Wireless Communication

Homework – 2

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1) Given X and Y are independent with, $X, Y \sim \mathcal{N}(0, \sigma^2)$

and $Z = |X + jY|$

$$Z = |X + jY| = \sqrt{X^2 + Y^2}$$

We know that –
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \times \exp\left(-\frac{(x-0)^2}{2\sigma^2}\right) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-x^2}{2\sigma^2}\right)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-0)^2}{2\sigma^2}\right) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{y^2}{2\sigma^2}\right)$$

Since X and Y are independent –

$$f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$$

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$$

CDF of Z :
$$F_Z(z) = \Pr(Z \le z)$$

$$F_Z(z) = \Pr(\sqrt{X^2+Y^2} \le z)$$

$$\Rightarrow F_Z(z) = \iint_{\sqrt{x^2+y^2} \le z} f_{XY}(x, y)\, dx\, dy$$

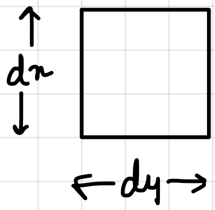
$$F_Z(z) = \iint_{\sqrt{x^2+y^2} \le z} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right) dx\, dy$$

Transform the coordinates as follows –

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \Rightarrow \begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$

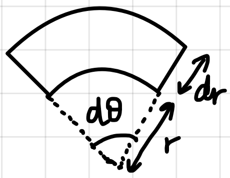
(polar coordinate transformation)

Area in Rectangular Coordinate



Area = $dx\,dy$

Area in Polar Coordinate



Area = $r\,d\theta\,dr$

$$\Rightarrow F_Z(z) = \int_0^{2\pi}\int_0^{z} \frac{1}{2\pi\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right) r\,dr\,d\theta$$

$$F_Z(z) = \frac{1}{2\pi\sigma^2}\underbrace{\int_0^{2\pi} d\theta}_{2\pi}\ \underbrace{\int_0^{z} r\exp\left(-\frac{r^2}{2\sigma^2}\right) dr}_{\text{take } t = -\frac{r^2}{2\sigma^2}}$$

$$\downarrow$$

$$r\,dr = -\sigma^2 dt$$

$$F_Z(z) = \frac{1}{2\pi\sigma^2}\cdot 2\pi \int_0^{-z^2/2\sigma^2} \exp(t)\,dt\,(-\sigma^2)$$

$$F_Z(z) = -\exp(t)\Big|_0^{-z^2/2\sigma^2} = 1 - \exp\left(-\frac{z^2}{2\sigma^2}\right)$$

$$F_Z(z) = 1 - e^{-z^2/2\sigma^2}$$ $\Rightarrow$ CDF of a Rayleigh distribution

$\Rightarrow Z = |X + jY|$ is Rayleigh distributed //

For Z^2 - $F_{Z^2}(z') = P_r(Z^2 \le z') = P_r(x^2+y^2 \le z')$

$\downarrow$ take z' as z^2

$$F_{Z^2}(z') = F_Z(\sqrt{z'}) \Leftarrow$$

This is from Rayleigh Distribution

$$F_{Z^2}(z') = 1 - \exp\left(-\frac{1}{2\sigma^2}z'\right)$$ $\Rightarrow$ CDF of Exponential Dist.

$\Rightarrow Z^2 = |X + jY|^2$ is Exponential distributed. //

2) Now given that X has non-zero mean $\Rightarrow X \sim N(\mu, \sigma^2)$

$Y \sim N(0, \sigma^2)$

$$\Rightarrow \quad f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$\Rightarrow \quad f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right)$$

X and Y are still independent, hence their joint PDF is given by -

$$f_{XY}(x,y) = f_X(x) f_Y(y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(x-\mu)^2+y^2}{2\sigma^2}\right)$$

CDF of Z: $F_Z(z) = Pr(Z \le z) = Pr(\sqrt{X^2+Y^2} \le z)$

$$F_Z(z) = \iint_{\sqrt{x^2+y^2} \le z} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(x-\mu)^2+y^2}{2\sigma^2}\right) dx\,dy$$

$$F_Z(z) = 1 - \iint_{\sqrt{x^2+y^2} \ge z} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(x-\mu)^2+y^2}{2\sigma^2}\right) dx\,dy$$

$$F_Z(z) = 1 - \iint_{\sqrt{x^2+y^2} \ge z} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right) \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \exp\left(\frac{\mu x}{\sigma^2}\right) \cdot dx\,dy$$

Here again we can perform the coordinate transformation as follows -

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \quad \begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$

and $dx\,dy \rightarrow r\,dr\,d\theta$

$$\therefore F_Z(z) = 1 - \int_z^{\infty} \int_{-\pi}^{\pi} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \exp\left(\frac{\mu r\cos\theta}{\sigma^2}\right) r\,d\theta\,dr$$

Rearranging the above integral -

$$F_Z(z) = 1 - \int_z^{\infty} \frac{r}{\sigma^2} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \exp\left(-\frac{r^2}{2\sigma^2}\right) \underbrace{\int_{-\pi}^{\pi} \frac{1}{2\pi} \exp\left(\frac{\mu r}{\sigma^2} \cos\theta\right) d\theta}_{\text{Bessel's Integral } = I_0\left(\frac{\mu r}{\sigma^2}\right)} \, dr$$

$$\therefore F_Z(z) = 1 - \int_z^{\infty} \frac{r}{\sigma^2} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \exp\left(-\frac{r^2}{2\sigma^2}\right) I_0\left(\frac{\mu r}{\sigma^2}\right) dr$$

Take $r = \sigma K \rightarrow$

$$F_Z(z) = 1 - \underbrace{\int_{z/\sigma}^{\infty} K \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \exp\left(-\frac{K^2}{2}\right) I_0\left(\frac{\mu}{\sigma} K\right) dK}_{\text{Marcum Q-function } = Q_1\left(\frac{\mu}{\sigma}, \frac{z}{\sigma}\right)}$$

$$F_Z(z) = 1 - Q_1\left(\frac{\mu}{\sigma}, \frac{z}{\sigma}\right)$$ $\Rightarrow$ CDF of Rice distribution (from wikipedia)

$\Rightarrow$ Z follows Rician distribution. //

3) Given power outage probability = 0.05

$$P_0 = -70 \text{ dBm}$$

For a Rayleigh fading channel-

$$Pr(z \leq P_0) = \int_0^{P_0} P_z(z)\,dz = \int_0^{P_0} \frac{1}{P_{avg}} \exp\left(-\frac{P}{P_{avg}}\right) dP$$

$$Pr(z \leq P_0) = -\exp\left(-\frac{P}{P_{avg}}\right)\Bigg|_0^{P_0} = 1 - \exp\left(-\frac{P_0}{P_{avg}}\right) = 0.05$$

(given)

$$P_0(\text{dBm}) = 10\log_{10}\left(\frac{P_0(\text{mW})}{1\text{mW}}\right) = -70 \Rightarrow P_0 = 10^{-10}\,W$$

$$\therefore -\frac{P_0}{P_{avg}} = \log_e(0.95) \Rightarrow P_{avg} = \frac{-10^{-10}}{-0.05129} = 19.6 \times 10^{-7} \text{ mW}$$

$$\therefore P_{avg} = 19.6 \times 10^{-7} \text{ mW} = -57.102 \text{ dBm}$$

4) Received power from two base stations is taken as -

$$\left.\begin{aligned} P_{r,1} &= W + Z_1 \\ P_{r,2} &= W + Z_2 \end{aligned}\right\}$$ W can be thought of as average received signal power //

$Z_1 \Rightarrow$ Shadowing component b/w base station 1 & mobile //

$Z_2 \Rightarrow$ Shadowing component b/w base station 2 & mobile //

From the given definition-

$$P_{outage} = Pr\left(P_{r,1} < T \text{ and } P_{r,2} < T \right)$$

Since both $P_{r,1}$ and $P_{r,2}$ are independent-

$$P_{outage} = Pr\left(P_{r,1} < T\right) \cdot Pr\left(P_{r,2} < T\right)$$

Both $P_{r,1}$ and $P_{r,2}$ are gaussians $\mathcal{N}(\omega, \sigma^2)$
$(\because z_1 \,\&\, z_2 \text{ are } \mathcal{N}(0, \sigma^2))$

$$Pr\left(P_{r,1} < T\right) = Pr\left(P_{r,2} < T\right) = \int_{-\infty}^{T} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\omega)^2}{2\sigma^2}\right) dx$$

Take $z = \dfrac{\omega - x}{\sigma} \Rightarrow dx = -\sigma dz$

$$Pr\left(P_{r,1} < z\right) = \int_{\infty}^{\frac{\omega - T}{\sigma}} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{z^2}{2}\right) (-\sigma)\, dz = Pr\left(P_{r,2} < z\right)$$

$$Pr\left(P_{r,1} < z\right) = \underbrace{\int_{\frac{\omega-T}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz}_{Q\left(\frac{\omega - T}{\sigma}\right) = Q\left(\frac{\Delta}{\sigma}\right) \text{ with } \Delta = \omega - T} = Pr\left(P_{r,2} < z\right)$$

$$\therefore P_{outage} = \left(Q\left(\frac{\omega - T}{\sigma}\right)\right)^2 = Q^2\left(\frac{\Delta}{\sigma}\right)$$

5) For two jointly distributed gaussian random variables X and Y as $\mathcal{N}(\omega, \sigma^2)$ and correlation coefficient as b, their joint PDF is given by –

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2\sqrt{1-b^2}} \exp\left(-\frac{1}{2(1-b^2)} z\right)$$

where $z = \dfrac{(x-\omega)^2 + (y-\omega)^2 - 2b(x-\omega)(y-\omega)}{\sigma^2}$

$$P_{outage} = P_r(P_{r,1} < T \text{ and } P_{r,2} < T)$$

$$P_{outage} = \int_{-\infty}^{T}\int_{-\infty}^{T} f_{XY}(x,y)\, dx\, dy$$

$$P_{outage} = \frac{1}{2\pi\sigma^2\sqrt{1-b^2}} \int_{-\infty}^{T}\int_{-\infty}^{T} \exp\left(-\frac{1}{2(1-b^2)\sigma^2}\left((x-\omega)^2 - 2b(x-\omega)(y-\omega) + (y-\omega)^2\right)\right) dx\, dy$$

Take $u = \dfrac{\omega - x}{\sigma\sqrt{1-b^2}} \Rightarrow dx = -\sigma\sqrt{1-b^2}\, du$

$v = \dfrac{\omega - y}{\sigma\sqrt{1-b^2}} \Rightarrow dy = -\sigma\sqrt{1-b^2}\, dv$

$$\therefore P_{outage} = \frac{\sqrt{1-b^2}}{2\pi} \int_{\beta}^{\infty}\int_{\beta}^{\infty} \exp\left(-\frac{u^2 - 2buv + v^2}{2}\right) du\, dv$$

with $\beta = \dfrac{\omega - T}{\sigma\sqrt{1-b^2}}$

6) a) Description of the code-

In-phase component: $r_I = \sum_{n=1}^{N(t)} \alpha_n(t) \cos \phi_n(t)$

Quadrature-phase component: $r_Q = \sum_{n=1}^{N(t)} \alpha_n(t) \sin \phi_n(t)$

$$\left[\phi_n(t) = 2\pi f_c T_n(t) - \phi_{D_n}\right]$$

In the code, I've initialized the variables and use it to find the sum.

Also, $\omega_d = 2\pi f_d$ → Doppler Frequency

The magnitude component, $r = \sqrt{r_I^2 + r_Q^2}$

Take the mean as, $\mu = \dfrac{\sum r}{Num}$

Now we plot $\underbrace{10 \log_{10}\left(\frac{r}{\mu}\right)}_{\text{Envelope in dB}}$ vs time

$$Z_Q(t) = \frac{1}{scale1}\left[2 \sum_{n=1}^{N_o} \sin \beta_n \cos \phi_n(t) + \sqrt{2} \sin\alpha \cos \phi_n\right]$$

$$Z_i(t) = \frac{1}{Scale2}\left[2 \sum_{n=1}^{N_o} \cos \beta_n \cos \phi_n(t) + \sqrt{2} \cos\alpha \cos \phi_n\right]$$

$$Z = Z_i + jZ_Q$$

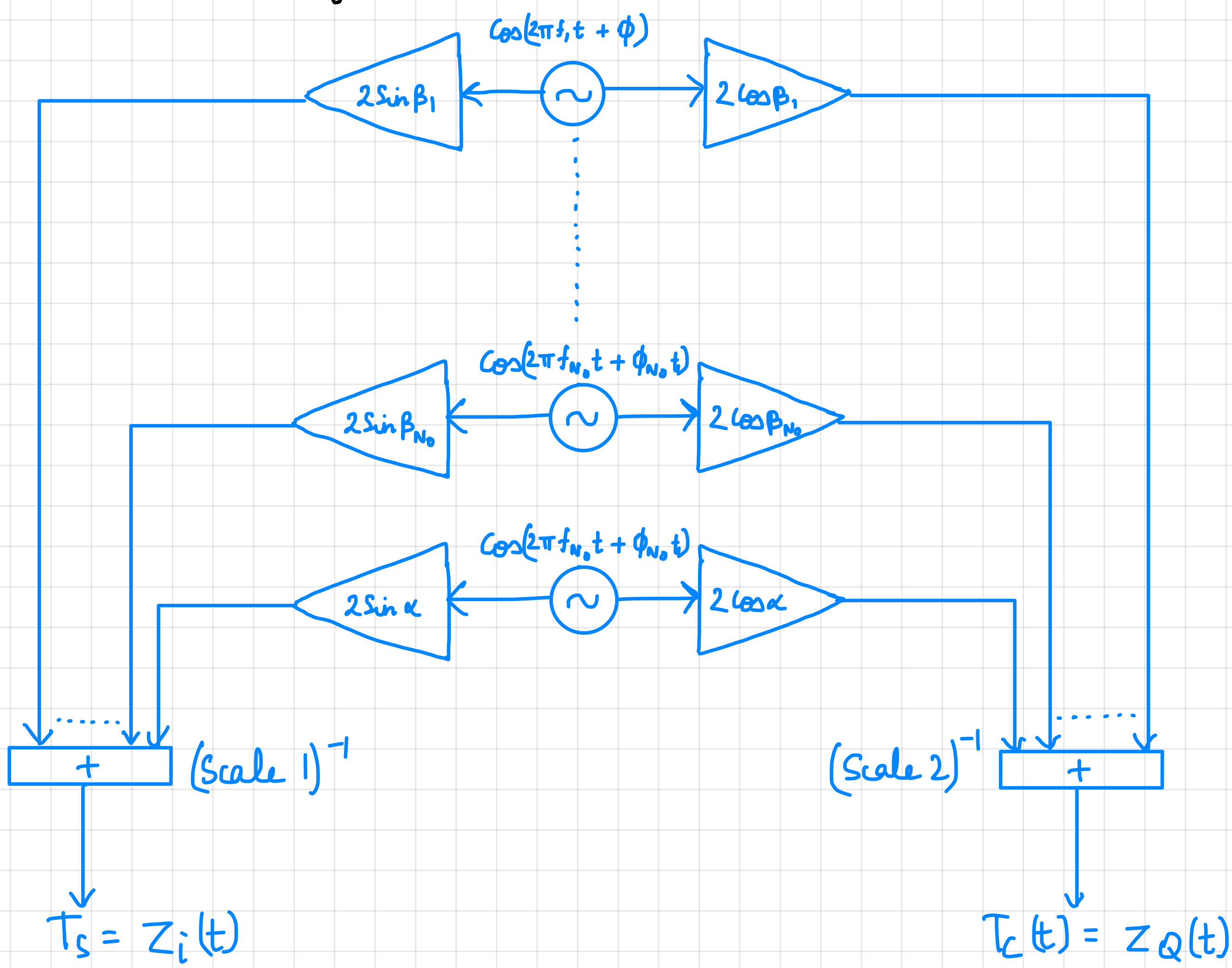
$$E[z(t)\, z^*(t+\Delta t)] = c\, J_o(2\pi f_D \Delta t)$$

Doppler's Frequency

Bessel's integral

constant

Block diagram -



6) c) For a Rayleigh Distribution - (from Q3)

$$\Pr(Z \le P_0) = 1 - \exp\left(-\frac{P_0}{P_{avg}}\right)$$

Given $P_0 = -70\,\text{dBm} \Rightarrow P_0(\text{W}) = 10^{-10}\,\text{W}$

$P_{avg} = 1\,\text{W}$

$\therefore \Pr(z \le P_0) = 1 - \exp(-10^{-10}) \approx 10^{-10}$ [from Taylor Series Expansion]

→ Very low probability of power outage

In [1]:

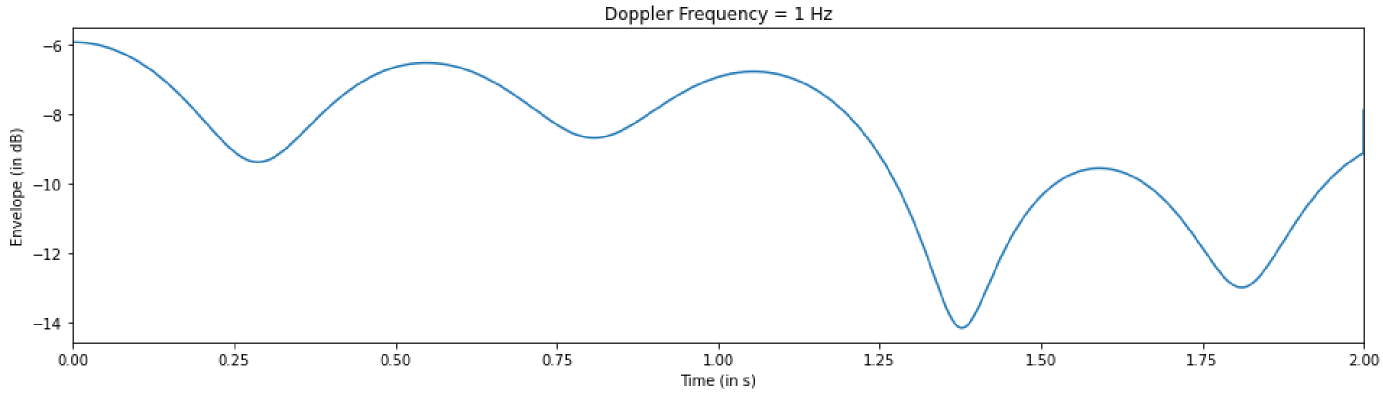
```
import numpy as np
import matplotlib.pyplot as plt
```

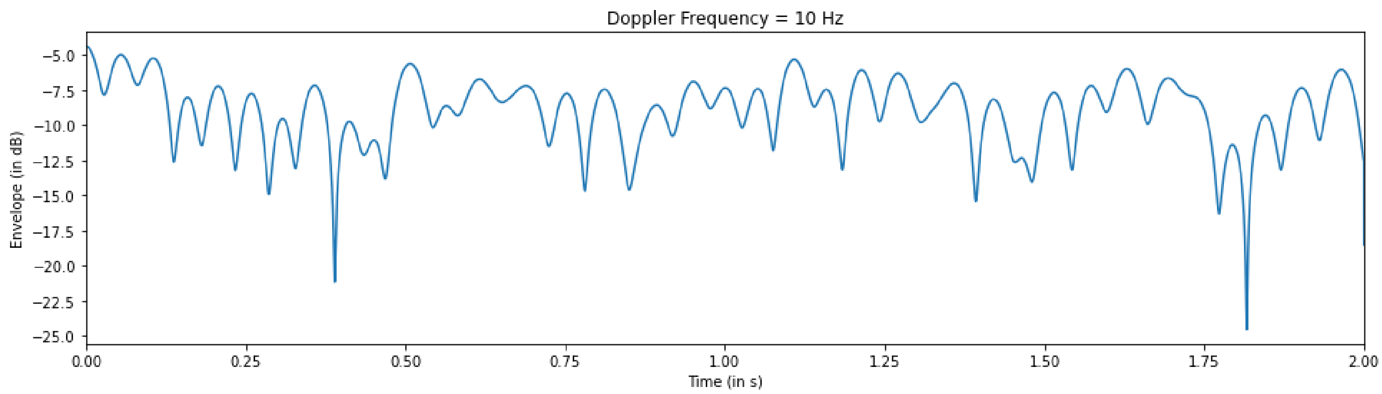
In [2]:

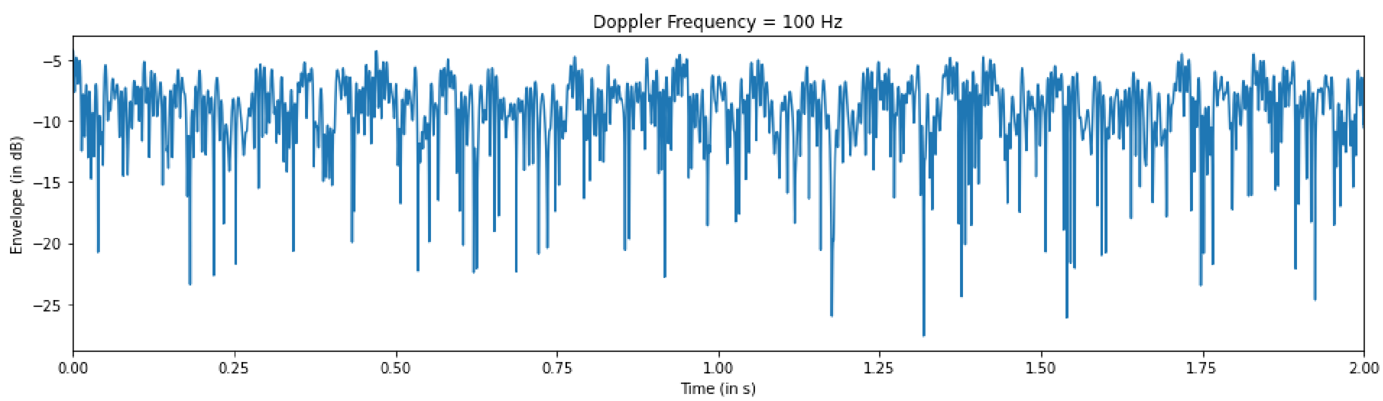
```
def rayleigh_fading_jakes_model(time, f_sampling, f_doppler, N):

    M = int(((N / 2) - 1) / 2)
    W_doppler = 2 * np.pi * f_doppler

    Tot_samples = time * f_sampling
    ri_tmp, rq_tmp, ri_alpha = np.zeros([M, Tot_samples]), np.zeros([M, Tot_samples]), n

    Ts = np.arange(0, time, (1 / f_sampling))

    for n in range(M):
        beta = np.pi * n / M
        Wn = W_doppler * np.cos(2 * np.pi * n / N)

        for t in Ts:
            ri_tmp[n, int(f_sampling * t)] = 2 * np.cos(beta) * np.cos(Wn * t)
            rq_tmp[n, int(f_sampling * t)] = 2 * np.sin(beta) * np.cos(Wn * t)

        ri_alpha[0, int(f_sampling * t)] = np.sqrt(2) * np.cos(W_doppler * t)

    # Finding out In-Phase component
    r_in = np.sum(ri_tmp, axis = 0) + ri_alpha
    # Finding out Quadrature-Phase component
    r_quad = np.sum(rq_tmp, axis = 0)

    # The Net magnitude
    r = np.sqrt(r_in ** 2 + r_quad ** 2)
    mean = np.sum(r) / Tot_samples

    z_dB = (10 * np.log10(r / mean))[0, :]

    return Ts, z_dB
```

In [3]:

```
time = 2
fs = 1000000
fD_list = [1, 10, 100]

for fD in fD_list:
    # Averaging over 30 samples
    N = 30
    Ts, z_dB = rayleigh_fading_jakes_model(time, fs, fD, N)

    fig = plt.figure(figsize = (16, 4))
    plt.plot(Ts, z_dB)
    plt.title(f"Doppler Frequency = {fD} Hz")
    plt.xlabel("Time (in s)")
    plt.ylabel("Envelope (in dB)")
    plt.xlim(0, 2)
    plt.show()
```


Doppler Frequency = 1 Hz
Envelope (in dB)
Time (in s)


Doppler Frequency = 10 Hz
Envelope (in dB)
Time (in s)


Doppler Frequency = 100 Hz
Envelope (in dB)
Time (in s)