

CS 105: DIC on Discrete Structures

Graph theory

Basic terminology, Eulerian walks

Lecture 24

Oct 12 2023

Topic 3: Graph theory

Topics covered in the last three lectures

- ▶ Pigeon-Hole Principle and its extensions.
- ▶ A glimpse of Ramsey theory

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Next topic

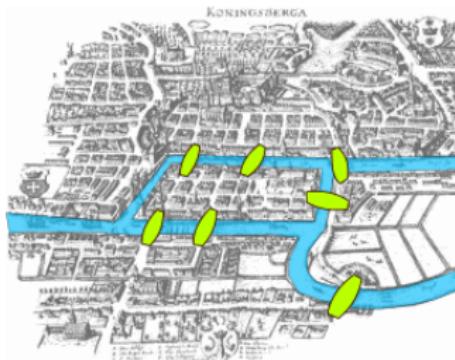
Graphs and their properties!

Topic 3: Graph theory

Textbook Reference

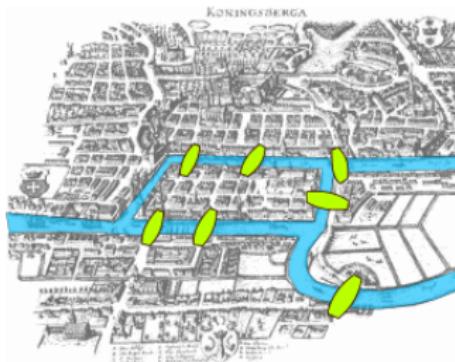
- ▶ **Introduction to Graph Theory, 2nd Ed., by Douglas West.**
- ▶ Low cost Indian edition available, published by PHI Learning Private Ltd.

Königsberg Bridge problem



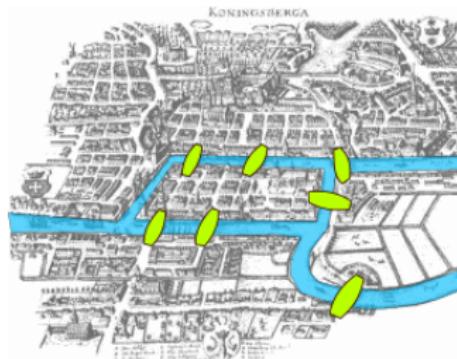
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- ▶ Find a walk from home, crossing every bridge exactly once and returning home.

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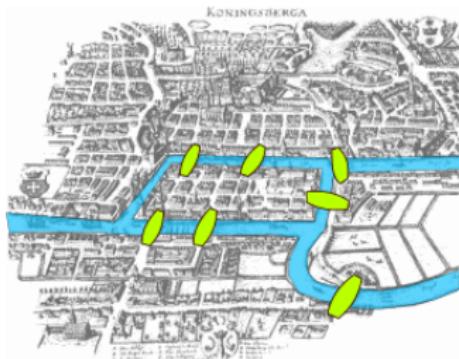
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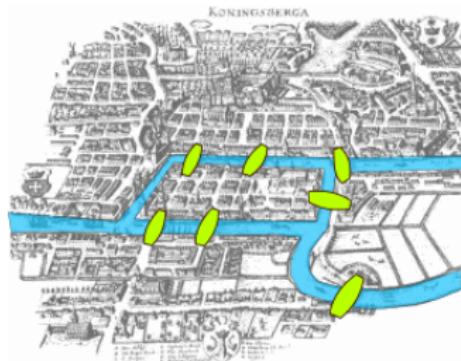
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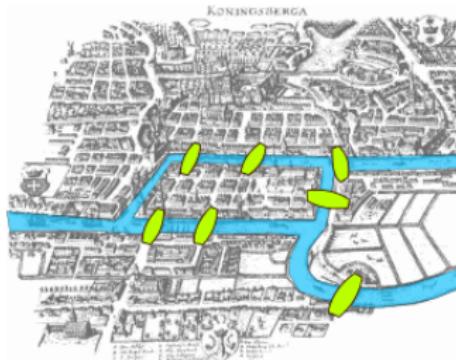
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- ▶ *“This question is so banal, but seemed to me worthy of attention in that [neither] geometry, nor algebra, nor even the art of counting was sufficient to solve it.”*

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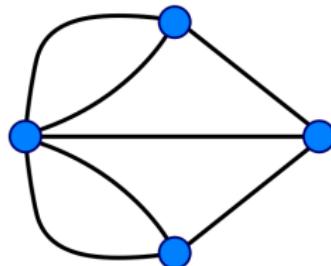
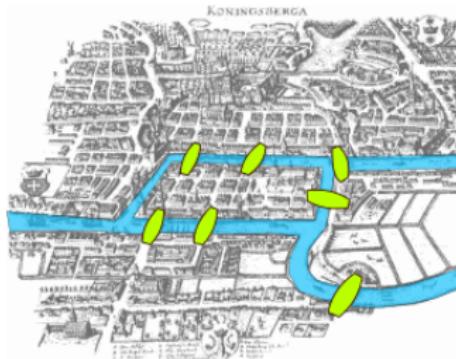


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- ▶ *“This question is so banal, but seemed to me worthy of attention in that [neither] geometry, nor algebra, nor even the art of counting was sufficient to solve it.”*
- ▶ Still, he wrote a paper showing that this is impossible!
- ▶ Thus, he “gave birth” to the area of graph theory.

Königsberg Bridge problem

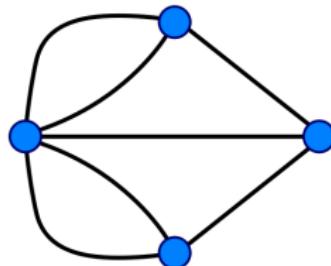
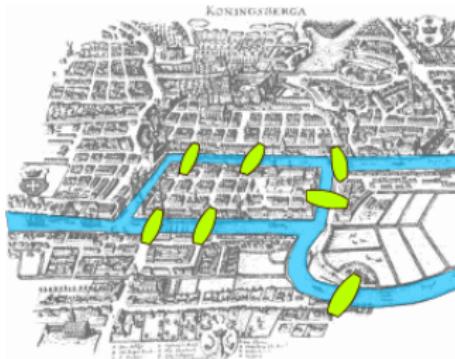


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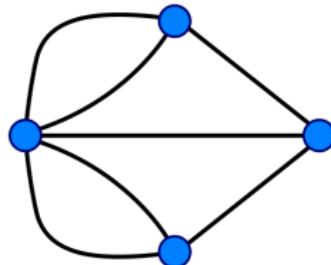
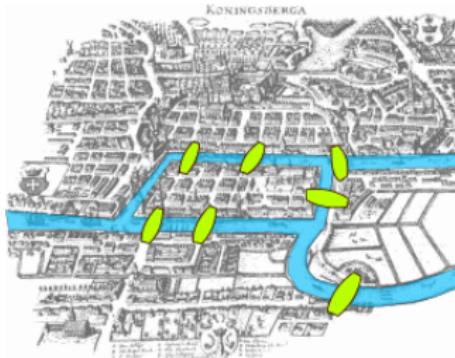
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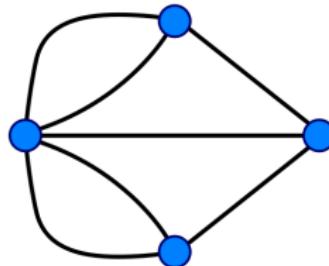
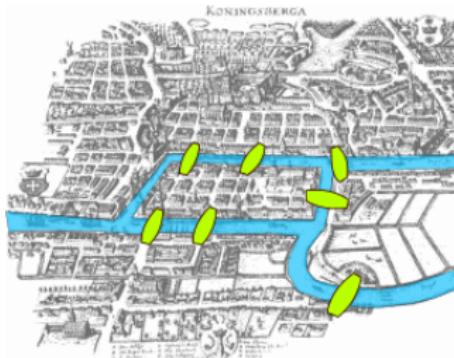
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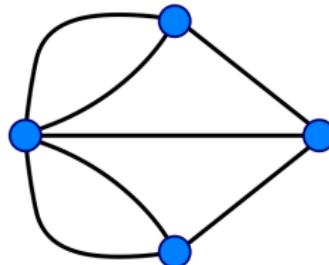
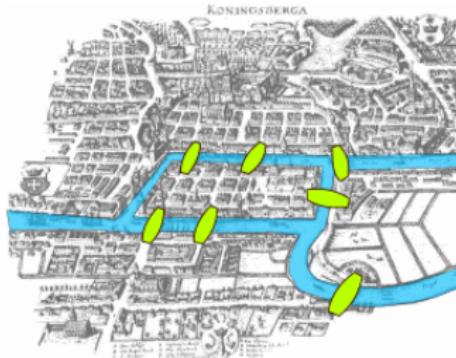
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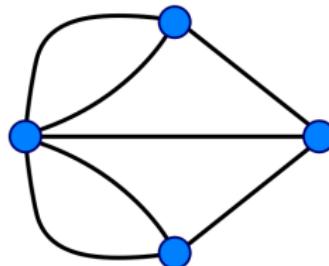
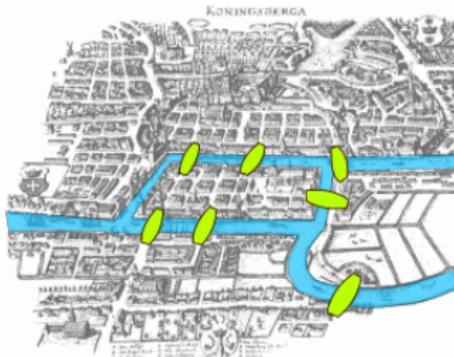
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 - ▶ So each vertex must be connected to an even no. of vertices.

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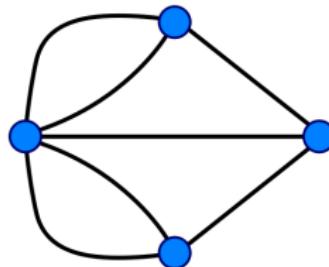
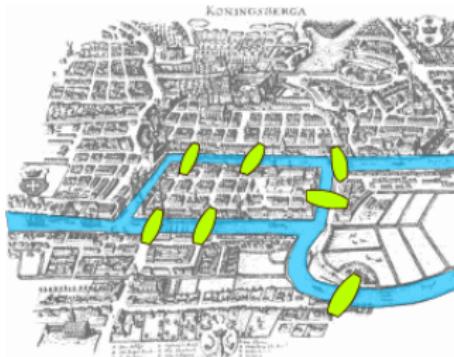
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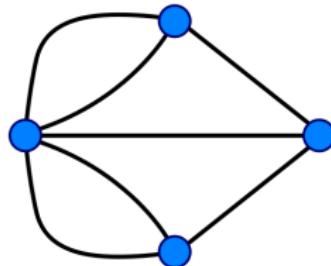
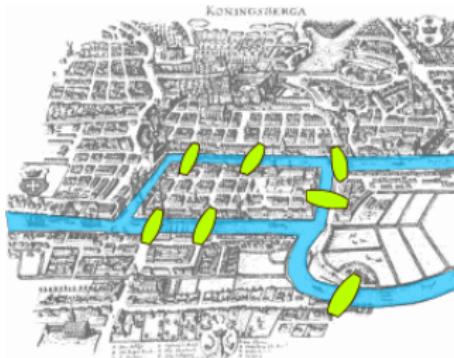
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- ▶ If every vertex is connected to an even no. of vertices in a graph, is there such a walk? This is called **Eulerian walk**.

What are graphs

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A simple graph G is a pair (V, E) of a set of vertices/nodes V and edges E between unordered pairs of vertices called end-points: $e = vu$ means that e is an edge between v and u ($u \neq v$).

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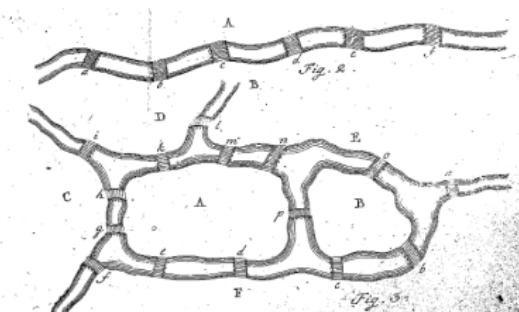
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General Definition

A graph G is a triple V, E, R where V is a set of vertices, E is a set of edges and $R \subseteq E \times V \times V$ is a relation that associates each edge with two vertices called its end-points.

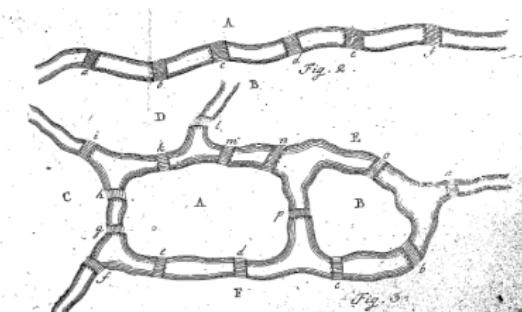
We will consider **only** finite graphs (i.e., $|V|, |E|$ are finite) and **often** deal with simple graphs.

Basic terminology



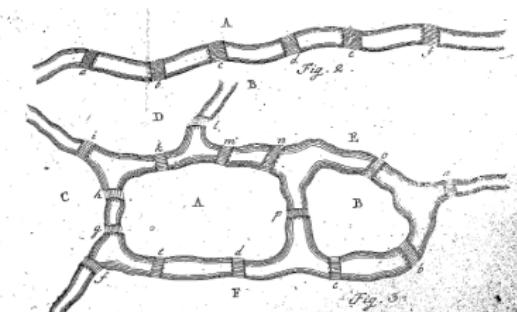
Ex. Draw the graphs!

Basic terminology



The **degree $d(v)$** of a vertex v (in an undirected loopless graph) is the number of edges incident to it, i.e., $|\{vw \in E \mid w \in V\}|$. A vertex of degree 0 is called an **isolated vertex**.

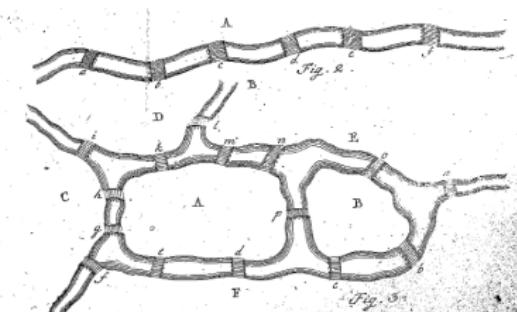
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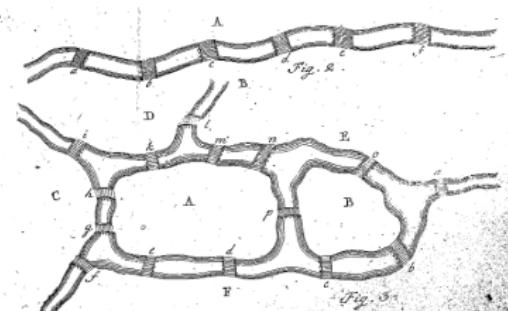
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Graph is **connected** if there is a walk between any two vertices.

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 - ▶ at the first vertex first edge is paired with last.
- ▶ Any two edges are in the same walk implies graph is connected (unless it has isolated vertices).