CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Aug 21, 2023

Lecture 06 – Basic Mathematical Structures

Sets and functions

A Quick Recap

Five lectures completed

A Quick Recap

Five lectures completed

- Week 1
  1. Propositions, Predicates, Theorems.
  2. Types of proofs; contradiction and contrapositive; axioms.
  3. Induction and the Well-Ordering Principle.
- Week 2
  4. Strong Induction and its applications.
  5. Basic mathematical structures: numbers and sets

A Quick Recap

Five lectures completed

- Week 1
  1. Propositions, Predicates, Theorems.
  2. Types of proofs; contradiction and contrapositive; axioms.
  3. Induction and the Well-Ordering Principle.
- Week 2
  4. Strong Induction and its applications.
  5. Basic mathematical structures: numbers and sets

Two problem-sheets released

1. 9 questions on Basic proofs, induction, WOP
2. 4 questions on More basic proofs and Strong Induction.

Is the fun done? How do we compare sets?

- For two finite sets, this is easy, just count the number of elements and compare them!

Is the fun done? How do we compare sets?

- For two finite sets, this is easy, just count the number of elements and compare them!
- But what about two infinite sets?
- Example: {set of all even natural numbers} vs $\mathbb{N}$ vs $\mathbb{Q}$ vs $\mathbb{R}$

Is the fun done? How do we compare sets?

- For two finite sets, this is easy, just count the number of elements and compare them!
- But what about two infinite sets?
- Example: $\{$set of all even natural numbers$\}$ vs $\mathbb{N}$ vs $\mathbb{Q}$ vs $\mathbb{R}$
- Turns out we need functions... but first...

Hilbert's hotel

- Suppose there is a hotel with infinitely many rooms.
- And suppose they are all full (like in IIT guest house).

Hilbert's hotel

- Suppose there is a hotel with infinitely many rooms.
- And suppose they are all full (like in IIT guest house).

1. Can you accommodate 1 or finitely many more guests, by shifting around the existing guests?

Hilbert's hotel

- Suppose there is a hotel with infinitely many rooms.
- And suppose they are all full (like in IIT guest house).

1. Can you accommodate 1 or finitely many more guests, by shifting around the existing guests?
2. What if infinitely many more guests arrive?

Hilbert's hotel

- Suppose there is a hotel with infinitely many rooms.
- And suppose they are all full (like in IIT guest house).

1. Can you accommodate 1 or finitely many more guests, by shifting around the existing guests?
2. What if infinitely many more guests arrive?
3. What if infinitely many more trains with infinitely many more guests arrive? (H.W)

Functions

What you did above was to define functions...

Definition

Let A, B be two sets. A function f from A to B is an assignment of exactly one element of B to each element of A.

Functions

What you did above was to define functions...

Definition

Let A, B be two sets. A function f from A to B is an assignment of exactly one element of B to each element of A.

Formally, $f : A \to B$ is a subset R of pairs $A \times B$ such that

(i) $\forall a \in A$, $\exists b \in B$ such that $(a, b) \in R$, and

(ii) if $(a, b) \in R$ and $(a, c) \in R$, then $b = c$.

Functions

What you did above was to define functions...

Definition

Let A, B be two sets. A function f from A to B is an assignment of exactly one element of B to each element of A.

Formally, $f : A \to B$ is a subset R of pairs $A \times B$ such that

(i) $\forall a \in A$, $\exists b \in B$ such that $(a, b) \in R$, and

(ii) if $(a, b) \in R$ and $(a, c) \in R$, then $b = c$.

- We write $f(a) = b$ and call b the image of a.
- $\mathit{Range}(f) = \{b \in B \mid \exists a \in A \text{ s.t. } f(a) = b\}$, $\mathit{Domain}(f) = A$

Functions

What you did above was to define functions...

Definition

Let A, B be two sets. A function f from A to B is an assignment of exactly one element of B to each element of A.

Formally, $f : A \to B$ is a subset R of pairs $A \times B$ such that

(i) $\forall a \in A, \exists b \in B$ such that $(a, b) \in R$, and

(ii) if $(a, b) \in R$ and $(a, c) \in R$, then $b = c$.

Composition of functions

- If $g : A \to B$ and $f : B \to C$, then $f \circ g : A \to C$ is defined by $f \circ g(x) = f(g(x))$.

- Defined only if $Range(g) \subseteq Domain(f)$.
- Qn: if $f(x) = x^2$, $g(x) = x - x^3$ with $f, g : \mathbb{R} \to \mathbb{R}$, what is $f \circ g(x), g \circ f(x)$?

Functions

What you did above was to define functions...

Definition

Let A, B be two sets. A function f from A to B is an assignment of exactly one element of B to each element of A.

Formally, $f : A \to B$ is a subset R of pairs $A \times B$ such that

(i) $\forall a \in A, \exists b \in B$ such that $(a, b) \in R$, and

(ii) if $(a, b) \in R$ and $(a, c) \in R$, then $b = c$.

Composition of functions is associative

- If $h : A \to B$ and $g : B \to C$ and $f : C \to D$, then $f \circ (g \circ h) = (f \circ g) \circ h$. doubt hai

Check it! (H.W.)

Functions

What you did above was to define functions...

Definition

Let A, B be two sets. A function f from A to B is an assignment of exactly one element of B to each element of A.

Formally, $f : A \to B$ is a subset R of pairs $A \times B$ such that

(i) $\forall a \in A$, $\exists b \in B$ such that $(a, b) \in R$, and

(ii) if $(a, b) \in R$ and $(a, c) \in R$, then $b = c$.

Inverse of a function

- If $f : A \to B$ is a function, then its inverse is the function $f^{-1} : B \to A$ defined by $f^{-1}(b) = a$ if $f(a) = b$.

Functions

What you did above was to define functions...

Definition

Let A, B be two sets. A function f from A to B is an assignment of exactly one element of B to each element of A.

Formally, $f : A \to B$ is a subset R of pairs $A \times B$ such that

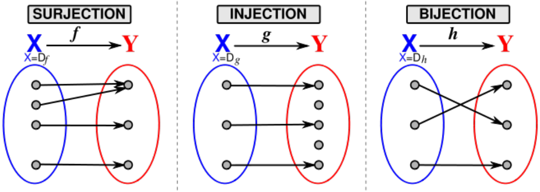
(i) $\forall a \in A, \exists b \in B$ such that $(a, b) \in R$, and

(ii) if $(a, b) \in R$ and $(a, c) \in R$, then $b = c$.

Inverse of a function

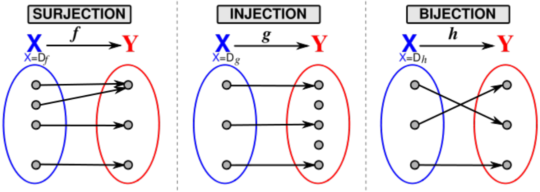
- If $f : A \to B$ is a function, then its inverse is the function $f^{-1} : B \to A$ defined by $f^{-1}(b) = a$ if $f(a) = b$. Does the inverse always exist?

Comparing (finite and infinite) sets



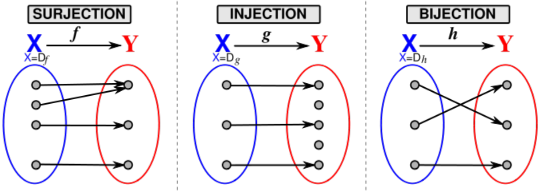
- Surjective or onto: $f : A \to B$ is surjective if $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$.

Comparing (finite and infinite) sets



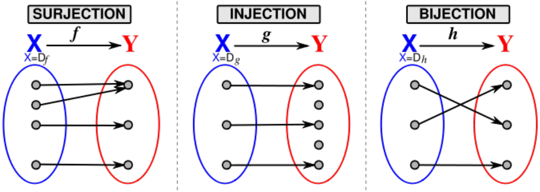
- Surjective or onto: $f : A \to B$ is surjective if $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$.
- Injective or 1-1: $f : A \to B$ is injective if $\forall x, y \in A$, if $f(x) = f(y)$, then $x = y$.

Comparing (finite and infinite) sets



- Surjective or onto: $f : A \to B$ is surjective if $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$.
- Injective or 1-1: $f : A \to B$ is injective if $\forall x, y \in A$, if $f(x) = f(y)$, then $x = y$.
- Bijective or 1-1 correspondence: A function is bijective if it is surjective and injective.

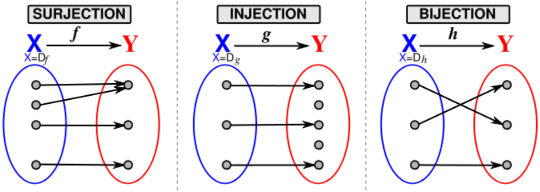
Comparing (finite and infinite) sets



- Surjective or onto: $f : A \to B$ is surjective if $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$.
- Injective or 1-1: $f : A \to B$ is injective if $\forall x, y \in A$, if $f(x) = f(y)$, then $x = y$.
- Bijective or 1-1 correspondence: A function is bijective if it is surjective and injective.

If f is a bijection, then its inverse function exists and $f \circ f^{-1} = f^{-1} \circ f = \text{id}$

Comparing (finite and infinite) sets

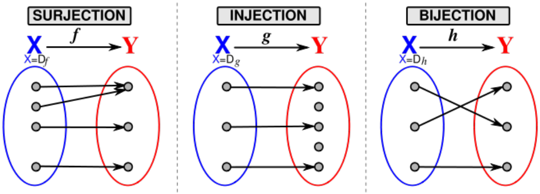


- Surjective or onto: $f : A \to B$ is surjective if $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$.
- Injective or 1-1: $f : A \to B$ is injective if $\forall x, y \in A$, if $f(x) = f(y)$, then $x = y$.
- Bijective or 1-1 correspondence: A function is bijective if it is surjective and injective.

Qns

1. $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = x^2$.
2. $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ such that $f(x) = x^2$.

Comparing (finite and infinite) sets



- Surjective or onto: $f : A \to B$ is surjective if $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$. – If A, B finite, $|A| \geq |B|$
- Injective or 1-1: $f : A \to B$ is injective if $\forall x, y \in A$, if $f(x) = f(y)$, then $x = y$. – If A, B finite, $|A| \leq |B|$
- Bijective or 1-1 correspondence: A function is bijective if it is surjective and injective. – If A, B finite, $|A| = |B|$

Qns

1. $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = x^2$.
2. $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ such that $f(x) = x^2$.

Properties of finite and infinite sets

Relative notion of "size" using bijections

Thus, two finite/infinite sets have the same "size" iff there is a bijection between them.

- For finite sets, this is a property that can be shown.
- For infinite sets, it is a definition!

Properties of finite and infinite sets

Relative notion of "size" using bijections

Thus, two finite/infinite sets have the same "size" iff there is a bijection between them.

Similarities between finite and infinite sets

- $\exists$ **bij** from A to B and B to C, implies $\exists$ **bij** from A to C.
- $\exists$ **bij** from A to B, then $\exists$ **bij** from B to A.
- $\exists$ **surj** from A to B and $\exists$ **surj** B to A, implies $\exists$ **bij** from A to B.

Properties of finite and infinite sets

Relative notion of "size" using bijections

Thus, two finite/infinite sets have the same "size" iff there is a bijection between them.

Similarities between finite and infinite sets

- $\exists$ **bij** from A to B and B to C, implies $\exists$ **bij** from A to C.
- $\exists$ **bij** from A to B, then $\exists$ **bij** from B to A.
- (Schröder-Bernstein Theorem:) $\exists$ **surj** from A to B and $\exists$ **surj** B to A, implies $\exists$ **bij** from A to B. (H.W: Read this!)

Properties of finite and infinite sets

Relative notion of "size" using bijections

Thus, two finite/infinite sets have the same "size" iff there is a bijection between them.

Similarities between finite and infinite sets

- $\exists$ **bij** from A to B and B to C, implies $\exists$ **bij** from A to C.
- $\exists$ **bij** from A to B, then $\exists$ **bij** from B to A.
- (Schröder-Bernstein Theorem:) $\exists$ **surj** from A to B and $\exists$ **surj** B to A, implies $\exists$ **bij** from A to B. (H.W: Read this!)

Differences between finite and infinite sets

- For finite sets, if A is a set and $b \notin A$, then $|A| \neq |A \cup \{b\}|$.

Properties of finite and infinite sets

Relative notion of "size" using bijections

Thus, two finite/infinite sets have the same "size" iff there is a bijection between them.

Similarities between finite and infinite sets

- $\exists$ **bij** from A to B and B to C, implies $\exists$ **bij** from A to C.
- $\exists$ **bij** from A to B, then $\exists$ **bij** from B to A.
- (Schröder-Bernstein Theorem:) $\exists$ **surj** from A to B and $\exists$ **surj** B to A, implies $\exists$ **bij** from A to B. (H.W: Read this!)

Differences between finite and infinite sets

- For finite sets, if A is a set and $b \notin A$, then $|A| \neq |A \cup \{b\}|$.
- What about infinite sets?

Difference between finite and infinite sets

Theorem

Let A be a set and $b \notin A$. Then A is infinite iff there is a bijection from A to $A \cup \{b\}$.

Difference between finite and infinite sets

Theorem

Let A be a set and $b \notin A$. Then A is infinite iff there is a bijection from A to $A \cup \{b\}$.

Proof: essentially Hilbert's hotel but be careful...