

# A Comparison of Global and Saturated Probabilistic Approximations

## Using Characteristic Sets in Mining Incomplete Data

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**Abstract**—In this paper, we discuss incomplete data sets with two interpretations of missing attribute values, lost values and “do not care” conditions. For such incomplete data sets, we apply data mining based on two probabilistic approximations, global and saturated. The main objective of our paper is to compare both approaches in terms of an error rate, evaluated by ten-fold cross validation. Saturated probabilistic approximations are closer to the concept than global probabilistic approximations, so the corresponding error rate should be smaller. Using a 5% level of significance, our main result shows that there are differences between both approaches. However, in general neither is better for all data sets and thus, both approaches should be tried for each data set with the best selected for rule induction.

**Keywords**—Data mining; rough set theory; probabilistic approximations; MLEM2 rule induction algorithm; lost values; “do not care” conditions.

## I. INTRODUCTION

In this paper, we use two interpretations of a missing attribute value: lost values and “do not care” conditions. Lost values indicate that the original values were erased, and as a result we should use only existing, specified attribute values for rule induction. A lost value is denoted by “?”. “Do not care” conditions mean that the missing attribute value may be replaced by any specified attribute value. A “do not care” condition is denoted by “\*”.

We use for data mining probabilistic approximations, a generalization of the idea of lower and upper approximations known in rough set theory. A probabilistic approximation is associated with a parameter (probability)  $\alpha$ , if  $\alpha = 1$ , a probabilistic approximation is reduced to the lower approximation; if  $\alpha$  is a small positive number, e.g., 0.001, the probabilistic approximation becomes the upper approximation. Usually, probabilistic approximations are applied to completely specified data sets [1]–[9], such approximations were generalized to incomplete data sets in [10].

Characteristic sets were introduced in [11] for incomplete data sets with any interpretation of missing attribute values. Recently, global probabilistic approximations were introduced

in [12]. In this paper, we introduce a new idea of saturated probabilistic approximations. Our main objective is to compare both approaches in terms of an error rate, evaluated by ten-fold cross validation. The Modified Learning from Examples Module, version 2 (MLEM2) [13] was used for rule induction.

This paper starts with a discussion on incomplete data in Section II where we define attribute-value blocks and characteristic sets. In Section III, we present two types of probabilistic approximations, global and saturated. Section IV contains the details of our experiments. Finally, conclusions are presented in Section V.

## II. INCOMPLETE DATA

We assume that the input data sets are presented in the form of a decision table. An example of a decision table is shown in Table I. Rows of the decision table represent cases, while columns are labeled by variables. The set of all cases will be denoted by  $U$ . In Table I,  $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ . Independent variables are called attributes and a dependent variable is called a decision and is denoted by  $d$ . The set of all attributes will be denoted by  $A$ . In Table I,  $A = \{\text{Temperature}, \text{Headache}, \text{Cough}\}$ . The value for a case  $x$  and an attribute  $a$  will be denoted by  $a(x)$ . For example,  $\text{Temperature}(1) = \text{high}$ .

The set  $X$  of all cases defined by the same value of the decision  $d$  is called a *concept*. For example, a concept associated with the value *yes* of the decision *Flu* is the set  $\{1, 2, 3, 4\}$ .

For a variable  $a$  and its value  $v$ ,  $(a, v)$  is called a variable-value pair. A *block* of  $(a, v)$ , denoted by  $[(a, v)]$ , is the set  $\{x \in U \mid a(x) = v\}$  [14]. For incomplete decision tables, the definition of a block of an attribute-value pair is modified in the following way:

- if for an attribute  $a$  and a case  $x$  we have  $a(x) = ?$ , the case  $x$  should not be included in any blocks  $[(a, v)]$  for all values  $v$  of attribute  $a$ ;
- if for an attribute  $a$  and a case  $x$  we have  $a(x) = *$ , the case  $x$  should be included in blocks  $[(a, v)]$  for all

TABLE I. A DECISION TABLE

Case	Attributes			Decision
	Temperature	Headache	Cough	
1	high	yes	*	yes
2	*	no	no	yes
3	very-high	*	yes	yes
4	high	no	*	yes
5	normal	yes	no	no
6	high	*	no	no
7	?	no	*	no
8	normal	yes	yes	no

specified values  $v$  of attribute  $a$ .

For the data set from Table I, the blocks of attribute-value pairs are:

$$\begin{aligned} [(\text{Temperature}, \text{normal})] &= \{2, 5, 8\}, \\ [(\text{Temperature}, \text{high})] &= \{1, 2, 4, 6\}, \\ [(\text{Temperature}, \text{very-high})] &= \{2, 3\}, \\ [(\text{Headache}, \text{no})] &= \{2, 3, 4, 6, 7\}, \\ [(\text{Headache}, \text{yes})] &= \{1, 3, 5, 6, 8\}, \\ [(\text{Cough}, \text{no})] &= \{1, 2, 4, 5, 6, 7\}, \text{ and} \\ [(\text{Cough}, \text{yes})] &= \{1, 3, 4, 7, 8\}. \end{aligned}$$

For a case  $x \in U$  and  $B \subseteq A$ , the characteristic set  $K_B(x)$  is defined as the intersection of the sets  $K(x, a)$ , for all  $a \in B$ , where the set  $K(x, a)$  is defined in the following way:

- if  $a(x)$  is specified, then  $K(x, a)$  is the block  $[(a, a(x))]$  of attribute  $a$  and its value  $a(x)$ ;
- if  $a(x) = ?$  or  $a(x) = *$ , then  $K(x, a) = U$ .

For example, for Table I and  $B = A$ ,

$$K(1, \text{Temperature}) = [(\text{Temperature}, \text{high})] = \{1, 2, 4, 6\},$$

$$K(1, \text{Headache}) = [(\text{Headache}, \text{yes})] = \{1, 3, 5, 6, 8\}$$

and

$$K(1, \text{Cough}) = U$$

$$\text{so } K_A(1) = \{1, 2, 4, 6\} \subseteq \{1, 3, 5, 6, 8\} \subseteq U = \{1, 6\}.$$

Similarly,

$$\begin{aligned} K_A(2) &= \{2, 4, 6, 7\}, \\ K_A(3) &= \{3\}, \\ K_A(4) &= \{2, 4, 6\}, \\ K_A(5) &= \{5\}, \\ K_A(6) &= \{1, 2, 4, 6\}, \\ K_A(7) &= \{2, 3, 4, 6, 7\}, \text{ and} \\ K_A(8) &= \{8\}. \end{aligned}$$

### III. PROBABILISTIC APPROXIMATIONS

In this section, we will discuss two types of probabilistic approximations: global and saturated.

#### A. Global Probabilistic Approximations

An idea of the global probabilistic approximation, though restricted only to lower and upper approximations, was introduced in [15][16], and presented in a general form in [12]. Let  $X$  be a concept,  $X \subseteq U$ . A  $B$ -global probabilistic approximation of the concept  $X$ , based on characteristic sets,

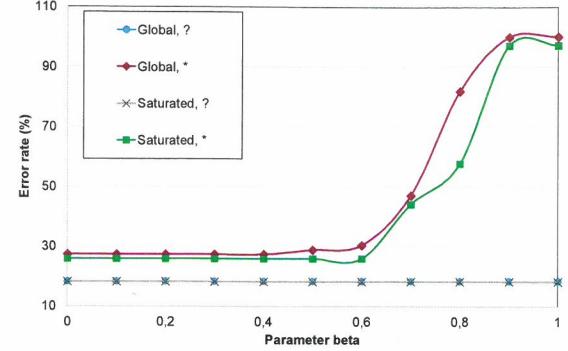


Figure 1. Error rate for the *Bankruptcy* data

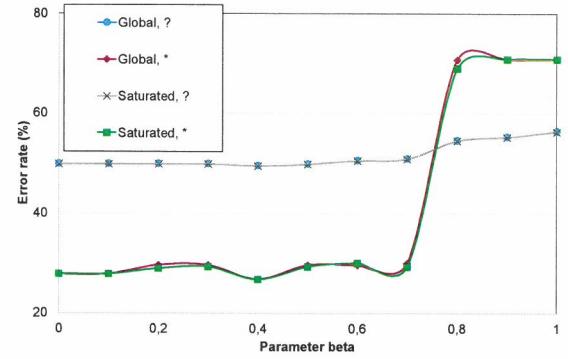


Figure 2. Error rate for the *Breast cancer* data set

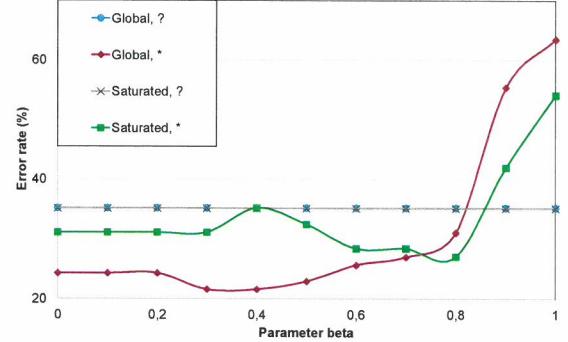


Figure 3. Error rate for the *Echocardiogram* data set

with the parameter  $\alpha$  and denoted by  $appr_{\alpha, B}^{global}(X)$  is defined as the following set

$$\bigcup \{K_B(x) \mid \exists Y \subseteq U \forall x \in Y, Pr(X|K_B(x)) \geq \alpha\}.$$

In general, for given sets  $B$  and  $X$  and the parameter  $\alpha$ , there exist many  $B$ -global probabilistic approximations of  $X$ . Additionally, the algorithm for computing  $B$ -global probabilistic approximations is of exponential computational complexity. Therefore, we decided to use a heuristic version of the definition of  $B$ -global probabilistic approximation,

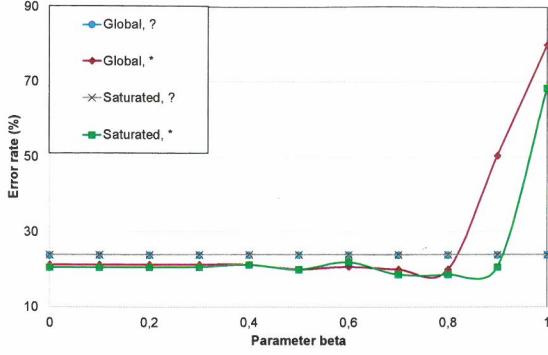


Figure 4. Error rate for the *Hepatitis* data set

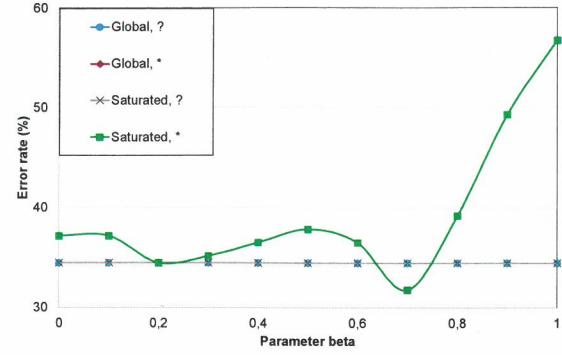


Figure 7. Error rate for the *Lymphography* data set

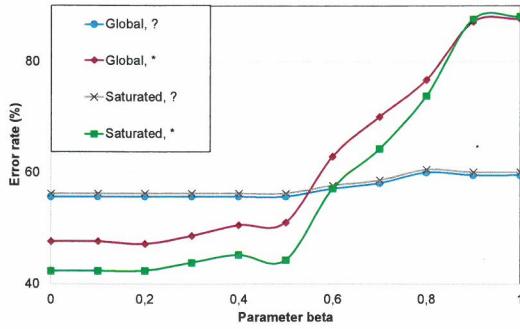


Figure 5. Error rate for the *Image segmentation* data set

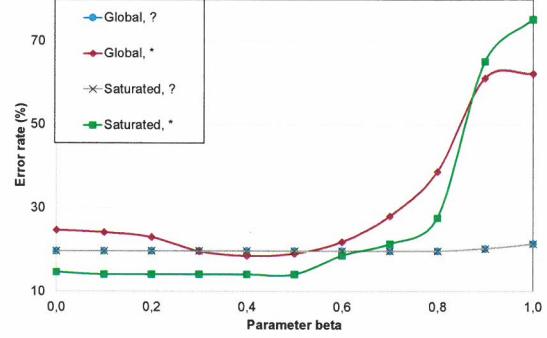


Figure 8. Error rate for the *Wine recognition* data set

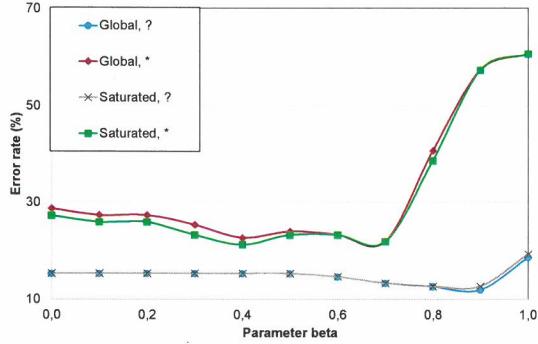


Figure 6. Error rate for the *Iris* data set

called a MLEM2  $B$ -global probabilistic approximation of the concept  $X$ , associated with a parameter  $\alpha$  and denoted by  $appr_{\alpha,B}^{mlem2}(X)$ , [12]. This definition is based on the rule induction algorithm MLEM2 (Modified Learning from Examples Module, version 2) [17]. The MLEM2 algorithm is used in the Learning from Examples using Rough Sets (LERS) data mining system [17]–[19]. The approximation  $appr_{\alpha,B}^{mlem2}(X)$  is constructed from characteristic sets  $K_B(y)$ , the most relevant to the concept  $X$ , i.e., with  $|X \cap K_B(y)|$  as large as possible and  $Pr(X|K_B(y)) \geq \alpha$ , where  $y \in U$ . If more than one characteristic set  $K_B(y)$  satisfies both conditions, we pick

the characteristic set  $K_B(y)$  with the largest  $Pr(X|K_B(y))$ . If this criterion ends up with a tie, a characteristic set is picked up heuristically, as the first on the list [12].

In this paper we study MLEM2  $B$ -global probabilistic approximations based on characteristic sets, with  $B = A$ , and calling them, for simplicity, *global probabilistic approximations* associated with the parameter  $\alpha$ , denoted by  $appr_{\alpha}^{mlem2}(X)$ . Similarly, for  $B = A$ , the characteristic set  $K_B(X)$  is denoted by  $K(x)$ .

Let  $E_{\alpha}(X)$  be the set of all eligible characteristic sets defined as follows

$$\{K(x) \mid x \in U, Pr(X|K(x)) \geq \alpha\}.$$

A heuristic version of the MLEM2 global probabilistic approximation is computed using the following algorithm.

#### MLEM2 global probabilistic approximation algorithm

**input:** a set  $X$  (a concept), a set  $E_{\alpha}(X)$ ,  
**output:** a set  $T$  ( $appr_{\alpha}^{mlem2}(X)$ )

**begin**

$G := X$ ;

$T := \emptyset$ ;

$Y := E_{\alpha}(X)$ ;

**while**  $G \neq \emptyset$  and  $Y \neq \emptyset$

**begin**

select a characteristic set  $K(x) \in Y$   
such that  $|K(x) \cap X|$  is maximum;

```

if a tie occurs, select  $K(x) \in Y$ 
with the smallest cardinality;
if another tie occurs, select the first  $K(x)$ ;
 $T := T \cup K(x)$ ;
 $G := G - T$ ;
 $Y := Y - K(x)$ 
end
end

```

For Table I, all distinct MLEM2 global probabilistic approximations are

$$\begin{aligned}
appr_1^{mlem2}(\{1, 2, 3, 4\}) &= \{3\}, \\
appr_{0.75}^{mlem2}(\{1, 2, 3, 4\}) &= \{1, 2, 3, 4, 6\}, \\
appr_{0.6}^{mlem2}(\{1, 2, 3, 4\}) &= \{1, 2, 3, 4, 6, 7\}, \\
appr_1^{mlem2}(\{5, 6, 7, 8\}) &= \{5, 8\}, \\
appr_{0.5}^{mlem2}(\{5, 6, 7, 8\}) &= \{2, 4, 5, 6, 7, 8\} \text{ and} \\
appr_{0.4}^{mlem2}(\{5, 6, 7, 8\}) &= \{2, 3, 4, 5, 6, 7, 8\}.
\end{aligned}$$

### B. Saturated Probabilistic Approximations

Another heuristic version of the probabilistic approximation is based on selection of characteristic sets while giving higher priority to characteristic sets with larger conditional probability  $Pr(X|K(x))$ . Additionally, if the approximation covers all cases from the concept  $X$ , we stop adding characteristic sets.

Let  $X$  be a concept and let  $x \in U$ . Let us compute all conditional probabilities  $Pr(X|K(x))$ . Then we sort the set

$$\{Pr(X|K(x)) \mid x \in U\}.$$

Let us denote the sorted list of such conditional probabilities by  $\alpha_1, \alpha_2, \dots, \alpha_n$ , where  $\alpha_1$  is the largest. For any  $i = 1, 2, \dots, n$ , the set  $E_i(x)$  is defined as follows

$$\{K(x) \mid x \in U, Pr(X|K(x)) = \alpha_i\}.$$

If we want to compute a saturated probabilistic approximation, denoted by  $appr_\alpha^{saturated}(X)$ , for some  $\alpha$ ,  $0 < \alpha \leq 1$ , we need to identify the index  $m$  such that

$$\alpha_m \geq \alpha > \alpha_{m+1},$$

where  $m \in \{1, 2, \dots, n\}$  and  $\alpha_{n+1} = 0$ . Then the saturated probabilistic approximation  $appr_{\alpha_m}^{saturated}(X)$  is computed using the following algorithm.

### Saturated probabilistic approximation algorithm

**input:** a set  $X$  (a concept), index  $m$ ,  
a set  $E_i(x)$  for  $i = 1, 2, \dots, n$  and  $x \in U$ ,  
**output:** a set  $T$  ( $appr_{\alpha_m}^{saturated}(X)$ )  
**begin**

```

 $T := \emptyset;$ 
 $Y_i(x) := E_i(x)$  for all  $i = 1, 2, \dots, m$  and  $x \in U$ ;
for  $j = 1, 2, \dots, m$  do
    while  $Y_j(x) \neq \emptyset$ 

```

```

begin
    select a characteristic set  $K(x) \in Y_j(x)$ 
    such that  $|K(x) \cap X|$  is maximum;
    if a tie occurs, select the first  $K(x)$ ;
     $Y_j(x) := Y_j(x) - K(x)$ ;
    if  $(K(x) - T) \cap X \neq \emptyset$ 
        then  $T := T \cup K(x)$ ;
    if  $X \subseteq T$  then exit
end

```

For Table I, all distinct saturated probabilistic approximations are

$$\begin{aligned}
appr_1^{saturated}(\{1, 2, 3, 4\}) &= \{3\}, \\
appr_{0.75}^{saturated}(\{1, 2, 3, 4\}) &= \{1, 2, 3, 4, 6\}, \\
appr_1^{saturated}(\{5, 6, 7, 8\}) &= \{5, 8\} \text{ and} \\
appr_{0.5}^{saturated}(\{5, 6, 7, 8\}) &= \{2, 4, 5, 6, 7, 8\}.
\end{aligned}$$

Note that  $appr_{0.6}^{mlem2}(\{1, 2, 3, 4\})$  covers the case 7 in spite of the fact that the case 7 is not a member of the concept  $\{1, 2, 3, 4\}$ . The set  $\{1, 2, 3, 4, 6, 7\}$  is not listed among saturated probabilistic approximations of the concept  $\{1, 2, 3, 4\}$ .

### C. Rule Induction

Once the global and saturated probabilistic approximations associated with a parameter  $\alpha$  are constructed, rule sets are induced using the rule induction algorithm based on another parameter, also interpreted as a probability, and denoted by  $\beta$ . This algorithm also uses MLEM2 principles [20].

### MLEM2 rule induction algorithm

**input:** a set  $Y$  (an approximation of  $X$ ) and a parameter  $\beta$ ,  
**output:** a set  $\mathcal{T}$  (a rule set),  
**begin**

```

 $G := Y;$ 
 $D := Y;$ 
 $\mathcal{T} := \emptyset;$ 
 $\mathcal{J} := \emptyset;$ 
while  $G \neq \emptyset$ 
    begin
         $T := \emptyset;$ 
         $T_s := \emptyset;$ 
         $T_n := \emptyset;$ 
         $T(G) := \{t \mid [t] \cap G \neq \emptyset\};$ 
        while  $(T = \emptyset \text{ or } [T] \not\subseteq D) \text{ and } T(G) \neq \emptyset$ 
            begin
                select a pair  $t = (a_t, v_t)$ 
                 $\in T(G)$  with maximum of
                 $|[t] \cap G|$ ; if a tie occurs,
                select a pair  $t \in T(G)$ 
                with the smallest cardinality
                of  $[t]$ ; if another tie occurs,
                select first pair;
                 $T := T \cup \{t\}$ ;
                 $G := [t] \cap G$ ;
                 $T(G) := \{t \mid [t] \cap G \neq \emptyset\}$ ;
                if  $a_t$  is symbolic let
                     $V_{a_t}$  be the domain of  $a_t$ 

```

```

then
   $T_s := T_s \cup \{(a_t, v) | v \in V_{a_t}\}$ 
else { $a_t$  is numerical,
  let  $t = (a_t, u..v)$ } and  $T_n :=$ 
   $T_n \cup \{(a_t, x..y) |$  disjoint
   $x..y$  and  $u..v\} \cup$ 
   $\{(a_t, x..y) | x..y \supseteq u..v\};$ 
   $T(G) := T(G) - (T_s \cup T_n);$ 
end {while};
if  $Pr(X | [T]) \geq \beta$ 
then
  begin
     $D := D \cup [T];$ 
     $\mathcal{T} := \mathcal{T} \cup \{T\};$ 
  end {then}
  else  $\mathcal{J} := \mathcal{J} \cup \{T\};$ 
   $G := D - \cup_{S \in \mathcal{T} \cup \mathcal{J}} [S];$ 
end {while};
for each  $T \in \mathcal{T}$  do
  for each numerical attribute  $a_t$  with
   $(a_t, u..v) \in T$  do
    while ( $T$  contains at least two different
    pairs  $(a_t, u..v)$  and  $(a_t, x..y)$  with
    the same numerical attribute  $a_t$ )
      replace these two pairs with a new pair
       $(a_t, \text{common part of } (u..v) \text{ and } (x..y));$ 
    for each  $t \in T$  do
      if  $[T - \{t\}] \subseteq D$  then  $T := T - \{t\};$ 
  for each  $T \in \mathcal{T}$  do
    if  $\cup_{S \in (\mathcal{T} - \{T\})} [S] = \cup_{S \in \mathcal{T}} [S]$  then  $\mathcal{T} := \mathcal{T} - \{T\};$ 
end {procedure}.

```

For example, for Table I and  $\alpha = \beta = 0.5$ , using the saturated probabilistic approximations, the MLEM2 rule induction algorithm induces the following rules:

(Cough, no) & (Headache, no)  $\rightarrow$  (Flu, no),  
 (Temperature, normal)  $\rightarrow$  (Flu, no),  
 (Temperature, high)  $\rightarrow$  (Flu, yes) and  
 (Temperature, very-high)  $\rightarrow$  (Flu, yes).

#### IV. EXPERIMENTS

For our experiments, we used eight data sets that are available in the University of California at Irvine *Machine Learning Repository*.

For every data set, a template was created. Such a template was formed by replacing randomly 35% of existing specified attribute values by *lost values*. The same templates were used for constructing data sets with “do not care” conditions, by replacing “?”’s with “\*”’s.

In our experiments, we used the MLEM2 rule induction algorithm. In all experiments, the parameter  $\alpha$  was equal to 0.5. Results of our experiments are presented in Figures 1–8, where “Global” denotes a MLEM2 global probabilistic approximation, “Saturated” denotes a saturated probabilistic approximation, “?” denotes lost values and “\*” denotes “do not care” conditions. In our experiments, four approaches for mining incomplete data sets were used, since we combined two options of probabilistic approximations: global and saturated with two interpretations of missing attribute values: lost and “do not care” conditions.

TABLE II. RESULTS OF STATISTICAL ANALYSIS

Data set	Friedman test results (5% significance level)
Bankruptcy	(Global, *) is better than (Global, ?) (Saturated, *) is better than (Global, *) (Global, *) is better than (Saturated, ?) (Saturated, *) is better than (Saturated, ?)
Iris	(Global, *) is better than (Global, ?) (Saturated, *) is better than (Global, ?) (Global, *) is better than (Saturated, ?) (Saturated, *) is better than (Saturated, ?)
Lymphography	(Global, *) is better than (Global, ?) (Saturated, *) is better than (Global, ?) (Global, *) is better than (Saturated, ?) (Saturated, *) is better than (Saturated, ?)
Echocardiogram	(Global, ?) is better than (Global, *) (Saturated, ?) is better than (Global, *)
Hepatitis	(Global, ?) is better than (Saturated, *) (Saturated, ?) is better than (Saturated, *)
Wine recognition	(Global, *) is better than (Saturated, *)

These four approaches were compared by applying the distribution free Friedman rank sum test and then by the post-hoc test (distribution-free multiple comparisons based on the Friedman rank sums), with a 5% level of significance. The null hypothesis  $H_0$  of the Friedman test, claiming that differences between these approaches are insignificant, was rejected for *Breast cancer* and *Image recognition* as the only data sets. Results of the post-hoc distribution free all-treatments multiple comparisons Wilcoxon-Nemenyi-McDonald-Thompson test for the remaining six data sets are presented in Table II. This table is divided into three parts. In the first part, for *Bankruptcy*, *Iris* and *Lymphography* data sets, data mining approaches based on “do not care” conditions are always better than approaches based on lost values. In the second part, for *Echocardiogram* and *Hepatitis*, it is the other way around. In the third part, for *Wine recognition*, the only conclusion is that for “do not care” conditions global probabilistic approximations are better than saturated ones.

#### V. CONCLUSIONS

We compared four approaches for mining incomplete data sets, combining two interpretations of missing attribute values with two types of probabilistic approximations. Our criterion of quality was an error rate computed as a result of ten-fold cross-validation. As follows from our experiments, there were significant differences between the four approaches. However, the best approach, associated with the smallest error rate, depends on a specific data set. Thus, for a given data set, the base approach to mining must be selected by running experiments involved with all four approaches.

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$\text{appr}_\alpha^{\text{mem}2}(X)$

Procedure for determining a single local probabilistic approximation

input: a set  $X$  (a subset of  $U$ ) and a parameter  $\alpha$ ,

output: a set  $T$  of the set  $X$ ,

begin

$G := X;$

$D := X;$

$T := \emptyset;$

$J := \emptyset;$

    while  $G \neq \emptyset$

        begin

$T := \emptyset;$

$T_s := \emptyset;$

$T_n := \emptyset;$

$T(G) := \{t \mid [t] \cap G \neq \emptyset\};$

            while ( $T = \emptyset$  or  $[T] \not\subseteq D$ ) and  $T(G) \neq \emptyset$

                begin

                    select a pair  $t = (a_t, v_t) \in T(G)$  such that

$|[t] \cap G|$  is maximum; if a tie occurs, select

                    a pair  $t \in T(G)$  with the smallest cardinality

                    of  $[t]$ ; if another tie occurs, select first pair;

$T := T \cup \{t\};$

$G := [t] \cap G;$

$T(G) := \{t \mid [t] \cap G \neq \emptyset\};$

                    if  $a_t$  is symbolic {let  $V_{a_t}$  be the domain of  $a_t$ }

                        then

$T_s := T_s \cup \{(a_t, v) \mid v \in V_{a_t}\}$

                        else { $a_t$  is numerical, let  $t = (a_t, u..v)\}$

$T_n := T_n \cup \{(a_t, x..y) \mid \text{disjoint } x..y$

                            and  $u..v\} \cup \{(a_t, x..y) \mid x..y \supseteq u..v\};$

$T(G) := T(G) - (T_s \cup T_n);$

                        end {while};

                        if  $Pr(X \mid [T]) \geq \alpha$

                            then

                                begin

$D := D \cup [T];$

$T := T \cup \{T\};$

                                end {then}

                                else  $J := J \cup \{T\};$

$G := D - \cup_{S \in T \cup J} [S];$

                        end {while};

                for each  $T \in T$  do

                        for each numerical attribute  $a_t$  with  $(a_t, u..v) \in T$  do

                                while ( $T$  contains at least two different

                                pairs  $(a_t, u..v)$  and  $(a_t, x..y)$  with

                                the same numerical attribute  $a_t$ )

  replace these two pairs with a new pair

  ( $a_t$ , common part of  $(u..v)$  and  $(x..y)$ );

                                for each  $t \in T$  do

  if  $[T - \{t\}] \subseteq D$  then  $T := T - \{t\};$

                                for each  $T \in T$  do

  if  $\cup_{S \in (T - \{T\})} [S] = \cup_{S \in T} [S]$  then  $T := T - \{T\};$

                end {procedure}.