MA 109 D3 Lecture 13

Ravi Raghunathan

Department of Mathematics

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Functions of severable variables

Limits and continuity

Differentiation

Functions with range contained in $\mathbb{R}$

We want to study functions from $\mathbb{R}^n \to \mathbb{R}^m$ (in our course $1 \leq m, n \leq 3$).

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function, we see that $f(x) \in \mathbb{R}^m$ for $x \in \mathbb{R}^n$, so $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$, where f_i, $1 \leq i \leq m$ is a function from $\mathbb{R}^n \to \mathbb{R}$. Thus, studying functions from $\mathbb{R}^n \to \mathbb{R}^m$ is the same as studying an m-tuple of functions of functions from $\mathbb{R}^n$ to $\mathbb{R}$.

For now, we study functions $f : \mathbb{R}^n \to \mathbb{R}$, or more commonly, functions $f : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^n$. The function f will often be given by a formula and there will be some natural domain on which the formula makes sense.

Level curves and contour lines

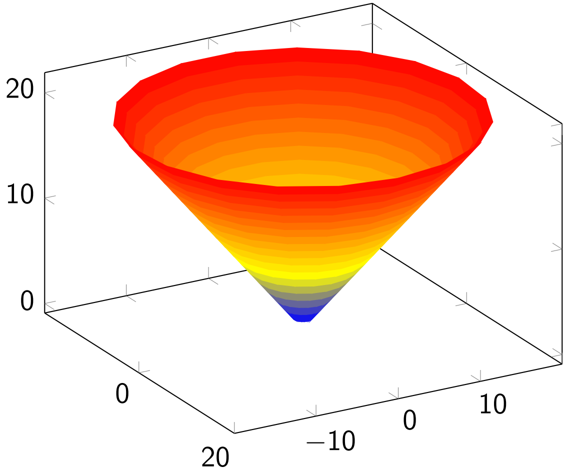
The second thing one should do with a function from $\mathbb{R}^2 \to \mathbb{R}$ is to study its range. This is done in different ways.

One way is to study the level sets of the functions. These are the sets of the form $\{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}$, where c is a constant. The level set "lives" in the xy-plane.

One can also plot (in three dimensions) the surface $z = f(x, y)$. By varying the value of c in the level curves one can get a good idea of what the surface looks like.

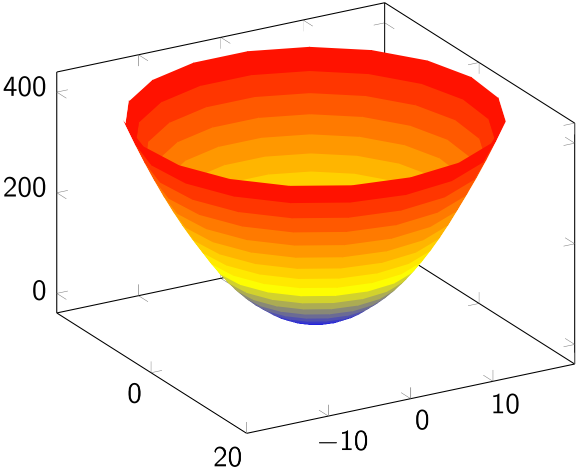
When one plots the $f(x, y) = c$ for some constant c one gets a curve. Such a curve is usually called a contour line (the contour "lives" in the $z = c$ plane).

I have a couple of pictures in the next two slides to illustrate the point.



This is the graph of the function $z = \sqrt{x^2 + y^2}$ lying above the xy-plane. It is a right circular cone.

The contour lines $z = c$ give circles lying on planes parallel to the xy-plane.The curves given by $z = f(x, 0)$ and $z = f(0, y)$ give pairs of straight lines in the planes $y = 0$ and $x = 0$.



This is the graph of the function $z = x^2 + y^2$ lying above the xy-plane. It is a paraboloid of revolution.

The contour lines $z = c$ give circles lying on planes parallel to the xy-plane. The curves $z = f(x, 0)$ or $z = f(y, 0)$ give parabolæ lying in the planes $y = 0$ and $x = 0$. Exercise 5.2.(ii).

Limits

We have already said what it means for a function of two or more variables to approach a limit. We simply have to replace the absolute value function on $\mathbb{R}$ by the distance function on $\mathbb{R}^m$. We will do this in two variables. The three variable definition is entirely analogous. We will denote by U a set in $\mathbb{R}^2$.

Definition: A function $f : U \to \mathbb{R}$ is said to tend to a limit l as $x = (x_1, x_2)$ approaches $c = (c_1, c_2)$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - l| < \epsilon,$$

whenever $0 < \|x - c\| < \delta$ with $x \in U$.

We recall that

$$\|x\| = \sqrt{x_1^2 + x_2^2}.$$

Notice that the set

$$B_\delta(c) = \{x \in \mathbb{R}^2 \mid \|x - c\| < \delta\}$$

is a circular disc/ball of radius δ. Thus, $f(x)$ is close to l (that is, within a distance ε) whenever x lies in a sufficiently small disc (of radius δ).

In the plane $\mathbb{R}^2$ it is possible to approach the point c from infinitely many different directions - not just from the right and from the left. In fact, one may not even be approaching the point c along a straight line! Hence, to say that a function from $\mathbb{R}^2$ to $\mathbb{R}$ possesses a limit is actually imposing a strong condition - for instance, the limits along all possible curves leading to the point must exist and all these (infinitely many) limits must be equal.

Continuity

Once we have the notion of a limit, the definition of continuity is just the same as for functions of one variable.

Definition: The function $f : U \to \mathbb{R}$ is said to be continuous at a point c if

$$\lim_{x \to c} f(x) = f(c).$$

Exercise: Formulate the definition of a limit and of continuity for functions from $\mathbb{R}^3$ to $\mathbb{R}$.

The rules for limits and continuity

The rules for addition, subtraction, multiplication and division of limits remain valid for functions of two variables (or three variables for that matter). Nothing really changes in the statements or the proofs.

Using these rules, we can conclude, as before, that the sum, difference, product and quotient of continuous functions are continuous (as usual we must assume that the denominator of the quotient is non zero).

The composition of continuous functions is also continuous.

Continuity through examples

Once again, we emphasise that continuity at a point c is a very powerful condition (since the existence of a limit is implicit).

Exercise 5.3.(i) asks whether the function

$$f(x,y) = \begin{cases} \frac{x^3 y}{x^6+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 \quad \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

Solution: Let us look at the sequence of points $z_n = (\frac{1}{n}, \frac{1}{n^3})$, which goes to 0 as $n \to \infty$. Clearly $f(z_n) = \frac{1}{2}$ for all n, so

$$\lim_{n\to\infty} f(z_n) = \frac{1}{2} \neq 0.$$

This shows that f is not continuous at 0.
But does the limit exist?

Iterated limits

When evaluating a limit of the form $\lim_{(x_1,x_2)\to(c_1,c_2)} f(x_1,x_2)$ one may naturally be tempted to let x_1 go to c_1 first, and then let x_2 go to c_2. Does this give the limit in the previous sense?

Exercise 5.5: Let

$$f(x,y) = \frac{x^2y^2}{x^2y^2 + (x-y)^2}.$$

we have

$$\lim_{x\to 0}\lim_{y\to 0}\frac{x^2y^2}{x^2y^2+(x-y)^2} = \lim_{x\to 0} 0 = 0$$

Similarly, one has $\lim_{y\to 0}\lim_{x\to 0} f(x,y) = 0$.

However, choosing $z_n = (\frac{1}{n}, \frac{1}{n})$, shows that $f(z_n) = 1$ for all $n \in \mathbb{N}$. Now choose $z_n = (\frac{1}{n}, \frac{1}{2n})$ to see that the limit cannot exist.

Partial Derivatives

As before, U will denote a subset of $\mathbb{R}^2$. Given a function $f : U \to \mathbb{R}$, we can fix one of the variables and view the function f as a function of the other variable alone. We can then take the derivative of this one variable function.

To make things precise, fix x_2.
Definition: The partial derivative of $f : U \to \mathbb{R}$ with respect to x_1 at the point (a, b) is defined by

$$\frac{\partial f}{\partial x_1}(a, b) := \lim_{x_1 \to a} \frac{f((x_1, b)) - f((a, b))}{x_1 - a}.$$

Similarly, one can define the partial derivative with respect to x_2. In this case the variable x_1 is fixed and f is regarded only as a function of x_2:

$$\frac{\partial f}{\partial x_2}(a, b) := \lim_{x_2 \to b} \frac{f((a, x_2)) - f((a, b))}{x_2 - b}.$$

Directional Derivatives

The partial derivatives are special cases of the directional derivative. Let $v = (v_1, v_2)$ be a unit vector. Then v specifies a direction in $\mathbb{R}^2$.

Definition: The directional derivative of f in the direction v at a point $x = (x_1, x_2)$ is denoted by $\nabla_v f(x)$ and is defined as

$$\lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} = \lim_{t \to 0} \frac{f((x_1 + tv_1, x_2 + tv_2)) - f((x_1, x_2))}{t}.$$

$\nabla_v f(x)$ measures the rate of change of the function f at x along the path $x + tv$.

If we take $v = (1, 0)$ in the above definition, we obtain $\frac{\partial f}{\partial x_1}(x)$, while $v = (0, 1)$ yields $\frac{\partial f}{\partial x_2}(x)$.

Consider the function

$$f(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 = 0 \text{ or if } x_2 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

It should be clear to you that since this function is constant along the two axes,

$$\frac{\partial f}{\partial x_1}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial x_2}(0,0) = 0$$

On the other hand, $f(x_1, x_2)$ is not continuous at the origin! Thus, a function may have both partial derivatives (and, in fact, any directional derivative - see the next slide) but still not be continuous. This suggests that for a function of two variables, just requiring that both partial derivatives exist is not a good or useful definition of “differentiability”.

Recall again, the following function from Exercise 5.5:

$$\frac{x^2y^2}{x^2y^2+(x-y)^2} \quad \text{for} \quad (x,y)\neq(0,0).$$

Let us further set $f(0,0)=0$. You can check that every directional derivative exists and is equal to 0, except along $y=x$ when the directional derivative is not defined. However, we have already seen that the function is not continuous at the origin since we have shown that $\lim_{(x,y)\to 0} f(x,y)$ does not exist. For an example with directional derivatives in all directions see Exercise 5.3(i).

Conclusion: All directional derivatives may exist at a point even if the function is discontinuous.

Let us go back and examine the notion of differentiability for a function of $f(x)$ of one variable. Suppose f is differentiable at the point x_0, What is the equation of the tangent line through $(x_0, f(x_0))$?

$$y = f(x_0) + f'(x_0)(x - x_0)$$

If we consider the difference $f(x) - f(x_0) - f'(x_0)(x - x_0)$ we get the distance of a point on the tangent line from the curve $y = f(x)$. Writing $h = (x - x_0)$, we see that the difference can be rewritten

$$f(x_0 + h) - f(x_0) - f'(x_0)h$$

The tangent line is close to the function f - how close?- so close that even after dividing by h the distance goes to 0. A few lectures ago we wrote this as

$$|f(x_0 + h) - f(x_0) - f'(x_0)h| = \varepsilon_1(h)|h|$$

where $\varepsilon_1(h)$ is a function that goes to 0 as h goes to 0. So $o(h) = \varepsilon_1(h)|h|$ is function that "goes to zero faster than h").

The preceding idea generalises to two (or more) dimensions. Let $f(x, y)$ be a function which has both partial derivatives. In the two variable case we need to look at the distance between the surface $z = f(x, y)$ and its tangent plane.

Let us first recall how to find the equation of a plane passing through the point $P = (x_0, y_0, z_0)$. It is the graph of the function

$$z = g(x, y) = z_0 + a(x - x_0) + b(y - y_0).$$

Let us determine the tangent plane to $z = f(x, y)$ passing through a point $P = (x_0, y_0, z_0)$ *on the surface*. In other words, we have to determine the constants a and b.

If we fix the y variable and treat $f(x, y)$ only as a function of x, we get a curve. Similarly, if we treat $g(x, y)$ as function only of x, we obtain a line. The tangent to the curve must be the same as the line passing through (x_0, y_0, z_0), and, in any event, their slopes must be the same. Thus, we must have

$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial g}{\partial x}(x_0, y_0) = a.$$

Arguing in exactly the same way, but fixing the x variable and varying the y variable we obtain

$$\frac{\partial f}{\partial y}(x_0, y_0) = \frac{\partial g}{\partial y}(x_0, y_0) = b.$$

Hence, the equation of the tangent plane to $z = f(x, y)$ at the point (x_0, y_0) is

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

Differentiability for functions of two variables

We now define differentiability for functions of two variables by imitating the one variable definition, but using the "$o(h)$" version.
We let $(x, y) = (x_0, y_0) + (h, k) = (x_0 + h, y_0 + k)$
Definition A function $f : U \to \mathbb{R}$ is said to be differentiable at a point (x_0, y_0) if $\frac{\partial f}{\partial x}(x_0, y_0)$, and $\frac{\partial f}{\partial y}(x_0, y_0)$ exist and

$$\lim_{(h,k)\to 0} \frac{\left| f(x_0 + h, y_0 + k) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)h - \frac{\partial f}{\partial y}(x_0, y_0)k \right|}{\|(h, k)\|} = 0,$$

This is saying that the distance between the tangent plane and the surface is going to zero even after dividing by $\|(h, k)\|$. We could rewrite this as

$$\left| f((x_0, y_0) + (h, k)) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)h - \frac{\partial f}{\partial y}(x_0, y_0)k \right| = \varepsilon(h, k)\|(h, k)\|$$

where $\varepsilon(h, k)$ is a function that goes to 0 as $\|(h, k)\| \to 0$.This form of differentiability now looks exactly like the one variable version case (put $o(h, k) = \varepsilon(h, k)\|(h, k)\|$).