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ROBUST CONTROL OF QUADRUPLE – TANK PROCESS

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Abstract. *The paper deals with a robust decentralized PID controller design for a nonlinear model of quadruple tank used in literature as a benchmark to study both minimum-phase and non-minimum phase system configuration. The proposed method applies LMI approach to the linearized state space model with polytopic uncertainties. The results are compared with decentralized controller design in frequency domain.*
Keywords: Robust control, Polytopic uncertainty, Decentralized PID controller, Quadruple tank process, LMI approach

1. **Introduction.** Robust stability of uncertain dynamical systems is of great importance when the real world system models are considered. The realistic approach includes uncertainties of various kinds into the system model and a basic required quality of the system is its stability in the whole uncertainty domain – this quality is called robust stability. In both time and frequency domains, various approaches to robust stability have been developed. In this paper we use results based on small gain theorem to controller design in frequency domain and polytopic description of uncertain system which is appropriate for using LMI approach to robust control design.

The quadruple-tank process presented recently in [3] provides possibility to study the multivariable dynamics both for minimum and non-minimum phase configurations. In this paper the decentralized controller is designed for a model of a quadruple-tank process in frequency domain and in time domain. Results of both approaches are compared and simulation results are presented.

2. **Quadruple-tank Control Problem Formulation.** The quadruple-tank process shown in Figure 1 has been introduced in [3] to study qualities of both minimum and non-minimum phase system on the same plant. The aim is to control the level in the lower two tanks with two pumps. The inputs v_1 and v_2 are pump 1 and 2 flows respectively; the controlled outputs y_1 and y_2 are levels in lower tanks 1 and 2 respectively. The plant can be shifted from minimum to non-minimum phase configuration and vice versa simply by changing a valve controlling the flow ratios γ_1 and γ_2 between lower and upper tanks. The minimum phase configuration corresponds to $1 < \gamma_1 + \gamma_2 < 2$ and a non-minimum phase one to $0 < \gamma_1 + \gamma_2 < 1$.

The nonlinear model of quadruple-tank can be described by state equations

$$
\begin{aligned}
\frac{dh_1}{dt} &= -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_1 k_1}{A_1} v_1 \\
\frac{dh_2}{dt} &= -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_2 k_2}{A_2} v_2 \\
\frac{dh_3}{dt} &= -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2) k_2}{A_3} v_2 \\
\frac{dh_4}{dt} &= -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_1) k_1}{A_4} v_1
\end{aligned}
\tag{1}
$$

where A_i is a cross-section of tank i, a_i is a cross-section of the outlet hole of tank i, h_i is water level in tank i, g is acceleration of gravity, the flow corresponding to pump i is $k_i v_i$. Parameter γ_1 denotes position of the valve dividing the pump 1 flow into the lower tank 1 and related upper tank 4 and similarly γ_2 divides flow from pump 2 to tanks 2 and 3. The flow to tank 1 is $\gamma_1 k_1 v_1$ and to tank 4 is $(1-\gamma_1)k_1 v_1$, similarly for tank 2 and 3.

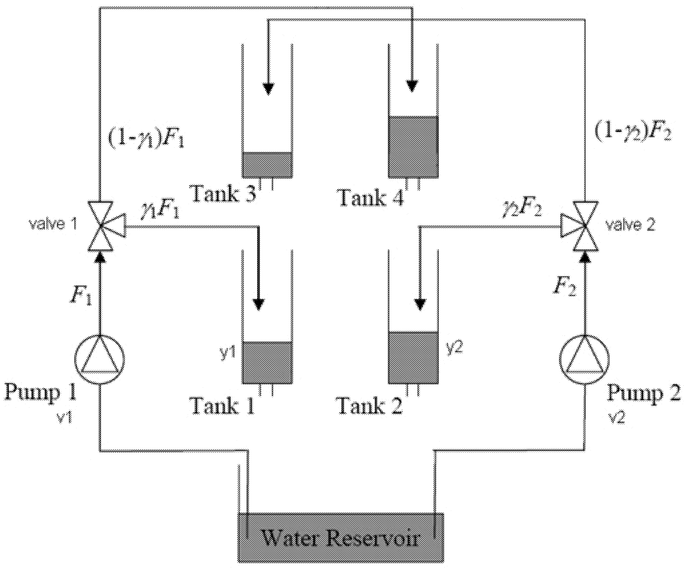


FIGURE 1. Quadruple tank process

The nonlinear model can be linearized around the working point given by the water levels in tanks h_{10}, h_{20}, h_{30}, h_{40} using first order term of Taylor series. For state space equations, state variables are defined as $x_i = h_i - h_{i0}$, the respective control variables are $u_i = v_i - v_{i0}$. The linearized state space model for (1) is then given by (2) and transfer function matrix having inputs v_1 and v_2 and outputs y_1 and y_2 is (3). In (2), the argument t has been omitted; the state variables corresponding to levels in tanks 2 and 3 has been interchanged in state vector so that subsystems respective to input u_1 from pump 1 (tanks 1 and 3) and u_2 from pump 2 (tanks 2 and 4) are more apparent. This decomposition into two subsystems is used later in decentralized control design.

The respective constants used in (2) and (3) are

$$
T_i = \frac{A_i}{a_i}\sqrt{\frac{2h_{i0}}{g}}, \quad i = 1,\dots,4 \quad \text{and} \quad c_i = \frac{T_i k_i}{A_i}\sqrt{\frac{2h_{i0}}{g}}, \quad i = 1,2
$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_3 \\ \dot{x}_2 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \frac{-1}{T_1} & \frac{A_3}{T_3A_1} & 0 & 0 \\ 0 & \frac{-1}{T_3} & 0 & 0 \\ 0 & 0 & \frac{-1}{T_2} & \frac{A_4}{T_4A_2} \\ 0 & 0 & 0 & \frac{-1}{T_4} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix} + \begin{bmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ 0 & \frac{\gamma_2 k_2}{A_1} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2}$$

$$G(s) = \begin{bmatrix} \frac{c_1\gamma_1}{T_1 s+1} & \frac{c_1(1-\gamma_2)}{(T_3 s+1)(T_1 s+1)} \\ \frac{c_2(1-\gamma_1)}{(T_4 s+1)(T_2 s+1)} & \frac{c_2\gamma_2}{T_2 s+1} \end{bmatrix} \tag{3}$$

For chosen values of model parameters we obtain

$$G(s) = \begin{bmatrix} \frac{3.7\gamma_1}{62s+1} & \frac{3.7(1-\gamma_2)}{(23s+1)(62s+1)} \\ \frac{4.7(1-\gamma_1)}{(30s+1)(90s+1)} & \frac{4.7\gamma_2}{90s+1} \end{bmatrix} \tag{4}$$

3. **Input-output Pairing and Structural Stabilizability.** Our aim is to keep the prescribed levels y_1 and $y_2$in lower tanks using decentralized control (two control loops). The pairing depends on configuration, to choose it, RGA (relative gain array) index value is used together with Niederlinski index *NI*, both given in (5), to check the structural stability condition (6) with the respective input – output pairing.

$$RGA = G(0) * \left[G(0)^{-1}\right]^T, \quad NI = \frac{\det(G(0))}{\Pi diag(G(0)} \tag{5}$$

$$NI > 0 \tag{6}$$

In our case study stabilizability condition (6) directly indicates the pairing for both configurations (Figure 2). As shown in Figure 2a, pairing $u_1 - y_1$, $u_2 - y_2$ can be used for the minimum phase case $1 < \gamma_1 + \gamma_2 < 2$, Figure 2b indicates that pairing $u_1 - y_2$, $u_2 - y_1$ is adequate for the non-minimum phase case $0 < \gamma_1 + \gamma_2 < 1$. RGA indicates the same pairing.

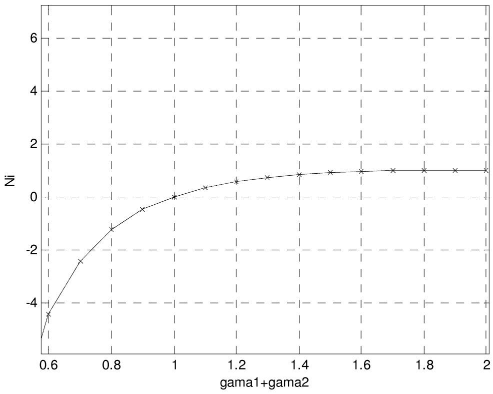
These pairings are used in decentralized control design in the next sections.

4. **PSD Controller Design in Frequency Domain.** In this section a discrete-time decentralized PSD controller is designed for both configurations of quadruple tank process. Sampling period has been chosen as $T_{vz} = 5$. Control algorithm for PSD controller is

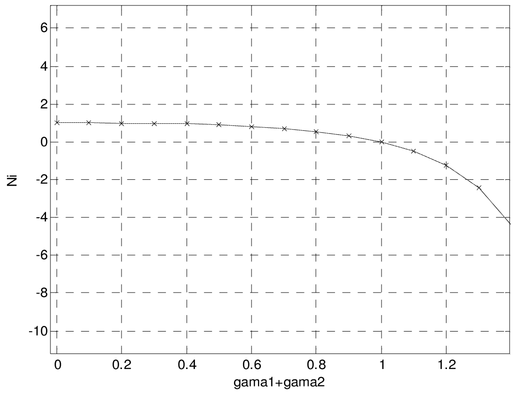
$$u(k) = k_P e(k) + k_I \sum_{i=0}^{k} e(k) + k_D[e(k) - e(k-1)] \tag{7}$$

where $u(k)$ is control variable, $y(k)$ is controlled (output) variable, $e(k)$ is control error $e(k) = w - y(k)$, w is reference value; k_P, k_I, k_D are controller parameters to be designed.

Several PSD controller design methods have been tested



(a)



(b)

FIGURE 2. a) Niederlinski index, pairing 1-1,2-2. b) Niederlinski index, pairing 1-2, 2-1.

- Inverse dynamic approach (Šulc, Vítečková, 2004) applied to discretized transfer function,
- PS design for prescribed first order closed loop system dynamics,
- PID controller design and its discretization to PSD, where PID is designed for prescribed closed loop system dynamics.

TABLE 1. Minimum phase configuration

$\gamma_1 = \gamma_2 = 0.8$; $h_{10} = h_{20} = 40$; $h_{30} = h_{40} = 20$	loop $u_1 - y_1$	loop $u_2 - y_2$
PS design for prescribed CL dynamics (time constant T_{CL}=20)	$\frac{0.9894 - 0.9464z^{-1}}{1 - z^{-1}}$	$\frac{1.123 - 1.089z^{-1}}{1 - z^{-1}}$
Inverse dynamic approach	$\frac{0.9938 - 0.9506z^{-1}}{1 - z^{-1}}$	$\frac{1.128 - 1.094z^{-1}}{1 - z^{-1}}$
state space PS design via (16)-(18)	$\frac{5.0363 - 3.8011z^{-1}}{1 - z^{-1}}$	$\frac{5.8632 - 4.3193z^{-1}}{1 - z^{-1}}$

TABLE 2. Non-minimum phase configuration

$\gamma_1 = \gamma_2 = 0.3; h_{10} = h_{20} = 40$; $h_{30} = h_{40} = 20$	loop $u_1 - y_2$	loop $u_2 - y_1$
PS design for prescribed CL dynamics (time const. T_{CL}=150), 1st order system model approximation is considered	$\frac{0.2713 - 0.2196z^{-1}}{1 - z^{-1}}$	$\frac{0.3214 - 0.2963z^{-1}}{1 - z^{-1}}$
Inverse dynamic approach, first order system model approximation is considered	$\frac{0.1873 - 0.183z^{-1}}{1 - z^{-1}}$	$\frac{0.2314 - 0.2239z^{-1}}{1 - z^{-1}}$
state space PS design – iterative solution to (20) + redesign	$\frac{0.0936 - 0.0871z^{-1}}{1 - z^{-1}}$	$\frac{0.1109 - 0.1075z^{-1}}{1 - z^{-1}}$

Two PSD control loops have been designed independently; the interaction between loops has been treated by small gain theorem approach to guarantee stability of the overall system. The results are summarized in Tables 1 and 2.

In both minimum and non-minimum phase configurations PI or PS controller structure provide better qualities than PID or PSD. The superiority of PS over PSD dominates mainly in simulations for original nonlinear systems.

5. **State Space PSD Controller Design.** Consider the uncertain state space system model

$$\begin{aligned} x(k+1) &= (A+\delta A)x(k) + (B+\delta B)u(k) \\ y(k) &= Cx(k) \end{aligned} \tag{8}$$

$x(k) \in R^n$, $u(k) \in R^m$, $y(k) \in R^l$ are state, control and output vectors respectively; A, B, C are known constant matrices of the respective dimensions corresponding to the nominal system, δA, δB are matrices of (affine) uncertainties. In decentralized control structure B, C and δB are considered as block diagonal. The decentralized feedback control law is

$$u(k) = FCx(k) \tag{9}$$

where F is a block diagonal matrix conforming to the structure of B and C.

The respective uncertain closed-loop polytopic system is then

$$x(k+1) = A_C(\alpha)x(k) \tag{10}$$

where

$$\begin{aligned} &A_C(\alpha) \in \left\{ \sum_{i=1}^{N} \alpha_i A_{Ci}, \quad \sum_{i=1}^{N} \alpha_i = 1, \quad \alpha_i \geq 0 \right\} \\ &A_{Ci} = A_i + B_i F C \end{aligned} \tag{11}$$

With control algorithm (7) for PSD the respective description in state space is

$$\begin{aligned} z(k+1) &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} z(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e(k) = A_R z(k) + B_R e(k) \\ u(k) &= \begin{bmatrix} k_D & k_I - k_D \end{bmatrix} z(k) + (k_P + k_I + k_D)e(k) \end{aligned} \tag{12}$$

Combining (7) and (20) the augmented closed loop system is received as

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = \begin{bmatrix} A+\delta A & 0 \\ -B_R C & A_R \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} B+\delta B \\ 0 \end{bmatrix} \begin{bmatrix} -K_2 & K_1 \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} \tag{13}$$

where $K_2 = (k_P + k_I + k_D)$, $K_1 = \begin{bmatrix} k_D & k_I - k_D \end{bmatrix}$.

In our case only PS controller is designed, then $K_2 = (k_P + k_I)$, $K_1 = \begin{bmatrix} k_I \end{bmatrix}$. To design a robust discrete time controller the Lyapunov stability condition (14) for closed loop system (10) is employed using parameter dependent Lyapunov function (15).

$$A_C^T(\alpha)P(\alpha)A_C(\alpha) - P(\alpha) < 0 \tag{14}$$

$$P(\alpha) = \sum_{i=1}^{N} \alpha_i P_i \qquad \text{where} \qquad P_i = P_i^T > 0 \tag{15}$$

The proposed PSD controller design scheme is based on results of [1] and [2]. The static output feedback controller (SOF) is obtained solving LMI (16)-(18) for unknown matrices F, M, G and P_i of appropriate dimensions, the P_i being symmetric, M, G are block diagonal with block dimensions conforming to subsystem dimensions (for quadruple tank system model has 4 states, two subsystems are 2×2).

$$\begin{bmatrix} -P_i & A_i^G + B_i K C \\ G^T A_i^T + C^T K^T B_i^T & -G - G^T + P_i \end{bmatrix} < 0, \quad i = 1, \dots, N \tag{16}$$

$$MC = CG \tag{17}$$

Compute the corresponding output feedback gain matrix

$$F = KM^{-1} \tag{18}$$

where

$$F = blockdiag\{ \begin{bmatrix} -(k_{Pi} + k_{Ii}) & k_{Ii} - k_{Di} \end{bmatrix} \} \tag{19}$$

The algorithm above is quite simple and often provides reasonable results. In our case study this method provides good results for minimum-phase configuration, however, for nonminimum-phase one it does not work. Therefore another possibility for SOF design has been employed: the iterative solution of robust stability condition (20) (for details see [7] alternatively for unknown P_i, G and F, G

$$\begin{bmatrix} -P_i & A_i^G + B_i F C G \\ G^T A_i^T + G^T C^T F^T B_i^T & -G - G^T + P_i \end{bmatrix} < 0, \qquad i = 1, \dots, N \tag{20}$$

In robust controller design we have used uncertainty domain with 4 vertices respective to different water levels. The respective results for both plant configurations are in Tables 1, 2. The closed loop system performance results are compared with the decentralized controller designed in frequency domain for both configurations (see also [6]).

In this case the results from inverse dynamics approach and small gain theorem use for prescribed closed loop dynamics are practically the same. We verified the controller on nonlinear system model, the relevant part is after water levels reach the steady state values. The following changes around the working point test the closed loop performance qualities (in dashed circles).

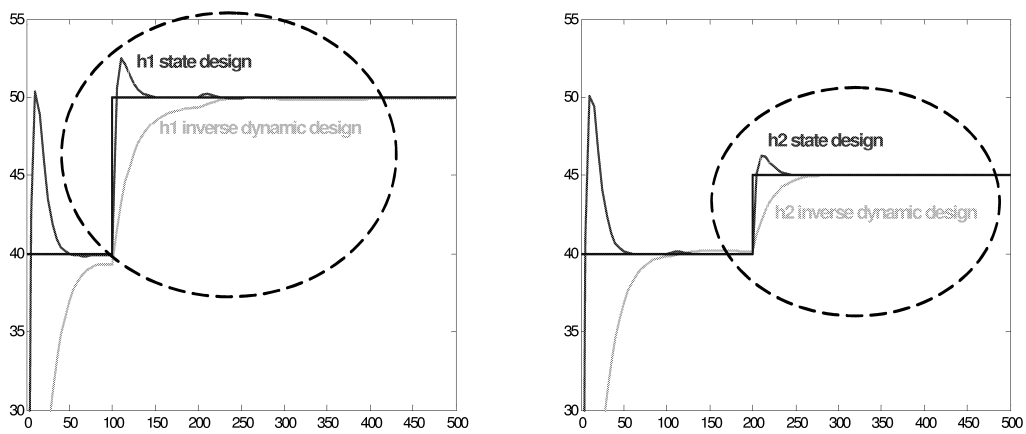


FIGURE 3. Comparison of step responses for **nonlinear** model with controllers from Table 1. (inverse dynamics design and robust state-space design); minimum-phase case

As shown in Figure 3, state space design provides quicker response, however with overshoot. Inverse dynamics yields significantly slower response. The situation changes rapidly for non-minimum phase configuration, which is much more difficult for controller design. In this case, inverse dynamics is superior to state space approach, which moreover required the redesign.

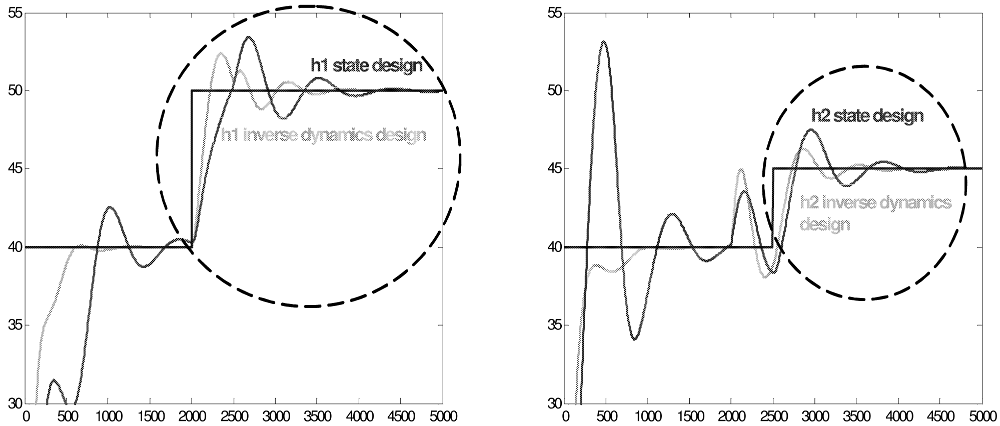


FIGURE 4. Comparison of step responses for **nonlinear** model with controllers from Table 2. (inverse dynamics design, robust state-space design); nonminimum-phase case

6. **Conclusions.** The robust decentralized PS controller has been designed both in frequency and time domain for quadruple-tank process model. The LMI based design of static output feedback controller provides good results for minimum phase configuration, verified on nonlinear process. The non-minimum phase case prefers inverse dynamics approach, state space design yields in this case too big integration constant, therefore oscillating response. The question of appropriate SOF design procedure for this case remains open.

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