भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Wavelets and Their Applications

EE6310: Image and Video Processing

April 13, 2023

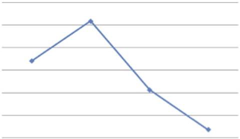
The Why, What, How of Wavelets

- Why: Time-frequency localization
- What: Wavelets are a tool that cuts up data or functions into different frequency components and studies each component with a resolution matched to its scale
- How: Digital filters

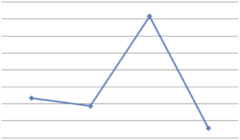
Orthogonal Wavelet Filters

- **Finite-length, orthogonal, discrete wavelets**
- Examples:
  - Haar: $h_0(i) = \begin{cases} \frac{1}{\sqrt{2}}; & i = 0 \\ \frac{1}{\sqrt{2}}; & i = 1 \\ 0; & \text{otherwise} \end{cases}$
  - Daubechies 4-tap filter: $h_0(i) = \frac{1}{4\sqrt{2}} \begin{cases} 1+\sqrt{3}; & i = 0 \\ 3+\sqrt{3}; & i = 1 \\ 3-\sqrt{3}; & i = 2 \\ 1-\sqrt{3}; & i = 3 \\ 0; & \text{otherwise} \end{cases}$

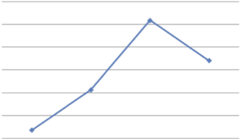
Daubechies 4-tap Wavelet



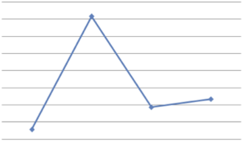
(a) Analysis LP



(b) Analysis HP



(c) Synthesis LP



(d) Synthesis HP

2-D Wavelet Decomposition

- 2-D analysis filters **decompose** images into high and low frequency bands
- Information in each band can be analyzed **separately**

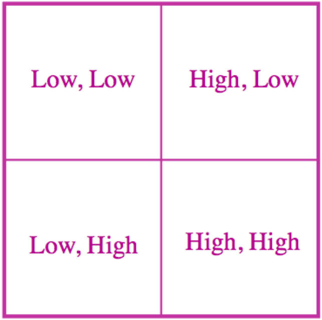


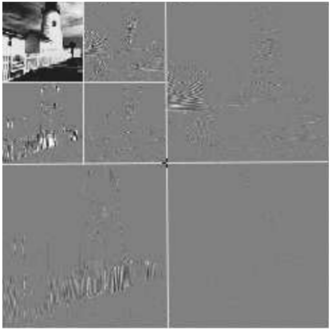
Figure: 2-D wavelet decomposition

Discrete Wavelet Transform

- Filter outputs are **wavelet coefficients**
- Image can be **prefectly** reconstructed from these coefficients
- Subsampling **heavier** at **lower** frequencies
- Multiple bands (> 2) created by **iterated filtering** on low-frequency subbands

Discrete Wavelet Transform - Example

(a) Image



(b) 2-level DWT

Image Processing Applications

Myriad of applications. A few popular ones...

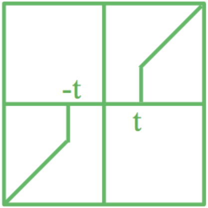
- Image restoration
- Image compression
- Image quality assessment
- Sparse representation
- Object detection

Wavelet Shrinkage

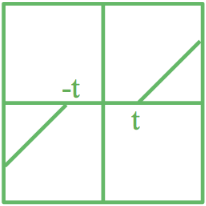
- Given $J = I + N$
- The process of **denoising** is accomplished simply by **thresholding the DWT coefficients** of J
- A **threshold is applied** to all non-LL subbands

Wavelet Shrinkage

- **Hard Thresholding:** signal zeroed if below threshold
$$J^t(i,j) = \begin{cases} J(i,j); & |J(i,j)| > \tau \\ 0; & |J(i,j)| \leq \tau \end{cases}$$
- **Soft Thresholding:** in addition to threshold, threshold is subtracted if above threshold
$$J^t(i,j) = \mathrm{sgn}[J(i,j)]\max\{0, |J(i,j)| - \tau\}$$



(a) Hard thresholding

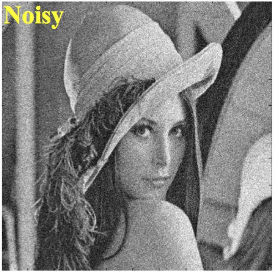


(b) Soft thresholding

Wavelet Shrinkage

- **Soft thresholding** is much more effective. It "shrinks" the wavelet coefficients
- Works since additive noise **increases coefficient energy**. Soft thresholding **reduces the energy**
- How to **find threshold**? Can be chosen to **optimize** many criteria such as MSE, Bayesian risk etc.
- **MMSE Threshold (SureShrink):** $\tau = \sigma\sqrt{2 \log NM}$ where σ^2 is the estimated noise variance, NM equals the number of pixels in the image
- **Bayesian threshold (BayesShrink):** $\tau = \sigma_{signal}/\sigma_{noise}$ is estimated for each subband, hence **adaptive**. Large SNR implies little thresholding and vice versa

Wavelet Shrinkage



Comments on Wavelet Shrinkage

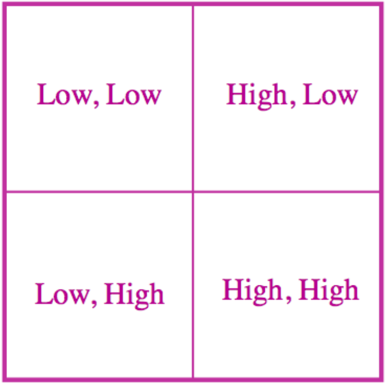
- Since DWTs can be computed **very fast**, so can **soft-threhold based denoising**
- The results are **optimal in wavelet space** and the appearance is **reasonably good**

Wavelet-based Image Coding

- **Idea:** Compress images in the wavelet domain
- Image blocks not used: **no blocking artifacts!**
- Forms the basis of **JPEG2000 standard**
- Requires **perfect reconstruction** filter banks

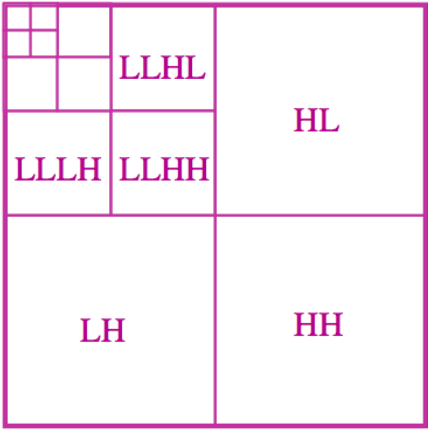
Wavelet Decomposition

- Recall that wavelet filters **separate or decompose** the image into high and low frequency **subbands**

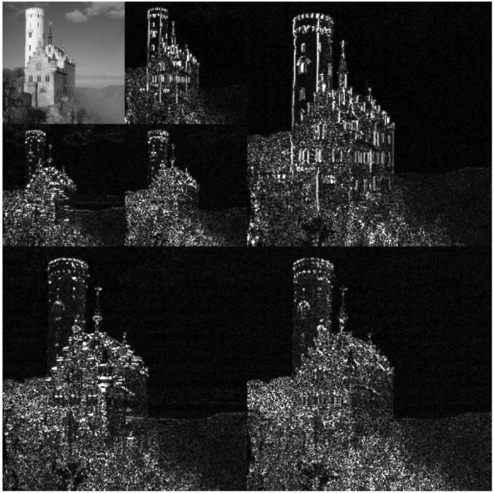


Wavelet Decomposition

- Typically frequency division is done iteratively on the **lowest-frequency bands**

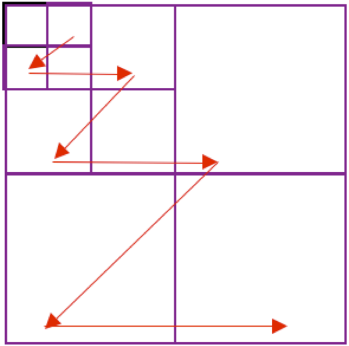


DWT of an Image

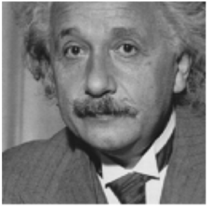


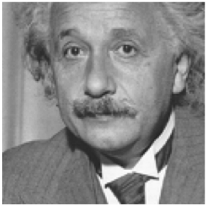
Zero-tree Encoding Concept

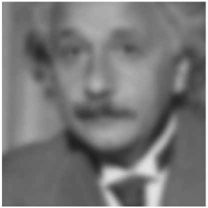
- Many **complex** wavelet-based compression techniques
- Popular technique is based on **zero-trees:** similar to JPEG, **scan from lower to higher frequencies** and **run-length code** the zero coefficients



Wavelet-based Image Quality Assessment


(a) Reference


(b) Mean


(c) Blurred


(d) JPEG

Figure: All images have same MSE (309)!

Wavelet-based Image Quality Assessment

Outline:

- Statistical modeling of wavelet coefficients - the Gaussian Scale Mixture model
- $X = xU$ where $z \geq 0$ is a scalar random variable, $U \sim \mathcal{N}(0, Q)$ is Gaussian random vector
- Model the IQA process as a communication channel: $Y = g * X + N$
- Image quality is measured as the mutual information between X and Y, $I(X; Y)$

Compressive sensing

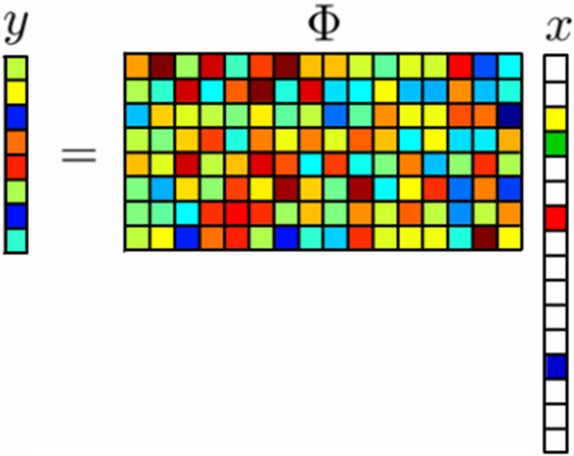


Image courtesy: http://nuit-blanche.blogspot.fr/

Compressive sensing

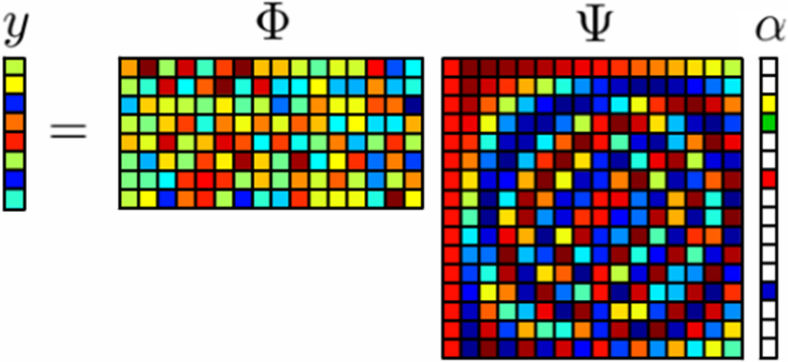
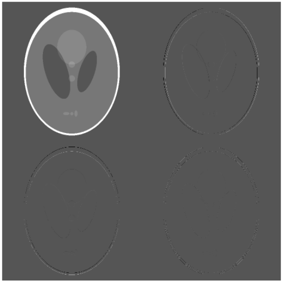


Image courtesy: http://nuit-blanche.blogspot.fr/

Sparse representation using wavelets

(a) Image



(b) Subband decomposition