# QM3

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## 1 Tensors: Introduction

Tensors are used to describe how quantities change on coordinate transforms. Consider a vector(?) space mapped by the coordinate system $\{x^1, x^2...\}$ that smoothly maps to a new set of coordinates $\{\bar{x}^1, \bar{x}^2..\}$.
If both spaces are $N$ dimensional, we can write

$$dx^i = \sum_{\alpha=1}^{N} \frac{\partial x^i}{\partial \bar{x}^\alpha} d\bar{x}^\alpha$$

$$d\bar{x}^i = \sum_{\alpha=1}^{N} \frac{\partial \bar{x}^i}{\partial x^\alpha} dx^\alpha$$

In the Einstein summation notation we can skip the $\sum$ sign and sum over all possible values of repeated (dummy) indices. All the free indices give us a tensor quantity. For example

$$\frac{dx^i}{dx^j} = \delta^i_j$$

$$\frac{d\bar{x}^\alpha}{d\bar{x}^\beta} = \delta^\alpha_\beta$$

Where $\delta$ is the usual kronecker delta tensor.
The way mathematical objects 'transform' is used to classify them:
**Rank 1 contravariant tensor**
A group of $N$ functions that depend on coordinates $A^i$ is a contravariant tensor if it transforms on changing coordinates as

$$\overline{A}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^j} A^j$$

Observe that on multiplying by $\frac{\partial x^i}{\partial \bar{x}^\alpha}$ and summing over all $\alpha$ we get

$$\frac{\partial x^i}{\partial \bar{x}^\alpha} \overline{A}^\alpha = \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\alpha}{\partial x^j} A^j$$

$$\frac{\partial x^i}{\partial \bar{x}^\alpha} \overline{A}^\alpha = \delta^i_j A^j$$

$$\frac{\partial x^i}{\partial \bar{x}^\alpha} \overline{A}^\alpha = A^i$$

Using the chain rule expansion to show $\frac{\partial x^i}{\partial x^j} = \frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial \bar{x}^\alpha}{\partial x^j}$. This shows us the transformation is consistent for both coordinate changes $\left\{x^1, x^2..\right\} \to \left\{\bar{x}^1, \bar{x}^2..\right\}$ and vice versa.

**Rank 1 covariant tensor**

A group of $N$ functions that depend on coordinates $A_i$ is a covariant tensor if it transforms on changing coordinates as

$$\overline{A}_\alpha = \frac{\partial x^j}{\partial \bar{x}^\alpha} A_j$$
$$A_i = \frac{\partial \bar{x}^\alpha}{\partial x^i} \overline{A}_\alpha$$

The gradient of a scalar field for example is a covariant vector. Let $\phi(\left\{x^i\right\}) = \phi(\left\{\bar{x}^i\right\})$ be a scalar field that is left unchanged on transforming to a new coordinate system. It's gradient is given by a covariant tensor.

$$A_i = \frac{\partial \phi}{\partial x^i}$$

It is easy to observe on transforming coordinates, using the chain rule that

$$\overline{A}_\alpha = \frac{\partial \phi}{\partial \bar{x}^i}$$
$$\overline{A}_\alpha = \frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial \phi}{\partial x^i}$$
$$\overline{A}_\alpha = \frac{\partial x^i}{\partial \bar{x}^\alpha} A_i$$

Thus verifying that it is a covariant tensor.

**General tensor**

$A$ is a general mixed tensor if it transforms as follows:

$$\overline{A}^{\alpha_1 \alpha_2 \ldots \alpha_p}_{\beta_1 \beta_2 \ldots \beta_q} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}}\frac{\partial \bar{x}^{\alpha_2}}{\partial x^{i_2}} \ldots \frac{\partial x^{j_1}}{\partial \bar{x}^{\beta_1}} \ldots A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$$

2 tensors can only be added or susbtracted if they are of the same type ie they both have the same no of co and contravaraint indices