AI-3001: Advanced Topics in Machine Learning **Aug-Nov 2022**

Lecture 2: Weighted Majority (WM) and Randomized WM Algorithms

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2.1 Introduction

Last time

- Formalized online learning with experts setting where experts recommend binary actions
- Explored deterministic algorithms (Naive and Majority) to identify the perfect expert with an assumption that the perfect expert exists
- We proved the mistake bound of each of the above three algorithms

This lecture

We first relax the perfect expert assumption and modify Majority algorithm

1. Weighted majority algorithm

We then show that any deterministic algorithm must suffer a linear regret (in terms of T).

2. Lower bound on worst case regret of deterministic algorithms

Finally, we show that the randomized algorithm achieves sublinear regret.

3. Randomized weighted Majority Algorithm

2.2 Weighted Majority algorithm

Weighted majority algorithm maintains weights (real valued numbers) for each expert and adjusts the weight in each time step according to the performance of that expert in that round. See Algorithm 2 for details.

Algorithm 1: WEIGHTED-MAJORITY

Input: # experts: N, $\alpha \in [0, 1)$
Initialize: $w_{i,0} = 1$ for all $i \leq N$;
for $t = 1, 2, \cdots$ **do**
- **Input:** Expert recommendation $(f_{i,t})_{i=1}^{N}$;
- **Predict:** $p_t = \mathbb{1}\{\sum_{i:f_{i,t}=1} w_{i,t-1} \geq \sum_{i:f_{i,t}=0} w_{i,t-1}\}$;
- **Observe:** y_t ;
- $w_{i,t} \leftarrow w_{i,t-1} \cdot \alpha^{\mathbb{1}(f_{i,t} \neq y_t)}$;

end

We prove the mistake bound using a potential function argument. The potential function is defined as the sum of weights of each expert i.e. $\Phi_t = \sum_{i=1}^{N} w_{i,t-1}$. We will use this proof technique in many algorithms to follow.

Lemma 2.1. *Let $\alpha \in [0, 1)$ and $\Phi_t = \sum_{i=1}^{N} w_{i,t-1}$. Then, $\Phi_{t+1} \leq \frac{1+\alpha}{2} \cdot \Phi_t$ for all t.*

Proof. We consider that $p_t \neq y_t$; i.e. the algorithm has made a mistake in time t. We have

$$\begin{aligned}
\Phi_{t+1} &= \sum_{i:f_{i,t} \neq y_t} w_{i,t} + \sum_{i:f_{i,t}=y_t} w_{i,t} \\
&= \sum_{i:f_{i,t} \neq y_t} w_{i,t-1} \cdot \alpha + \sum_{i:f_{i,t}=y_t} w_{i,t-1} && \text{(from weight update rule)} \\
&= \alpha \cdot (\Phi_t - \sum_{i:f_{i,t}=y_t} w_{i,t-1}) + \sum_{i:f_{i,t}=y_t} w_{i,t-1} \\
&\leq \alpha \cdot \Phi_t + (1-\alpha) \cdot \Phi_t/2 && \text{(the weight of correct experts is at-most half of total weight)} \\
&= \frac{1+\alpha}{2} \cdot \Phi_t
\end{aligned}$$

□

Furthermore, let t' be a time instance such that algorithm does not make a mistake in t'. We have $\Phi_{t'+1} \leq \Phi_{t'}$. This is true since each individual weights only decrease. Let M_t be the number of mistakes made by the algorithm till time t, and T be the stopping time. We have

$$\Phi_{T+1} \leq \left(\frac{1+\alpha}{2}\right)^{M_T} \cdot \Phi_1 = \left(\frac{1+\alpha}{2}\right)^{M_T} \cdot N. \tag{2.1}$$

Next, let $M_{i,t}$ denote the number of mistakes made by expert i till time t.

$$\alpha^{M_{i,T}} \leq \Phi_{T+1}. \tag{2.2}$$

The above two equations together give,

$$\begin{aligned} \alpha^{M_{i,t}} &\leq \Big(\frac{1+\alpha}{2}\Big)^{M_t} \cdot N && \text{(Since } M_t \leq t\text{)} \\ \implies M_{i,t} &\leq \frac{M_t \cdot \log(\frac{1+\alpha}{2}) + \log(N)}{\log(\alpha)} \\ \iff M_t &\leq \frac{\log(1/\alpha) \cdot M_{i,t} + \log(N)}{\log(\frac{2}{1+\alpha})} \end{aligned}$$

The above bound holds for every i. Let the best expert be $i^\star$ i.e. $i^\star \in \arg\min_i M_{i,t}$. We have

$$M_t \leq \frac{\log(1/\alpha) \cdot M_{i^\star,t} + \log(N)}{\log(\frac{2}{1+\alpha})}.$$

When $M_{i^\star,t} = 0$ i.e. there is a perfect expert then we achieve the required bound of $\log(N)$ with $\alpha = 0$.

2.3 Worst case lower bound on the mistakes made by Weighted-Majority

Claim 2.2. *Let $M_{i^\star,T} \leq T/2$. Then there is an instance on which* Weighted-Majority *algorithm makes at-least $2M_{i^\star,T}$ mistakes and incurs a linear regret.*

Proof. Consider a two experts example where first expert predicts 1 and second expert predicts 0 in each time step. The sequence $(y_t)_{t=1}^T$ is chosen adversarially. That is, if algorithm predicts 1 adversary chooses 0 and vice-versa. The total number of mistakes of algorithm is T whereas the best expert makes $M_{i^*,T} \leq T/2$ mistakes (least of the two experts' wrong predictions). This gives the regret of $T - M_{i^*,T} \geq T/2$. That is, the algorithm incurs linear regret. □

Above result is more general and applies to any deterministic algorithm. One way to get competitive results is to introduce randomness. The other way is to reduce the class of problems we consider either by restricting the outcome/prediction sets or considering some special structural assumptions on loss function.

2.4 Randomized Weighted Majority

We return to the mistake upper bound of the weighted majority algorithm given by

$$\beta_1(\alpha) \cdot M_{i^\star,t} + \beta_2(\alpha) \log(N) \tag{2.3}$$

with, $\beta_1(\alpha) = \frac{\log(1/\alpha)}{\log(\frac{2}{1+\alpha})}$ and $\beta_2(\alpha) = \frac{1}{\log(\frac{2}{1+\alpha})}$.

The first term shows the dependence of upper bound on mistakes made by the best expert. Note that this dependence is unavoidable. That is, no combination of expert advice can do better than the best expert in this setting (why?). The second term is additional mistakes an algorithm makes to discover the best expert (additional loss incurred in terms of mistakes by algorithms to identify the best expert). However, the dependence on the constants can be improved. First note the following

Observation 1. *$\beta_1(\alpha) \to 2$ when $\alpha \to 1$.*

The question is, can we improve the value of $\beta_1(\alpha)$. Let $\Phi_t^1 = \sum_{i:f_{i,t}=1} w_{i,t-1}$ be the weight of experts who recommended 1 at time step t and $\Phi_t^0 = \Phi_t - \Phi_t^t$ be the weight of experts who recommended 0 at time t. The randomized weighted algorithm predicts 1 (that is $p_t = 1$), with probability $\frac{\Phi_t^1}{\Phi}$. Equivalently, the algorithm predicts based on an outcome of a coin toss with bias Φ_t^1/Φ.

Algorithm 2: RANDOMIZED WEIGHTED-MAJORITY

Input: # experts: N, $\alpha \in [0, 1)$
Initialize: $w_{i,0} = 1$ for all $i \leq N$;
for $t = 1, 2, \cdots$ **do**
- **Input:** Expert recommendations $(f_{i,t})_{i=1}^N$;
- **Predict:** $p_t = \begin{cases} 1 & \text{w. p. } \frac{\sum_{i=1}^n w_{i,t-1} \cdot f_{i,t}}{\sum_{i=1}^n w_{i,t-1}} \\ 0 & \text{Otherwise} \end{cases}$;
- **Observe:** y_t ;
- **Update**
  - $w_{i,t} \leftarrow w_{i,t-1} \cdot \alpha^{\mathbb{1}(f_{i,t} \neq y_t)}$;

end

Theorem 2.3. *The expected number of mistakes of the randomized weighted majority algorithm is given by*

$$\beta_1(\alpha) M_{i^\star,T} + \beta_2(\alpha) \log(N)$$

where $\beta_1(\alpha) = \frac{\log(1/\alpha)}{1-\alpha}$ *and* $\beta_2(\alpha) = \frac{1}{1-\alpha}$.

Proof. We will follow similar technique we used for WEIGHTED-MAJORITY algorithm. Recall the potential function was $\Phi_T = \sum_{i=1}^N w_{i,t-1}$ and that $M_{i,T}$ denoted the total number of mistake by expert i till round T. Similar to the WEIGHTED-MAJORITY algorithm we have, $\alpha^{M_{i,T}} \leq \Phi_T$ (prove!). We now prove the upper bound on Φ_T.

Claim 2.4. $\Phi_{t+1} = (1 - \ell_t(1-\alpha)) \cdot \Phi_t$ *for all* t. *Here* $\alpha \in [0, 1)$ *and* $\ell_t := \frac{\sum_{i:f_{i,t} \neq y_t} w_{i,t-1}}{\sum_{i=1}^N w_{i,t-1}}$.

Proof.

$$\Phi_{t+1} = \sum_{i:f_{i,t} \neq y_t} w_{i,t-1} \cdot \alpha + \sum_{i:f_{i,t}=y_t} w_{i,t-1} = \Phi_t \cdot \alpha + (1-\alpha)(\Phi_t - \ell_t \cdot \Phi_t) = (1 - (1-\alpha)\ell_t)\Phi_t$$

□

The above claim gives $\Phi_T = \Pi_{t=1}^T (1 - (1-\alpha)\ell_t) \cdot N \leq e^{-(1-\alpha)\cdot\sum_{t=1}^T \ell_t} \cdot N$. For the inequality we use the fact that $\log(1+x) \leq x$ (Taylor series expansion). Combining lower and upper bounds we have

$$\alpha^{M_{i,T}} \leq \Phi_T \leq e^{-(1-\alpha)L} \cdot N \text{ (here } \textstyle\sum_{t=1}^{T+1} \ell_t = L = \mathbb{E}[M_t]).$$

Simplifying, we get $M_{i,T} \log(\alpha) \leq -(1-\alpha)L + \log(N)$. This gives $L \leq \frac{\log(1/\alpha)}{1-\alpha} M_{i,T} + \frac{1}{1-\alpha}\log(N)$. This upper bound holds for every i and hence it should also hold for $i^\star \in \arg\min_i M_{i,T}$. We are done. □

Remarks:

1. $\beta_1(\alpha) \to 1$ as $\alpha \to 1$. We can get optimal value of the constant.

2. If there is a perfect expert we recover the $\log(N)$ bound of MAJORITY algorithm.

Assume: It is known beforehand that $M_{i^*,T} \leq M$ for some M. With these two assumptions we have the following claim.

Claim 2.5. $L \leq M_{i^*,T} + (\sqrt{2} + 1/\sqrt{2})\sqrt{M \log(N)} + \log(N)$.

Proof. We have,

$$L \leq \frac{\log(1/\alpha)}{(1-\alpha)} M_{i^*,T} + \frac{1}{(1-\alpha)} \log(N)$$

Choose $\alpha = \frac{1}{1+\sqrt{\frac{2\log(N)}{M}}}$ (why?)

$$\begin{aligned}
\implies L &\leq \frac{1+\sqrt{2\log(N)/M}}{\sqrt{2\log(N)/M}} \cdot \log\left(1+\sqrt{2\log(N)/M}\right) \cdot M_{i^*,T} + \frac{1+\sqrt{2\log(N)/M}}{\sqrt{2\log(N)/M}} \cdot \log(N) \\
&\leq (1+\sqrt{2\log(N)/M}) M_{i^*,T} + \log(N) + \sqrt{\frac{M\log(N)}{2}} \\
&\leq M_{i^*,T} + (\sqrt{2} + 1/\sqrt{2}) \cdot \sqrt{M\log(N)} + \log(N).
\end{aligned}$$

□

Next we answer the question; how we compute optimal value of α?
We need to minimize the right hand side. Differentiate and equate the right hand side in the upper bound of L to zero. we get

$$\frac{1-\alpha}{\alpha} - \log(1/\alpha) = \frac{\log(N)}{M}$$

Approximate the $\log(1/\alpha)$ by $(1/\alpha - 1) - (1/\alpha - 1)^2/2$ by using inequality $\log(1+x) \geq x - x^2/2$ for all $x \geq 0$ (prove this inequality!). This gives the required value of α. Note that one can do better by getting closer approximation of logarithmic function but this works just fine for us.

2.4.1 Next time

We will continue with the review of some results in convex analysis and standard inequalities useful throughout this class.