Integral Solutions

Scaling analysis does not provide details on the local values of τ and h (for a specific x) and the average values of τ_{0-L} and h_{0-L};

Where,

$$\tau_{0-L} = \frac{1}{L}\int \tau dx \ ; \quad h_{0-L} = \frac{1}{L}\int h dx$$

In the integral approach, we focus on the actual definition of

$$\tau = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} \qquad \text{and } h = -\frac{1}{\Delta T} k \frac{\partial T}{\partial y}\bigg|_{y=0}$$

So, we are interested in the variation of u and T close to the boundary. This can be accomplished by integrating each term of the boundary layer equation from y=0 to y=Y,

where Y> max (δ, δ_T) is situated in the free stream.

BL equations

Momentum balance

$$u\frac{\partial u}{\partial x} + v\frac{\partial (u)}{\partial x} = -\frac{1}{\rho}\frac{dP_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

Rewriting in integral form,

$$\frac{\partial u^2}{\partial x} + \frac{\partial (uv)}{\partial x} = -\frac{1}{\rho}\frac{dP_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \qquad \text{(i)}$$

(Since, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ =0)

and similarly,

$$\frac{\partial uT}{\partial x} + \frac{\partial (vT)}{\partial x} = \alpha\frac{\partial^2 T}{\partial y^2} \qquad \text{(ii)}$$

From the momentum balance equation,

$$\int_0^Y \frac{\partial u^2}{\partial x} dy + \int_0^Y \frac{\partial (uv)}{\partial x} dy = -\frac{1}{\rho}\frac{dP_\infty}{dx}\int_0^Y dy + \nu\int_0^Y \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) dy$$

$$\frac{d}{dx}\int_0^Y u^2 dy + [uv]_0^Y = -\frac{1}{\rho}\frac{dP_\infty}{dx}Y + \left[\nu\frac{\partial u}{\partial y}\right]_0^Y \qquad \text{[Leibnitz rule: } \frac{d}{dx}\int_a^b f(x,y)dy = \int_a^b \frac{\partial}{\partial x}f(x,y)dy \text{]}$$

$$\frac{d}{dx}\int_0^Y u^2 dy + [u_Y v_Y - u_0 v_0] = -\frac{1}{\rho}\frac{dP_\infty}{dx}Y + \left[\nu\left.\frac{\partial u}{\partial y}\right|_{y=Y} - \nu\left.\frac{\partial u}{\partial y}\right|_{y=0}\right] \qquad \text{(iii)}$$

Now, $\nu\left.\dfrac{\partial u}{\partial y}\right|_{y=Y} = 0$ [as free stream is uniform]

$u_Y = u_\infty; v_0 = 0$ [impermeable wall]

So, equation (iii) reduces to

$$\therefore \frac{d}{dx}\int_0^Y u^2 dy + v_Y u_\infty = -\frac{1}{\rho}\frac{dP_\infty}{dx}Y - \nu\left.\frac{\partial u}{\partial y}\right|_{y=0} \qquad \text{(iv)}$$

From, continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\int_0^Y \frac{\partial u}{\partial x}dy + \int_0^Y \frac{\partial v}{\partial y}dy = 0$$

$$\frac{d}{dx}\int_0^Y u dy + v_y - v_0 = 0$$

$$v_y = v_0 - \frac{d}{dx}\int_0^Y u dy$$

$$v_0 \approx 0$$

$$v_y = -\frac{d}{dx}\int_0^Y u dy \qquad \text{(v)}$$

Substituting v_y into equation (iv)

$$\therefore \frac{d}{dx}\int_0^Y u^2 dy + [v_0 - \frac{d}{dx}\int_0^Y u dy]u_\infty = -\frac{1}{\rho}\frac{dP_\infty}{dx}Y - \nu\left.\frac{\partial u}{\partial y}\right|_{y=0}$$

$$v_0 \approx 0$$

$$\therefore \frac{d}{dx}\int_0^Y u^2 dy + [0 - \frac{d}{dx}\int_0^Y u dy] u_\infty = -\frac{1}{\rho}\frac{dP_\infty}{dx}Y - \nu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

$$\frac{d}{dx}\int_0^Y u^2 dy - [\frac{d}{dx}\int_0^Y u dy - \frac{du_\infty}{dx}\int_0^Y u dy] = -\frac{1}{\rho}\frac{dP_\infty}{dx}Y - \nu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

$$\therefore \frac{d}{dx}\int_0^Y (u^2 - u v_\infty) dy = -\frac{1}{\rho}\frac{dP_\infty}{dx}Y - \nu \left.\frac{\partial u}{\partial y}\right|_{y=0} - \frac{dv_\infty}{dx}\int_0^Y u dy$$

Or,

$$\frac{d}{dx}\int_0^Y u(v_\infty - u) dy = \frac{1}{\rho}\frac{dP_\infty}{dx}Y + \nu \left.\frac{\partial u}{\partial y}\right|_{y=0} + \frac{dv_\infty}{dx}\int_0^Y u dy \qquad \text{(vi)}$$

Similarly, for the energy equation, we will get,

$$\frac{d}{dx}\int_0^Y u(T_\infty - T) dy = \alpha \left.\frac{\partial T}{\partial y}\right|_{y=0} + \frac{dT_\infty}{dx}\int_0^Y u dy \qquad \text{(vii)}$$