MA 105: D3 Lecture 16

Ravi Raghunathan

Department of Mathematics

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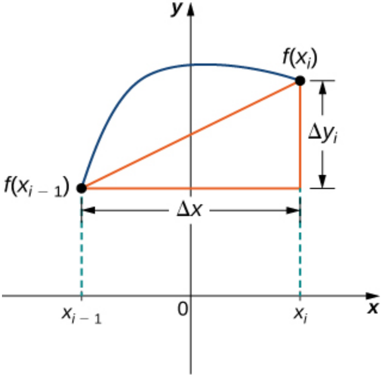
Multivariable calculus: Supplementary exercises (mostly from Stewart)

Arc Length



The formula for arc length

Let us denote the arc length of the curve $y = f(x)$ by S. The length of any given hypotenuse in the previous slide is given by the Pythagorean Theorem: $\sqrt{\Delta x^2 + \Delta y^2}$.
Intuitively, the sum of the lengths of the n hypotenuses appears to approximate S:

$$S \sim \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\, \Delta x_i,$$

where "$\sim$" means approximately equal. We can use this idea to define the arc length as

$$S := \lim_{\Delta x_i \to 0} \sum_{i=1}^{\infty} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\, \Delta x_i = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

provided this limit exists (in particular, we demand that the limit is a finite number).

Exercise 4.10.(ii) Find the length of the curve

$$y(x) = \int_0^x \sqrt{\cos 2t}\, dt,\ 0 \le x \le \pi/4.$$

Solution: The formula for the arc length of a curve $y = f(x)$ between the points $x = a$ and $x = b$ is given by

$$\int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

For the problem at hand this gives

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 2x} dx = \sqrt{2} \int_0^{\pi/4} \cos(x) dx = 1.$$

Rectifiable curves

Not all curves have finite arc length! Here is an example of a curve with infinite arc length.

Example: Let $\gamma : [0,1] \to \mathbb{R}^2$ be the curve given by $\gamma(t) = (t, f(t))$, where

$$f(t) = \begin{cases} t\cos\left(\frac{\pi}{2t}\right), & \text{if } t \neq 0 \\ 0 \text{ if } t = 0 \end{cases}$$

If
http://math.stackexchange.com/questions/296397/nonrectifiable-curve
is correct, you should be able to check that this curve has infinite arc length. Try it as an exercise.

Notice that the curve above is given by a continuous function. Curves for which the arc length S is finite are called rectifiable curves. You can easily check that the graphs of piecewise $\mathcal{C}^1$ functions are rectifiable.

Things can get even stranger

In fact, there exist space filling curves, that is curves $\gamma : [0,1] \to [0,1] \times [0,1]$ which are continuous and surjective (but it is not injective!). Obviously the graph of this curve "fills up" the entire square. Such curves are not rectifiable (can you prove this - see
`https://en.wikipedia.org/wiki/Peano_curve`
for an example.

The existence of such curves should make you question whether your intuitive notion of dimension actually has any mathematical basis. If a line segment can be mapped continuously onto a square, is it reasonable to say that they have different dimensions? After all, this means we can describe any point on the square using just one number.

We will answer this question (without a proof) later in this course. We will also come back to arc length of a curve when studying multivariable calculus.

Comments

General comments based on today's interaction in class. To show that a limit exists you must either use the $\varepsilon - \delta$ definition or use the rules for limits when applicable. Choosing particular curves and approaching the limit point along these curves is a good strategy for showing a limit does not exist (or that a function is not continuous). It cannot be used to show limits exist.

Using the rules for limits will not work if the function you are given has a denominator which goes to 0 as (x, y) approaches the limit point. In this case, the only way to show that a limit exists to use some kind of inequality which shows that the numerator goes to zero at least as fast as the denominator.

The natural domain

Exercise 1. What is the natural domain of the following functions (try to describe the domain geometrically):
(a) $f(x,y) = \frac{\sqrt{x+y+1}}{x-1}$, (b) $f(x,y) = x\ln(y^2 - x)$.

Solution. (a)

$$D = \{(x,y) \,|\, x+y+1 \geq 0, x \neq 1\}.$$

This is the set of points that lie above the line $x+y+1=0$ but not on the line $x=1$.

(b)

$$D = \{(x,y) \,|\, x < y^2\}.$$

This this the set of points to the left of the parabola $x = y^2$.

Limits and Continuity

Exercise 2. Determine if the following limits exist. If they exist find them.

(a) $\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right)$,

(b) $\lim_{(x,y)\to(0,0)} \frac{x^2ye^x}{x^2+4y}$,

(c) $\lim_{(x,y)\to(0,0)} \frac{x^2\sin y}{x^2+2y^2}$,

(d) $\lim_{(x,y)\to(1,-1)} e^{xy}\cos(x+y)$

(e) $\lim_{(x,y,z))\to(0,0,0)} \frac{xy+yz}{x^2+y^2+z^2}$.

Exercise 2 (a)

Solution. Determine if the following limits exist. If they exist find them.

(a) $\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right)$:

The limit of a quotient is the quotient of the limits and $x \to \ln x$ is continuous. It follows that the limit exists.

Exercise 2 (b)

$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin y}{x^2+2y^2}$:

If (x, y) such that $x^2 + y^2 < 1$,

$$|x^2 \sin y| < x^2|y| < (x^2 + 2y^2)|y|.$$

Hence, the quotient

$$\left|\frac{x^2 \sin y}{x^2 + 2y^2}\right| < |y|.$$

Thus, if $x^2 + y^2 < \delta = \varepsilon^2$, $|y| < |\varepsilon$, so $\left|\frac{x^2 \sin y}{x^2+2y^2} - 0\right| < \varepsilon$.

This shows that the limit is 0.

Exercise 2 (c)

(b) $\lim_{(x,y)\to(0,0)} \frac{x^2ye^x}{x^2+4y}$:

Let $f(x,y) = \frac{x^2ye^x}{x^2+4y}$. One solution offered in class was to say that the function is not defined for points such that $4y - x^2 = 0$. This is correct (although we did not pursue this fully in class). After all, if the function is not defined at points arbitrarily close to $(0,0)$, one can argue that the inequality $|f(x,y) - \ell| < \varepsilon$ cannot be satisfied for $\|x\| < \delta$ for any $\delta > 0$.

If this line of argument makes you uncomfortable, we can use the strategy from the class. The limit along the line $y = x$ is 0. Now take points (x,y) that lie on a curve close to the curve $x^2 + 4y = 0$. For instance, we can look at the curve $x^2 + 4y = x^2y$ or $y = \frac{x^2}{x^2-4}$. Along this curve $f(x,y) = e^x$, so the limit is simply 1 as $x \to 0$ (one of you gave me a similar curve in class, but I think that my example is a little simpler).

Thus, this limit does not exist.

Exercise 2 (d)

$\lim_{(x,y)\to(1,-1)} e^{xy} \cos(x+y)$

Again, use the rules for limits!

Exercise 2 (e)

(e) $\lim_{(x,y,z))\to(0,0,0)} \frac{xy+yz}{x^2+y^2+z^2}$.

Let $x = y$ and $z = y$. Then the quotient is $2/3$.
Let $x = y$ and $z = 0$. Then, the quotient is $1/2$.

Hence, the limit does not exist.