Information Theory
Practice Set 8

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Solutions are not to be returned

Practice Set

1. Recall that the MAP rule (maximum a posteriori decision) minimizes the average probability of error in a Bayesian detection problem. In the binary hypothesis testing situation, let $\theta = 0$ denote the hypothesis $H_0 : X \sim P_0$ and let $\theta = 1$ denote the hypothesis $H_1 : X \sim P_1$. Let $P[\theta = 0] = \pi_0 = 1 - P[\theta = 1]$. Then the MAP rule is

$$\text{Choose } \hat{\theta} = 0 \text{ if } P[\theta = 0|X = x] \geq P[\theta = 1|X = x]$$
$$\text{Choose } \hat{\theta} = 1 \text{ otherwise.}$$

   This MAP rule minimizes the probability of error $P[\theta \neq \hat{\theta}]$.

   Use this information to solve Problem 11.16(b) in Cover & Thomas (exercise problem at the end of Chapter 11).

2. From Cover & Thomas, Exercise problems at the end of Chapter 11: 11.12

3. Consider the binary hypothesis testing problem, with notation as used in the lectures. Show that the likelihood ratio $L(X) = P_1(X)/P_2(X)$ is a sufficient statistic of X for hypothesis testing. You can use the Fisher factorization theorem to prove this result.

4. (From Polyanskiy & Wu) Consider P_1 vs. P_2 hypothesis testing problem. For each $\alpha \in [0, 1]$, let $\beta^*(\alpha)$ denote the minimum possible type-II error under the condition that the type-I error is at the most α.

   (a) We say that two distributions P_1 and P_2 are *mutually singular* with respect to each other, if there exists a subset $A \subset \mathcal{X}$ such that $P_1[A] = 1$ and $P_2[A] = 0$. Plot β^* as a function of α if P_1 and P_2 are mutually singular.

   (b) Plot $\beta^*(\alpha)$ if $P_1 = P_2$.

5. Show that for any binary hypothesis testing problem and for any choice of $\alpha \in [0, 1]$ there exists a randomized test with both type-I and type-II errors equal to α.

6. *Bound on β under likelihood ratio test (Polyanskiy & Wu).* Let A be the acceptance region under the deterministic likelihood ratio test $A = \{x : P_1(x)/P_2(x) > T\}$. Then, $\alpha = P_1[A^c]$. The following derivation gives a bound on β. You must verify the correctness of each step.

$$\begin{aligned}\beta &= P_2[A] \\ &= \sum_{x \in A} P_2(x) \\ &\leq \sum_{x \in A} P_1(x)/T \\ &= \frac{(1-\alpha)}{T}.\end{aligned}$$

   Note that this implies $\beta \leq \frac{1}{T}$.

7. Can you provide examples of binary hypothesis testing problems for which

   (a) $\alpha = 0$, $\beta = 1$ is a Pareto-optimal point.

   (b) $\alpha = 0$, $\beta = 1$ is NOT a Pareto-optimal point.

8. Consider the P_1 versus P_2 hypothesis testing problem based on n iid observations $X_1, \ldots, X_n$. Consider a test for n observations with acceptance region $A_n \subset \mathcal{X}^n$. Let α_n and β_n be the type-I and type-II errors of this test.

   You must show that there exists a test based on $n+1$ observations with type-I and type-II errors equal to α_n and β_n, respectively. That is, identify an acceptance region $A_{n+1} \subset \mathcal{X}^{n+1}$ with the same value of type-I and type-II errors as the test based on n observations.
