Black-Derman-Toy (BDT)

(Discrete setup)

Volatility of interest rate is defined by $\sigma = \frac{1}{2}\log\left(\frac{r_u}{r_d}\right)$

r_u = interest rate if it goes up at 1st year

r_d = ——— ,, ——— down ——— ,, ———

Assumptions:

i) Volatilities depend on time but not on state.

ii) Volatility at a node in a given time period is the same as at any other node in that same time period.

* Find out interest rate tree?

Find out bond prices?

Maturity	Coupon rate (%)	Volatility (%)
1	10	20
2	11	19
3	12	18

Initial price of 1 year maturity bond $= \frac{100}{1.1} = 90.91$

——— 2 year ——— $= \frac{100}{(1.11)^2} = 81.16$

——— 3 year ——— $= \frac{100}{(1.12)^3} = 71.18$

For 2 year Maturity

r_u, r_d be the short rate after 1 year.

After 1 year, 2 year maturity bond will become 1 year maturity bond.

$$\frac{100}{1+r_u} \text{ or } \frac{100}{1+r_d}$$

We can assume that, under risk neutral probability, each state (either up or down) achieved is equally likely.

1 + Gross expected return under Risk Neutral Probability from investing in a 2 year zero coupon bond for 1 year $= \left\{ \dfrac{\frac{1}{2}\left(\frac{100}{1+r_u}\right) + \frac{1}{2}\left(\frac{100}{1+r_d}\right)}{81.16} \right\}$

$$1 + \text{expected return} = \frac{1}{81.16}\left\{\frac{1}{2}\left(\frac{100}{1+r_u}\right) + \frac{1}{2}\left(\frac{100}{1+r_d}\right)\right\}$$

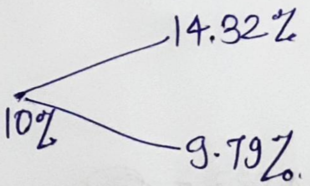
$$1 + 10\% = 1.10 = \frac{1}{81.16}\left\{\frac{1}{2}\left(\frac{100}{1+r_u}\right) + \frac{1}{2}\left(\frac{100}{1+r_d}\right)\right\}$$

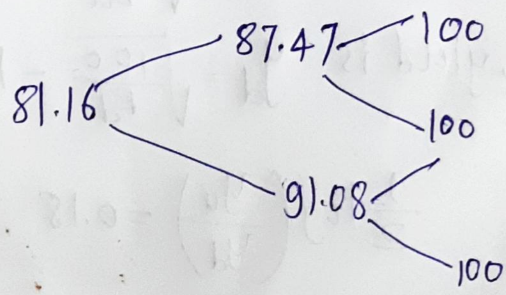
$$1.7855 = \left(\frac{1}{1+r_u}\right) + \left(\frac{1}{1+r_d}\right) \quad \text{—} ①$$

using BDT, $\frac{1}{2}\log\left(\frac{r_u}{r_d}\right) = 19\%$ — ②

$$\frac{r_u}{r_d} = e^{0.19\times 2} = 1.4627$$

Solving ① & ②, $r_u = 14.32\%$, $r_d = 9.79\%$

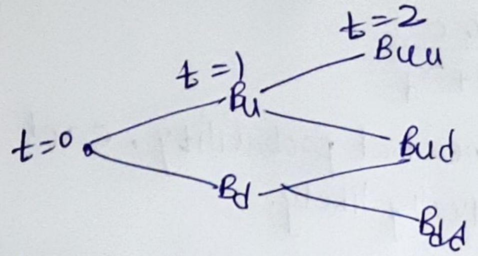




For 3 year Maturity

After 2 year, 3 year maturity bond will become 1 year maturity (1) bond.

$$B_{uu} = \frac{100}{1+r_{uu}}, \quad B_{ud} = \frac{100}{1+r_{ud}}, \quad B_{dd} = \frac{100}{1+r_{dd}}$$



Under risk neutral probability,

$$B_u = \frac{1}{1+r_u}\left(\frac{1}{2}B_{uu} + \frac{1}{2}B_{ud}\right)$$

$$B_d = \frac{1}{1+r_d}\left(\frac{1}{2}B_{ud} + \frac{1}{2}B_{dd}\right)$$

Taking expectations of these values under risk neutral probability and discounting back at the risk free interest rate.

$$71.18 = \frac{1}{1+10\%}\left[\frac{1}{2}B_u + \frac{1}{2}B_d\right] = \frac{1}{2(1+10\%)}\left[\frac{1}{1+r_u}\left(\frac{1}{2}B_{uu}+\frac{1}{2}B_{ud}\right) + \frac{1}{1+r_d}\left(\frac{1}{2}B_{ud} + \frac{1}{2}B_{dd}\right)\right] \quad (1)$$

We can rewrite equation (1) in terms of r_{uu}, r_{ud}, r_{dd} using B_{uu}, B_{ud}, B_{dd}

Under Assumption of BDT model,

$$\frac{1}{2}\log\left(\frac{r_{uu}}{r_{ud}}\right) = \frac{1}{2}\log\left(\frac{r_{ud}}{r_{dd}}\right)$$

$$(r_{ud})^2 = r_{uu} \cdot r_{dd} \quad (2)$$

At up state, yield is $y_u = \sqrt{\frac{100}{B_u}} - 1$

At down state, yield is $y_d = \sqrt{\frac{100}{B_d}} - 1$

$$\frac{1}{2}\log\left(\frac{y_u}{y_d}\right) = 0.18 \qquad \frac{y_u}{y_d} = 1.4333 \quad (3)$$

Solving eqn 1, 2, 3, we get.