Information Theory
Practice Set 7

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Solutions are not to be returned

Practice Set

1. Exercise Problems from Cover and Thomas: 8.7

2. You must prove that among all non-negative random variables with mean equal to $\mu > 0$, the exponential distribution has the maximum differential entropy. The following steps will help with this proof.

   (a) Let $g(x) = \frac{1}{\mu} \exp\left(-\frac{x}{\mu}\right)$, $x \geq 0$, be the exponential distribution. Find its differential entropy.

   (b) Let $f(x)$ be the density function of any non-negative random variable with mean μ, i.e.,

   $$f(x) = 0 \text{ for all } x < 0 \quad \text{and} \quad \int_{x=0}^{\infty} x f(x)\, dx = \mu.$$

   Use the fact $0 \leq D(f\|g)$, where g is as above, to show that

   $$\int f \log(1/f) \leq \int f \log(1/g) = \int g \log(1/g).$$

3. *The entropy power.* Let $X \in \mathbb{R}$ be any continuous random variable with differential entropy $h(X)$. Now consider a *Gaussian* random variable $\tilde{X}$ with the same differential entropy as X, i.e., $h(\tilde{X}) = h(X)$. Since $\tilde{X}$ is Gaussian, we know that

   $$h(\tilde{X}) = \frac{1}{2} \log 2\pi e \mathrm{Var}(\tilde{X}), \quad \text{i.e.,} \quad \mathrm{Var}(\tilde{X}) = \frac{1}{2\pi e} 2^{2h(\tilde{X})} = \frac{1}{2\pi e} 2^{2h(X)}.$$

   Definition. The entropy power of X, denoted as $N(X)$, is the variance of the Gaussian random variable with the same differential entropy as X, and this is equal to $\frac{1}{2\pi e} 2^{2h(X)}$.

   Questions.

   (a) For any continuous random variable X, which is larger $N(X)$ or $\mathrm{Var}(X)$? Why?

   (b) When is $N(X) = \mathrm{Var}(X)$?

   `Remark`: *Entropy Power Inequality (EPI).* The EPI states that for any two independent continuous random variables X and Y, we have

   $$N(X + Y) \geq N(X) + N(Y).$$

   This is an important inequality that relates the entropy of the sum of two independent random variables to their individual entropies. We have not included the proof of this result in this course.

   It is interesting to compare the EPI with the following basis property of variance: for any two independent random variables X and Y, we have

   $$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

4. Reading exercise:
   Statement of proof of Theorems 17.9.1 and 17.9.2 in Cover & Thomas.
