AI3001: Advanced Topics in Machine Learning
Classroom Quiz 1
September 27, 2022

Instructions:

- *All the problems are compulsory.*
- *Total duration of the quiz is 30 minutes.*

Problem 1 (1+1+1 = 3 marks)**.** Let the regret upper bound guarantee of some algorithm ALG is written in terms of tunable parameters η, α as given below. Find the optimal values of these parameters and also the corresponding tight regret upper bound. Here, assume all other parameters T, N, M are strictly positive and treated as constants.

1. $\mathcal{R}_T(\text{ALG}; \eta) \leq M\eta + \frac{T}{\eta}$
2. $\mathcal{R}_T(\text{ALG}; \eta) \leq T\eta + \frac{\log(N)}{\eta} + \eta^2$
3. $\mathcal{R}_T(\text{ALG}; \eta) \leq \frac{T\alpha}{\eta} + \frac{N}{\alpha} + T\eta$

Problem 2 (7 marks)**.** Consider a portfolio optimization setup with two stocks; cash and hot-stock. Assume that we begin with a unit wealth and the stopping time T is an even integer. Let the sequence of market returns be as given in table below.

	Day 1	Day 2	Day 3	Day 4	$\cdots$
cash	1	1	1	1	$\cdots$
hot-stock	$\frac{1}{3}$	4	$\frac{1}{3}$	4	$\cdots$

1. What is the optimal buy-and-hold strategy and what is wealth generated by optimal buy-and-hold strategy after T days? (1/2 + 1/2 = 1 mark)
2. What is the wealth generated by the CRP strategy $b := \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ after T days? (1 mark)
3. What is the optimal CRP strategy b^* and what is the wealth generated by optimal CRP strategy after T days? (2+ 1 = 3 marks)
4. For $T = 4$ and $\delta = 1/T$, does the CRP strategy b given above satisfies the condition $W_T(b, r^T) \geq (1-\delta)^T W_T(b^*, r^T)$? show your work. (2 mark)

Solution of Problem 1.

1. Take the derivative of RHS and equate it to 0.

$$M - \frac{T}{\eta^2} = 0 \implies \eta = \sqrt{\frac{T}{M}}$$

   This gives $\mathcal{R}_T(\textsc{Alg}) \leq 2\sqrt{TM}$.

2. This one is slightly tricky. First, we do similar to what we did in the first question,

$$T - \frac{\log(N)}{\eta^2} + 2\eta = 0$$

   From above we have $\sqrt{\frac{\log(N)}{2T}} \leq \eta^* \leq \sqrt{\frac{\log(N)}{T}}$. Where η^* is the solution of the above 3 degree polynomial. To see how we get the lower bound observe that

$$\eta_1^2 = \frac{\log(N)}{2T} \leq \frac{\log(N)}{T}(1 - \frac{2\eta_1}{T}) \leq \frac{\log(N)}{T(1+2\eta_1/T)} = \frac{\log(N)}{T+2\eta_1}$$

   The first inequality assumes that $T \geq 2\log(N)$ wehreas the second ineqality uses Taylor series expansion. The above inequalities imply that $T - \log(N)/\eta_1^2 + 2\eta_1 \leq 0$. On the other hand we have $T - \log(N)/\eta_2^2 + 2\eta_2 \geq T - \log(N)/\eta_2^2 = 0$ for upper bound $\eta_2 = \sqrt{\frac{\log(N)}{T}}$. Apply intermediate value theorem to locate the solution. This gives us a slightly loose bound as follows.

$$\mathcal{R}_T(\textsc{Alg}) \leq \min(\sqrt{2T\log(N)} + \log(N)/2T, \sqrt{T\log(N)} + \log(N)/T)$$

   I understand that this manipulation was tricky and hence everyone will get full points for this question.

3. Taking partial derivatives wrt each independent parameter and setting to 0 we get $\alpha^* = \eta^{*2} = (N/T)^{2/3}$. Putting these values we get $\mathcal{R}_T(\textsc{Alg}) \leq 3T^{2/3}N^{1/3}$.

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Solution of Problem 2.

1. Optimal Buy-and-hold strategy is to invest entire wealth in hot-stock which gives the returns $(4/3)^{T/2}$.
2. $(5/3)^{T/2}$
3. Optimal CRP : $(5/12, 7/12)$. Wealth from optimal CRP: $(121/72)^{T/2}$
4. Yes. Note the following

$$W_T(b, r^T) = (5/3)^2 \overset{?}{\geq} (1-\delta)^T W_T(b^*, r^T) = (1-1/4)^4(121/72)^2 \iff 5/3 \overset{?}{\geq} 9/16 \cdot 121/72$$

A simple calcupation show that the last inequality holds. ■