Symbolic Models for Interconnected Impulsive Systems

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***Abstract*— In this paper, we present a compositional methodology for constructing symbolic models of nonlinear interconnected impulsive systems. Our approach relies on the concept of "alternating simulation function" to establish a relationship between concrete subsystems and their symbolic models. Assuming some small-gain type conditions, we develop an alternating simulation function between the symbolic models of individual subsystems and those of the nonlinear interconnected impulsive systems. To construct symbolic models of nonlinear impulsive subsystems, we propose an approach that depends on incremental input-to-state stability and forward completeness properties. Finally, we demonstrate the advantages of our framework through a case study.**

I. Introduction

The symbolic model (a.k.a abstraction) of dynamical systems involves representing complex systems using finite sets of states, inputs, and transition relations that capture the essential dynamics of the concrete system. The resulting abstract model must be related to the concrete system by some formal inclusion relation (e.g., simulation relation or alternating simulation relation) to enable model checking and controller design. For example, supervisory control [1], and algorithmic game theory [2], can be used for these purposes. The abstraction-based controller synthesis techniques, commonly used in this context, have the advantage of addressing high-level specifications that are usually expressed as temporal logic formulae [3]. Abstraction-based approaches, in general, rely on state-space and input-space discretization. The need for state space discretization results in an exponential increase in computational complexity with the dimension of state space in the concrete system, and, hence, these techniques suffer severely from the issue of so-called curse of dimensionality, especially for the systems with high-dimensional state space.

When dealing with complex, interconnected systems, the use of compositional abstraction becomes essential. In this approach, the abstraction process is broken down into smaller subsystem level construction of abstraction, allowing for a more manageable construction of the abstraction of the concrete system. A significant amount of research has been devoted to developing compositional abstractions for different classes of large-scale interconnected dynamical systems. The results include the construction of compositional abstraction for acyclic interconnected linear [4] and nonlinear [5] systems, compositional frameworks based on the notion of an (alternating) simulation function and small-gain type conditions [6], [7], compositional frameworks based on dissipativity properties [8], and compositional abstraction for interconnected switched system [9]. Authors in [10] propose a compositional approach using the concept of assume-guarantee contracts [11]. Finally, authors in [12], [13] proposed compositional abstraction frameworks using the concept of approximate composition.

However, none of the proposed approaches in the literature makes it possible to compositionally construct abstractions for the class of impulsive systems. Indeed, although [14] addressed the abstraction of impulsive systems, it focuses on providing a monolithic abstraction of impulsive systems, which can result in a high computational burden when applied to large-scale interconnected systems. Therefore, this paper aims to address this gap in the literature by developing novel results for the compositional abstraction of interconnected impulsive systems.

This paper establishes a novel compositional scheme for the construction of symbolic models of interconnected impulsive systems. In particular, we adapt the notion of alternating approximate simulation functions in [15] to establish a relation between each subsystem and its symbolic model. Based on some small gain-type conditions, we compositionally construct an overall alternating simulation function as a relation between an interconnection of symbolic models and that of the original interconnected subsystems. Furthermore, under certain stability and forward completeness properties, we present the construction of symbolic models for each subsystem of the original model. In our case study, we demonstrate the effectiveness of our approach by comparing the computational efficiency of compositional and monolithic methods for constructing symbolic models of systems while varying the number of interconnected subsystems.

II. Notations and Preliminaries

A. Notations

We denote by $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ the set of real numbers, integers, and non-negative integers, respectively. These symbols are annotated with subscripts to restrict them in an obvious way, e.g., $\mathbb{R}_{>0}$ denotes the positive real numbers. We denote the closed, open, and half-open intervals in $\mathbb{R}$ by $[a, b]$, (a, b), $[a, b)$, and $(a, b]$, respectively. For $a, b \in \mathbb{N}$ and $a \leqslant b$, we

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use $[a;b]$, $(a;b)$, $[a;b)$, and $(a;b]$ to denote the corresponding intervals in $\mathbb{N}$. Given any $a \in \mathbb{R}$, $|a|$ denotes the absolute value of a. Given any $u = [u_1;\ldots;u_n] \in \mathbb{R}^n$, the infinity norm of u is defined by $\|u\| = \max_{i\in[1;n]} \|u_i\|$. Given a function $\nu : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, the supremum of ν is denoted by $\|\nu\|_\infty$; we recall that $\|\nu\|_\infty := \sup_{t\in\mathbb{R}_{\geq 0}} \|\nu(t)\|$. Given $\mathbf{x} : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^n, \forall t, s \in \mathbb{R}_{\geqslant 0}$ with $t \geqslant s$, we define $\mathbf{x}(^-t) = \lim_{s\to t} \mathbf{x}(s)$ as the left limit operator. For a given constant $\tau \in \mathbb{R}_{\geq 0}$ and a set $\mathcal{W} := \{\mathbf{x} : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^n\}$, we denote the restriction of $\mathcal{W}$ to the interval $[0,\tau]$ by $\mathcal{W}|_{[0,\tau]} := \{\mathbf{x} : [0,\tau] \to \mathbb{R}^n\}$. We denote by $\mathcal{C}(\cdot)$ the cardinality of a given set and by $\varnothing$ the empty set. Given sets U and $S \subset U$, the complement of S with respect to U is defined as $U\backslash S = \{x : x \in U, x \notin S\}$. Given a family of finite or countable sets $S_i, i \in \mathcal{N} \subset \mathbb{N}$, the j^{th} element of the set S_i is denoted by s_{i_j}. For any set $S \subseteq \mathbb{R}^n$ of the form $S = \bigcup_{j=1}^{M} S_j$ for some $M \in \mathbb{N}_{>0}$, where $S_j = \prod_{i=1}^{n}[c_i^j, d_i^j] \subseteq \mathbb{R}^n$ with $c_i^j < d_i^j$, and non-negative constant $\eta \leqslant \tilde{\eta}$, where $\tilde{\eta} = \min_{j=1,\ldots,M} \eta_{S_j}$ and $\eta_{S_j} = \min\{|d_1^j - c_1^j|,\ldots,|d_n^j - c_n^j|\}$, we define $[S]_\eta = \{a \in S \mid a_i = k_i\eta, k_i \in \mathbb{Z}, i = 1,\ldots,n\}$ if $\eta \neq 0$, and $[S]_\eta = S$ if $\eta = 0$. The set $[S]_\eta$ will be used as a finite approximation of the set S with precision $\eta \neq 0$. Note that $[S]_\eta \neq \varnothing$ for any $\eta \leqslant \tilde{\eta}$. We use notations $\mathcal{K}$ and $\mathcal{K}_\infty$ to denote different classes of comparison functions, as follows: $\mathcal{K} = \{\alpha : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}|\ \alpha$ is continuous, strictly increasing, and $\alpha(0) = 0\}$; $\mathcal{K}_\infty = \{\alpha \in \mathcal{K}|\ \lim\limits_{s\to\infty} \alpha(s) = \infty\}$. For $\alpha, \gamma \in \mathcal{K}_\infty$ we write $\alpha \leq \gamma$ if $\alpha(r) \leq \gamma(r)$, $\forall r \in \mathbb{R}_{\geqslant 0}$, and, by abuse of notation, $\alpha = c$ if $\alpha(r) = cr$ for all $c, r \geqslant 0$. Finally, we denote by id the identity function over $\mathbb{R}_{\geq 0}$, i.e. $\mathsf{id}(r) = r, \forall r \in \mathbb{R}_{\geq 0}$.

B. Interconnected Impulsive System

1) Characterization of Impulsive Subsystems: We consider a set of impulsive subsystems indexed by $i \in \mathcal{N}$, where $\mathcal{N} = [1;N]$ and $N \in \mathbb{N}_{\geqslant 1}$. The i^{th} subsystem can be formally defined by,

Definition 2.1: A nonlinear impulsive subsystem Σ_i, $i \in \mathcal{N}$, is defined by the tuple $\Sigma_i = (\mathbb{R}_i^{n_i}, \mathbb{W}_i, \mathsf{W}_i, \mathbb{U}_i, \mathsf{U}_i, f_i, g_i, \mathbb{Y}_i, h_i, \Omega_i)$, where

- $\mathbb{R}_i^{n_i}$ is the state set;
- $\mathbb{W}_i \subseteq \mathbb{R}^{q_i}$ is the internal input set;
- W_i is the set of all measurable bounded internal input functions $\omega_i : \mathbb{R}_{\geqslant 0} \to \mathbb{W}_i$;
- $\mathbb{U}_i \subseteq \mathbb{R}^{m_i}$ is the external input set;
- U_i is the set of all measurable bounded external input functions $\nu_i : \mathbb{R}_{\geqslant 0} \to \mathbb{U}_i$;
- $f_i, g_i : \mathbb{R}^{n_i} \times \mathbb{W}_i \times \mathbb{U}_i \to \mathbb{R}^{n_i}$ are locally Lipschitz functions;
- $\mathbb{Y}_i \subseteq \mathbb{R}^{p_i}$ is the output set;
- $h_i : \mathbb{R}_i \to \mathbb{Y}_i$ is the output map;
- $\Omega_i = \{t_i^k\}_{k\in\mathbb{N}}$ is a set of strictly increasing sequence of impulsive times in $\mathbb{R}_{\geq 0}$ comes with $t_i^{k+1} - t_i^k \in \{\underline{z}_i\tau_i,\ldots,\overline{z}_i\tau_i\}$ for fixed jump parameters $\tau_i \in \mathbb{R}_{>0}$ and $\underline{z}_i, \overline{z}_i \in \mathbb{N}_{\geq 1}$, $\underline{z}_i \leq \overline{z}_i$.

The non-linear flow and jump dynamics, f_i and g_i are described by differential and difference equations of the form,

$$\Sigma_i : \begin{cases} \dot{\mathbf{x}}_i(t) = f_i(\mathbf{x}_i(t), \omega_i(t), \nu_i(t)), & t \in \mathbb{R}_{\geqslant 0}\backslash\Omega_i, \\ \mathbf{x}_i(t) = g_i(\mathbf{x}_i(^-t), \omega_i(^-t), \nu_i(t)), & t \in \Omega_i, \\ \mathbf{y}_i(t) = h_i(\mathbf{x}_i(t)), & t \in \mathbb{R}_{\geqslant 0}, \end{cases} \tag{II.1}$$

where $\mathbf{x}_i : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{n_i}$ and $\omega_i : \mathbb{R}_{\geqslant 0} \to \mathbb{W}_i$ are the state and internal input signals, respectively, and assumed to be right-continuous for all $t \in \mathbb{R}_{\geqslant 0}$. Function $\nu_i : \mathbb{R}_{\geqslant 0} \to \mathbb{U}_i$ is the external input signal. We will use $\mathbf{x}_{x_i,\omega_i,\nu_i}(t)$ to denote a point reached at time $t \in \mathbb{R}_{\geqslant 0}$ from initial state x_i under input signals $\omega_i \in \mathsf{W}_i$ and $\nu_i \in \mathsf{U}_i$. We denote by Σ_{c_i} and Σ_{d_i} the continuous and discrete dynamics of subsystem Σ_i, i.e., $\Sigma_{c_i} : \dot{\mathbf{x}}_i(t) = f_i(\mathbf{x}_i(t), \omega_i(t), \nu_i(t))$, and $\Sigma_{d_i} : \mathbf{x}_i(t) = g_i(\mathbf{x}_i(^-t), \omega_i(^-t), \nu_i(t))$.

2) Interconnections among Impulsive Subsystems: We assume that the input-output structure of each impulsive subsystem Σ_i, $i \in \mathbb{N}$, is general and formally given by,

$$\omega_i = [\omega_{i1};\ldots;\omega_{i(i-1)};\omega_{i(i+1)};\ldots;\omega_{iN}], \mathbb{W}_i = \prod_{\substack{j=1,\\ j\neq i}}^{N} \mathbb{W}_{ij}, \tag{II.2}$$

$$y_i = [y_{i1};\ldots;y_{iN}], \quad \mathbb{Y}_i = \prod_{j=1}^{N} \mathbb{Y}_{ij}, \tag{II.3}$$

where $\omega_{ij} \in \mathbb{W}_{ij}$, $y_{ij} = h_{ij}(x_i) \in \mathbb{Y}_{ij}$, and output function,

$$h_i(x_i) = [h_{i1}(x_i);\ldots;h_{iN}(x_i)], \tag{II.4}$$

and x_i denotes the state vector of the i^{th} subsystem. The outputs y_{ii} are considered as external, while y_{ij} with $i \neq j$ are internal and are used to define the connections between the subsystems. In fact, we consider that the dimension of the vector ω_i is equal to that of the vector y_i. If there is no connection between the subsystems Σ_i and Σ_j, h_{ij} is fixed as zero, i.e. $h_{ij} \equiv 0$.

Assumption 2.2: The interconnections are constrained by $\omega_{ij} = y_{ji}$, $\mathbb{Y}_{ji} \subseteq \mathbb{W}_{ij}$, $\forall i, j \in \mathcal{N}, i \neq j$.

3) Interconnected Impulsive Systems: The formal definition of the interconnected impulsive system is given by,

Definition 2.3: Consider $N \in \mathbb{N}_{\geqslant 1}$ impulsive subsystems

$$\Sigma_i = (\mathbb{R}^{n_i}, \mathbb{W}_i, \mathsf{W}_i, \mathbb{U}_i, \mathsf{U}_i, f_i, g_i, \mathbb{Y}_i, h_i, \Omega_i)$$

with input-output structure given by (II.2)-(II.4). The interconnected impulsive system is a tuple $\Sigma = (\mathbb{X}, \mathbb{U}, f, \mathcal{G}, \Omega)$, denoted by $\mathcal{I}(\Sigma_1,\ldots,\Sigma_N)$ and described by the differential, difference equation of the form,

$$\Sigma : \begin{cases} \dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \nu(t)), & \forall t \in \mathbb{R}_{\geqslant 0}\backslash\Omega \\ \mathbf{x}(t) = \mathcal{G}(\mathbf{x}(^-t), \nu(t)) & \forall t \in \Omega \end{cases} \tag{II.5}$$

with $x \in \mathbb{X} = \prod_{i=1}^{N} \mathbb{R}^{n_i}$, $\nu \in \mathbb{U} = \prod_{i=1}^{N} \mathbb{U}_i$, $\Omega = \bigcup_{i=1}^{N} \Omega_i$ and

$$f(\mathbf{x}(t), \nu(t)) = [f_1(x_1(t), \omega_1(t), v_1(t)),\ldots,f_n(x_n(t), \omega_n(t), v_n(t))]$$
$$\mathcal{G}(\mathbf{x}(^-t), \nu(t)) = [\beta_1(x_1(^-t), \omega_1(^-t), v_1(t)),\ldots,\beta_n(x_n(^-t), \omega_n(^-t), v_n(t))]$$

where,

$$\beta_i(x_i(^-t), \omega_i(^-t), v_i(t)) = \begin{cases} x_i(^-t) & if\ t \notin \Omega_i \\ g_i(x_i(^-t), \omega_i(^-t), v_i(t)) & if\ t \in \Omega_i \end{cases}.$$

C. Transition systems

1) Transition Subsystems: Now, we will introduce the class of transition subsystems [16], which will be later interconnected to form an interconnected transition system. Indeed, the concept of transition subsystems permits to model impulsive subsystems and their symbolic models in a common framework.

Definition 2.4: A transition subsystem is a tuple $T_i = (X_i, X_{0_i}, W_i, \mathcal{W}_i, U_i, \mathcal{U}_i, \mathcal{F}_i, Y_i, \mathcal{H}_i)$, $i \in \mathcal{N}$, consisting of:

- a set of states X_i;
- a set of initial states $X_{0_i} \subseteq X_i$;
- a set of internal inputs values W_i;
- a set of internal inputs signals $\mathcal{W}_i := \{\omega_i : \mathbb{R}_{\geq 0} \to W_i\}$;
- a set of external inputs values U_i;
- a set of external inputs signals $\mathcal{U}_i := \{u_i : \mathbb{R}_{\geq 0} \to U_i\}$;
- transition function $\mathcal{F}_i : X_i \times \mathcal{W}_i \times \mathcal{U}_i \rightrightarrows X_i$;
- an output set Y_i;
- an output map $\mathcal{H}_i : X_i \to Y_i$.

The transition $x_i^+ \in \mathcal{F}_i(x_i, \omega_i, u_i)$ means that the system can evolve from state x_i to state x_i^+ under the input signals ω_i and u_i. Thus, the transition function defines the dynamics of the transition system. Let $\mathsf{x}_{x_i,\omega_i,u_i}$ denotes an infinite state run of T_i associated with external input signal u_i, internal input signal ω_i, and initial state x_i. Correspondingly, define $\mathsf{y}_{x_i,\omega_i,u_i} := \mathcal{H}_i(\mathsf{x}_{x_i,\omega_i,u_i})$ as an infinite output run of T_i. Sets X_i, W_i, U_i, and Y_i are assumed to be subsets of normed vector spaces with appropriate finite dimensions. If for all $x_i \in X_i, \omega_i \in \mathcal{W}_i, u_i \in \mathcal{U}_i$, $\mathcal{C}(\mathcal{F}_i(x_i, \omega_i, u_i)) \leq 1$, we say that T_i is deterministic, and non-deterministic otherwise. Additionally, T_i is called finite if X_i, ω_i, U_i are finite sets and infinite otherwise. Furthermore, if for all $x_i \in X_i$ there exists $\omega_i \in \mathcal{W}_i$ and $u_i \in \mathcal{U}_i$ such that $\mathcal{C}(\mathcal{F}_i(x_i, \omega_i, u_i)) \neq 0$ we say that T_i is non-blocking.

2) Interconnections among transition subsystems: We assume that the input-output structure of each transition subsystem T_i, $i \in \mathcal{N}$, is formally defined as the interconnection structure for the impulsive subsystems in part II-B.2 and is formally defined by,

$$\omega_i{=}[\omega_{i1}; \ldots; \omega_{i(i-1)}; \omega_{i(i+1)}; \ldots; \omega_{iN}], W_i{=}\prod_{\substack{j=1,\\ j\neq i}}^{N} W_{ij}, \quad \text{(II.6)}$$

$$y_i = [y_{i1}; \ldots; y_{iN}], \quad Y_i = \prod_{j=1}^{N} Y_{ij}, \quad \text{(II.7)}$$

where $\omega_{ij} \in W_{ij}$, $y_{ij} = h_{ij}(x_i) \in Y_{ij}$, and the output map,

$$\mathcal{H}_i(x_i){=}[\mathcal{H}_{i1}(x_i); \ldots; \mathcal{H}_{iN}(x_i)]. \quad \text{(II.8)}$$

Assumption 2.5: The input-output interconnection variables of transition systems are constrained by,

$$\|\omega_{ij} - \mathcal{H}_{ji}(x_j)\| \leqslant \Phi_{ij}, \quad \Phi_{ij} \in \mathbb{R}_{\geq 0} \quad \text{(II.9)}$$

Remark 2.6: When abstracting subsystems, it is possible that the output of the subsystem does not match the internal input of the direct system. The assumption 2.5 allows taking into account this mismatch under a certain tolerance limit Φ_{ij}.

3) Composed transition system: We define the composed transition system by $\mathcal{I}(T_1, \ldots, T_N)$ and we define it formally by,

Definition 2.7: Consider $N \in \mathbb{N}_{\geqslant 1}$ transition subsystems

$$T_i = (X_i, X_{0_i}, W_i, \mathcal{W}_i, U_i, \mathcal{U}_i, \mathcal{F}_i, Y_i, \mathcal{H}_i)$$

with input-output structure given by (II.6)-(II.4). The interconnected transition system is a tuple $T = (X, X_0, U, \mathcal{F}, Y, \mathcal{H})$, denoted by $\mathcal{I}(T_1, \ldots, T_N)$, where $X = \prod_{i=1}^{N} X_i$, $X_0 = \prod_{i=1}^{N} X_{0_i}$, $U = \prod_{i=1}^{N} U_i$, $Y = \prod_{i=1}^{N} Y_i$. Moreover, the transition relation $\mathcal{F}$ and the output map $\mathcal{H}$ are defined by,

$$\mathcal{F}(x, u){:=}\{[x_1^+; \ldots; x_N^+] \,|x_i^+ \in \mathcal{F}_i(x_i, u_i, \omega_i) \ \forall i{\in}\mathcal{N}\}, \quad \text{(II.10)}$$

$$\mathcal{H}(x){:=}[\mathcal{H}_{11}(x_1); \ldots; \mathcal{H}_{NN}(x_N)] \quad \text{(II.11)}$$

where $x = [x_1; \ldots; x_N] \in X$, $u = [u_1; \ldots; u_N] \in U$.

D. Alternating Simulation Function

In this section, we recall the so-called notion of $\varepsilon-$approximate alternating simulation function in [7].

Definition 2.8: Let $T = (X, X_0, U, \mathcal{F}, Y, \mathcal{H})$ and $\hat{T} = (\hat{X}, \hat{X}_0, \hat{U}, \hat{\mathcal{F}}, \hat{Y}, \mathcal{H})$ with $\hat{Y} \subseteq Y$. A function $\tilde{\mathcal{S}} : X \times \hat{X} \to \mathbb{R}_{\geqslant 0}$ is called an alternating simulation function from $\hat{T}$ to $\hat{T}$ if there exist $\tilde{\alpha} \in \mathcal{K}_\infty$, $0 < \tilde{\sigma} < 1$, $\tilde{\rho}_u \in \mathcal{K}_\infty \cup \{0\}$, and some $\tilde{\varepsilon} \in \mathbb{R}_{\geqslant 0}$ so that the following hold:

1) For every $x \in X, \hat{x} \in \hat{X}$, we have,

$$\tilde{\alpha}(\|\mathcal{H}(x) - \hat{\mathcal{H}}(\hat{x})\|){\leqslant}\tilde{\mathcal{S}}(x, \hat{x}); \quad \text{(II.12)}$$

2) For every $x \in X, \hat{x} \in \hat{X}, \hat{u} \in \hat{U}$ there exists $u \in U$ such that for every $x^+ \in \mathcal{F}(x, u)$ there exists $\hat{x}^+ \in \hat{\mathcal{F}}(\hat{x}, \hat{u})$ so that,

$$\tilde{\mathcal{S}}(x^+, \hat{x}^+) \leqslant \max\{\tilde{\sigma}\tilde{\mathcal{S}}(x, \hat{x}), \tilde{\rho}_u(\|\hat{u}\|_\infty), \tilde{\varepsilon}\}; \quad \text{(II.13)}$$

It was shown in [7] that the existence of an approximate alternating simulation function implies the existence of an approximate alternating relation from T to $\hat{T}$. This relation guarantees that for any output behavior of T there exists one of $\hat{T}$ such that the distance between these two outputs is uniformly bounded by $\hat{\varepsilon} = \tilde{\alpha}^{-1}(\max\{\tilde{\rho}_u(r), \tilde{\varepsilon}\})$. For local abstraction, the notion of ε-approximately alternating simulation function from T_i to $\hat{T}_i$, $\forall i \in \mathcal{N}$, is formally defined by,

Definition 2.9: Let $T_i = (X_i, X_{0_i}, W_i, U_i, \mathcal{F}_i, Y_i, \mathcal{H}_i)$ and $\hat{T}_i = (\hat{X}_i, \hat{X}_{0_i}, \hat{W}_i, \hat{U}_i, \hat{\mathcal{F}}_i, \hat{Y}_i, \hat{\mathcal{H}}_i)$ be transition subsystems with $\hat{Y}_i \subseteq Y_i$, $\hat{\omega}_i \subseteq W_i$. A function $\mathcal{S}_i : X_i \times \hat{X}_i \to \mathbb{R}_{\geqslant 0}$ is called a local alternating simulation function from $\hat{T}_i$ to T_i if there exist $\alpha_i, \rho_{\omega_i} \in \mathcal{K}_\infty$, $0 < \sigma_i < 1$, $\rho_{u_i} \in \mathcal{K}_\infty \cup \{0\}$, and some $\varepsilon_i \in \mathbb{R}_{\geqslant 0}$ so that the following hold:

1) For every $x_i \in X_i, \hat{x}_i \in \hat{X}_i$, we have,

$$\alpha_i(\|\mathcal{H}_i(x_i) - \hat{\mathcal{H}}_i(\hat{x}_i)\|){\leqslant}\mathcal{S}_i(x_i, \hat{x}_i); \quad \text{(II.14)}$$

2) For every $x_i \in X_i, \hat{x}_i \in \hat{X}_i, \hat{u}_i \in \hat{U}_i$ there exists $u_i \in U_i$ such that for every $\omega_i \in W_i, \hat{\omega}_i \in \hat{W}_i, x_i^+ \in \mathcal{F}_i(x_i, \omega_i, u_i)$ there exists $\hat{x}_i^+ {\in} \hat{\mathcal{F}}_i(\hat{x}_i, \hat{\omega}_i, \hat{u}_i)$ so that,

$$\begin{aligned}\mathcal{S}_i(x_i^+, \hat{x}_i^+) \leqslant & \bar{\sigma}_i \mathcal{S}_i(x_i, \hat{x}_i) + \bar{\rho}_{\omega_i}(\|\omega_i {-} \hat{\omega}_i\|) \\ & + \bar{\rho}_u(\|\hat{u}_i\|_\infty) + \bar{\varepsilon}_i. \end{aligned} \quad \text{(II.15)}$$

The goal is to construct alternating simulation functions for the compound transition systems $T = \mathcal{I}(T_1, \ldots, T_N)$ and $\hat{T} = \mathcal{I}(\hat{T}_1, \ldots, \hat{T}_N)$ from the local alternating simulation functions of the subsystems. To achieve this goal, the following lemmas are recalled.

Lemma 2.10: [17, Theorem 1] Let $\mathcal{S}_i : X_i \times \hat{X}_i \to \mathbb{R}_{\geqslant 0}$ be a local alternating simulation function from $\hat{T}_i$ to T_i then, for every $x_i \in X_i, \hat{x}_i \in \hat{X}_i, \hat{u}_i \in \hat{U}_i$ there exists $u_i \in U_i$ such that for every $\omega_i \in W_i, \hat{\omega}_i \in \hat{W}_i, x_i^+ \in \mathcal{F}_i(x_i, \omega_i, u_i)$ there exists $\hat{x}_i^+ \in \hat{\mathcal{F}}_i(\hat{x}_i, \hat{\omega}_i, \hat{u}_i)$ so that,

$$\mathcal{S}_i(x_i^+, \hat{x}_i^+) \leqslant \max\{\sigma_i \mathcal{S}_i(x_i, \hat{x}_i), \rho_{\omega_i}(\|\omega_i - \hat{\omega}_i\|), \rho_{u_i}(\|\hat{u}_i\|_\infty), \varepsilon_i\}; \tag{II.16}$$

where $\sigma_i = 1 - (1 - \psi)(1 - \bar{\sigma}_i)$, $\rho_{\omega_i} = \frac{1}{(1-\bar{\sigma})\psi}\bar{\rho}_{\omega_i}$, $\rho_{u_i} = \frac{1}{(1-\bar{\sigma})\psi}\bar{\rho}_{u_i}$, and $\varepsilon_i = \frac{\bar{\varepsilon}}{(1-\bar{\sigma}_i)\psi}$, for an arbitrarily chosen positive constant $\psi < 1$, and $\bar{\sigma}, \bar{\varepsilon}, \bar{\rho}_w, \bar{\rho}_u$ are constants and function appearing in Definition 2.9.

Lemma 2.11: [18] Consider $\alpha \in \mathcal{K}$ and $\chi \in \mathcal{K}_\infty$, where $(\chi - \mathcal{I}_d) \in \mathcal{K}_\infty$. Then for any $a, b \in \mathbb{R}_{\geqslant 0}$

$$\alpha(a + b) \leqslant \alpha \circ \chi(a) + \alpha \circ \chi \circ (\chi - \mathcal{I}_d)^{-1}(b).$$

Lemma 2.12: [7] For any $a, b \in \mathbb{R}_{>0}$, the following holds

$$a + b \leq \max\left\{(\mathcal{I}_d + \lambda)(a), \left(\mathcal{I}_d + \lambda^{-1}\right)(b)\right\}$$

for any $\lambda \in \mathcal{K}_\infty$.

III. Compositionality Result

The goal of this section is to provide a method for the compositional construction of an alternating simulation function for the interconnected transition system $T = \mathcal{I}(T_1, \ldots, T_N)$ to $\hat{T} = \mathcal{I}(\hat{T}_1, \ldots, \hat{T}_N)$ as defined in Definition 2.7. For the functions σ_i, α_i, and ρ_{wi} associated with $\mathcal{S}_i$, $i \in \mathcal{N}$, given in Lemma 2.10, we define $\forall i, j \in \mathcal{N}$,

$$\gamma_{ij} := \begin{cases} \sigma_i & \text{if} \quad i, j \in \mathcal{N} | i = j, \\ \rho_{\omega_i} \circ \alpha_j^{-1} & \text{if} \quad i, j \in \mathcal{N} | i \neq j, \end{cases} \tag{III.1}$$

and we set γ_{ij} equal to zero if there is no connection from T_j to T_i, i.e., $\omega_{ij} = 0$.

To establish the compositionality results of the paper, we make the following scaled small-gain assumption.

Assumption 3.1: Assume that functions γ_{ij} defined in (III.1) satisfy,

$$\gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \cdots \circ \gamma_{i_{r-1} i_r} \circ \gamma_{i_r i_1} < \mathsf{id}, \tag{III.2}$$

$\forall (i_1, \ldots, i_r) \in \{1, \ldots, N\}^r$, where $r \in \{1, \ldots, N\}$.

Remark 3.2: Note that from Theorem 5.2 in [19], the scaled small-gain condition (III.2) implies the existence of $\psi_i \in \mathcal{K}_\infty$ $\forall i \in \mathcal{N}$, satisfying,

$$\max_{j \in \mathcal{N}}\{\psi_i^{-1} \circ \gamma_{ij} \circ \psi_j\} < \mathsf{id}. \tag{III.3}$$

Remark 3.3: To compute the $\mathcal{K}_\infty$ functions $\psi_i, i \in \mathcal{N}$ it is possible to consider three cases:

(i) If $N \in \{2, 3\}$, i.e., the system is made of two or three subsystems, then one can use the simple construction techniques provided by [20] and [19, Section 9];

(ii) If $N \in \mathbb{N} \setminus \{2, 3\}$, functions $\psi_i, i \in \mathcal{N}$, can be constructed numerically using the algorithm proposed by [21] and the technique provided by [19, Proposition 8.8], see [22, Chapter 4];

(iii) If $\gamma_{ij} < \mathsf{id}$, $\forall\, i, j \in \mathcal{N}$, then the functions $\psi_i, i \in \mathcal{N}$, can be chosen as identity. Moreover, inequality (III.4) reduces to $\mathcal{S}(x, \hat{x}) := \max_{i \in \mathcal{N}}\{\mathcal{S}_i(x_i, \hat{x}_i)\}$.

The next theorem provides a compositional approach to construct an alternating simulation function from $\hat{T} = (\hat{T}_1, \ldots, \hat{T}_N)$ to $T = (T_1, \ldots, T_N)$ via local alternating simulation functions from $\hat{T}_i$ to T_i, $i \in \mathcal{N}$.

Theorem 3.4: Consider the interconnected transition system $T = \mathcal{I}(T_1, \ldots, T_N)$. Assume that each T_i and its abstraction $\hat{T}_i$ admit a local alternating simulation function $\mathcal{S}_i$ as in Lemma 2.10. Suppose Assumption 3.1 holds. Then, function $\tilde{\mathcal{S}} : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ defined as,

$$\tilde{\mathcal{S}}(x, \hat{x}) := \max_{i \in \mathcal{N}}\{\psi_i^{-1}(\mathcal{S}_i(x_i, \hat{x}_i))\} \tag{III.4}$$

is an alternating simulation function from $T = \mathcal{I}(T_1, \ldots, T_N)$ to $\hat{T} = \mathcal{I}(\hat{T}_1, \ldots, \hat{T}_N)$.

Proof: First, we show that (II.12) holds for some $\mathcal{K}_\infty$ function $\tilde{\alpha}$. Define $\hat{\alpha} := \max_{i \in \mathcal{N}}\{\alpha_i^{-1} \circ \psi_i\}$ and $\tilde{\alpha} := \hat{\alpha}^{-1}$, and consider any $x_i \in X_i$, $\hat{x}_i \in \hat{X}_i$, $\forall i \in \mathcal{N}$. Then, one gets,

$$\begin{aligned} \|\mathcal{H}(x) - \hat{\mathcal{H}}(\hat{x})\| &= \max_{i \in \mathcal{N}}\{\|\mathcal{H}_{ii}(x_i) - \hat{\mathcal{H}}_{ii}(\hat{x}_i)\|\} \\ &\leqslant \max_{i,j \in \mathcal{N}}\{\|\mathcal{H}_{ij}(x_i) - \hat{\mathcal{H}}_{ij}(\hat{x}_i)\|\} \\ &= \max_{i \in \mathcal{N}}\{\|\mathcal{H}_i(x_i) - \hat{\mathcal{H}}_i(\hat{x}_i)\|\} \\ &\leqslant \max_{i \in \mathcal{N}}\{\alpha_i^{-1} \circ \mathcal{S}_i(x_i, \hat{x}_i)\} \\ &\leqslant \hat{\alpha} \circ \max_{i \in \mathcal{N}}\{\psi_i^{-1} \circ \mathcal{S}_i(x_i, \hat{x}_i)\} \\ &= \hat{\alpha} \circ \tilde{\mathcal{S}}(x, \hat{x}). \end{aligned}$$

where the first equality comes from (II.11) and the second equality follows from (II.8). Hence, one obtains,

$$\tilde{\alpha}(\|\mathcal{H}(x) - \hat{\mathcal{H}}(\hat{x})\|) \leqslant \tilde{\mathcal{S}}(x, \hat{x}),$$

satisfying (II.12).

Now, we show that (II.13) holds. Let $\tilde{\sigma} = \max_{i,j \in \mathcal{N}}\{\psi_i^{-1} \circ \gamma_{ij} \circ \psi_j\}$. It follows from (III.3) that $\tilde{\sigma} < \mathsf{id}$. Let us define $\tilde{\rho}_u$ and $\tilde{\varepsilon}$ as $\tilde{\rho}_u := \max_{i \in \mathcal{N}}\{\psi_i^{-1}\} \circ \max_{i \in \mathcal{N}}\{\rho_{u_i}\}$ $\tilde{\rho}_u := \max_{i \in \mathcal{N}}\{\psi_i^{-1} \circ \rho_{u_i}\}$ and $\tilde{\varepsilon} := \max_{i \in \mathcal{N}}\{\psi_i^{-1}(\varepsilon_i)\}$. Based on (II.16), for all $i, j \in \mathcal{N}$, one gets the chain of inequalities in (III.5) which satisfies (II.13). In (III.5), the first and second inequalities use (II.16) and resp., (II.9). The third, fourth, fifth and sixth inequalities result from using (II.14), lemma 2.11, lemma 2.12 and resp., III.1. The seventh inequality arises from the fact that γ_{ij} and ψ_i are $\mathcal{K}_\infty$ maps. The remaining inequalities use the definitions of $\tilde{\sigma}$, $\tilde{\rho}_u$ and $\tilde{\varepsilon}$ and the notations,

$$\bar{\Phi}_i = (\mathsf{id} + \lambda^{-1}) \circ \Big(\rho_{\omega_i} \circ \mathcal{X} \circ (\mathcal{X} - \mathsf{id})^{-1} \max_{j, j \neq i} \Phi_{ij}\} + \varepsilon_i\Big),$$

$$\tilde{\varepsilon} = \max_i \left\{\psi_i^{-1}(\bar{\Phi}_i)\right\}.$$

$$
\begin{aligned}
\tilde{\mathcal{S}}(x^+,\hat{x}^+) &= \max_i\{\psi_i^{-1}\circ\mathcal{S}_i(x_i^+,\hat{x}_i^+)\}\\
&\leqslant \max_i\Big\{\psi_i^{-1}\big(\max\{\sigma_i\mathcal{S}_i(x_i,\hat{x}_i),\rho_{\omega_i}(\|\omega_i-\hat{\omega}_i\|),\rho_{u_i}(\|\hat{u}_i\|_\infty),\varepsilon_i\}\big)\Big\}\\
&= \max_i\Big\{\psi_i^{-1}\big(\max\{\sigma_i\mathcal{S}_i(x_i,\hat{x}_i),\rho_{\omega_i}(\max_{j,j\neq i}\{\|\omega_{ij}-\hat{\omega}_{ij}\|\}),\rho_{u_i}(\|\hat{u}_i\|_\infty),\varepsilon_i\}\big)\Big\}\\
&= \max_i\Big\{\psi_i^{-1}\big(\max\{\sigma_i\mathcal{S}_i(x_i,\hat{x}_i),\rho_{\omega_i}(\max_{j,j\neq i}\{\|y_{ji}-\hat{y}_{ji}+\hat{y}_{ji}-\hat{\omega}_{ij}\|\}),\rho_{u_i}(\|\hat{u}_i\|_\infty),\varepsilon_i\}\big)\Big\}\\
&\leqslant \max_i\Big\{\psi_i^{-1}\big(\max\{\sigma_i\mathcal{S}_i(x_i,\hat{x}_i),\rho_{\omega_i}(\max_{j,j\neq i}\{\|\mathcal{H}_{ji}(x_j)-\hat{\mathcal{H}}_{ji}(\hat{x}_j)\|+\Phi_{ij}\}),\rho_{u_i}(\|\hat{u}_i\|_\infty),\varepsilon_i\}\big)\Big\}\\
&= \max_i\Big\{\psi_i^{-1}\big(\max\{\sigma_i\mathcal{S}_i(x_i,\hat{x}_i),\rho_{\omega_i}(\max_{j,j\neq i}\{\|\mathcal{H}_{j}(x_j)-\hat{\mathcal{H}}_{j}(\hat{x}_j)\|+\Phi_{ij}\}),\rho_{u_i}(\|\hat{u}_i\|_\infty),\varepsilon_i\}\big)\Big\}\\
&\leqslant \max_i\Big\{\psi_i^{-1}\big(\max\{\sigma_i\mathcal{S}_i(x_i,\hat{x}_i),\rho_{\omega_i}(\max_{j,j\neq i}\{\alpha_j^{-1}\circ\mathcal{S}_j(x_j,\hat{x}_j)+\Phi_{ij}\}),\rho_{u_i}(\|\hat{u}_i\|_\infty),\varepsilon_i\}\big)\Big\}\\
&\leqslant \max_i\Big\{\psi_i^{-1}\big(\max\{\sigma_i\mathcal{S}_i(x_i,\hat{x}_i),\rho_{\omega_i}\circ\mathcal{X}(\max_{j,j\neq i}\{\alpha_j^{-1}\circ\mathcal{S}_j(x_j,\hat{x}_j)\})\\
&\quad +\rho_{\omega_i}\circ\mathcal{X}\circ(\mathcal{X}-\mathsf{id})^{-1}\max_{j,j\neq i}\{\Phi_{ij}\}\Big),\rho_{u_i}(\|\hat{u}_i\|_\infty),\varepsilon_i\}\big)\Big\}\\
&\leqslant \max_i\Big\{\psi_i^{-1}\big(\max\{\sigma_i\mathcal{S}_i(x_i,\hat{x}_i),(\mathsf{id}+\lambda)\circ\rho_{\omega_i}\circ\mathcal{X}(\max_{j,j\neq i}\{\alpha_j^{-1}\circ\mathcal{S}_j(x_j,\hat{x}_j)\}),\\
&\quad \rho_{u_i}(\|\hat{u}_i\|_\infty)\}\big),(\mathsf{id}+\lambda^{-1})\circ\Big(\rho_{\omega_i}\circ\mathcal{X}\circ(\mathcal{X}-\mathsf{id})^{-1}\max_{j,j\neq i}\{\Phi_{ij}\}+\varepsilon_i\Big)\Big\}\\
&\leqslant \max_{i,j}\Big\{\psi_i^{-1}\big(\max\{\gamma_{ij}\circ\mathcal{S}_j(x_j,\hat{x}_j),\rho_{u_i}(\|\hat{u}_i\|_\infty),\bar{\Phi}_i\}\big)\Big\}\\
&= \max_{i,j}\Big\{\psi_i^{-1}\big(\max\{\gamma_{ij}\circ\psi_j\circ\psi_j^{-1}\circ\mathcal{S}_j(x_j,\hat{x}_j),\rho_{u_i}(\|\hat{u}_i\|_\infty),\bar{\Phi}_i\}\big)\Big\}\\
&\leqslant \max_{i,j,l}\Big\{\psi_i^{-1}\big(\max\{\gamma_{ij}\circ\psi_j\circ\psi_l^{-1}\circ\mathcal{S}_l(x_l,\hat{x}_l),\rho_{u_i}(\|\hat{u}_i\|_\infty),\bar{\Phi}_i\}\big)\Big\}\\
&= \max_{i,j}\Big\{\psi_i^{-1}\big(\max\{\gamma_{ij}\circ\psi_j\circ\tilde{\mathcal{S}}(x,\hat{x}),\rho_{u_i}(\|\hat{u}_i\|_\infty),\bar{\Phi}_i\}\big)\Big\}\\
&= \max\Big\{\tilde{\sigma}\circ\tilde{\mathcal{S}}(x,\hat{x}),\max_i\big\{\psi_i^{-1}\circ\rho_{u_i}(\|\hat{u}_i\|_\infty),\max_i\big\{\psi_i^{-1}(\bar{\Phi}_i)\big\}\Big\}\\
&\leqslant \max\Big\{\tilde{\sigma}\circ\tilde{\mathcal{S}}(x,\hat{x}),\max_i\big\{\psi_i^{-1}\circ\max_i\{\rho_{u_i}(\|\hat{u}_i\|_\infty)\},\max_i\big\{\psi_i^{-1}(\bar{\Phi}_i)\big\}\Big\}\\
&\leqslant \max\Big\{\tilde{\sigma}\circ\tilde{\mathcal{S}}(x,\hat{x}),\max_i\big\{\psi_i^{-1}\circ\max_i\{\rho_{u_i}(\max_i\{\|\hat{u}_i\|_\infty\})\},\max_i\big\{\psi_i^{-1}(\bar{\Phi}_i)\big\}\Big\}\\
&= \max\Big\{\tilde{\sigma}\circ\tilde{\mathcal{S}}(x,\hat{x}),\tilde{\rho}_u(\|\hat{u}\|_\infty),\tilde{\varepsilon}\Big\},
\end{aligned}
\tag{III.5}
$$

Hence, $\tilde{\mathcal{S}}$ is indeed an alternating simulation function from $T=\mathcal{I}(T_1,\ldots,T_N)$ to $\hat{T}=\mathcal{I}(\hat{T}_1,\ldots,\hat{T}_N)$. ■

Remark 3.5: If there exist at least one pair $a,b\in\mathcal{N}$ satisfying $\gamma_{ab}<\mathsf{id}$ and for any $i,j\in\mathcal{N}\backslash(a,b)$ $\gamma_{ij}=\mathsf{id}$, then, assumption (III.2) still holds and we can consider, $\psi_i=\mathsf{id}$ for all $i\in\mathcal{N}$. Moreover, inequality (III.4) reduces to $\mathcal{S}(x,\hat{x}):=\max_{i\in\mathcal{N}}\{\mathcal{S}_i(x_i,\hat{x}_i)\}$. ◇

IV. Construction of Symbolic Models

In the previous section, we showed how to construct an abstraction of a system from the abstractions of its subsystems. In this section, our focus is on constructing a symbolic model for an impulsive subsystem using an approximate alternating simulation. To ease readability, in the sequel, the index $i\in\mathcal{N}$ is omitted.

Consider an impulsive subsystem $\Sigma=(\mathbb{R}^n,\mathbb{W},\mathsf{W},\mathbb{U},\mathsf{U}_\tau,f,g,\mathbb{Y},h,\Omega)$, as defined in Definition 2.1. We restrict our attention to sampled-data impulsive systems, where the input curves belong to U_τ containing only curves of constant duration τ, i.e.,

$$
\mathsf{U}_\tau=\{\nu:\mathbb{R}_{\geq 0}\to\mathbb{U}|\nu(t)=\nu((k-1)\tau),\quad t\in[(k-1)\tau,k\tau),k\in\mathbb{N}_{\geqslant 1}\}. \tag{IV.1}
$$

Moreover, we assume that there exist constant φ such that for all $\omega\in\mathsf{W}$ the following holds,

$$
\|\omega(t)-\omega((k-1)\tau)\|\leqslant\varphi,\forall t\in[(k-1)\tau,k\tau),k\in\mathbb{N}_{\geqslant 1}. \tag{IV.2}
$$

We also have the following Lipschitz continuity assumption on the output map h.

Assumption 4.1: There exist positive constant L, such that the output maps h satisfy the following Lipschitz assumption is satisfied,

$$
\|h(x)-h(y)\|\leqslant L\|x-y\|\ \forall x,y\in\mathbb{R}^n. \tag{IV.3}
$$

Remark 4.2: Since the internal input ω results from the concatenation of the neighbors' outputs. It suffices to show that $\forall x \in X$ the term $\|y((k+1)\tau) - y(k\tau)\|$ is bounded.

Note that the physical variables are restricted to a compact set, thus, the states and inputs have bounded quantities, which implies from the continuity of the maps f and g that there exists a positive constant C^{te} such that $\forall x \in X$ we have, $\|x((k+1)\tau) - x(k\tau)\| \leqslant C^{te}\tau$, which gives,

$$\begin{aligned}\|y((k+1)\tau) - y(k\tau)\| &\leqslant \|h(x((k+1)\tau)) - h(x(k\tau))\| \\ &\leqslant L\|x((k+1)\tau) - x(k\tau)\| \\ &\leqslant LC^{te}\tau = \varphi\end{aligned}$$

hence, condition (IV.2) is not restrictive.

Next, we define sampled-data impulsive systems as transition subsystems. Such transition subsystems would be the bridge that relates impulsive systems to their symbolic models.

Definition 4.3: Given an impulsive system $\Sigma = (\mathbb{R}^n, \mathbb{W}, \mathsf{W}, \mathbb{U}, \mathsf{U}_\tau, f, g, \mathbb{Y}, h, \Omega)$, we define the associated transition system $T_\tau(\Sigma) = (X, X_0, W, \mathcal{W}, U, \mathcal{U}, \mathcal{F}, Y, \mathcal{H})$ where:

- $X = \mathbb{R}^n \times \{0, \ldots, \overline{z}\}$;
- $X_0 = \mathbb{R}^n \times \{0\}$;
- $U = \mathbb{U}$;
- $\mathcal{U} = \mathsf{U}_\tau$;
- $W = \mathbb{W}$;
- $\mathcal{W} = \mathsf{W}$;
- $(x^+, c^+) \in \mathcal{F}((x,c), \omega, u)$ if and only if one of the following scenarios hold:
  - Flow scenario: $0 \leq c \leq \overline{z} - 1$, $x^+ = \mathbf{x}_{x,\omega,u}({}^-\tau)$, and $c^+ = c + 1$;
  - Jump scenario: $\underline{z} \leq c \leq \overline{z}$, $x^+ = g(x, \omega(0), u(0))$, and $c^+ = 0$;
- $Y = \mathbb{Y}$;
- $\mathcal{H} : X \to Y$, defined as $\mathcal{H}(x,c) = h(x)$.

For later use, define $\mathcal{W}_\tau$ as,

$$\begin{aligned}\mathcal{W}_\tau = \{\omega : \mathbb{R}_{\geq 0} \to W | \omega(t) = \omega((k-1)\tau), \\ t \in [(k-1)\tau, k\tau), k \in \mathbb{N}_{\geqslant 1}\}.\end{aligned} \tag{IV.4}$$

In order to construct a symbolic model for $T_\tau(\Sigma)$, we introduce the following assumptions and lemmas.

Assumption 4.4: Consider impulsive system $\Sigma = (\mathbb{R}^n, \mathbb{W}, \mathsf{W}, \mathbb{U}, \mathsf{U}_\tau, f, g, \mathbb{Y}, h, \Omega)$. Assume that there exist a locally Lipschitz function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, $\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}, \rho_{\omega_c}, \rho_{\omega_d}, \rho_{u_c}, \rho_{u_d}$, and constants $\kappa_c \in \mathbb{R}, \kappa_d \in \mathbb{R}$, such that the following hold,

- $\forall x, \hat{x} \in \mathbb{R}^n$,
$$\underline{\alpha}(\|x - \hat{x}\|) \leqslant V(x, \hat{x}) \leqslant \overline{\alpha}(\|x - \hat{x}\|); \tag{IV.5}$$
- $\forall x, \hat{x} \in \mathbb{R}^n$ a.e, $\forall \omega, \hat{\omega} \in W$, and $\forall u, \hat{u} \in \mathbb{U}$,
$$\begin{aligned}&\frac{\partial V(x,\hat{x})}{\partial x} f(x, \omega, u) + \frac{\partial V(x,\hat{x})}{\partial \hat{x}} f(\hat{x}, \hat{\omega}, \hat{u}) \\ &\leqslant -\kappa_c V(x,\hat{x}) + \rho_{\omega_c}(\|w - \hat{\omega}\|) + \rho_{u_c}(\|u - \hat{u}\|);\end{aligned} \tag{IV.6}$$
- $\forall x, \hat{x} \in \mathbb{R}^n$, $\forall \omega, \hat{\omega} \in W$, and $\forall u, \hat{u} \in \mathbb{U}$,
$$\begin{aligned}&V(g(x, \omega, u), g(\hat{x}, \hat{\omega}, \hat{u})) \\ &\leqslant \kappa_d V(x, \hat{x}) + \rho_{\omega_d}(\|\omega - \hat{\omega}\|) + \rho_{u_d}(\|u - \hat{u}\|).\end{aligned} \tag{IV.7}$$

Assumption 4.5: There exist $\mathcal{K}_\infty$ function $\hat{\gamma}$ such that for all $x, y, z \in \mathbb{R}^n$,

$$V(x, y) \leqslant V(x, z) + \hat{\gamma}(\|y - z\|). \tag{IV.8}$$

Remark 4.6: Assumptions 4.4 has different implications based on the values of κ_c and κ_d as the following:

(i) $\kappa_c \leqslant 0$ and $\kappa_d \geqslant 1$ results in incremental forward completeness of the continuous and discrete dynamics of Σ, respectively, and we say Σ_c and Σ_d are δ-FC [23].

(ii) $\kappa_c > 0$ and $\kappa_d < 1$ results in incremental input-to-state stability of the continuous and discrete dynamics of Σ, respectively, and we say Σ_c and Σ_d are δ-ISS [24]. $\diamond$

Remark 4.7: In condition (IV.6), "$\forall x, \hat{x} \in \mathbb{R}^n$ a.e." should be interpreted as "for every $x, \hat{x} \in \mathbb{R}^n$ except on a set of zero Lebesgue-measure in $\mathbb{R}^n$". From Rademacher's theorem [25], the local Lipschitz assumption on function V ensures that $\frac{\partial V(x,\hat{x})}{\partial x} f(x, \omega, u) + \frac{\partial V(x,\hat{x})}{\partial \hat{x}} f(\hat{x}, \hat{\omega}, \hat{u})$ is well defined, except on a set of measure zero. $\diamond$

Remark 4.8: Assumptions 4.5 and 4.1 are non-restrictive conditions provided that one wants to work on a compact subset C of $\mathbb{R}^n$ and that V and h are $\mathcal{C}^1$ on C, [26]. $\diamond$

The following lemma provides an estimate of the evolution of function V in Assumption 4.4, which is needed in the proof of Theorem 4.11.

Lemma 4.9: Consider an impulsive subsystem $\Sigma = (\mathbb{R}^n, W, \mathsf{W}, \mathbb{U}, \mathsf{U}, f, g, \mathbb{Y}, h, \Omega)$. Let (IV.6) and (IV.7) in Assumption 4.4 holds. Then for all $x, \hat{x} \in \mathbb{R}^n$, for all $\omega, \hat{\omega} \in \mathsf{W}$, for all $\nu, \hat{\nu} \in \mathsf{U}$, and for any two consecutive impulses $(t^k, {}^-t^{k+1})$, one has,

$$\begin{aligned}&V(\mathbf{x}_{x,\omega,\nu}({}^-t^{k+1}), \mathbf{x}_{\hat{x},\hat{\omega},\hat{\nu}}({}^-t^{k+1})) \\ &\leqslant e^{-\kappa_c({}^-t^{k+1} - t^k)} V(\mathbf{x}_{x,\omega,\nu}(t^k), \mathbf{x}_{\hat{x},\hat{\omega},\hat{\nu}}(t^k)) \\ &\quad + \frac{1 - e^{-\kappa_c({}^-t^{k+1} - t^k)}}{\kappa_c} \rho_{\omega_c}\Big(\sup_{t \in [t^k, {}^-t^{k+1}]} (\|\omega(t) - \hat{\omega}(t)\|)\Big) \\ &\quad + \frac{1 - e^{-\kappa_c({}^-t^{k+1} - t^k)}}{\kappa_c} \rho_{u_c}\Big(\sup_{t \in [t^k, {}^-t^{k+1}]} (\|\nu(t) - \hat{\nu}(t)\|)\Big).\end{aligned} \tag{IV.9}$$

We now have all the ingredients to construct a symbolic model $\hat{T}_\tau(\Sigma)$ of transition system $T_\tau(\Sigma)$ associated with the impulsive system Σ admitting a function V that satisfies Assumption 4.4 as follows.

Definition 4.10: Consider a transition system $T_\tau(\Sigma) = (X, X_0, W, \mathcal{W}, U, \mathcal{U}, \mathcal{F}, Y, \mathcal{H})$, associated to the impulsive system $\Sigma = (\mathbb{R}^n, W, \mathsf{W}, \mathbb{U}, \mathsf{U}_\tau, f, g, \mathbb{Y}, h, \Omega)$. Assume Σ admits a function V that satisfies Assumption 4.4. One can construct symbolic model $\hat{T}_\tau(\Sigma) = (\hat{X}, \hat{X}_0, \hat{W}, \hat{\mathcal{W}}, \hat{U}, \hat{\mathcal{U}}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ where:

- $\hat{X} = \hat{\mathbb{R}}^n \times \{0, \ldots, \overline{z}\}$, where $\hat{\mathbb{R}}^n = [\mathbb{R}^n]_{\eta^x}$ and η^x is the state set quantization parameter;
- $\hat{X}_0 = \hat{X} \times \{0\}$;
- $\hat{W} = [W]_{\eta^\omega}$, where η^ω is the internal input set quantization parameter;

- $\hat{\mathcal{W}} = \{\hat{\omega} : [0,\tau] \to \hat{W} | \hat{\omega} \in \mathcal{W}_\tau|_{[0,\tau]}\}$;
- $\hat{U} = [U]_{\eta^u}$, where η^u is the external input set quantization parameter;
- $\hat{\mathcal{U}} = \{\hat{u} : [0,\tau] \to \hat{U} | \hat{u} \in \mathcal{U}|_{[0,\tau]}\}$;
- $(\hat{x}^+, c^+) \in \hat{\mathcal{F}}((\hat{x}, c), \hat{\omega}, \hat{u})$ iff one of the following scenarios hold:
  - Flow scenario: $0 \leqslant c \leqslant \overline{z}-1$, $|\hat{x}^+ - \mathbf{x}_{\hat{x},\hat{\omega},\hat{\nu}}(\tau)| \leqslant \eta^x$, and $c^+ = c+1$;
  - Jump scenario: $\underline{z} \leqslant c \leqslant \overline{z}$, $|\hat{x}^+ - g(\hat{x}, \hat{\omega}(0), \hat{u}(0))| \leqslant \eta^x$, and $c^+ = 0$;
- $\hat{Y} = Y$;
- $\hat{\mathcal{H}} = \mathcal{H}$.

In the definition of the transition function, and in the remainder of the paper, we abuse notation by identifying $\hat{u}$ (respectively $\hat{\omega}$) with the constant external (respectively internal) input curve with domain $[0,\tau)$ and value $\hat{u}$ (respectively $\hat{\omega}$). Now, we establish the relation from $T_\tau(\Sigma)$ to $\hat{T}_\tau(\Sigma)$, introduced above, via the notion of alternating simulation function as in Definition 2.8.

Theorem 4.11: Consider an impulsive system $\Sigma = (\mathbb{R}^n, W, \mathsf{W}, \mathbb{U}, \mathsf{U}, f, g, \mathbb{Y}, h, \Omega)$ with its associated transition system $T_\tau(\Sigma) = (X, X_0, W, \mathcal{W}, U, \mathcal{U}, \mathcal{F}, Y, \mathcal{H})$. Suppose Assumptions 4.4, 4.5, and 4.1 hold. Consider symbolic model $\hat{T}_\tau(\Sigma) = (\hat{X}, \hat{X}_0, \hat{\omega}, \hat{\mathcal{W}}, \hat{U}, \hat{\mathcal{U}}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ constructed as in Definition 4.10. If inequality,

$$\ln(\kappa_d) - \kappa_c \tau c < 0, \tag{IV.10}$$

holds for $c \in \{\underline{z}, \overline{z}\}$, then function $\mathcal{V}$ defined as,

$$\mathcal{V}((x,c),(\hat{x},c)) := \begin{cases} V(x,\hat{x}) & \text{if} \quad \kappa_d < 1 \,\&\, \kappa_c > 0, \\[2ex] \dfrac{V(x,\hat{x})}{e^{-\kappa_c \tau \epsilon c}} & \text{if} \quad \kappa_d \geqslant 1 \,\&\, \kappa_c > 0, \\[2ex] \dfrac{V(x,\hat{x})}{\kappa_d^{-\frac{c}{\delta}}} & \text{if} \quad \kappa_d < 1 \,\&\, \kappa_c \leqslant 0, \end{cases} \tag{IV.11}$$

for some $0 < \epsilon < 1$ and $\delta > \overline{z}$, is an alternating simulation function from $\hat{T}_\tau(\Sigma)$ to $T_\tau(\Sigma)$.

After proving Theorem 4.11, we will provide additional insight into condition (IV.10). Note that for the case in which $\kappa_d \geq 1$ and $\kappa_c \leqslant 0$, this condition cannot hold at all. Hence this case is excluded from the definition of $\mathcal{V}$ in (IV.11).

Proof: By using (IV.5) and Assumption 4.1, $\forall (x,c) \in X$ and $\forall (\hat{x}, c) \in \hat{X}$, we have,

$$\begin{aligned} &\|\mathcal{H}(x,c) - \hat{\mathcal{H}}(\hat{x},c)\| = \|h(x) - \hat{h}(\hat{x})\| \leqslant L\|x - \hat{x}\| \\ &\leqslant L\underline{\alpha}^{-1}(V(x,\hat{x})) \leqslant \hat{\alpha}\left(\mathcal{V}((x,c),(\hat{x},c))\right), \end{aligned}$$

where,

$$\hat{\alpha}(s) = \begin{cases} L\underline{\alpha}^{-1}(s) & \text{if} \quad \kappa_d < 1 \,\&\, \kappa_c > 0, \\ L\underline{\alpha}^{-1}(e^{-\kappa_c \tau \epsilon \underline{z}} s) & \text{if} \quad \kappa_d \geqslant 1 \,\&\, \kappa_c > 0, \\ L\underline{\alpha}^{-1}(\kappa_d^{-\frac{\overline{z}}{\delta}} s), & \text{if} \quad \kappa_d < 1 \,\&\, \kappa_c \leqslant 0, \end{cases}$$

for all $s \in \mathbb{R}_{\geqslant 0}$. Hence, (II.14) holds with $\alpha = \hat{\alpha}^{-1}$.

Now we show that inequality (II.15) holds as well. Consider any $\hat{u} \in \hat{\mathcal{U}}$ and choose $u = \hat{u}$. Then, using (IV.8), for all $x \in X$, $\hat{x} \in \hat{X}$, for all $\hat{u} \in \hat{\mathcal{U}}$, for all $\omega \in \mathcal{W}$, $\hat{\omega} \in \hat{\mathcal{W}}$ we have in the flow scenario the following inequality,

$$\begin{aligned} V(\mathbf{x}_{x,\omega,\hat{u}}({}^-\tau), \hat{x}^+) \leqslant\; & V(\mathbf{x}_{x,\omega,\hat{u}}({}^-\tau), \mathbf{x}_{\hat{x},\hat{\omega},\hat{u}}({}^-\tau)) \\ & + \hat{\gamma}(\|\hat{x}^+ - \mathbf{x}_{\hat{x},\hat{\omega},\hat{u}}({}^-\tau)\|). \end{aligned}$$

Now, from Definition 4.10, the above inequality reduces to,

$$\begin{aligned} & V(\mathbf{x}_{x,\omega,\hat{u}}({}^-\tau), \hat{x}^+) \\ & \leqslant V(\mathbf{x}_{x,\omega,\hat{u}}({}^-\tau), \mathbf{x}_{\hat{x},\hat{\omega},\hat{u}}({}^-\tau)) + \hat{\gamma}(\eta^x). \end{aligned}$$

for any $\hat{x}^+$ such that $(\hat{x}^+, c^+) \in \hat{\mathcal{F}}((\hat{x}, c), \hat{\omega}, \hat{u})$.
From (IV.9) for all $x \in \mathbb{R}^n$, $\hat{x} \in \hat{\mathbb{R}}^n$, for all $\hat{u} \in \hat{\mathcal{U}}$, for all $\omega \in \mathcal{W}$, $\hat{\omega} \in \hat{\mathcal{W}}$ with $t^{k+1} = \tau, t^k = 0$, one gets,

$$\begin{aligned} & V(\mathbf{x}_{x,\omega,\hat{u}}({}^-\tau), \mathbf{x}_{\hat{x},\hat{\omega},\hat{u}}({}^-\tau)) \\ & \leqslant e^{-\kappa_c \tau} V(\mathbf{x}_{x,\omega,\hat{u}}(0), \mathbf{x}_{\hat{x},\hat{\omega},\hat{u}}(0)) \\ & \quad + \frac{1 - e^{-\kappa_c \tau}}{\kappa_c} \rho_{\omega_c}(\sup_{t \in [0,\tau]} (\|\omega(t) - \hat{\omega}(t)\|)) \\ & = e^{-\kappa_c \tau} V(\mathbf{x}_{x,\omega,\hat{u}}(0), \mathbf{x}_{\hat{x},\hat{\omega},\hat{u}}(0)) \\ & \quad + \frac{1 - e^{-\kappa_c \tau}}{\kappa_c} \rho_{\omega_c}(\sup_{t \in [0,\tau]} (\|\omega(t) - w(0) \\ & \qquad + w(0) - \hat{\omega}(0)\|)) \\ & \leqslant e^{-\kappa_c \tau} V(\mathbf{x}_{x,\omega,\hat{u}}(0), \mathbf{x}_{\hat{x},\hat{\omega},\hat{u}}(0)) \\ & \quad + \frac{1 - e^{-\kappa_c \tau}}{\kappa_c} \rho_{\omega_c}(\sup_{t \in [0,\tau]} (\|\omega(t) - w(0)\| \\ & \qquad + \|w(0) - \hat{\omega}(0)\|) \\ & = e^{-\kappa_c \tau} V(\mathbf{x}_{x,\omega,\hat{u}}(0), \mathbf{x}_{\hat{x},\hat{\omega},\hat{u}}(0)) \\ & \quad + \frac{1 - e^{-\kappa_c \tau}}{\kappa_c} \rho_{\omega_c}(\varphi + \|w(0) - \hat{\omega}(0)\|). \end{aligned}$$

Since $\mathbf{x}_{x,\omega,\hat{u}}(0) = x$ and $\mathbf{x}_{\hat{x},\hat{\omega},\hat{u}}(0) = \hat{x}$, we have

$$\begin{aligned} & V(\mathbf{x}_{x,\omega,\hat{u}}({}^-\tau), \mathbf{x}_{\hat{x},\hat{\omega},\hat{u}}({}^-\tau)) \\ & \leqslant e^{-\kappa_c \tau} V(x, \hat{x}) + \frac{1 - e^{-\kappa_c \tau}}{\kappa_c} \rho_{\omega_c}(\varphi + \|w(0) - \hat{\omega}(0)\|). \end{aligned}$$

Hence, for all $x \in X$, $\hat{x} \in \hat{X}$, for all $\hat{u} \in \hat{\mathcal{U}}$ and choosing $u = \hat{u}$, for all $\omega \in \mathcal{W}$, $\hat{\omega} \in \hat{\mathcal{W}}$, one obtains,

$$\begin{aligned} & V(\mathbf{x}_{x,\omega,\hat{u}}({}^-\tau), \hat{x}^+) \leqslant e^{-\kappa_c \tau} V(x, \hat{x}) \\ & \quad + \frac{1 - e^{-\kappa_c \tau}}{\kappa_c} \rho_{\omega_c}(\varphi + \|w(0) - \hat{\omega}(0)\|) + \hat{\gamma}(\eta^x). \end{aligned}$$

for any $\hat{x}^+$ such that $(\hat{x}^+, c^+) \in \hat{\mathcal{F}}((\hat{x}, c), \hat{\omega}, \hat{u})$. Using Lemma 2.11, we have another upper bound for the previous inequality as follows,

$$\begin{aligned} & V(\mathbf{x}_{x,\omega,\hat{u}}({}^-\tau), \hat{x}^+) \\ & \leqslant e^{-\kappa_c \tau} V(x, \hat{x}) + \hat{\rho}_{\omega_c}(\mathcal{L}(\omega, \hat{\omega})) + \hat{\varphi}, \end{aligned}$$

with,

$$\hat{\rho}_{\omega_c} := \frac{1-e^{-\kappa_c\tau}}{\kappa_c}\rho_{\omega_c}\circ\chi,$$
$$\tilde{\rho}_{\omega_c} := \frac{1-e^{-\kappa_c\tau}}{\kappa_c}\rho_{\omega_c}\circ\chi\circ(\chi-\mathsf{id})^{-1},$$
$$\hat{\varphi} := \tilde{\rho}_{\omega_c}(\varphi)+\hat{\gamma}(\eta^x),$$
$$\mathcal{L}(\omega,\hat{\omega}) := \|w(0)-\hat{\omega}(0)\|,$$

for an arbitrary $\mathsf{id} < \chi \in \mathcal{K}_\infty$.

By following a similar argument to the previous one and using (IV.7), one also obtains the following inequality in the jump scenario for all $x\in X$, $\hat{x}\in\hat{X}$, for all $\hat{u}\in\hat{\mathcal{U}}$, for all $\omega\in\mathcal{W}$, $\hat{\omega}\in\hat{\mathcal{W}}$,

$$\begin{aligned} &V(g(x,\omega,\hat{u}),\hat{x}^+) \\ &\leqslant \kappa_d V(x,\hat{x})+\hat{\rho}_{\omega_d}(\mathcal{L}(\omega,\hat{\omega}))+\hat{\gamma}(\eta^x) \\ &\leqslant \kappa_d V(x,\hat{x})+\hat{\rho}_{\omega_d}(\mathcal{L}(\omega,\hat{\omega}))+\hat{\varphi}, \end{aligned} \tag{IV.12}$$

for any $\hat{x}^+$ such that $(\hat{x}^+,c^+)\in\hat{\mathcal{F}}((\hat{x},c),\hat{\omega},\hat{u})$.

Now, in order to show function $\mathcal{V}$ defined in (IV.11) satisfies (II.15), we consider the different scenarios in Definition 4.10 and different cases for values of κ_d and κ_c as follows:

- **case 1**: $\kappa_d<1$ & $\kappa_c>0$
  - Flow scenario ($c^+=c+1$):

$$\begin{aligned} &\mathcal{V}((x^+,c^+),(\hat{x}^+,c^+)) = V(x^+,\hat{x}^+) \\ &\leqslant e^{-\kappa_c\tau}V(x,\hat{x})+\hat{\rho}_{\omega_c}(\mathcal{L}(\omega,\hat{\omega}))+\hat{\varphi} \\ &= e^{-\kappa_c\tau}\mathcal{V}((x,c),(\hat{x},c))+\hat{\rho}_{\omega_c}(\mathcal{L}(\omega,\hat{\omega}))+\hat{\varphi}. \end{aligned}$$

  - Jump scenario ($c^+=0$):

$$\begin{aligned} &\mathcal{V}((x^+,c^+),(\hat{x}^+,c^+)) = V(x^+,\hat{x}^+) \\ &\leqslant \kappa_d V(x,\hat{x})+\rho_{\omega_d}(\mathcal{L}(\omega,\hat{\omega}))+\hat{\varphi}, \\ &\leqslant \kappa_d \mathcal{V}((x,c),(\hat{x},c))+\rho_{\omega_d}(\mathcal{L}(\omega,\hat{\omega}))+\hat{\varphi}. \end{aligned}$$

  Let $\lambda_f=\max\{e^{-\kappa_c\tau},\kappa_d\}$, $\rho_{\omega_f}=\max\{\hat{\rho}_{\omega_c},\rho_{\omega_d}\}$, and $\varphi_f=\hat{\varphi}$, then

$$\begin{aligned} &\mathcal{V}((x^+,c^+),(\hat{x}^+,c^+)) \\ &\quad\leqslant \lambda_f\mathcal{V}((x,c),(\hat{x},c))+\rho_{\omega_f}(\mathcal{L}(\omega,\hat{\omega}))+\varphi_f. \end{aligned}$$

- $\kappa_d\geqslant 1$ & $\kappa_c>0$ (**case 2**):
  - Flow scenario ($c^+=c+1$):

$$\begin{aligned} &\mathcal{V}((x^+,c^+),(\hat{x}^+,c^+)) = V(x^+,\hat{x}^+)e^{\kappa_c\tau\epsilon c^+} \\ &= V(x^+,\hat{x}^+)e^{\kappa_c\tau\epsilon(c+1)} \\ &\leqslant (e^{-\kappa_c\tau}V(x,\hat{x})+\hat{\rho}_{\omega_c}(\mathcal{L}(\omega,\hat{\omega}))+\hat{\varphi})e^{\kappa_c\tau\epsilon(c+1)} \\ &= e^{-\kappa_c\tau}e^{\kappa_c\tau\epsilon}e^{\kappa_c\tau\epsilon c}V(x,\hat{x})+\frac{\hat{\rho}_{\omega_c}(\mathcal{L}(\omega,\hat{\omega}))+\hat{\varphi}}{e^{-\kappa_c\tau\epsilon(c+1)}} \\ &= e^{-\kappa_c\tau(1-\epsilon)}\mathcal{V}((x,c),(\hat{x},c))+\frac{\hat{\rho}_{\omega_c}(\mathcal{L}(\omega,\hat{\omega}))+\hat{\varphi}}{e^{-\kappa_c\tau\epsilon(c+1)}}. \end{aligned}$$

  - Jump scenario ($c^+=0$):

$$\begin{aligned} &\mathcal{V}((x^+,c^+),(\hat{x}^+,c^+)) = V(x^+,\hat{x}^+)e^{\kappa_c\tau\epsilon c^+} \\ &= V(x^+,\hat{x}^+) \leqslant \kappa_d V(x,\hat{x})+\rho_{\omega_d}(\mathcal{L}(\omega,\hat{\omega}))+\hat{\varphi} \\ &= \frac{e^{\kappa_c\tau\epsilon c}}{e^{\kappa_c\tau\epsilon c}}\kappa_d V(x,\hat{x})+\rho_{\omega_d}(\mathcal{L}(\omega,\hat{\omega}))+\hat{\varphi} \\ &= e^{-\kappa_c\tau\epsilon c}\kappa_d\mathcal{V}((x,c),(\hat{x},c))+\rho_{\omega_d}(\mathcal{L}(\omega,\hat{\omega}))+\hat{\varphi}. \end{aligned}$$

  Let $\lambda_f=\max\{e^{-\kappa_c\tau(1-\epsilon)},e^{-\kappa_c\tau\epsilon\underline{z}}\kappa_d\}$, $\rho_{\omega_f}=\max\{e^{\kappa_c\tau\epsilon(\overline{z}+1)}\hat{\rho}_{\omega_c},\rho_{\omega_d}\}$, and $\varphi_f=e^{\kappa_c\tau\epsilon(\overline{z}+1)}\hat{\varphi}$, then,

$$\begin{aligned} &\mathcal{V}((x^+,c^+),(\hat{x}^+,c^+)) \\ &\quad\leqslant \lambda_f\mathcal{V}((x,c),(\hat{x},c))+\rho_{\omega_f}(\mathcal{L}(\omega,\hat{\omega}))+\varphi_f. \end{aligned}$$

- $\kappa_d<1$ & $\kappa_c\leqslant 0$ (**case 3**):
  - Flow scenario ($c^+=c+1$):

$$\begin{aligned} &\mathcal{V}((x^+,c^+),(\hat{x}^+,c^+)) = V(x^+,\hat{x}^+)\kappa_d^{\frac{c^+}{\delta}} \\ &= V(x^+,\hat{x}^+)\kappa_d^{\frac{(c+1)}{\delta}} \\ &\leqslant (e^{-\kappa_c\tau}V(x,\hat{x})+\hat{\rho}_{\omega_c}(\mathcal{L}(\omega,\hat{\omega}))+\hat{\varphi})\kappa_d^{\frac{(c+1)}{\delta}} \\ &= e^{-\kappa_c\tau}\kappa_d^{\frac{c}{\delta}}\kappa_d^{\frac{1}{\delta}}V(x,\hat{x})+(\hat{\rho}_{\omega_c}(\mathcal{L}(\omega,\hat{\omega}))+\hat{\varphi})\kappa_d^{\frac{(c+1)}{\delta}} \\ &= e^{-\kappa_c\tau}\kappa_d^{\frac{1}{\delta}}\mathcal{V}((x,c),(\hat{x},c))+(\hat{\rho}_{\omega_c}(\mathcal{L}(\omega,\hat{\omega}))+\hat{\varphi})\kappa_d^{\frac{(c+1)}{\delta}}. \end{aligned}$$

  - Jump scenario ($c^+=0$):

$$\begin{aligned} &\mathcal{V}((x^+,c^+),(\hat{x}^+,c^+)) = V(x^+,\hat{x}^+)\kappa_d^{\frac{c^+}{\delta}} \\ &= V(x^+,\hat{x}^+) \leqslant \kappa_d V(x,\hat{x})+\rho_{\omega_d}(\mathcal{L}(\omega,\hat{\omega}))+\hat{\varphi} \\ &= \frac{\kappa_d^{\frac{c}{\delta}}}{\kappa_d^{\frac{c}{\delta}}}\kappa_d V(x,\hat{x})+\rho_{\omega_d}(\mathcal{L}(\omega,\hat{\omega}))+\hat{\varphi} \\ &= \kappa_d^{\frac{\delta-c}{\delta}}\mathcal{V}((x,c),(\hat{x},c))+\rho_{\omega_d}(\mathcal{L}(\omega,\hat{\omega}))+\hat{\varphi}. \end{aligned}$$

  Let $\lambda_f=\max\{e^{-\kappa_c\tau}\kappa_d^{\frac{1}{\delta}},\kappa_d^{\frac{\delta-\overline{z}}{\delta}}\}$, $\rho_{\omega_f}=\max\{\kappa_d^{\frac{(\underline{z}+1)}{\delta}}\hat{\rho}_{\omega_c},\rho_{\omega_d}\}$, and $\varphi_f=\hat{\varphi}$, then

$$\begin{aligned} &\mathcal{V}((x^+,c^+),(\hat{x}^+,c^+)) \\ &\quad\leqslant \lambda_f\mathcal{V}((x,c),(\hat{x},c))+\rho_{\omega_f}(\mathcal{L}(\omega,\hat{\omega}))+\varphi_f. \end{aligned}$$

To continue with the proof, we need to show that $\lambda_f<1$ for **case 2** and **case 3** (**case 1** is trivial). In **case 2**, note that $e^{-\kappa_c\tau(1-\epsilon)}<1$ since $0<\epsilon<1$ and $\kappa_c>0$. Additionally, $e^{-\kappa_c\tau\epsilon\underline{z}}\kappa_d<1 \Leftrightarrow \ln(\kappa_d)-\kappa_c\tau\epsilon\underline{z}<0$. By density of the real numbers[1], we can always find some $0<\epsilon<1$ such that $\ln(\kappa_d)-\kappa_c\tau c<0$, $c\in\{\underline{z},\overline{z}\}$, implies $\ln(\kappa_d)-\kappa_c\tau\epsilon\underline{z}<0$. Hence, $\lambda_f<1$. Similarly, in **case 3**, we have $\kappa_d^{\frac{\delta-\overline{z}}{\delta}}<1$ since $\delta>\overline{z}$ and $\kappa_d<1$. Moreover, $e^{-\kappa_c\tau}\kappa_d^{\frac{1}{\delta}}<1 \Leftrightarrow \ln(\kappa_d)-\kappa_c\tau\delta<0$. By density of the real numbers, we can always find some $\delta>\overline{z}$ such that $\ln(\kappa_d)-\kappa_c\tau c<0$, $c\in\{\underline{z},\overline{z}\}$, implies $\ln(\kappa_d)-\kappa_c\tau\delta<0$. Hence, $\lambda_f<1$. Consequently, for all $((x,c),(\hat{x},c))\in X\times\hat{X}$, for all $\hat{u}\in\hat{U}$, for any $\hat{x}^+$ such that $(\hat{x}^+,c^+)\in\hat{\mathcal{F}}((\hat{x},c),\hat{u})$, $\mathcal{V}$ satisfies inequality (II.15) with $\nu=\hat{u}$, $\bar{\sigma}=\lambda_f$, $\bar{\varepsilon}=\varphi_f(\eta)$, $\bar{\rho}_w=\rho_{\omega_f}$, and $\bar{\rho}_u=0$. Therefore, $\mathcal{V}$ is an alternating simulation function from $\hat{T}_\tau(\Sigma)$ to $T_\tau(\Sigma)$. ■

Remark 4.12: The symbolic model $\hat{T}_\tau(\Sigma)$ has a countably infinite set of states. However, in practical applications, the physical variables are restricted to a compact set. Hence, we are usually interested in the dynamics of the impulsive system only on a compact subset $X\subseteq\mathbb{R}^n$. Then, we can restrict the set of states of $\hat{T}_\tau(\Sigma)$ to the sets $([\mathbb{R}^n]_{\eta^x}\cap X)\times\{0,\ldots,\overline{z}\}$ which is finite. We refer interested readers to

[1] For any $a,b\in\mathbb{R}$ satisfying $a<b$, there always exists $\epsilon\in\mathbb{R}$ such that $a+\epsilon<b$, e.g., choose $\epsilon<b-a$.

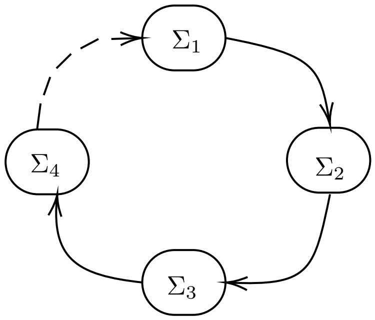

Fig. 1: Circular interconnections of subsystems

the explanation provided after Remark 4.1 in [23] for more details. $\diamond$

Finally, we would like to provide a discussion on condition (IV.10) in Theorem 4.11. In the case when $\kappa_d < 1$ and $\kappa_c > 0$, the continuous and discrete dynamics of Σ are Incrementally input to state stable, and, clearly, (IV.10) always holds. For the case when $\kappa_c > 0$ and $\kappa_d \geq 1$, the continuous dynamic Σ_c is Incrementally input to state stable while the discrete dynamic Σ_d is δ-FC. In order for condition (IV.10) to hold in this case, κ_c should be large enough to account for the instability of the jump mode and that the impulses do not happen too frequently. Finally, $\kappa_c \leqslant 0$ and $\kappa_d < 1$ corresponds to the case that the continuous dynamic Σ_c is δ-FC while the discrete one Σ_d is Incrementally input to state stable. Here, we require the impulses to happen very often and κ_d to be small enough to account for the instability of the flow mode. Note that condition (IV.10) ensures that an increase in the value of function V in Assumption 4.4 during flows is compensated by a decrease at jumps and vice versa. A similar argument was used in [27, Sections 4,5,6] to reason about input-to-state stability of impulsive systems, and we expect that by utilizing Assumption 4.4 with condition (IV.10), one can get Incrementally input to state stable for system Σ in (II.1).

V. Case study

Consider the exchange problems between N interconnected warehouses adapted from the storage-delivery process in [28]. Denote by $\mathbf{x}_i \in \mathbb{R}_{\geq 0}$, the number of goods in the warehouse i. The interconnections between the warehouses is supposed to be circular as depicted in Figure 1.

Under the flow mode: When $t \in \mathbb{R}_{\geqslant 0} \backslash \Omega_i$, for each warehouse the state x_i is continuously controlled through a delivery and picking-up process with a quantity d_i and input signal $\nu_i(t) \in \{-1, 1\}, t \in [0, \tau)$.

Under the jump mode: At each time $t \in \Omega_i = \{t_k^i\}_{k\in\mathbb{N}, i=1,2,3}$, with $t_{k+1}^i - t_k^i \in \{\underline{z}_i\tau_i, \ldots, \bar{z}_i\tau_i\}$ for fixed jump parameters $\tau_i \in \mathbb{R}_{>0}$ and $\underline{z}_i, \bar{z}_i \in \mathbb{N}_{\geq 1}, \underline{z}_i \leq \bar{z}_i$,, a truck enters warehouse i and the state x_i becomes controlled through a delivery and picking-up process with a quantity $\bar{d}_i$ and input signal $\nu_i(t) \in \{-1, 1\}, t \in [0, \tau)$.

The full state of each warehouse x_i is observable and we assume that the interconnected system is realisable. The dynamic motion of this process in the case $N = 3$ is modeled by,

$$\Sigma_1 : \begin{cases} \dot{\mathbf{x}}_1(t) = a_1\mathbf{x}_1(t) + b_1\mathbf{x}_3(t) + d_1\nu_1(t), & t \in \mathbb{R}_{\geq 0}\backslash\Omega_1, \\ \mathbf{x}_1(t) = r_1\mathbf{x}_1(t^-) + q_1\mathbf{x}_3(t) + \bar{d}_1\nu_1(t), & t \in \Omega_1, \\ \mathbf{y}_1(t) = \mathbf{x}_1(t). \end{cases}$$

$$\Sigma_2 : \begin{cases} \dot{\mathbf{x}}_2(t) = a_2\mathbf{x}_2(t) + b_2\mathbf{x}_1(t) + d_2\nu_2(t), & t \in \mathbb{R}_{\geq 0}\backslash\Omega_2, \\ \mathbf{x}_2(t) = r_2\mathbf{x}_2(t^-) + q_2\mathbf{x}_1(t) + \bar{d}_2\nu_2(t), & t \in \Omega_2, \\ \mathbf{y}_2(t) = \mathbf{x}_2(t). \end{cases}$$

$$\Sigma_3 : \begin{cases} \dot{\mathbf{x}}_3(t) = a_3\mathbf{x}_3(t) + b_3\mathbf{x}_2(t) + d_3\nu_3(t), & t \in \mathbb{R}_{\geq 0}\backslash\Omega_3, \\ \mathbf{x}_3(t) = r_3\mathbf{x}_3(t^-) + q_3\mathbf{x}_2(t) + \bar{d}_3\nu_3(t), & t \in \Omega_3, \\ \mathbf{y}_3(t) = \mathbf{x}_3(t). \end{cases}$$

In order to construct a symbolic model for the interconnected impulsive systems, we have to check Assumptions 3.1, 4.4, 4.5 and 4.1.

In the sequel, we will only detail the shell for the case $N = 3$. It can be shown that conditions (IV.5), (IV.6) and (IV.7) hold for each subsystem Σ_i with $V_i(x_i, x_i') = \|x_i - x_i'\|$, $i = 1, 2, 3$, with, $\underline{\alpha}_i = \bar{\alpha}_i = \mathcal{I}_d, \kappa_{c_i} = -a_i, \kappa_{d_i} = |r_i|$, $\rho_{u_c,1} = |d_1|$, $\rho_{u_d,1} = |\bar{d}_1|$, $\rho_{\omega_c,1} = |b_1|$, $\rho_{\omega_d,1} = |q_1|$, $\rho_{u_c,2} = |d_2|$, $\rho_{u_d,2} = |\bar{d}_2|$, $\rho_{\omega_c,2} = |b_2|$, $\rho_{\omega_q,2} = |q_2|$, $\rho_{u_c,3} = |d_3|$, $\rho_{u_d,3} = |\bar{d}_3|$, $\rho_{\omega_c,3} = |b_3|$ and $\rho_{\omega_d,3} = |q_3|$. From these functions, we can drive the expressions of the γ_{ij} functions in Assumption 3.1. Thus, $\gamma_{31} = \max\{|b_1|, |q_1|\}$, $\gamma_{12} = \max\{|b_2|, |q_2|\}$ and $\gamma_{23} = \max\{|b_3|, |q_3|\}$.

Remark 5.1: To verify Assumption 3.1, we can project the bounded-real lemmas for both continuous-time and discrete-time systems, as described in [29], in the form of convex linear matrix inequality on each subsystem Σ_i.

- For the flow mode, we have $\left\|\Sigma_i^{t\in\mathbb{R}_{\geqslant 0}\backslash\Omega_i}\right\|_\infty < \gamma_{ij}^F$ iff there exists $P_i \geqslant 0$ such that,
$$\begin{bmatrix} a_i^{\mathrm{T}}P_i + P_ia_i + I & P_ib_i \\ b_i^{\mathrm{T}}P_i & -(\gamma_{ij}^F)^2I \end{bmatrix} \preccurlyeq 0; \tag{V.1}$$
- For the jump mode, the constraint $\left\|\Sigma_i^{t\in\Omega_i}\right\|_\infty < \gamma_i^J$ is equivalent to the existence of $P_i > 0$ such that,
$$\begin{bmatrix} r_i^{\mathrm{T}}P_ir_i - P_i + I & r_i^{\mathrm{T}}P_iq_i \\ q_i^{\mathrm{T}}P_i\bar{A} & q_i^{\mathrm{T}}P_iq_i - (\gamma_{ij}^J)^2I \end{bmatrix} \preccurlyeq 0. \tag{V.2}$$
Note that γ_{ij} for each Σ_i is defined by $\gamma_{ij} = \max\{\gamma_{ij}^F, \gamma_{ij}^J\}$.

$\diamond$

Assumption 4.5 holds with $\hat{\gamma} = \mathcal{I}_d$ and Assumption 4.1, is satisfied with $L = 1$. Now, given τ_i and c_i satisfying (IV.10) for $c_i \in \{\underline{z}_i, \bar{z}_i\}$, and, with a proper choices of ϵ_i and δ_i, functions $\mathcal{V}_i(x_i, \hat{x}_i)$ given by (IV.11) are local alternating simulation functions from $\hat{T}_\tau(\Sigma_i)$, constructed as in Definition 4.10 for each i^{th} subsystem $i = 1, 2, 3$, to $T_\tau(\Sigma_i)$. In particular, each $\mathcal{V}_i$ satisfies conditions (II.14) and (II.15) with functions α_i, $\bar{\rho}_{\omega_i}, \bar{\rho}_{u_i}$, and constants $\bar{\sigma}_i, \varepsilon_i$ given below based on the values of a_i and r_i, with $\psi = 0.99$.

- $|r_i| < 1$ & $a_i < 0$: $\alpha_i = \mathcal{I}_d, \tilde{\sigma}_i = \max\{e^{a_i\tau_i}, r_i\}, \bar{\rho}_{\omega_i} = \max\{b_i, q_i\}, \rho_{u_i} = 0, \varepsilon_i = \hat{\varphi}_i$.

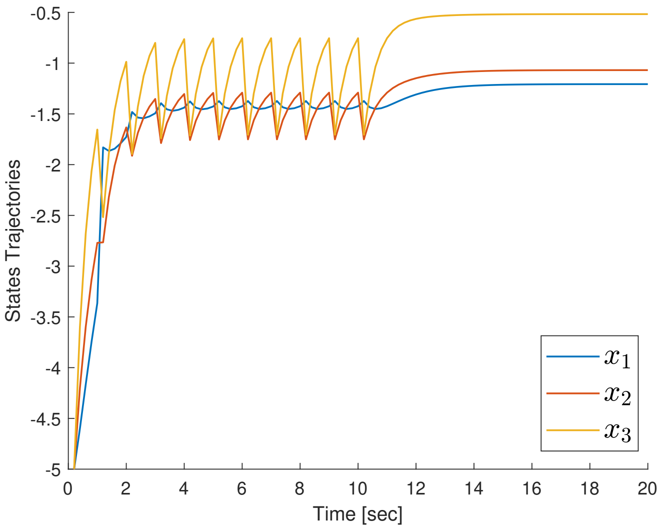

Fig. 2: State trajectories under fixed point controller.

- $|r_i| \geqslant 1 \ \& \ a_i < 0 : \alpha_i = \mathcal{I}_d, \rho_{u_i} = \rho_{\omega_i} = 0, \bar{\sigma}_i = \max\left\{e^{a_i\tau_i(1+\epsilon_i c_i)}, e^{a_i\tau_i\epsilon_i c_i}|r_i|\right\}, \varepsilon_i = e^{\kappa_c\tau\epsilon(\overline{z}+1)}\hat{\varphi}$.
- $|r_i| < 1 \ \& \ a_i \geqslant 0 : \alpha_i = \mathcal{I}_d, \rho_{u_i} = \rho_{\omega_i} = 0, \bar{\sigma}_i = \max\left\{e^{a_i\tau_i}|r_i|^{\frac{c_i}{\delta_i}}, |r_i|^{\frac{\delta_i+c_i}{\delta_i}}\right\}, \varepsilon_i = \hat{\varphi}_i$.

The control objective is to maintain the number of items of each warehouse i in a desired range O_i given by $O_i = [\ominus_{min}, \ominus_{min}]$ (a safety specification). We set up the system with the following parameters $a1 = -1,\ b_1 = 0.4,\ d_1 = 1,\ r_1 = 0.05,\ q_1 = 0.4,\ \bar{d}_1 = 1,\ a_2 = -1.5,\ b_2 = 0.5,\ d_2 = 1,\ r_2 = 0.03,\ q_2 = 0.5,\ \bar{d}_2 = 1,\ a_3 = -2,\ b_3 = 0.5,\ d_3 = 0.5,\ r_3 = 0.08,\ q_3 = 0.5,\ \bar{d}_3 = 1,$ and consider the following, for $i = 1, \ldots, 3$,

$$\Omega_i = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{bmatrix};$$

Each system state is expected to operate around an equilibrium point within the range of $\begin{bmatrix} -5 & 5 \end{bmatrix}$. With the defined system parameters, the sampling period for the controller to be designed is set $\tau = 0.2$, which satisfies condition (IV.10) for all the subsystems. We discretize the state by $n^x = 0.6667$. We conducted both monolithic and compositional abstractions, with the former taking 3589 seconds and the latter taking 1546 seconds to compute. Figure 2 displays the state trajectories using the designed fixed-point controller [1]. It is evident from the figure that the designed controller successfully keeps the states within the required safe region.

We performed a comparison of computation time between monolithic and compositional abstractions for different numbers of subsystems. Table I presents the results of this comparison, showing the computation time in seconds for each abstraction and the corresponding number of subsystems, for a discretization parameter $n^x = 2.5$. From the comparison table, it can be observed that the compositional abstraction generally requires less computational time than the monolithic abstraction for all the numbers of subsystems considered. As the number of subsystems increases, the computational time of both abstractions increases, and the difference between them remains significant. For instance,

TABLE I: Comparison of Computation Time in Seconds for Abstractions.

Number of subsystems / Abstraction	2	3	4	5
Monolithic	0.3107	1.2285	13.0902	5453.65
Compositional	0.2108	0.3147	2.2348	975.4288
ratio	1.4739	3.9037	5.8574	5.5910

when there are five subsystems, the computational time of the monolithic abstraction is almost six times faster than that of the compositional abstraction. Therefore, using compositional abstraction offers significant advantages in terms of computational efficiency, especially when dealing with complex systems with many subsystems.

VI. Conclusion

In conclusion, this paper proposes a novel compositional approach for constructing symbolic models of interconnected impulsive systems by utilizing the concept of approximate alternating simulation function in [7]. Based on some small gain type conditions, the proposed method compositionally constructs an overall alternating simulation function as a relation between an interconnection of symbolic models and that of original impulsive subsystems. In addition, under some stability and forward completeness properties, we introduce a technique to construct symbolic models together with their corresponding alternating simulation functions for impulsive subsystems. The effectiveness of the approach is demonstrated in a case study involving the construction of a symbolic model for a system consisting of different numbers of interconnected warehouses using both compositional and monolithic methods. The results show that the compositional abstraction is more computationally efficient than the monolithic abstraction.

As a future research direction, we aim to extend the proposed compositional approach to stochastic impulsive systems, where the flow and jump mode functions are characterized by probabilistic distributions.

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