Information Theory, Exam 1 (02 Feb, 2023)

Serial Number: ______________ **Roll Number:** ________________________

Important

- Any malpractice will lead to instant fail grade.
- For Part A: provide only the final answers in this question paper itself. For Part B: provide complete solutions in the answer booklets.
- **Write your roll number and "serial number" both in this question paper and in the answer sheet.**
- No breaks during the exam. No books, notes, laptops, calculators, mobile devices etc. are allowed.

Part A (7 questions × 1 mark)

Instructions: Provide your **answers in this question paper** only. Write only the final answers in the space provided here.

1. If T is the number of fair coin tosses needed by an optimal algorithm to generate a discrete random variable X, give upper and lower bounds on $\mathbb{E}(T)$.
   `Answer:`
2. Write the expression for the log-sum inequality.
   `Answer:`
3. Write the expression for data processing inequality if $X \to Y \to Z$.
   `Answer:`
4. $H(X_1, \ldots, X_n) = \sum_{i=1}^{n} H(X_i)$ if and only if
   `Answer:`
5. If X and Y are Bernoulli random variables and $\alpha = P[X \neq Y]$. Then Y contains the least information about X for what value of α ?
   `Answer:`
6. Give an upper bound on $P\big[p_X(X) \leq a\big]$, where $a < 1$ and X is a discrete random variable. The upper bound must be in terms of $H(X)$ and a.
   `Answer:`
7. If $X_1, \ldots, X_n$ are i.i.d with probability mass function p_X, what is the limit, in probability, of $(p(X_1, \ldots, X_n))^{\frac{1}{n}}$?
   `Answer:`

Part B (3 questions × 5 marks)

Instructions: Provide **mathematically precise answers, with complete proofs,** in the answer booklet. Any result proved during the lectures can be stated and used in your answer without giving a proof.

8. Let X be a discrete random variable with probability mass function $p(x)$, $x \in \mathcal{X}$. Let $q(x)$, $x \in \mathcal{X}$ be any probability mass function defined on $\mathcal{X}$.

   Show that $\displaystyle\sum_{x \in \mathcal{X}} p(x) \log \frac{1}{q(x)} \geq H(X)$. When is equality attained in this inequality?

9. Let $\mathcal{N}(\mu, \sigma^2)$ denote the normal (Gaussian) distribution with mean μ and variance σ^2. Let the parameter $\theta \in \{0, 1\}$, i.e., it assumes only two possible values. Let $X = (X_1, X_2)$ and

   $$\text{if } \theta = 0 : X_1 \sim \mathcal{N}(1,1), X_2 \sim \mathcal{N}(0,1), \text{ and}$$
   $$\text{if } \theta = 1 : X_1 \sim \mathcal{N}(0,1), X_2 \sim \mathcal{N}(1,1).$$

   In both cases X_1, X_2 are independent.

   Is $T(X) = X_1 - X_2$ a sufficient statistic of X for θ? Support your answer with a proof.

10. *Scenario:* $M = (X_1, \ldots, X_n) \in \mathcal{X}^n$ where X_i are i.i.d discrete random variables each with probability mass function p_X. We now describe two (deterministic) functions:

    (i) f maps M into a string of ℓ binary digits $C = f(M) \in \{0,1\}^\ell$, i.e., $f\big((x_1, \ldots, x_n)\big) \in \{0,1\}^\ell$.

    (ii) g is a function used for decoding M from C, i.e., $g(C) = g\big(f(M)\big) \in \mathcal{X}^n$ is an estimate of M.

    Let $P_e = P\big[g(C) \neq M\big]$ denote the probability of detection error.

    Question: Assuming that $P_e \leq \epsilon$ where $0 < \epsilon < 0.5$, obtain a good lower bound on the ratio $\frac{\ell}{n}$ for all sufficiently large n. Your lower bound must be independent of the functions f and g.

    Bonus: (3 marks) If n is allowed grow, and ℓ also grows with n, and $P_e \leq \epsilon$ for all n, find a lower bound on $\lim_{\epsilon \to 0} \left(\lim_{n \to \infty} \frac{\ell}{n}\right)$, assuming that this limit exists. Can you identify functions f and g that attain this limit?