A Brief Introduction to Wavelets

1 Introduction

These notes are meant to summarize the material covered in class. The notation and the formulation is adapted from the book "A first course in wavelets with Fourier analysis," by A. Boggess, and F. J. Narcowich [1] (available in the library).

1.1 Motivation

Traditional frequency analysis tools such as Fourier series and Fourier transforms do not provide any localization in time. In other words, they are provide a global picture of the frequency content of a signal. In many practical applications it is important to localize a signal is time and frequency or space and frequency. For e.g., if one were analyzing the notes played by a pianist using Fourier-based techniques, it would not be possible to find or localize in time when a particular note was played. The same extends to image data as well – if we were asked to find where in an image a particular spatial frequency occurred, Fourier-based techniques would not give us the answer. While a short-time Fourier transform (STFT) does provide time-frequency localization, it is limited in the amount of localization it can provide (due to the uncertainty principle). Further, since Fourier-based techniques use complex sinusoid bases, they have trouble representing functions with discontinuities.

Wavelet bases are defined to address these drawbacks. For simplicity, the following analysis will be done using the most basic wavelet basis - the Haar wavelet. The results hold for other more complicated bases as well.

2 Mathematical Preliminaries

In the section, basic definitions are presented and the notation from Kovacevic and Chebira [2] is used. These definitions will help us with the rest of the development. For the purpose of this discussion, we will consider finite dimensional real or complex spaces ($\mathbb{R}^n$ or $\mathbb{C}^n$).

2.1 Bases

A subset $\Phi = \{\varphi_i\}_{i \in I}$ of a finite-dimensional vector space $\mathbb{V}$ (where I is some index set) is called a *basis* for $\mathbb{V}$ if $\mathbb{V} = \text{span}(\Phi)$ and the vectors in Φ are linearly independent (given $S \subset \mathbb{V}$, the span of S is the subspace of $\mathbb{V}$ consisting of all finite linear combinations of vectors in S). If $I = 1, \ldots, n$, we say that $\mathbb{V}$ has dimension n.

Exercise: An example basis we discussed in class was the DFT basis: $\varphi_i = \frac{1}{\sqrt{n}}[W_n^0, W_n^i, \ldots, W_n^{i(n-1)}]^T$, where $W_n = e^{j2\pi/n}$. What is dimension of this basis?

2.1.1 Inner product

The inner product between two vectors $x, y \in \mathbb{C}^n$ (or any other Hilbert space) is defined as

$$\langle x, y \rangle = \sum_{i \in I} x_i^* y_i, \tag{1}$$

where x_i^* is the complex conjugate of x_i.

Exercise: For the DFT basis, verify that the inner product between $\langle \varphi_i, \varphi_j \rangle$ $(i \neq j)$ is indeed 0.

2.1.2 Norm

The norm of a vector x is defined as

$$||x|| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i \in I} |x_i|^2}. \tag{2}$$

Exercise: For the DFT basis, verify that $\langle \varphi_i, \varphi_i \rangle = 1$, for all $i \in I$.

2.2 Orthonormal Bases (ONB)

A basis $\Phi = \{\varphi_i\}_{i \in I}$ where the vectors are orthonormal:

$$\langle \varphi_i, \varphi_j \rangle = \delta_{i-j}, \tag{3}$$

is called an *orthonormal basis (ONB)*. In other words, an orthonormal system is called an ONB for a given space of vectors $\mathbb{H}$, if for every x in $\mathbb{H}$,

$$x = \sum_{i \in I} X_i \varphi_i, \tag{4}$$

for some scalars X_i. These scalars are called the transform or expansion coefficients of x with respect to Φ and they are given by

$$X_i = \langle \varphi_i, x \rangle, \tag{5}$$

for all $i \in I$.

Exercise: Verify that the DFT basis is an ONB.

3 The Haar Wavelet

We will discuss wavelets and their properties with the Haar example. This simplifies our analysis without diluting any of the results. We operate in $L^2(\mathbb{R})$ where the notions of basis and inner product can be defined as in finite dimensional vector spaces.

3.1 The Haar Approximation Function

$$\phi(x) = \begin{cases} 1, & 0 \leq x < 1, \\ 0, & \text{elsewhere.} \end{cases} \tag{6}$$

The set of all functions defined using $\phi(x)$ is given by

$$V_0 = \sum_{k \in \mathbb{Z}} a_k \phi(x-k), a_k \in \mathbb{R}. \tag{7}$$

The basic function $\phi(x)$ can be scaled and shifted to achieve the fundamental properties of wavelet basis. This is captured by the function $\phi(2^j x - k)$ where $j, k \in \mathbb{Z}$ are the scaling and shifting parameters.

Exercise: From the basic definition of $\phi(x)$, sketch $\phi(2^j x - k)$.

The set of all functions defined using $\phi(2^j x - k)$ is given by

$$V_j = \sum_{k \in \mathbb{Z}} a_k \phi(2^j x - k), a_k \in \mathbb{R} \tag{8}$$

From the definition of $\phi(x)$ and $\phi(2^j x - k)$ it follows that

$$V_0 \subset V_1 \subset \ldots V_j \subset V_{j+1} \tag{9}$$

3.1.1 Examples

Given a function $f(x) = \begin{cases} -1 & -1 \leq x < 0, \\ 1 & 0 \leq x < 1, \\ -1 & 1 \leq x < 2 \end{cases}$. Express $f(x)$ in terms of the approximation function $\phi(x)$.

Answer: From the definition of $\phi(x)$ in (6), it follows in a straight-forward manner that

$$f(x) = -\phi(x+1) + \phi(x) - \phi(x-1). \tag{10}$$

3.2 The Haar Wavelet Function

The approximation function family $\phi(2^j x - k)$ has the ability to perfectly represent any function of the type discussed in the previous example. Why do we need the wavelet function? The answer to this can be explained intuitively as follows. The representation using approximation functions provides an all-pass view of the signal. If we want a bandpass view of the signal, we resort to the wavelet function. In other words, if we want to isolate a part of the function that is present in V_j but not in V_{j-1}, we need the wavelet function as defined in the following:

$$\psi(x) = \phi(2x) - \phi(2x - 1). \tag{11}$$

At a scale j, the relationship generalizes to

$$\phi(2^j x) = \frac{[\psi(2^{j-1}x) + \phi(2^{j-1}x)]}{2},\ \phi(2^j x - 1) = \frac{[\phi(2^{j-1}x) - \psi(2^{j-1}x)]}{2}. \tag{12}$$

The set of all functions defined using $\psi(x)$ is defined as

$$W_0 = \sum_{k \in \mathbb{Z}} a_k \psi(x - k), a_k \in \mathbb{R}. \tag{13}$$

From the definition of $\phi(x), \psi(x)$ it follows that

$$V_1 = V_0 \oplus W_0. \tag{14}$$

This relation can be generalized to

$$V_j = V_0 \oplus W_0 \oplus W_1 \ldots \oplus W_{j-1}. \tag{15}$$

3.2.1 Examples

Given a function $f(x) = 2\phi(4x) + 2\phi(4x - 1) + \phi(4x - 2) - \phi(4x - 3)$. Find the highest scale j for this function. Express $f(x)$ in terms of the approximation function at scale 0 and wavelet functions scales $j - 1 \ldots 0$. In other words, analyze the signal at various scales $V_j = V_0 \oplus W_0 \oplus W_1 \ldots \oplus W_{j-1}$.

Solution: Examining $f(x)$, it is clear that the highest scale

$$j = 2. \tag{16}$$

Using the recursive relations in (12), $f(x)$ can be written as:

$$f(x) = \phi(x) + \psi(x) + \psi(2x - 1). \tag{17}$$

4 Wavelet Analysis

In this section we derive a recursive relation between the analysis coefficients at successive scales of the function representation.

$$\begin{aligned}
f^j(x) &= \sum_{k \in \mathbb{Z}} a_k \phi(2^j x - k) && (18)\\
&= \sum_{k \in \mathbb{Z}} [a_{2k}\phi(2^j x - 2k) + a_{2k+1}\phi(2^j x - (2k+1))] && (19)\\
&= \sum_{k \in \mathbb{Z}} [a_{2k}\phi(2(2^{j-1}x - k)) + a_{2k+1}\phi(2(2^{j-1}x - k) - 1))] && (20)\\
&= \sum_{k \in \mathbb{Z}} a_{2k} \left[\frac{\phi(2^{j-1}x - k) + \psi(2^{j-1}x - k)}{2}\right] + a_{2k+1}\left[\frac{\phi(2^{j-1}x - k) - \psi(2^{j-1}x - k)}{2}\right] \text{(from (12))} && (21)\\
&= \sum_{k \in \mathbb{Z}} \left(\frac{a_{2k} + a_{2k+1}}{2}\right)\phi(2^{j-1}x - k) + \left(\frac{a_{2k} - a_{2k+1}}{2}\right)\psi(2^{j-1}x - k). && (22)
\end{aligned}$$

Exercise: Show how this relation can be implemented using the perfect reconstruction filter bank we derived in class.

References

[1] A. Boggess and F. J. Narcowich, *A First Course in Wavelets with Fourier Analysis*. John Wiley & Sons, 2009.

[2] J. Kovacevic and A. Chebira, "Life beyond bases: The advent of frames (part i)," *Signal Processing Magazine, IEEE*, vol. 24, no. 4, pp. 86–104, 2007.