) and (x,H) exchange heat by convection. The heat transfer coefficient is h and ambient temperature is T_∞. Side (x,0) is insulated. Half the surface at x = L is cooled at a flux q_1'' while the other half is heated at a flux q_2''. Determine the heat transfer along (0,y).



(1) Observations. (i) This is a steady state two-dimensional conduction problem. (ii) Heat transfer rate along surface (0,*y*) is determined using temperature distribution and Fourier's law. (iii) Four boundary conditions are needed. (iv) Two conditions can be made homogeneous by defining a new temperature variable $\theta(x,y) = T(x,y) - T_\infty$. (v) The heat flux along boundary (*L*,*y*) is not uniform. This requires special consideration.

(2) Origin and Coordinates. The origin and coordinate axes are selected as shown.

(3) Formulation.

(i) Assumptions. (1) Two-dimensional, (2) steady, (3) no energy generation and (4) constant thermal conductivity.

(ii) Governing Equations. To determine the rate of heat transfer we must first determine the temperature distribution in the plate. Introducing the above assumptions into eq. (1.8) gives the heat equation for this case

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \tag{a}$$

where

$$\theta(x,y) = T(x,y) - T_\infty \tag{b}$$

(iii) Independent Variable with Two Homogeneous Boundary Conditions. The two boundaries at $y = 0$ and $y = H$ have homogeneous conditions. Thus, the *y*-variable has two homogeneous boundary conditions.

(iv) Boundary Conditions. The boundary conditions are

(1) $\dfrac{\partial\theta(x,0)}{\partial y} = 0$, homogeneous

(2) $-k\dfrac{\partial\theta(x,H)}{\partial y} = h\theta(x,H)$, homogeneous

(3) $-k\dfrac{\partial\theta(0,y)}{\partial x} = -h\theta(0,y)$, homogeneous

(4) $-k\dfrac{\partial\theta(L,y)}{\partial x} = f(y) = \begin{cases} q_1'', & 0 < y < H/2 \\ -q_2'', & H/2 < y < H \end{cases}$ non-homogeneous

This condition represents a heat flux, which varies along the boundary according to $f(y)$.

PROBLEM 3.5 (continued)

(4) Solution.

(i) Assumed Product Solution. Assume a solution in the form

$$\theta(x, y) = X(x)\, Y(y) \tag{c}$$

Substituting into (a), separating variables and setting the resulting equation equal to constant

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \pm\lambda_n^2$$

Assuming that λ_n is multi-valued, the above gives

$$\frac{d^2X_n}{dx^2} \mp \lambda_n^2 X_n = 0 \tag{d}$$

and

$$\frac{d^2Y_n}{dy^2} \pm \lambda_n^2 Y_n = 0 \tag{e}$$

(ii) Selecting the Sign of the λ_n^2 Terms. Since the *y*-variable has two homogeneous boundary conditions the $\lambda_n^2 Y_n$ term in (e) takes the positive sign. Therefore, (d) and (e) become

$$\frac{d^2X_n}{dx^2} - \lambda_n^2 X_n = 0 \tag{f}$$

and

$$\frac{d^2Y_n}{dy^2} + \lambda_n^2 Y_n = 0 \tag{g}$$

For the important case of $\lambda_n = 0,$ equations (f) and (g) become

$$\frac{d^2X_0}{dx^2} = 0 \tag{h}$$

and

$$\frac{d^2Y_0}{dy^2} = 0 \tag{i}$$

(iii) Solutions to the Ordinary Differential Equations. The solutions to eqs. (f)-(i) are

$$X_n(x) = A_n \sinh\lambda_n x + B_n \cosh\lambda_n x \tag{j}$$

$$Y_n(y) = C_n \sin\lambda_n y + D_n \cos\lambda_n y \tag{k}$$

$$X_0(x) = A_0 x + B_0 \tag{l}$$

$$Y_0(y) = C_0 y + D_0 \tag{m}$$

Corresponding to each value of λ_n there is a temperature solution $\theta_n(x, y)$. Thus

$$\theta_n(x, y) = X_n(x) Y_n(y) \tag{n}$$

and

$$\theta_0(x,y) = X_0(x)Y_0(y) \tag{o}$$

The complete solution becomes

$$\theta(x,y) = X_0(x)Y_0(y) + \sum_{n=1}^{\infty} X_n(x)Y_n(y) \tag{p}$$

(iv) Application of Boundary Conditions. Boundary condition (1) applied to solutions (k) and (m) gives

$$C_n = C_0 = 0 \tag{q}$$

Boundary condition (2) applied to (k) gives the characteristic equation for λ_n

$$\lambda_n H \tan \lambda_n H = (hH/k) \tag{r}$$

Similarly, boundary condition (2) applied to (m) gives

$$D_0 = 0$$

With $C_0 = D_0 = 0$, the solution corresponding to $\lambda_n = 0$ vanishes. Application of boundary condition (3) to (j) gives

$$B_n = (\lambda_n H)(k/hH)A_n$$

Solution (p) becomes

$$\theta(x,y) = \sum_{n=1}^{\infty} a_n \left[\sinh \lambda_n x + (\lambda_n H)(k/hH)\cosh \lambda_n x\right] \cos \lambda_n y \tag{s}$$

where $a_n = A_n D_n$. The only remaining unknown is the set of constants a_n. Application of condition (4) gives

$$f(y) = -k \sum_{n=1}^{\infty} a_n \lambda_n \left[\cosh \lambda_n L + (\lambda_n H)(k/hH)\sinh \lambda_n L\right] \cos \lambda_n y \tag{t}$$

where $f(y)$ is defined in boundary condition (4).

(v) Orthogonality. To determine a_n in equation (t) we apply orthogonality. Note that the characteristic functions $\phi_n(y) = \cos \lambda_n y$ in (t) are solutions to equation (g). Comparing (g) with eq. (3.5a) shows that it is a Sturm-Liouville equation with

$$a_1(y) = a_2(y) = 0 \text{ and } a_3(y) = 1$$

Thus eq. (3.6) gives

$$p(y) = w(y) = 1 \text{ and } q(y) = 0$$

Since the boundary conditions at $y = 0$ and $y = H$ are homogeneous, it follows that the characteristic functions $\phi_n(y) = \cos \lambda_n y$ are orthogonal with respect to the weighting function $w(y) = 1$. Multiplying both sides of (t) by $w(y)\cos \lambda_m y dy$ and integrating from $y = 0$ to $y = H$

$$\int_0^H f(y)w(y)\cos\lambda_m y dy = -k\int_0^H \left\{\sum_{n=1}^{\infty} a_n\lambda_n\left[\cosh\lambda_n L + (\lambda_n H)(k/hH)\sinh\lambda_n L\right]\cos\lambda_n y\right\} w(y)\cos\lambda_m y\, dy$$

Interchanging the summation with integration, recalling that $w(y)=1$ and invoking orthogonality, eq. (3.7), gives

$$\int_0^H f(y)\cos\lambda_n y dy = -ka_n\lambda_n\left[\cosh\lambda_n L\lambda_n H + (k/hH)\sinh\lambda_n L\right]\int_0^H \cos^2\lambda_n y\, dy$$

Solving for a_n the above gives

$$a_n = \frac{-\int_0^H f(y)\cos\lambda_n y\, dy}{k\lambda_n\left[\cosh\lambda_n L + (\lambda_n H)(k/hH)\sinh\lambda_n L\right]\int_0^H \cos^2\lambda_n y\, dy}$$

Evaluating the integral in the denominator, the above becomes

$$a_n = \frac{-4\int_0^H f(y)\cos\lambda_n y\, dy}{k\left[\cosh\lambda_n L + (\lambda_n H)(k/hH)\sinh\lambda_n L\right](2\lambda_n H + \sin 2\lambda_n H)} \tag{u}$$

Using the definition of $f(y)$ in condition (4) the integral in (u) is evaluated

$$\int_0^H f(y)\sin\lambda_n y\, dy = q_1''\int_0^{H/2}\cos\lambda_n y\, dy - q_2''\int_{H/2}^H \cos\lambda_n y\, dy =$$
$$(q_1''/\lambda_n)\sin\lambda_n H/2 - (q_2''/\lambda_n)(\sin\lambda_n H - \sin\lambda_n H/2)$$

Substituting into (u)

$$a_n = \frac{-4\left[(q_1''H/k)\sin\lambda_n H/2 - (q_2''H/k)(\sin\lambda_n H - \sin\lambda_n H/2)\right]}{(\lambda_n H)\left[\cosh\lambda_n L + (\lambda_n H)(k/hH)\sinh\lambda_n L\right](2\lambda_n H + \sin 2\lambda_n H)} \tag{v}$$

Substituting (v) into (s) and rearranging the result gives

$$\frac{T(x,y)-T_\infty}{q_1''H/k} = -4\sum_{n=1}^{\infty}\frac{\left[\sin(\lambda_n H/2) - (q_2''/q_1'')(\sin\lambda_n H - \sin\lambda_n H/2)\right]\left[\sinh\lambda_n x + (\lambda_n H)(k/hH)\cosh\lambda_n x\right]}{(\lambda_n H)\left[\cosh\lambda_n L + (\lambda_n H)(k/hH)\sinh\lambda_n L\right](2\lambda_n H + \sin 2\lambda_n H)}\cos\lambda_n y \tag{w}$$

Having determined the temperature distribution in the plate, the heat transfer rate through surface (0,*y*) can be determined using Fourier's law as follows

$$q'(0) = -k\int_0^H \frac{\partial T(0,y)}{\partial x}\, dy$$

where $q'(0)$ is heat transfer rate through (0,y) per unit plate depth. Substituting (s) into the above

PROBLEM 3.5 (continued)

$$q'(0) = -k\int_0^H a_n \lambda_n \cos\lambda_n y dy = -k\sum_{n=1}^{\infty} a_n \sin\lambda_n H$$

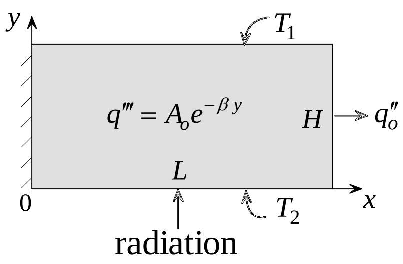
(5) Checking. *Dimensional check*: (i) The arguments of sin, cos, sinh and cosh are dimensionless. (ii) The Biot number, *hH*/*k*, is dimensionless. (iii) According to (s), the coefficient a_n should have units of temperature. The term $q''/\lambda_n k$ in (v) gives a_n the correct units.

Limiting check: If $q_1'' = q_2'' = 0$, the entire plate should be at a uniform temperature T_∞. Setting $q_1'' = q_2'' = 0$ in (w) gives $T(x, y) = T_\infty$.

Boundary conditions check: Conditions (1), (2) and (3) can be shown to be satisfied by direct substitution.

(6) Comments. (i) If the direction of q_1'' and/or q_2'' is reversed, solution (w) is still applicable. However, the signs of q_1'' and q_2'' must be changed accordingly. The same approach can be used to solve the more general problem where the heat flux at the boundary is variable, $q_o'' = q_o''(y)$. (ii) Equation (r) gives the dimensionless roots $\lambda_k H$ in terms of the Biot number *hH*/*k*. However, the argument $\lambda_k L$ is determined by multiplying and dividing by *L*. This introduces an additional parameter H/L. Thus examining solution (w) we note that the temperature distribution depends on three parameters: *hH*/*k*, *H*/*L* and q_2''/q_1''.

PROBLEM 3.6



Radiation strikes one side of a rectangular element $L \times H$ and is partially absorbed. The absorbed energy results in a non-uniform volumetric energy generation given by

$$q''' = A_o\, e^{-\beta y}$$

where A_o and β are constant. The surface which is exposed to radiation, (x,0), is maintained at T_2 while the opposite surface (x,H) is at T_1. The element is insulated along surface (0,y) and cooled along (L,y) at a uniform flux q_o''. Determine the steady state temperature of the insulated surface.

(1) Observations. (i) This is a steady state two-dimensional conduction problem. (ii) Four boundary conditions are needed. (iii) Only one boundary condition is homogeneous. (iv) The energy generation term makes the differential equation non-homogeneous. (v) Energy generation varies in the *y*-direction.

(2) Origin and Coordinates. The origin and coordinate axes are selected as shown.

(3) Formulation.

` **(i) Assumptions.** (1) Two-dimensional, (2) steady and (3) constant thermal conductivity.

(ii) Governing Equations. Introducing the above assumptions into eq. (1.8) gives

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{q'''}{k} = 0 \tag{a}$$

where

$$q''' = A_o\, e^{-\beta y} \tag{b}$$

Substituting (b) into (a)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{A_o}{k} e^{-\beta y} = 0 \tag{c}$$

(iii) Independent Variable with Two Homogeneous Boundary Conditions. Neither variable has two homogeneous conditions. Only the *y*-variable can have two homogeneous conditions.

(iv) Boundary Conditions.

(1) $T(x,0) = T_2$, non-homogeneous

(2) $T(x,H) = T_1$, non-homogeneous

(3) $\dfrac{\partial T(0,y)}{\partial x} = 0$, homogeneous

(4) $-k\dfrac{\partial T(L,y)}{\partial x} = q_o''$, non-homogeneous

(4) Solution. Equation (c) is non-homogeneous due to the heat generation term. Thus we can not proceed directly with the application of the method of separation of variables. Noting that the heat generation term is a function of *y*, we assume a solution of the form

$$T(x,y) = \psi(x,y) + \phi(y) \tag{d}$$

Note that $\psi(x,y)$ depends on two variables while $\phi(y)$ depends on a single variable. Substituting (d) into (c)

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{d^2\phi}{dy^2} + \frac{A_o}{k}e^{-\beta y} = 0 \tag{e}$$

The next step is to split (c) into two equations, one for $\psi(x,y)$ and the other for $\phi(y)$. We select

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \tag{f}$$

Thus, the second part must be

$$\frac{d^2\phi}{dy^2} + \frac{A_o}{k}e^{-\beta y} = 0 \tag{g}$$

Boundary conditions on $\psi(x,y)$ and $\phi(y)$ are obtained by substituting (d) into the four boundary conditions. In this process the function $\psi(x,y)$ is assigned homogeneous conditions whenever possible. Using (d) and boundary condition (1)

$$\psi(x,0) + \phi(0) = T_2$$

Let

$$\psi(0,y) = 0 \tag{f-1}$$

Thus

$$\phi(0) = T_2 \tag{g-1}$$

Boundary condition (2) and (b) give

$$\psi(x,H) + \phi(H) = T_1$$

Let

$$\psi(x,H) = 0 \tag{f-2}$$

Thus

$$\phi(H) = T_1 \tag{g-2}$$

Boundary condition (3) and (b) give

$$\frac{\partial\psi(0,y)}{\partial x} = 0 \tag{f-3}$$

Boundary condition (4) and (b) give

$$-k\frac{\partial\psi(L,y)}{\partial x} = q_o'' \tag{f-4}$$

Thus, non-homogeneous equation (c) is replaced by two equations: (f), which is a homogeneous partial differential equation and (g) which is a non-homogeneous ordinary differential equation. Equation (f) has three homogeneous conditions, (f-1), (f-2) and (f-3) and one non-homogeneous condition, (f-4). The solution to (g) is

$$\phi(y) = -\frac{A_o}{\beta^2 k} e^{-\beta y} + E\, y + F \tag{h}$$

where E and F are constants of integration. Application of boundary conditions (g-1) and (g-2) gives the two constants

$$E_1 = (1/H)\left[(T_1 - T_2) + \frac{A_o}{\beta^2 k}\left(e^{-\beta H} - 1\right)\right] \text{ and } F = T_2 + \frac{A_o}{\beta^2 k} \tag{i}$$

Substituting (i) into (h)

$$\phi(x) = -\frac{A_o}{\beta^2 k} e^{-\beta y} + \left[(T_1 - T_2) + \frac{A_o}{\beta^2 k}\left(e^{-\beta H} - 1\right)\right](y/H) + T_2 + \frac{A_o}{\beta^2 k} \tag{j}$$

(i) Assumed Product Solution. To solve partial differential equation (f) we assume a product solution. Let

$$\psi(x, y) = X(x)Y(y) \tag{k}$$

Substituting (k) into (f), separating variables and setting the resulting two equations equal to a constant, $\pm\lambda_n^2$, we obtain

$$\frac{d^2 X_n}{d x^2} \pm \lambda_n^2 X_n = 0 \tag{l}$$

and

$$\frac{d^2 Y_n}{d y^2} \mp \lambda_n^2 Y_n = 0 \tag{m}$$

(ii) Selecting the Sign of the λ_n^2 Terms. Since the *y*-variable has two homogeneous boundary conditions, the plus sign is selected in (m). Thus (l) and (m) become

$$\frac{d^2 X_n}{d x^2} - \lambda_n^2 X_n = 0 \tag{n}$$

and

$$\frac{d^2 Y_n}{d y^2} + \lambda_n^2 Y_n = 0 \tag{o}$$

For the special case of $\lambda_n = 0$ the above equations take the form

$$\frac{d^2 X_0}{d x^2} = 0 \tag{p}$$

and

$$\frac{d^2 Y_0}{d y^2} = 0 \tag{q}$$

PROBLEM 3.6 (continued)

(iii) Solutions to the Ordinary Differential Equations. The solutions to equations (n)-(q) are

$$X_n(x) = A_n \sinh\lambda_n x + B_n \cosh\lambda_n x \tag{r}$$

$$Y_n(y) = C_n \sin\lambda_n y + D_n \cos\lambda_n y \tag{s}$$

$$X_0(x) = A_0 x + B_0 \tag{t}$$

and

$$Y_0(y) = C_0 y + D_0 \tag{u}$$

Corresponding to each value of λ_n there is a solution $\psi_n(x, y)$. Thus

$$\psi_n(x, y) = X_n(x) Y_n(y) \tag{v}$$

and

$$\psi_0(x, y) = X_0(x) Y_0(y) \tag{w}$$

The complete solution to $T(x, y)$ becomes

$$T(x, y) = \phi(x) + X_0(x) Y_0(y) + \sum_{n=1}^{\infty} X_n(x) Y_n(y) \tag{x}$$

(iv) Application of Boundary Conditions. Applying boundary condition (f-1) to equation (s) and (u) gives

$$D_n = D_0 = 0$$

Condition (f-2) applied to (s) gives the characteristic equation for λ_n

$$\sin\lambda_n H = 0$$

Thus

$$\lambda_n = n\pi / H, \quad (n = 1, 2, 3....) \tag{y}$$

Condition (f-2) applied to (u)

$$C_0 = 0$$

Boundary condition (f-3) and (r) give

$$A_n = 0$$

With $C_0 = D_0 = 0$, the solution $X_0 Y_0$ vanishes. Substituting into (x) and using (r) and (s), the solution to $\psi(x, y)$ becomes

$$\psi(x, y) = \sum_{n=1}^{\infty} a_n (\cosh\lambda_n x) \sin\lambda_n y \tag{z}$$

where $a_n = B_n C_n$. Thus the only remaining unknown is a_n. Application of boundary condition (f-4) gives

$$q_o'' = -k \sum_{n=1}^{\infty} a_n (\lambda_n \sinh\lambda_n L) \sin\lambda_n y \tag{z-1}$$

To determine a_n from this result we apply orthogonality.

PROBLEM 3.6 (continued)

(v) Orthogonality. The characteristic functions, $\sin\lambda_n y$, in (z-1) are solutions to equation (o). Comparing (o) with eq. (3.5a) shows that it is a Sturm-Liouville equation with

$$a_1(y) = a_2(y) = 0 \text{ and } a_3(y) = 1$$

Thus eq. (3.6) gives

$$p(y) = w(y) = 1 \text{ and } q(y) = 0$$

Since boundary conditions (f-1) and (f-2) at y = 0 and y = H are homogeneous, it follows that the characteristic functions $\sin\lambda_n y$ are orthogonal with respect to $w(y)$ = 1. Multiplying both sides of (z-1) by $w(y)\sin\lambda_m y\, dy$ and integrating from y = 0 to $y = H$

$$\int_0^H q_o'' w(y)\sin\lambda_m y\, dy = -k\int_0^H \left\{\sum_{n=1}^{\infty}(a_n\lambda_n \sinh\lambda_n L)\sin\lambda_n y\right\} w(y)(\sin\lambda_m y)\, dy$$

Interchanging integration the integration and summation signs, noting that $w(y) = 1$ and applying orthogonality, the above gives

$$\int_0^H q_o'' \sin\lambda_n y dy = -a_n\, \lambda_n \sinh\lambda_n H \int_0^H \sin^2\lambda_n y\, dy$$

Evaluating the integrals, solving the above for a_n and using (y) gives

$$a_n = \frac{2(q_o'' H/k)\left[(-1)^n - 1\right]}{(n\pi)^2 \sinh\lambda_n L} \tag{z-2}$$

The solution to the temperature distribution is obtained by substituting (j), (z) and (z-2) into (d)

$$T(x,y) = -\frac{A_o}{\beta^2 k}e^{-\beta y} + \left[(T_1 - T_2) + \frac{A_o}{\beta^2 k}\left(e^{-\beta H} - 1\right)\right](y/H) + T_2 + \frac{A_o}{\beta^2 k} +$$

$$2(q_o'' H/k)\sum_{n=1}^{\infty}\frac{\left[(-1)^n - 1\right]}{(n\pi)^2 \sinh\lambda_n L}(\cosh\lambda_n x)\sin\lambda_n y \tag{z-3}$$

Expressed in dimensionless form the becomes

$$\frac{T(x,y) - T_2}{(A_o/\beta^2 k)} = 1 - e^{-\beta y} + \left[\frac{(T_1 - T_2)}{A_o/\beta^2 k} + \left(e^{-\beta H} - 1\right)\right](y/H) +$$

$$2\frac{q_o'' H}{A_o/\beta^2}\sum_{n=1}^{\infty}\frac{\left[(-1)^n - 1\right]}{(n\pi)^2 \sinh\lambda_n L}(\cosh\lambda_n x)\sin\lambda_n y \tag{z-4}$$

(5) Checking. *Dimensional checks*: (i) The arguments of sin, sinh and cosh are dimensionless. (ii). Each term in (z-4) is dimensionless.

Limiting check: For the special case of $q_o'' = 0$ temperature distribution should be one-dimensional. Setting $q_o'' = 0$ in (z-3) or (z-4) gives the one-dimensional solution to the problem. If we further assume that there is no energy generation, (z-3) gives

$$T(y) = T_2 + (T_1 - T_2)(y / H)$$

This is recognized as the linear one-dimensional solution in a plate with two boundaries at specified temperatures.

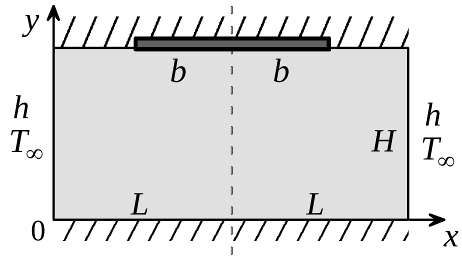
Boundary conditions check: Conditions (1), (2) and (3) can be shown to be satisfied by direct substitution.

(6) Comments. (i) Selecting ϕ in the assumed solution (d) to be a function of *y* rather than *x* is motivated by the fact that the energy generation term is a function of *y*. (ii) Equation (y) gives the dimensionless roots $\lambda_k H$. However, the argument $\lambda_k L$ is determined by multiplying and dividing by *L*. This introduces H/L as a parameter, Thus examining solution (z-4) we note that the temperature distribution depends on four parameters:

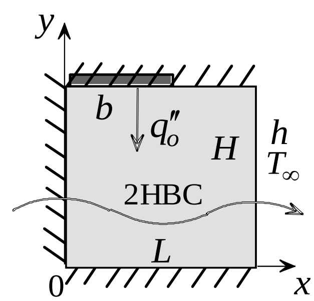
$$\frac{T_1 - T_2}{A_o / \beta^2 k}, \quad \frac{q_o'' H}{A_o / \beta^2}, \ \beta H \ \text{and} \ \frac{H}{L}$$

PROBLEM 3.7

An electric strip heater of width 2b dissipates energy at a flux q''_o. The heater is mounted symmetrically on one side of a plate of height H and width 2L. Heat is exchanged by convection along surfaces (0,y) and (2L,y). The ambient temperature is T_∞ and the heat transfer coefficient is h. Surfaces (x,0) and (x,H) are insulated. Determine the temperature of the plate-heater interface for two-dimensional steady state conduction.



(1) Observations. (i) To determine the plate-heater interface temperature it is necessary to first determine the temperature distribution in the plate. (ii) This is a steady state two-dimensional conduction problem. (iii) Four boundary conditions are needed. (iv) Temperature distribution is symmetrical about the vertical centerline. This makes it possible to analyze half the plate. (v) Only the *x*-variable can have two homogeneous conditions. (vi) Defining a new temperature variable $\theta(x,y) = T(x,y) - T_\infty$ makes the convection condition at (*L*,*y*) homogeneous. (vii) The heat flux along (*x*,*H*) is non-uniform. This requires special consideration.



(2) Origin and Coordinates. Because half the plate is to be considered, the origin is selected at the centerline and the coordinate axes are directed as shown.

(3) Formulation.

(i) Assumptions. (1) Two-dimensional, (2) steady, (3) no energy generation and (4) constant thermal conductivity.

(ii) Governing Equations. Introducing the above assumptions into eq. (1.8) gives the heat equation

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \tag{a}$$

where

$$\theta(x,y) = T(x,y) - T_\infty \tag{b}$$

(iii) Independent Variable with Two Homogeneous Boundary Conditions. The two boundaries at $x = 0$ and $x = L$ have homogeneous conditions. Thus, the *x*-variable has two homogeneous boundary conditions.

(iv) Boundary Conditions. The required four boundary conditions are

(1) $\dfrac{\partial\theta(0,y)}{\partial x} = 0$, homogeneous

(2) $-\dfrac{\partial\theta(L,y)}{\partial x} = h\theta(L,y)$, homogeneous

(3) $\dfrac{\partial\theta(x,0)}{\partial y} = 0$, homogeneous

(4) $$-k\frac{\partial\theta(x,H)}{\partial y}=f(x)=\begin{cases}-q_o'', & 0<x<b\\ 0, & b<x<L\end{cases}$$

This condition represents a heat flux which varies along the boundary according to $f(x)$.

(4) Solution.

(i) Assumed Product Solution. Assume a solution in the form

$$\theta(x,y) = X(x)\,Y(y) \tag{c}$$

Substituting into (a), separating variables and setting the resulting equation equal to constant

$$\frac{1}{X}\frac{d^2X}{dx^2}=-\frac{1}{Y}\frac{d^2Y}{dy^2}=\pm\lambda_n^2$$

Assuming that λ_n is multi-valued, the above gives

$$\frac{d^2X_n}{dx^2}\mp\lambda_n^2X_n=0 \tag{d}$$

and

$$\frac{d^2Y_n}{dy^2}\pm\lambda_n^2Y_n=0 \tag{e}$$

(ii) Selecting the Sign of the λ_n^2 Terms. Since the x-variable has two homogeneous boundary conditions the $\lambda_n^2X_n$ term in (d) takes the positive sign. Therefore, (d) and (e) become

$$\frac{d^2X_n}{dx^2}+\lambda_n^2X_n=0 \tag{f}$$

and

$$\frac{d^2Y_n}{dy^2}-\lambda_n^2Y_n=0 \tag{g}$$

For the important case of $\lambda_n=0,$ equations (f) and (g) become

$$\frac{d^2X_0}{dx^2}=0 \tag{h}$$

and

$$\frac{d^2Y_0}{dy^2}=0 \tag{i}$$

(iii) Solutions to the Ordinary Differential Equations. The solutions to eqs. (f)-(i) are

$$X_n(x)=A_n\sin\lambda_nx+B_n\cos\lambda_nx \tag{j}$$

$$Y_n(y)=C_n\sinh\lambda_ny+D_n\cosh\lambda_ny \tag{k}$$

PROBLEM 3.7 (continued)

$$X_0(x) = A_0 x + B_0 \tag{l}$$

$$Y_0(y) = C_0 y + D_0 \tag{m}$$

Corresponding to each value of λ_n there is a temperature solution $\theta_n(x, y)$. Thus

$$\theta_n(x, y) = X_n(x) Y_n(y) \tag{n}$$

and

$$\theta_0(x, y) = X_0(x) Y_0(y) \tag{o}$$

The complete solution becomes

$$\theta(x, y) = X_0(x) Y_0(y) + \sum_{n=1}^{\infty} X_n(x) Y_n(y) \tag{p}$$

(iv) Application of Boundary Conditions. Boundary condition (1) applied to solutions (j) and (l) gives

$$A_n = A_0 = 0 \tag{q}$$

Boundary condition (2) applied to (j) gives the characteristic equation for λ_n

$$\lambda_n L \tan \lambda_n L = hL/k = Bi \tag{r}$$

Similarly, boundary condition (2) applied to (l) gives

$$B_0 = 0$$

With $A_0 = B_0 = 0,$ the solution corresponding to $\lambda_n = 0$ vanishes. Application of boundary condition (3) to (k) gives

$$C_n = 0 \tag{s}$$

Solution (p) becomes

$$\theta(x, y) = \sum_{n=1}^{\infty} a_n (\cos \lambda_n x)(\cosh \lambda_n y) \tag{t}$$

where $a_n = B_n D_n$. The only remaining unknown is the set of constants a_n. Application of condition (4) gives

$$f(x) = -k \sum_{n=1}^{\infty} (a_n \lambda_n \sinh \lambda_n H) \cos \lambda_n x \tag{u}$$

where $f(x)$ is defined in boundary condition (4).

(v) Orthogonality. To determine a_n in equation (u) we apply orthogonality. Note that the characteristic functions $\phi_n(x) = \cos \lambda_n x$ in (u) are solutions to equation (f). Comparing (f) with eq. (3.5a) shows that it is a Sturm-Liouville equation with

$$a_1(x) = a_2(x) = 0 \text{ and } a_3(x) = 1$$

Thus eq. (3.6) gives

$$p(x) = w(x) = 1 \text{ and } q(x) = 0$$

PROBLEM 3.7 (continued)

Since the boundary conditions at $x = 0$ and $x = L$ are homogeneous, it follows that the characteristic functions $\phi_n(x) = \cos\lambda_n x$ are orthogonal with respect to the weighting function $w(x) = 1$. Multiplying both sides of (u) by $w(x)\sin\lambda_m x\,dx$ and integrating from $x = 0$ to $x = L$

$$\int_0^L f(x)w(x)\cos\lambda_m x\,dx = -k\int_0^L \left\{\sum_{n=1}^{\infty}(a_n\lambda_n \sinh\lambda_n H)\cos\lambda_n x\right\}w(x)\cos\lambda_m x\,dx$$

Interchanging the summation with integration and invoking orthogonality gives

$$\int_0^L f(x)w(x)\cos\lambda_n x\,dx = -k\,a_n\lambda_n \sinh\lambda_n H\int_0^L w(x)\cos^2\lambda_n x$$

Solving for a_n, recalling that $w(x) = 1$ and evaluating the integral on the right-hand-side, the above gives

$$a_n = \frac{4\int_0^L f(x)\cos\lambda_n x\,dx}{-k\sinh\lambda_n H(2\lambda_n L + \sin 2\lambda_n L)} \tag{v}$$

Using the definition of $f(x)$ in condition (4) the integral in (v) is evaluated

$$\int_0^L f(x)\cos\lambda_n x\,dx = \int_0^b (-q_o'')\cos\lambda_n x\,dx - \int_b^L (0)\lambda_n x\,dx = -(q_o''/\lambda_n)\sin\lambda_n b$$

Substituting into (v)

$$a_n = \frac{4q_o''\sin\lambda_n b}{k\lambda_n \sinh\lambda_n H(2\lambda_n L + \sin 2\lambda_n L)} \tag{w}$$

The solution to the temperature distribution is obtained by substituting (b) and (v0 into (t)

$$\frac{T(x,y) - T_\infty}{(q_o''L/k)} = 4\sum_{n=1}^{\infty}\frac{\sin\lambda_n b}{\lambda_n L(\sinh\lambda_n H)(2\lambda_n L + \sin 2\lambda_n L)}(\cos\lambda_n x)\cosh\lambda_n y \tag{x}$$

To determine the plate-heater interface temperature set $y = H$ in (x)

$$\frac{T(x,H) - T_\infty}{(q_o''L/k)} = 4\sum_{n=1}^{\infty}\frac{(\coth\lambda_n H)(\sin\lambda_n b)}{\lambda_n L(2\lambda_n L + \sin 2\lambda_n L)}\cos\lambda_n x \qquad 0 < x < b \tag{y}$$

(5) Checking. *Dimensional check*: (i) The arguments of sin, cos, sinh and cosh are dimensionless. (ii) The term $q_o''L/k$ in (x) has units of temperature. Thus both sides of (x) are dimensionless.

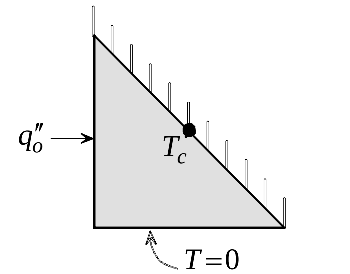
Limiting check: If $q_o'' = 0$, the entire plate should be at a uniform temperature T_∞. Multiplying (x) through by $(q_o''L/k)$ and setting $q_o'' = 0$ gives $T(x,y) = T_o$.

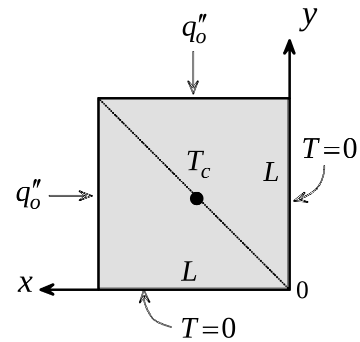
Boundary conditions check: Conditions (1), (2) and (3) can be shown to be satisfied by direct substitution.

(6) Comments. (i) The same approach can be used to solve the more general problem where the heat flux at the boundary is variable, $q''_o = q''_o(x)$. This will only affect the integral in (v). (ii) Taking advantage of symmetry simplifies the problem. (iii) Equation (r) gives the dimensionless roots $\lambda_n L$. However, the argument $\lambda_k H$ is determined by multiplying and dividing by L according to $\lambda_k H = (\lambda_k L)(H/L)$. This introduces an additional dimensionless parameters H/L. Similarly, the argument $\lambda_k b$ introduces the parameter b/L. Thus, three parameters characterize the problem: Bi, b/L, and H/L.

PROBLEM 3.8

The cross section of a long prism is a right angle isosceles triangle of side L. One side is heated with uniform flux q_o'' while the second side is maintained at zero temperature. The hypotenuse is insulated. Determine the temperature T_c at the mid-point of the hypotenuse.





(1) Observations. (i) This is a steady state two-dimensional conduction problem. (ii) The geometry does not lend itself to analytic solution by the method of separation of variables. However, due to symmetry the temperature distribution in the right-angled isosceles triangle is the same as that in a square with symmetrical boundary conditions about the diagonal as shown. (iii) To determine the temperature at the mid-point of the hypotenuse (center of square) it is necessary to determine the temperature distribution in the square. (iv) There are two non-homogeneous boundary conditions. Since no direction has two homogeneous conditions a modified procedure for applying the separation of variables method is needed.

(2) Origin and Coordinates. The origin and coordinate axes for the square geometry are selected as shown.

(3) Formulation.

(i) Assumptions. (1) Two-dimensional, (2) steady, (3) no energy generation and (4) constant thermal conductivity.

(ii) Governing Equations. Introducing the above assumptions into eq. (1.8) gives

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{a}$$

(iii) Independent Variable with Two Homogeneous Boundary Conditions. No variable has two homogeneous conditions. Thus the method of separation of variables can not be applied directly to obtain a solution. A modified procedure will be used in which a two-dimensional problem with two homogeneous conditions in the *x*-variable will represent one part of the solution. A second part which is one-dimensional will satisfy the non-homogeneous conditions in the *x*-variable.

(iv) Boundary Conditions. Consider the square formed by the triangle and its mirror image. The four boundary conditions are

(1) $T(0,y) = 0$, homogeneous

(2) $k\dfrac{\partial T(L,y)}{\partial x} = q_o''$, non-homogeneous

(3) $T(x,0) = 0$, homogeneous

(4) $k\dfrac{\partial T(x,L)}{\partial y} = q_o''$, non-homogeneous

(4) Solution. Since there are two non-homogenous boundary conditions we assume a solution of the form

$$T(x, y) = \psi(x, y) + \phi(x) \tag{b}$$

Substituting (b) into (a)

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{d^2 \phi}{\partial x^2} = 0 \tag{c}$$

Let

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{d}$$

Thus

$$\frac{d^2 \phi}{d x^2} = 0 \tag{e}$$

Boundary conditions on (d) and (e) are obtained by substituting (b) into the four conditions. Substituting (b) in condition (1)

$$\psi(0, y) + \phi(0) = 0$$

Let

$$\psi(0, y) = 0 \tag{d-1}$$

Thus

$$\phi(0) = 0 \tag{e-1}$$

Condition (2) gives

$$\frac{\partial \psi(L, y)}{\partial x} + \frac{d\phi(L)}{dx} = q_o''$$

Let

$$\frac{\partial \psi(L, y)}{\partial x} = 0 \tag{d-2}$$

Thus

$$\frac{d\phi(L)}{dx} = q_o'' \tag{e-2}$$

Condition (3) gives

$$\psi(x, 0) + \phi(x) = 0$$

Or

$$\psi(x, 0) = -\phi(x) \tag{d-3}$$

Boundary condition (4) gives

$$k \frac{\partial \psi(x, L)}{\partial y} = q_o'' \tag{d-4}$$

Thus partial differential equation (c) has two homogeneous boundary conditions in the x-variable, (d-1) and (d-2). The solution to (e) with boundary conditions (e-1) and (e-2) is

$$\phi(x) = \frac{q_o''}{k} x \tag{f}$$

We proceed now to determine the solution to (d) using separation of variables.

PROBLEM 3.8 (continued)

(i) Assumed Product Solution. Assume a solution in the form

$$\psi(x,y) = X(x)\,Y(y) \tag{g}$$

Substituting into (d), separating variables and setting the resulting equation equal to constant

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \pm\lambda_n^2 \tag{h}$$

Assuming that λ_n is multi-valued, the above gives

$$\frac{d^2X_n}{dx^2} \mp \lambda_n^2 X_n = 0 \tag{i}$$

and

$$\frac{d^2Y_n}{dy^2} \pm \lambda_n^2 Y_n = 0 \tag{j}$$

(ii) Selecting the Sign of the λ_n^2 Terms. Since the *x*-variable has two homogeneous boundary conditions the $\lambda_n^2 X_n$ term in (i) takes the positive sign. Therefore, (i) and (j) become

$$\frac{d^2X_n}{dx^2} + \lambda_n^2 X_n = 0 \tag{k}$$

and

$$\frac{d^2Y_n}{dy^2} - \lambda_n^2 Y_n = 0 \tag{l}$$

For the important case of $\lambda_n = 0,$ equations (k) and (l) become

$$\frac{d^2X_0}{dx^2} = 0 \tag{m}$$

and

$$\frac{d^2Y_0}{dy^2} = 0 \tag{n}$$

(iii) Solutions to the Ordinary Differential Equations. The solutions to eqs. (k)-(n) are

$$X_n(x) = A_n \sin\lambda_n x + B_n \cos\lambda_n x \tag{o}$$

$$Y_n(y) = C_n \sin\lambda_n y - D_n \cos\lambda_n y \tag{p}$$

$$X_0(x) = A_0 x + B_0 \tag{q}$$

$$Y_0(y) = C_0 y + D_0 \tag{r}$$

Corresponding to each value of λ_n there is a temperature solution $\psi(x,y)$. Thus

$$\psi(x,y) = X_n(x)Y_n(y) \tag{s}$$

and

$$\psi_0(x,y) = X_0(x)Y_0(y) \tag{t}$$

The complete solution becomes

$$\psi(x,y) = X_0(x)Y_0(y) + \sum_{n=1}^{\infty} X_n(x)Y_n(y) \tag{u}$$

(iv) Application of Boundary Conditions. Boundary condition (d-1) applied to solutions (o) and (q) gives

$$B_n = B_o = 0 \tag{v}$$

Boundary condition (d-2) applied to (o) gives the characteristic equation for λ_n

$$\cos\lambda_n L = 0 \tag{w}$$

Or

$$\lambda_n L = \frac{(2n-1)\pi}{2} \qquad (n = 1,2,3\ldots.) \tag{x}$$

Similarly, boundary condition (d-2) applied to (q) gives

$$A_0 = 0$$

With $A_0 = B_0 = 0,$ the solution corresponding to $\lambda_n = 0$ vanishes. Thus (u) becomes

$$\psi(x,y) = \sum_{n=1}^{\infty}(a_n \sinh\lambda_n y + b_n \cosh\lambda_n y)\sin\lambda_n x \tag{y}$$

where $a_n = A_n C_n$ and $b_n = A_n D_n$. Condition (d-3) applied to (y) and using (f) gives

$$-q_o''x = k\sum_{n=1}^{\infty} b_n \sin\lambda_n x \tag{z-1}$$

Similarly, condition (d-4) and (y) give

$$q_o'' = k\sum_{n=1}^{\infty}\lambda_n(a_n \cosh\lambda_n L + b_n \sinh\lambda_n L)\sin\lambda_n x \tag{z-2}$$

(v) Orthogonality. To determine a_n and b_n in equations (z-1) and (z-2) we apply orthogonality to each equation. Note that the characteristic functions $\phi_n(x) = \sin\lambda_n x$ in (z) are solutions to equation (k). Comparing (k) with eq. (3.5a) shows that it is a Sturm-Liouville equation with

$$a_1(x) = a_2(x) = 0 \text{ and } a_3(x) = 1$$

Thus eq. (3.6) gives

$$p(x) = w(x) = 1 \text{ and } q(x) = 0$$

Since the boundary conditions at $x = 0$ and $x = L$ are homogeneous, it follows that the characteristic functions $\phi_n(x) = \sin\lambda_n x$ are orthogonal with respect to the weighting function $w(x) = 1.$ Multiplying both sides of (z-1) by $w(x)\sin\lambda_m x\,dx$ and integrating from $x = 0$ to $x = L$ gives

$$-q_o''\int_0^L x\,w(x)\sin\lambda_m x\,dx \;=\; k\int_0^L \left\{\sum_{n=1}^{\infty} b_n \sin\lambda_n x\right\} w(x)\sin\lambda_m x\,dx$$

Interchanging the summation with integration and noting that $w(x)$ = 1, orthogonality gives

$$-q_o''\int_0^L x\sin\lambda_n x\,dx = k\,b_n\int_0^L \sin^2\lambda_n x\,dx$$

Evaluation the integrals, using equation (x) and solving for b_n gives

$$b_n = \frac{8(q_o''L/k)}{\pi^2}\frac{(-1)^n}{(2n-1)^2} \qquad \text{(z-3)}$$

Similarly, multiplying both sides of (z-2) by $w(x)\sin\lambda_m x\,dx$ and integrating from $x = 0$ to $x = L$ gives

$$q_o''\int_0^L w(x)\sin\lambda_m x\,dx \;=\; k\int_0^L \lambda_n\left\{\sum_{n=1}^{\infty}(a_n\cosh\lambda_n L + b_n\sinh\lambda_n L)\sin\lambda_n x\right\} w(x)\sin\lambda_m x\,dx$$

Interchanging the summation with integration and noting that $w(x)$ = 1, orthogonality gives

$$q_o''\int_0^L \sin\lambda_n x\,dx \;=\; k\lambda_n(a_n\cosh\lambda_n L + b_n\sinh\lambda_n L)\int_0^L \sin^2\lambda_n x\,dx$$

Evaluation the integrals using equation (x) and (z-3) and solving for a_n gives

$$a_n = \frac{8(q_o''L/k)}{\pi^2}\frac{1-(-1)^n\sinh\lambda_n L}{(2n-1)^2\cosh\lambda_n L} \qquad \text{(z-4)}$$

The solution to the temperature distribution is obtained by substituting (f), (y), (z-3) and (z-4) into (b) and rearranging the result

$$\frac{T(x,y)}{(q_o''L/k)} = \frac{x}{L} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left[\frac{1-(-1)^n\sinh\lambda_n L}{\cosh\lambda_n L}\sinh\lambda_n y + (-1)^n\cosh\lambda_n y\right]\sin\lambda_n x \qquad \text{(z-5)}$$

Evaluating (z-5) at the center, $x = L/2$ and $y = L/2$, gives T_c

$$\frac{T_c}{(q_o''L/k)} = \frac{1}{2} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left[\frac{1-(-1)^n\sinh\lambda_n L}{\cosh\lambda_n L}\sinh\lambda_n L/2 + (-1)^n\cosh\lambda_n L/2\right]\sin\lambda_n L/2$$

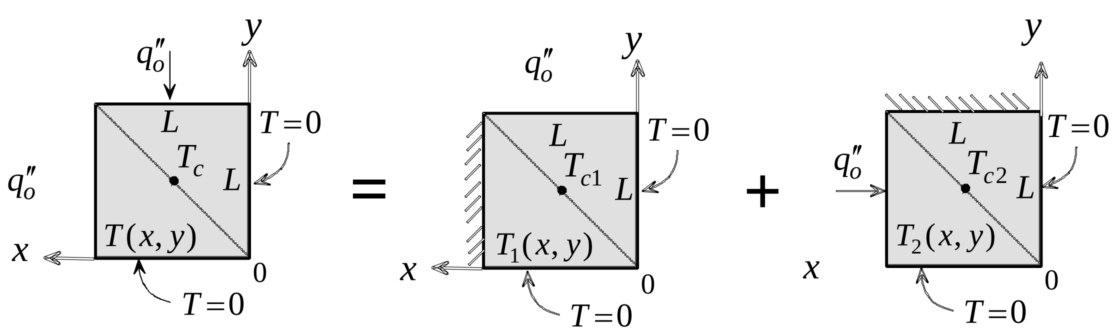
(z-6)

(5) Checking. *Dimensional check*: (i) The arguments of sin, cos, sinh and cosh are dimensionless. (ii) Each term in (z-5) and (z-6) is dimensionless.

Limiting check: If $q_o'' = 0$, the entire plate should be at zero temperature. Setting $q_o'' = 0$ in (z-5) gives $T(x,y) = 0$.

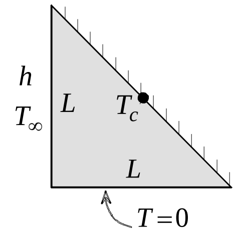
Boundary conditions check: Conditions (1), (2) and (3) can be shown to be satisfied by direct substitution.

(5) Comments. (i) The same approach can be used to solve the more general problem where the heat flux at the side of the right-angled isosceles triangle is non-uniform as long as the corresponding square problem is assigned the same non-uniform flux on two of its adjacent sides such that there is symmetry with respect to the diagonal. (ii) The method used to solve this problem is limited to a right-angled isosceles triangle which forms a square with its mirror image. It does not work for a right-angled triangle which forms a rectangle with its mirror image because of asymmetry. (iii) An alternate approach to solving the square problem is to use the method of superposition by decomposing the problem into two simpler problems each having a single non-homogeneous condition as illustrated be below. Each problem is solved using separation of variables directly.

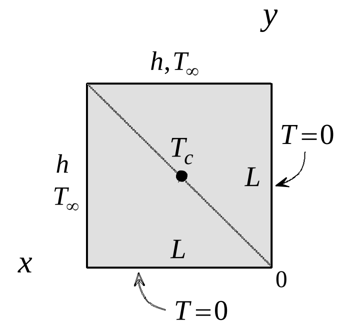


PROBLEM 3.9

The cross section of a long prism is a right angle isosceles triangle of side L. One side exchanges heat by convection with the surroundings while the second side is maintained at zero temperature. The heat transfer coefficient is h and ambient temperature is T_∞. The hypotenuse is insulated. Determine the temperature T_c at the mid-point of the hypotenuse.



(1) Observations. (i) This is a steady state two-dimensional conduction problem. (ii) The geometry does not lend itself to analytic solution by the method of separation of variables. However, due to symmetry the temperature distribution in the right-angled isosceles triangle is the same as that in a square with boundary conditions symmetrical about the diagonal as shown. (iii) To determine the temperature at the mid-point of the hypotenuse (center of square) it is necessary to determine the temperature distribution in the square. (vi) There are two non-homogeneous boundary conditions. Since no direction has two homogeneous conditions the method of superposition should be used.



(2) Origin and Coordinates. The origin and coordinate axes for the square geometry are selected as shown.

(3) Formulation.

(i) Assumptions. (1) two-dimensional, (2) steady, (3) no energy generation, (4) constant conductivity and (5) uniform heat transfer coefficient h and ambient temperature T_∞.

(ii) Governing Equations. Introducing the above assumptions into eq. (1.8) gives

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{a}$$

(iii) Independent Variable with Two Homogeneous Boundary Conditions. No variable has two homogeneous conditions. Thus the method of separation of variables can not be applied directly to obtain a solution. Since there are two non-homogenous boundary conditions, the problem will be decomposed into two simpler problems each having a single non-homogeneous condition.

(iv) Boundary Conditions. Consider the square formed by the triangle and its mirror image. The four boundary conditions are

(1) $T(0, y) = 0$, homogeneous

(2) $-k\dfrac{\partial T(L, y)}{\partial x} = h[T(L, y) - T_\infty]$, non-homogeneous

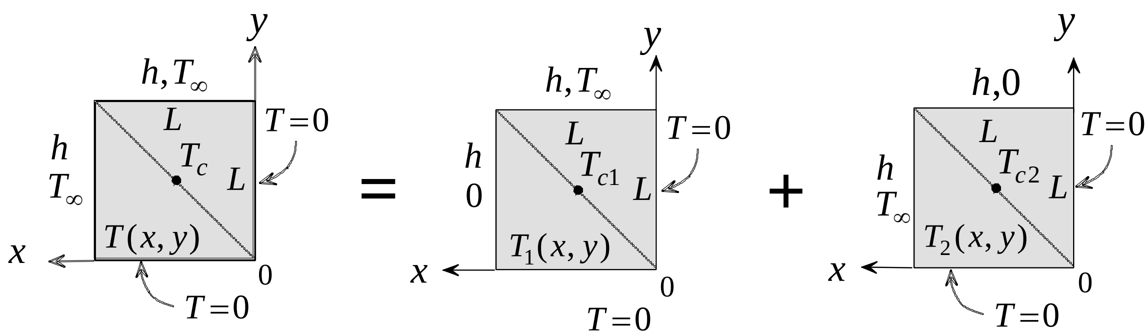
(3) $T(x, 0) = 0$, homogeneous

(4) $-k\dfrac{\partial T(x, L)}{\partial y} = h[T(x, L) - T_\infty]$, non-homogeneous

(4) Solution. Since there are two non-homogenous boundary conditions, the problem will be decomposed into two simpler problems each having a single non-homogeneous condition. Let

$$T(x,y) = T_1(x,y) + T_2(x,y) \tag{b}$$

where $T_1(x,y)$ and $T_2(x,y)$ are the solutions to two problems each governed by (a) with boundary conditions as shown.



Substituting (b) into (a)

$$\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} + \frac{\partial^2 T_2}{\partial x^2} + \frac{\partial^2 T_2}{\partial y^2} = 0 \tag{c}$$

Let

$$\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} = 0 \tag{d}$$

Thus

$$\frac{\partial^2 T_2}{\partial x^2} + \frac{\partial^2 T_2}{\partial y^2} = 0 \tag{e}$$

The temperature at the center is given by

$$T_c = T_{c1} + T_{c2} \tag{f}$$

We recognize that the center temperature is the same for each problem. Therefore

$$T_c = 2T_{c1} \tag{g}$$

Thus it is only necessary to solve the temperature distribution in one problem. Working with the first problem, $T_1(x,y)$, the boundary conditions are

(1) $T_1(0,y) = 0$, homogeneous

(2) $-k\dfrac{\partial T_1(L,y)}{\partial x} = hT_1(L,y)$, homogeneous

(3) $T_1(x,0) = 0$, homogeneous

(4) $-k\dfrac{\partial T_1(x,L)}{\partial y} = h\left[T_1(x,L) - T_\infty\right]$, non-homogeneous

Thus the x-variable has two homogeneous boundary conditions.

(i) Assumed Product Solution. Assume a solution in the form

$$T_1(x, y) = X(x)\, Y(y) \tag{h}$$

Substituting into (d), separating variables and setting the resulting equation equal to constant

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \pm\lambda_n^2$$

Assuming that λ_n is multi-valued, the above gives

$$\frac{d^2X_n}{dx^2} \mp \lambda_n^2 X_n = 0 \tag{i}$$

and

$$\frac{d^2Y_n}{dy^2} \pm \lambda_n^2 Y_n = 0 \tag{j}$$

(ii) Selecting the Sign of the λ_n^2 Terms. Since the *x*-variable has two homogeneous boundary conditions the $\lambda_n^2 X_n$ term in (i) takes the positive sign. Therefore, (d) and (e) become

$$\frac{d^2X_n}{dx^2} + \lambda_n^2 X_n = 0 \tag{k}$$

and

$$\frac{d^2Y_n}{dy^2} - \lambda_n^2 Y_n = 0 \tag{l}$$

For the important case of $\lambda_n = 0,$ equations (k) and (l) become

$$\frac{d^2X_0}{dx^2} = 0 \tag{m}$$

and

$$\frac{d^2Y_0}{dy^2} = 0 \tag{n}$$

(iii) Solutions to the Ordinary Differential Equations. The solutions to eqs. (k)-(n) are

$$X_n(x) = A_n \sin\lambda_n x + B_n \cos\lambda_n x \tag{o}$$

$$Y_n(y) = C_n \sinh\lambda_n y + D_n \cosh\lambda_n y \tag{p}$$

$$X_0(x) = A_0 x + B_0 \tag{q}$$

$$Y_0(y) = C_0 y + D_0 \tag{r}$$

Corresponding to each value of λ_n there is a temperature solution $T_{1n}(x, y).$ Thus

$$T_{1n}(x,y) = X_n(x)Y_n(y) \tag{s}$$

and

$$T_{10}(x,y) = X_0(x)Y_0(y) \tag{t}$$

The complete solution becomes

$$T_1(x,y) = X_0(x)Y_0(y) + \sum_{n=1}^{\infty} X_n(x)Y_n(y) \tag{u}$$

(iv) Application of Boundary Conditions. Boundary condition (1) applied to solutions (o) and (q) gives

$$B_n = B_0 = 0 \tag{v}$$

Boundary condition (2) applied to (o) gives the characteristic equation for λ_n

$$-\lambda_n L = (hL/k)\tan\lambda_n L \tag{w}$$

Similarly, boundary condition (2) applied to (q) gives

$$A_0 = 0$$

With $A_0 = B_0 = 0,$ the solution corresponding to $\lambda_n = 0$ vanishes. Application of boundary condition (3) to (p) gives

$$D_n = 0$$

Solution (u) becomes

$$T_1(x,y) = \sum_{n=1}^{\infty} a_n(\sinh\lambda_n y)(\sin\lambda_n x) \tag{x}$$

where $a_n = A_n C_n$. The only remaining unknown is the set of constants a_n. Application of condition (4) gives

$$-k\sum_{n=1}^{\infty}(a_n\lambda_n \cosh\lambda_n L)\sin\lambda_n x = h\sum_{n=1}^{\infty}(a_n \sinh\lambda_n L)\sin\lambda_n x - hT_\infty$$

Rearranging the above

$$hT_\infty = \sum_{n=1}^{\infty} a_n\left[h\sinh\lambda_n L + k\lambda_n \cosh\lambda_n L\right]\sin\lambda_n x \tag{y}$$

(v) Orthogonality. To determine a_n in equation (y) we apply orthogonality. Note that the characteristic functions $\phi_n(x) = \sin\lambda_n x$ in (y) are solutions to equation (k). Comparing (k) with eq. (3.5a) shows that it is a Sturm-Liouville equation with

$$a_1(x) = a_2(x) = 0 \text{ and } a_3(x) = 1$$

Thus eq. (3.6) gives

$$p(x) = w(x) = 1 \text{ and } q(x) = 0$$

Since the boundary conditions at x = 0 and x = L are homogeneous, it follows that the characteristic functions $\phi_n(x) = \sin\lambda_n x$ are orthogonal with respect to the weighting function $w(x) = 1$. Multiplying both sides of (y) by $w(x)\sin\lambda_m x\,dx$ and integrating from x = 0 to $x = L$

$$\int_0^L hT_\infty w(x)\sin\lambda_m x dx = \int_0^L \sum_{n=1}^{\infty} a_n\left[(h\sinh\lambda_n L + k\lambda_n \cosh\lambda_n L)\sin\lambda_n x\right]w(x)\sin\lambda_m x dx$$

Interchanging the summation with integration and invoking orthogonality give

$$hT_\infty \int_0^L w(x)\sin\lambda_n x dx = a_n(h\sinh\lambda_n L + k\lambda_n \cosh\lambda_n L)\int_0^L w(x)\sin^2\lambda_n x dx$$

Solving for a_n, recalling that $w(x)$ = 1 and evaluating the integrals in the above gives a_n

$$a_n = 2T_\infty \frac{1-\cos\lambda_n L}{\left[\sinh\lambda_n L + (k/hL)(\lambda_n L)\cosh\lambda_n L)\right]\left[\lambda_n L - (\cos\lambda_n L)\sin\lambda_n L\right]} \tag{z}$$

Substituting into (x)

$$\frac{T_1(x,y)}{T_\infty} = 2\sum_{n=1}^{\infty} \frac{(1-\cos\lambda_n L)(\sin\lambda_n x)\sinh\lambda_n y}{\left[\sinh\lambda_n L + (k/hL)(\lambda_n L)\cosh\lambda_n L)\right]\left[\lambda_n L - (\cos\lambda_n L)\sin\lambda_n L\right]} \tag{z-1}$$

Evaluating (s-1) at the center, $x = L/2$ and $y = L/2$ and substituting into (g) gives T_c

$$\frac{T_c}{T_\infty} = 4\sum_{n=1}^{\infty} \frac{(1-\cos\lambda_n L)(\sin\lambda_n L/2)\sinh\lambda_n L/2}{\left[\sinh\lambda_n L + (k/hL)(\lambda_n L)\cosh\lambda_n L)\right]\left[\lambda_n L - (\cos\lambda_n L)\sin\lambda_n L\right]} \tag{z-2}$$

(5) Checking. *Dimensional check*: (i) The arguments of sin, cos, sinh and cosh are dimensionless. (ii) Each term in solution (z-1) is dimensionless.

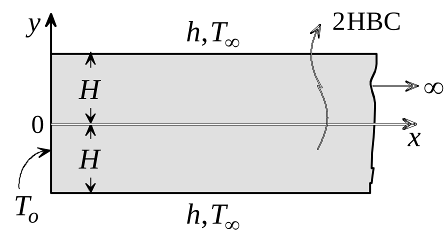
Limiting check: If $T_\infty = 0$, the entire plate should be at zero temperature. Setting $T_\infty = 0$ in (z-1) gives $T_1(x,y) = 0$.

Boundary conditions check: Conditions (1) and (3) can be readily shown to be satisfied by direct substitution in equation (z-1).

(6) Comments. (i) The solution is expressed in terms of the Biot number *hL*/*k*. This is characteristic of all problems involving surface convection, (ii) Equation (z-2) gives the temperature at the center of a square with convection along two adjacent sides while the other two sides are at zero temperature. If three sides exchange heat by convection, equation (z-2) still applies. However, the constant 4 is replaced by 6. This results from decomposing the problem into three simpler problems whose solutions are given by (z-1). (iii) The method used to solve this problem is limited to a right-angled triangle which forms a square with its mirror image. It does not work with a triangle which forms a rectangle with its mirror image because of asymmetry.

PROBLEM 3.10

By neglecting lateral temperature variation in the analysis of fins, two-dimensional conduction is modeled as a one-dimensional problem. To examine this approximation, consider a semi-infinite plate of thickness 2H. The base is maintained at uniform temperature T_o. The plate exchanges heat by convection at its semi-infinite surfaces. The heat transfer coefficient is h and the ambient temperature is T_∞. Determine the heat transfer rate at the base.



(1) Observations. (i) This is a steady state two-dimensional conduction problem. (ii) At $x = \infty$ the plate reaches ambient temperature. (iii) All four boundary conditions are non-homogenous. However, by defining a temperature variable $\theta = T - T_\infty$, three of the four conditions become homogeneous. (iv) Temperature distribution is symmetrical about $y = 0$ and thus the boundary (x,0) is insulated. Therefore, consideration is given to half the plate.

(2) Origin and Coordinates. The origin and coordinate axes are selected as shown.

(3) Formulation.

(i) Assumptions. (1) Two-dimensional, (2) steady, (3) uniform heat transfer coefficient and ambient temperature, (4) constant conductivity and (5) no energy generation.

(ii) Governing Equations. Introducing the above assumptions into eq. (1.8) gives

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \tag{a}$$

where θ is defined as

$$\theta(x,y) = T(x,y) - T_\infty$$

(iii) Independent Variable with Two Homogeneous Boundary Conditions. In terms of θ, the y-variable has two homogeneous conditions.

(iv) Boundary Conditions. We consider the upper half of the plate. The boundary conditions for this half are

(1) $\dfrac{\partial\theta(x,0)}{\partial y} = 0$, homogeneous

(2) $-k\dfrac{\partial\theta(x,H)}{\partial y} = h\theta(x,H)$, homogeneous

(3) $\theta(\infty,y) = 0$, homogeneous

(4) $\theta(0,y) = T_o - T_\infty$, non-homogeneous

(4) Solution.

(i) Assumed Product Solution. Let

(ii)

$$\theta(x,y) = X(x)Y(y)$$

Substituting into (a), separating variables and setting the resulting equation equal to constant

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \pm\lambda_n^2$$

Assuming that λ_n is multi-valued, the above gives

$$\frac{d^2X_n}{dx^2} \mp \lambda_n^2 X_n = 0 \tag{b}$$

and

$$\frac{d^2Y_n}{dy^2} \pm \lambda_n^2 Y_n = 0 \tag{c}$$

(ii) Selecting the Sign of the λ_n^2 Terms. Since the *y*-variable has two homogeneous conditions, the $\lambda_n^2 Y_n$ term in (c) takes the positive sign. Thus

$$\frac{d^2X_n}{dx^2} - \lambda_n^2 X_n = 0 \tag{d}$$

and

$$\frac{d^2Y_n}{dy^2} + \lambda_n^2 Y_n = 0 \tag{e}$$

For the special case of $\lambda_n = 0,$ equations (d) and (e) become

$$\frac{d^2X_0}{dx^2} = 0 \tag{f}$$

and

$$\frac{d^2Y_0}{dy^2} = 0 \tag{g}$$

(iii) Solutions to the Ordinary Differential Equations. The solutions to eqs. (d)-(g) are

$$X_n(x) = A_n \exp(-\lambda_n x) + B_n \exp(\lambda_n x) \tag{h}$$

$$Y_n(y) = C_n \sin\lambda_n y + D_n \cos\lambda_n y \tag{i}$$

$$X_0(x) = A_0 x + B_0 \tag{j}$$

$$Y_0(y) = C_0 y + D_0 \tag{k}$$

Corresponding to each value of λ_n there is a temperature solution $\theta_n(x,y).$ Thus

$$\theta_n(x,y) = X_n(x)Y_n(y) \tag{l}$$

and

PROBLEM 3.10 (continued)

$$\theta_0(x,y) = X_0(x)Y_0(y) \tag{m}$$

The complete solution becomes

$$\theta(x,y) = X_0(x)Y_0(y) + \sum_{n=1}^{\infty} X_n(x)Y_n(y) \tag{n}$$

(iv) Application of Boundary Conditions. Boundary condition (1) applied to (i) and (k) gives

$$C_n = C_0 = 0 \tag{o}$$

Applying boundary condition (2) to (i) gives the characteristic equation for λ_n

$$\lambda_n H \tan \lambda_n L = hH/k = Bi \tag{p}$$

where $Bi = hH/k$ is the Biot number. Boundary conditions and (2) applied to (k) gives

$$D_0 = 0$$

Thus, the solution corresponding to $\lambda_n = 0$ vanishes. Boundary condition (3) gives

$$B_n = 0$$

The temperature solution (n) becomes

$$\theta(x,y) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n x) \cos \lambda_n y \tag{q}$$

where $a_n = A_n D_n$. Finally, we apply boundary condition (4) to (q)

$$T_o - T_\infty = \sum_{n=1}^{\infty} a_n \cos \lambda_n y \tag{r}$$

(v) Orthogonality. To determine a_n in equation (r) we apply orthogonality. Note that the characteristic functions $\phi_n(y) = \cos \lambda_n y$ in (r) are solutions to equation (e). Comparing (e) with equation (3.5a) shows that it is a Sturm-Liouville equation with

$$a_1(y) = a_2(y) = 0 \text{ and } a_3(y) = 1$$

Thus eq. (3.6) gives

$$p(y) = w(y) = 1 \text{ and } q(y) = 0$$

Since the boundary conditions at y = 0 and y = H are homogeneous, it follows that the characteristic functions $\phi_n = \cos \lambda_n y$ are orthogonal with respect to $w(y)$ = 1. Multiplying both sides of (r) by $w(y) \cos \lambda_m y \, dy$ and integrating from y = 0 to y = H, we obtain

$$(T_o - T_\infty)\int_0^H w(y) \cos \lambda_m y dy = \int_0^H \left[\sum_{n=1}^{\infty} a_n \cos \lambda_n y \right] w(y) \cos \lambda_m y dy$$

Interchanging the summation with integration, setting $w(y) = 1$ and applying orthogonality, the above gives

$$(T_o - T_\infty)\int_0^H \cos\lambda_n y dy = a_n \int_0^H \cos^2 \lambda_n y dy$$

Evaluating the integrals and solving for a_n

$$a_n = \frac{2(T_o - T_\infty)\sin\lambda_n H}{\lambda_n H + (\sin\lambda_n H)\cos\lambda_n H} \tag{s}$$

Substituting (s) into (r)

$$\frac{T(x,y) - T_\infty}{T_o - T_\infty} = 2\sum_{n=1}^{\infty} \frac{\sin\lambda_n H}{\lambda_n H + (\sin\lambda_n H)\cos\lambda_n H} e^{-\lambda_n x}\cos\lambda_n y \tag{t}$$

(5) Checking. *Dimensional check*: (ii) The arguments of sin, cos and the exponential are dimensionless. (ii) Each term in solution (t) is dimensionless.

Limiting check: If $T_o = T_\infty$, no heat transfer takes place within the plate and consequently the entire plate should be at T_∞. Setting $T_o = T_\infty$ in (t) gives $T(x,y) = T_\infty$.

Boundary conditions check: Conditions (1) and (3) can be readily shown to be satisfied by direct substitution into solution (q).

(6) Comments. (i) By introducing the definition $\theta = T - T_\infty$ it was possible to transform three of the four boundary conditions from non-homogeneous to homogeneous. (ii) The solution is in terms of the Biot number hH/k which appears in the characteristic equation (p). This parameter appears in problems with surface convection. (iii) In the fin model, temperature variation in the y-direction is neglected and a one-dimensional solution is obtained. This approximation is valid for small Biot number compared to unity. That is for

$$Bi = hH/k << 1 \tag{u}$$

Thus the two-dimensional solution (t) should reduce to the one-dimensional fin solution for small Biot number. Numerical solution to the characteristic equation (p) shows that for $Bi << 1$ the first root is $\lambda_1 H << 1$. Thus for $Bi << 1$

$$\lambda_1 H << 1, \quad \sin\lambda_1 L \approx \lambda_1 L, \quad \cos\lambda_1 H \approx \cos\lambda_1 y \approx 1 \tag{v}$$

When this is substituted into (p) we obtain

$$\lambda_1 H \approx \sqrt{hH/k} \tag{w}$$

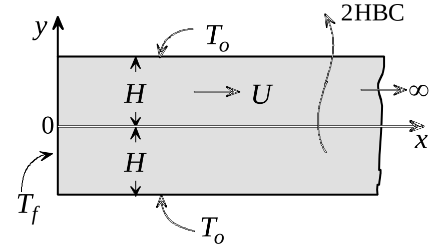
Substituting (v) and (w) into (t) and taking the first term of the series gives

$$T_1(x) - T_\infty = (T_o - T_\infty)e^{-\sqrt{hk/H}\,x} \tag{x}$$

This result is identical to the one-dimensional temperature solution for a semi-infinite fin with surface convection and a specified temperature at the base.

PROBLEM 3.11

A very long plate of thickness 2H leaves a furnace at temperature T_f. The plate is cooled as it moves with velocity U through a liquid tank. Because of the high heat transfer coefficient, the surface of the plate is maintained at temperature T_o. Determine the steady state two-dimensional temperature distribution in the plate.



(1) Observations. (i) This is a two-dimensional steady state conduction problem. (ii) Temperature distribution is symmetrical about the x-axis. (iii) At $x = \infty$ the plate reaches a uniform temperature T_o. (iv) All four boundary conditions are non-homogenous. However, by defining a new variable, $\theta = T - T_o$, three of the four boundary conditions become homogeneous. (v) Temperature distribution is symmetrical about the x-axis. Thus no heat crosses the boundary (x,0).

(2) Origin and Coordinates. The origin and coordinate axes are selected as shown.

(3) Formulation.

(i) Assumptions. (1) Two-dimensional, (2) steady, (3) constant properties, (4) uniform velocity and (5) no energy generation.

(ii) Governing Equations. Introducing the definition $\theta(x,y) = T(x,y) - T_o$, eq. (1.7) gives

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} - 2\beta\frac{\partial\theta}{\partial x} = 0 \qquad \text{(a)}$$

where β is defined as

$$\beta = \frac{\rho c_p U}{2k} \qquad \text{(b)}$$

(iii) Independent Variable with Two Homogeneous Boundary Conditions. The y-direction has two homogeneous conditions in the θ variable.

(iv) Boundary Conditions. Because of symmetry we consider half the plate. The boundary conditions for the upper half are

(1) $\dfrac{\partial\theta(x,0)}{\partial y} = 0$, homogeneous

(2) $\theta(x,H) = 0$, homogeneous

(3) $\theta(\infty, y) = 0$, homogeneous

(4) $\theta(0,y) = T_f - T_o$, non-homogeneous

(4) Solution.

(i) Assumed Product Solution.

Assume a product solution $\theta = X(x)Y(y)$. Substituting into (a), separating variables and setting the resulting equation equal to constant gives

$$\frac{d^2 X_n}{dx^2} - 2\beta \frac{dX_n}{dx} \pm \lambda_n^2 X_n = 0 \tag{c}$$

and

$$\frac{d^2 Y_n}{dy^2} \mp \lambda_n^2 Y_n = 0 \tag{d}$$

(ii) Selecting the Sign of the λ_n^2 Terms. Since the y-variable has two homogeneous conditions, the correct signs in (c) and (d) are

$$\frac{d^2 X_n}{dx^2} - 2\beta \frac{dX_n}{dx} - \lambda_n^2 X_n = 0 \tag{e}$$

and

$$\frac{d^2 Y_n}{dy^2} + \lambda_n^2 Y_n = 0 \tag{f}$$

For $\lambda_n = 0,$ equations (e) and (f) become

$$\frac{d^2 X_0}{dx^2} - 2\beta \frac{dX_0}{dx} = 0 \tag{g}$$

and

$$\frac{d^2 Y_0}{dy^2} = 0 \tag{h}$$

(iii) Solutions to the Ordinary Differential Equations. The solutions to equations (e)-(h) are

$$X_n(x) = [A_n \exp(\beta x + \sqrt{\beta^2 + \lambda_n^2}\, x) + B_n \exp(\beta x - \sqrt{\beta^2 + \lambda_n^2}\, x)] \tag{i}$$

$$Y_n(y) = C_n \sin\lambda_n y + D_n \cos\lambda_n y \tag{j}$$

$$X_0(x) = A_0 \exp(2\beta x) + B_0 \tag{k}$$

and

$$Y_0(y) = C_0 y + D_0 \tag{l}$$

The complete solution becomes

$$\theta(x, y) = X_0(x)Y_0(y) + \sum_{n=1}^{\infty} X_n(x)Y_n(y) \tag{m}$$

(iv) Application of Boundary Conditions. Condition (1) applied to (j) and (l) gives

$$C_n = C_0 = 0$$

Boundary condition (2) applied to (j) gives the characteristic equation

PROBLEM 3.11 (continued)

$$\cos\lambda_n H = 0$$

This gives

$$\lambda_n H = \frac{2n-1}{2}\pi \qquad (n = 1, 2, 3....) \tag{n}$$

Boundary condition (2) applied to (l) gives

$$D_0 = 0$$

Condition (3) gives

$$A_n = 0$$

With $C_0 = D_0 = 0,$ it follows that $X_0 Y_0 = 0.$ Thus equation (m) becomes

$$\theta(x,y) = \sum_{n=1}^{\infty} a_n \left[\exp(\beta - \sqrt{\beta^2 + \lambda_n^2}\,)x\right]\cos\lambda_n y \tag{o}$$

where $a_n = B_n D_n$. Finally, we apply boundary condition (4) to determine a_n

$$T_f - T_o = \sum_{n=1}^{\infty} a_n \cos\lambda_n y \tag{p}$$

(v) Orthogonality. Note that the characteristic functions $\phi_n = \cos\lambda_n y$ are solutions to equation (f). Comparing (f) with equation (3.5a) shows that it is a Sturm-Liouville equation with

$$a_1(y) = a_2(y) = 0 \text{ and } a_3(y) = 1$$

Thus eq. (3.6) gives

$$p(y) = w(y) = 1 \text{ and } q(y) = 0$$

Since the boundary conditions at y = 0 and y = H are homogeneous, it follows that the characteristic functions $\phi_n = \cos\lambda_n y$ are orthogonal with respect to $w(y)$ = 1. Multiplying both sides of (p) by $w(y)\cos\lambda_m y dy$ and integrating from y = 0 to y = H gives

$$(T_f - T_o)\int_0^H w(y)\cos\lambda_m y dy = \int_0^H \left[\sum_{n=1}^{\infty} a_n \cos\lambda_n y\right] w(y)\cos\lambda_m y dy$$

Interchanging summation with integration, noting that $w(y)$ = 1 and invoking orthogonality, the above becomes

$$(T_f - T_o)\int_0^H \cos\lambda_n y dy = a_n \int_0^H \cos^2\lambda_n y dy$$

Evaluating the integrals and solving for a_n

$$a_n = 2(T_f - T_\infty)\frac{\sin\lambda_n H}{\lambda_n H} \tag{q}$$

Substituting into (o)

$$\frac{T(x,y)-T_o}{T_f-T_o}=2\sum_{n=1}^{\infty}\frac{\sin\lambda_n H}{\lambda_n H}\left[\exp(\beta-\sqrt{\beta^2+\lambda_n^2}\,)x\right]\cos\lambda_n y \qquad \text{(r)}$$

(5) Checking. *Dimensional check*: (i) The arguments of sin and cos are dimensionless. (ii) Each term in solution (r) is dimensionless. (iii) Solution (r) requires that units of β be the same as that of λ_n. According to (d), λ_n is measured in (1/m). From the definition of β in (b) we have

$$\beta=\frac{\rho(\text{kg}/\text{m}^3)c_p(\text{J/kg–}^\text{o}\text{C})U(\text{m/s})}{k(\text{W/m-}^\text{o}\text{C})}=(1/\text{m})$$

Limiting check: If $T_f=T_o$, no heat transfer takes place in the plate and consequently the temperature remains uniform. Setting $T_f=T_o$ in (r) gives $T(x,y)=T_o$.

Boundary conditions check: conditions (1), (2) and (3) are readily shown to be satisfied by direct substitution in (o).

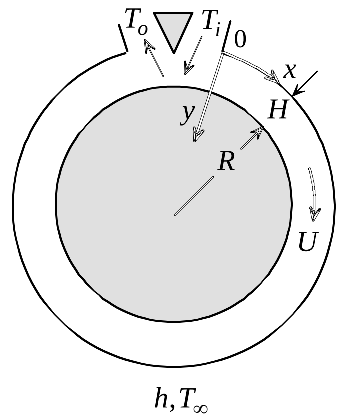
(6) Comments. (i) For a stationary plate ($U=\beta=0$), the solution to this special case is obtained by setting $\beta=0$ in (r)

$$\frac{T(x,y)-T_o}{T_f-T_o}=2\sum_{n=1}^{\infty}\frac{\sin\lambda_n H}{\lambda_n H}\exp(-\lambda_n x)\cos\lambda_n y$$

(ii) The solution depends on two parameters. The first parameter is the Biot number hH/k which appears in the characteristic equation for λ_n. The second is $\beta H=\rho c_p UH/k$ which appears in the exponent of the exponential in equation (r) when the exponent is multiplied and divided by H.

PROBLEM 3.12

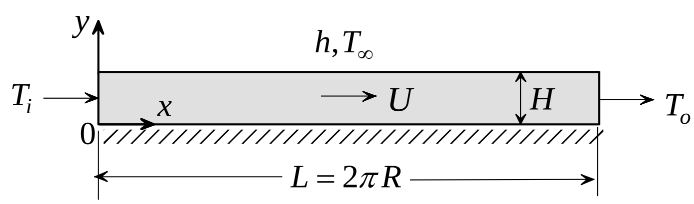
Liquid metal flows through a cylindrical channel of width H and inner radius R. It enters at T_i and is cooled by convection along the outer surface to T_o. The ambient temperature is T_∞ and heat transfer coefficient is h.. The inner surface is insulated. Assume that the liquid metal flows with a uniform velocity U and neglect curvature effect (H/R << 1), determine the steady state temperature of the insulated surface.



(1) Observations. (i) This is a two-dimensional steady state conduction problem. (ii) To determine the temperature of the insulated surface it is necessary to determine the temperature distribution in the moving liquid metal. (iii) The velocity is uniform and inlet and outlet temperatures are specified. (iv) Three boundary conditions are non-homogenous. However, by defining a new variable, $\theta = T - T_\infty$, the y-direction will have two homogeneous conditions.

(2) Origin and Coordinates. The origin and coordinate axes are selected as shown.

(3) Formulation.



(i) Assumptions. (1) Steady state, (2) Two-dimensional, (3) constant properties, (4) uniform velocity, (5) negligible curvature effect (H/R<<1) and (6) no energy generation.

(ii) Governing Equations. Introducing the definition $\theta(x,y) = T(x,y) - T_\infty$, eq. (1.7) gives

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} - 2\beta\frac{\partial\theta}{\partial x} = 0 \tag{a}$$

where β is defined as

$$\beta = \frac{\rho c_p U}{2k} \tag{b}$$

(iii) Independent Variable with Two Homogeneous Boundary Conditions. The y-direction has two homogeneous conditions in the θ variable.

(iv) Boundary Conditions. The boundary conditions for the upper half are

(1) $\dfrac{\partial\theta(x,0)}{\partial y} = 0$, homogeneous

(2) $-k\dfrac{\partial\theta(x,H)}{\partial y} = h\theta(x,H)$, homogeneous

(3) $\theta(0,y) = T_i - T_\infty$, non-homogeneous

(4) $\theta(L,y) = T_o - T_\infty$, non-homogeneous

where L is the length of channel given by

$$L = 2\pi R \tag{c}$$

(4) Solution.

(i) Assumed Product Solution. Assume a product solution $\theta = X(x)Y(y)$. Substituting into (a), separating variables and setting the resulting equation equal to constant gives

$$\frac{d^2 X_n}{dx^2} - 2\beta \frac{dX_n}{dx} \pm \lambda_n^2 X_n = 0 \tag{d}$$

and

$$\frac{d^2 Y_n}{dy^2} \mp \lambda_n^2 Y_n = 0 \tag{e}$$

(ii) Selecting the Sign of the λ_n^2 Terms. Since the y-variable has two homogeneous conditions, the correct signs in (d) and (e) are

$$\frac{d^2 X_n}{dx^2} - 2\beta \frac{dX_n}{dx} - \lambda_n^2 X_n = 0 \tag{f}$$

and

$$\frac{d^2 Y_n}{dy^2} + \lambda_n^2 Y_n = 0 \tag{g}$$

For $\lambda_n = 0,$ equations (f) and (g) become

$$\frac{d^2 X_0}{dx^2} - 2\beta \frac{dX_0}{dx} = 0 \tag{h}$$

and

$$\frac{d^2 Y_0}{dy^2} = 0 \tag{i}$$

(iii) Solutions to the Ordinary Differential Equations. The solutions to equations (f)-(i) are

$$X_n(x) = A_n \exp(\beta x + \sqrt{\beta^2 + \lambda_n^2}\, x) + B_n \exp(\beta x - \sqrt{\beta^2 + \lambda_n^2}\, x) \tag{j}$$

$$Y_n(y) = C_n \sin \lambda_n y + D_n \cos \lambda_n y \tag{k}$$

$$X_0(x) = A_0 \exp(2\beta x) + B_0 \tag{l}$$

and

$$Y_0(y) = C_0 y + D_0 \tag{m}$$

The complete solution is

$$\theta(x, y) = X_0(x) Y_0(y) + \sum_{n=1}^{\infty} X_n(x) Y_n(y) \tag{n}$$

(iv) Application of Boundary Conditions. Condition (1) applied to (k) and (m) gives

PROBLEM 3.12 (continued)

$$C_n = C_0 = 0$$

Boundary condition (2) applied to (k) gives

$$\lambda_n H \tan \lambda_n H = hH / k = Bi \tag{o}$$

where Bi is the Biot number. This is the characteristic equation whose roots give λ_n. Boundary condition (2) applied to (m) gives

$$D_0 = 0$$

With $C_0 = D_0 = 0,$ it follows that $X_0 Y_0 = 0.$ Thus equation (n) becomes

$$\theta(x, y) = \sum_{n=1}^{\infty} \left[a_n e^{(\beta + \sqrt{\beta^2 + \lambda_n^2}\,)x} + b_n e^{(\beta - \sqrt{\beta^2 + \lambda_n^2}\,)x} \right] \cos \lambda_n y \tag{p}$$

where $a_n = A_n D_n$ and $b_n = B_n D_n$. Application of condition (3) and (4) to (p) gives

$$T_i - T_\infty = \sum_{n=1}^{\infty} (a_n + b_n) \cos \lambda_n y \tag{q}$$

and

$$T_o - T_\infty = \sum_{n=1}^{\infty} \left[a_n e^{(\beta + \sqrt{\beta^2 + \lambda_n^2}\,)L} + b_n e^{(\beta - \sqrt{\beta^2 + \lambda_n^2}\,)L} \right] \cos \lambda_n y \tag{r}$$

(v) Orthogonality. Note that the characteristic functions $\phi_n = \cos \lambda_n y$ in (q) and (r) are solutions to equation (g). Comparing (g) with equation (3.5a) shows that it is a Sturm-Liouville equation with

$$a_1(y) = a_2(y) = 0 \text{ and } a_3(y) = 1$$

Thus eq. (3.6) gives

$$p(y) = w(y) = 1 \text{ and } q(y) = 0$$

Since the boundary conditions at $y = 0$ and $y = H$ are homogeneous, it follows that the characteristic functions $\phi_n = \sin \lambda_n y$ are orthogonal with respect to $w(y) = 1$. Multiplying both sides of (q) by $w(y) \sin \lambda_m y dy$ and integrating from $y = 0$ to $y = H$ gives

$$(T_i - T_\infty) \int_0^H w(y) \cos \lambda_m y dy = \int_0^H \left[\sum_{n=1}^{\infty} (a_n + b_n) \cos \lambda_n y \right] w(y) \cos \lambda_m y dy$$

Interchanging summation with integration, noting that $w(y) = 1$ and invoking orthogonality, the above becomes

$$(T_i - T_\infty) \int_0^H \cos \lambda_n y dy = (a_n + b_n) \int_0^H \cos^2 \lambda_n y dy$$

Evaluating the integral and solving for $(a_n + b_n)$

$$(a_n + b_n) = \frac{2(T_i - T_\infty)\sin\lambda_n H}{\lambda_n H + (\sin\lambda_n H)\cos\lambda_n H} \tag{s}$$

Similarly, multiplying both sides of (r) by $w(y)\sin\lambda_m y dy$ and integrating from y = 0 to y = H gives

$$(T_o - T_\infty)\int_0^H w(y)\cos\lambda_m y dy = \int_0^H \sum_{n=1}^{\infty}\left\{a_n e^{(\beta+\sqrt{\beta^2+\lambda_n^2})L} + b_n e^{(\beta-\sqrt{\beta^2+\lambda_n^2})L}\right\} w(y)\cos^2\lambda_m y dy \tag{t}$$

Interchanging summation with integration, noting that $w(y)$ = 1 and invoking orthogonality, the above becomes

$$(T_o - T_\infty)\int_0^H \cos\lambda_n y dy = \left\{a_n e^{(\beta+\sqrt{\beta^2+\lambda_n^2})L} + b_n e^{(\beta-\sqrt{\beta^2+\lambda_n^2})L}\right\}\int_0^H \cos^2\lambda_n y dy$$

Evaluating the integrals and rearranging

$$a_n e^{(\beta+\sqrt{\beta^2+\lambda_n^2})L} + b_n e^{(\beta-\sqrt{\beta^2+\lambda_n^2})L} = \frac{2(T_o - T_\infty)\sin\lambda_n H}{\lambda_n H + (\sin\lambda_n H)\cos\lambda_n H} \tag{u}$$

Equation (s) and (u) are solved for the coefficients a_n and b_n

$$\frac{a_n}{(T_i - T_\infty)} = \left\{\frac{2\sin\lambda_n H}{\lambda_n H + (\sin\lambda_n H)\cos\lambda_n H}\right\}\frac{\left[(T_o - T_\infty)/(T_i - T_\infty)\right] - e^{(\beta-\sqrt{\beta^2+\lambda_n^2})L}}{e^{(\beta+\sqrt{\beta^2+\lambda_n^2})L} - e^{(\beta-\sqrt{\beta^2+\lambda_n^2})L}} \tag{v}$$

and

$$\frac{b_n}{T_i - T_\infty} = \left\{\frac{2\sin\lambda_n H}{\lambda_n H + (\sin\lambda_n H)\cos\lambda_n H}\right\}\left\{1 - \frac{\left[(T_o - T_\infty)/(T_i - T_\infty)\right] - e^{(\beta-\sqrt{\beta^2+\lambda_n^2})L}}{e^{(\beta+\sqrt{\beta^2+\lambda_n^2})L} - e^{(\beta-\sqrt{\beta^2+\lambda_n^2})L}}\right\} \tag{w}$$

Rewriting solution (r) in dimensionless form gives

$$\frac{T(x,y) - T_\infty}{T_i - T_\infty} = \sum_{n=1}^{\infty}\left[\frac{a_n}{T_i - T_\infty} e^{(\beta+\sqrt{\beta^2+\lambda_n^2})x} + \frac{b_n}{T_i - T_\infty} e^{(\beta-\sqrt{\beta^2+\lambda_n^2})x}\right]\cos\lambda_n y \tag{x}$$

where $a_n/(T_i - T_\infty)$ and $b_n/(T_i - T_\infty)$ are given in (v) and (w). The temperature of the insulated surface is obtained by evaluating (x) at y = 0

$$\frac{T(x,0) - T_\infty}{T_i - T_\infty} = \sum_{n=1}^{\infty}\left[\frac{a_n}{T_i - T_\infty} e^{(\beta+\sqrt{\beta^2+\lambda_n^2})x} + \frac{b_n}{T_i - T_\infty} e^{(\beta-\sqrt{\beta^2+\lambda_n^2})x}\right] \tag{y}$$

(5) Checking. *Dimensional check*: (i) The arguments of the sin, cos and exponentials are dimensionless. (ii) Solution (x) requires that units of β be the same as that of λ_n. According to (g), λ_n is measured in (1/m). From the definition of β in (b) we have

$$\beta = \frac{\rho(\text{kg/m}^3)c_p(\text{J/kg–}^\text{o}\text{C})U(\text{m/s})}{k(\text{W/m-}^\text{o}\text{C})} = (1/\text{m})$$

Limiting check: If $T_i = T_o = T_\infty$, no heat transfer takes place and consequently the temperature in the entire region remains uniform. Setting $T_i = T_o = T_\infty$ in (v) and (w) gives $a_n = b_n = 0$. When this is substituted into (p) gives $T(x, y) = T_\infty$.

Boundary conditions check: Solution (p) can be readily shown to satisfy condition (1).

(6) Comments. (i) For a stationary material ($U = \beta = 0$), the solution to this special case is obtained by setting $\beta = 0$ in (p)

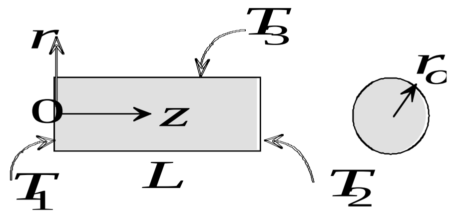
$$\theta(x, y) = \sum_{n=1}^{\infty} \left[a_n e^{\lambda_n x} + b_n e^{-\lambda_n x} \right] \cos \lambda_n y$$

where a_n and b_n are obtained by setting $\beta = 0$ in (v) and (w).

(ii) Examination of equations (o), (v), (w) and (x) shows that the solution depends on four parameters. The first parameter is the Biot number hH/k which appears in the characteristic equation for λ_n. The second is $(T_o - T_\infty)/(T_i - T_\infty)$ shown in (v) and (w). The third parameter is $\beta H = \rho c_p UH/k$ and the fourth is *L/H* which appears in $\lambda_n L = \lambda_n H(L/H)$.

PROBLEM 3.13

A solid cylinder of radius r_o and length L is maintained at T_1 at one end. The other end is maintained at T_2 and the cylindrical surface at T_3. Determine the steady state two-dimensional temperature distribution in the cylinder.



(1) Observations. (i) This is a steady state two-dimensional conduction problem. (ii) There are three non-homogeneous boundary conditions. Defining a new temperature variable, $\theta = T - T_3$, makes the boundary condition at the cylindrical surface homogeneous. This gives the radial direction two homogeneous conditions. (iii) A cylindrical coordinate system should be used.

(2) Origin and Coordinates. The origin and coordinate axes are selected as shown.

(3) Formulation.

(i) Assumptions. (1) Two-dimensional, (2) steady, (3) constant conductivity and (4) no energy generation.

(ii) Governing Equations. Defining $\theta = T - T_3$ and applying the above assumptions to eq. (1.11) gives

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} = 0 \qquad \text{(a)}$$

(iii) Independent Variable with Two Homogeneous Boundary Conditions. The r-variable has two homogeneous conditions.

(iv) Boundary Conditions. The boundary conditions on $\theta(r,z)$ are

(1) $\dfrac{\partial\theta(0,z)}{\partial r} = 0$, or $\theta(0,z) = \text{finite}$, homogeneous

(2) $\theta(r_o, z) = 0$, homogeneous

(3) $\theta(r,0) = (T_1 - T_3)$, non-homogeneous

(4) $\theta(r,L) = (T_2 - T_3)$, non-homogeneous

(4) Solution.

(i) Assumed Product Solution. To solve equation (a) we assume a product solution. Let

$$\theta(r,z) = R(r)Z(z) \qquad \text{(b)}$$

Substituting (b) into (a), separating variables and setting the resulting two equations equal to a constant, $\pm\lambda_k^2$, we obtain

$$\frac{d^2 Z_k}{dz^2} \pm \lambda_k^2 Z_k = 0 \qquad \text{(c)}$$

and

$$r^2\frac{d^2R_k}{dr^2}+r\frac{dR_k}{dr}\mp\lambda_k^2 r^2R_k=0 \tag{d}$$

(ii) Selecting the Sign of the λ_k^2 Terms. Since the r-variable has two homogeneous boundary conditions, the plus sign is selected in (d). Thus (c) and (d) become

$$\frac{d^2Z_k}{dz^2}-\lambda_k^2Z_k=0 \tag{e}$$

and

$$r^2\frac{d^2R_k}{dr^2}+r\frac{dR_k}{dr}+\lambda_k^2 r^2R_k=0 \tag{f}$$

For the special case of $\lambda_k=0$ the above equations take the form

$$\frac{d^2Z_0}{dz^2}=0 \tag{g}$$

and

$$r\frac{d^2R_0}{dr^2}+\frac{dR_0}{dr}=0 \tag{h}$$

(iii) Solutions to the Ordinary Differential Equations. The solution to (e) is

$$Z_k(z)=A_k\sinh\lambda_k z+B_k\cosh\lambda_k z \tag{i}$$

Equation (f) is a Bessel differential equation. Comparing (f) with eq. (2.26) shows that $A=B=n=0$, C = 1 and $D=\lambda_k$. Thus the solution to (f) is given by eq. (2.27)

$$R_k(r)=C_kJ_0(\lambda_k r)+D_kY_0(\lambda_k r) \tag{j}$$

The solutions to (g) and (h) are

$$Z_0(z)=A_0z+B_0 \tag{k}$$

and

$$R_0(r)=C_0\ln r+D_0 \tag{l}$$

The complete solution to $\theta(r,z)$ becomes

$$\theta(r,z)=R_0(r)Z_0(z)+\sum_{k=1}^{\infty}R_k(r)Z_k(z) \tag{m}$$

(iv) Application of Boundary Conditions. Applying boundary condition (1) to equation (j) and (l) gives

$$D_k=C_0=0$$

Condition (2) applied to (j) gives the characteristic equation for λ_k

$$J_0(\lambda_k r_o)=0 \tag{n}$$

Condition (2) applied to (l) gives

$$D_0 = 0$$

With $C_0 = D_0 = 0$ the solution $R_0 Z_0$ vanishes and the temperature solution (m) becomes

$$\theta(r,z) = \sum_{k=1}^{\infty}\left[a_k \sinh\lambda_k z + b_k \cosh\lambda_k z\right] J_0(\lambda_k r) \tag{o}$$

where $a_k = A_k C_k$ and $b_k = B_k C_k$. Application of boundary conditions (3) and (4) to (o) gives

$$T_1 - T_3 = \sum_{k=1}^{\infty} b_k\, J_0(\lambda_k r) \tag{p}$$

and

$$T_2 - T_3 = \sum_{k=1}^{\infty}\left[a_k \sinh\lambda_k L + b_k \cosh\lambda_k L\right] J_0(\lambda_k r) \tag{q}$$

(v) Orthogonality. Note that the characteristic functions $J_0(\lambda_k r)$ in (p) and (q) are solutions to equation (f). Comparing (f) with eq. (3.5a) shows that it is a Sturm-Liouville equation with

$$a_1(r) = 1/r,\ \ a_2(r) = 0 \ \text{ and } \ a_3(r) = 1$$

Thus eq. (3.6) gives

$$p(r) = w(r) = r \ \text{ and } \ q(r) = 0$$

Since the boundary conditions at $r = 0$ and $r = r_o$ are homogeneous, it follows that the characteristic functions $J_0(\lambda_k r)$ are orthogonal with respect to $w(r) = r$. Multiplying both sides of (p) by $J_0(\lambda_i r) w(r) dr$ and integrating from $r = 0$ to $r = r_o$

$$(T_1 - T_3)\int_0^{r_o} J_0(\lambda_i r) w(r) dr = \int_0^{r_o}\left[\sum_{k=1}^{\infty} b_k\, J_0(\lambda_k r)\right] J_0(\lambda_i r) w(r) dr$$

Interchanging summation with integration, noting that $w(r) = r$ and invoking orthogonality, eq. (3.7), the above gives

$$(T_1 - T_3)\int_0^{r_o} J_0(\lambda_k r) r dr = b_k \int_0^{r_o} J_0^2(\lambda_k r) r dr$$

Evaluating the integrals using Appendix B and Table 3.1 gives

$$b_k = \frac{2(T_1 - T_3)}{\lambda_k r_o J_1(\lambda_k r_o)} \tag{r}$$

Similarly, multiplying both sides of (q) by $J_0(\lambda_i r) w(r) dr$ and integrating from $r = 0$ to $r = r_o$

$$(T_2 - T_3)\int_0^{r_o} J_0(\lambda_i r) w(r) dr = \int_0^{r_o}\left[\sum_{k=1}^{\infty}\left\{a_k \sinh\lambda_k L + b_k \cosh\lambda_k L\right\} J_0(\lambda_k r)\right] J_0(\lambda_i r) w(r) dr$$

PROBLEM 3.13 (continued)

Interchanging summation with integration, noting that $w(r) = r$ and invoking orthogonality, eq. (3.7), the above gives

$$(T_2 - T_3)\int_0^{r_o} J_0(\lambda_k r) w(r) dr = (a_k \sinh\lambda_k L + b_k \cosh\lambda_k L)\int_0^{r_o} J_0^2(\lambda_k r) r dr$$

Evaluating the integrals using Appendix B and Table 3.1

$$(a_k \sinh\lambda_k L + b_k \cosh\lambda_k L) = \frac{2(T_1 - T_3)}{\lambda_k r_o J_1(\lambda_k r_o)} \tag{s}$$

Solving for a_k and using (r) to eliminate b_k

$$a_k = \frac{2}{(\sinh\lambda_k L)\lambda_k r_o J_1(\lambda_k r_o)}\left[(T_2 - T_3) - (T_1 - T_3)\cosh\lambda_k L\right] \tag{t}$$

Substituting (r) and (t) into (o) gives the temperature solution in the cylinder

$$\frac{T(r,z) - T_3}{(T_1 - T_3)} = 2\sum_{k=1}^{\infty}\frac{1}{\lambda_k r_o}\left\{\left[\frac{(T_2 - T_3)}{(T_1 - T_3)} - \cosh\lambda_k L\right]\frac{\sinh\lambda_k z}{\sinh\lambda_k L} + \cosh\lambda_k z\right\}\frac{J_0(\lambda_k r)}{J_1(\lambda_k r_o)} \tag{u}$$

(5) Checking. *Dimensional checks*: (i) The arguments of sinh, cosh, J_0 and J_1 are dimensionless. (ii) Each term in (u) is dimensionless.

Limiting check: If $T_1 = T_2 = T_3$, no heat transfer can take place and the entire cylinder should be at uniform temperature. Setting $T_1 = T_2 = T_3$ in (u) gives $T(r,z) = T_\infty$.

Boundary conditions check: Setting r = 0 and $r = r_o$ in solution (u) shows that boundary conditions (1) and (2) are satisfied.

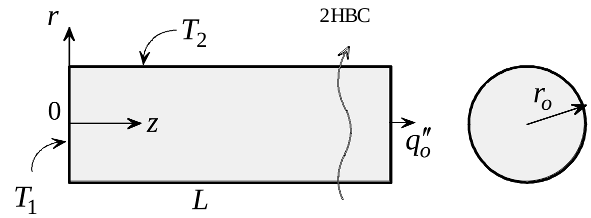
(6) Comments. (i) An alternate approach is to assume a solution of the form

$$T(r,z) = \psi(r,z) + \phi(z)$$

The function $\phi(z)$ can be assigned two non-homogeneous boundary conditions in the z-direction leaving $\psi(r,z)$ with two homogeneous conditions in the z-direction and one in the r-direction. Another approach is to apply the method of superposition by decomposing the problem into two simpler problems each having one non-homogeneous boundary condition. (ii) The solution depends on two dimensionless parameter: $(T_2 - T_3)/(T_1 - T_3)$ and L/r_o.

PROBLEM 3.14

A solid cylinder of radius r_o and length L is maintained at T_1 at one end and is cooled with uniform flux q_o'' at the other end. The cylindrical surface is maintained at a uniform temperature T_2. Determine the steady state two-dimensional temperature distribution in the cylinder.



(1) Observations. (i) This is a steady state two-dimensional conduction problem. (ii) There are three non-homogeneous boundary conditions. Defining a new temperature variable, $\theta = T - T_2$ makes the boundary condition at the cylindrical surface homogeneous. This gives the radial direction two homogeneous conditions. (iii) Use a cylindrical coordinate system.

(2) Origin and Coordinates. The origin and coordinate axes are selected as shown.

(3) Formulation.

(i) Assumptions. (1) Two-dimensional, (2) steady, (3) constant conductivity and (4) no energy generation.

(ii) Governing Equations. Defining $\theta = T - T_2$ and applying the above assumptions to eq. (1.11) gives

$$\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{\partial z^2} = 0 \tag{a}$$

(iii) Independent Variable with Two Homogeneous Boundary Conditions. The r-variable has two homogeneous conditions.

(iv) Boundary Conditions. The boundary conditions on $\theta(r,z)$ are

(1) $\dfrac{\partial \theta(0,z)}{\partial r} = 0$, or $\theta(0,z) = \text{finite}$, homogeneous

(2) $\theta(r_o, z) = 0$, homogeneous

(3) $\theta(r,0) = (T_1 - T_2)$, non-homogeneous

(4) $-k\dfrac{\partial \theta(r,L)}{\partial z} = q_o''$ non-homogeneous

(4) Solution.

(i) Assumed Product Solution. To solve equation (a) we assume a product solution. Let

$$\theta(r,z) = R(r)Z(z) \tag{b}$$

Substituting (b) into (a), separating variables and setting the resulting two equations equal to a constant, $\pm\lambda_k^2$, we obtain

$$\frac{d^2 Z_k}{d z^2} \pm \lambda_k^2 Z_k = 0 \tag{c}$$

and

$$r^2 \frac{d^2R_k}{dr^2} + r\frac{dR_k}{dr} \mp \lambda_k^2 r^2 R_k = 0 \qquad \text{(d)}$$

(ii) Selecting the Sign of the λ_k^2 Terms. Since the r-variable has two homogeneous boundary conditions, the plus sign is selected in (d). Thus (c) and (d) become

$$\frac{d^2Z_k}{dz^2} - \lambda_k^2 Z_k = 0 \qquad \text{(e)}$$

and

$$r^2 \frac{d^2R_k}{dr^2} + r\frac{dR_k}{dr} + \lambda_k^2 r^2 R_k = 0 \qquad \text{(f)}$$

For the special case of $\lambda_k = 0$ the above equations take the form

$$\frac{d^2Z_0}{dz^2} = 0 \qquad \text{(g)}$$

and

$$r\frac{d^2R_0}{dr^2} + \frac{dR_0}{dr} = 0 \qquad \text{(h)}$$

(iii) Solutions to the Ordinary Differential Equations. The solution to (e) is

$$Z_k(z) = A_k \sinh\lambda_k z + B_k \cosh\lambda_k z \qquad \text{(i)}$$

Equation (f) is a Bessel differential equation. Comparing (f) with eq. (2.26) shows that $A = B = n = 0$, $C = 1$ and $D = \lambda_k$. Thus the solution to (f) is given by eq. (2.27)

$$R_k(r) = C_k J_0(\lambda_k r) + D_k Y_0(\lambda_k r) \qquad \text{(j)}$$

The solutions to (g) and (h) are

$$Z_0(z) = A_0 z + B_0 \qquad \text{(k)}$$

and

$$R_0(r) = C_0 \ln r + D_0 \qquad \text{(l)}$$

The complete solution to $\theta(r,z)$ becomes

$$\theta(r,z) = R_0(r)Z_0(z) + \sum_{k=1}^{\infty} R_k(r)Z_k(z) \qquad \text{(m)}$$

(iv) Application of Boundary Conditions. Applying boundary condition (1) to equation (j) and (l) gives

$$D_k = C_0 = 0$$

Condition (2) applied to (j) gives the characteristic equation for λ_k

$$J_0(\lambda_k r_o) = 0 \qquad \text{(n)}$$

Condition (2) applied to (l) gives

$$D_0 = 0$$

With $C_0 = D_0 = 0$ the solution $R_0 Z_0$ vanishes and the temperature solution (m) becomes

$$\theta(r,z) = \sum_{k=1}^{\infty}\left[a_k \sinh\lambda_k z + b_k \cosh\lambda_k z\right]J_0(\lambda_k r) \tag{o}$$

where $a_k = A_k C_k$ and $b_k = B_k C_k$. Application of boundary conditions (3) and (4) to (o) gives

$$T_1 - T_2 = \sum_{k=1}^{\infty} b_k \, J_0(\lambda_k r) \tag{p}$$

and

$$q_o'' = -k\sum_{k=1}^{\infty}\lambda_k\left[a_k \cosh\lambda_k L + b_k \sinh\lambda_k L\right]J_0(\lambda_k r) \tag{q}$$

(v) Orthogonality. Note that the characteristic functions $J_0(\lambda_k r)$ in (p) and (q) are solutions to equation (f). Comparing (f) with eq. (3.5a) shows that it is a Sturm-Liouville equation with

$$a_1(r) = 1/r,\ \ a_2(r) = 0 \ \text{ and } \ a_3(r) = 1$$

Thus eq. (3.6) gives

$$p(r) = w(r) = r \ \text{ and } \ q(r) = 0$$

Since the boundary conditions at $r = 0$ and $r = r_o$ are homogeneous, it follows that the characteristic functions $J_0(\lambda_k r)$ are orthogonal with respect to $w(r) = r$. Multiplying both sides of (p) by $J_0(\lambda_i r)w(r)dr$ and integrating from $r = 0$ to $r = r_o$

$$(T_1 - T_2)\int_0^{r_o} J_0(\lambda_i r)w(r)dr = \int_0^{r_o}\left[\sum_{k=1}^{\infty} b_k \, J_0(\lambda_k r)\right]J_0(\lambda_i r)w(r)dr$$

Interchanging summation with integration, noting that $w(r) = r$ and invoking orthogonality, eq. (3.7), the above gives

$$(T_1 - T_2)\int_0^{r_o} J_0(\lambda_k r)rdr = b_k\int_0^{r_o} J_0^2(\lambda_k r)rdr$$

Evaluating the integrals using Appendix B and Table 3.1 the above gives b_k

$$b_k = \frac{2(T_1 - T_2)}{\lambda_k r_o J_1(\lambda_k r_o)} \tag{r}$$

Similarly, multiplying both sides of (q) by $J_0(\lambda_i r)w(r)dr$ and integrating from $r = 0$ to $r = r_o$

$$q_o''\int_0^{r_o} J_0(\lambda_i r)w(r)dr = -k\int_0^{r_o}\left[\sum_{k=1}^{\infty}\lambda_k\{a_k \cosh\lambda_k L + b_k \sinh\lambda_k L\}J_0(\lambda_k r)\right]J_0(\lambda_i r)w(r)dr$$

Interchanging summation with integration, noting that $w(r) = r$ and invoking orthogonality, eq. (3.7), the above gives

$$q_o''\int_0^{r_o} J_0(\lambda_k r) r dr = -k\lambda_k (a_k \cosh\lambda_k L + b_k \sinh\lambda_k L)\int_0^{r_o} J_0^2(\lambda_k r) r dr$$

Evaluating the integrals using Appendix B and Table 3.1 the above gives

$$(a_k \cosh\lambda_k L + b_k \sinh\lambda_k L) = -2\frac{(q_o'' r_o / k)}{(\lambda_k r_o)^2 J_1(\lambda_k r_o)} \tag{s}$$

Solving for a_k and using (r) to eliminate b_k

$$a_k = \frac{-2}{(\cosh\lambda_k L)(\lambda_k r_o) J_1(\lambda_k r_o)}\left[\frac{(q'' r_o / k)}{\lambda_k r_o} + (T_1 - T_2)\sinh\lambda_k L\right] \tag{t}$$

substituting (s) and (t) into (o) and rearranging the result

$$\frac{T(r,z) - T_2}{T_2 - T_1} = \sum_{k=1}^{\infty}\frac{2}{(\lambda_k r_o)}\left\{\left[\frac{(q'' r_o / k)}{\lambda_k r_o (T_2 - T_1)} + \sinh\lambda_k L\right]\frac{\sinh\lambda_k z}{\cosh\lambda_k L} + \cosh\lambda_k z\right\}\frac{J_0(\lambda_k r)}{J_1(\lambda_k r_o)} \tag{u}$$

(5) Checking. *Dimensional check*: (i) The arguments of sinh, cosh, J_0 and J_1 are dimensionless. (ii) Each term in (u) is dimensionless.

Limiting check: If $T_1 = T_2$, and $q_o'' = 0$ no heat transfer can take place and the entire cylinder should be at uniform temperature. Setting $T_1 = T_2$ in (u) gives $T(r,z) = T_2$.

Boundary conditions check: Setting $r = 0$ and $r = r_o$ in solution (u) shows that boundary conditions (1) and (2) are satisfied.

(6) Comments. (i) One of parameters which appears in the solution is $q_o'' r_o / k(T_2 - T_1)$. In addition, roots of (n) give $\lambda_k r_o$. However, the argument $\lambda_k L = \lambda_k r_o (L / r_o)$ introduces a third parameter which is (L / r_o). (ii) An alternate approach is to assume a solution of the form

$$T(r,z) = \psi(r,z) + \phi(z)$$

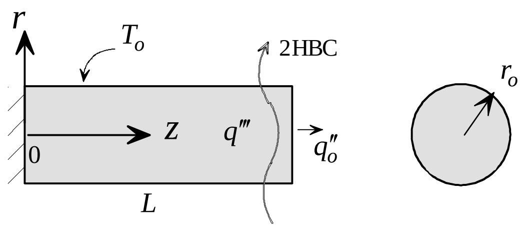
The function $\phi(z)$ can be assigned two non-homogeneous boundary conditions in the z-direction leaving $\psi(r,z)$ with two homogeneous conditions in the z-direction and one in the r-direction. Another approach is to apply the method of superposition by decomposing the problem into two simpler problems each having one non-homogeneous boundary condition.

PROBLEM 3.15

The volumetric heat generation rate in a solid cylinder of radius r_o and length L varies along the radius according to

$$q''' = q_o''' \, r$$

where q_o''' is constant. One end is insulated while the other end is cooled with uniform flux q_o''. The cylindrical surface is maintained at uniform temperature T_o. Determine the steady state temperature of the insulated surface.



(1) Observations. (i) This is a steady state two-dimensional conduction problem. (ii) There are two non-homogeneous boundary conditions. (iii) Energy generation makes the heat equation non-homogeneous. (iv) A cylindrical coordinate system should be used.

(2) Origin and Coordinates. The origin and coordinate axes are selected as shown.

(3) Formulation.

(i) Assumptions. (1) Two-dimensional, (2) steady and (3) constant conductivity.

(ii) Governing Equations. Applying the above assumptions to eq. (1.11) gives

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{q'''}{k} = 0$$

Energy generation q''' is given by

$$q''' = q_o''' r$$

where q_o''' is constant. Substituting into the above heat equation gives

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{q_o'''}{k} r = 0 \tag{a}$$

(iii) Independent Variable with Two Homogeneous Boundary Conditions. Neither variable has two homogeneous conditions. However, the problem can be decomposed into two problems one of which has two homogeneous conditions in the r-direction.

(iv) Boundary Conditions. The boundary conditions are

(1) $\dfrac{\partial T(0,z)}{\partial r} = 0$, or $\theta(0,z) = \text{finite}$, homogeneous

(2) $T(r_o, z) = T_o$, non-homogeneous

(3) $\dfrac{\partial T(r,0)}{\partial z} = 0$, homogeneous

(5) $-k\dfrac{\partial T(r,L)}{\partial z} = q_o''$ non-homogeneous

(4) Solution. Equation (a) is non-homogeneous due to the heat generation term. Thus we can not proceed directly with the application of the separation of variables method. Instead, we assume a solution of the form

$$T(r,z) = \psi(r,z) + \phi(r) \tag{b}$$

Note that $\psi(r,z)$ depends on two variables while $\phi(r)$ depends on a single variable r. the function $\phi(r)$ is introduced to satisfy the non-homogeneous part of equation (a) which also depends on the variable r. Substituting (b) into (a)

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} + \frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + \frac{q_o'''}{k}r = 0 \tag{c}$$

The next step is to split (c) into two equations, one for $\psi(r,z)$ and one for $\phi(r)$. We let

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = 0 \tag{d}$$

Thus

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + \frac{q_o'''}{k}r = 0 \tag{e}$$

Boundary conditions on (d) and (e) are obtained by substituting (b) into the four boundary conditions on T. In this process the function $\psi(r,z)$ is assigned homogeneous conditions whenever possible. Using (b) and condition (1)

$$\frac{\partial\psi(0,z)}{\partial r} + \frac{d\phi(0)}{dr} = 0$$

Let

$$\frac{\partial\psi(0,z)}{\partial r} = 0 \tag{d-1}$$

Thus

$$\frac{d\phi(0)}{dr} = 0 \tag{e-1}$$

Boundary condition (2) and (b) give

$$\psi(r_o,z) + \phi(r_o) = T_o$$

Let

$$\psi(r_o,z) = 0 \tag{d-2}$$

Thus

$$\phi(r_o) = T_o \tag{e-2}$$

Boundary condition (3) and (b) give

$$\frac{\partial\psi(r,0)}{\partial z} = 0 \tag{d-3}$$

Boundary condition (4) and (b) give

$$-k\frac{\partial\psi(r,L)}{\partial z} = q_o'' \tag{d-4}$$

Thus non-homogeneous equation (a) is replaced by two equations: (d) which is a homogeneous partial differential equation and (e) which is a non-homogeneous ordinary differential equation. Equation (d) has three homogeneous conditions, (d-1), (d-2) and (d-3) and one non-homogeneous condition (d-4). The solution to (e) is

$$\phi(r) = -\frac{q_o'''}{9k} r^3 + E \ln r + F \tag{f}$$

where E and F are constants of integration. Boundary conditions (e-1) and (e-2) give E and F. The above solution becomes

$$\phi(r) = \frac{q_o'''}{9k}(r_o^3 - r^3) + T_o \tag{g}$$

(i) Assumed Product Solution. To solve equation (d) we assume a product solution. Let

$$\psi(r,z) = R(r)Z(z) \tag{h}$$

Substituting (h) into (d), separating variables and setting the resulting two equations equal to a constant, $\pm\lambda_k^2$, we obtain

$$\frac{d^2 Z_k}{d z^2} \pm \lambda_k^2 Z_k = 0 \tag{i}$$

and

$$r^2 \frac{d^2 R_k}{d r^2} + r\frac{dR_k}{dr} \mp \lambda_k^2 r^2 R_k = 0 \tag{j}$$

(ii) Selecting the Sign of the λ_k^2 Terms. Since the r-variable has two homogeneous boundary conditions, the plus sign is selected in (j). Thus (i) and (j) become

$$\frac{d^2 Z_k}{d z^2} - \lambda_k^2 Z_k = 0 \tag{k}$$

and

$$r^2 \frac{d^2 R_k}{d r^2} + r\frac{dR_k}{dr} + \lambda_k^2 r^2 R_k = 0 \tag{l}$$

For the special case of $\lambda_k = 0$ the above equations take the form

$$\frac{d^2 Z_0}{d z^2} = 0 \tag{m}$$

and

$$r\frac{d^2 R_0}{d r^2} + \frac{dR_0}{dr} = 0 \tag{n}$$

(iii) Solutions to the Ordinary Differential Equations. The solution to (k) is

$$Z_k(z) = A_k \sinh\lambda_k z + B_k \cosh\lambda_k z \tag{o}$$

PROBLEM 3.15 (continued)

Equation (l) is a Bessel differential equation. Comparing (l) with eq. (2.26) shows that $A = B = n = 0$, $C = 1$ and $D = \lambda_k$. Thus the solution to (l) is given by eq. (2.27)

$$R_k(r) = C_k J_0(\lambda_k r) + D_k Y_0(\lambda_k r) \tag{p}$$

The solutions to (m) and (n) are

$$Z_0(z) = A_0 z + B_0 \tag{q}$$

and

$$R_0(r) = C_0 \ln r + D_0 \tag{r}$$

The complete solution to $\psi(r,z)$ becomes

$$\psi(r,z) = R_0(r)Z_0(z) + \sum_{k=1}^{\infty} R_k(r)Z_k(z) \tag{s}$$

(iv) Application of Boundary Conditions. Applying boundary condition (d-1) to equation (p) and (r) gives

$$D_k = C_0 = 0$$

Condition (2) applied to (p) gives the characteristic equation for λ_k

$$J_0(\lambda_k r_o) = 0 \tag{t}$$

Condition (2) applied to (r) gives

$$D_0 = 0$$

With $C_0 = D_0 = 0$ the solution $R_0 Z_0$ vanishes. Boundary condition (d-3) gives

$$A_k = 0$$

Substituting into (s) gives

$$\psi(r,z) = \sum_{k=1}^{\infty} (b_k \cosh\lambda_k z)\, J_0(\lambda_k r) \tag{u}$$

where $b_k = B_k C_k$. Application of boundary conditions (4) to (u) gives

$$q_o'' = -k \sum_{k=1}^{\infty} b_k (\sinh\, \lambda_k L)\, J_0(\lambda_k r) \tag{v}$$

(v) Orthogonality. Note that the characteristic functions $J_0(\lambda_k r)$ in (v) are solutions to equation (l). Comparing (l) with eq. (3.5a) shows that it is a Sturm-Liouville equation with

$$a_1(r) = 1/r,\ \ a_2(r) = 0 \ \text{ and } \ a_3(r) = 1$$

Thus eq. (3.6) gives

$$p(r) = w(r) = r \ \text{ and } \ q(r) = 0$$

Since the boundary conditions at $r = 0$ and $r = r_o$ are homogeneous, it follows that the characteristic functions $J_0(\lambda_k r)$ are orthogonal with respect to $w(r) = r$. Multiplying both sides of (v) by $J_0(\lambda_i r)w(r)dr$ and integrating from $r = 0$ to $r = r_o$

PROBLEM 3.15 (continued)

$$q_o'' \int_0^{r_o} J_0(\lambda_i r)w(r)dr = -k\int_0^{r_o} \left[\sum_{k=1}^{\infty} b_k \,(\sinh\ \lambda_k L) J_0(\lambda_k r) \right] J_0(\lambda_i r)w(r)dr$$

Interchanging summation with integration, noting that $w(r) = r$ and invoking orthogonality, eq. (3.7), the above gives

$$q_o'' \int_0^{r_o} J_0(\lambda_k r)rdr = -kb_k(\sinh\ \lambda_k L)\int_0^{r_o} J_0^2(\lambda_k r)rdr$$

Evaluating the integrals using Appendix B and Table 3.1 the above gives b_k

$$b_k = \frac{-2(q_o'' r_o / k)}{(\lambda_k r_o)^2 (\sinh\ \lambda_k L)\ J_1(\lambda_k r_o)} \tag{w}$$

Substituting (w) into (u) gives the solution $\psi(r,z)$

$$\psi(r,z) = -2(q_o'' r_o / k)\sum_{k=1}^{\infty} \frac{1}{(\lambda_k r_o)^2} \frac{\cosh\lambda_k z}{\sinh\lambda_k L} \frac{J_0(\lambda_k r)}{J_1(\lambda_k r_o)} \tag{x}$$

The complete solution to the temperature distribution in the cylinder is obtained by substituting (g) and (x) into (b) and rearranging the result

$$\frac{T(r,z) - T_o}{\dfrac{q_o''' r_o^3}{k}} = \frac{1}{9}\left[1 - (r/r_o)^3\right] - \frac{2q_o''}{q_o''' r_o^2} \sum_{k=1}^{\infty} \frac{1}{(\lambda_k r_o)^2} \frac{\cosh\lambda_k z}{\sinh\lambda_k L} \frac{J_0(\lambda_k r)}{J_1(\lambda_k r_o)} \tag{y}$$

(5) Checking. *Dimensional checks*: (i) The arguments of sinh, cosh, J_0 and J_1 are dimensionless. (ii) Each term in (y) is dimensionless.

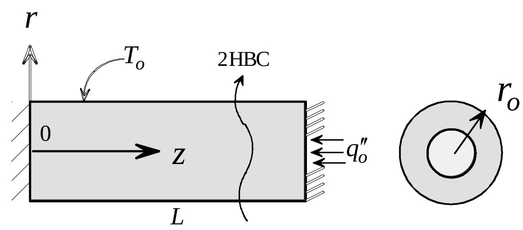
Limiting check: If $T_1 = T_2$ and $q_o''' = q_o'' = 0$ no heat transfer can take place and the entire cylinder should be at uniform a temperature T_o. Setting $q_o''' = q_o'' = 0$ in (y) gives $T(r,z) = T_o$.

Boundary conditions check: Setting $r = 0$ and $r = r_o$ in solution (y) shows that boundary conditions (1) and (2) are satisfied.

(6) Comments. (i) The key to solving this problem is to recognize that the non-homogeneous term in differential equation (a) is a function of *r*. This suggests that the function ϕ in the assumed solution (b) should also be a function of *r*. (ii) Two parameters characterize the solution: $\dfrac{q_o''}{q_o''' r_o^2}$ and $\dfrac{L}{r_o}$.

PROBLEM 3.16

Heat is added at a uniform flux q_o'' over a circular area of radius b at one end of a solid cylinder of radius r_o and length L. The remaining surface of the heated end and the opposite surface are insulated. The cylindrical surface is maintained at uniform temperature T_o. Determine the steady state two-dimensional temperature distribution.



(1) Observations. (i) This is a steady state two-dimensional conduction problem. (ii) There are two non-homogeneous boundary conditions. Defining a new temperature variable, $\theta = T - T_o$, makes the boundary condition at the cylindrical surface homogeneous. This gives the radial direction two homogeneous conditions. (iii) Part of boundary (r,L) is heated while the rest is insulated. Thus the heat flux along this boundary is non-uniform. (iv) A cylindrical coordinate system should be used.

(2) Origin and Coordinates. The origin and coordinate axes are selected as shown.

(3) Formulation.

(i) Assumptions. (1) Two-dimensional, (2) steady, (3) constant conductivity and (4) no energy generation.

(ii) Governing Equations. Defining $\theta = T - T_o$ and applying the above assumptions to eq. (1.11) gives

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} = 0 \tag{a}$$

(iii) Independent Variable with Two Homogeneous Boundary Conditions. The r-variable has two homogeneous conditions.

(iv) Boundary Conditions. The boundary conditions on $\theta(r,z)$ are

(1) $\dfrac{\partial\theta(0,z)}{\partial r} = 0$, or $\theta(0,z) = \text{finite}$, homogeneous

(2) $\theta(r_o, z) = 0$, homogeneous

(3) $\dfrac{\partial\theta(r,0)}{\partial z} = 0$, homogeneous

(4) $k\dfrac{\partial\theta(r,L)}{\partial z} = f(r) = \begin{cases} q_o'', & 0 < r < b \\ 0 & b < r < r_o \end{cases}$ non-homogeneous

This represents a specified heat flux which varies with r according to $f(r)$ which is defined in condition (4).

(4) Solution.

(i) Assumed Product Solution. To solve equation (a) we assume a product solution. Let

$$\theta(r,z) = R(r)Z(z) \tag{b}$$

PROBLEM 3.16 (continued)

Substituting (b) into (a), separating variables and setting the resulting two equations equal to a constant, $\pm\lambda_k^2$, we obtain

$$\frac{d^2 Z_k}{d\,z^2} \pm \lambda_k^2 Z_k = 0 \tag{c}$$

and

$$r^2\frac{d^2 R_k}{d r^2} + r\frac{dR_k}{dr} \mp \lambda_k^2\, r^2 R_k = 0 \tag{d}$$

(ii) Selecting the Sign of the λ_k^2 Terms. Since the *r*-variable has two homogeneous boundary conditions, the plus sign is selected in (d). Thus (c) and (d) become

$$\frac{d^2 Z_k}{d\,z^2} - \lambda_k^2 Z_k = 0 \tag{e}$$

and

$$r^2\frac{d^2 R_k}{d r^2} + r\frac{dR_k}{dr} + \lambda_k^2\, r^2 R_k = 0 \tag{f}$$

For the special case of $\lambda_k = 0$ the above equations take the form

$$\frac{d^2 Z_0}{d\,z^2} = 0 \tag{g}$$

and

$$r\frac{d^2 R_0}{d r^2} + \frac{dR_0}{dr} = 0 \tag{h}$$

(iii) Solutions to the Ordinary Differential Equations. The solution to (e) is

$$Z_k(z) = A_k \sinh\lambda_k z + B_k \cosh\lambda_k z \tag{i}$$

Equation (f) is a Bessel differential equation. Comparing (f) with eq. (2.26) shows that $A = B = n = 0$, $C = 1$ and $D = \lambda_k$. Thus the solution to (f) is given by eq. (2.27)

$$R_k(r) = C_k J_0(\lambda_k r) + D_k Y_0(\lambda_k r) \tag{j}$$

The solutions to (g) and (h) are

$$Z_0(z) = A_0 z + B_0 \tag{k}$$

and

$$R_0(r) = C_0 \ln r + D_0 \tag{l}$$

The complete solution becomes

$$\theta(r,z) = R_0(r)Z_0(z) + \sum_{k=1}^{\infty} R_k(r)Z_k(z) \tag{m}$$

PROBLEM 3.16 (continued)

(iv) Application of Boundary Conditions. Applying boundary condition (1) to equation (j) and (l) gives

$$D_k = C_0 = 0$$

Condition (2) applied to (j) gives the characteristic equation for λ_k

$$J_0(\lambda_k r_o) = 0 \tag{n}$$

Condition (2) applied to (l) gives

$$D_0 = 0$$

With $C_0 = D_0 = 0$ the solution $R_0 Z_0$ vanishes. Boundary condition (3) gives

$$A_k = 0$$

Substituting into (m) gives

$$\theta(r,z) = \sum_{k=1}^{\infty} (b_k \cosh \lambda_k z) J_0(\lambda_k r) \tag{o}$$

where $b_k = B_k C_k$. Application of boundary conditions (4) to (u) gives

$$f(r) = k \sum_{k=1}^{\infty} \lambda_k b_k (\sinh \lambda_k L) J_0(\lambda_k r) \tag{p}$$

where $f(r)$ is defined in boundary condition (4).

(v) Orthogonality. To determine b_k in (p) we apply orthogonality. Note that the characteristic functions $J_0(\lambda_k r)$ in (p) are solutions to equation (f). Comparing (f) with eq. (3.5a) shows that it is a Sturm-Liouville equation with

$$a_1(r) = 1/r, \ a_2(r) = 0 \ \text{and} \ a_3(r) = 1$$

Thus eq. (3.6) gives

$$p(r) = w(r) = r \ \text{and} \ q(r) = 0$$

Since the boundary conditions at $r = 0$ and $r = r_o$ are homogeneous, it follows that the characteristic functions $J_0(\lambda_k r)$ are orthogonal with respect to $w(r) = r$. Multiplying both sides of (p) by $J_0(\lambda_i r) w(r) dr$ and integrating from $r = 0$ to $r = r_o$

$$\int_0^{r_o} f(r) J_0(\lambda_i r) w(r) dr = k \int_0^{r_o} \left[\sum_{k=1}^{\infty} \lambda_k b_k (\sinh \lambda_k L) J_0(\lambda_k r) \right] J_0(\lambda_i r) w(r) dr$$

Interchanging summation with integration, noting that $w(r) = r$ and invoking orthogonality, eq. (3.7), the above gives

$$\int_0^{r_o} f(r)\,J_0(\lambda_k r) r dr = k\lambda_k b_k (\sinh\ \lambda_k L)\int_0^{r_o} J_0^2(\lambda_k r) r dr$$

Using the definition of $f(r)$, the above becomes

PROBLEM 3.16 (continued)

$$q_o''\int_0^{b} J_0(\lambda_k r) r dr + \int_b^{r_o} (0)\ J_0(\lambda_k r) r dr = k\lambda_k b_k (\sinh\ \lambda_k L)\int_0^{r_o} J_0^2(\lambda_k r) r dr$$

Noting that the second integral vanishes, the above gives

$$q_o''\int_0^{b} J_0(\lambda_k r) r dr = k\lambda_k b_k (\sinh\ \lambda_k L)\int_0^{r_o} J_0^2(\lambda_k r) r dr$$

Evaluating the integrals using Appendix B and Table 3.1 the above gives b_k

$$b_k = \frac{2(q_o'' r_o / k)}{(\lambda_k r_o)^2 (\sinh\ \lambda_k L)}\,\frac{J_1(\lambda_k b)}{J_1^2(\lambda_k r_o)} \tag{q}$$

Substituting (q) into (o) and rearranging gives the solution to the temperature distribution in the cylinder

$$\frac{T(r,z) - T_o}{\dfrac{q_o'' r_o}{k}} = 2\sum_{k=1}^{\infty} \frac{J_1(\lambda_k b)}{(\lambda_k r_o)^2 (\sinh \lambda_k L) J_1(\lambda_k r_o)} (\cosh \lambda_k z) J_0(\lambda_k r) \tag{r}$$

(5) Checking. *Dimensional checks*: (i) The arguments of sinh, cosh, J_0 and J_1 are dimensionless. (ii) Each term in (r) is dimensionless.

Limiting check: If $q_o'' = 0$, no heat transfer can take place and the entire cylinder should be at uniform temperature T_o. Setting $q_o'' = 0$ in (r) gives $T(r,z) = T_o$.

Boundary conditions check: Setting r = 0 and $r = r_o$ in solution (r) shows that boundary conditions (1) and (2) are satisfied.

(6) Comments. Equation (r) shows that the temperature solution depends on two dimensionless parameters L/r_o and b/r_o.