MA 105 D3 Lecture 3

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Recap

Cauchy sequences: the definition

Limits of functions

The rigourous definition of a limit

Definition: A sequence a_n tends to a limit l/converges to a limit l, or that $\lim_{n\to\infty} a_n = l$, if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - l| < \epsilon$$

whenever $n > N$.

We also stated rules for limits and the Sandwich Theorems and proved the product rule. Once we have these rules, we can handle more complicated limits.

Lemma: Every convergent sequence is bounded.

Theorem 3: A montonically increasing (resp. decreasing) sequence which is bounded above (resp. below) converges to its supremum (resp. infimum).

Example

Let us look at Exercise 1.5.(i) which considers the sequence

$$a_1 = 3/2 \quad \text{and} \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right).$$

$$\begin{aligned} a_{n+1} \quad < a_n \quad &\iff \quad \tfrac{1}{2}\left(a_n + \tfrac{2}{a_n}\right) < a_n \\ &\iff \quad \sqrt{2} < a_n. \end{aligned}$$

On the other hand,

$$\frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \geq \sqrt{2}, \quad \text{(Why is this true?)}$$

so $a_{n+1} \geq \sqrt{2}$ for all $n \geq 1$ and $a_1 > \sqrt{2}$ is given.
Hence, $\{a_n\}_{n=1}^{\infty}$ is a monotonically decreasing sequence, bounded below by $\sqrt{2}$. By Theorem 3, it converges.
Exercise 1. What do you think is the limit of the above sequence (Refer to the supplement to Tutorial 1)?

Cauchy sequences

As we have seen, it is not easy to tell whether a sequence converges or not because we have to first guess what the limit might be, and then try and prove that the sequence actually converges to this limit. For a monotonic sequence, we have a criterion, but what about more general sequences?

There is another very useful notion which allows us to decide whether the sequence converges by looking only at the elements of the sequence itself. We describe this below.
Definition: A sequence a_n in $\mathbb{R}$ is said to be a Cauchy sequence if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon,$$

for all $m, n > N$.

Cauchy sequences: the theorem

Theorem 4: Every Cauchy sequence in $\mathbb{R}$ converges to a limit in $\mathbb{R}$.

Remark 1: One can now check the convergence of a sequence just by looking at the sequence itself!

Remark 2: One can easily check the converse:

Theorem 5: Every convergent sequence is Cauchy.
Remark 3: Remember that when we defined sequences we defined them to be functions from $\mathbb{N}$ to X, for any set X. So far we have only considered $X = \mathbb{R}$, but as we said earlier we can take other sets, for instance, susbets of $\mathbb{R}$. For instance, if we take $X = \mathbb{R} \setminus 0$, Theorem 4 is not valid. The sequence $1/n$ is a Cauchy sequence in this X but obviously does not converge in X. If we take $X = \mathbb{Q}$, the example given in 1.5.(i) ($a_{n+1} = (a_n + 2/a_n)/2$) is a Cauchy sequence in $\mathbb{Q}$ which does not converge in $\mathbb{Q}$. Thus Theorem 4 is really a theorem about real numbers.

The completeness of $\mathbb{R}$

A set in which every Cauchy sequence converges (to a limit which is also in the set) is called a a complete set. Thus Theorem 4 is sometimes rewritten as

Theorem 4': The real numbers are complete.

You will see other examples of complete sets in MA 110.

Sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$

Most of our definitions for sequences in $\mathbb{R}$ are actually valid for sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$. Indeed, the only thing we really need to define the limit is the notion of distance. Thus if we replace the modulus function $|\ |$ on $\mathbb{R}$ by the distance functions in $\mathbb{R}^2$ and $\mathbb{R}^3$ all the definitions of convergent sequences and Cauchy sequences remain the same.

For instance, a sequence $a(n) = (x_n, y_n)$ in $\mathbb{R}^2$ is said to converge to a point $l = (l_1, l_2)$ (in $\mathbb{R}^2$) if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sqrt{(x_n - l_1)^2 + (y_n - l_2)^2} < \epsilon$$

whenver $n > N$. A similar defintion can be made in $\mathbb{R}^3$ using the distance function on $\mathbb{R}^3$. Some of our earlier rules will make sense and some won't.

Infinite series - a more rigourous treatment

Let us recall what we mean when we write

$$a + ar + ar^2 + \ldots = \frac{a}{1-r}.$$

Another way of writing the same expression is

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

The precise meaning is the following. Form the partial sums

$$s_n = \sum_{k=0}^{n} ar^k.$$

These partial sums $s_1, s_2, \ldots s_n, \ldots$ form a sequence and by $\sum_{k=0}^{\infty} ar^k = a/(1-r)$, we mean $\lim_{n\to\infty} s_n = a/1-r$.
So when we speak of the sum of an infinite series, what we really mean is the limit of its partial sums.

Convergence of the geometric series

So to justify our formula we should show that $\lim_{n\to\infty} s_n = a/1-r$, that is, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| s_n - \frac{a}{1-r} \right| < \epsilon,$$

for all $n > N$.

In other words we need to show that

$$\left| \frac{a(1-r^{n+1})}{1-r} - \frac{a}{1-r} \right| = \left| \frac{ar^{n+1}}{1-r} \right| < \epsilon$$

if n is chosen large enough.

But $\lim_{n\to\infty} r^n = 0$, so there exists N such that $r^{n+1} < (1-r)\epsilon/a$ for all $n > N$, so for this N, if $n > N$,

$$\left| s_n - \frac{a}{1-r} \right| < \epsilon.$$

This shows that the geometric series converges to the given expression.

Limits of functions of a real variable

Since we have already defined the limit of a sequence rigourously, it will not be hard to define the limit of a real valued function $f : (a, b) \to \mathbb{R}$.

Definition: A function $f : (a, b) \to \mathbb{R}$ is said to tend to (or converge to) a limit l at a point $x_0 \in [a, b]$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - l| < \epsilon$$

for all $x \in (a, b)$ such that $0 < |x - x_0| < \delta$.In this case, we write

$$\lim_{x \to x_0} f(x) = l,$$

or $f(x) \to l$ as $x \to x_0$ which we read as "$f(x)$" tends to l as x tends to x_0". This is just the rigourous way of saying that the

distance between $f(x)$ and l can be made as small as one pleases by making the distance between x and x_0 sufficiently small.

A subtle point and the rules for limits

Notice that in the definition above, the point x_0 can be one of the end points a or b.

Thus the limit of a function may exist even if the function is not defined at that point.
The rules and formulæ for limits of functions are the same as those for sequence and can be proved in almost exactly the same way. If $\lim_{x\to x_0} f(x) = l_1$ and $\lim_{x\to x_0} g(x) = l_2$, then

1. $\lim_{x\to x_0} f(x) \pm g(x) = l_1 \pm l_2$.
2. $\lim_{x\to x_0} f(x)g(x) = l_1 l_2$.
3. $\lim_{x\to x_0} f(x)/g(x) = l_1/l_2$. provided $l_2 \neq 0$

As before, implicit in the formulæ is the fact that if the limits on the left hand side exist. We prove the first rule below.

The proof of the addition formula for limits

Proof: We first show that $\lim_{x\to x_0} f(x) + g(x) = l_1 + l_2$. Let $\epsilon > 0$ be arbitrary.
Since $\lim_{x\to x_0} f(x) = l_1$ and $\lim_{x\to x_0} g(x) = l_2$, there exist δ_1, δ_2 such that

$$|f(x) - l_1| < \frac{\epsilon}{2} \quad \text{and} \quad |g(x) - l_2| < \frac{\epsilon}{2}$$

whenever $0 < |x - x_0| < \delta_1$ and $0 < |x - x_0| < \delta_2$. If we choose $\delta = \min\{\delta_1, \delta_2\}$ and if $0 < |x - x_0| < \delta$ then both the above inequalities hold. Thus, if $|x - x_0| < \delta$, then

$$\begin{aligned} |f(x) + g(x) - (l_1 + l_2)| &= |f(x) - l_1 + g(x) - l_2| \\ &\leq |f(x) - l_1| + |g(x) - l_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

which is what we needed to prove. If we replace $g(x)$ by $-g(x)$ we get the second part of the first rule. $\square$

The Sandwich Theorem(s) for limits of functions

Theorem 5: As $x \to x_0$, if $f(x) \to l_1$, $g(x) \to l_2$ and $h(x) \to l_3$ for functions f, g, h on some interval (a, b) such that
$f(x) \leq g(x) \leq h(x)$ for all $x \in (a, b)$, then

$$l_1 \leq l_2 \leq l_3.$$

As before, we have a second version.

Theorem 6: Suppose $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = l$ and If $g(x)$ is a function satisfying $f(x) \leq g(x) \leq h(x)$ for all $x \in (a, b)$, then $g(x)$ converges to a limit as $x \to x_0$ and

$$\lim_{x \to x_0} g(x) = l$$

Once again, note that we do not assume that $g(x)$ converges to a limit in this version of the theorem - we get the convergence of $g(x)$ for free .

Some examples

Let us look at Exercise 1.11. We will use this exercise to explore a few subtle points.
Let $c \in [a, b]$ and $f, g : (a, b) \to \mathbb{R}$ be such that $\lim_{x \to c} f(x) = 0$.
Prove or disprove the following statements.

(i) $\lim_{x \to c}[f(x)g(x)] = 0$.

(ii) $\lim_{x \to c}[f(x)g(x)] = 0$, if g is bounded.($g(x)$ is said to be bounded on (a, b) if there exists $M > 0$ such that $|g(x)| < M$ for all $x \in (a, b)$.

doubt

(iii) $\lim_{x \to c}[f(x)g(x)] = 0$, if $\lim_{x \to c} g(x)$ exists.

Before getting into proofs, let us guess whether the statements above are true or false.

(i) false

(ii) true

(iii) true.

(i) Notice that $g(x)$ is not given to be bounded - if this was not obvious before, you should suspect that such a condition is needed after looking at part (ii). So the most natural thing to do is to look for a counter-example to this statement by taking $g(x)$ to be an unbounded function. What is the simplest example of an unbounded function $g(x)$ on an open interval?

How about $g(x) = \frac{1}{x}$ on $(0, 1)$?
What would a candidate for $f(x)$ be - what is the simplest example of a function $f(x)$ which tends to 0 for some value of c in $[0, 1]$.

$f(x) = x$, and $c = 0$ is a pretty simple candidate.

Clearly $\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0} 1 = 1 \neq 0$, which shows that (i) is not true in general.

Exercise 1: Can you find a counter-example to (i) with c in (a, b) (that is, c should not be one of the end points)? (Hint: Can you find an unbounded function on a closed interval $[a, b]$?)

Let us move to part (ii).
Suppose $g(x)$ is bounded on (a, b). This means that there is some real number $M > 0$ such that $|g(x)| < M$. Let $\epsilon > 0$. We would like to show that

$$|f(x)g(x) - 0| = |f(x)g(x)| < \epsilon,$$

if $0 < |x - c| < \delta$ for some $\delta > 0$.
Since $\lim_{x \to c} f(x) = 0$, there exists $\delta > 0$ such that $|f(x)| < \epsilon / M$ for all $|x - c| < \delta$. It follows that

$$|f(x)g(x)| = |f(x)||g(x)| < \frac{\epsilon}{M} \cdot M = \epsilon$$

for all $0 < |x - c| < \delta$, and this is what we had to show.

Part (iii) follows immediately from the product rule, but can one deduce part (iii) from (ii) instead?
Hint: Think back to the lemma on convergent sequences that we proved in Lecture 1: Every convergent sequence is bounded. What is the analogue for functions which converge to a limit at some point? Indeed, you can easily show the following

doubt

Lemma 7: Let $f : (a, b) \to \mathbb{R}$ be a function such that $\lim_{x \to c} f(x)$ exists for some $c \in [a, b]$. If $c \in (a, b)$, there exists an (open) interval $I = (c - \eta, c + \eta) \subset (a, b)$ such that $f(x)$ is bounded on I. If $c = a$, then there is an open interval $I_1 = (a, a + \eta)$ such that $f(x)$ is bounded on I_1. Similarly if $c = b$, there exists an open interval $I_2 = (b - \eta, b)$ such that $f(x)$ is bounded on I_2.

The proof of the lemma above is almost the same as the the lemma for convergent sequences. Basically, replace "N" by "δ" in the proof.

If one applies the Lemma above to $g(x)$, we see that $g(x)$ is bounded in some (possibly) smaller interval $(0, \eta)$. Now apply part (ii) to this interval to deduce that (iii) is true.