Process Dynamics & Control

State Space realization from Transfer Function

- Process of converting transfer function to state space form is not unique.
- Various realizations possible
- All realizations are equivalent
- One realization may have some advantages over others for a particular task

Possible realizations :

- ➢First Companion form (Controllable Canonical Form)
- ➢Jordan Canonical form
- ➢Alternate first companion form (Toeplitz form)
- ➢Second Companion form (Observable canonical form)

Controllable Canonical Form

Consider Laplace domain transfer function

$$g(s) = \frac{y(s)}{u(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots\dots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \dots\dots\dots + a_{n-1}s + a_n}$$

$$= \frac{y(s)}{z(s)}\frac{z(s)}{u(s)} = \left(b_0 s^n + b_1 s^{n-1} + \dots\dots + b_{n-1}s + b_n\right)\left(\frac{1}{s^n + a_1 s^{n-1} + \dots\dots + a_{n-1}s + a_n}\right)$$

$$Considering, \frac{z(s)}{u(s)} = \frac{1}{s^n + a_1 s^{n-1} + \dots\dots\dots + a_{n-1}s + a_n}$$

$$i.e, \quad \frac{d^n z}{dt^n} + a_1 \frac{d^{n-1}z}{dt^{n-1}} + \dots\dots\dots + a_{n-1}\frac{dz}{dt} + a_n z = u$$

Controllable Canonical Form

Let us choose the states x_1 to x_n as,

$$x_1 = z;\, x_2 = \frac{dz}{dt};\, x_3 = \frac{d^2z}{dt^2} \ldots\ldots \quad x_n = \frac{d^{n-1}z}{dt^{n-1}}$$

Therefore, the state equations are:

$$\dot{x}_1 = x_2;$$
$$\dot{x}_2 = x_3;$$
$$.$$
$$.$$
$$\dot{x}_{n-1} = x_n;$$
$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \ldots\ldots - a_2 x_{n-1} - a_1 x_n + u;$$

Controllable Canonical Form

Now for the output map,

$$\frac{y(s)}{z(s)} = b_0 s^n + b_1 s^{n-1} + \ldots\ldots + b_{n-1} s + b_n$$

$$i.e, y = b_0 \frac{d^n z}{dt^n} + b_1 \frac{d^{n-1} z}{dt^{n-1}} + \ldots\ldots + b_{n-1} \frac{dz}{dt} + b_n z$$

$$= b_0 \dot{x}_n + b_1 x_n + b_2 x_{n-1} + \ldots\ldots + b_{n-1} x_2 + b_n x_1$$

$$= b_0 \left(-a_n x_1 - a_{n-1} x_2 - \ldots\ldots - a_2 x_{n-1} - a_1 x_n \right)$$

$$+ b_1 x_n + b_2 x_{n-1} + \ldots\ldots + b_{n-1} x_2 + b_n x_1 + b_0 u$$

$$= \left(b_n - a_n b_0 \right) x_1 + \left(b_{n-1} - a_{n-1} b_0 \right) x_2 + \ldots\ldots$$

$$+ \left(b_2 - a_2 b_0 \right) x_{n-1} + \left(b_1 - a_1 b_0 \right) x_n + b_0 u$$

Controllable Canonical Form

So, in standard vector-matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ . \\ . \\ . \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & 0 & 0 \\ 0 & 0 & 1 & 0 & . & . & 0 & 0 \\ 0 & 0 & 0 & 1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & . & . & . & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{bmatrix} u$$

and

$$y = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & . & . & b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} + [b_0]u$$

Jordan Canonical Form

Consider Laplace domain transfer function

$$g(s) = \frac{y(s)}{u(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots\ldots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \ldots\ldots + a_{n-1} s + a_n}$$

$$= b_0 + \frac{r_1}{(s-\lambda_1)} + \frac{r_2}{(s-\lambda_2)} + \ldots\ldots + \frac{r_n}{(s-\lambda_n)}$$

Therefore,

$$y(s) = b_0 u(s) + \frac{r_1 u(s)}{(s-\lambda_1)} + \frac{r_2 u(s)}{(s-\lambda_2)} + \ldots\ldots + \frac{r_n u(s)}{(s-\lambda_n)}$$

$$= b_0 u(s) + r_1 x_1(s) + r_2 x_2(s) + \ldots\ldots + r_n x_n(s)$$

Where, $x_1, x_2 \ldots\ldots x_n$ are considered as the states of the system

Jordan Canonical Form

Therefore, the state equations are

$$x_1(s) = \frac{u(s)}{(s-\lambda_1)} \qquad \dot{x}_1 = \lambda_1 x_1 + u$$

$$x_2(s) = \frac{u(s)}{(s-\lambda_2)} \qquad \dot{x}_2 = \lambda_2 x_2 + u$$

....

$$x_n(s) = \frac{u(s)}{(s-\lambda_n)} \qquad \dot{x}_n = \lambda_n x_n + u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ . \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & . & . & 0 \\ 0 & \lambda_2 & 0 & & & \\ 0 & 0 & \lambda_3 & 0 & . & 0 \\ 0 & 0 & 0 & \lambda_4 & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix} u$$

and output map

$$y = r_1 x_1 + r_2 x_2 + \ldots\ldots + r_n x_n + b_0 u$$

What will happen in case of repeated roots?

Jordan Canonical Form (repeated roots)

For repeated roots, the partial fraction expression may be written as

$$g(s)=\frac{y(s)}{u(s)}=\frac{b_0 s^n+b_1 s^{n-1}+......+b_{n-1}s+b_n}{s^n+a_1 s^{n-1}+.......+a_{n-1}s+a_n}$$

$$=b_0+\frac{r_{11}}{\left(s-\lambda_1\right)^3}+\frac{r_{12}}{\left(s-\lambda_1\right)^2}+\frac{r_{13}}{\left(s-\lambda_1\right)}+\frac{r_4}{\left(s-\lambda_2\right)}+.......+\frac{r_n}{\left(s-\lambda_n\right)}$$

Therefore,

$$y(s)=b_0 u(s)+\frac{r_{11}\,u(s)}{\left(s-\lambda_1\right)^3}+\frac{r_{12}\,u(s)}{\left(s-\lambda_1\right)^2}+\frac{r_{13}\,u(s)}{\left(s-\lambda_1\right)}+\frac{r_4\,u(s)}{\left(s-\lambda_2\right)}+.......+\frac{r_n\,u(s)}{\left(s-\lambda_n\right)}$$

$$=b_0 u(s)+r_{11}x_1(s)+r_{12}x_2(s)+r_{13}x_3(s)+r_4 x_4(s)+.......+r_n x_n(s)$$

Jordan Canonical Form (repeated roots)

Now the state equations are

$$x_1(s) = \frac{x_2(s)}{(s-\lambda_1)} \qquad \dot{x}_1 = \lambda_1 x_1 + x_2$$

$$x_2(s) = \frac{x_3(s)}{(s-\lambda_1)} \qquad \dot{x}_2 = \lambda_1 x_2 + x_3$$

$$x_3(s) = \frac{u(s)}{(s-\lambda_1)} \qquad \dot{x}_3 = \lambda_1 x_3 + u$$

$$x_4(s) = \frac{u(s)}{(s-\lambda_2)} \qquad \dot{x}_4 = \lambda_2 x_4 + u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ . \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & 0 & . & . & 0 \\ 0 & \lambda_1 & 1 & & & \\ 0 & 0 & \lambda_1 & 0 & . & 0 \\ 0 & 0 & 0 & \lambda_2 & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix} u$$

....

$$x_n(s) = \frac{u(s)}{(s-\lambda_n)} \qquad \dot{x}_n = \lambda_n x_n + u$$

Output map

$$y = r_{11}x_1 + r_{12}x_2 + r_{13}x_3 + r_4 x_4 + \ldots\ldots + r_n x_n + b_0 u$$

Alternate Canonical form (Toeplitz form)

Consider the transfer function as

$$g(s) = \frac{y(s)}{u(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots\ldots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \ldots\ldots + a_{n-1}s + a_n}$$

$$i.e, \quad \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \ldots\ldots + a_{n-1}\frac{dy}{dt} + a_n y = b_0 \frac{d^n u}{dt^n} + b_1 \frac{d^{n-1}u}{dt^{n-1}} + \ldots\ldots + b_{n-1}\frac{du}{dt} + b_n u$$

Define State equations and output map as the following:

$$y = x_1 + p_0 u$$

$$\dot{x}_1 = x_2 + p_1 u$$

$$\dot{x}_2 = x_3 + p_2 u$$

$$\ldots\ldots$$

$$\dot{x}_{n-1} = x_n + p_{n-1}u$$

$$\dot{x}_n = -a_1 x_n - a_2 x_{n-1} - \ldots\ldots - a_n x_1 + p_n u$$

Alternate Canonical form (Toeplitz form)

From the above definition, we can write

$$y = x_1 + p_0 u$$

$$\dot{y} = \dot{x}_1 + p_0 \dot{u} = x_2 + p_1 u + p_0 \dot{u}$$

$$\ddot{y} = \dot{x}_2 + p_1 \dot{u} + p_0 \ddot{u} = x_3 + p_2 u + p_1 \dot{u} + p_0 \ddot{u}$$

....

$$\frac{d^{n-1}y}{dt^{n-1}} = x_n + p_{n-1}u + p_{n-2}\dot{u} + \ldots.. + p_1 \frac{d^{n-2}u}{dt^{n-2}} + p_0 \frac{d^{n-1}u}{dt^{n-1}}$$

$$\frac{d^n y}{dt^n} = \dot{x}_n + p_{n-1}\dot{u} + p_{n-2}\ddot{u} + \ldots.. + p_1 \frac{d^{n-1}u}{dt^{n-1}} + p_0 \frac{d^n u}{dt^n}$$

$$= -a_1 x_n - a_2 x_{n-1} - \ldots\ldots - a_n x_1 + p_n u + p_{n-1}\dot{u} + p_{n-2}\ddot{u} + \ldots.. + p_1 \frac{d^{n-1}u}{dt^{n-1}} + p_0 \frac{d^n u}{dt^n}$$

Alternate Canonical form (Toeplitz form)

Therefore,

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots\ldots + a_{n-1} \frac{dy}{dt} + a_n y$$

$$= \left(-a_1 x_n - a_2 x_{n-1} - \ldots\ldots - a_n x_1 + p_n u + p_{n-1} \dot{u} + p_{n-2} \ddot{u} + \ldots\ldots + p_1 \frac{d^{n-1} u}{dt^{n-1}} + p_0 \frac{d^n u}{dt^n} \right)$$

$$+ a_1 \left(x_n + p_{n-1} u + p_{n-2} \dot{u} + \ldots\ldots + p_1 \frac{d^{n-2} u}{dt^{n-2}} + p_0 \frac{d^{n-1} u}{dt^{n-1}} \right) + \ldots\ldots\ldots$$

$$+ a_{n-1} \left(x_2 + p_1 u + p_0 \dot{u} \right) + a_n (x_1 + p_0 u)$$

$$= (p_n + a_1 p_{n-1} + \ldots\ldots + a_{n-1} p_1 + a_n p_0) u + (p_{n-1} + a_1 p_{n-2} + \ldots\ldots + a_{n-2} p_1 + a_{n-1} p_0) \dot{u} + \ldots\ldots$$

$$+ (p_1 + a_1 p_0) \frac{d^{n-1} u}{dt^{n-1}} + p_0 \frac{d^n u}{dt^n}$$

$$= b_0 \frac{d^n u}{dt^n} + b_1 \frac{d^{n-1} u}{dt^{n-1}} + \ldots\ldots + b_{n-1} \frac{du}{dt} + b_n u$$

Alternate Canonical form (Toeplitz form)

Equating the coefficients of $u, \dot{u}, \ldots, \frac{d^{n-1}u}{dt^{n-1}}, \frac{d^n u}{dt^n}$

$$b_0 = p_0;$$

$$b_1 = p_1 + a_1 p_0;$$

$$\ldots\ldots$$

$$b_{n-1} = p_{n-1} + a_1 p_{n-2} + \ldots\ldots + a_{n-2} p_1 + a_{n-1} p_0$$

$$b_n = p_n + a_1 p_{n-1} + \ldots\ldots + a_{n-1} p_1 + a_n p_0$$

In vector-Matrix form,

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & . & . & 0 & 0 \\ a_1 & 1 & 0 & . & . & 0 & 0 \\ a_2 & a_1 & 1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{n-1} & a_{n-2} & . & . & . & 1 & 0 \\ a_n & a_{n-1} & a_{n-2} & . & . & a_1 & 1 \end{bmatrix}}_{Toeplitz\ \ Matrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ . \\ . \\ p_{n-1} \\ p_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ . \\ . \\ b_{n-1} \\ b_n \end{bmatrix}$$

2^{nd} Companion form (Observer Canonical form)

Consider the transfer function

$$g(s) = \frac{y(s)}{u(s)} = \frac{b_0 s^n + b_1 s^{n-1} + ... + b_{n-1} s + b_n}{\left(s^n + a_1 s^{n-1} + ... + a_{n-1} s + a_n\right)}$$

$$i.e, \quad \left(s^n + a_1 s^{n-1} + ... + a_{n-1} s + a_n\right) y(s) = \left(b_0 s^n + b_1 s^{n-1} + ... + b_{n-1} s + b_n\right) u(s)$$

Rearranging the terms,

$$s^n \left[y(s) - b_0 u(s)\right] + s^{n-1}\left[a_1 y(s) - b_1 u(s)\right] + ... + \left[a_n y(s) - b_n u(s)\right] = 0$$

Simplify :

$$y(s) - b_0 u(s) = \frac{1}{s}\left[b_1 u(s) - a_1 y(s)\right] + \frac{1}{s^2}\left[b_2 u(s) - a_2 y(s)\right] + ... + \frac{1}{s^n}\left[b_n u(s) - a_n y(s)\right]$$

$$y(s) = b_0 u(s) + \frac{1}{s}\left[b_1 u(s) - a_1 y(s)\right] + \frac{1}{s^2}\left[b_2 u(s) - a_2 y(s)\right] + ... + \frac{1}{s^n}\left[b_n u(s) - a_n y(s)\right]$$

2nd Companion form (Observer Canonical form)

Rearranging the terms,

$$y(s) = b_0 u(s) + \underbrace{\frac{1}{s}\left[[b_1 u(s) - a_1 y(s)] + \underbrace{\frac{1}{s}\left[[b_2 u(s) - a_2 y(s)] + \ldots\ldots + \underbrace{\frac{1}{s}[b_n u(s) - a_n y(s)]}_{x_n(s)} \right]}_{x_2(s)} \right]}_{x_1(s)}$$

The equations now can be written as

$$y = x_1 + b_0 u$$

$$\dot{x}_1 = x_2 - a_1 y + b_1 u = -a_1 x_1 + x_2 + (b_1 - a_1 b_0) u$$

$$\dot{x}_2 = x_3 - a_2 y + b_2 u = -a_2 x_1 + x_3 + (b_2 - a_2 b_0) u$$

.....

.....

$$\dot{x}_{n-1} = x_n - a_{n-1} y + b_{n-1} u = -a_{n-1} x_1 + x_n + (b_{n-1} - a_{n-1} b_0) u$$

$$\dot{x}_n = -a_n y + b_n u = -a_n x_1 + (b_n - a_n b_0) u$$

2^{nd} Companion form (Observer Canonical form)

In vector Matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ . \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & . & . & 0 \\ -a_2 & 0 & 1 & & & \\ -a_3 & 0 & 0 & 1 & . & 0 \\ . & 0 & 0 & 0 & . & 0 \\ . & . & . & . & . & . \\ -a_{n-1} & . & . & . & . & 1 \\ -a_n & 0 & 0 & . & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ . \\ . \\ b_{n-1} - a_{n-1} b_0 \\ b_n - a_n b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & . & . & . & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

2nd Companion form (Observer Canonical form)

On the other hand, if we formulate

$$y(s) = b_0 u(s) + \underbrace{\frac{1}{s}\left[\left[b_1 u(s) - a_1 y(s) \right] + \underbrace{\frac{1}{s}\left[\left[b_2 u(s) - a_2 y(s) \right] + \ldots\ldots + \underbrace{\frac{1}{s}\left[b_n u(s) - a_n y(s) \right]}_{x_1(s)} \right]}_{x_{n-1}(s)} \right]}_{x_n(s)}$$

Then,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ . \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & . & . & -a_n \\ 1 & 0 & 0 & 0 & 0 & -a_{n-1} \\ 0 & 1 & 0 & 0 & . & -a_{n-2} \\ 0 & 0 & 1 & 0 & . & -a_{n-3} \\ . & . & . & . & . & . \\ 0 & . & . & . & . & -a_2 \\ 0 & 0 & 0 & . & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ b_{n-2} - a_{n-2} b_0 \\ . \\ . \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u$$