20/4/23

Introduction to Wavelets

- Recall basic definitions: basis, inner product, norm, orthonormal basis, transform coefficients
- Haar wavelet
  - Approximation functions - scaling and shifting, nested spaces
  - Example
  - Wavelet function - motivation and definition
  - Scaling & shifting
  - Relating approximation & wavelet functions at scale j
  - Example
  - Recursive relationship

- <u>Basis</u>: A set $\Phi = \{\phi_i\}_{i \in I}$ is called a basis of a vector space $\mathbb{V}$ (where $\phi_i \in \mathbb{V}$) if every element of $\mathbb{V}$ can be expressed as a unique linear combination of ϕ_i's, ϕ_i's are linearly independent. Further $|I| = n$. Example $\mathbb{V} = \mathbb{R}^n$; $\Phi = \{e_i\}_{i=1}^{n}$, $e_i = [0, \ldots \underset{\uparrow_i}{1}, 0, 0, \ldots]^T$.

- <u>Inner product</u>: If $x, y \in \mathbb{C}^n$, $\langle x, y \rangle = \sum_{i \in I} x_i y_i^*$; (simple defn)

$$f, g \in L^2(\mathbb{R}) \quad \langle f, g \rangle = \int_{x \in \mathbb{R}} f(x) g(x) dx.$$

(In general, we operate in a Hilbert Space, which is a complete inner product space)

- <u>Norm</u>: If $x \in \mathbb{C}^n$, $\|x\| = \sqrt{\langle x, x \rangle}$

- <u>Orthonormal basis</u>: An orthonormal system $\Phi = \{\phi_i\}_{i \in I}$ defined on a vector space $\mathbb{V}$ is an ONB if $\langle \phi_i, \phi_j \rangle = \delta_{i-j}$. Also every $x \in \mathbb{V}$ can be written as

$$x = \sum_{i \in I} X_i \phi_i \ ; \quad \text{where} \quad X_i = \langle x, \phi_i \rangle$$

Exercise: Show that the DFT is an ONB.

Recall: $x = [5 \ 8 \ 2 \ 4 \ 0 \ 1 \ 3 \ 6 \ ]$ (we subsample the filter outputs to maintain dimension).

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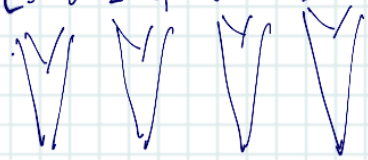
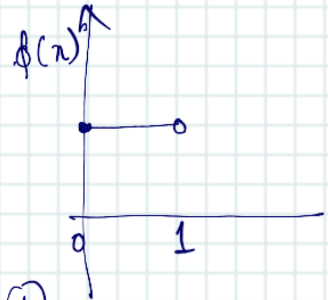


- The Haar Wavelet: Let's operate in $L^2(\mathbb{R})$.

- The Haar approximation function:

$$\phi(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & \text{elsewhere} \end{cases} \quad - (1)$$



$$f(x) = \begin{cases} -2 & -1 \le x < 0 \\ 1 & 0 \le x < 1 \\ -4 & 1 \le x < 2 \end{cases}$$

$$f(x) = -2\phi(x+1) + \phi(x) - 4\phi(x-1)$$

Wavelets allow for scaling & shifting of $\phi(x)$ i.e. we can generate scaled and shifted version of $\phi(x)$. E.g. $\phi(2^j x - k) = \begin{cases} 1, & \frac{k}{2^j} \le x < \frac{k+1}{2^j} \\ 0 \end{cases} \quad - (2)$

Let $V_0 = \sum_{k \in \mathbb{Z}} a_k \phi(2^0 x - k)$; $a_k \in \mathbb{R}$,

$|||^{ly}$ $V_j = \sum_{k \in \mathbb{Z}} a_k \phi(2^j x - k)$; $a_k \in \mathbb{R}$, $j \in \mathbb{Z}$

$$V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots V_{j-1} \subset V_j \subset \cdots \subseteq L^2(\mathbb{R})$$

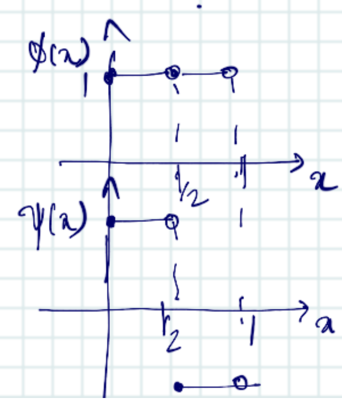
Q: How do we a portion of a function that is in V_1 but not in V_0?

A: Let's define the Haar wavelet function

$$\psi(x) = \phi(2x) - \phi(2x-1)$$

If $W_0 = \sum_{k \in \mathbb{Z}} a_k \psi(x-k)$; $a_k \in \mathbb{R}$,

claim $V_1 = \underbrace{V_0 \oplus W_0}$, i.e.



any $v_1 \in V_1$ can be written as a unique combination of $v_0 \in V_0$ and $w_0 \in W_0$

$$f(x) = \phi(2x) + (-2)\,\phi(2x-1)$$

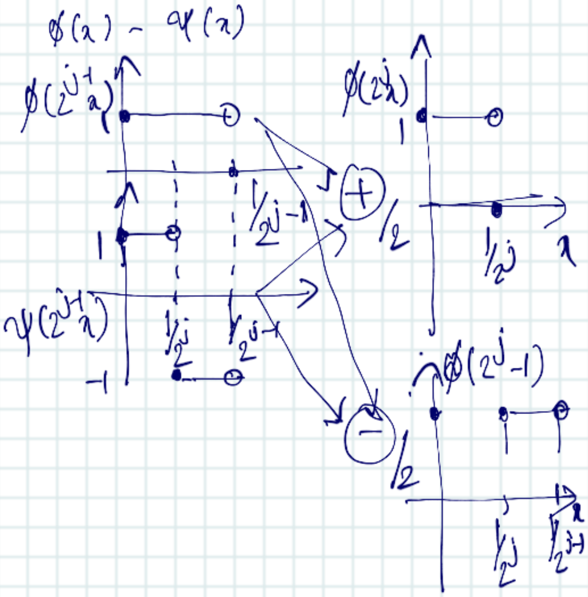
$$= 2\phi(2x) - 2\phi(2x-1) - \phi(2x)$$

$$= 2\underbrace{\psi(x)}_{W_0} - \underbrace{\phi(2x)}_{V_1}$$

Note. $V_1 = V_0 \oplus W_0$ $\Rightarrow V_1$ is composed of subspaces V_0 and W_0 such that

$$V_0 \cap W_0 = \emptyset \,! \qquad V_1 = V_0 + W_0$$

$$f(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \leftarrow \frac{\psi(x) + \phi(x)}{2} \\ -2 & \frac{1}{2} \le x < 1 \leftarrow \phi(x) - \psi(x) \end{cases}$$



- $$\phi(2^j x) = \frac{1}{2}\left[\phi(2^{j-1}x) + \psi(2^{j-1}x)\right]$$

- $$\phi(2^j x - 1) = \frac{1}{2}\left[\phi(2^{j-1}x) - \psi(2^{j-1}x)\right]$$

- $$\Rightarrow V_j = V_{j-1} \oplus W_{j-1}$$

$$= V_{j-2} \oplus W_{j-2} \oplus W_{j-1}$$

$$\vdots$$

$$V_j = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{j-1}$$

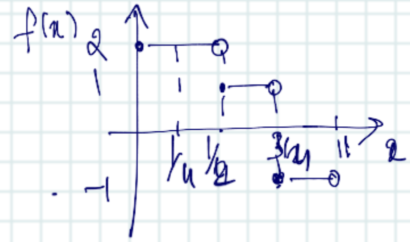
$$2\phi(2^2 x - 0\cdot 2) + 2\phi(2^2 x - (2\cdot 0 + 1))$$

- $$f(x) = 2\phi(4x) + 2\phi(4x-1) + \phi(4x-2) - \phi(4x-3)$$

$$= 2\cdot\left[\frac{\phi(2x) + \psi(2x)}{2}\right] + 2\cdot\left[\frac{\phi(2x) - \psi(2x)}{2}\right] +$$

$$\phi(4x-2) = \frac{1}{2}\left[\phi(2\ , + \psi(2x-1)\right] - \frac{1}{2}\left[\phi(2x-1) - \psi(2x-1)\right]$$

$$f(x) = \phi(x) + \psi(x) + \psi(2x-1)$$



- Recall: $\phi(x)$ & its scaling and shifting
  $\psi(x)$ & its scaling and shifting } Haar wavelet

- $$\phi(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & \text{else} \end{cases}$$

- $$\phi(2^j x - k) = \begin{cases} 1 & 0 \leq 2^j x - k < 1 \\ 0 & \text{else} \end{cases}$$

- $$V_j = \sum_{k \in \mathbb{Z}} a_k \phi(2^j x - k) \qquad a_k \in \mathbb{R}$$

- $$\psi(x) = \phi(2x) - \phi(2x-1)$$
  $$\psi(x) \in V_1$$

- $$\psi(2^{j-1}x - k) \in V_j$$

- $$\{0\} \subset \dots V_{-1} \subset V_0 \subset V_1 \subset \dots \subset L^2(\mathbb{R})$$

- $$V_j = V_{j-1} \oplus W_{j-1}$$
  $$= V_0 \oplus W_0 \oplus \dots \oplus W_{j-1}$$

- $$\phi(2^j x) = \frac{1}{2}\left[\phi(2^{j-1}x) + \psi(2^{j-1}x)\right] \quad - (1)$$

  $$\phi(2^j x - 1) = \frac{1}{2}\left[\phi(2^{j-1}x) - \psi(2^{j-1}x)\right] \quad - (2)$$

- Let $f^j(x) = \sum_{k \in \mathbb{Z}} a_k \phi(2^j x - k) \quad a_k \in \mathbb{R}$.

$$= \sum_{k \in \mathbb{Z}} a_{2k} \phi(2^j x - 2k) + a_{2k+1} \phi(2^j x - (2k+1))$$

$$= \sum_{k \in \mathbb{Z}} a_{2k} \phi(2(2^{j-1}x - k)) + a_{2k+1} \phi(2(2^{j-1}x - k) - 1)$$

$$= \sum_{k \in \mathbb{Z}} \underbrace{\left(\frac{a_{2k} + a_{2k+1}}{2}\right)}_{LPF} \cdot \underbrace{\phi(2^{j-1}x - k)}_{\in V_{j-1}} + \underbrace{\left(\frac{a_{2k} - a_{2k+1}}{2}\right)}_{HPF} \underbrace{\psi(2^{j-1}x - k)}_{\in W_{j-1}}$$

Example: $f^j(x) = (-1)\phi(4x) + (4)\phi(4x-1) + (2)\phi(4x-2) + (-3)\phi(4x-3)$

$j = 2$. i.e. $f^j(x) \in V_2$

$$= -\phi(2^2 x) + 4\phi(2^2 x - 1) + 2\phi(2^2 x - 2) - 3 \cdot \phi(2^2 x - 3)$$

$$= \underbrace{\overset{3/2}{\left(\frac{-1+4}{2}\right)} \cdot \phi(2x) + \overset{-1/2}{\left(\frac{2-3}{2}\right)} \phi(2x-1)}_{V_1} +$$

$$\underbrace{\left(\frac{-1-4}{2}\right) \psi(2x) + \left(\frac{2+3}{2}\right) \psi(2x-1)}_{W_1}$$

$$\therefore f^j(x) = \underbrace{\left(\frac{\frac{3}{2} - \frac{1}{2}}{2}\right)}_{V_0} \phi(x) + \underbrace{\left(\frac{\frac{3}{2} + \frac{1}{2}}{2}\right)}_{W_0} \cdot \psi(x) + \underbrace{\left(-\frac{5}{2}\right)\psi(2x) + \frac{5}{2}\psi(2x-1)}_{W_1}$$

Q: How do we generalize this analysis to any function space in $L^2(\mathbb{R})$? i.e. not just piecewise linear functions.

A: • Ingrid Daubechies showed how we can construct a multiresolution analysis (MRA) using compactly supported functions.

• She also showed they can be realised using filters

• MRA is a sequence of subspaces $\{\emptyset\} \subset \cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \cdots \subset L^2(\mathbb{R})$

• Self similarity on time and scale

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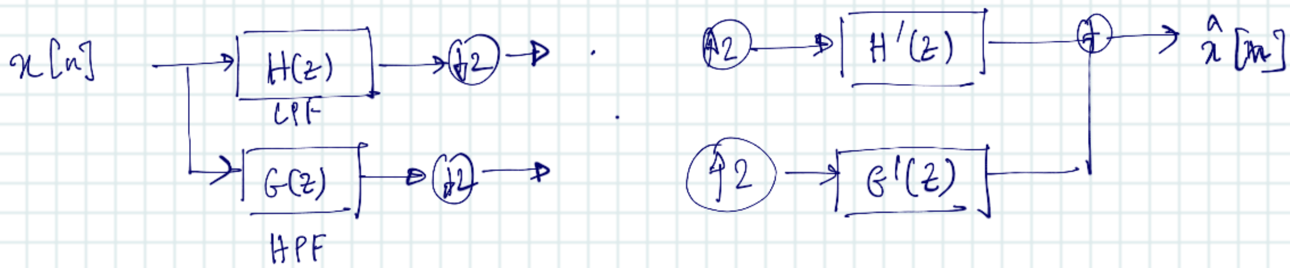
• <u>Perfect Reconstruction filters</u>



• <u>Orthogonality</u> : Recall $\langle \phi(x-j), \phi(x-k) \rangle = \delta_{j-k}$

<u>Daubechies</u> proposed a recursive algorithm for the design of filters that satisfied the above conditions. This lead to the design of <u>maximally flat</u> filters.

* Not on the final.

• <u>Convolutional Neural Networks</u> (CNNs)

$$J = I * h$$

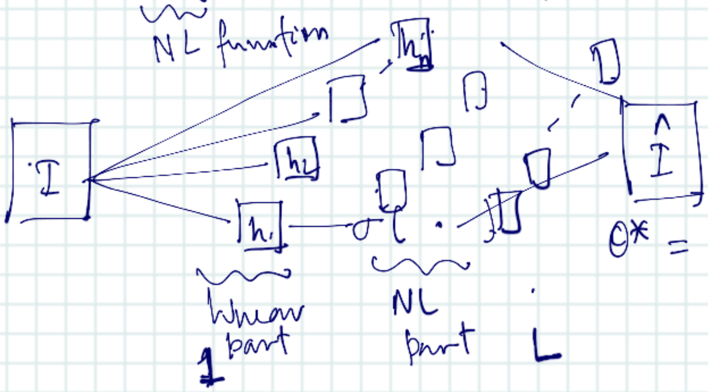
$$J(i,j) = \sum_{m=0}^{K-1} \sum_{n=0}^{K-1} \underbrace{h(m,n)} \cdot I(i-m, j-n) \quad - \text{①}$$

linear

$K \times K$

$$L = \underbrace{\sigma}_{\text{NL function}}(J) = \sigma(I * h) \quad - \text{②}$$

output of a non linearity applied to ①



$$\theta^* = \underset{\theta \in \mathbb{R}^N}{\operatorname{argmin}}\, d(\hat{I}, I_g)$$

- We can use a model like above to solve a number of image processing tasks. – denoising, segmentation, edge detection, binarization etc.

- In our past discussions, the filters h were deterministic or hand-crafted. For e.g. the Sobel filter

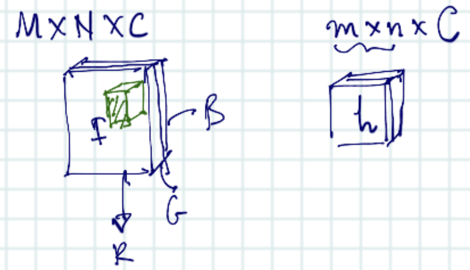
- Here, the filters are learnt from the training data to minimize loss.

$$\mathcal{L}(\theta) = \sum_{i=1}^{N_t} d(\underset{\text{model prediction}}{\hat{I}_i}, \underset{\text{ground truth label}}{I_{gi}})$$

$$\hat{I}_i = f(I_i, \theta) = f_1 \circ f_2 \circ \cdots \circ f_L$$

- Iterative approach $\quad \theta^{(r)} = \theta^{(r-1)} - \underset{\text{learning rate}}{\eta} \underbrace{\nabla_{r-1} \mathcal{L}(\theta^{(r-1)})}_{\text{found using back propagation of the error}}$

- Typically images have multiple channels in them (colour)



- Transformer (Vision)

Recall: $$J(i,j) = \sum_m \sum_n \underbrace{W(i-m, j-n)}\, \underbrace{I(m,n)}$$

$$J(i,j) = \sum\sum_{mn \in \mathcal{N}} \text{softmax}_{mn}(\underline{q}_{ij}^T \cdot \underline{k}_{mn}) \cdot \underline{v}_{mn}$$

$$\underline{q}_{ij} = \underset{\sim}{W_Q} \cdot \underline{x}_{ij} \qquad \underline{v}_{mn} = \underset{\sim}{W_V}\, x_{mn}$$

$$\underline{k}_{mn} = \underset{\sim}{W_K}\, x_{mn}$$

learnt