Heat Transfer → energy in transit due to Temp. difference (chemical, nuclear, metallurgical and electrical engg)

Heat Transfer v/s Thermodynamics

heat flow in direction of $-\nabla T$ (high T to low T) — rate of heat transfer (heat & work interactions of a system)

Modes of Heat Transfer

Convection = Cond^n + Radiation

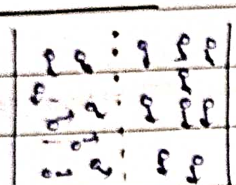
Convection: heat transfer is augmented along with the movement of material

Convect^n → cond^n (and radiation) in moving fluids.

Cond^n: heat transfer by molecular motion supplemented in some cases by flow of free e⁻s and lattice vib^ns, through a body from high T to low T.

Radiation: Transfer of heat in the form of electromagnetic waves

Conduction :-



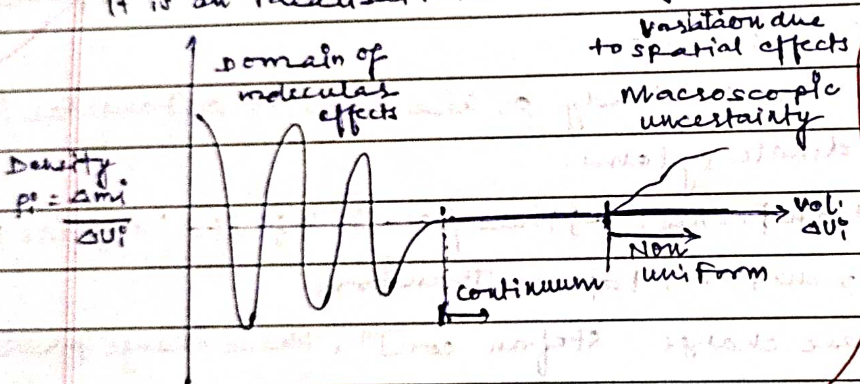
$T_1 > T_2$ q^n → T_2

transfer of KE of the molecular movement. Heat cond^n is transfer of KE of more active molecules by successive collisions to the molecules in the low molecular K.E. regions

In general T distrib^n in a body is controlled by comb^n of all 3 modes.

Continuum concept :-

It is an idealisation even though on microscopic level, we have individual molec.



Continuum concept may be used provided that the size and mean free path of the molecules are small compared with other dimensions existing in the medium, so that a statistical avg. is meaningful

This approach is called Phenomenological approach

3 scientific approaches for problem solving :-

- Experimental approach — realistic, but expensive, technical hurdles, not extrapolatable
- analytical approach — elegant but different in applic^n of adv. physics, complex geometries, non linearity
- computational approach — getting cheaper by the days, accurate, need validation by experiments

Temp. distribution f^n :-

Instantaneous values of Temperatures at all pts. in the medium of interest

It can be :-

Steady v/s Unsteady (Transient)

$T = f(\underline{r})$, pos. $\frac{\partial T}{\partial t} = 0$

$T = f(r, t)$ — pos^n, Time — can be 1D / 2D / 3D

f^n of x space coordinates → x-dimension

Law of conservation of mass :- in the absence of any mass-energy conversion, the mass of the system remains const.

$\frac{dm}{dt} = 0$ or m = constant where m = mass

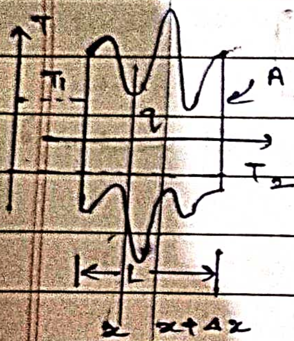
$$\int_{cv}\left[\frac{\partial \rho}{\partial t}+\nabla(\rho V)\right]dv = 0.$$

$\frac{\partial \rho}{\partial t}+\nabla(\rho V) = 0 \Rightarrow$ continuity eqn

First Law of Thd. :– When a system undergoes a cyclic process $\oint \delta Q = \oint \delta W$.
Both heat & work are path fns, i.e., the net amt. of heat transferred to, and net amt. of work done by, a system when sys. undergoes a change of state depend on the path that the sys. follow. This is why inexact differentials are used.
otherwise → for infinitesimal change of state during time dt :– $dE = \delta Q - \delta W$

Second Law of Thd. :– for reversible :– $ds = \left(\frac{\delta Q}{T}\right)_{rev}$
for irreversible process :– $ds > (\delta Q/T)_{irr} \rightarrow ds/dt \geq \frac{1}{T}\frac{\delta Q}{dt}$

Fourier's Law of Heat Condn :– It is an observed phenomenon (experimental)



Consider a solid flat plate of thickness L such that the other 2D are very large compared to the thickness L. Let A be the surface area of the plate, and T_1 and T_2 ($<T_1$) be the temps. of the 2 surfaces. Since a temp. difference of (T_1-T_2) exists b/w the surfaces, heat will flow through the plate

$q \propto A\frac{T_1-T_2}{L} \rightarrow q = kA\frac{T_1-T_2}{L} \rightarrow q = kA\frac{T(x)-T(x+\Delta x)}{\Delta x}$

where $\boxed{k = \frac{(q/A)}{(T_1-T_2)/L}}$ → heat flux; → temp. gradient; Thermal conductivity

for a 1D SS state sys. : $= -kA \lim_{\Delta x \to 0}\frac{T(x+\Delta x)-T(x)}{\Delta x}$

units = W/m·K.

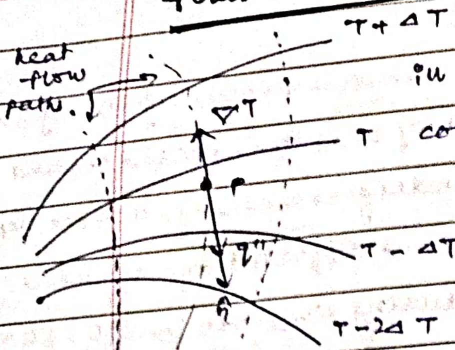
$\rightarrow q = -kA\frac{dT}{dx} \rightarrow \boxed{\frac{q}{A} = -k\frac{dT}{dx} = q''}$ → heat flux

Diff. types of medium based on k :–

- Homogenous } Depending on whether k varies with position
- Heterogenous }
- Isotropic } depending on whether k is same / different in any dirn.
- Anisotropic } → exhibits directional variation.

Materials with porous structure → cork & glass wool → heterogenous media
Materials with fibrous structure → wood or asbestos → anisotropic media

Fourier's Law of heat condⁿ : Vector form :-



Considering isothermal surfaces in a body, each differing in temperature by a small amt. ΔT. The heat flux due to conduction across the isothermal surface at pt. P :-

$$q_n'' = -k\frac{\partial T}{\partial n} \quad \text{where } \frac{\partial}{\partial n} \text{ rep. the differentiation}$$

along the normal to the isothermal surface, which is characterized by the unit vector $\hat{n}$ pointing in the dirⁿ of ↓ Temp.

For rectangular coordinate sys. :-

$$q_n'' = -k\left[\frac{\partial T}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial n} + \frac{\partial T}{\partial z}\cdot\frac{\partial z}{\partial n}\right]$$

$$q_n'' = -k\left[\frac{\partial T}{\partial x}\cos\alpha + \frac{\partial T}{\partial y}\cos\beta + \frac{\partial T}{\partial z}\cos\gamma\right]$$

where (α, β, γ) are the dirⁿ cosines of the unit vector $\hat{n}$, i.e.,

$$\hat{n} = \hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma$$

Using vector calculus :-

$$q_n'' = -k\left(\hat{i}\frac{\partial T}{\partial x} + \hat{j}\frac{\partial T}{\partial y} + \hat{k}\frac{\partial T}{\partial z}\right)\hat{n} \quad \text{or} \quad q_n'' = -k\nabla T\cdot\hat{n}$$

where $\nabla T = \hat{i}\frac{\partial T}{\partial x} + \hat{j}\frac{\partial T}{\partial y} + \hat{k}\frac{\partial T}{\partial z}$ is the gradient of the Temp. distribution in rectangular coordinate system.

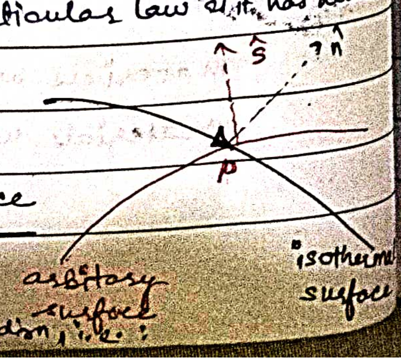
$$q_n'' = -k\nabla T\cdot\hat{n}$$

We now define a heat flux vector normal to the isothermal surface at P and pointing in the dirⁿ of decreasing Temp., as shown :-

$$q'' = \hat{n}\, q_n''$$

Use :- $\boxed{q'' = -k\nabla T}$ → diff. forms in diff. coordinate systems.

also used in unsteady state problems as a valid particular law as it has never been refuted.



Extension of Fourier's Law to non-isothermal surface

Magnitude of heat flux across any arbitary surface passing through P and having $\hat{s}$ as its normal unit vector will be equal to the component of q'' in $\hat{s}$ dirⁿ, i.e. :

$q_s'' = q''\cdot \hat{s} = k\nabla T \cdot \hat{s}$

Since $\nabla T\cdot\hat{s} = \frac{\partial T}{\partial s}$; we have $q_s'' = -k\frac{\partial T}{\partial s}$ where $\partial/\partial s$ represents the differentiation in the dir^n of the normal $\hat{s}$

In the ~~react~~ rectangular coordinate sys., for Ex., the 3 components of the heat flux vector q'' are given by :- $q_x'' = -k\frac{\partial T}{\partial x}$, $q_y'' = -k\frac{\partial T}{\partial y}$ and $q_z'' = -k\frac{\partial T}{\partial z}$ which are the magnitudes of the heat fluxes at P across the surfaces $\perp^r$ to the dir^ns x, y, z, respectively.

Thermal conductivity : Anisotropic solids.

Flux vector may not necessarily be parallel to ∇T, i.e., the heat flux due to cond^n in a given dir^n can also be proportional to the temp. gradients in other dir^ns and therefore Fourier's Law can be generalized for anisotropic media by assuming each component of the heat flux vector at a pt. to be linearly dependent on all components of the ∇T at the pt.

$$q_1'' = -k_{11}\frac{\partial T}{\partial x_1} - k_{12}\frac{\partial T}{\partial x_2} - k_{13}\frac{\partial T}{\partial x_3}$$

$$q_2'' = -k_{21}\frac{\partial T}{\partial x_1} - k_{22}\frac{\partial T}{\partial x_2} - k_{23}\frac{\partial T}{\partial x_3}$$

$$q_3'' = -k_{31}\frac{\partial T}{\partial x_1} - k_{32}\frac{\partial T}{\partial x_2} - k_{33}\frac{\partial T}{\partial x_3}$$

$\Rightarrow k_{ij} \rightarrow$ thermal conductivity coeffs. They are comp. of thermal conductivity tensor – $k_{ij} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$

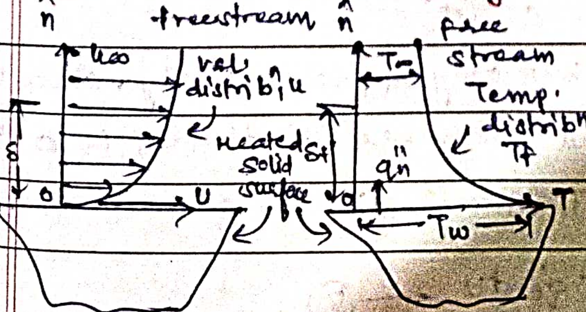
The above 3 eq^ns can be written in more compact form in Cartesian tensor notation is :- $q_i'' = -k_{ij}\frac{\partial T}{\partial x_j}$, $i, j = 1, 2, 3$

For isotropic materials, $k_{11} = k_{22} = k_{33} = k$; other terms ($k_{12}, k_{13}\cdots$) are zero.

Thermal conductivity :– Thermophysical property

Newton's Law of cooling :–

if $T_w > T_\infty$, heat flow from solid to fluid particles at surf.



velocity BL

Thermal BL

$\because$ fluid particles at the surface are stationary

$\rightarrow q_n'' = -k\left(\frac{\partial T_f}{\partial n}\right)_w \rightarrow q_n'' = h(T_w - T_\infty)$

to be evaluated at the surface

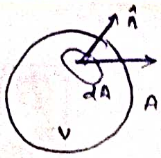
h – HT coeff.

$$h = \frac{q_n''}{T_w - T_\infty} = \frac{-k_f(\partial T_f/\partial n)_w}{T_w - T_\infty}$$

T_f is the temp. distrib^n in the fluid.

Total heat flow by cond^n :— $q = \int_A q$

$q = -kA\frac{dT}{dx}$



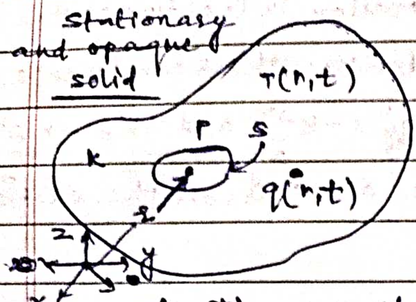
rate of heat flow out of the solid through surface element $dA = q''\hat{n}\, dA$

rate of heat flowing out of the entire boundary surface $= \int_A q''\hat{n}\, dA$

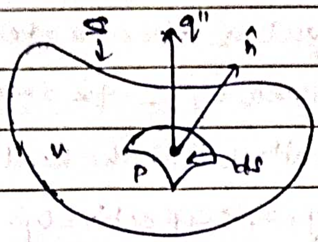
Heat flowing out of the boundary surf in time interval t_1 to t_2 $= \int_{t_1}^{t_2}\int_A q''\hat{n}\, dA$

Heat Cond^n Eq^n :—

General heat Cond^n Eq^n :— imp



control volume v for the derivation of general heat cond^n eq^n



Let $T(r,t)$ represent the Temp distrib^n in the solid, and k and ρ be the usual, respectively both of which may be f^{n}s of space coordinates and / or Temp. Consider a pt. P at any loc^n r in the solid. Let v be the vol. of the space enclosed by S. Assume int. energy gen. in the solid due to power drawn from an ext. electric circuit at a rate of $\dot{q}_e = \dot{q}_e(r,t)$ per unit Volume.

Since $V = 0$, $\dot{W}_{shaft} = 0$ and $\dot{W}_{shear} = 0$

First Law of Thd. reduces to :- $(dE = dQ - \cancel{dW}_0 \rightarrow dE = dQ)$ Heat

E may be considered as sum of many small vol. element $\frac{dE}{dt}$

(1a) — $\int \rho\frac{\partial e}{\partial t}dv = q_s + \int \dot{q}_e\, dv$ where e = total energy per unit mass (or the specific energy) of the solid

q_s = net rate of heat conducted into the vol. v across its bounding surface S

∵ the solid is stationary and there are no nuclear and chemical rx^{n}s, $de = du$ where u is the int. energy per unit mass of the solid. Therefore (1) :—

(1b) — $\int_v \rho\frac{\partial u}{\partial t}dv = q_s + \underbrace{\int_v \dot{q}_e\, dv}_{\text{heat gen}^n\text{ term in vol } v}$,

In gen. for a substance that is homogenous and invariable in composition

$du = \left(\frac{\partial u}{\partial v}\right)_T dv + c_v dT$ — (2a) and $dh = \left(\frac{\partial h}{\partial p}\right)_T dp + c_p dT$ — (2b)

where h = enthalpy per unit mass and p is pressure. $v (= 1/\rho)$ denote specific vol, and c_v and c_p are the specific heats at const. vol. and pr., respectively. Therefore, for solids (and incompressible fluids) the specific volume v = const. On the other hand, if the pressure p = const. then we have $dh = du$. — (3)

Therefore, we get :— $c_v = c_p = c$ — (4)

If p is not const., then (4) still holds, but only approximately, since the difference $(c_p - c_v)$ for solids (and incompressible fluids) is negligibly small. Thus, introducing $du = c\,dT$ in eq^n (1b) we get

$$\int_v \rho c\frac{\partial T}{\partial t}dv = q_s + \int_v \dot{q}_e\, dv \quad \text{— (5)}$$

→ In a fissionable material, int. energy is generated as a result of nuclear rx's which consists of cont. changes in the composⁿ of fissionable material as it is turned into int. energy. Since these composⁿ changes are small, the effect on thermophysical properties of such a material can be assumed to be insignificant. Although this int. energy genⁿ cannot be identified as power i/p from an ext. power src., the time rate of change of int. energy per unit mass, in the absence of chemical rx's can be written as :- $\rho \frac{\partial u}{\partial t} = \rho c \frac{\partial T}{\partial t} = \dot{q}_n$ where $\dot{q}_n = \dot{q}_n(r,t)$ rep. rate of int energy genⁿ per unit vol. due to nuclear rx's.

If the solid under considerⁿ is a fissionable material and the int. energy genⁿ is solely due to nuclear rx's, then substituting above expression in (1b) we get :- $\int \rho c \frac{\partial T}{\partial t} dv = q_s + \int_v \dot{q}_n dv$

→ A similar argument can be made for the case of int. energy sources or sinks resulting from exothermic and endothermic chemical rx's. Hence (5) can be $\int_v \rho c \frac{\partial T}{\partial t} dv = q_s + \int_v \dot{q}\, dv$ where $\dot{q}(r,t)$ represents the rate of int. energy genⁿ in the solid per unit vol., and this genⁿ may be due to multiple other srces

The term q_s which rep. net rate of heat conducted into the vol. v across its bounding surface s, can be written as :- $q_s = -\int_s q''.\hat{n}\, ds$ where $\hat{n}$ is the outward drawn vector normal to the surface element ds and q'' is the heat flux vector due to condⁿ. So :-

$$\int_v \rho c \frac{\partial T}{\partial t} dv = -\int_s q''\hat{n}\, ds + \int_v \dot{q}\, dv \quad \text{—} (6)$$

The surface integral in the above eqⁿ can be converted into a volume integral by using the **Divergence Theorem** ($\oiint_S \vec{F}.d\vec{s} = \iiint_v \nabla.\vec{F} dv$) (converts surface integral to vol. integral)

$$\int_s q''\, \hat{n}\, ds = \int_v \nabla q''\, dv$$

Putting above in eqⁿ (6) :- $\int_v \rho c \frac{\partial T}{\partial t} dv = -\int_v \nabla q'' dv + \int_v \dot{q}\, dv$

or $\int_v \left[\rho c \frac{\partial T}{\partial t} + \nabla \cdot q'' - \dot{q} \right] dv = 0$ → Since the integral vanishes for every vol. element v integrand must vanish everywhere

$$\rightarrow -\nabla q'' + \dot{q} = \rho c \frac{\partial T}{\partial t} \quad \text{—} (7)$$

Assume, isotropic material. So, Fourier's law gives

$$\boxed{q'' = -k\nabla T}$$

So, we obtain :– $\nabla(k\nabla T) + \dot{q} = \rho c \frac{\partial T}{\partial t}$ {general heat cond^n for isotropic solid}

$\boxed{k\nabla^2 T + \nabla k \cdot \nabla T + \dot{q} = \rho c\, \partial T/\partial t}$ where $\nabla^2 = \nabla \cdot \nabla$ is a Laplacian operator.

if all thermophysical prop. (k, ρ and c) are f^n of space coordinates only then above eq^n → Linear PDE. if depend on T → nonlinear PDE.

for a homogenous isotropic solid, k const and general heat cond^n eq^n reduces to

→ Fourier's Biot Eq^n :– const. Thermophysical properties

$$\boxed{\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

$\alpha = \frac{k}{\rho c_p}$ = Thermal diffusivity

In the absence of int. energy srces and power drawn to the sys. from an ext. electric ckt. (Both will also be named int. heat srces), the heat cond^n eq^n :–

→ Heat Diffusion Eq^n :– const. Thermophysical prop., No internal sources

$$\boxed{\nabla^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

for SS cond^ns and in the presence of Int. heat srces we get :–

→ Poisson Eq^n :– SS, k const

$$\boxed{\nabla^2 T + \dot{q}/k = 0}$$

Under SS cond^n and in absence of int heat srces :–

→ Laplace Eq^n :– SS, no int. heat srces, k = const

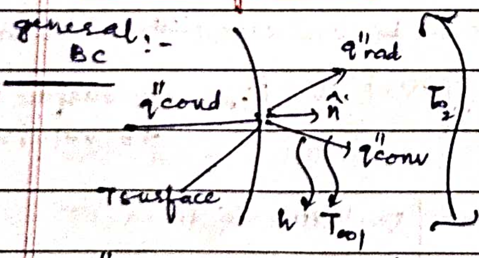
$\boxed{\nabla^2 T = 0}$ → k const, T distrb^n depends only on solid's shape and boundary

Initial cond^n :– for a time dep. problem (unsteady state) usually at t = 0 some rel^n or value is known. if initial cond^n is given by $T_0(r)$ then the sol^n $T(r,t)$ of the problem must be such that, at all pts. of the medium :–

$T(r,t)\big|_{t\to 0} = T_0(r)$ where r = position vector.

Boundary cond^ns :– $T(r,t)\big|_{r=r_s} = T_s(r_s, t)$

general BC :–



consider conserv^n of energy at the surface
> surface is assumed to be stationary
> No energy can be accumulated at an only thin surface.

$q''_{in} = q''_{out} \Rightarrow q''_{cond} = q''_{rad} + q''_{conv}$

$$\Rightarrow -k\frac{dT}{dn}\Big|_{surf} = h(T|_{surf.} - T_{\infty_1}) + \varepsilon\sigma(T^4|_{surf.} - T^4_{\infty_2})$$

Fourier's Law (+ve flux in the coordinate dir^n)

Newton's Law (+ve flux in dir^n of surf. normal)

Stefan Boltzmann Law (+ve flux away from surface when $T_s > T_\infty$)

Heat Cond^n Eq^n : Moving Solids

Assume rectangular coordinate sys. Assume ρ, C_p const.

vel. components : u, v, w

Motion of solid will add convective (enthalpy) fluxes: ρcTu, ρcTv, ρcTw

Modify comp. heat flux vectors :

$$q''_x = -k\frac{\partial T}{\partial x} + \rho cTu \qquad q''_y = -k\frac{\partial T}{\partial y} + \rho cTv \qquad q''_z = -k\frac{\partial T}{\partial z} + \rho cTw$$

Substitute in $\rho c\frac{\partial T}{\partial t} + \nabla q'' - q''' = 0$

$$\Rightarrow \nabla(k\nabla T) + q''' = \rho c\left[\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right]$$

$$\Rightarrow \nabla(k\nabla T) + q''' = \rho c\frac{DT}{Dt}$$

Laplacian of various coordinate systems :-

rectangular :- $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$ cylindrical :- $\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$

spherical :- $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}$

or $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial T}{\partial\mu}\right) + \frac{1}{r^2(1-\mu^2)}\frac{\partial^2 T}{\partial\phi^2}$ where $\mu = \cos\theta$

General heat cond^n eq^n with variable Thermal conductivity in various coordinate systems :-

Coordinate system	$\nabla(k\nabla T) + \dot{q} = \rho c\,\partial T/\partial t$
rectangular	$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c\frac{\partial T}{\partial t}$
cylindrical	$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\phi}\left(k\frac{\partial T}{\partial\phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c\frac{\partial T}{\partial t}$
spherical	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(k\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(k\frac{\partial T}{\partial\phi}\right) + \dot{q} = \rho c\frac{\partial T}{\partial t}$

15-01-24 **Linear / Non Linear / Quasilinear PDE :-**

A PDE is said to be linear if it is linear in the unknown f^n and all its derivatives.

Linear :- $\frac{\partial u}{\partial x} + a(x,y)\frac{\partial u}{\partial x} = f(x,y)$, $\frac{\partial^2 u}{\partial x \partial y} = 2y - x$, $\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} + u(x,y)$

Non linear :- $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 0$ $\frac{\partial u}{\partial x} + a(x,y)\frac{\partial u}{\partial y} = u^2$

A nonlinear eq^n is quasilinear if it is linear in all highest order derivatives of the unknown f^n. Ex: $\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} + u^2 = 0$.

The free term in a PDE is the term that contains no unknown f^n and its partial derivatives.

If the free term is identically zero, a linear eq^n is called a homogenous PDE, otherwise it is called a non homogenous PDE.

The most general 2^{nd} order linear PDE in n independent variables has the form :- $\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu = f$

The above eq^n is homogenous if $f = 0$; otherwise non-homogenous.

Note → The def^n of homogeneity is only for linear PDE.

Principle of superposition :-

Let L be a linear operator $L = \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + a$

The most general homogenous or non homogenous 2^{nd} order linear PDE in n independent variables. which may be written of the form $Lu = 0$

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu = f$$

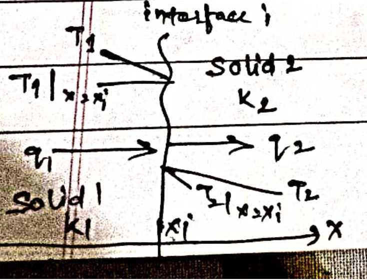
Property I :- A linear $comb^n$ of 2 sol^{ns} of a homogenous eq^n is also a sol^n of the eq^n.

That is $L(c_1 u_1 + c_2 u_2) = 0$, if $Lu_1 = 0$ and $Lu_2 = 0$

Here c_1 and c_2 are arbitrary constants.

B.C. at interface :- $q_i'' = -k_1 \frac{\partial T_1}{\partial x}\Big|_i = h_c(T_1 - T_2) = -k_2 \frac{\partial T_2}{\partial x}\Big|_i$

interface ; contact conductance for the interface



$h_c \to \infty \to T_1|_i = T_2|_i$

For perfect Thermal contact :- $-k_1 \frac{\partial T_1}{\partial x}\Big|_i = -k_2 \frac{\partial T_2}{\partial x}\Big|_i$; at the surface interface

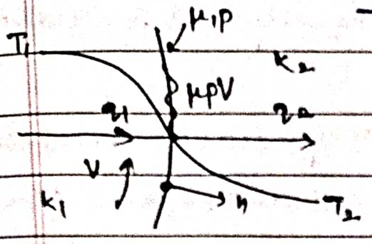
"Transform"

Phase change eq^n is nonlinear

$x_i \to$ loc^n of interface.

Anisotropic $\quad g \to$ gen^n term (heat src)

BCs at Interface | Solids at rel. motion :-



The local pressure on the common boundary is p, the coefficient of dry friction μ and the rel. vel. V.

$$q_1 + \mu P V - q_2 = 0$$

Heat cond^n Eq^n : NonLinearity :-

Non linearity in cond^n problems arise when thermophysical properties are temp dependent or when BCs are non linear.

Surface rad^n & free convec^n $\to$ Exs. of nonlinear BCs.

In phase change problems the interface Energy eq^n is non linear

Srcs. of Non-Linearity : Non linear Differential Eq^{n}s :-

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + q''' = \rho c_p \frac{\partial T}{\partial t}$$

if ρ and/or c_p vary with T, the transient term is non linear. Similarly, if $k = k(T)$ the First term becomes non linear

This is evident if the eq^n :-

$$k\frac{\partial^2 T}{\partial x^2} + \frac{dk}{dT}\left[\frac{\partial T}{\partial x}\right]^2 + q''' = \rho c_p \frac{\partial T}{\partial t}$$

Another Ex. — The governing eq^n for a fin exchanging heat by convec^n and rad^n

$$\frac{d^2T}{dx^2} - \frac{hC}{KA}(T - T_\infty) - \frac{\sigma \varepsilon C}{KA}(T^4 - T^4_{surface}) = 0$$

Srcs of Non Linearity : Non Linear BCs :-

Free convec^n BCs :- $-k\frac{\partial T}{\partial x} = \beta(T - T_\infty)^{5/4}$

Radiation BC :- $-k\frac{\partial T}{\partial x} = \varepsilon\sigma(T^4 - T^4_{sur})$

Phase Change BC

$$k_S\frac{\partial T}{\partial x} - k_L\frac{\partial T}{\partial x} = \rho_S L \frac{\partial x_i}{\partial t}$$

The eq^n of conduc^n for hetergenous isotropic solids (and also for frictionless incompressible fluids): $\rho c \frac{dT}{dt} = \nabla(k\nabla T) + q'''$

rearrange :- $\rho c \frac{dT}{dt} = \nabla k \cdot \nabla T + k\nabla T^2 + q'''$

when k depends on temp. $\nabla K = \frac{dk}{dT} \cdot \nabla T$

Then :- $\rho c \frac{dT}{dt} = \frac{dk}{dT}(\nabla T)^2 + k\nabla^2 T + q'''$

This is non linear

For heterogenous anisotropic solid, k becomes a tensor :-

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k_{11}\frac{\partial T}{\partial x} + k_{12}\frac{\partial T}{\partial y} + k_{13}\frac{\partial T}{\partial z}\right) + \frac{\partial}{\partial y}\left(k_{21}\frac{\partial T}{\partial x} + k_{22}\frac{\partial T}{\partial y} + k_{23}\frac{\partial T}{\partial z}\right) + \frac{\partial}{\partial z}\left(k_{31}\frac{\partial T}{\partial x} + k_{23}\frac{\partial T}{\partial y} + k_{33}\frac{\partial T}{\partial z}\right) + q''$$

for homogenous anisotropic solids, (k independent of space) :-

$$\rho c\,\partial T/\partial t = q'' + k_{11}\,\partial^2T/\partial x^2 + k_{22}\,\partial^2T/\partial y^2 + k_{33}\,\partial^2T/\partial z^2 + (k_{12}+k_{21})\,\partial^2T/\partial x\partial y + (k_{23}+k_{32})\,\partial^2T/\partial y\partial z + (k_{13}+k_{31})\,\partial^2T/\partial x\partial z$$

Heat Condn Eqn : Anisotropic Solids :-

Many bodies of engg. interest do not conduct heat equally well in all dirns and are called anisotropic bodies.

Ex. → Laminates, crystals, composites, graphite, molybdenum disulphide, and wood are among the materials that have preferred dirns of heat flow.

Thermal conductivity matrix in rectangular coordinate :-

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

The components of heat flux vector are given by $q_i = \sum_{j=1}^{3} k_{ij} \frac{\partial T}{\partial x_j}$,

The energy eqn for anisotropic bodies contains cross derivatives.

Orthotropic Bodies :-

The conductivity matrix depends on the orientation of the coordinate system in the body. If the coordinate system is parallel to 3 mutually ⊥r preferred dirns of heat condn, then the geometry is said to be orthotropic and the coordinate sys. lies along the principle axes of heat condn.

An orthotropic body has dirn dependent thermal conductivity whose principal values ~~to~~ are aligned with the coordinate axes. In an orthotropic body the conductivity matrix has a diagonal form.

$$\begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{bmatrix}$$

wood → Ex. of orthotropic body.

The energy eqn for orthotropic bodies does not contain any cross derivatives and it can be transformed into the std. isotropic energy eqn by a suitable choice of new spatial coordinates.

Transformation: orthotropic solids

The following transformn converts orthotropic heat condn eqn to the usual heat condn eqn. The heat condn eqn in cartesian coordinates for an orthotropic body is given by :-

$$k_{11}\frac{\partial^2 T}{\partial x^2} + k_{22}\frac{\partial^2 T}{\partial y^2} + k_{33}\frac{\partial^2 T}{\partial z^2} + g(x,y,z,t) = \rho c\frac{\partial T}{\partial t} \quad \text{Eq}^n \text{ A}$$

Define stretched coordinate axes of the form

$$x_1 = x\left(\frac{k}{k_{11}}\right)^{1/2} ; y_1 = y\left(\frac{k}{k_{22}}\right)^{1/2} ; z_1 = z\left(\frac{k}{k_{33}}\right)^{1/2}$$

where k is a reference conductivity.
Replace these scaled coordinates into Eqⁿ (A) to show that the orthotropic heat condⁿ eqⁿ can be written into the familiar heat condⁿ eqⁿ for isotropic medium.

$$k\left(\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial y_1^2} + \frac{\partial^2 T}{\partial z_1^2}\right) + g(x_1, y_1, z_1, t) = \rho c \frac{\partial T}{\partial t}$$

The reference conductivity is not arbitrary, it must be chosen so that the original differential volume is equal to the scaled differential volume.
For the 3D cartesian case, the differential volume scales according to

$$dx\,dy\,dz = \frac{(k_{11}k_{22}k_{33})^{1/2}}{k^{3/2}}\,dx_1\,dy_1\,dz_1$$

and the requirement that $dV = dV_1$ causes $k = (k_{11}k_{22}k_{33})^{1/3}$

Temperature dependent Thermal Conductivity : Kirchoff Transformation

Consider :

$$\nabla[k(T)\nabla T] + \dot{q}(r,T) = \rho(T)c(T)\frac{\partial T}{\partial t} \quad \text{-- Eq (A)}$$

bcoz of dependence of k, ρ, c on $T \rightarrow$ Eqⁿ (A) is nonlinear D.E. Provided that thermal diffusivity is independent of T, Eqⁿ (A) can be reformulated as :-

$$\theta(r,T) = \frac{1}{k_R}\int_{T_R}^{T(r,t)} k(T')\,dT' \quad \text{— (B)}$$

where T_R is ref. Temp. and $k_R = k(T_R)$ and it follows

$$\nabla\theta = \frac{k(T)}{k_R}\nabla T \quad \text{and} \quad \frac{\partial\theta}{\partial t} = \frac{k(T)}{k_R}\frac{\partial T}{\partial t}$$

Using these with eqⁿ (A) :- $\nabla^2\theta + \frac{\dot{q}(r,t)}{k_R} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$ — (C) where $\alpha(T) = \frac{k(T)}{\rho(T)c(T)}$

Since α is temp. dependent. Eqⁿ is still nonlinear. For many solids, however the dependence of α on temp. can usually be neglected compared to that of k. If the variation of α with T is not significant and hence, it can be approximated to be constant, then eqⁿ becomes linear. For SS problems, since the RHS vanishes identically Eqⁿ (C) is linear D.E. regardless of whether α is T dependent or not

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Variable Thermal Conductivity $k(T)$: Kirchoff Transform^n

Consider 1D case :- $\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + q''' = \rho c_p \frac{\partial T}{\partial t}$

Introducing a new temp. variable $\theta(T)$ defined as $\theta(T) = \frac{1}{k_0}\int_0^T k(T)\,dT$ (f^n of x)

$T = f(x)$
$T(x=L) = T(x=0)$

compute : $\frac{d\theta}{dT} = \frac{k}{k_0}$, $\frac{\partial T}{\partial t} = \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial t} = \frac{k_0}{k}\frac{\partial \theta}{\partial t}$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{k_0}{k}\frac{\partial \theta}{\partial x}$$

$$\Rightarrow k\frac{\partial^2 T}{\partial x^2} + \frac{\partial T}{\partial x}\frac{\partial k}{\partial x} + q''' = \frac{\rho c_p k_0}{k}\frac{\partial \theta}{\partial t}$$

$$\Rightarrow k\frac{\partial}{\partial x}\left(\frac{k_0}{k}\frac{\partial \theta}{\partial x}\right) + \frac{k_0}{k}\frac{\partial \theta}{\partial x}\cancel{\frac{\partial k}{\partial x}}^{\,0} + q''' = \frac{\rho c_p k_0}{k}\frac{\partial \theta}{\partial t}$$

on substituting → $\frac{\partial^2 \theta}{\partial x^2} + \frac{q'''}{k_0} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}$, where $\alpha = \alpha(T) = \frac{k}{\rho c_p}$ {Thermal diffusivity}

[k is only T dependendent. So $\partial k/\partial x = 0$.]

Ex: 1D slab → heat flow in only one dir^n.

Suppose :- $\left.\frac{dT}{dx}\right|_{x=0} = 0$, $\left.T\right|_{x=L} = T_0$ (ref T)

BC :- $\left.\frac{k_0}{k}\frac{d\theta}{dx}\right|_{x=0} = 0 \rightarrow \left.\frac{d\theta}{dx}\right|_{x=0} = 0$ | $\left.\theta\right|_{x=L} = 0$

governing eq^n (given) → $\frac{d}{dx}\left[k(T)\frac{dT}{dx}\right] + \dot{q} = 0 \rightarrow k\frac{d^2T}{dx^2} + \dot{q} = 0$

Kirchoff Transform^n → $k_0\frac{\partial T}{\partial x} + \dot{q} = 0 \rightarrow \frac{\partial T}{\partial x} + \frac{\dot{q}}{k_0} = 0$

Variable Thermal Conductivity : Linear Variation :-

Consider $k = k_0(1 + b\theta) \rightarrow q = -k_0(1 + b\theta) A \frac{d\theta}{dx}$ — (i)

$$\rightarrow \frac{q}{A}\int_{x_1}^{x_2} dx = -k_0\int_{\theta_1}^{\theta_2}(1 + b\theta)\,d\theta \rightarrow \frac{q}{A}(x_2 - x_1) = k_0\left[\theta_1 - \theta_2 + \frac{b}{2}(\theta_1^2 - \theta_2^2)\right]$$

$$\rightarrow \frac{q(x_2 - x_1)}{A} = k_0(\theta_1 - \theta_2)\left[1 + \frac{b}{2}(\theta_1 + \theta_2)\right] = k_0(\theta_1 - \theta_2)\cdot(1 + b\theta_m)$$

where $k_m = k_0(1 + b\theta_m)$ and $\theta_m = \frac{\theta_1 + \theta_2}{2}$

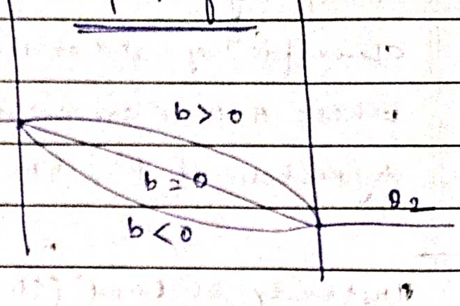
$$\frac{q(x_2 - x_1)}{A} = -k_m(\theta_2 - \theta_1)$$

from eqⁿ ① upon diff. w.r.t. x:

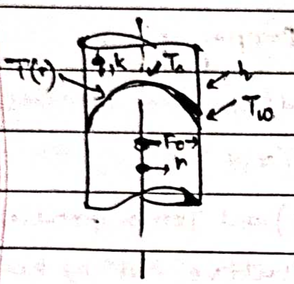
$$\frac{d^2\theta}{dx^2} = -\left(\frac{b}{1+b\theta}\right)\left(\frac{d\theta}{dx}\right)^2$$

if $b>0$, $\left(\frac{-b}{1+b\theta}\right) < 0$ and $T(x)$ is concave

Temp. Profile



Q. Find the rate of heat genⁿ per unit vol. in a rod that will produce a center line T of 2000°C for the following condⁿs: $r_0 = 1\,cm$, $T_w = 350°C$ and $k = \frac{3167}{T+273}$ where T is in °C and k in W/(m·K). Also, cal. the surface flux.



Differential eqⁿ :-

$$\frac{1}{r}\frac{d}{dr}\left[r k(T)\frac{dT}{dr}\right] + \dot{q} = 0$$

BC : $\left(\frac{dT}{dr}\right)_{r=0} = 0$ and $T(r_0) = T_w$

{T symmetry}

$$\theta(T) = \frac{1}{k_w}\int_{T_w}^{T(r)} k(T')\,dT' = \frac{1}{k_w}\int_{T_w}^{T(r)} \frac{3167}{T+273}\,dT \quad \text{where } k_w = k(T_w)$$

Transformⁿ :- $\frac{1}{r}\frac{d}{dr}\left[\frac{r\,d\theta}{dr}\right] + \frac{\dot{q}}{k_w} = 0$ $\quad \left(\frac{d\theta}{dr}\right)_{r=0} = 0$ and $\theta(r_0) = 0$

Solⁿ → $\theta(r) = \frac{\dot{q} r_0^2}{4k_w}\left[1 - \left(\frac{r}{r_0}\right)^2\right]$ $\quad \theta(r) = \frac{1}{k_w}\int_{T(w)}^{T(r)} k(T')\,dT'$

$$\int_{T_w}^{T(r)} k(T')\,dT' = \frac{\dot{q} r_0^2}{4}\left[1 - \left(\frac{r}{r_0}\right)^2\right]$$

This solⁿ can be written explicitly for T(r) when the relⁿ k = k(T) is given.

At r = 0, this eqⁿ yields :-

$$\int_{T_w}^{T_c} k(T)\,dT = \frac{\dot{q} r_0^2}{4} \quad \text{where } T_c \text{ is the centerline T}$$

Now :- $\dot{q} = \frac{4}{r_0^2}\int_{350}^{2000}\frac{3167\,dT}{T+273} = \frac{4\times 3167}{(0.01)^2}\ln\left(\frac{2273}{623}\right) = 1.64\times10^8\,W/m^3$

Surface heat flux :- $q_s'' = \dot{q}V/A$

$$q_s'' = \frac{\dot{q}\pi r_0^2 L}{2\pi r_0 L} = \frac{\dot{q} r_0}{2} = \frac{1.64\times10^8\times0.01}{2} = 8.2\times10^5\,W/m^3.$$

classific^n of PDEs :-

classific^n of 2nd order eq^{n}s :- $A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G(x,y)$

where A, B, C are constant. It is said to be :-

hyperbolic if $B^2 - 4AC > 0$ parabolic → $B^2 - 4AC = 0$ elliptic → $B^2 - 4AC < 0$

Unsteady SS Cond^n (1D, 2D, 3D) : Parabolic → $\rho c \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$

S.S heat Cond^n (2D, 3D) : Elliptic $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$

Wave Eq^n : Hyperbolic $\frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$

Non - Fourier Heat Cond^n : Finite speed of Heat Propagation.

For heat cond^n in a homogenous and isotropic medium, the fourier law of heat cond^n :- $\rho c \frac{\partial T}{\partial t} + \nabla q'' - q''' = 0$ $\quad q(r,t) = -k\nabla T(r,t)$

$$\boxed{\frac{1}{\alpha}\frac{\partial T}{\partial t} = \nabla^2 T + \frac{q'''}{k}}$$

rel^n b/w q (heat flux) and Temp. gradient ∇T is called the constitutive rel^n of heat flux.

Parabolic - heat cond^n Eq^n.

→ ~~Fourier's~~ law is an early empirical law. It assumes that q and ∇T appear at the same time instant t and consequently implies that thermal signals propagate ~~at~~ with an ∞ speed.

→ If the material is subjected to a thermal disturbance, the effects of the disturbance will be felt instantantaneously at distances only far from its source.

→ Although this result is physically unrealistic, it has been confirmed by many expts. that the Fourier Law of heat cond^n holds for many media in the usual range of heat flux q and T gradient.

Non Fourier heat Cond^n

Technology → Ultrafast pulse - laser heating on metal films → heat cond^n appears in the range of high heat flux and high unsteadiness.

∞ heat propagation speed in the Fourier law becomes unacceptable. This has inspired the work of searching for new constitutive rel^{n}s.

New constitutive rel^n proposed by Cattaneo and Vernotte :-

$$q(r,t) + \tau_0 \frac{\partial q(r,t)}{\partial t} = -k\nabla T(r,t)$$

Here $\tau_0 > 0$ is a material property and is called relx^n time, the avg. time b/w heat carriers collisions.

if an ∞ speed of heat propagation i.e. instantaneous collisions assumed ($\tau = 0$) Eq^n reduces to Fourier's Law.

CV constitutive rel^n :- $q(r,t) + \tau_0 \dfrac{\partial q(r,t)}{\partial t} = -k\nabla T(r,t)$

Substitute q in :- $\rho c \dfrac{\partial T}{\partial t} + \nabla q'' - q''' = 0$

The corresponding heat cond^n eq^n :- $\dfrac{1}{\alpha}\dfrac{\partial T}{\partial t} + \dfrac{\tau_0}{\alpha}\dfrac{\partial^2 T}{\partial t^2} = \nabla^2 T + \dfrac{1}{k}\left(q''' + \tau_0 \dfrac{\partial q'''}{\partial t}\right)$

if $\tau_0 = 0$ it is Fourier's Law

$\tau_0 \uparrow$ deviation from Fourier's Law $\uparrow$.

The eq^n is of hyperbolic type, characterizes the combined diffusion and wave like behaviour of heat cond^n, and predicts a finite speed for heat propagation

$V_{CV} = \sqrt{\dfrac{k}{\rho c \tau_0}} = \sqrt{\dfrac{\alpha}{\tau_0}}$ Consider no heat gen^n Std. wave Eq^n $\dfrac{\partial^2 q}{\partial t^2} = c^2\left(\dfrac{\partial^2 q}{\partial x^2} + \dfrac{\partial^2 q}{\partial y^2} + \dfrac{\partial^2 q}{\partial z^2}\right)$

Single Phase Lagging Model :-

Consider the constitutive rel^n proposed by Cattaneo and Vernotte

$q(r,t) + \tau_0 \dfrac{\partial q(r,t)}{\partial t} = -k\nabla T(r,t)$

Note - CV constitutive rel^n is actually a first order approxim^n of a more general constitutive rel^n :-

$q(r, t+\tau_0) = -k\nabla T(r,t)$ → Single Phase Lagging model (Tzou 1992)

This eq^n suggest that the Temp. grad. established at a pt. r at a time t gives rise to a heat flux vector at r at a later time $t+\tau$. There is a finite built up time for τ, for the onset of heat flux at r after a temp. grad is imposed there.

The value of τ_0 is material-dependent

for most solid materials, τ_0 varies from 10^{-10}s to 10^{-14}s

for gases, τ_0 is normally in the range of $10^{-8} \sim 10^{-10}$s.

The value of τ_0 for some biological materials and materials with non homogenous inner structures can be up to 10^2 s.

Dual Phase Lagging Model :-

Single-Phase-Lagging-Model : $q(r, t+\tau_0) = -k\nabla T(r,t)$

The CV constitutive rel^n generates a more accurate prediction than the classical Fourier law. However, some of its predictions do not agree with experimental results.

The CV constitutive rel^n has only taken account of the fast-transient effects, but not the micro-structural interactions

These 2 effects can be reasonably represented by the dual-phase-lag b/w q and ∇T:

$$q(r, t + \tau_0) = -k \nabla T(r, t + \tau_T)$$

Acc. to this rel^n, the Temp. gradient at a pt. r of the material at time $t + \tau_T$ corresponds to the heat flux density vector at r at time $t + \tau_0$.

The delay time τ_T is interpreted as being caused by the micro-structural inter^{n}s (small scale heat transport mechanisms occuring in the micro-scale, or small-scale effects of heat transport in space) such as phonon-electron inter^n or phonon scattering, and is called the phase-lag of the Temp. gradient.

Expanding both sides by using the Taylor series and retaining only the first-order terms of τ_0 and τ_T, we obtain the following constitutive rel^n that is valid at pt. r and time t:

$$q(r,t) + \tau_0 \frac{\partial q(r,t)}{\partial t} = -k \left\{ \nabla T(r,t) + \tau_T \frac{\partial}{\partial t}[\nabla T(r,t)] \right\}$$

This is known as the Jeffreys-type constitutive eq^n of heat flux (Joseph and Preziosi 1989). In literature this relation is also called the dual-phase-lagging constitutive relation.

When $\tau_0 = \tau_T$, this rel^n reduces to classical Fourier's Law. When $\tau_T = 0$, it reduces to the CV constitutive relation.

<u>Constitutive rel^n:-</u>

$$q(r,t) + \tau_0 \frac{\partial q(r,t)}{\partial t} = -k \left\{ \nabla T(r,t) + \tau_T \frac{\partial}{\partial t}[\nabla T(r,t)] \right\}$$

Substitute q in: $\rho c \frac{\partial T}{\partial t} + \nabla q'' - q''' = 0$ to obtain:-

Proof

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} + \frac{\tau_0}{\alpha}\frac{\partial^2 T}{\partial t^2} = \nabla^2 T + \tau_T \frac{\partial}{\partial t}(\nabla^2 T) + \frac{1}{k}\left(q''' + \tau_0 \frac{\partial q'''}{\partial t}\right)$$

Imp.

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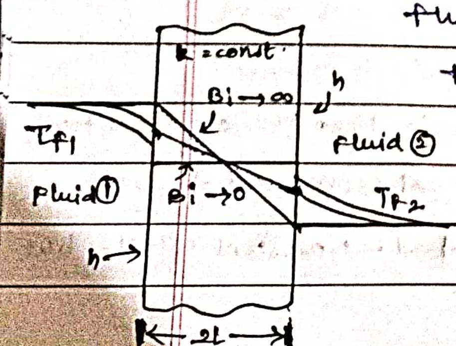
<u>Temp. Distribution in a Plane: Biot No.: 2 Limiting cases:-</u>

Homogenous wall seperating 2 fluids. Heat is convected from the fluid of higher Temp. to the wall, conducted through the wall, and then convected again from the wall to the fluid of lower T. Under SS cond^{n}s and in absence of internal energy sources, the rate of heat transfer will be the same on both surfaces and through the wall.



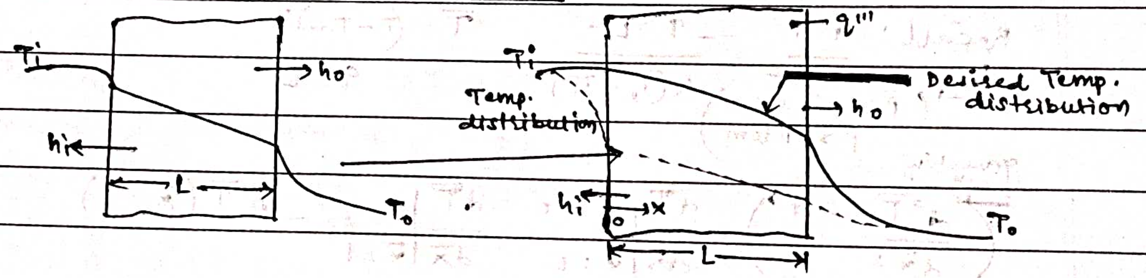
$q = \dfrac{T_{f_1} - T_{f_2}}{\sum R_t}$ $\quad \sum R_t = \underbrace{\dfrac{2}{hA}}_{\text{from fluid ①} \text{ surface resistance}} + \underbrace{\dfrac{2L}{kA}}_{\text{from fluid ② ! internal resistance}}$

Bi = Biot No. = $\dfrac{\text{Internal resistance}}{\text{surface resistance}} = \dfrac{hL}{k}$

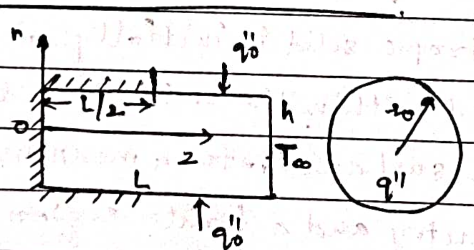
Case 1 :- Bi no. may be very large $Bi \to \infty$. In this case, Total ~~resista~~ int resistance is very large compared to total surface resistance i.e. $\sum R_t \cong 2L/kA$. Thus, there will be almost no Temp. drop on the surfaces. The T of the surfaces and fluids will be same on both sides.

Case 2 :- Bi no. may be very small; i.e. $Bi \to 0$. In this case, the total surface resistance is very large compared to total internal resistance, i.e., $\sum R_t \cong \dfrac{2}{hA}$. Thus, Temp. drop in the wall will be negligible.

Eliminate Heat Loss from one side ($x = 0$)



25/01/24 Problem formulation : 2D



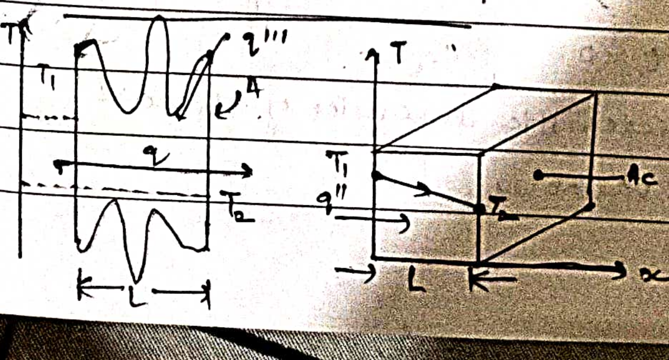
$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{k\partial T}{\partial z}\right) + q'' = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} + \frac{q''}{k} = 0$$

$\dfrac{\partial T(0,r)}{\partial z} = 0 \qquad -k\dfrac{\partial T(L,r)}{\partial z} = h\left[T(r,L) - T_\infty\right]$

$\dfrac{\partial T(0,z)}{\partial r} = 0$, or $T(0,z)$ = finite $\qquad k\dfrac{\partial T(r_o,z)}{\partial r} = f(z) = \begin{cases} 0 & 0 \le z \le L/2 \\ q_o'' & L/2 \le z \le L \end{cases}$

Nondimensional form :-



$\dfrac{d^2T}{dx^2} = -\dfrac{q'''}{k}$ BCs ?

define
> reference length
> reference temperature
> reference temp. difference

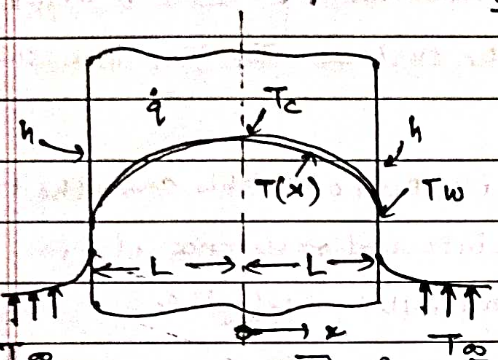
$$\bar{T} = \frac{T - T_1}{T_2 - T_1} \qquad \bar{x} = \frac{x}{L} \qquad \frac{d^2\bar{T}}{d\bar{x}^2} = \frac{q''' L^2}{k(T_2 - T_1)} = -S$$

$$\bar{T}(\bar{x} = 0) = 0, \quad \bar{T}(\bar{x} = 1) = 1$$

$$\bar{T} = -\frac{S\bar{x}^2}{2} + c_1\bar{x} + c_2 \implies \bar{T} = \bar{x} + \frac{S\bar{x}}{2}(1 - \bar{x})$$

Non dimensional form ! The correct way :—

Symmetric – solves half the problem.



$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0 \qquad \left.\frac{dT}{dx}\right|_{x=0} = 0 \qquad -k\left.\frac{dT}{dx}\right|_{x=L} = h\left(T\big|_{x=L} - T_\infty\right)$$

define :—

> ref length (L) > ref. Temp. (T_∞)

> ref. Temp. difference (ΔT_c ; = ?)

$$\bar{x} = \frac{x}{L} \qquad \bar{T} = ? \qquad \bar{T} = \frac{T - T_\infty}{\Delta T_c}$$

(if we do $T - T_\infty$, q term will be ~~there, we'll lose the~~ benefit of nondimensionalizn)

Recall : $\dfrac{d^2\bar{T}}{d\bar{x}^2} = -\dfrac{q''' L^2}{k(T_2 - T_1)}$, $\quad \bar{T} = \dfrac{(T - T_\infty)k}{q''' L^2}$

↳ (previous problem)

governing eqn

$$\frac{d^2\bar{T}}{d\bar{x}^2} = -1 \qquad \left.\frac{d\bar{T}}{d\bar{x}}\right|_{\bar{x}=0} = 0 \qquad \left.\frac{d\bar{T}}{d\bar{x}}\right|_{\bar{x}=1} = -B_i\bar{T}(\bar{x} = 1)$$

Non dimensional form ! 3D heat eqn :—

A stationary, homogenous, isotropic solid is initially at a const. T, T_0 for time $t > 0$, heat is generated within the solid and dissipated by convection from the bounding surfaces into a medium at const. Temp T_∞. Assume a rectangular geometry and a finite region, R.

BVP of heat conducn eqn :—

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \text{in region } R,\ t > 0$$

$$k_i \frac{\partial T}{\partial n_i} = h_i(T_i - T_\infty) \quad \text{on } S_i \text{ boundary of } R,\ t > 0 \quad i = 1, 2 \ldots \text{no. of continuous bounding surfaces of the solid.}$$

$$T = T_0 \quad \text{in region } R,\ t = 0.$$

Ref. length = L = a characteristic dimension of the solid.

Define dimensionless length variables..

$\xi = \frac{x}{L}$, $\eta = \frac{y}{L}$, $\psi = \frac{z}{L}$

$\frac{\partial}{\partial N}$ = differentiation along outward drawn normal in the new dimensionless coordinate sys. (ξ, η, ψ)

choose ref. Temp. $= T_\infty$
net temp. difference $= (T_0 - T_\infty)$

Define dimensionless excess Temp. , $\theta = \frac{T - T_\infty}{T_0 - T_\infty}$

on substitution :-

$$\frac{T_0 - T_\infty}{L^2}\left(\frac{\partial^2\theta}{\partial\psi^2} + \frac{\partial^2\theta}{\partial\eta^2} + \frac{\partial^2\theta}{\partial\xi^2}\right) + \frac{\dot q}{k} = \frac{T_0 - T_\infty}{\alpha}\frac{\partial\theta}{\partial t} \quad \text{in } R,\ t>0$$

$-\frac{k_i}{L}\frac{\partial\theta}{\partial N_i} = h_i\theta$ on s_i , $t > 0$

$i = 1, 2 \ldots$ = no. of continuous bounding surfaces

$\theta = 1$ in R , $t = 0$

$$\frac{\partial^2\theta}{\partial\xi^2} + \frac{\partial^2\theta}{\partial\eta^2} + \frac{\partial^2\theta}{\partial\psi^2} + \frac{\dot q L^2}{(T_0 - T_\infty)k} = \frac{L^2}{\alpha}\frac{\partial\theta}{\partial t} = \frac{\partial\theta}{\partial\tau} \quad \text{here } \tau = \frac{\alpha t}{L^2}$$

$$\frac{\partial^2\theta}{\partial\psi^2} + \frac{\partial^2\theta}{\partial\eta^2} + \frac{\partial^2\theta}{\partial\xi^2} + \phi = \frac{\partial\theta}{\partial\tau} \quad \text{in } R,\ t>0$$

$$\frac{\partial\theta}{\partial N_i} + \frac{h_i L\theta}{k_i} = 0$$

$$\Rightarrow \frac{\partial\theta}{\partial N_i} + Bi\,\theta = 0 \text{ on } s_i,\ \tau > 0$$

$\theta = 1$ in R, ~~...~~ $\tau = 0$.

$\tau = \frac{\alpha t}{L^2} = \frac{kt}{\rho c_p L^2} = \frac{ktL}{\rho c_p L^3} = \frac{kL}{\rho c_p L^3/t} = \frac{\left(\frac{k}{L}\right)L^2}{\rho c_p L^3/t}$ — rate of heat cond^n in solid / rate of heat storage in the sys. of vol.

ϕ = Non dimensional heat gen^n term $= \frac{\dot q L^2}{k(T_0 - T_\infty)} = \frac{\dot q L^3}{\left(\frac{k}{L}\right)L^2(T_0 - T_\infty)}$

$Bi = hL/k = h/(k/L)$ = resistance to convec^n / resistance to heat conduc^n.

Sol^n to homogenous problem by sep^n of variables :-

$\frac{1}{\alpha}\frac{\partial T}{\partial t}(r,t) = \nabla^2 T(r,t)$ $\qquad T(r,t) = \Gamma(t)\,\psi(r)$

$\frac{1}{\alpha}\frac{\Gamma'(t)}{\Gamma} = \frac{\nabla^2\psi(r)}{\psi(r)} = -\lambda^2$

$k_i\frac{\partial T}{\partial n_i} + h_i T(r,t) = 0$

$T(r, t=0) = F(r)$

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Time :- $\dot{\Gamma}(t) + \alpha\lambda^2\Gamma(t) = 0$

$\dot{\Gamma}(t) : e^{-\alpha\lambda^2 t}$

at $t \to \infty$ medium $T \to$ surface T.

Helmholtz eq^n

$$\begin{cases} \nabla^2\psi(r) + \lambda^2\psi(r) = 0 \\ k_i \dfrac{\partial\psi(r)}{\partial\eta_i} + h\psi(r) = 0 \end{cases}$$

$T(r,t) = C_m\psi_m(r)e^{-\alpha\lambda_m^2 t}$

$F(r) = \sum_{m=1}^{\infty} C_m\psi_m(r)$

$\underbrace{\int \psi_m(r)\psi_n(r)\,dr = 0}_{m \neq n}$

$\psi_m(r)$

$$C_m = \frac{\int_R \psi_m(r)F(r)\,dr}{\int_R \psi_m^2(r)\,dr}$$

Uniqueness of sol^n :- for 3D Heat Eq^n

Prove that the sol^n of the foll. 3D Heat problem is unique (Take $u = T$)

$u_t = \nabla^2 u, \quad x \in D$

$u(x,t) = 0, \quad x \in \partial D$ on the boundary

$u(x,0) = f(x), \quad x \in D$

Let u_1, u_2 be 2 sol^{n}s. Define $v = u_1 - u_2$. Then v satisfies

$v_t = \nabla^2 v, \quad x \in D$

$v(x,t) = 0, \quad x \in \partial D$

$v(x,0) = 0, \quad x \in D$

Define : $V(t) = \iiint_D v^2\,dV \geq 0$ $V(t) \geq 0$ ∵ the integrand $v^2(x,t) \geq 0$ ~~always~~ for all (x,t)

$\dfrac{dV(t)}{dt} = \iiint_D 2vv_t\,dV \qquad v_t = \nabla^2 v$

Substituting for v_t from the PDE yields $\dfrac{dV(t)}{dt} = \iiint_D 2v\nabla^2 v\,dV$

Now, we :- $\iint_S v\nabla v\cdot\hat{n}\,dS = \iiint_V (v\nabla^2 v + \nabla v\cdot\nabla v)\,dV$

$= \iiint_V (v\nabla^2 v + |\nabla v|^2)\,dV$

we get :

$\dfrac{dV(t)}{dt} = 2\iint_{\partial D} v\nabla v\cdot\hat{n}\,dS - 2\iiint_D |\nabla v|^2\,dV$

But on ∂D ; $v = 0$, so that the first integral on the RHS vanishes Thus,

$\dfrac{dV(t)}{dt} = -2\iiint_D |\nabla v|^2\,dV \leq 0$

At $t = 0$: $V(0) = \iiint_D v^2(x,0)\,dV = 0$ Recall IC : $v(x,0) = 0$

Thus $V(0) = 0$, $V(t) \geq 0$ and $dV/dt \leq 0$

3 types of BC :— ① Known T at fin tip.
② Adiabatic fin tip (base)
③ Convecⁿ at fin tip.

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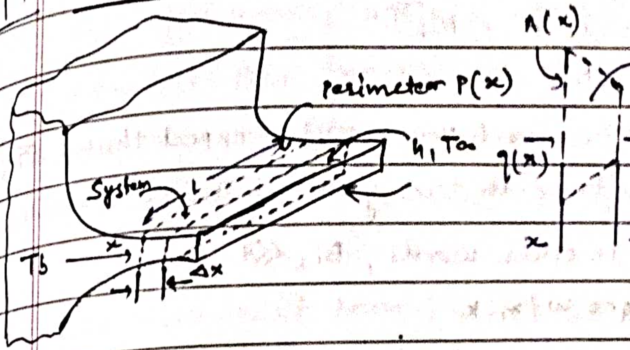
$v(t)$ is a non-ve, non increasing fⁿ that starts at zero.
Thus $v(t)$ must be zero for all time t, so that $v(x,t)$ must be identically zero throughout the volume D for all time, implying the 2 solⁿs are the same, $u_1 = u_2$.
Thus, the solⁿ to the 3D heat problem is unique.

19/01/24 Extended surfaces : fin Eqⁿ :— consider 1D fin



cond" convecⁿ

$q(x) = q(x+\Delta x) + q_{conv.}$

$q(x) = q(x) + \frac{dq}{dx}\Delta x + q_{conv.}$

$\{\because \Delta x \to 0\}$

Also, $q_{conv} = hP(x)\Delta x(T-T_\infty)$

$$\Rightarrow \frac{dq}{dx} + hP(x)(T-T_\infty) = 0$$

Use : $q(x) = -kA(x)\frac{dT}{dx} \Rightarrow \frac{d}{dx}\left[A(x)\frac{dT}{dx}\right] - \frac{hP(x)}{k}(T-T_\infty) = 0$ (Fin Eqⁿ)

fin Eqⁿ :- $$\frac{d}{dx}\left[A(x)\frac{dT}{dx}\right] - \frac{hP(x)}{k}(T-T_\infty) = 0$$

$\left\{\bar{T} = \frac{T-T_\infty}{T_B - T_\infty},\ \bar{x} = \frac{x}{L}\right\} \Rightarrow$ Non dimensional fin eqⁿ :- $\frac{d}{d\bar{x}}\left(A_c\frac{d\bar{T}}{d\bar{x}}\right) - \frac{hPL^2}{k}\bar{T} = 0$

It is not non dimensional

Each term has a unit of area. Further redⁿ cannot be made until the specific form of A_c has been set

One BC at Fin base : $T = T_b$ $\bar{T}(\bar{x}=0) = 1$

3 types of BC at Fin Tip :-

$T(\bar{x}=1) = \bar{T}_t,$ fixed tip T

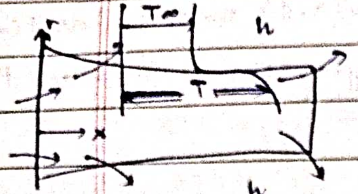
$\left.\frac{d\bar{T}}{d\bar{x}}\right|_{\bar{x}=1} = 0,$ insulated tip

$\left.-\frac{d\bar{T}}{d\bar{x}}\right|_{\bar{x}=1} = Bi_t\,\bar{T}(\bar{x}=1)$ tip convecⁿ

If y denotes the dirⁿ normal to the surf. area, the energy balance at the surface would give :- $\left.-k\frac{\partial T}{\partial y}\right|_{y=b} = h(T-T_\infty)$ where b = thickness of the fin at a particular position x.

$$k\frac{\Delta T}{b} = h(T-T_\infty) \to \Delta T = \frac{hb}{k}(T-T_\infty) \to \underbrace{\frac{\Delta T}{T_B - T_\infty}}_{\overline{\Delta T}} = \frac{hb}{k}\underbrace{\left(\frac{T-T_\infty}{T_B-T_\infty}\right)}_{\overline{T}}$$

Thus, a temp. gradient must exist in the y dirⁿ



$$-k\frac{\partial T}{\partial y}\bigg|_{y=b} = h(T-T_\infty)$$

Approximate the derivative as :- $\frac{\partial T}{\partial y}\bigg|_{y=b} \approx \frac{\Delta T}{b}$

ΔT represents the avg. temp. difference across the fin in the y direction

If the ~~energy~~ surf. ~~bal~~ energy balance is divided by $T_B - T_\infty$ and rearranged: $\overline{\Delta T} = \frac{\Delta T}{T_B - T_\infty} \approx \frac{bh\overline{T}}{k} = Bi_b\overline{T}$

for the 1-D assumpⁿ to be correct we would expect that $\overline{\Delta T} < \overline{T}$ i.e. the variation in the temperature in the y dirⁿ is much smaller than the variation in the x dirⁿ. In other words, $Bi_b \ll 1$

Consider Aluminium $k \simeq 400\ W/m\text{-}K$

final thickness 1cm $h \not> 10\ W/m^2K$

$$\frac{d}{d\bar{x}}A_c\frac{d\overline{T}}{d\bar{x}} - \frac{hPL^2\overline{T}}{k} = 0$$

if A_c is constant $\overline{T}'' - N^2\overline{T} = 0 \to N^2 = \frac{hPL^2}{kA_c}$

solⁿ: $\overline{T} = Ae^{N\bar{x}} + \cancel{e^{N(1-\bar{x})}}\, Be^{-N\bar{x}}$

$$\overline{T} = \frac{e^{N(1-\bar{x})} + e^{-N(1-\bar{x})}}{e^{N} + e^{-N}} = \frac{\cosh[N(1-\bar{x})]}{\cosh N}$$

for adiabatic fin tip.

$A = e^{-N}/(e^{N}+e^{-N})$

$B = 1 - A = e^{N}/(e^{N}+e^{-N})$

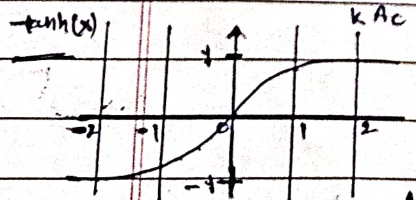
- All the heat removed from the fin must be transported into the fin at the base by condⁿ. This gives :-

$$q = -kA_{c,B}\frac{dT}{dx}\bigg|_{x=0} = -\frac{kA_{c,B}(T_B - T_\infty)}{L}\frac{d\overline{T}}{d\bar{x}}\bigg|_{\bar{x}=0}$$

$$q = \frac{kA_c(T_B - T_\infty)}{L}N\tanh(N) = \sqrt{hPkA_c}\,(T_B - T_\infty)\tanh(N)$$

$N^2 = \frac{hPL^2}{kA_c}$ $\tanh(N) \to 1$ for $N \gg 1$ Longer the fin - higher is the heat removal

$\tanh(3) = 0.995$



But how long is LONG?

consequently a fin with $N > 3$ is essentially ∞ in length

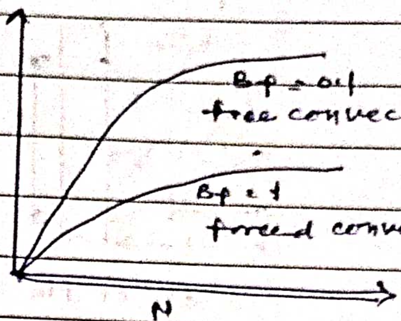
Adding additional length to the fin (and thus increasing N) will not significantly ↑ the heat transfer from the fin.

from a design viewpt. rule of thumb: $N > 2$ to 2.5

Fin effectiveness (constant A_c):-

$$\epsilon = \frac{\dot{q}_{fin}}{q_{w/o\ fin}} = \frac{\sqrt{hPkA_c}\,(T_B - T_\infty)\tanh(N)}{hA_c(T_B - T_\infty)} = \frac{N}{Bi}\tanh(N)$$

Rule of thumb: Fin is justified if $\epsilon > 2$



performance measurement!

fin efficiency:-

The efficiency defined as the ratio of the actual to the theoretical max. heat transfer from the fin.

for the specific case of the const. cross section fin; the heat transfer

$$q = \frac{kA_c(T_B - T_\infty)N\tanh(N)}{L} = \sqrt{hPkA_c}\,(T_B - T_\infty)\tanh(N)$$

The max. heat transfer would occur if the fin was entirely at the temp. of the fin base.

$q_{max} = hA_{fin}(T_B - T_\infty) = hPL(T_B - T_\infty)$; uniform A_c

for const A_c: $\eta = \frac{\tanh(N)}{N}$ $\quad N^2 = \frac{hPL^2}{kA_c}$

$$\frac{d}{dx}\left[A(x)\frac{dT}{dx}\right] - \frac{hP(x)}{k}(T - T_\infty) = 0$$

defining a new temp. f^n by $\theta(x) = T(x) - T_\infty$

$$\frac{d}{dx}\left[A(x)\frac{d\theta}{dx}\right] - \frac{hP(x)\theta}{k} = 0$$

Extended surfaces with const. cross sections $\quad$ $C_1, C_2, C_3, C_4 \rightarrow$ consts of integr^n det. by BC

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad \theta(x) = C_1 e^{-mx} + C_2 e^{mx}$$

where $m^2 = \frac{hP}{kA}$ $\qquad$ or

$$\theta(x) = C_3 \sinh mx + C_4 \cosh mx$$

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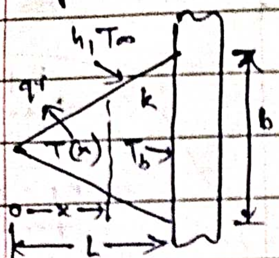
Fins : Non uniform cross section : Triangular fin :-

fins of non uniform cross section can usually transfer more heat for a given mass than those of const. cross section.



$A(x) = \frac{b x L}{L}$ and $P(x) = 2\left(\frac{bx}{L}+1\right)$

$$\frac{d}{dx}\left[A(x)\frac{d\theta}{dx}\right] - \frac{hP(x)}{k}\theta = 0$$

width of fin $= L$

If we assume that $b \ll 1$, then $P(x) \cong 2L$.

$$\frac{d}{dx}\left(x\frac{d\theta}{dx}\right) - m^2\theta = 0 \quad \text{where } m^2 = 2hL/kb$$

$$\rightarrow x^2\frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - m^2 x\theta = 0 \quad \text{(multiply both sides by } x)$$

Define: $\eta = \sqrt{x}$, $\Rightarrow \eta^2\frac{d^2\theta}{d\eta^2} + \eta\frac{d\theta}{d\eta} - 4m^2\eta^2\theta = 0$ (Modified Bessel Eqn)

sol^n: $\theta(\eta) = C_1 I_0(2m\eta) + C_2 K_0(2m\eta)$

$\theta(x) = C_1 I_0(2m\sqrt{x}) + C_2 K_0(2m\sqrt{x})$

$T(0) =$ finite $\Rightarrow \theta(0) =$ finite

$T(L) = T_b \Rightarrow \theta(L) = T_b - T_\infty = \theta_b$

Since $K_0(0) \rightarrow \infty$, $C_2 = 0$

$C_1 = \frac{\theta_b}{I_0(2m\sqrt{L})}$ $\quad \frac{\theta(x)}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{I_0(2m\sqrt{x})}{I_0(2m\sqrt{L})}$

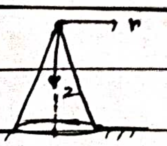
Triangular fins : rate of heat transfer

$$q_f = kA\left(\frac{dT}{dx}\right)_{x=L} = kA\left(\frac{d\theta}{dx}\right)_{x=L}$$

$$\frac{d}{dx}[I_0(\alpha x)] = \alpha I_1(\alpha x) \qquad q_f = L\sqrt{2hkb}\,\theta_b\,\frac{I_1(2m\sqrt{L})}{I_0(2m\sqrt{L})}$$

Fin Optimiz^n

Ex.



A spine attached to a wall maintained at a uniform temp. T_0 has the shape of a circular cone with base radius r_0 and height L. It is exposed to a fluid at a uniform Temp. T_∞

Assume :- > const. k > const. h

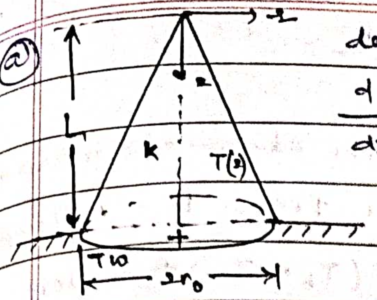
> the variation of the T in r dir^n is negligible.

1) obtain an expression for the SS temp. distribution $T(z)$ in the spine.

2) obtain an expression for the rate of heat loss from the spine to the surrounding fluid

(b) $q_L = k\pi r_0^2 \frac{dT(L)}{dz} = k\pi r_0^2 \frac{d\theta(L)}{dz} = k\pi r_0^2 \theta_w \frac{m I_2(2m\sqrt{L})}{I_1(2m\sqrt{L})}$

(a)



defining a new temp. fn : $\theta(z) = T(z) - T_\infty$

$\frac{d}{dx}\left[A(x)\frac{d\theta}{dx}\right] - \frac{hP(x)}{k}\theta = 0$. $A(z) = \pi\left(\frac{r_0 z}{L}\right)^2 = \frac{\pi r_0^2 z^2}{L^2}$

$\underline{x \to z}$

$P(z) = 2\pi r = 2\pi\left(\frac{r_0 z}{L}\right)$

$\Downarrow$

$z^2\frac{d^2\theta}{dz^2} + 2z\frac{d\theta}{dz} - m^2 z\theta = 0$, $m^2 = \frac{2hL}{kr_0}$

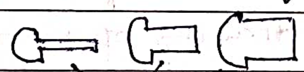
$\Rightarrow \theta(z) = z^{-1/2}\left[A I_1(2m z^{1/2}) + B K_1(2m z^{1/2})\right]$ ∞ at $z=0$.

BC :- $\theta(0)$ = finite → $B = 0$. $\theta(L) = T_w - T_\infty = \theta_w$ → $A = \sqrt{L}\,\theta_w / I_1(2m\sqrt{L})$

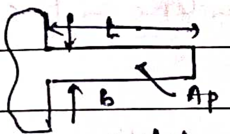
$$\frac{\theta(z)}{\theta(w)} = \frac{T(z) - T_\infty}{T_w - T_\infty} = \sqrt{\frac{L}{z}}\,\frac{I_1(2m\sqrt{z})}{I_1(2m\sqrt{L})}$$

fin optimizn : rectangular fin :-

for a given fin shape, fin material and convecn condns, there exists an optimized design which transfers the max. amt. of heat for a given mass of the fin.



all have same A_p
which one transfers max. amt. of heat



consider :- adiabatic fin tip

$q = \sqrt{hPkA_c}\,(T_b - T_\infty)\tanh(N)$ $\quad N^2 = \frac{hPL^2}{kA_c}$

for a long fin ($w \gg b$), $P \simeq 2w$ and $A_c = bw$. Thus :-

$q' = \frac{q}{w} = \sqrt{2bhk}\,(T_b - T_\infty)\tanh N$ $\quad N^2 = \frac{2hL^2}{kb}$

length L can be eliminated using $A_p = bL$. The formula for N becomes

$N^2 = \frac{2hA_p^2}{kb^3}$ → $b = \left(\frac{2hA_p^2}{kN^2}\right)^{1/3}$ (which value of N maximizes the fn)

$q' = (4h^2kA_p)^{1/3}(T_b - T_\infty)\,N^{-1/3}\tanh N$

$f(N) = N^{-1/3}\tanh N$ → Set $\frac{df}{dN} = 0$ → $\cosh N \sinh N - 3N = 0$
Solve for N
or plot

$N_{opt} = 1.419 = \left(\frac{2hA_p^2}{k b_{opt}^3}\right)^{1/2}$

once A_p is fixed, use the above formula to obtain b_{opt}, and find L from $L = A_p / b_{opt}$.

rate of heat transfer :- $q'_{opt} = \left(\frac{4h^2kA_p}{N_{opt}}\right)^{1/3}(T_b - T_\infty)\tanh N_{opt}$
$= 1.256\,(h^2kA_p)^{1/3}(T_b - T_\infty)$

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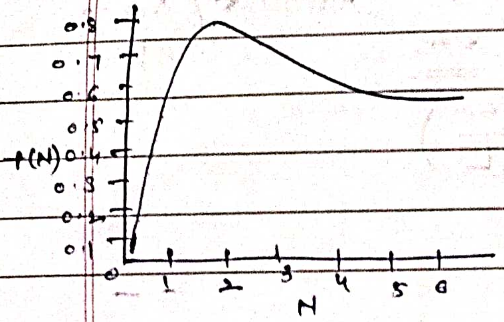
$f(N) = N^{-1/3} \tanh N$ $\quad N_{opt} = 1.419 \left(\dfrac{2hA_p^2}{Kb_{opt}^3}\right)^{1/2}$



once A_p is fixed use the above formula to obtain b_{opt} and find L from $L = A_p/b_{opt}$.

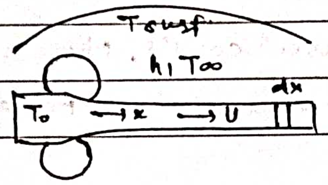
$$q'_{opt} = \left(\frac{4h^2KA_p}{N_{opt}}\right)^{1/3} (T_b - T_\infty) \tanh N_{opt}$$

$$= 1.258 (h^2KA_p)^{1/3} (T_b - T_\infty)$$

Triangular fins :- $q'_{opt} = 1.422 (h^2KA_p)^{1/3} (T_b - T_\infty)$

So Triangular fins are better

8/02/2024 Ex. : Moving fin → These are applications where a material exchanges heat with the surrounding while moving through a surface or a channel.



Ex. :- > Extrusion of plastics
> drawing of wires and sheets.

Such problems can be modeled as moving fins as long as the criterion for fin approxim^n is satisfied.

> fig. shows a sheet being drawn with vel. U through rollers
> The sheet exchanges heat with the surroundings by radiation.
> It also exchanges heat with an ambient fluid by convection.
> Thus its temp. varies with dist. from the rollers. find T(x)

Assume steady state

$$\frac{d^2T}{dx^2} - \frac{\rho C_p U}{K}\frac{dT}{dx} - \frac{hP}{KA_c}(T - T_\infty) - \frac{\epsilon\sigma P}{KA_c}(T^4 - T^4_{sur}) = 0$$

ϵ → emissivity σ → stefan - Boltzmann constant P = Perimeter A_c = Area

$$q(x) = q(x + \Delta x) + q_{conv} + q_{rad}.$$

$$\cancel{q(x)} = \cancel{q(x)} + \frac{dq}{dx}\Delta x + hP\Delta x (T - T_\infty) + \epsilon\sigma P\Delta x (T^4 - T^4_{surr})$$

$$\Rightarrow \frac{dq}{dx} + hP(T - T_\infty) + \epsilon\sigma P(T^4 - T^4_{surr}) = 0$$

$$\frac{d}{dx}\left(-KA_c\frac{dT}{dx}\right)$$

for non moving

for moving → $-KA_c\dfrac{dT}{dx} + \rho C_p U A_c T$

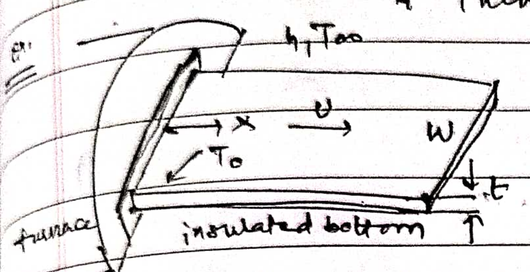
$$-KA\frac{d^2T}{dx^2} + \rho C_p U A_c \frac{dT}{dx} + hP(T - T_\infty) + \epsilon\sigma P \Delta T^4 = 0$$

$$\Rightarrow \frac{d^2T}{dx^2} - \frac{\rho C_p U}{k}\frac{dT}{dx} - \frac{hP}{kA_c}(T - T_\infty) - \frac{\varepsilon \sigma P}{kA_c}(T^4 - T_{surr}^4) = 0$$

~~Ex~~

Ex:

A thin plastic sheet of thickness t and width W is heated in a furnace to temp. T_0. The sheet moves on a conveyor belt travelling with const. vel. U. It is cooled by convection outside the furnace by an ambient fluid at T_∞.



Assumptions :— 1-D, const k. $Bi << 0.1$

const. h, no radiation.

Det. the SS Temp. distribution in the sheet.

$$\frac{d^2T}{dx^2} - \frac{\rho C_p U}{k}\frac{dT}{dx} - \frac{hP}{kA_c}(T - T_\infty) - \frac{\varepsilon \sigma P}{kA_c}(T^4 - T_{surr}^4) = 0.$$