CS 105: DIC on Discrete Structures

Graph theory

Matchings!

Lecture 30
Oct 30 2023

Topic 3: Graph theory

Recap:

1. Basics: graphs, paths, cycles, walks, trails.
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A General Recipe

1. Consider an "interesting" problem.
2. Model it as a "special" class of graphs.
3. Characterize them using properties on vertices, cycles etc.
4. Not done in this course: Build algorithms based on the characterization, analyze and implement them!

This lecture, we will consider a new problem:

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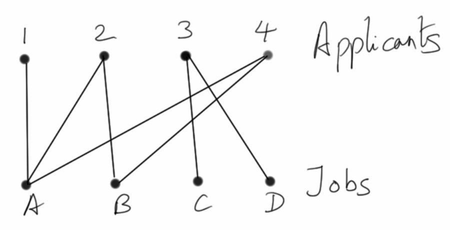

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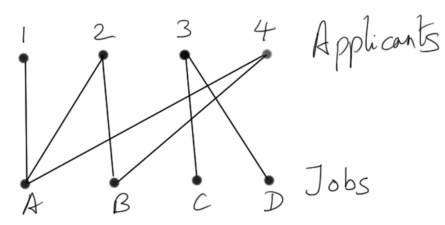

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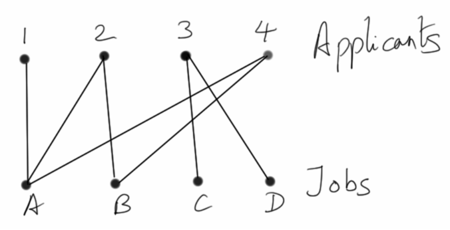

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- Another practical example: the dating scene!

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- A matching in a graph G is a set of (non-loop) edges with no shared end-points.
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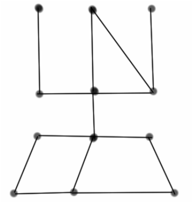

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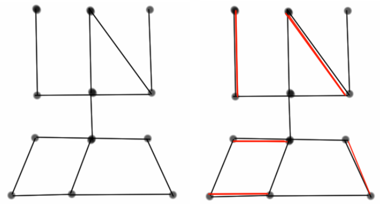

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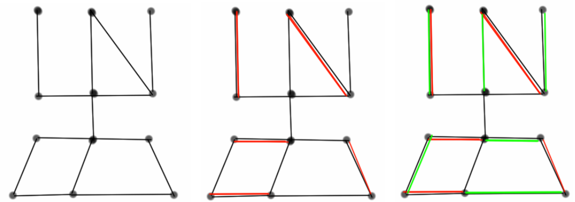

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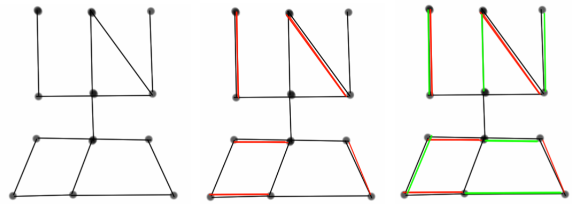

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If a matching has an M-augmenting path, then can it be a maximum matching?

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Theorem

A matching M in G is a maximum matching iff G has no M-augmenting path.