MA 105: Review of Taylor series and integration

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General Advice

1. Concentrate on understanding the statements of the theorems. You will not be asked to reproduce long proofs.
2. When trying to understand a definition, make sure you know plenty of examples.
3. When trying to understand a theorem, make sure you know counter-examples to the conclusion of the theorem when you drop some of the hypotheses.
4. In general, the statement of the theorem is more important than its proof. And examples are more important than theorems!

Taylor's theorem

Taylor's theorem: Know how to compute the Taylor polynomials. Know the form of the remainder term. Recall that there are smooth functions for which the Taylor series about a point converges but does not converge to the function ($e^{-1/x}$).

Theorem 19: Let I be an open interval and suppose that $[a, b] \subset I$. Suppose that $f \in \mathcal{C}^n(I)$ ($n \geq 0$) and suppose that $f^{(n)}$ is differentiable on I. Then there exists $c \in (a, b)$ such that

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1},$$

where

$$P_n(b) = f(a) + f^{(1)}(a)(b-a) + \frac{f^{(2)}(a)}{2!}(b-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(b-a)^n.$$

denotes the Taylor polynomial of degree n at a.

The term $R_n(b) = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$ is sometimes called the remainder term.

Taylor series

If we have a smooth or $\mathcal{C}^\infty$-function f (that is, a function for which all derivatives exist) we can form its Taylor series about the point a.

$$T_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

For sufficiently nice functions f,

1. $T_f(x)$ will be a convergent series for every vaule of x in some interval $(a-r, a+r)$, $r > 0$.
2. $T_f(x) = f(x)$ for all $x \in (a-r, a+r)$.

In general, neither 1. nor 2. need hold.

Exercise 1.

Find the first three terms of the Taylor series of the function $1/x^2$ at 1.

Solution: If the Taylor series of the function f at $x = a$ is $\sum_{n=0}^{\infty} b_n(x-a)^n$, then $b_n = \frac{f^{(n)}(a)}{n!}$.

Using these notations, for $f(x) = 1/x^2$ and $a = 1$, we get $b_0 = 1$, $b_1 = -2$ and $b_2 = 3$.

Exercise 2.

True or False (justify): Let $P_2(x)$ denote the Taylor polynomial of degree 2 about the point $a = 0$ for the function $f(x) = \log(1 + x)$. The inequality $|f(x) - P_2(x)| < 0.05$ holds for all x in $[0, 1/2]$.

True.

Applying Taylor's theorem for $n = 2$, we know that $R_2(x) = \frac{f^{(3)}(c)}{3!}(x - a)^3$ for some point $c \in [0, 1/2]$.

In our case $f(x) = \log(1 + x)$, so $f^{(3)}(c) = 2/(1 + c)^3$. If $c \in (0, 1/2)$, clearly $2/(1 + c)^3 < 2$.

If $x \in [0, 1/2]$ the maximum value that $x^3/3!$ can take is when $x = 1/2$.

Thus $0 \leq R_2(x) < 1/24 < 0.05$.

Integration

Remember what partitions and tagged partitions are.

Recall the definitions of the (Darboux) lower sums, upper sums, lower integrals, upper integrals and Riemann sums.

Learn all three definitions of the Riemann integral.

Basic fact: Bounded functions on closed intervals with at most a finite number of discontinuities are Riemann/Darboux integrable.

The Fundamental Theorem of calculus.

The basic definitions

Definition: Given a closed interval $[a, b]$, a partition P of $[a, b]$ is simply a collections of points

$$P = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}.$$

If we are also giventags $t = \{t_0, \ldots, t_{n-1}\}$ with $t_j \in I_j = [x_{j-1}, x_j]$, the pair (P, t) is called a tagged partition.

Definition: A partition P' is called a refinement of a partition P if $P \subseteq P'$ (every point of P is also a point of P').

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \quad \text{and} \quad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \quad 1 \le i \le n$$

Defintion: We define the (Darboux) Lower sum and Upper sum by

$$L(f, P) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}) \text{ and } U(f, P) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}).$$

We always have $L(f, P_1) \le U(f, P_2)$ for any two partitions P_1 and P_2 of $[a, b]$ and if $P \subseteq P'$,

$$L(f, P) \le L(f, P') \le U(f, P') \le U(f, P).$$

The Darboux integrals

We now define the lower Darboux integral of f by

$$L(f) = \sup\{L(f, P)\colon P \text{ is a partition of } [a, b]\},$$

where the supremum is taken over all partitions of $[a, b]$.

and similarly the upper Darboux integral of f by

$$U(f) = \inf\{U(f, P)\colon P \text{ is a partition of } [a, b]\},$$

and again the infimum is over all partitions of $[a, b]$.

If $L(f) = U(f)$, then we say that f is Darboux-integrable and define

$$\int_a^b f(t)\, dt := U(f) = L(f).$$

This common value of the two integrals is called the Darboux integral.

Riemann integration

For a tagged partition (P, t), we define the Riemann sum to be

$$R(f, P, t) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}).$$

Definition 1: A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|R(f, P, t) - R| < \epsilon,$$

whenever $\|P\| < \delta$. In this case R is called the Riemann integral of the function f on the interval $[a, b]$.

Definition 2: A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$ there exists a partition P such that for every tagged refinement (P', t') of P ~~with~~ $\cancel{\|P'\| < \delta}$,

$$|R(f, P', t') - R| < \epsilon.$$

(Recall that the struck out portion was initially part of the definition above, but later I pointed out that it was not necessary.)

Exercise 3

3. For the function $f(x) = 3x^2$ and the partition

$$P_n = \left\{0 < \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n-1}{n} < 1\right\}$$

of $[0, 1]$ find the lower sum, $L(f, P_n)$, upper sum, $U(f, P_n)$. Compute $\sup_n L(f, P_n)$ and $\inf_n U(f, P_n)$.

Solution:

$$L(f, P_n) = \sum_{i=0}^{n-1} 3\frac{i^2}{n^2}\frac{1}{n} = 3\frac{1}{n^3}\frac{n(n-1)(2n-1)}{6}$$

So

$$L(f, P_n) = \frac{2n^2 - 3n + 1}{2n^2} \quad \text{and} \quad U(f, P_n) = \frac{2n^2 + 3n + 1}{2n^2}$$

and

$$\sup_n L(f, P_n) = 1 \quad \text{and} \quad \inf_n U(f, P_n) = 1.$$

Exercise 4

Evaluate $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{i^2+n^2}$ by identifying it as a Riemann sum for a certain continuous function on a certain interval and with respect to a certain partition.

Solution: We observe that

$$\sum_{i=1}^{n}\frac{n}{i^2+n^2}=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{(i/n)^2+1}.$$

Thus, the given sum is the Riemann sum for the function $\dfrac{1}{x^2+1}$ over the interval $[0,1]$ with respect to the partition $0<\dfrac{1}{n}<\dfrac{2}{n}<\cdots<\dfrac{n-1}{n}<1.$

Since the function $1/(1+x^2)$ is continuous on $[0,1]$, it is Riemann integrable.

Hence the limit of the given sum is $\displaystyle\int_0^1\frac{1}{x^2+1}dx=\pi/4.$

The Fundamental Theorem of Calculus

If f is continuous in $[a, b]$ and differentiable in (a, b), and we define

$$F(x) = \int_a^x f(t)dt,$$

then $F'(x) = f(x)$. This is (part I of) the Fundamental Theorem of Calculus (FTC).

Suppose $G(x) = \int_{p(x)}^{q(x)} f(t)dt$, where the range of $p(x)$ and $q(x)$ lies in $[a, b]$. We can write

$$G(x) = \int_a^{q(x)} f(t)dt - \int_a^{p(x)} f(t)dt = G_1(x) + G_2(x).$$

We can write $G_1(x) = (F \circ q)(x)$ and $G_2(x) = (F \circ p)(x)$. Using the Chain rule and the FTC, we see that

$$G'(x) = f(q(x))q'(x) - f(p(x))p'(x).$$

More exercises:

1. Write the n-th term of the Taylor series around 0 for
(a) $\sin \pi x$, (b) $\cosh x$, (c) $(1-x^2)^{1/2}$, (d) $(1+x)^{1/4}$.

2. For the functions $f(x)$ and the point a given below write down the Taylor series around a.
(a) $f(x) = \sin x, a = \pi/2$, (b) $f(x) = x - x^3, a = -2$,
(c) $f(x) = \sqrt{x}, a = 16$, (d) $f(x) = \ln x, a = 2$.

3. In 2. (d) above, write down an expression for $R_n(x)$. Show that $\lim_{n\to\infty} R_n(x) = 0$ if $x \in (2, 4)$. Is there a larger interval in which the Taylor series converges?

4. Let $P_n(x)$ denote the Taylor polynomial of degree n for the function e^x around the point 0. Find an n so that $P_n(-1)$ calculates e^{-1} to five decimal place accuracy?

5. Use 1. (c) and term by term integration of power series to get the Taylor series for $\arcsin x$.

Still more exercises

1. For $f(x) = x$, the partition P_n, and the tags $T_n = \{1/2n, 3/2n, \ldots, 2n-1/2n\}$, write down $R(f, P_n, T_n)$.

2. Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^k}{n^{k+1}}$ by identifying it as a Riemann sum for a certain continuous function on a certain interval and with respect to a certain partition.

3. Find the derivative of $h(x) = \int_0^{\sqrt{x}} \frac{z^2}{z^4+1} dz$.

4. Find the derivative of $g(x) = \int_0^{\tan x} (\sqrt{t + \sqrt{t}}) dt$.

5. True or false:

$$\left(\frac{\pi}{22}\right)\cos\left(\frac{\pi}{22}\right) + \left(\frac{2\pi}{11}\right)\cos\left(\frac{5\pi}{22}\right) + \left(\frac{2\pi}{11}\right)\cos\left(\frac{9\pi}{22}\right) + \left(\frac{\pi}{22}\right)\cos\left(\frac{5\pi}{11}\right) < \left(\frac{\pi}{26}\right) + \left(\frac{3\pi}{13}\right)\cos\left(\frac{\pi}{26}\right) + \left(\frac{3\pi}{13}\right)\cos\left(\frac{7\pi}{26}\right)$$