Name - Gopal Gupta        Roll no - 20CH30008

Assignment - 2 (PDC)

Prob 1

Solution:-

$$\frac{dC_b}{dt} = (\mu - F_d) C_b = f_1(C_b, C_s, F_d)$$

$$\frac{dC_s}{dt} = F_d(C_{sf} - C_s) - \frac{\mu C_b}{P} = f_2(C_b, C_s, F_d)$$

$$\mu = \frac{\mu_m C_s}{K_m + C_s + K_1 C_s^2}$$

at steady state -

$$(\mu_s - F_{ds}) C_{bs} = 0 \quad \text{---} (1) \Rightarrow \mu_s - F_{ds} = 0 \quad \because C_{bs} \neq 0$$

$$\boxed{\mu_s = F_{ds}} \text{ --- } (3)$$

$$F_{ds}(C_{sf} - C_{ss}) - \frac{\mu_s C_{bs}}{P} = 0 \text{ --- } (2) \Rightarrow \boxed{C_{sf} - C_{ss} = \frac{C_{bs}}{P}} \text{ --- } (4)$$

where $\mu_s = \dfrac{\mu_m C_{ss}}{K_m + C_{ss} + K_1 C_{ss}^2}$ & $C_{ss}$, $C_{bs}$ & $F_{ds}$ = steady state values.

(a) Linear state space model for the given system to control $C_b$ by manipulating $F_d$ —

Linear state space model —

$$\dot{X} = AX + Bu$$

$$y = CX + Du$$

where, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} C_b - C_{bs} \\ C_s - C_{ss} \end{bmatrix}$

$u = F_d - F_{ds}$

$$B = \begin{bmatrix} \frac{df_1}{du} \\ \frac{df_2}{du} \end{bmatrix} = \begin{bmatrix} \frac{df_1}{dF_d} \\ \frac{df_2}{dF_d} \end{bmatrix} = \begin{bmatrix} -C_{bs} \\ C_{sf} - C_{ss} \end{bmatrix}$$

$$B = \begin{bmatrix} -C_{bs} \\ \frac{C_{bs}}{P} \end{bmatrix} \quad [\text{from eq}^n\ 4]$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial C_b} & \frac{\partial f_1}{\partial C_s} \\ \frac{\partial f_2}{\partial C_b} & \frac{\partial f_2}{\partial C_s} \end{bmatrix}$$

$$\frac{\partial f_1}{\partial C_b} = \mu_s - F_{ds} = 0 \quad [eq^n 3] \quad \Big| \quad \frac{\partial f_1}{\partial C_s} = \mu' C_{bs} \qquad \text{where } \mu' = \frac{d\mu}{dC_s}\Big|_{C_{ss}}$$

$$\frac{\partial f_2}{\partial C_b} = -\frac{\mu_s}{P} = -\frac{F_{ds}}{P} \quad \Big| \quad \frac{\partial f_2}{\partial C_s} = -F_{ds} - \frac{\mu' C_{bs}}{P}$$

$$\therefore \quad A = \begin{bmatrix} 0 & \mu' C_{bs} \\ -\frac{F_{ds}}{P} & -F_{ds} - \frac{\mu' C_{bs}}{P} \end{bmatrix}$$

~~Contro~~

$$\therefore \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \mu' C_{bs} \\ -\frac{F_{ds}}{P} & -\left(F_{ds} + \frac{\mu' C_{bs}}{P}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -C_{bs} \\ \frac{C_{bs}}{P} \end{bmatrix} u$$

(b)

Controllability Matrix $(C_B) = [\, B \quad AB \,]$

$$C_B = \begin{bmatrix} -C_{bs} & \frac{\mu' C_{bs}^2}{P} \\ \frac{C_{bs}}{P} & \frac{F_{ds} C_{bs}}{\cancel{P}} - \frac{F_{ds} C_{bs}}{\cancel{P}} - \frac{\mu' C_{bs}^2}{P^2} \end{bmatrix}$$

$$C_B = \begin{bmatrix} -C_{bs} & \frac{\mu' C_{bs}^2}{P} \\ \frac{C_{bs}}{P} & -\frac{\mu' C_{bs}^2}{P^2} \end{bmatrix}$$

$$\det(C_B) = \frac{\mu' C_{bs}^3}{P^2} - \frac{\mu' C_{bs}^3}{P^2} = 0$$

$$\therefore \quad \text{Rank}(C_B) \neq 2$$

Name - Gopal Gupta    Roll no - 20CH30008

The given system is not fully state - controllable because $Rank(K_B) \neq 2$ irrespective of pameter and steady-state values.

## Prob2:

$$\frac{dC_A}{dt} = \frac{F}{V}(C_{Af} - C_A) - k_1 C_A - k_3 C_A^2$$

$$\frac{dC_B}{dt} = -\frac{F}{V} C_B + k_1 C_A - k_2 C_B$$

Data: $k_1 = 50$, $k_2 = 100$
$k_3 = 10$, $C_{Af} = 10$
$V = 1$

Solⁿ:

(a) for steady values of $C_A$ & $C_B$ at $F_s = 60$

at steady state --

$$0 = \frac{F_s}{V}(C_{Af} - C_{As}) - k_1 C_{As} - k_3 C_{As}^2 \quad \text{----} (1)$$

$$\frac{60}{1}(10 - C_{As}) - 50 C_{As} - 10 C_{As}^2 = 0$$

$$C_{As}^2 + 11 C_{As} - 60 = 0$$

$$(C_{As} + 15)(C_{As} - 4) = 0$$

$$C_{As} = 4, -15$$

($C_{As} \neq -15$, concentration of species cannot be negative)

$\therefore C_{As} = 4$

$$0 = -\frac{F_s}{V} C_{Bs} + k_1 C_{As} - k_2 C_{Bs}$$

$$0 = -60 C_{Bs} + 50 C_{As} - 100 C_{Bs}$$

$$C_{Bs} = \frac{5}{16} C_{As} = 1.25$$

$\therefore$ steady state values, $C_{As} = 4$ & $C_{Bs} = 1.25$ at $F_s = 60$.

(b) $$\frac{dC_A}{dt} = \frac{F}{V}(C_{Af} - C_A) - k_1 C_A - k_3 C_A^2 = f_1(C_A, F)$$

$$\frac{dC_B}{dt} = -\frac{F}{V} C_B + k_1 C_A - k_2 C_B = f_2(C_A, C_B, F)$$

Linear state space model –

$$\dot{X} = AX + BU \; \& \; Y = CX + DU$$

where, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} C_A - C_{As} \\ C_B - C_{Bs} \end{bmatrix}$ ; $u = F - F_s$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial C_A} & \frac{\partial f_1}{\partial C_B} \\ \frac{\partial f_2}{\partial C_A} & \frac{\partial f_2}{\partial C_B} \end{bmatrix} \; \& \; B = \begin{bmatrix} \frac{df_1}{du} \\ \frac{df_2}{du} \end{bmatrix} = \begin{bmatrix} \frac{df_1}{dF} \\ \frac{df_2}{dF} \end{bmatrix}$$

$$\frac{df_1}{dC_A} = -\frac{F_s}{V} - k_1 - 2k_3 C_{As} = -190 \;\Big|\; \frac{df_1}{dC_B} = 0$$

$$\frac{df_2}{dC_A} = k_1 = 50 \;\Big|\; \frac{df_2}{dC_B} = -\frac{F_s}{V} - k_2 = -160$$

$$\frac{df_1}{dF} = \frac{1}{V}(C_{Af} - C_{As}) = \frac{1}{1}(10 - 4) = 6 \;\Big|\; \frac{df_2}{dF} = -\frac{C_{Bs}}{V} = -1.25$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -190 & 0 \\ 50 & -160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ -1.25 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\, u$$

where $x_1 = C_A - C_{As} = C_A -$

$x_2 = C_B - C_{Bs} = C_B$

$u = F - F_s = F.$

(c) Now, for ~~desing~~ designing a state feedback controller using Bass-Gura method –

$$\dot{X} = AX + BU$$
$$U = -KX$$

Controllability Matrix :

$$C_B = [B \quad AB] = \begin{bmatrix} 6 & -1140 \\ -1.25 & 500 \end{bmatrix}$$

$\det(C_B) \neq 0 \quad \therefore \quad \text{Rank}(C_B) = 2$ [Fully state controllable]

$\therefore$ system is controllable.

Eigenvalues of $A \rightarrow \quad |\lambda I - A| = \begin{vmatrix} \lambda+190 & 0 \\ -50 & \lambda+160 \end{vmatrix} \Rightarrow (\lambda+190)(\lambda+160) = 0$

$$\lambda_{1,2} = -190, -160$$

$\therefore$ Let the desired location of closed loop poles, $\mu_{1,2} = -180, -180$

<u>Bass-Gura Method</u>

$$|sI - A| = s^2 + 350s + 30400 \longrightarrow a_1 = 350,\ a_2 = 30400$$

$$(s-\mu_1)(s-\mu_2) = (s+180)(s+180) = s^2 + 360s + 32400 \rightarrow \alpha_1 = 360, \alpha_2 = 32400$$

We know,

$$T = C_B W = \begin{bmatrix} 6 & -1140 \\ -1.25 & 500 \end{bmatrix} \begin{bmatrix} 350 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 960 & 6 \\ 62.5 & -1.25 \end{bmatrix}$$

$$T^{-1} = \frac{1}{-1575} \begin{bmatrix} -1.25 & -6 \\ -62.5 & 960 \end{bmatrix} = \frac{1}{1575} \begin{bmatrix} 1.25 & 6 \\ 62.5 & -960 \end{bmatrix}$$

$$K = [\alpha_2 - a_2 \quad \alpha_1 - a_1]\, T^{-1} = [32400 - 30400 \quad 360 - 350] \frac{1}{1575} \begin{bmatrix} 1.25 & 6 \\ 62.5 & -960 \end{bmatrix}$$

$$= \frac{1}{1575} [2000 \quad 10] \begin{bmatrix} 1.25 & 6 \\ 62.5 & -960 \end{bmatrix}$$

$$K = [1.984 \quad 1.524]$$

$\therefore$ state feedback controller for the reactor ---

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -190 & 0 \\ 50 & -160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ -1.25 \end{bmatrix} u$$

$$u = -\begin{bmatrix} 1.984 & 1.524 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

when $x_1 = C_A - 4$ & $x_2 = C_B - 1.25$