Information Theory
Solutions to Practice Sets 11 & 12

Lakshmi Prasad Natarajan

Remarks and Solutions to selected questions from Practice Sets 11 & 12

Practice Set 11

- 1(a). $f(y|x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-x)^2}{2\sigma^2}\right)$
- 1(b). $f(\boldsymbol{y}|\boldsymbol{x}) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y_i-x_i)^2}{2\sigma^2}\right)$.
- 1(c). The first option is the optimal decoder.
- 1(d). This is rather straightforward given the answer to 1(c). Diagrams like these can be found in digital communications textbooks as well.

  `Remark`: These regions $\mathcal{D}_i$, $i = 1, \ldots, M$, are also known as *Voronoi regions*.
- 2. Suppose $\mathcal{C} = \{\boldsymbol{x}(1), \ldots, \boldsymbol{x}(M)\}$ is the codebook, and $\boldsymbol{y} \in \{0, 1, ?\}^n$ is the received vector. The following is the optimal decoding rule.

  Identify the list L of all codewords that match $\boldsymbol{y}$ at the unerased coordinates, and output any codeword from the list L. That is, $L = \{\boldsymbol{x}(w) \ : \ x_i(w) = y_i \text{ for all } i \text{ such that } y_i \neq ?\}$.

  `Remark`: Note that the decoder identifies the input message without any ambiguity if and only if the list L contains exactly one codeword. If $|L| > 1$, the decoder is wrong with probability $1 - \frac{1}{|L|}$. Thus,

  $$P[\hat{W} \neq W] = \mathbb{E}\left[\frac{|L| - 1}{|L|}\right] \leq P[\,|L| \geq 2\,].$$
- 3(a). $\sum_{i=0}^{t} \binom{n}{i}$.
- 3(b). $\sum_{i=0}^{t} \delta^i (1-\delta)^{n-i} \binom{n}{i} = 1 - \sum_{i=t+1}^{n} \delta^i (1-\delta)^{n-i} \binom{n}{i}$.

  `Remark`: It is a good exercise to prove the triangle inequality of the Hamming distance. It follows from a simple counting argument.
- 4(a). 2.
- 4(b). n.
- 4(c). 2.
- 5(a). This follows from the definition of $d_{\min}$.
- 5(c). Use the fact that the optimal decoder outputs the codeword closest to $\boldsymbol{y}$ (in terms of the Hamming distance).
- 5(d). Use the answer from Question 3(b).

Practice Set 12

- 2(c). Reliable communication (that is, $P_e \to 0$) is possible as long as $P > 0$. It is not necessary to have $P > \sigma^2$ (this is a common misconception). When the signal power is smaller than the noise power the capacity is less than $\frac{1}{2}$, but the capacity is non-zero as long as $P \neq 0$.

  `Remark`: *Capacity limit in terms of minimum required energy per bit E_b.* Denote σ^2 by $N_o/2$; this is standard notation is digital communications. Define E_b to be the energy spent by the transmitter per 1 bit of information. That is, if we are transmitting reliably at some rate $R < C$, we are encoding nR bits of information or message, and we are using an energy of nP. Therefore,

$$E_b = \frac{nP}{nR} = \frac{P}{R} \text{ and } \frac{E_b}{N_o} = \frac{P}{2\sigma^2 R}.$$

We want to determine the smallest value of E_b/N_o required for reliable communication to take place (without imposing any restrictions on rate R).

First note that E_b/N_o is a decreasing function of R. Since the supremum of all achievable rates is C, we observe that E_b/N_o is minimized when $R = C$ (to be mathematically precise, the infimum of E_b/N_o corresponds to $R = C$). Thus, for a fixed P, the minimum value of E_b/N_o is

$$\frac{P}{2\sigma^2 C} = \frac{\text{SNR}}{2C} = \frac{\text{SNR}}{\log_2(1+\text{SNR})},$$

where SNR is P/σ^2.

Question. For what value of SNR does E_b/N_o take the minimum possible value (to be precise, this is infimum not minimum)? What is this minimum value of $\mathrm{E_b}/N_o$?

This minimum value of E_b/N_o is a fundamental limit in digital communications, and is especially relevant when the available SNR is limited (such as communications from a far away satellite to an earth station).

- 4. You must use the data processing inequality as done in the class.
- 5(a). This is a cascade of two BSC(p) channels. The resulting channel is also a BSC. Its bit flip probability is $2p(1-p)$.
- 5(b). Typo in question: reliable communication is possible at any rate $R < 1 - h_2(p)$.
