Information Theory
Practice Set 5

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Solutions are not to be returned

Independent Reading

It will be a good exercise to explore these questions independently and understand their proofs. Please use any appropriate textbook or good online lecture materials.
This is NOT a part of course syllabus/evaluation.

1. Are all convex functions continuous? Why do you think a discontinuity will make a function non-convex?

2. Are all convex functions differentiable?

Practice Set

See the last page for hints for questions 1, 3, 4, 5, 6.

1. Exercise Problems from Cover and Thomas: 2.28

2. Reading exercise from Cover and Thomas: Lemma 2.10.1 and the following corollary.

3. Are the following sets convex?

   (a) Let $\mathcal{X} = \{0, 1, 2, 3, \dots\}$. All probability mass functions on $\mathcal{X}$ with mean equal to 5.

   (b) Let $\mathcal{X} = \{0, 1, 2, 3, \dots\}$. All probability mass functions on $\mathcal{X}$ with $P(X = 0) = 0.5$.

4. Show that $D(p\|q)$ is convex in the pair (p, q).

5. Use the Jensen's inequality to prove the arithmetic-mean-geometric-mean (AM-GM) inequality: for positive real numbers $x_1, \dots, x_n$
$$\frac{1}{n}(x_1 + \cdots + x_n) \geq \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}.$$

6. Can you argue that the sum of two convex functions is convex?

7. If f and g are convex functions and g is a non-decreasing function, can you explain why the following steps are correct, and what does this result mean (as a summary/theorem). Here $g \circ f(x) \triangleq g(f(x))$.
$$\begin{aligned} g \circ f(\alpha x + (1-\alpha)y) &= g\left(f(\alpha x + (1-\alpha)y)\right) \\ &\leq g\left(\alpha f(x) + (1-\alpha) f(y)\right) \\ &\leq \alpha g(f(x)) + (1-\alpha) g(f(y)) \\ &= \alpha g \circ f(x) + (1-\alpha) g \circ f(y). \end{aligned}$$

8. Use the second derivative test to check if the following functions are convex, strictly convex, concave, strictly concave:
$$e^x, \quad x^2, \quad ax + b, \quad \sqrt{x}.$$

9. Can you construct a numerical example to highlight the fact that $I(X;Y)$ is concave in $p(x)$ and convex in $p(y|x)$? You might have to use a calculator, or a plotting software for visualization.

   You can assume that X and Y are Bernoulli, and consider following scenarios:

(a) to observe concavity with respect to $p(x)$: assume $p(y|x) = \epsilon$ if $x \neq y$ and $p(y|x) = 1 - \epsilon$ if $x = y$, for some $\epsilon < 0.5$. Consider two choices for $p(x)$:

$$P(X = 0) = 1, \text{ and } P(X = 0) = 0.5.$$

(b) to observe convexity with respect to $p(y|x)$: vary the value of ϵ from the previous point, and assume $P(X = 0) = P(X = 1) = 1/2$.

Hints for selected questions
in the next page

Hints

- Q.1

  As a first step, find two probability mass functions q_X and r_X such that

  $$H(q_X) = H(r_X) = H\left((p_1, \ldots, p_i, \ldots, p_j, \ldots, p_m)\right), \text{ and } (q_X + r_X)/2 \quad = \left(p_1, \ldots, \frac{p_i + p_j}{2}, \ldots, \frac{p_i + p_j}{2}, \ldots, p_m\right).$$

- Q.3

  Both sets are convex.

- Q.4

  Expand $D(\alpha p_1 + (1 - \alpha p_2) \| \alpha q_1 + (1 - \alpha) q_2)$, and use the log-sum inequality.

- Q.5

  Apply log on both sides, and use Jensen's inequality.

- Q.6

  This is not hard, we simply need to use the definition of convexity of functions on

  $$(f + g)(\alpha x + (1 - \alpha) y) = f(\alpha x + (1 - \alpha) y) + g(\alpha x + (1 - \alpha) y).$$
