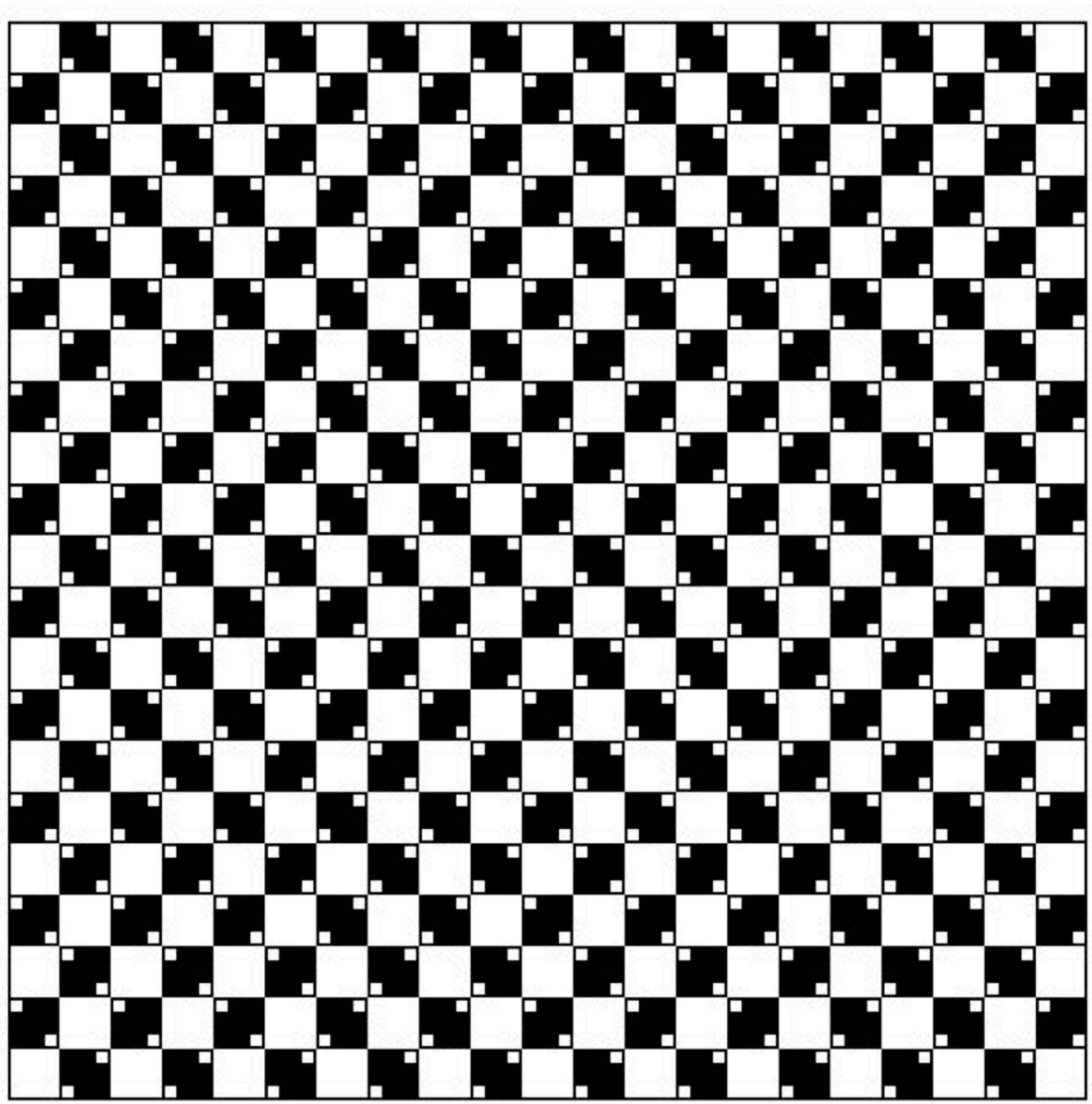
EE6310: Image and Video Processing Spring 2023

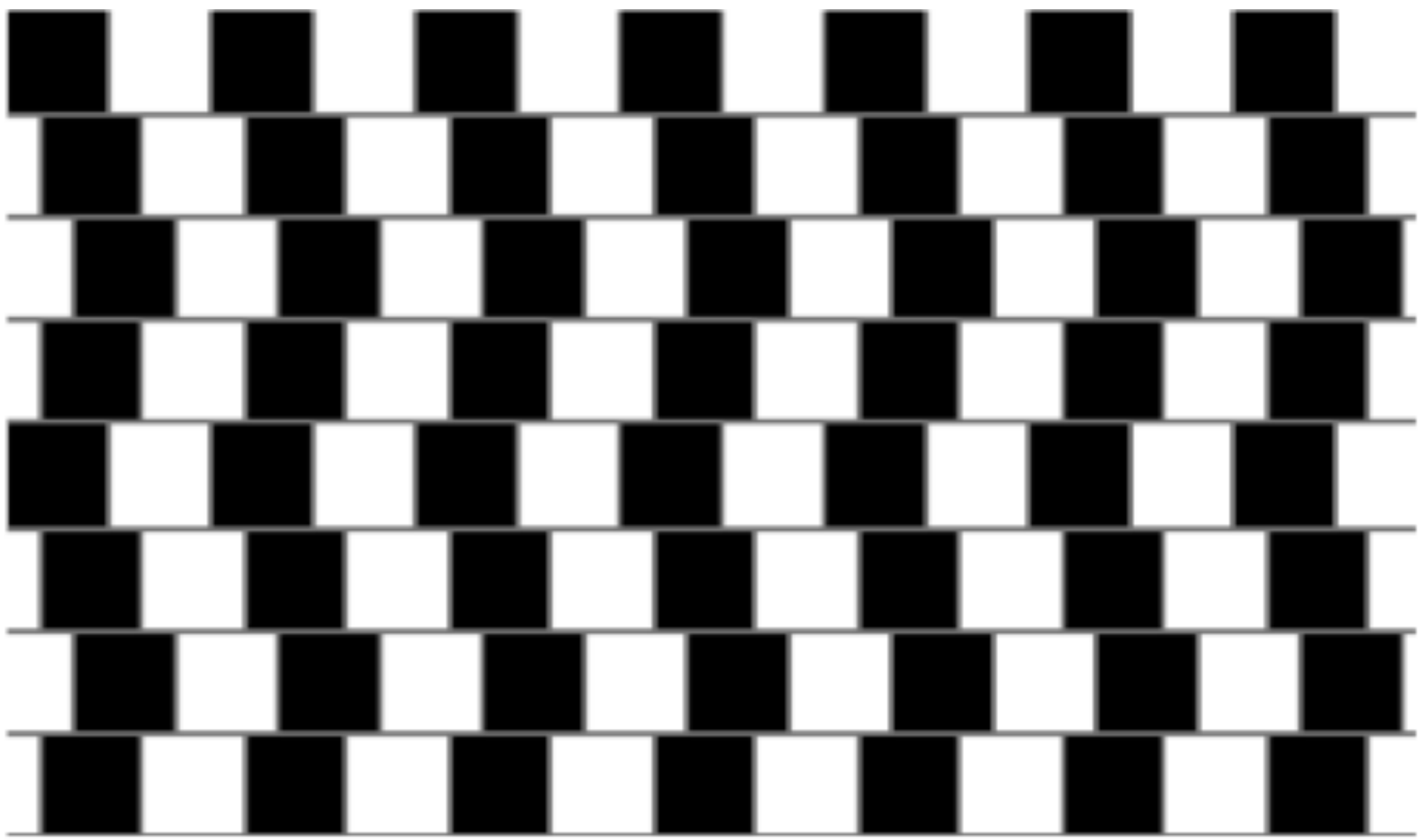
Discrete Fourier Transform

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

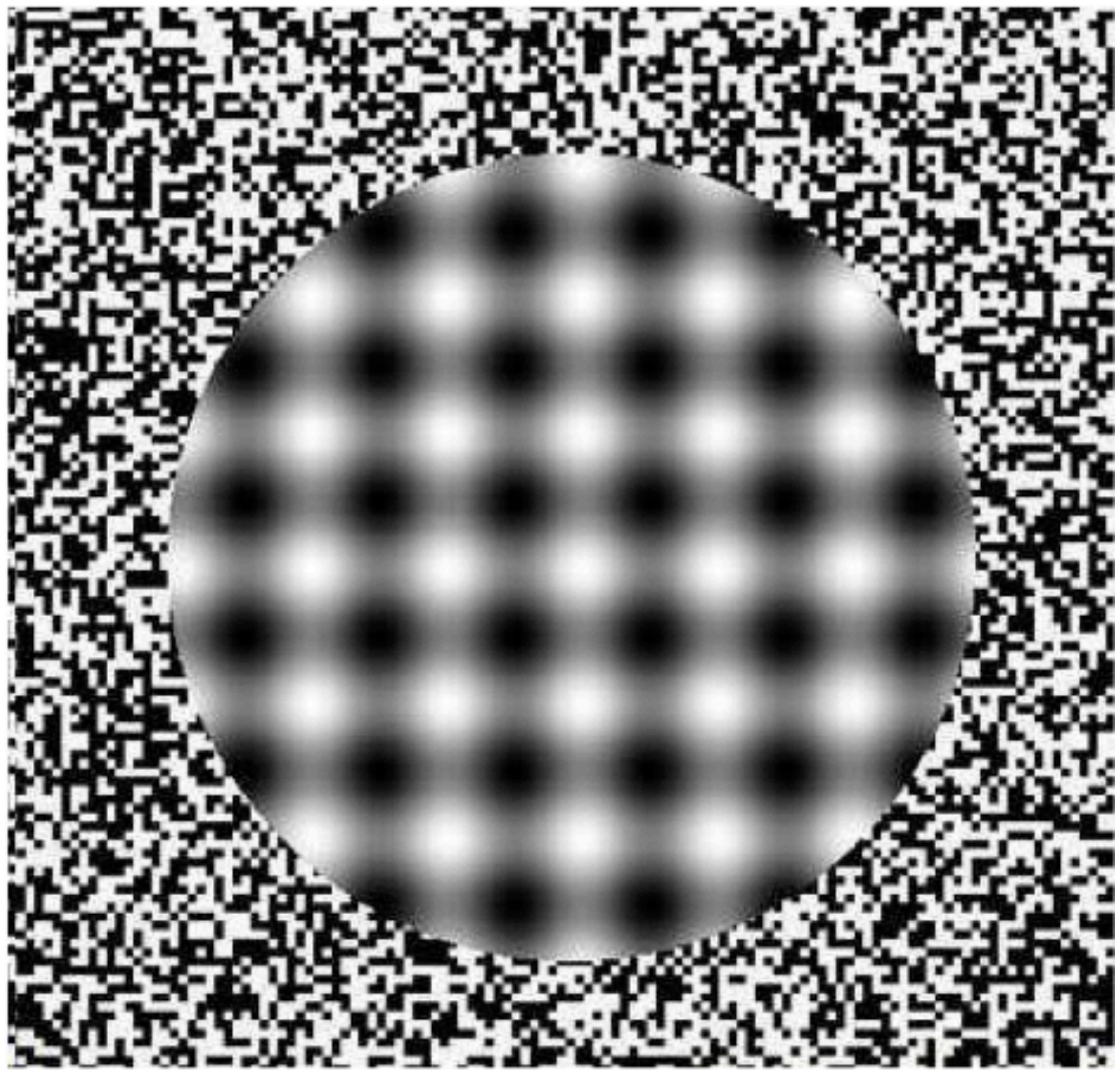
Discrete Fourier Transform



Discrete Fourier Transform

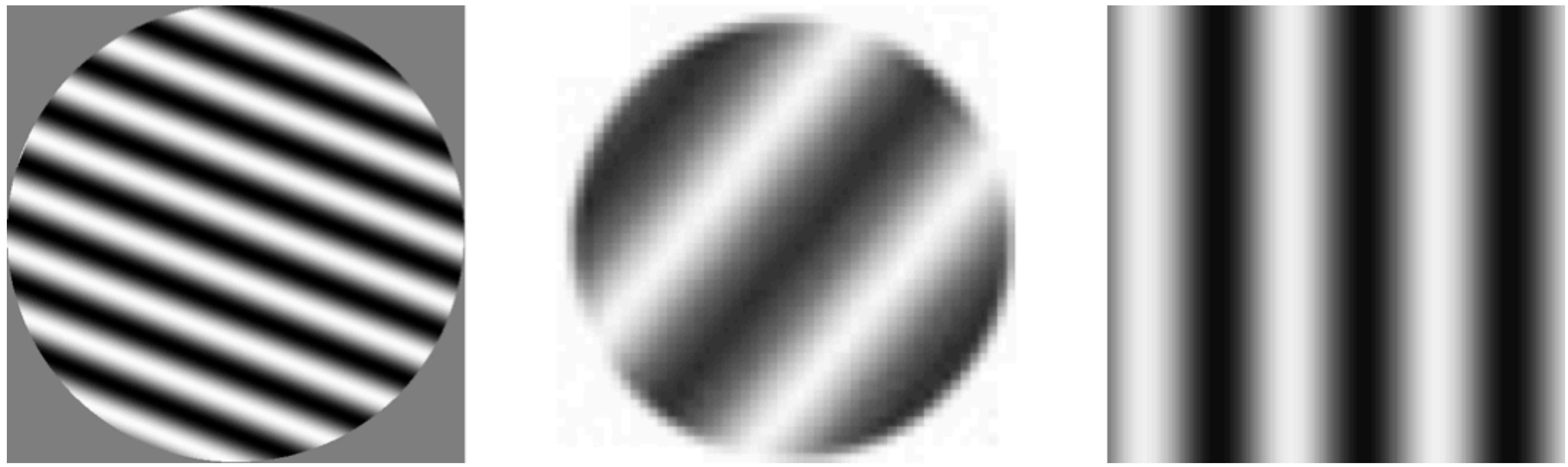


Discrete Fourier Transform



Discrete Fourier Transform

Sinusoidal Images



- Image with the **simplest** frequency content is the **sinusoidal** image
- A discrete **sine** image**:** $I(i,j) = \textbf{sin}[2\pi(\frac{u}{N}i + \frac{v}{M}j)]; 0 \le i \le (N-1), 0 \le j \le (M-1)$
- A discrete **cosine** image:
  $I(i,j) = \textbf{cos}[2\pi(\frac{u}{N}i + \frac{v}{M}j)]; 0 \le i \le (N-1), 0 \le j \le (M-1)$
- u, v are **spatial integer** frequencies along i, j respectively measured is cycles/image
- **Orientation** = $\tan^{-1}\frac{v}{u}$, **radial frequency** = $\sqrt{u^2 + v^2}$

Discrete Fourier Transform

Complex Exponential

- The **complex exponential** is used to define the **Discrete Fourier Transform**
- The 2D complex exponential is defined as:
  $\exp[-2\pi\sqrt{-1}(\frac{u}{N}i + \frac{v}{M}j)]$
- Note that we explicitly use $\sqrt{-1}$ instead of i or j to avoid confusion with image axes
- Convenient **representation** and **manipulation** of frequencies

Discrete Fourier Transform

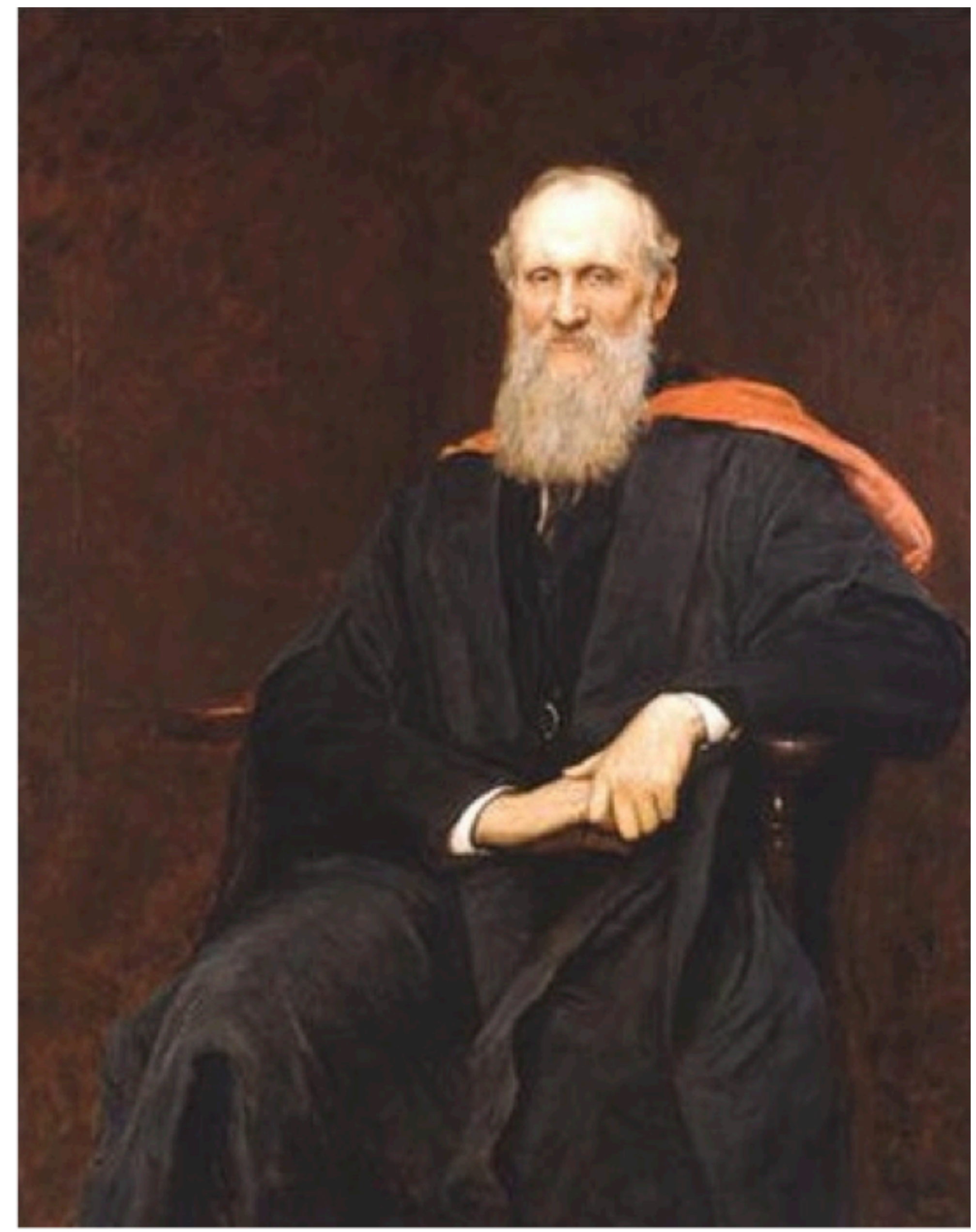
Complex Exponential Notation and Properties

- Let $W_N = \exp(-\sqrt{-1}\frac{2\pi}{N})$
- From **Euler's identity**:
  $W_N = \cos(\frac{2\pi}{N}) - \sqrt{-1}\sin(\frac{2\pi}{N}), W_N^{ui} = \cos(\frac{2\pi ui}{N}) - \sqrt{-1}\sin(\frac{2\pi ui}{N})$
- Minimum frequency when $u = kN$, maximum frequency when $u = (k + \frac{1}{2})N$
- The complex exponential can therefore be written as
  $\exp[-2\pi\sqrt{-1}(\frac{u}{N}i + \frac{v}{M}j)] = W_N^{ui}W_M^{vj}$
- The powers of W_N correspond to the spatial frequencies of the component sinusoids
- Useful to think of $W_N^{ui}W_M^{vj}$ as the **representation of direction & frequency of oscillation**

Discrete Fourier Transform

History

- “Fourier’s theorem is not only one of the most beautiful results of modern analysis, but it may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics.” – **Lord Kelvin**



Discrete Fourier Transform

History

- "Yesterday was my 21st birthday, at that age **Newton** and **Pascal** had already acquired many claims to immortality." – **Joseph Fourier**

Discrete Fourier Transform

Definition of Synthesis Equation

- Any image I of size $M \times N$ can be uniquely expressed as a weighted sum of a finite number of complex exponential images:

$$I(i,j) = \frac{1}{MN} \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} \tilde{I}(u,v) W_N^{-ui} W_M^{-vj}$$

- The weights $\tilde{I}(u,v)$ are unique

- Known as the **Inverse Discrete Fourier Transform (IDFT)** or the **synthesis equation**

Discrete Fourier Transform

Definition of Analysis Equation

- The forward transform: $\tilde{I}(u, v) = \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} I(i, j) W_N^{ui} W_M^{vj}$

- Known as the **Forward Discrete Fourier Transform (DFT)** or the **analysis equation**

Discrete Fourier Transform

Notes

- $\mathbf{I}$ and $\tilde{\mathbf{I}}$ can be uniquely obtained from each other
- Remember:
  - (i, j) are **spatial** indices
  - (u, v) are **spatial frequency** indices
- The DFT $\tilde{I}$ has the same dimension as the image:
  $\tilde{\mathbf{I}} = [\tilde{I}(u, v); 0 \leq u \leq (N-1), 0 \leq v \leq (M-1)]$
- Linear: DFT$[a_1\mathbf{I}_1 + a_2\mathbf{I}_2 + \ldots + a_n\mathbf{I}_n] = a_1\tilde{\mathbf{I}}_1 + a_2\tilde{\mathbf{I}}_2 + \ldots + a_n\tilde{\mathbf{I}}_n$

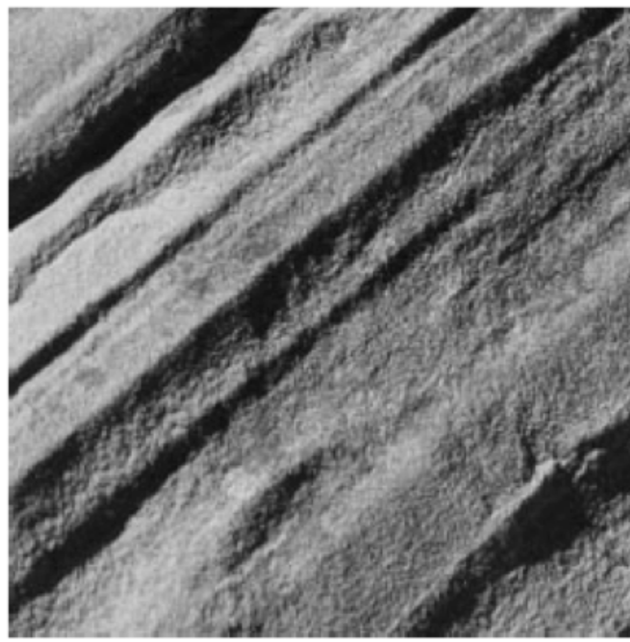
Discrete Fourier Transform

Notes

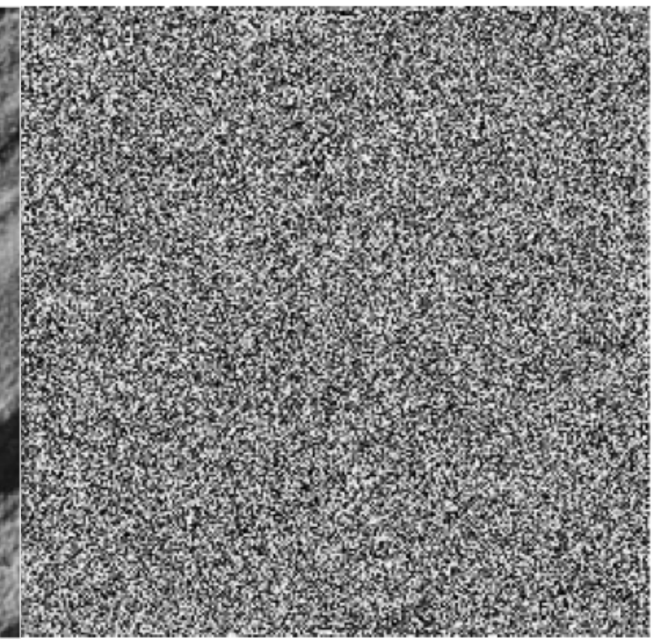
- $\tilde{\mathbf{I}} = \tilde{\mathbf{I}}_{\text{real}} + \sqrt{-1}\ \tilde{\mathbf{I}}_{\text{img}}$ where:
  - $\tilde{\mathbf{I}}_{\text{real}} = [\tilde{I}_{\text{real}}(u,v) = \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} I(i,j)\cos[2\pi(\frac{ui}{N} + \frac{vj}{M})]; 0 \le u \le (N-1), 0 \le v \le (M-1)]$
  - $\tilde{\mathbf{I}}_{\text{imag}} = [\tilde{I}_{\text{imag}}(u,v) = -\sum_{i=0}^{N-1}\sum_{j=0}^{M-1} I(i,j)\sin[2\pi(\frac{ui}{N} + \frac{vj}{M})]; 0 \le u \le (N-1), 0 \le v \le (M-1)]$
- Magnitude image:
  $|\tilde{\mathbf{I}}| = [\,|\tilde{I}(u,v)| = \sqrt{I^2_{\text{real}}(u,v) + I^2_{\text{imag}}(u,v)}; 0 \le u \le (N-1), 0 \le v \le (M-1)]$
- Phase image:
  $\angle\tilde{\mathbf{I}} = [\angle\tilde{I}(u,v) = \text{arctan}[\frac{I_{\text{imag}}(u,v)}{I_{\text{real}}(u,v)}]; 0 \le u \le (N-1), 0 \le v \le (M-1)]$

Discrete Fourier Transform

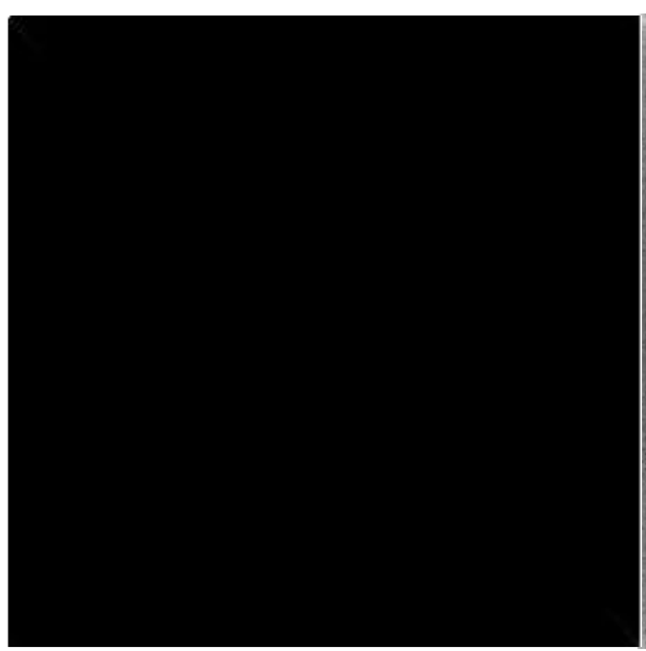
Displaying the DFT



$\mathbf{I}$



$\angle\tilde{\mathbf{I}}$



$|\mathbf{I}|$

$\log(1 + |\tilde{\mathbf{I}}|)$

Discrete Fourier Transform

Displaying the DFT

- Image DFT is usually displayed as images of **magnitude** and **phase**
- Magnitude and phase represented by **gray scale image**
- Visually, phase does not convey much, however, **very important**
- Magnitude is usually **log transformed** followed by **FSCS** i.e., FSCS[log$(1 + |\tilde{\mathbf{I}}|)$]

Discrete Fourier Transform

Properties

- **Conjugate symmetry**:
  $\tilde{I}(N-u, M-v) = \tilde{I}^*(u, v); 0 \leq u \leq (N-1), 0 \leq v \leq (M-1)$
  - $W_N^{(N-u)i} = W_N^{-ui} = W_N^{ui*}$
- This implies the following:
  - $\tilde{I}_{\text{real}}(N-u, M-v) = \tilde{I}_{\text{real}}(u, v); 0 \leq u \leq (N-1), 0 \leq v \leq (M-1)$
  - $\tilde{I}_{\text{imag}}(N-u, M-v) = -\tilde{I}_{\text{imag}}(u, v); 0 \leq u \leq (N-1), 0 \leq v \leq (M-1)$
  - $|\tilde{I}(N-u, M-v)| = |\tilde{I}(u, v)|; 0 \leq u \leq (N-1), 0 \leq v \leq (M-1)$
  - $\angle\tilde{I}(N-u, M-v) = -\angle\tilde{I}(u, v); 0 \leq u \leq (N-1), 0 \leq v \leq (M-1)$
- Symmetry implies **redundancy**

Discrete Fourier Transform

Properties

- The DFT matrix is **finite**:
  $\tilde{\mathbf{I}} = [\tilde{I}(u, v); 0 \leq u \leq (N-1), 0 \leq v \leq (M-1)]$
- What happens if we let u, v go outside this range?
  - $W_N^{(u+nN)i} = W_N^{ui} W_N^{nNi} = W_N^{ui}$
  - $\tilde{I}(u + nN, v + mM) = \tilde{I}(u, v); \forall m, n \in \mathbb{Z}$
- This is called **periodic extension** of the DFT

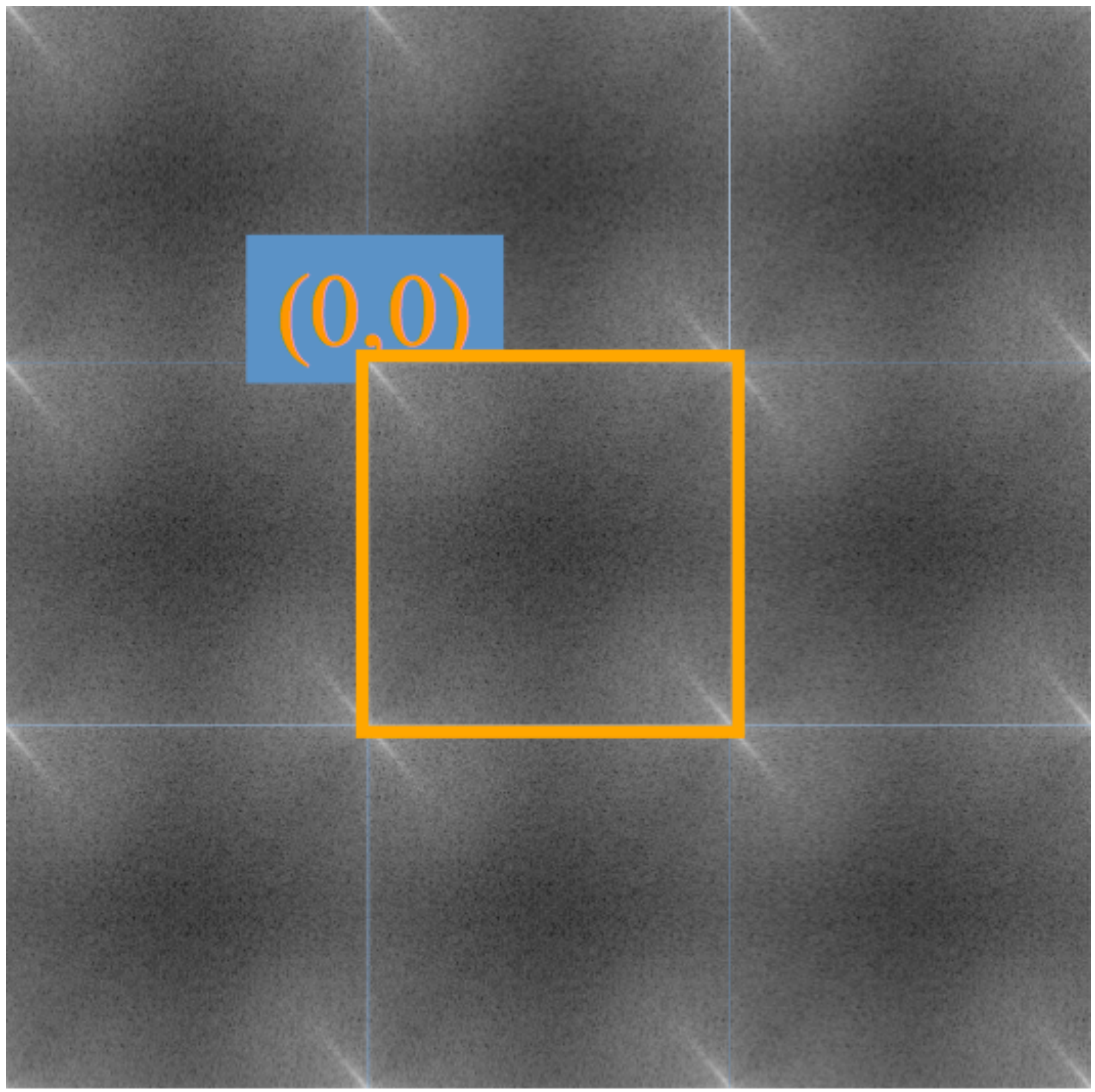
Discrete Fourier Transform

Properties

- Recall: $I(i, j) = \frac{1}{MN} \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} \tilde{I}(u, v) W_N^{-ui} W_M^{-vj}$
- This implies $I(i + nN, j + mM) = I(i, j)$
  - $W_N^{-(u+nN)i} = W_N^{-ui} W_N^{-Ni} = W_N^{-ui}$
- When DFT is used, image periodicity is implied. **Important** in convolution implementations

Discrete Fourier Transform

Properties



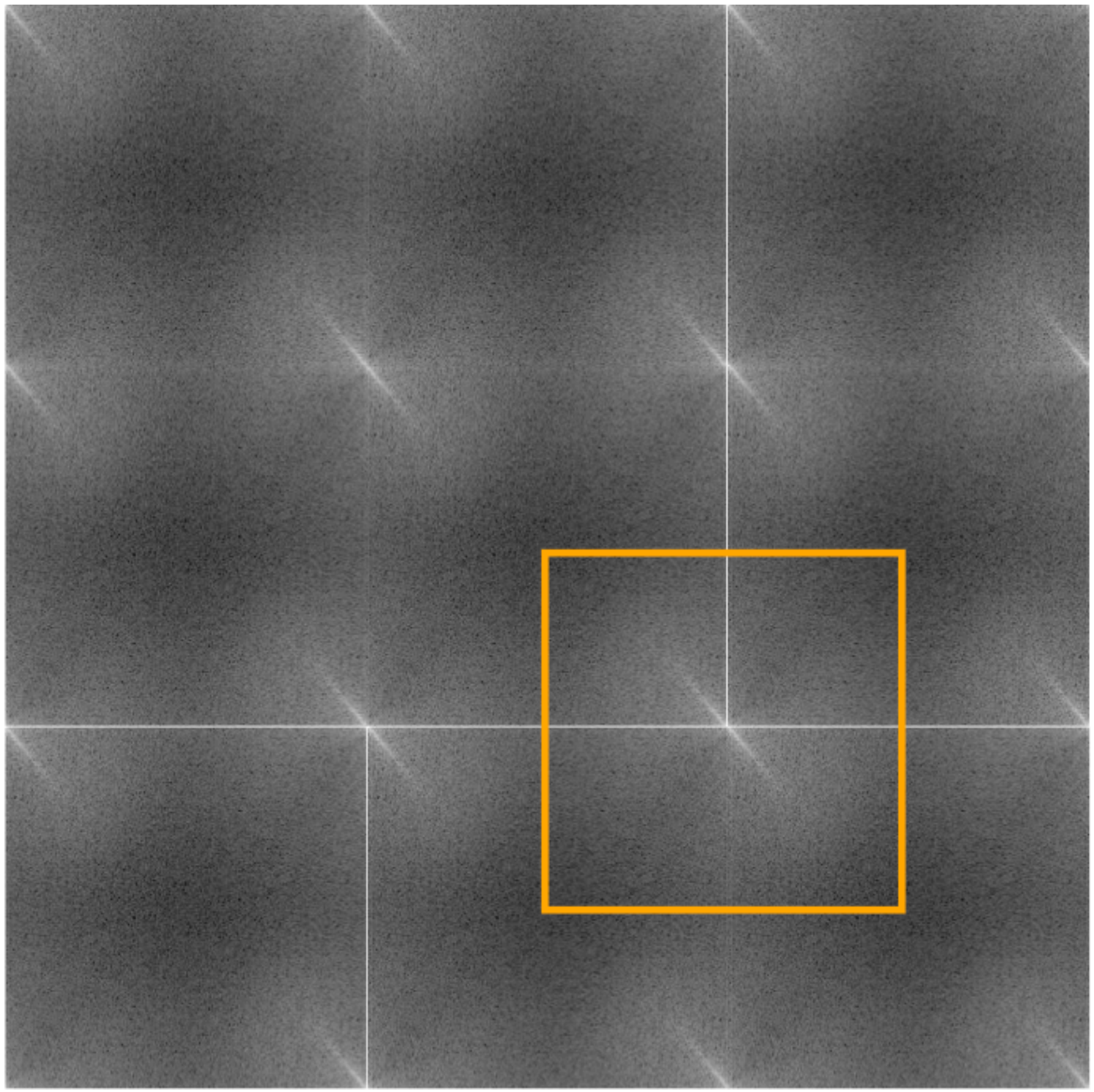
Discrete Fourier Transform

Properties

- Ideally, would like to have $(u, v) = (0,0)$ at the center of the image
- Low frequency components cluster around the center
- How can we achieve this?
  - Modulation or frequency shifting
  - DFT$[(-1)^{(i+j)}I(i,j)] =$ DFT$[I(i,j)W_N^{-Ni/2}W_M^{-Mj/2}] = \tilde{I}(u - N/2, v - M/2)$
- **This is only for display!**

Discrete Fourier Transform

Properties



Discrete Fourier Transform

Interpreting Image Frequencies

- Image DFT or **spectrum** reveals much about image
- DFT magnitude can be regarded as **image of frequency content**
- Bright regions correspondto frequencies having **large magnitude**
- Think of image frequency in terms of **granularity** and **orientation**
- Large DFT coefficients near the origin correspond to **smooth** image
- The distribution of frequencies relative to the origin correspond to **granularity**

Discrete Fourier Transform

Properties

- Matrix implementation of DFT
  - $$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & W_N & \dots & W_N^{N-1} \\ \vdots & & & \\ 1 & W_N^{N-1} & \dots & W_N^{(N-1)^2} \end{bmatrix}$$
  - $$\mathbf{W}_N^{-1} = \frac{1}{N}\mathbf{W}_N^* = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & W_N^{-1} & \dots & W_N^{-(N-1)} \\ \vdots & & & \\ 1 & W_N^{-(N-1)} & \dots & W_N^{-(N-1)^2} \end{bmatrix}$$
- Assuming $N \times N$ image
  - $\tilde{\mathbf{I}} = \mathbf{W}_N \mathbf{I} \mathbf{W}_N$
  - $\mathbf{I} = ?$