MA 105 Part II (IIT Bombay) Tutorial Sheet 1 : Multiple integrals

1. (a) Let $R := [0,1] \times [0,1]$ and $f(x,y) := [x] + [y] + 1$ for all $(x,y) \in R$, where $[u]$ is the greatest integer less than equal to u, for any $u \in \mathbb{R}$. Using the definition of integration over rectangles, show that f is integrable over R. Also, find its value.

   (b) Let $R := [0,1] \times [0,1]$ and $f(x,y) := (x+y)^2$ for all $(x,y) \in R$. Show that f is integrable over R and find its value using Riemann sum.

   (c) Let $R := [a,b] \times [c,d]$ be a rectangle in $\mathbb{R}^2$ and let $f : R \to \mathbb{R}$ be integrable. Show that $|f|$ is also integrable over R.

   (d) Check the integrability of the function f over $[0,1] \times [0,1]$;

$$f(x,y) := \begin{cases} 1 & \text{if both } x \text{ and } y \text{ are rational numbers,} \\ -1 & \text{otherwise.} \end{cases}$$

   What do you conclude about the integrability of $|f|$?

2. (a) Sketch the solid bounded by the surface $z = \sin y$, the planes $x = -1$, $x = 0$, $y = 0$ and $y = \frac{\pi}{2}$ and the xy plane and compute its volume.

   (b) The integral $\int\int_R \sqrt{9 - y^2}\, dxdy$, where $R = [0,3] \times [0,3]$, represents the volume of a solid. Sketch the solid and find its volume.

3. Consider the function $f : [0,1] \times [0,1] \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} 1 - 1/q & \text{if } x = p/q \quad \text{where} \quad p, q \in \mathbb{N} \quad \text{are relatively prime and } y \quad \text{is rational,} \\ 1 & \text{otherwise.} \end{cases}$$

   Show that f is integrable but the iterated integrals do not always exist.

4. Consider the function $f : [0,1] \times [0,1] \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < y < x < 1, \\ -\frac{1}{y^2} & \text{if } 0 < x < y < 1, \\ 0 & \text{otherwise} \end{cases}$$

   Is f integrable over the rectangle? Do both iterated integrals exist? If they exist, do they have the same value?

5. For the following, write an equivalent iterated integral with the order of integration reversed and verify if their values are equal:

   (a) $\displaystyle\int_0^1 \Big(\int_0^1 \log[(x+1)(y+1)]\, dx \Big)\, dy.$

   (b) $\displaystyle\int_0^1 \Big(\int_0^1 (xy)^2 \cos(x^3)\, dx \Big)\, dy.$

6. (a) Let $R = [a,b] \times [c,d]$ and $f(x,y) = \phi(x)\psi(y)$ for all $(x,y) \in R$, where ϕ is continuous on $[a,b]$ and ψ is continuous on $[c,d]$. Show that

$$\int\int_R f(x,y)\, dxdy = \Big(\int_a^b \phi(x)\, dx \Big) \Big(\int_c^d \psi(y)\, dy \Big).$$

(b) Compute $\int\int_{[1,2]\times[1,2]} x^r y^s \, dxdy$, for any given $r \geq 0$ and $s \geq 0$.

(c) Compute $\int\int_{[0,1]\times[0,1]} xye^{x+y} \, dxdy$.

7. Evaluate the following integrals:

(a) $\displaystyle\int\int_R (x+2y)^2 \, dxdy$, where $R = [-1,2] \times [0,2]$.

(b) $\displaystyle\int\int_R \left[xy + \frac{x}{y+1}\right] dxdy$, where $R = [1,4] \times [1,2]$.

8. Consider the function f over $[-1,1] \times [-1,1]$:

$$f(x,y) = \begin{cases} x+y & \text{if } \ x^2+y^2 \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Determine the set of points at which f is discontinuous. Is f integrable over $[-1,1]\times[-1,1]$?

MA 105 Part II (IIT Bombay) Tutorial Sheet 2 : Multiple integrals

1. For the following, write an equivalent iterated integral with the order of integration reversed:

   (a) $\displaystyle\int_0^1 \left[\int_1^{e^x} dy\right] dx$

   (b) $\displaystyle\int_0^1 \left[\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)dx\right] dy$

2. Evaluate the following integrals

   (a) $\displaystyle\int_0^\pi \left[\int_x^\pi \frac{\sin y}{y} dy\right] dx$

   (b) $\displaystyle\int_0^1 \left[\int_y^1 x^2 e^{xy} dx\right] dy$

   (c) $\displaystyle\int_0^2 (\tan^{-1} \pi x - \tan^{-1} x)dx.$

3. Find $\displaystyle\iint_D f(x,y)d(x,y)$, where $f(x,y) = e^{x^2}$ and D is the region bounded by the lines $y = 0,\ x = 1$ and $y = 2x$.

4. (a) Compute the volume of the solid enclosed by the ellipsoid:

   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

   where a, b, c are given real numbers.

   (b) Find the volume of the region under the graph of $f(x,y) = e^{x+y}$ over the region

   $$D := \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| \leq 1\}.$$

5. Find

   $$\lim_{r\to\infty} \iint_{D(r)} e^{-(x^2+y^2)} d(x,y),$$

   where $D(r)$ equals:

   (a) $\{(x,y) : x^2 + y^2 \leq r^2\}$.

   (b) $\{(x,y) : x^2 + y^2 \leq r^2,\ x \geq 0,\ y \geq 0\}$.

   (c) $\{(x,y) : |x| \leq r,\ |y| \leq r\}$.

   (d) $\{(x,y) : 0 \leq x \leq r,\ 0 \leq y \leq r\}$.

6. Find the volume common to the cylinders $x^2+y^2 = a^2$ and $x^2+z^2 = a^2$ using double integral over a region in the plane. (Hint: Consider the part in the first octant.)

7. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ above the region $x^2 + y^2 = 2x$ in $x - y$ plane.

8. Express the solid $D = \{(x, y, z) | \sqrt{x^2 + y^2} \leq z \leq 1\}$ as

$$\{(x, y, z) \mid a \leq x \leq b, \quad \phi_1(x) \leq y \leq \phi_2(x), \quad \xi_1(x, y) \leq z \leq \xi_2(x, y)\}.$$

9. Evaluate

$$I = \int_0^{\sqrt{2}} \left(\int_0^{\sqrt{2-x^2}} \left(\int_{x^2+y^2}^{2} xdz \right) dy \right) dx.$$

Sketch the region of integration and evaluate the integral by expressing the order of integration as $dxdydz$.

10. Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

11. Describe the solid whose volume is given by the integral

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^2 \sin \phi d\rho d\phi d\theta,$$

and evaluate the integral.

MA 105 Part II Tutorial Sheet 3 :
Change of variables, Line integrals, October 16, 2023

I Multiple integrals and change of variables

2. Using a suitable change of variables, evaluate the integral $\int\int_D y dxdy$, where D is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.
4. Use cylindrical coordinates to evaluate $\int\int\int_W (x^2 + y^2) dzdydx$, where

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid -2 \leq x \leq 2, \quad -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}, \quad \sqrt{x^2 + y^2} \leq z \leq 2\}.$$

6. Find $\iiint_F \frac{1}{(x^2 + y^2 + z^2)^{n/2}} dV$, where F is the region bounded by the spheres with center the origin and radii r and R, $0 < r < R$.
7. Evaluate the integral

$$\iint_D (x - y)^2 \sin^2(x + y) d(x, y),$$

where D is the parallelogram with vertices at $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$.

8. Let D be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Find $\iint_D dxdy$ by transforming it to $\iint_E dudv$, where $x = \frac{u}{v}$, $y = uv$, $v > 0$.
9. Using suitable change of variables, evaluate the following:

   i.
   $$I = \iiint_D (z^2x^2 + z^2y^2) dxdydz$$
   where D is the cylindrical region $x^2 + y^2 \leq 1$ bounded by $-1 \leq z \leq 1$.

   ii.
   $$I = \iiint_D \exp(x^2 + y^2 + z^2)^{3/2} dxdydz$$
   over the region enclosed by the unit sphere in $\mathbb{R}^3$.

II Vector analysis and line integrals

1. Let f, g be differentiable functions on $\mathbb{R}^2$. Show that
   A. $\nabla(fg) = f\nabla g + g\nabla f$;
   B. $\nabla f^n = nf^{n-1}\nabla f$;
   C. $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$ whenever $g \neq 0$.
2. Let $\mathbf{a}, \mathbf{b}$ be two fixed vectors, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r^2 = x^2 + y^2 + z^2$. Prove the following:
   (i) $\nabla(r^n) = nr^{n-2}\mathbf{r}$ for any integer n.
   (ii) $\mathbf{a} \cdot \nabla\left(\frac{1}{r}\right) = -\left(\frac{\mathbf{a} \cdot \mathbf{r}}{r^3}\right)$.
   (iii) $\mathbf{b} \cdot \nabla\left(\mathbf{a} \cdot \nabla\left(\frac{1}{r}\right)\right) = \frac{3(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})}{r^5} - \frac{\mathbf{a} \cdot \mathbf{b}}{r^3}$.

3. Calculate the line integral of the vector field
$$\mathbf{F}(x, y) = (x^2 - 2xy)\mathbf{i} + (y^2 - 2xy)\mathbf{j}$$
from $(-1, 1)$ to $(1, 1)$ along $y = x^2$.
4. Calculate the line integral of
$$\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} + (x - y)\mathbf{j}$$
once around the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ in the counter clockwise direction.
Remark Often line integral of a vector field $\mathbf{F}$ along a 'geometric curve' C is represented by $\int_C \mathbf{F}.\mathbf{ds}$. A geometric curve C is a set of points in the plane or in the space that can be traversed by a parametrized path in the given direction.
To evaluate $\int_C \mathbf{F}.\mathbf{ds}$, choose a convenient parametrization $\mathbf{c}$ of C traversing C in the given direction and then
$$\int_C \mathbf{F}.\mathbf{ds} := \int_{\mathbf{c}} \mathbf{F}.\mathbf{ds}.$$
'$\oint_C$' means the line integral over a closed curve C.
5. Calculate the value of the line integral
$$\oint_C \frac{(x+y)dx - (x-y)dy}{x^2 + y^2}$$
where C is the curve $x^2 + y^2 = a^2$ traversed once in the counter clockwise direction.
6. Calculate
$$\oint_C ydx + zdy + xdz$$
where C is the intersection of two surfaces $z = xy$ and $x^2+y^2 = 1$ traversed once in a direction that appears counter clockwise when viewed from high above the xy-plane.
7. Let the curve C be given by $x^2 + y^2 = 1, z = 0$. Let $\mathbf{c}_1$ be a parametrization defined by $\mathbf{c}_1(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$. Find the line integral of $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$ along this curve. Also find the line integral along the curve parametrized by $\mathbf{c}_2(t) = (\cos t, -\sin t)$, for $t \in [0, \pi]$.
8. Show that a constant force field does zero work on a particle that moves once uniformly around the circle: $x^2+y^2 = 1$. Is this also true for a force field $\mathbf{F}(x, y, z) = \alpha(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$, for some constant α.
9. Let $C : x^2 + y^2 = 1$. Find
$$\oint_C \text{grad } (x^2 - y^2) \cdot \mathbf{ds}.$$
10. Evaluate
$$\int_C \text{grad } (x^2 - y^2) \cdot \mathbf{ds},$$
where C is $y = x^3$, joining $(0, 0)$ and $(2, 8)$.
11. Compute the line integral
$$\oint_C \frac{dx + dy}{|x| + |y|}$$
where C is the square with vertices $(1, 0), (0, 1)$, $(-1, 0)$ and $(0, -1)$ traversed once in the counter clockwise direction.
12. A force $F = xy\mathbf{i} + x^6y^2\mathbf{j}$ moves a particle from $(0, 0)$ onto the line $x = 1$ along $y = ax^b$ where $a, b > 0$. If the work done is independent of b find the value of a.