MA 105 Part II Week 1

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Welcome to MA 105 Part II!

There is a total of 50 marks to be earned in this part of the course. The following breakup is tentative*.

Quiz	10 marks
Final	40 marks
Total	50 marks

Academic Honesty: It is obligatory on your part to be honest and not to violate the academic integrity of the Institute. Any form of academic dishonesty, including, but not limited to cheating, plagiarism, submitting as one's own the same or substantially similar work of another, will not be tolerated, and will invite the harshest possible penalties as per institute norms.

Disclaimer: The instructors reserve the right to modify the schedules and procedures announced in this syllabus. Any such change will be announced in the class. It is the responsibility of the student to keep informed of such details.

Course objectives

Calculus can be broadly divided into two parts: differential calculus and integral calculus. This course will be focused on integral calculus of several variables and vector analysis, mainly,

- double and triple integration, Jacobians and change of variables.
- parametrisation of curves, vector fields, line integrals.
- parametrisation of surfaces and surface integrals.
- gradient of functions, divergence and curl of vector fields, theorems of Green, Gauss, and Stokes and their applications.

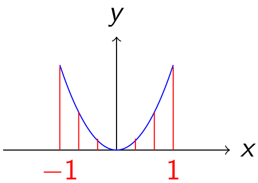
References:

1. [MR] Debanjana Mitra and Ravi Raghunathan, *Lecture slides for MA 105*.
2. [MTW] J.E Marsden, A. J. Tromba, A. Weinstein. *Basic Multivariable Calculus*, South Asian Edition, Springer (2017).
3. [CJ] R. Courant and F. John, *Introduction to Calculus and Analysis, Volumes 1 and 2*, Springer-Verlag (1989).
4. [Apo] T.M. Apostol, *Calculus, Volumes 1 and 2*, 2nd ed., Wiley (2007).

Recall : One variable Integration

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $a, b \in \mathbb{R}$.

- The area enclosed by the graph of a non-negative function over the region of the interval is $\int_a^b f(t)\, dt$.



The area in the figure on the left is $\int_{-1}^{1} x^2\, dx = 2/3$.

- A partition of the interval $[a, b]$ is a set of points $P = \{a = x_0 \le x_1 \le \dots x_n = b\}$ for some $n \in \mathbb{N}$.
- The lower Darboux integral and upper Darboux integral of f are $L(f) = \sup\{L(f, P) \colon P \text{ is a partition of } [a, b]\}$, and $U(f) = \inf\{U(f, P) \colon P \text{ is a partition of } [a, b]\}$, respectively.
- When $L(f) = U(f)$ then f is Darboux integrable and $\int_a^b f := L(f) = U(f)$.

- Tagged partition: partition P with a set of points $t = \{t_1, \dots t_n\}$, $t_j \in [x_{j-1}, x_j]$ for all $j = 1, \dots, n$. Define $S(f, P, t) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1})$ and define the *norm* of a partition P as $\|P\| = \max_j\{|x_j - x_{j-1}|\}, \quad 1 \leq j \leq n.$
- $f : [a, b] \to \mathbb{R}$ is said to be *Riemann integrable* if for some $S \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $|S(f, P, t) - S| < \epsilon$, whenever $\|P\| < \delta$. The Riemann integral of f is then S.
- Theorem: The Riemann integral exists iff the Darboux integral exists. Further, the two integrals are equal.
- Unlike the Darboux integral, Riemann integral can be computed as a limit: clearly advantageous in computations!
- $f : [a, b] \to \mathbb{R}$ is bounded, and continuous at all but finitely many points of $[a, b]$. Then f is Riemann integrable on $[a, b]$.
- For computing integrals, we use the Fundamental theorem of calculus. If $f : [a, b] \to \mathbb{R}$ is continuous and $f = g'$ for some continuous function $g : [a, b] \to \mathbb{R}$ which is differentiable on (a, b), then $\displaystyle\int_a^b f = g(b) - g(a)$.

Integrating functions on two variables

Any *closed, bounded rectangle R* in $\mathbb{R}^2$:

$$R = [a, b] \times [c, d],$$

the Cartesian product of two closed intervals $[a, b]$ and $[c, d]$.

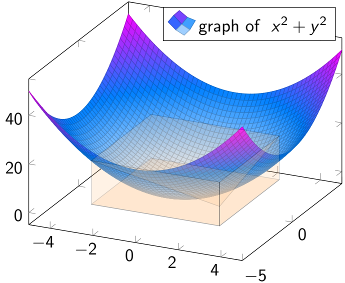
Consider a real valued function f defined on R i.e.,

$$f : R \subset \mathbb{R}^2 \to \mathbb{R}.$$

- Graph of f: The subset $\{(x, y, f(x, y)) \in \mathbb{R}^3 \mid (x, y) \in R\}$ in $\mathbb{R}^3$ is called the graph of f.
- Contour line: Fix $c \in \mathbb{R}$. Then the set $\{(x, y, c) \in \mathbb{R}^3 \mid f(x, y) = c, (x, y) \in R\}$ in $\mathbb{R}^3$ is called the contour line of f. It is the intersection of the graph of f by the horizontal plane $z = c$ in $\mathbb{R}^3$. In other words, it is the image of the c-level set of f.

In particular, let $f(x,y) = x^2 + y^2$, for all $(x,y) \in \mathbb{R}^2$. We can use contour lines to draw the graph of this function by drawing $f(x,y) = c$ for varying values of c.

We want to compute volume of the region below the graph of f over the rectangle $[-3,3] \times [-3,3]$. The volume of the figure in the shaded region is $V := \{(x,y,z) \mid (x,y) \in [-3,3] \times [-3,3], \quad 0 \le z \le f(x,y)\}$.



The integral of the non-negative function f over $[-3,3] \times [-3,3]$ can be defined as the volume V; $\int\int_{[-3,3]\times[-3,3]} f(x,y)\,dxdy :=$ Volume of V.

Example: Let $g(x, y) = \alpha$, for some non-zero constant $\alpha \in \mathbb{R}$. Then for any rectangle $[a, b] \times [c, d]$ it is easy to see that
$\int\int_{[a,b]\times[c,d]} g(x, y)\, dxdy = (b - a)(d - c)\alpha$.

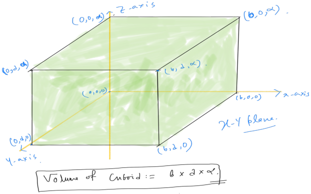


Figure: Cuboid: $[0, b] \times [0, d] \times [0, \alpha]$

Clearly for $f(x, y) = x^2 + y^2$, the computing of the volume is not that simple and we want to be able to define integral for all bounded functions instead of only non-negative ones.

Partitions for rectangles

Partition of R: A partition P of a rectangle $R = [a, b] \times [c, d]$ is the Cartesian product of a partition P_1 of $[a, b]$ and a partition P_2 of $[c, d]$. Let

$$P_1 = \{x_0, x_1, \cdots x_m\}, \quad \text{with} \quad a = x_0 < x_1 < x_2 < \cdots < x_m = b\},$$

$$P_2 = \{y_0, y_1, \cdots y_n\}, \quad \text{with} \quad c = y_0 < y_1 < y_2 < \cdots < y_n = d\},$$

and $P = P_1 \times P_2$ be defined by

$$P = \{(x_i, y_j) \mid i \in \{0, 1, \cdots m\}, \quad j \in \{0, 1, \cdots, n\}\}.$$

The points of P divide the rectangle R into *nm non-overlapping sub-rectangles* denoted by

$$R_{ij} := [x_i, x_{i+1}] \times [y_j, y_{j+1}], \quad \forall\, i = 0, \cdots m-1, \quad j = 1, \cdots, n-1.$$

Note $R = \cup_{i,j} R_{ij}$.

Partitions for rectangles: continued

Example: Let P_1 denote a partition of $[-3,3]$ into 3 equal intervals and P_2 the partition of $[-3,3]$ into 2 equal intervals. Describe the rectangles in the partition $P_1 \times P_2$.

Note $P_1 = \{-3,-1,1,3\}$ and $P_2 = \{-3,0,3\}$ and thus $[-3,3]\times[-3,3]$ is devided into 6 sub-rectangles $R_{00} = [-3,-1]\times[-3,0]$,
$R_{01} = [-3,-1]\times[0,3]$, $R_{10} = [-1,1]\times[-3,0]$, $R_{11} = [-1,1]\times[0,3]$,
$R_{20} = [1,3]\times[-3,0]$, $R_{21} = [1,3]\times[0,3]$.

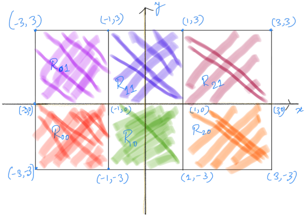


Figure: Partition of $[-3,3]\times[-3,3]$

Partitions for rectangles: continued

The area of each R_{ij}*:* $\Delta_{ij} := (x_{i+1} - x_i) \times (y_{j+1} - y_j)$, for all $i = 0, \cdots, m-1$, $j = 0, \cdots, n-1$.

Norm of the partition P*:*
$\|P\| := \max\{(x_{i+1} - x_i), (y_{j+1} - y_j) \mid i = 0, \cdots, m-1, \quad j = 0, \cdots, n-1\}$.

Question: Why do we not define the norm by
$\max\{(x_{i+1} - x_i) \times (y_{j+1} - y_j) \mid i = 0, \cdots, m-1, \quad j = 0, \cdots, n-1\}$?

Darboux integral

Let $f : R \to \mathbb{R}$ be a bounded function where R is a rectangle . Let $m(f) = \inf\{f(x,y) \mid (x,y) \in R\}$, $M(f) = \sup\{f(x,y) \mid (x,y) \in R\}$. For all $i = 0, 1, \cdots, m-1, \quad j = 0, 1, \cdots, n-1$, let,
$m_{ij}(f) := \inf\{f(x,y) \mid (x,y) \in R_{ij}\}$, and
$M_{ij}(f) := \sup\{f(x,y) \mid (x,y) \in R_{ij}\}$.

Lower double sum: $L(f,P) := \displaystyle\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} m_{ij}(f)\Delta_{ij}$, and

Upper double sum: $U(f,P) := \displaystyle\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} M_{ij}(f)\Delta_{ij}$,

Note that for any partition P of R

$$m(f)(b-a)(d-c) \le L(f,P) \le U(f,P) \le M(f)(b-a)(d-c).$$

Lower Darboux integral: $L(f) := \sup\{L(f,P) \mid P \text{ is any partition of } R\}$.

Upper Darboux integral: $U(f) := \inf\{U(f,P) \mid P \text{ is any partition of } R\}$.

Note $L(f) \le U(f)$.

Darboux integral continued

Definition (Darboux integral)

A bounded function $f : R \to \mathbb{R}$ is said to be *Darboux integrable* if $L(f) = U(f)$. The Double integral of f is the common value $U(f) = L(f)$ and is denoted by

$$\int\int_R f, \quad \text{or} \quad \int\int_R f(x,y)dA, \quad \text{or} \quad \int\int_R f(x,y)dxdy.$$

Theorem (Riemann condition)

Let $f : R \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if for every $\epsilon > 0$ there is a partition P_ϵ of R such that

$$|U(f, P_\epsilon) - L(f, P_\epsilon)| < \epsilon.$$

Recall the Dirichlet function for one variable:

$$f(x) := \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \cap [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Is f integrable over $[0,1]$? Ans. No!

Exercise: Check the integrability of bivariate Dirichlet function over $[0,1] \times [0,1]$

$$f(x,y) := \begin{cases} 1 & \text{if both } x \text{ and } y \text{ are rational numbers,} \\ 0 & \text{otherwise.} \end{cases}$$

Riemann Integral

Riemann integral: Let P be any partition of a rectangle $R = [a, b] \times [c, d]$. We define a tagged partition (P, t) where

$$t = \{t_{ij} \mid t_{ij} \in R_{ij}, \quad i = 0, 1, \ldots m-1, \quad j = 0, 1, \ldots n-1\}.$$

The *Riemann sum* of f associated to (P, t) is defined by

$$S(f, P, t) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(t_{ij})\Delta_{ij} \text{ where, } \Delta_{ij} = (x_{i+1} - x_i)(y_{j+1} - y_j)$$

Definition (Riemann integral)

A bounded function $f : R \to \mathbb{R}$ is said to be *Riemann integrable* if there exists a real number S such that for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|S(f, P, t) - S| < \epsilon,$$

for every tagged partition (P, t) satisfying $\|P\| < \delta$ and S is the value of Riemann integral of f.

Riemann Integral contd.

- For any rectangle $R \subseteq \mathbb{R}^2$, let $f : R \to \mathbb{R}^2$ be bounded. The Darboux integrability and Riemann integrability are equivalent.
- A function $f : R \to \mathbb{R}^2$ is called integrable on R if (Darboux or) Riemann integrability condition holds on R.

Examples: Let $R = [a, b] \times [c, d]$.

• The constant function is integrable.

• The projection functions $p_1(x, y) = x$ and $p_2(x, y) = y$ are both integrable on any rectangle $R \subset \mathbb{R}^2$. Why?

• Let $f : R \to \mathbb{R}$ be defined as $f(x, y) = \phi(x)$ where $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous function. Is f integrable? What is $\int\int_R f \, dxdy$?

Regular partitions

However, the current definition isn't truly helpful in making computations. We define *regular* partitions.

The regular partition of R of order any $n \in \mathbb{N}$ is defined by $x_0 = a$ and $y_0 = c$, and for $i = 0, 1, \cdots, n-1, \quad j = 0, 1, \cdots, n-1$,

$$x_{i+1} = x_i + \frac{b-a}{n}, \quad y_{j+1} = y_j + \frac{d-c}{n}.$$

We take $t = \{t_{ij} \in R_{ij} \mid i, j \in \{0, 1, \cdots, n-1\}\}$ any arbitrary tag.

To check the integrability of a function f, it is enough to consider a sequence of regular partitions P_n of R.

Theorem

A bounded function $f : R \to \mathbb{R}$ is Riemann integrable if and only if the Riemann sum

$$S(f, P_n, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(t_{ij}) \Delta_{ij},$$

tends to the same limit $S \in \mathbb{R}$ as $n \to \infty$, for any choice of tag t.

An Example

Example: Let $f(x,y) = x^2 + y^2$. Is it a continuous function on $\mathbb{R}^2$?
Ans. Yes!
Suppose the function is integrable on $[0,1] \times [0,1]$. Compute the integral using the theorem.

Let $R = [0,1] \times [0,1]$ and P_n be a regular partition. Then for tag $t = \{(\frac{i}{n}, \frac{j}{n}) \mid i = 0, \dots n-1, j = 0, \dots, n-1\}$,

$$S(f, P_n, t) = (\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (\frac{i}{n})^2 + (\frac{j}{n})^2) \frac{1}{n^2}.$$

Compute $\lim_{n \to \infty} S(f, P_n, t)$. How would you go about it?

Conventions

Based on our definition, we make the following convention: Let $a, b, c, d \in \mathbb{R}$

- If $a = b$ or $c = d$, then $\int\int_{[a,b]\times[c,d]} f(x,y)dxdy := 0$.
- If $a < b$ and $c < d$:
  $\int\int_{[b,a]\times[c,d]} f(x,y)dxdy := -\int\int_{[a,b]\times[c,d]} f(x,y)dxdy$,
  $\int\int_{[a,b]\times[d,c]} f(x,y)dxdy := -\int\int_{[a,b]\times[c,d]} f(x,y)dxdy$,
  $\int\int_{[b,a]\times[d,c]} f(x,y)dxdy := \int\int_{[a,b]\times[c,d]} f(x,y)dxdy$.

Properties of integrals over rectangles

Domain Additivity Property: $f : R \to \mathbb{R}$ is a bounded function. Partition R into finitely many non-overlapping sub-rectangles. Then f is integrable on R iff it is integrable on each sub-rectangle. When it exists, the integral of f on R is the sum of the integrals of f on the sub-rectangles.

Algebraic properties :

Let $R := [a, b] \times [c, d]$. Let f and g are integrable on R.

- If f is defined as $f(x, y) = \alpha \in \mathbb{R}$ for all $(x, y) \in \mathbb{R}^2$ then $\int\int_R f = \alpha A(R)$ where A is the area of R.
- The function $f + g$ is integrable, and $\int\int_R f + g = \int\int_R f + \int\int_R g$.
- For all $\alpha \in \mathbb{R}$, αf is integrable and $\int\int_R \alpha f = \alpha \int\int_R f$.
- If $f(x, y) \leq g(x, y)$ for all $(x, y) \in R$, then $\int\int_R f \leq \int\int_R g$.
- $|f|$ is integrable and $|\int\int_R f| \leq \int\int_R |f|$.
- The function $f.g$ is integrable.
- If $\frac{1}{f}$ is well defined and bounded on R, then $\frac{1}{f}$ is integrable on R.

All these follow by applying the definition and properties of limits. An immediate consequence is that all polynomial functions are integrable.

Calculating integrals

While we have now given a reasonable definition of the integral for functions of two variables, actually calculating integrals using the definition proves much too complicated. After all, even in the one variable method, integrating any but the simplest functions using the definition of the Riemann integral is more or less impossible. Instead, we proved the fundamental theorem of calculus and used the fact that the integral and the antiderivative were the same in order to evaluate the integrals of various standard functions.

The key idea is to reduce integration in two variables to integrating in one variable (but doing it twice, that is, iteratedly).
In fact, this idea goes back all the way to Archimedes, but was perhaps first extensively used by Cavalieri, a student of Galileo (note that this was before Newton and Liebniz developed the Fundamental Theorem of Calculus).

Iterated Integrals

For $f : R \to \mathbb{R}$ we can define the iterated integrals as follows. We can first define functions of y and x respectively as follows, provided the integrands below are Riemann integrable as functions of one variable:

$$h(y) = \int_a^b f(x,y)dx \quad \text{and} \quad g(x) = \int_c^d f(x,y)dy.$$

We then consider the integrals

$$\int_c^d h(y)dy = \int_c^d \left[\int_a^b f(x,y)dx\right] dy \quad \text{and}$$

$$\int_a^b g(x)dx = \int_a^b \left[\int_c^d f(x,y)dy\right] dx.$$

These integrals (if they exist) are called iterated integrals

If you think about it, it is not obvious that either of the integrals above should be equal to the double integral, but, in fact, they will be in the most common situations we encounter.

Fubini theorem and Iterated integrals

Theorem

Let $R := [a, b] \times [c, d]$ and $f : R \to \mathbb{R}$ be integrable. Let I denote the integral of f on R.

1. *If for each $x \in [a, b]$, the Riemann integral $\int_c^d f(x, y)\, dy$ exists, then the iterated integral $\int_a^b (\int_c^d f(x, y)\, dy)\, dx$ exists and is equal to I.*
2. *If for each $y \in [c, d]$, the Riemann integral $\int_a^b f(x, y)\, dx$ exists, then the iterated integral $\int_c^d (\int_a^b f(x, y)\, dx)\, dy$ exists and is equal to I.*

As a consequence, if f is integrable on R and if both iterated integrals exist in 1. and 2. in above theorem, then

$$\int_a^b \Big(\int_c^d f(x, y)\, dy \Big)\, dx = I = \int_c^d \Big(\int_a^b f(x, y)\, dx \Big)\, dy.$$

Sketch of the proof

The proof is using Riemann condition. Since f is double integrable over R, for any given $\epsilon > 0$, there exists a partition $P_\epsilon = \{(x_i, y_j) \mid i = 0, 1, \cdots k-1, \quad j = 0, \cdots n-1\}$ of R such that

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

Assume for each fixed $x \in [a, b]$, the Riemann integral $\int_c^d f(x, y)\, dy$ exists. Define

$$A(x) := \int_c^d f(x, y)\, dy, \quad \forall\, x \in [a, b].$$

Claim: The function A is integrable over $[a, b]$. Note that $m(f)(d-c) \leq A(x) \leq M(f)(d-c)$ for all $x \in [a, b]$ and hence A is bounded. Also by domain additivity, $A(x) = \sum_{j=0}^{n-1} \int_{y_j}^{y_{j+1}} f(x, y)\, dy$, for all $x \in [a, b]$.

Thus for each fixed $i \in \{0, \cdots k-1\}$, for $x \in [x_i, x_{i+1}]$, we obtain

$$\sum_{j=0}^{n-1} m_{ij}(f)(y_{j+1} - y_j) \leq A(x) \leq \sum_{j=0}^{n-1} M_{ij}(f)(y_{j+1} - y_j).$$

Sketch of the proof continued

Denoting $m_i(A) := \inf\{A(x) \mid x \in [x_i, x_{i+1}]\}$ and $M_i(A) := \sup\{A(x) \mid x \in [x_i, x_{i+1}]\}$, we have

$$\sum_{j=0}^{n-1} m_{ij}(f)(y_{j+1} - y_j) \leq m_i(A) \leq M_i(A) \leq \sum_{j=0}^{n-1} M_{ij}(f)(y_{j+1} - y_j).$$

Multiplying by $(x_{i+1} - x_i)$ and summing over $i = 0, \ldots, k-1$, we obtain

$$L(f, P_\epsilon) \leq \sum_{i=0}^{k-1} m_i(A)(x_{i+1} - x_i) \leq \sum_{i=0}^{k-1} M_i(A)(x_{i+1} - x_i) \leq U(f, P_\epsilon).$$

and it yields that there exists a partition $P_1 := \{x_0, \cdots, x_{k-1}\}$ of $[a, b]$ such that

$$U(A, P_1) - L(A, P_1) < \epsilon.$$

Thus the function of A is integrable and

$$\int\int_R f\,dx\,dy = \int_a^b A(x)\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx.$$

Remarks on Fubini's theorem

The both iterated integrals may exist but the function f may not be double integrable.

Example 1: $R := [0,1] \times [0,1]$, $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{(x^2+y^2)^3}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$

Compute both the iterated integrals. Are they same? Is f integrable?

The function f may be double integrable. But one of the iterated integrals may not exist (check Tutorial problems).

Let R be a rectangle in $\mathbb{R}^2$ and let $f : R \to \mathbb{R}$ be a continuous function. Then both iterated integrals of f exist and are equal to the double integral of f over R.

Examples:

Example : Find the integral of $f(x,y) = x^2 + y^2$ on the rectangle $[0,1] \times [0,1]$ if it exists.

Solution: Check the integrability of f using the definition. Let us now compute the integral using iterated integrals.

$$\begin{aligned}
\int\int_{[0,1]\times[0,1]} x^2 + y^2 \ dxdy &= \int_0^1 \int_0^1 x^2 + y^2 \ dxdy \\
&= \int_0^1 \frac{x^3}{3} + xy^2 \Big|_0^1 \\
&= \int_0^1 (\frac{1}{3} + y^2) \\
&= \frac{y}{3} + \frac{y^3}{3} \Big|_0^1 = \frac{2}{3}
\end{aligned}$$

Example (Marsden, Tromba and Weinstein page 288): Compute $\int\int_R \sin(x+y)dxdy$, where $R = [0,\pi] \times [0, 2\pi]$.

Solution:

$$\begin{aligned}
\int\int_R \sin(x+y)dxdy \quad &= \int_0^{2\pi} \left[\int_0^{\pi} \sin(x+y)dx\right] dy \\
&= \int_0^{2\pi} [-\cos(x+y)|_{x=0}^{\pi}]dy \\
&= \int_0^{2\pi} [\cos y - \cos(y+\pi)]dy \\
&= [\sin y - \sin(y+\pi)]|_{y=0}^{2\pi} = 0
\end{aligned}$$

Example (Marsden, Tromba and Weinstein, page 289): If D is a plate defined by $1 \leq x \leq 2, 0 \leq y \leq 1$ (measured in centimeters), and the mass density $\rho(x,y) = ye^{xy}$ grams per square centimeter. Find the mass of the plate.

Solution: The total mass of the plate is got by integrating over the rectangular region covered by D:

$$\begin{aligned} \int\int_D \rho(x,y)dxdy &= \int_0^1\int_0^2 ye^{xy}dxdy = \int_0^1 (e^{xy}|_{x=1}^2 dy \\ &= \int_0^1 (e^{2y} - e^y)dy = \frac{e^2}{2} - e + \frac{1}{2} \end{aligned}$$

Special case Let $\phi : [a,b] \to \mathbb{R}$ and $\psi : [c,d] \to \mathbb{R}$ be Riemann integrable. Define $f(x,y) := \phi(x)\psi(y)$, for all $(x,y) \in R = [a,b] \times [c,d]$. Then f is integrable on R and

$$\int\int_R f(x,y)\,dx\,dy = \Big(\int_a^b \phi(x)\,dx\Big)\Big(\int_c^d \psi(y)\,dy\Big).$$

Example Let $0 < a < b$ and $0 < c < d$ and $r \geq 0$ and $s \geq 0$. Denote $R = [a,b] \times [c,d]$. Compute $\int\int_R x^r y^s\,dx\,dy$.

Cavalieri's Principle

The volumes of two solids are equal if the areas of their corresponding cross sections are equal.

The Slice Method

Cavalieri's basic idea is that we can find the volume of a given solid by slicing it into thin cross sections, calculating the areas of the slices and then adding up these areas.

Let S be a solid and P_x be a family of planes perpendicular to the x-axis with x as x-coordinate such that
1. S lies between P_a and P_b,
2. the area of the slice of S cut by P_x is $A(x)$.
Then the volume of S is given by

$$\int_a^b A(x)dx.$$

Applying this to the solid graph of $z = f(x, y)$ above a rectangle R in the plane, we see that we get exactly the second of our iterated integrals.

Thus Cavalieri's principle is actually a generalization of the method of iterated integrals. Note that in order to apply the principle we do not require the solid to necessarily lie above a rectangular region in the plane.

Cavalieri's principle is particularly useful in computing the volumes of solids of revolution. These are obtained by taking a region B lying between the lines $x = a$ and $x = b$ on the x-axis and the graph of a function $y = f(x)$ and rotating it through an angle 2π around the x-axis.

Solids of revolution

In this case, we can easily compute the cross-sectional area $A(x)$, since each cross section is nothing but a disc. The radius of the circle is nothing but $f(x)$. Hence, the area $A(x)$ is given by

$$A(x) = \pi[f(x)]^2,$$

and the volume V of the solid is given by

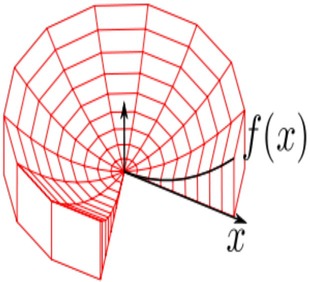
$$V = \pi \int_a^b [f(x)]^2 dx.$$

Solids of revolution may also arise by rotating the graph of a function $f(x)$. around the y-axis. In this case, we can follow the procedure above, replacing x by y and the function $f(x)$ by its inverse.

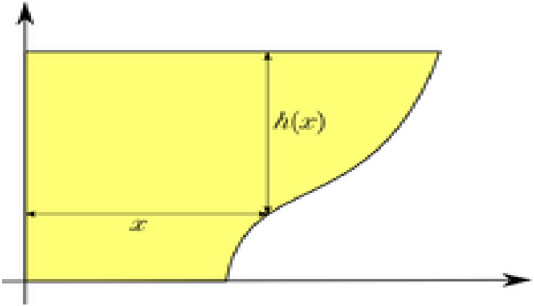
The shell method

There is another way to compute the volume of a solid of revolution obtained by rotating the graph of a function around the y-axis. It is called the shell method.

In this case, rather than slicing the solid by cross sections, we view the solid as being made of cylindrical shells.



The shell method continued



As you can see, from the picture above, the radius of the cylindrical shell above the point $(x, f(x))$ is x and the height is $h(x)$. (The point is $h(x)$ can be determined: it is constant initially and later $= f(b) - f(x)$.) Hence its surface area is $2\pi x h(x)$. To get the volume we must integrate, and this yields

$$2\pi \int_a^b x h(x) dx.$$

The washer method

This is a variant on the previous methods. Sometimes we have have calculate the volume of a solid of revolution which is hollow, where the shape of the hollow part of the solid is also given as a solid of revolution. Thus, we can think of the solid as being obtained by rotating the region that lies between the graphs of two functions $f_1(x)$ and $f_2(x)$ on an interval $[a, b]$ around an axis. If we are rotating around the x-axis, we get

$$\pi \int_a^b [f_2(x)^2 - f_1(x)^2] dx.$$

When we use the shell method, we get the formula

$$2\pi \int_a^b x[h_2(x) - h_1(x)] dx.$$

Above, we assume that $f_2(x)$ lies further away from the axis of rotation than $f_1(x)$. This method of calculating the volumes of hollow solids of revolution is called the washer method.

Exercise 4.15: A round hole of radius $\sqrt{3}$ cms is bored through the center of a solid ball of radius 2 cms. Find the volume cut out.

Solution: We may describe the desired volume as the difference of the volume of the sphere of radius 2 and a certain hollow solid of revolution.

Let us use the slice method first.

The hollow solid of revolution may be described as being obtained by rotating the region between the line $x = \sqrt{3}$ and $x = \sqrt{4 - y^2}$ around the y-axis. The two curves intersect at the points $(\sqrt{3}, \pm 1)$.

The volume of the hollow solid is given by

$$\int_{-1}^{1} \pi x^2 dy - \pi(\sqrt{3})^2 2] = 2\pi[\int_{0}^{1} (4 - y^2) dy - 3] = \frac{4}{3}\pi.$$

The volume of the sphere is $\frac{32}{3}\pi$. Hence the required volume is

$$\frac{32}{3}\pi - \frac{4}{3}\pi = \frac{28}{3}\pi.$$

We could also use the shell method to solve this problem. In the case we will get

$$\begin{aligned} 32\pi/3 - \int_{\sqrt{3}}^{2} 2\pi x(2y)dx \quad &= 32\pi/3 - 4\pi \int_{\sqrt{3}}^{2} x\sqrt{4-x^2}dx \\ &= 32\pi/3 - 4\pi(1/3) = 28\pi/3 \end{aligned}$$

Existence of integrals on $R = [a, b] \times [c, d]$ - Part I

All our statements so far depend on f being integrable on R. Is there any characterisation to determine if f is integrable?

Let $f : R \to \mathbb{R}$ be bounded. 'f is monotonic in each of two variables' means that for each fixed x, $f(x, y)$ is a monotonic function in y variable and similarly, for each fixed y, $f(x, y)$ is a monotonic function in x variable.

Theorem

If f is bounded and monotonic in each of two variables, then f is integrable on R.

Again the proof follows by using Riemann condition.

Example: Let $f(x, y) := [x + y]$, for all $(x, y) \in R$, where $[u]$ means the greatest integer less than equal to u, for any $u \in \mathbb{R}$. Since f is monotonic in each of two variables, f is integrable on R.

However, the previous condition is not that common.

Surely what worked in one variable should work here. In fact, a proof similar to the case of one variable will show the following theorem.

Theorem

If a function $f : R \to \mathbb{R}$ is bounded and continuous on R except possibly finitely many points in R, then f is integrable on R.

Example. Let $R := [-1, 1] \times [-1, 1]$,

$$f(x, y) = \begin{cases} \frac{xy}{(x^2+y^2)}, & (x, y) \in R, \quad (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

What are points of discontinuity for f on R?

In the one variable case, we saw that a bounded function with at most a finite number of discontinuities on a closed bounded interval is Riemann integrable.

The reason that a finite number of discontinuities do not matter is that points have length zero. What might be the analogous result in two variables?

In other words what sets have "zero area"?

A bounded subset E of $\mathbb{R}^2$ has 'zero area' if for every $\epsilon > 0$, there are finitely many rectangles whose union contains E and the sum of whose areas is less than ϵ.

It turns out graph of a continuous function, that is, set of the form $\{(x, \phi(x)) \mid x \in [a, b]\}$ for a continuous function $\phi : [a, b] \to [c, d]$ has 'zero area' or has *content zero.*

Theorem

If a function f is bounded and continuous on a rectangle $R = [a, b] \times [c, d]$ except possibly along a finite number of graphs of continuous functions, then f is integrable on R.

Example: Let $R = [0, 1] \times [0, 1]$ and

$$f(x, y) = \begin{cases} 1, & 0 \le x < y, \quad y \in [0, 1], \\ 0, & y \le x \le 1, \quad y \in [0, 1]. \end{cases}$$

Is f integrable over R?

Theorem

(Slightly more general) Given a rectangle R and a bounded function $f : R \to \mathbb{R}$, the function is integrable over R if the points of discontinuity of f is a set of 'content zero'.

However the converse of the above statement is not true. There are integrable functions whose points of discontinuity is not a set of 'content zero'. (Check Tutorial)

Counter example: Bivariate Thomae function: $f : [0,1] \times [0,1] \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} 1, & \text{if} \quad x = 0, \quad y \in \mathbb{Q} \cap [0,1], \\ \frac{1}{q}, & x, y \in \mathbb{Q} \cap [0,1] \quad \text{and} \quad x = \frac{p}{q}, \\ & \qquad\qquad p, q \in \mathbb{N} \quad \text{are relatively prime}, \\ 0, & \text{otherwise.} \end{cases}$$

MA 105 Calculus II

Week 2

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1. Integrals over any bounded region in $\mathbb{R}^2$
2. Evaluating integrals over Elementary regions
3. The integral in polar coordinate
4. The mean value theorem for double integrals
5. Triple integral

Integrals over any bounded region in $\mathbb{R}^2$

So far we have learnt to integrate bounded functions on any rectangle in $\mathbb{R}^2$.

Let D be any bounded subset (not necessarily rectangle) of $\mathbb{R}^2$.

How to define integral of $f : D \to \mathbb{R}$ on D?

Remedy: If D is a bounded subset of $\mathbb{R}^2$, then there exists a rectangle R in $\mathbb{R}^2$ containing D, i.e., $D \subset R$. Why?
Since D is a bounded subset of R^2, there exists $a > 0$ such that any $(x, y) \in D$ satisfies $x^2 + y^2 < a^2$, i.e, $D \subset B_a = \{(x, y) \mid x^2 + y^2 \leq a^2\}$.

Note $B_a \subset [-a, a] \times [-a, a]$

Then the rectangle $R := [-a, a] \times [-a, a]$ contains D.

Extend f from D to R by defining

$$f^*(x,y) := \begin{cases} f(x,y), & (x,y) \in D, \\ 0, & (x,y) \notin D. \end{cases}.$$

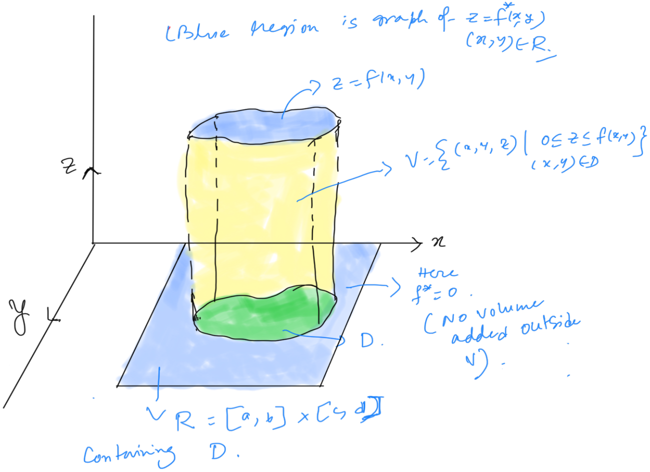
Definition

The function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be integrable on bounded $D \subset \mathbb{R}^2$, if f^* is integrable on R and the integral of f on D is defined by

$$\int\int_D f(x,y)\,dx\,dy := \int\int_R f^*(x,y)\,dx\,dy.$$

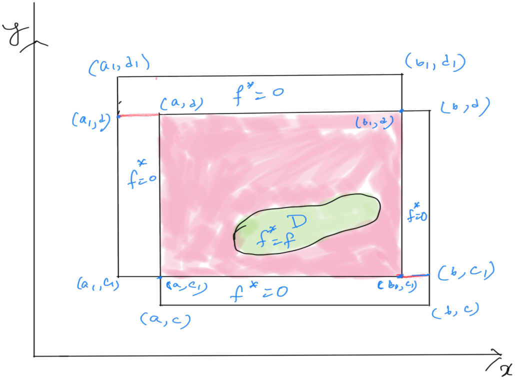
- If $f \geq 0$ on $D \subset \mathbb{R}^2$ and f is integrable on D, then the double integral of f on D is the volume of the solid that lies above D in the x-y plane and below the graph of the surface $z = f(x,y)$ for all $(x,y) \in D$.

$\int\int_D f$ = volume of V



Independent of choice of rectangle

- The choice of rectangle R containing D is not unique.
- But the value of the integral of f on D does not depend on the choice of the rectangle R containing D.
- Use the additivity property of integrals on rectangle and note that only 'zero' is getting added outside D.



Properties of Integrals over bounded sets in $\mathbb{R}^2$

Let D be a bounded subset of $\mathbb{R}^2$. Let $f : D \to \mathbb{R}$ be an integrable function.

- The algebraic properties for integrals on any bounded set D in $\mathbb{R}^2$ hold similarly to those of the case of integrals on rectangle.

Domain additivity property: Let $D \subseteq \mathbb{R}^2$ be a bounded set. Let $D_1, D_2 \subseteq D$ such that $D = D_1 \cup D_2$. Let $f : D \to \mathbb{R}^2$ be a bounded function. If f is integrable over D_1 and D_2 and $D_1 \cap D_2$ has content zero then f is integrable on D and

$$\int\int_D f = \int\int_{D_1} f + \int\int_{D_2} f.$$

Boundary of a bounded region

Let $D \subseteq \mathbb{R}^2$ be a bounded set. A point in the boundary of D is one which has a sequence in D and a sequence in $\mathbb{R}^2 - D$ converging to it. The set of boundary points of D is denoted by ∂D.

Example. $D = \{(x, y) \mid x^2 + y^2 \leq r^2\}$. The boundary of D, $\partial D = \{(x, y) \mid x^2 + y^2 = r^2\}$.

Example. $R = [a, b] \times [c, d]$. The boundary of rectangle R, $\partial R = \{(a, y) \in \mathbb{R}^2 \mid c \leq y \leq d\} \cup \{(b, y) \in \mathbb{R}^2 \mid c \leq y \leq d\} \cup \{(x, c) \in \mathbb{R}^2 \mid a \leq x \leq b\} \cup \{(x, d) \in \mathbb{R}^2 \mid a \leq x \leq b\}$.

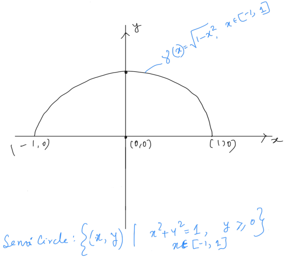
What is the boundary of the set $S = \{(x, y) \mid x, y \in \mathbb{Q}\}$? $\partial S = \mathbb{R}^2$.

Therefore for $f : D \to \mathbb{R}$ to be integrable we need ∂D to be content zero and same should be true for the points of discontinuity of f on D.

Path and Curve

Convention : A *path* γ in $\mathbb{R}^2$ (or $\mathbb{R}^3$) will mean a continuous function $\gamma : [a, b] \to \mathbb{R}^2$ (or $\gamma : [a, b] \to \mathbb{R}^3$) for $a, b \in \mathbb{R}$. It is said to be *closed* if $\gamma(a) = \gamma(b)$.

By a *curve* γ we mean the image of a path γ in $\mathbb{R}^2$ (or $\mathbb{R}^3$).



Existence of Integrals over bounded sets in $\mathbb{R}^2$

Theorem

Let $D \subset \mathbb{R}^2$ be a bounded set whose boundary ∂D is given by finitely many continous closed curve then any bounded and continuous function $f : D \to \mathbb{R}$ is integrable over D.

Example. Let $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$ and $f(x, y) = x^2 + y^2, \quad \forall (x, y) \in D$. Then f is integrable over D.

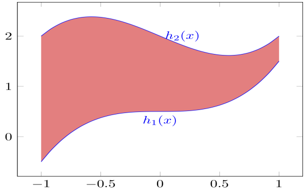
A slightly more general theorem is as follows:
Let D be a bounded set in $\mathbb{R}^2$ such that ∂D is of content zero. Let $f : D \to \mathbb{R}$ be a bounded function whose points of discontinuity have 'content zero'. Then f is integrable over D.

Elementary region: Type 1

Let $h_1, h_2 : [a, b] \to \mathbb{R}$ be two continuous functions such that $h_1(x) \leq h_2(x)$ for all $x \in [a, b]$. Consider the set of points

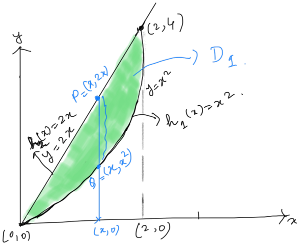
$$D_1 = \{(x, y) \,|\, a \leq x \leq b \text{ and } h_1(x) \leq y \leq h_2(x)\}$$

Such a region is said to be of *Type 1* and for every $x \in \mathbb{R}$ vertical cross-section of D_1 is an interval.



Type 1 contd.

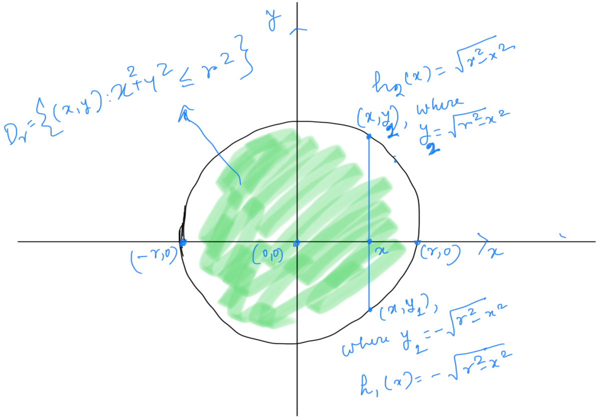
Example I: $D_1 = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 2, \quad x^2 \le y \le 2x\}$. Here for all $x \in [0, 2]$, $h_1(x) = x^2$ and $h_2(x) = 2x$. Note $h_1(x) \le h_2(x)$ for $x \in [0, 2]$.



Example II: The closed disc D_r of radius r around the origin,

$$D_r := \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\}.$$

Take $h_1(x) = -\sqrt{r^2 - x^2}$ and $h_2(x) = \sqrt{r^2 - x^2}$. We see that D_r is of Type 1.



Integrability on Type 1 region

For Type 1, when $D_1 = \{(x, y) \,|\, a \le x \le b \text{ and } h_1(x) \le y \le h_2(x)\}$, the boundary

$$\begin{aligned} \partial D_1 = \{(a, y) \mid h_1(a) \le y \le h_2(a)\} \cup \{(b, y) \mid h_1(b) \le y \le h_2(b)\} \\ \cup\{(x, h_1(x)) \mid a \le x \le b\} \cup \{(x, h_2(x)) \mid a \le x \le b\} \end{aligned}.$$

The region D_1 is bounded by continuous curves (the straight lines $x = a$ and $x = b$ and the graphs of the curves $y = h_1(x)$ and $y = h_2(x)$).

Thus ∂D_1 is of 'content zero' in $\mathbb{R}^2$.

Hence any continuous function defined on D_1 is integrable over the elementary region D_1.

Evaluating integrals on regions of Type 1

Let D be a region of Type 1 and assume that $f : D \to \mathbb{R}$ is continuous. Let $D \subset R = [\alpha, \beta] \times [\gamma, \delta]$ and let f^* be the corresponding function on R (obtained by extending f by zero).

The region D is bounded by continuous curves (the straight lines $x = a$ and $x = b$ and the graphs of the curves $y = h_1(x)$ and $y = h_2(x)$). Hence we can conclude that f^* is integrable on R. Applying Fubini's theorem on f^* we get,

$$\int\int_D f(x,y)dxdy := \int\int_R f^*(x,y)dxdy = \int_\alpha^\beta \left[\int_\gamma^\delta f^*(x,y)dy\right] dx.$$

In turn, this gives

$$\int_\alpha^\beta \left[\int_{h_1(x)}^{h_2(x)} f^*(x,y)dy\right] dx = \int_a^b \left[\int_{h_1(x)}^{h_2(x)} f(x,y)dy\right] dx,$$

since $f^*(x,y) = 0$ if $y < h_1(x)$ or $y > h_2(x)$.

Examples

Example Let $D = \{(x,y) \mid 0 \leq x \leq 2, \quad x^2 \leq y \leq 2x\}$ and $f(x,y) = x+y$. Find $\int\int_D f(x,y)dxdy$.

Ans Since D is a Type 1 region and f is continuous over D, f is integrable over D.

$$\int\int_D f(x,y)\,dxdy = \int_0^2 \Big(\int_{x^2}^{2x} (x+y)\,dy\Big)dx = \int_0^2 [xy + \frac{y^2}{2}]_{y=x^2}^{y=2x}\,dx$$
$$= \int_0^2 [2x^2 + 4\frac{x^2}{2} - x^3 - \frac{x^4}{2}]\,dx$$

Example Let $D = \{(x,y) \mid x^2+y^2 \leq 1, \quad x \geq 0, \quad y \geq 0\}$ and $f(x,y) = \sqrt{1-y^2}$. Find $\int\int_D f(x,y)dxdy$.

Ans Type 1, i.e, $D = \{(x,y) \mid 0 \leq x \leq 1, \quad 0 \leq y \leq \sqrt{1-x^2}\}$. Then

$$\int\int_D f(x,y)\,dxdy = \int_0^1 \Big(\int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\,dy\Big)\,dx.$$

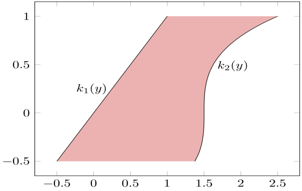
Not easy to compute!

Elementary region: Type 2

Similarly, if $k_1, k_2 : [c, d] \to \mathbb{R}$ are two continuous functions such that $k_1(y) \leq k_2(y)$, for all $y \in [c, d]$. The set of points

$$D_2 = \{(x, y) \mid c \leq y \leq d \text{ and } k_1(y) \leq x \leq k_2(y)\}$$

is called a region of Type 2 and for every $y \in \mathbb{R}$ horizontal cross-section of D_2 is an interval.



Example $D_2 = \{(x, y) \mid x^2 + y^2 \leq 1\}$. If we take $k_1(y) = -\sqrt{1 - y^2}$ and $k_2(y) = \sqrt{1 - y^2}$, we see that D_2 is of Type 2.

Evaluating integrals on regions of type 2

Note that the boundary of D_2 is of content zero in $\mathbb{R}^2$. Hence any continuous function defined on D_2 is integrable over the elementary region.

Using exactly the same reasoning as in the previous case (basically, interchanging the roles of x and y) we can obtain a formula for regions of Type 2.

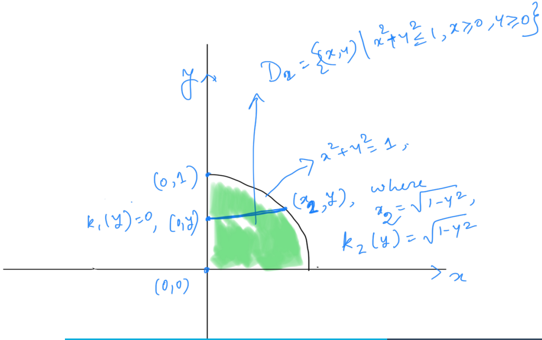
Let D be a bounded set of Type 2 in $\mathbb{R}^2$. Let $f : D \to \mathbb{R}$ be a continuous function on D. We get

$$\int\int_D f(x,y)dxdy = \int_c^d \left[\int_{k_1(y)}^{k_2(y)} f(x,y)dx\right] dy.$$

Example

Example: Let $D = \{(x,y) \mid x^2+y^2 \leq 1, x \geq 0, y \geq 0\}$. Evaluate $\int\int_D \sqrt{1-y^2}dxdy$.

Ans. $$\int\int_D \sqrt{1-y^2}dxdy = \int_0^1 \left(\int_0^{\sqrt{1-y^2}} \sqrt{1-y^2}\,dx\right) dy$$
$$= \int_0^1 [x\sqrt{1-y^2}]_{x=0}^{\sqrt{1-y^2}}\,dy = \int_0^1 (1-y^2)\,dy = \frac{2}{3}.$$



Remark

Both of these formulæ can be viewed as special cases of Cavalieri's principle when $f(x, y) \geq 0$. In the first case we are slicing by planes perpendicular to the x-axis, while in the second case, we are slicing by planes perpendicular to the y-axis.

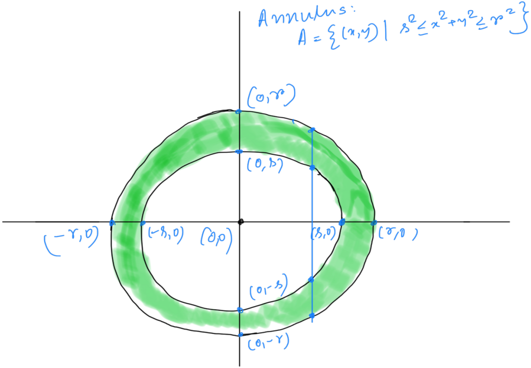
Caution! There exist bounded subsets of $\mathbb{R}^2$ which are not elementary regions; for example, *star-shaped subset* of $\mathbb{R}^2$ or an *annulus*.

Often we can write D as a union of regions of Types 1 and 2 and then we call it a region of type 3.

We could also view the disc as a region of type 3, by dividing it into four quadrants.

What about the *annulus* $A = \{(x,y) \in \mathbb{R}^2 \mid s^2 \leq x^2 + y^2 \leq r^2\}$?

Is it a type 3 region? yes



The mean value theorem for double integrals

Theorem

If D is an elementary region in $\mathbb{R}^2$, and $f : D \to \mathbb{R}$ is continuous. There exists (x_0, y_0) in D such that

$$f(x_0, y_0) = \frac{1}{A(D)} \int \int_D f(x, y) dA.$$

The proof follows using the boundedness of $f(x, y)$ and mean value theorem for continuous functions . How does one interpret the above statement geometrically?

In summary,

- If D is a bounded region in $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ is bounded, then consider any rectangle R containing the region D in $\mathbb{R}^2$ and extend f to the rectangle by 0 outside D and denote it by f^*. The integral of f over D is defined by the integral of f^* on the rectangle R.
- The above definition is consistent because the definition of integral of f on D is independent of the choice of rectangle R.
- To determine the integrability of f over region D, conditions on f and D? The boundary of D should be 'well-behaved'. The set containing points of discontinuity of f is of 'content zero'.
- Algebraic properties of integrals on D are similar to that of the integrals on rectangle.
- To evaluate the value, use Fubini's theorem.
- To apply Fubini's theorem, hardest part is to determine the lower limit and upper limit of the integration: elementary regions Type 1 and Type 2 or combination of both.

Example 1: Compute the integral of $f(x, y) = x^2 + y^2$ on $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

Can we compute this integral using iterated integrals ?

Example 2: Compute the integral of $g(x, y) = e^{x^2+y^2}$ on $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

Can we use substitution like we did in one variable?

Let us see what happens when we use polar coordinates.

Polar Coordinates

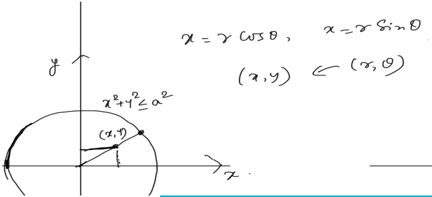
Change of variables from Cartesian coordinate system to polar coordinate system, any $(x, y) \in \mathbb{R}^2$ in Cartesian coordinate can be written as

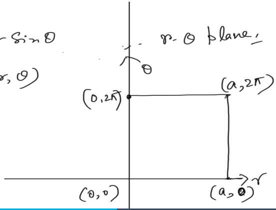
$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad r > 0, \theta \in [0, 2\pi].$$

Transformation of region under change of variables:

Ex. $D := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq a^2\}$ is transformed in polar coordinate system as a rectangle

$$D^* = \{(r, \theta) \mid 0 \leq r \leq a, \quad \theta \in [0, 2\pi]\}.$$





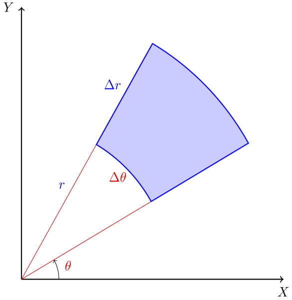
The integral in polar coordinates

Let D^* be a subset of $\mathbb{R}^2$ in polar coordinate system, such that for all $(r,\theta) \in D^*$, $(r\cos(\theta), r\sin(\theta)) \in D$, for $0 \le r \le 1$, and

$$g(r,\theta) := f(r\cos(\theta), r\sin(\theta)), \quad (r,\theta) \in D^*.$$

To integrate the function g on a domain D^* we need to cut up D^* into small rectangles, but these will be rectangles in the r-θ coordinate system.

What shape does a rectangle $[r, r+\Delta r] \times [\theta, \theta + \Delta\theta]$ represent in the x-y plane? A part of a sector of a circle.



Then we will be integrating over this sector instead of rectangle.

What is the area of this part of a sector?

Ans: It is $\frac{1}{2} \cdot [(r+\Delta r)^2 \Delta\theta - r^2 \Delta\theta] \sim r^* \Delta r \Delta\theta$, $r \leq r^* \leq r + \Delta r$.

Partitioning the region into subrectangles is equivalent to partitioning the region into parts of sectors as shown earlier.

It follows that the integral we want is approximated by a sum of the form

$$\sum_i \sum_j g(r_i^*, \theta_j^*) r_i^* \Delta r_i \Delta\theta_j,$$

where $\{(r_i^*, \theta_j^*\}$ is a tag for the partition of the "rectangle" in polar coordinates and

$$\int\int_D f(x,y)dxdy = \int\int_{D^*} f(r\cos\theta, r\sin\theta)rdrd\theta,$$

where D is the image of the region D^*.

This is the change of variable formula for polar coordinates.

Examples

Example1: Integrate $f(x,y) = x^2 + y^2$ on $D = \{(x,y) \mid x^2 + y^2 \leq 1\}$.

Solution: Let us use polar coordinates. Let

$$D^* = \{(r,\theta) \mid 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi\}.$$

Denoting $x = r\cos\theta$ and $y = r\sin\theta$, the polar coordinates will transform D^* to D and

$$g(r,\theta) = f(r\cos\theta, r\sin\theta) = r^2.$$

$$\begin{aligned}\int\int_D f(x,y)\ dxdy &= \int\int_{D^*} g(r,\theta)\ r\ drd\theta = \int\int_{[0,1]\times[0,2\pi]} r^2.r\ drd\theta \\ &= \int_0^{2\pi}\int_0^1 r^3\ drd\theta = \int_0^{2\pi} \frac{r^4}{4}\Big|_0^1\ d\theta = \frac{\pi}{2}\end{aligned}$$

Examples contd.

Example 2: Integrate $f(x,y) = e^{x^2+y^2}$ on $D = \{(x,y) \mid x^2+y^2 \leq 1\}$.
Solution: Using the same transformation as above

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we get

$$\begin{aligned} \int\int_D f(x,y)\ dxdy &= \int\int_{D^*} g(r,\theta)\ r\ drd\theta = \int\int_{[0,1]\times[0,2\pi]} e^{r^2} r\ drd\theta \\ &= \int_0^{2\pi}\int_0^1 e^{r^2} r\ drd\theta = \int_0^{2\pi} \frac{e^{r^2}}{2}\Big|_0^1\, d\theta = \pi(e-1) \end{aligned}$$

An Applicaton:The integral of the Gaussian

We would like to evaluate the following integral:

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx.$$

What does this integral mean? - so far we have only looked at Riemann integrals inside closed bounded intervals, so the end points were always finite numbers a and b.

An integral like the one above is called an improper integral. We can assign it a meaning as follows. It is defined as

$$\lim_{T \to \infty} \int_{-T}^{T} e^{-x^2} dx,$$

provided, of course, this limit exists. We will see how to evaluate this.

The most amazing trick ever

Consider
$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \cdot \int_{-\infty}^{\infty} e^{-y^2} dy.$$
We view this product as an iterated integral!
$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dxdy.$$

Now under polar coordinates, the plane is sent to the plane. Hence, we can write this as
$$\int_0^{2\pi} \left[\int_0^{\infty} e^{-r^2} rdr \right] d\theta.$$
But we can now evaluate the inner integral. Hence, we get
$$\int_0^{2\pi} \left[-\frac{1}{2} e^{-r^2} \Big|_0^{\infty} \right] d\theta = \int_0^{2\pi} \frac{1}{2} d\theta = \pi$$

The answer

Since $I^2 = \pi$, we see that $I = \sqrt{\pi}$.

Using the above result you can easily conclude that

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}.$$

The integral above arises in a number of places in mathematics - in probability, the study of the heat equation, the study of the Gamma function and in many other contexts.

There are many other ways of evaluating the integral I, but the method above is easily the cleverest.

Example Continued

Example: Evaluate $\int\int_D (3x + 4y^2)\, dxdy$, where D is the region in the upper half-plane bounded by the circled $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
Ans: The region

$$D = \{(x, y) \mid y \geq 0, \quad 1 \leq x^2 + y^2 \leq 4\}.$$

In polar coordinate, after using change of variables $x = r\cos\theta$ and $y = r\sin\theta$, in $r - \theta$ plane, D becomes

$$D^* = \{(r, \theta) \mid 1 \leq r \leq 2, \quad 0 \leq \theta \leq \pi\}.$$

$$\int\int_D (3x + 4y^2)\, dxdy = \int_{\theta=0}^{\pi}\int_{r=1}^{2} (3r\cos\theta + 4r^2\sin^2\theta) r\, drd\theta$$
$$= \int_0^{\pi} [r^3\cos\theta + r^4\sin^2\theta]_{r=1}^{2}\, d\theta = \int_0^{\pi} [7\cos\theta + 15\sin^2\theta]\, d\theta = \frac{15\pi}{2}.$$

The mean value theorem for double integrals

Theorem

If D is an elementary region in $\mathbb{R}^2$, and $f : D \to \mathbb{R}$ is continuous. There exists (x', y') in D such that

$$f(x', y') = \frac{1}{A(D)} \int \int_D f(x, y) dA.$$

The proof follows using the boundedness of $f(x, y)$ and mean value theorem for continuous functions .

Sketch of Proof Since D is closed and bounded and f is continuous, the function attains its maximum and minimum at some points $(x_0, y_0) \in D$ and $(x_1, y_1) \in D$ respectively. Since D is an elementary region, there exists a path $\gamma : [0, 1] \to \mathbb{R}^2$ such that $\gamma(0) = (x_0, y_0) \in D$ and $\gamma(1) = (x_1, y_1)$.

Now apply the intermediate value theorem function $f \circ \gamma : [0, 1] \to \mathbb{R}$.

Average value contd.

How does one interpret the above statement geometrically?

If $f(x,y) \geq 0$, $f(x_0,y_0)$, the solid region under the graph of f and over the region D is same as the volume of the region over D whose height is the average value or mean value of f defined above.i.e.,

$$f(x_0,y_0) \times A(D) = \int\int_D f(x,y)dxdy.$$

Application: Center of Mass of a thin plate: (Weighted average): Let a plate occupies a region D of the $x-y$ plane and $\rho(x,y)$ be its density at a point (x,y) in D. Let ρ be a positive continuous function on D. The the coordinate of the center of mass $(\bar{x},\bar{y})$ is given by

$$\bar{x} = \frac{\int\int_D x\rho(x,y)\,dxdy}{\int\int_D \rho(x,y)\,dxdy}, \quad \bar{y} = \frac{\int\int_D y\rho(x,y)\,dxdy}{\int\int_D \rho(x,y)\,dxdy}.$$

Note that for $\rho \equiv 1$, $\bar{x}$ is the average of $f(x,y) = x$ over the region D and $\bar{y}$ is the average of $g(x,y) = y$ over the region D.

Generalising integration for $n \geq 3$

Recall our definition of Darboux integrals and Riemann integrals. Both these definitions have an analogue in dimensions $n \geq 3$.

In this course, we only extend these ideas to functions on 3 variables. Note we already cannot imagine the graph of a function of 3 variables and much of the geometry is lost.

As an exercise you can think about which of the following definitions are specific to $n = 3$ and which can be generalized further.

If we have a bounded function $f : B = [a, b] \times [c, d] \times [e, f] \to \mathbb{R}$, we can integrate it over this rectangular cuboid (which we often refer to as a cuboid.) We divide the rectangular cuboid into smaller ones B_{ijk}, making sure that the length, breadth and height of the subcuboids are all small.

Integrals over rectangular cuboids

In particular, we can use the regular partition of order n to obtain the Riemann sum

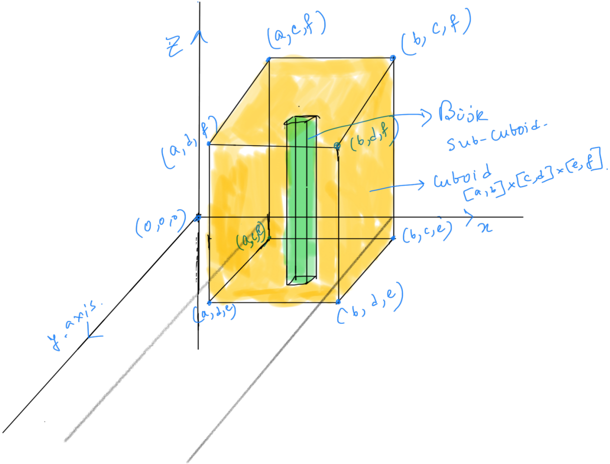
$$S(f, P_n, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} f(t_{ijk}) \Delta B_{ijk},$$

where ΔB_{ijk} is the volume of B_{ijk}, and $t = \{t_{ijk} \in B_{ijk}\}$ is an arbitrary tag.

As before we say that f is integrable if $\lim_{n \to \infty} S(f, P_n, t)$ converges to some fixed $S \in \mathbb{R}$ for any choice of tag t. The value of this limit is denoted by

$$\iiint_B f dV, \iiint_B f(x, y, z) dV \quad \text{or} \quad \iiint_B f(x, y, z) dxdydz.$$

All the theorems for integrals over rectangles go through for integrals over rectangular cuboids.



Z
(a,c,f)
(b,c,f)
(a,d,f)
(b,d,f)
Brick
sub-cuboid.
cuboid
$[a,b]\times[c,d]\times[e,f]$.
$(0,0,0)$
(a,c,e)
(b,c,e)
x
y-axis.
(a,d,e)
(b,d,e)

Integrating over bounded regions B in $\mathbb{R}^3$

First, if $f : B \subset \mathbb{R}^3 \to \mathbb{R}$ is bounded and continuous in B, except possibly on (a finite union of) graphs of continuous functions of the form $z = a(x, y)$, $y = b(x, z)$ and $x = c(y, z)$, then it is integrable.

This allows us to define the integral of (say) a continuous function on any bounded region B whose boundary is a set of content zero in $\mathbb{R}^3$. Let B^* be a cuboid enclosing the bounded region and $f^* : B^* \to \mathbb{R}$ be defined as f on B and 0 elsewhere.

Then integral of f over B exists if integral of f^* over B^* exists and

$$\iiint_{B^*} f^* = \iiint_B f.$$

Once we have defined the triple integral in this way, it remains to evaluate it.

Evaluating triple integrals: Fubini's Theorem

Fubini's Theorem can be generalized - that is, triple integrals can usually be expressed as iterated integrals, this time by integrating functions of a single variable three times.

Let f be integrable on the cuboid B. Then any iterated integral that exists is equal to the triple integral; i.e.,

$$\iiint_B f(x,y,z)dxdydz = \int_a^b \int_c^d \int_e^f f(x,y,z)dzdydx,$$

provided the right hand side iterated integral exists.
There are, in fact, five other possibilities for the iterated integrals.

We have a theorem saying if f is integrable, whenever any of these iterated integral exists, it is equal to the value of the integral of f over B. If f is continuous on B, then f is integrable on B and all iterated integrals exist and their values are equal to the integral of f on B.

Elementary regions in $\mathbb{R}^3$

The triple integrals that are easiest to evaluate are those for which the region W in space can be described by bounding z between the graphs of two functions in x and y with the domain of these functions being an elementary region in two variables.

For example,

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid \gamma_1(x, y) \leq z \leq \gamma_2(x, y), (x, y) \in D\},$$

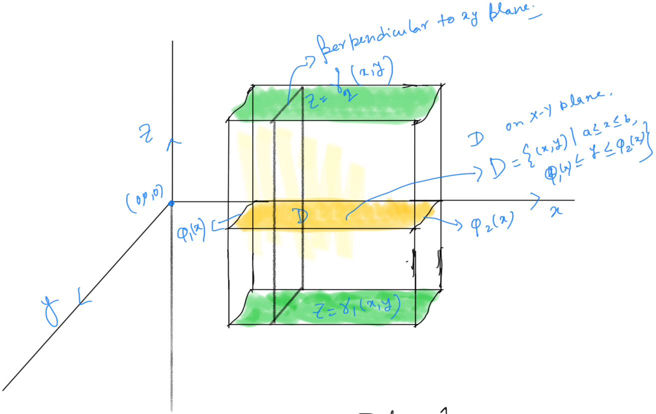
where γ_1 and γ_2 are continuous on $D \subset \mathbb{R}^2$ and D is an elementary region in $\mathbb{R}^2$. For rxample, if D is Type 1, then

$$D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \quad \phi_1(x) \leq y \leq \phi_2(x)\},$$

where $\phi_1 : [a, b] \to \mathbb{R}$ and $\phi_2 : [a, b] \to \mathbb{R}$ are continuous functions. The region D can be Type 2 also.

Example:

- The region W between the paraboloid $z = x^2 + y^2$ and the plane $z = 2$.
- The region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 4$ and $x = z - y - 1$.



When D is Type 1.

Elementary regions (Example)

Suppose that the region W lies between $z = \gamma_1(x, y)$ and $z = \gamma_2(x, y)$. Suppose that the projection of W on the xy plane is bounded by the curves $y = \phi_1(x)$ and $y = \phi_2(x)$ and the straight lines $x = a$ and $x = b$, then for a continuous function f defined over W, we have

$$\iiint_W f(x, y, z)dxdydz = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x, y, z)dzdydx.$$

Example: Let us find the volume of the sphere using the above formula. In other words, let us integrate the function 1 on the region W, where W is the unit sphere, i.e.,

$$\int\int\int_W 1dxdydz = ?, \quad \text{where} \quad W = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}.$$

The volume of the unit sphere

The sphere can be described as the region lying between $z = -\sqrt{1-x^2-y^2}$ and $z = \sqrt{1-x^2-y^2}$.

The projection of the sphere onto the xy plane gives a disc of unit radius. This can be described as the set of points lying between the curves $-\sqrt{1-x^2}$ and $\sqrt{1-x^2}$ and the lines $x = \pm 1$. Thus our triple integral reduces to the iterated integral

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dzdydx.$$

This yields

$$2\int_{-1}^{1} \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2)^{1/2} dy \right] dx.$$

After evaluating the inner integral we obtain

$$2\pi \int_{-1}^{1} \frac{1-x^2}{2} dx = \frac{4}{3}\pi.$$

MA 105 Calculus II

Week 3

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Change of variables in $\mathbb{R}^2$

Let Ω be an open subset of $\mathbb{R}^2$ and $h : \Omega \to \mathbb{R}^2$ be an one-one transformation denoted by

$$h(u, v) := (h_1(u, v), h_2(u, v)), \quad \forall\,(u, v) \in \Omega.$$

We now want to make a general change of coordinates given by

$$x = h_1(u, v), \quad y = h_2(u, v).$$

What conditions do we need on h to be able to do a change of coordinates?

Can we compute the area of the image of a rectangle in the u-v plane?

Suppose we have a change in coordinates given by linear functions composed with translations (such functions are called affine linear functions):

$$x = au + bv + t_1 \quad \text{and} \quad y = cu + dv + t_2.$$

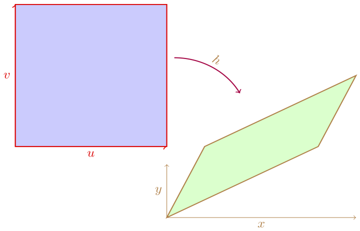
A linear change of coordinates

How does the area of the image of a rectangle under this map compare with the area of the original rectangle?

First, let us write down the affine map in a more compact notation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

Clearly, a rectangle $[0,1] \times [0,1]$ in the u-v plane is mapped to a parallelogram in the x-y plane. The vertices of the parallelogram are given by (t_1, t_2), $(a + t_1, c + t_2)$, $(b + t_1, d + t_2)$ and $(a + b + t_1, c + d + t_2)$.

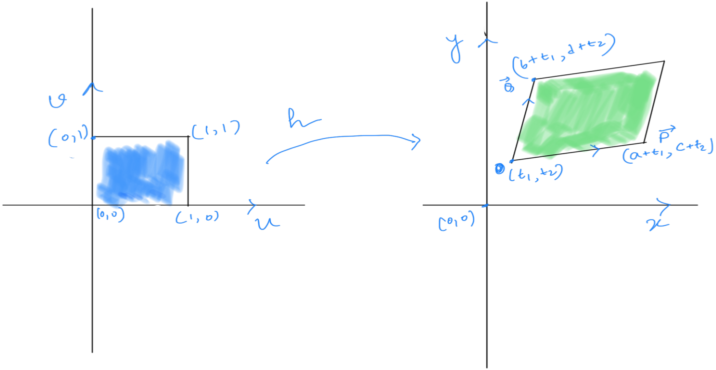


How does one compute the area of this parallelogram?

The area element for a change of coordinates

This is given by the absolute value of the cross product of the vectors,

$$(a, c, 0) \times (b, d, 0) = (ad - bc) \cdot \mathbf{k} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \cdot \mathbf{k}.$$



The area element for a change of coordinates

Let us now suppose that we have a general (not linear any more) change of coordinates given by $x = h_1(u, v)$ and $y = h_2(u, v)$.
How does the area of a rectangle in the u-v plane change? In order to compute the change we need to know the partial derivatives exist.
Let us assume h is a one-one continuously differentiable function .
Noting

$$\Delta x = h_1(u+\Delta u, v+\Delta v) - h_1(u, v), \quad \Delta y = h_2(u+\Delta u, v+\Delta v) - h_2(u, v),$$

and using Taylor's theorem for functions of two variables we see that

$$\Delta x \sim \frac{\partial h_1}{\partial u}\Delta u + \frac{\partial h_1}{\partial v}\Delta v \ \text{ and } \Delta y \sim \frac{\partial h_2}{\partial u}\Delta u + \frac{\partial h_2}{\partial v}\Delta v.$$

Using our previous notation, we can write

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}$$

The Jacobian

You may recognize the matrix

$$J(h) = \begin{pmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{pmatrix}$$

that appears in the preceding formula. The derivative matrix for the function $h = (h_1, h_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is called the Jacobian.

In a neighbourhood of the point (u_0, v_0), the function h and the function $J(h)$, behave very similarly (that is, they are the same upto the first order terms - use Taylor's theorem!). In fact, the derivative matrix is the *linear approximation* to the function h, at least in a neighbourhood of a point, say (u_0, v_0).

In particular, it is easy to see how the area of a small rectangle changes under h, since we have already done so in the case of a linear map. It simply changes by the (absolute value of) determinant of J!

Change of Variables Formula

Theorem (Change of Variables Formula)

- *Let D be a closed and bounded subset of $\mathbb{R}^2$ such that ∂D has content zero. Let $f : D \to \mathbb{R}$ be continuous.*
- *Suppose Ω is an open subset of $\mathbb{R}^2$ and $h : \Omega \to \mathbb{R}^2$ is a one-one differentiable function such that $h := (h_1, h_2)$, where h_1 and h_2 have continuous partial derivatives in Ω and $det(J(h)(u, v)) \neq 0$ for all $(u, v) \in \Omega$.*
- *Let $D^* \subset \Omega$ be such that $h(D^*) = D$.*

Then D^* is a closed and bounded subset of Ω, and ∂D^* is of content zero. Moreover, $f \circ h : D^* \to \mathbb{R}$ is continuous, and

$$\int\int_D f(x, y)\, dxdy = \int\int_{D^*} (f \circ h)(u, v)|det(J(h)(u, v))|\, dudv.$$

Notation

Often we write $x = x(u, v)$ and $y = y(u, v)$. In this case we use the notation $\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$, for the Jacobian determinant.

Let D be a region in the xy plane and D^* a region in the uv plane such that $\phi(D^*) = D$. Then

$$\int\int_D f(x, y)dxdy = \int\int_{D^*} f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv.$$

Remark: Note what we get in the familiar case of polar coordinates: We have $x = r\cos\theta$, $y = r\sin\theta$ and

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r,$$

which is what we obtained previously.

How to choose the change of variables

- Aim: Find h such that a rectangle D^* in $u-v$ plane is getting mapped to the given area D in the xy plane. If D^* is not a rectangle, at least try to have it in the form of the elementary region Type 1 or Type 2.
- Presumably, the boundary D^* in $u-v$ plane should go to the boundary of D in $x-y$ plane.
- The non-vanishing Jacobian determinant of h assures that the properties of D^* is preserved under the transformation and D has similar properties as of D^*.
- In some cases, h can be chosen in a way such that the expression of the integrand becomes simpler after the change of variables.

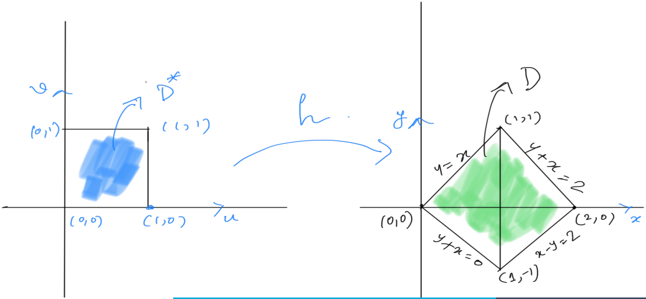
Example: Evaluate the integral

$$\iint_D (x^2 - y^2)\,dxdy$$

where D is the square with vertices at $(0,0)$, $(1,-1)$, $(1,1)$ and $(2,0)$.

Solution: Note D is the region in $x-y$ plane bounded by lines $y=x$, $y+x=0$, $x-y=2$ and $y+x=2$.
Put

$$x = u+v, \quad y = u-v,$$



Example Contd.

Then the rectangle

$$D^* = \{(u, v) \in \mathbb{R}^2 \mid 0 \leq u \leq 1,\ \ 0 \leq v \leq 1\}$$

in the uv-plane gets mapped to D, in the xy-plane.
Further,

$$\frac{\partial(x, y)}{\partial(u, v)} = \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2.$$

$$\int\int_D (x^2 - y^2)\, dxdy = \int\int_{D^*} (4uv) \times 2\, dudv$$

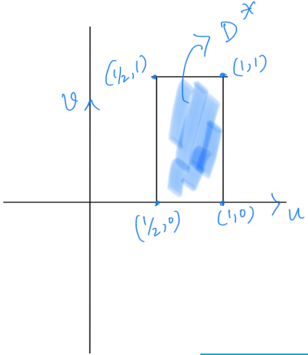
$$= 8\left(\int_0^1 udu\right)\left(\int_0^1 vdv\right) = 2.$$

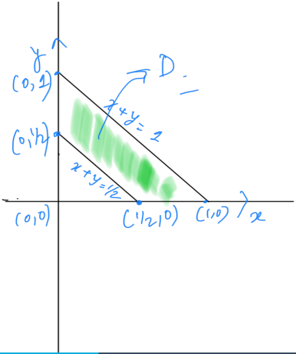
Example

Example: Let D be the region in the first quadrant of the xy-plane bounded by the lines $x+y=\frac{1}{2}$ and $x+y=1$. Find $\iint_D dA$ by transforming it to $\iint_{D^*} dudv$, where $u=x+y, v=\frac{y}{x+y}$.

Solution: Put

$$x=u(1-v),\ \ y=uv.$$





Example Contd.

Then the rectangle $D^* = \{(u,v) \in \mathbb{R}^2 \mid \frac{1}{2} \leq u \leq 1, \quad 0 \leq v \leq 1\}$ in the uv-plane gets mapped to D in the xy-plane. Further,

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1-v & -u \\ v & u \end{pmatrix} = u \neq 0.$$

Hence,

$$\text{Area}(D) = \int\int_D dA = \int\int_{D^*} |u| du dv$$
$$= \left(\int_{\frac{1}{2}}^1 \frac{u^2}{2} du\right)\left(\int_0^1 dv\right) = \frac{3}{4}.$$

The change of variables formula in three variables

In three variables, we once again have a formula for a change of variables. The formula has the same form as in the two variable case:

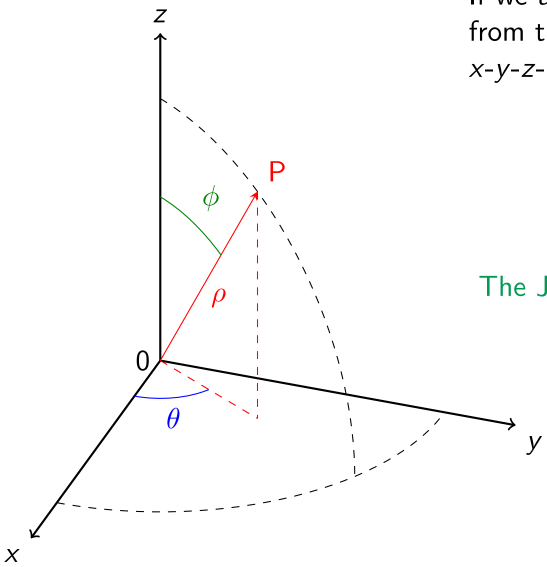
$$\iiint_P f(x,y,z)dxdydz = \iiint_{P^*} g(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw,$$

where $h(P^*) = P$. If the change in coordinates is given by $h = (h_1, h_2, h_3)$, the function g is defined as $g = f(h_1, h_2, h_3)$. The expression

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

is just the Jacobian determinant for a function of three variables.

Spherical Coordinates



If we use (ρ, θ, ϕ) what is the map from these coordinates to the x-y-z-planes?

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi. \end{aligned}$$

The Jacobian determinant is

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}x = \rho^2 \sin\phi.$$

Example

Example: It should be much easier computing the volume of the unit sphere now. Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$.

Then $W^* = \{(\rho, \theta, \phi) \in \mathbb{R}^3 \mid 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}$.
Then,

$$\begin{aligned}
\iiint_W dxdydz &= \iiint_{W^*} \rho^2 \sin\phi \; d\rho d\theta d\phi \\
&= \int_0^{\pi} \int_0^{2\pi} \int_0^1 \rho^2 \sin\phi \; d\rho d\theta d\phi \\
&= \frac{2\pi}{3} \int_0^{\pi} \sin\phi \; d\phi = \frac{4\pi}{3}.
\end{aligned}$$

Cylindrical coordinates in formulae

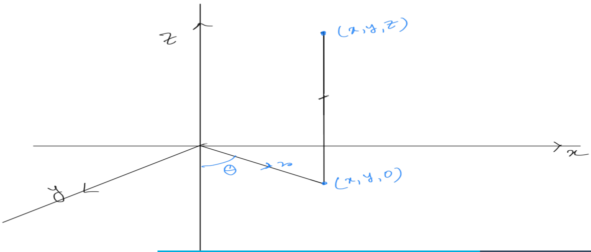
We can also consider a generalization of the polar coordinates. In this case, we use the change of transformation from (r, θ, z) coordinates to $P = (x, y, z) \in \mathbb{R}^3$ given by

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{and} \quad z = z.$$

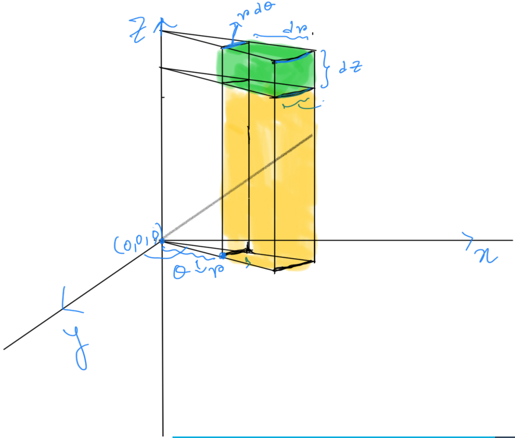
Here $r \geq 0$ and $0 \leq \theta \leq 2\pi$ and the (r, θ, z) are

It is very easy to see that

$$\left|\frac{\partial(x, y, z)}{\partial(r, \theta, z)}\right| = r.$$



The good thing about our convention is that θ means the same thing in both the cylindrical and spherical coordinate systems as well as in the (two-dimensional) polar coordinate system, and r means the same thing in both the cylindrical and (two-dimensional) polar coordinate systems.



Example

Evaluate $\int\int\int_W z^2(x^2+y^2)\,dxdydz$, where W is the cylindrical region determined by $x^2+y^2\leq 1$ and $-1\leq z\leq 1$.

Solution. The region W is described in cylindrical coordinates as W^*

$$W^*=\{(r,\theta,z)\mid 0\leq r\leq 1,\quad 0\leq\theta\leq 2\pi,\quad -1\leq z\leq 1\}.$$

$$\int\int\int_W z^2(x^2+y^2)\,dxdydz=\int_{z=-1}^{1}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} z^2r^2\,r\,drd\theta dz$$
$$=\int_{-1}^{1}\frac{2\pi}{4}z^2\,dz=\frac{\pi}{3}.$$

Let $n \in \mathbb{N}$ and $\mathbb{R}^n$ be the Euclidean space defined by

$$\mathbb{R}^n := \{(x_1, x_2, \cdots, x_n) \mid x_j \in \mathbb{R}; \quad \forall\, j = 1, 2, \cdots, n\},$$

equipped with the norm

$$\|x\| := \Big(\sum_{j=1}^{n} |x_j|^2\Big)^{\frac{1}{2}}.$$

- Any real number is called a scalar.
- For $n \in \mathbb{N}$, any element from $\mathbb{R}^n$ is called vector. Note this means elements of $\mathbb{R}$ can be thought of both as a scalar and vector. To avoid confusion we will talk about vectors in $\mathbb{R}^n$ for $n > 1$.

Basic structure:

For any $x := (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and $y := (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ and any $a \in \mathbb{R}$:

$$x + y := (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n) \in \mathbb{R}^n, \quad \text{sum of two elements in} \quad \mathbb{R}^n$$

$$ax := (ax_1, ax_2, \cdots, ax_n) \in \mathbb{R}^n, \quad \text{Scalar multiplication.}$$

Scalar fields and Vector fields

Let D be a subset of $\mathbb{R}^n$.

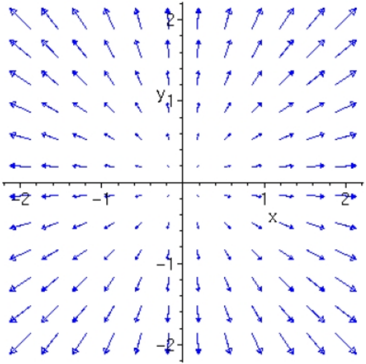
Definition: A scalar field on D is a map $f : D \to \mathbb{R}$.

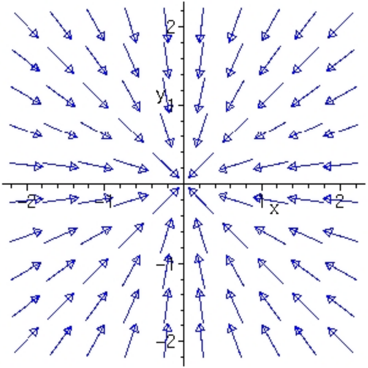
Definition A vector field on D is a map $\mathbf{F} : D \to \mathbb{R}^n$. We choose $n \geq 2$.

- A scalar field associates a number to each point of D, whereas a vector field associates a vector (of the same space) to each point of D.
- The temperature at a point on the earth is a scalar field.
- The velocity field of a moving fluid, a field describing heat flow, the gravitational field, the magnetic field etc are examples of various vector fields.

Vector fields: Examples

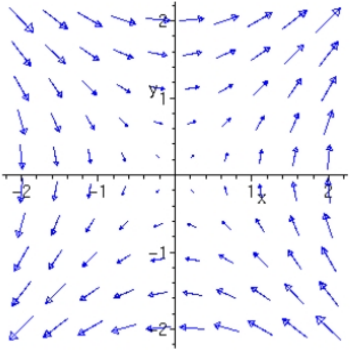
$$F_1(x,y) = (2x, 2y) \qquad F_2(x,y) = \left(\frac{-x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$$

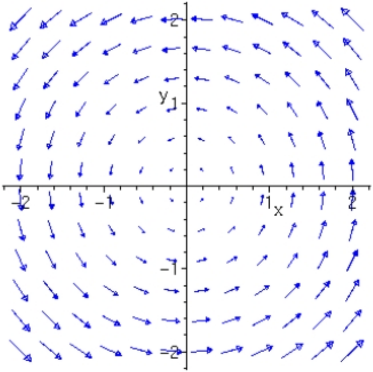




Vector fields: Examples

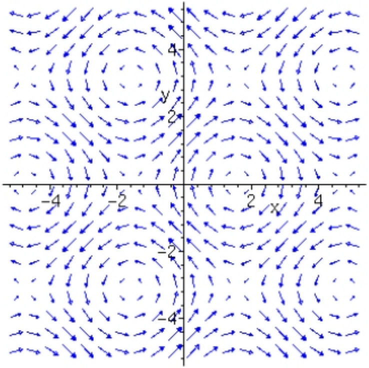
$F_3(x, y) = (y, x)$ $\qquad\qquad$ $F_4(x, y) = (-y, x)$





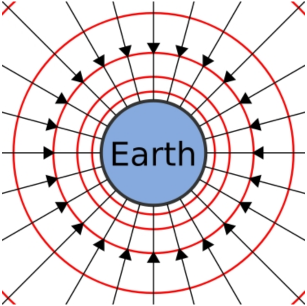
Vector fields: Examples

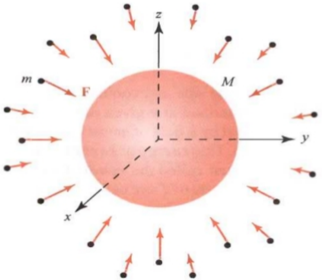
$$F_5(x, y) = (\sin y, \cos x)$$



The vector fields also occur in nature. Some of this you may have seen in MA 109 as well.

Gravitation fields





The first figure describes the gravitational field of the earth whereas the second one describes that of a body with mass M. The red lines denote the direction of the force exerted on the small particles around the body.

Del operator on Functions

We will assume from now on that our vector fields are smooth wherever they are defined.

One important class of vector fields are those that are given by the gradient of a scalar function. We will study these in some detail later.

The del operator on functions: We define the del operator restricting ourselves to the case $n = 3$:

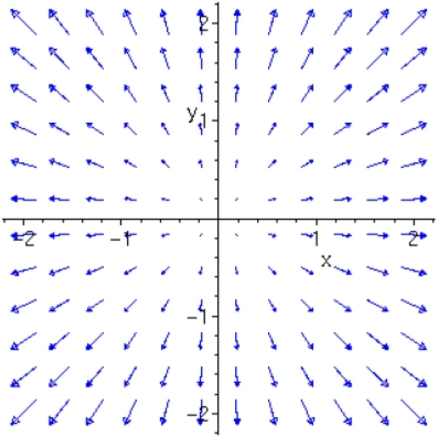
$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

The del operator acts on functions $f : \mathbb{R}^3 \to \mathbb{R}$ to give a gradient vector field :

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

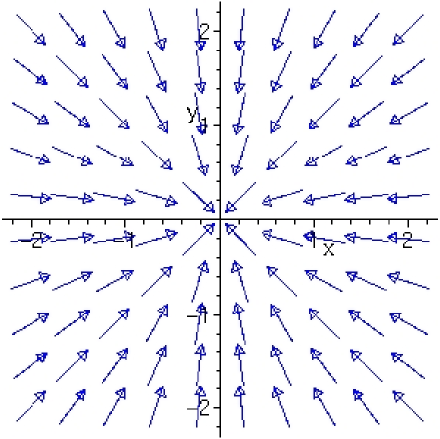
Thus the del operator takes scalar functions to vector fields.

Gradient fields



$$F_1(x,y) = (2x, 2y) = \nabla(x^2 + y^2)$$

Gradient fields



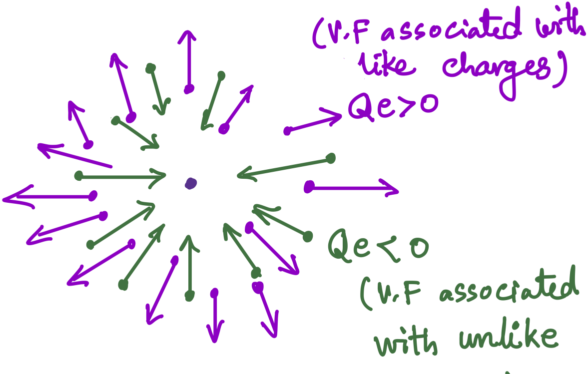
$$F_2(x,y) = \left(\frac{-x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) = \nabla\left(-\sqrt{x^2+y^2}\right)$$

Gradient Vector fields

Coulomb's law says that the force acting on a charge e at a point r due to a charge Q at the origin is

$$F = -\nabla V$$

where $V = \epsilon Qe/r$ is the potential. For like charges $Qe > 0$ force is repulsive and for unlike charges $Qe < 0$ the force is attractive.

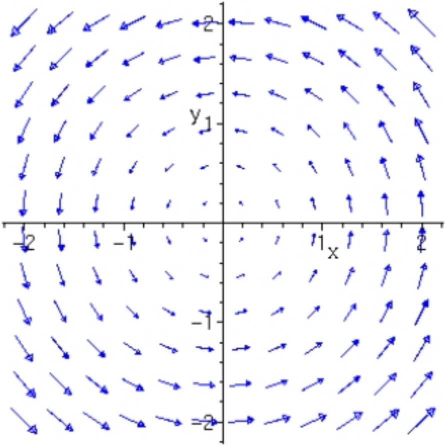


Such a force is called conservative. Conservative forces are important as work done along a path will be only dependent on the end points.

Several of the examples we have seen turn out to be gradient vector fields. The natural question to ask is which vector field is a gradient field.

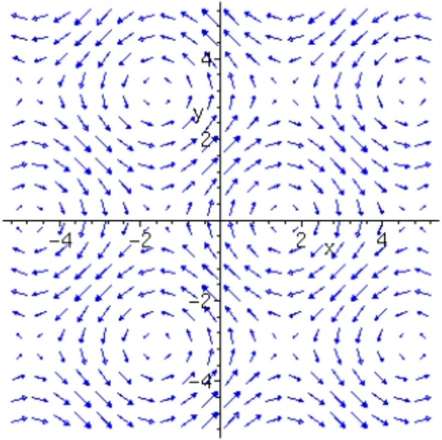
There is a neat answer to the above question, which we will see later. Not all vector fields will turn out to be gradient vector field.

Not gradient fields



$F_4(x, y) = (-y, x)$, this vector field is not ∇f for any f.

Not gradient fields



$F_5(x,y) = (\sin y, \cos x)$, this vector field is not ∇f for any f.

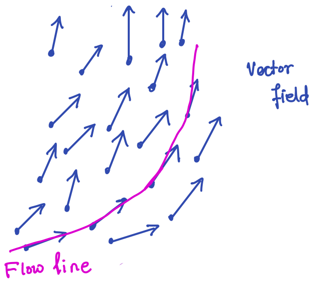
Flow lines for vector field

Vector fields also arise as the tangent vectors to the fluid flow.
Or conversely, given a vector field we can talk about its flow lines.

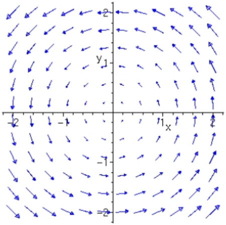
Definition If $\mathbf{F}$ is a vector field defined from $D \subset \mathbb{R}^n$ to $\mathbb{R}^n$, a flow line or integral curve is a path i.e., a map $\mathbf{c} : [a, b] \to D$ such that

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)), \quad \forall t \in [a, b].$$

In particular, $\mathbf{F}$ yields the velocity field of the path $\mathbf{c}$.



Example: Show that $c(t) = (\cos t, \sin t)$ is a flow line for the vector field $F(x, y) = -y\mathbf{i} + x\mathbf{j}$. Does it have other flow lines? Can you guess by looking at the vector field?



Finding the flow line for a given vector field involves solving a system of differential equations, if $c(t) = (x(t), y(t), z(t))$ then

$$\begin{aligned} x'(t) &= P(x(t), y(t), z(t)) \\ y'(t) &= Q(x(t), y(t), z(t)) \\ z'(t) &= R(x(t), y(t), z(t)), \end{aligned}$$

where the vector field is given by $F = (P, Q, R)$.

Such questions are dealt with in MA108.

Curve and path

Recall a path in $\mathbb{R}^n$ is a continuous map $\mathbf{c} : [a, b] \to \mathbb{R}^n$.

A curve in $\mathbb{R}^n$ is the image of a path $\mathbf{c}$ in $\mathbb{R}^n$.

Both the curve and path are denoted by the same symbol $\mathbf{c}$.

• Let $n = 3$ and $\mathbf{c}(t) = (x(t), y(t), z(t))$, for all $t \in [a, b]$. The path $\mathbf{c}$ is continuous iff each component x, y, z is continuous. Similarly, $\mathbf{c}$ is a C^1 path, i.e., continuously differentiable if and only if each component is C^1.

• A path $\mathbf{c}$ is called closed if $\mathbf{c}(a) = \mathbf{c}(b)$.

• A path $\mathbf{c}$ is called simple if $\mathbf{c}(t_1) \neq \mathbf{c}(t_2)$ for any $t_1 \neq t_2$ in $[a, b]$ other than $t_1 = a$ and $t_2 = b$ endpoints.

• If we write $\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ in vector notation, the tangent vector to $\mathbf{c}(t)$ is $\mathbf{c}'(t)$, i.e.,

$$\mathbf{c}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

• If a C^1 curve $\mathbf{c}$ is such that $\mathbf{c}'(t) \neq 0$ for all $t \in [a, b]$, the curve is called a regular or non-singular parametrised curve.

Examples of curves

• Let $\mathbf{c}(t) = (\cos 2\pi t, \sin 2\pi t)$ where $0 \leq t \leq 1$. This is a simple closed C^1 (actually smooth) curve.

• Let $\mathbf{c}(t) = (t, t^2)$ where $-1 \leq t \leq 5$ is a simple curve but not closed.

• Let $\mathbf{c}(t) = (\sin(2t), \sin t)$ where $-\pi \leq t \leq \pi$. It traces out a figure 8. It is not a simple but a closed C^1 curve.

• Let $\mathbf{c}(t) = (t^3, t)$ where $-1 \leq t \leq 1$ for some real numbers a, b is a part of the graph of the function $y = x^{1/3}$. This is simple but not a closed curve. Though the function $y = x^{\frac{1}{3}}$ is not a smooth function at origin, but this parametrization is regular.

Work done along a curve

• Recall from Physics, that work done by a particle on which force $\mathbf{F}$ is applied is given by the $\mathbf{F}.ds$ where ds is the displacement.

• If this is in one variable it is just the product and given by dot-product when it is in 2D or 3D space. This idea works when the displacement is straight line.

• If the particle is moving along a curve $\mathbf{c}$ then locally the curve can be approximated by a straight line.

• For a path $\mathbf{c} : [0, 1] \to \mathbb{R}^n$ for $n = 2$ or 3 if $\Delta t = t_2 - t_1$ is *very very* small then

$$\Delta s = \mathbf{c}(t_2) - \mathbf{c}(t_1) = \mathbf{c}'(\widehat{t})(t_2 - t_1)$$

for some $\widehat{t} \in [t_1, t_2]$ by mean value theorem.

• Then work done will have to be computed over these small intervals $[t_i, t_{i+1}]$ for $i = 1, \ldots, n$.

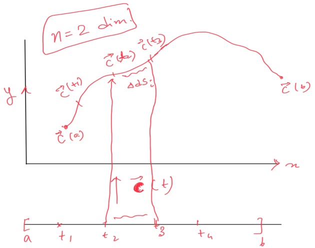
Work done along a curve contd.

- Total work done $= \sum_{i=1}^{n} \mathbf{F}(\widehat{t_i}) \cdot \mathbf{c}(t_{i+1}) - \mathbf{c}(t_i)$

$= \sum_{i=1}^{n} (\mathbf{F}(\widehat{t_i}) \cdot \mathbf{c}'(\widehat{t_i}))(t_{i+1} - t_i)$.

Does this remind you of something? Riemann sum: The limit of these Riemann sums as the length of the subintervals tends to zero, if it exists, is defined to be the line integral of the vector field $\mathbf{F}$ over the curve $\mathbf{c}$ and is denoted by

$$\int_{\mathbf{c}} \mathbf{F}.\mathbf{ds}.$$



Line integrals of vector fields

Assume that the vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, for $n = 1, 2$, is continuous and the curve $\mathbf{c} : [a, b] \to D$ is C^1.

Then we define the line integral of $\mathbf{F}$ over $\mathbf{c}$ as:

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} := \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt.$$

If $\mathbf{F} = (F_1, F_2, F_3)$ and $\mathbf{c}(t) = (x(t), y(t), z(t))$, we see that

$$\begin{aligned}
&\int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt \\
&= \int_a^b \left(F_1(\mathbf{c}(t))\frac{dx(t)}{dt} + F_2(\mathbf{c}(t))\frac{dy(t)}{dt} + F_3(\mathbf{c}(t))\frac{dz(t)}{dt} \right) dt.
\end{aligned}$$

Because of the form of the right hand side the line integral is sometimes written as

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b F_1 dx + F_2 dy + F_3 dz.$$

The expression on the right hand side is just alternate notation for the line integral. It does not have any independent meaning.

Examples

Example 1: Evaluate

$$\int_{\mathbf{c}} x^2 dx + xydy + dz,$$

where $\mathbf{c} : [0,1] \to \mathbb{R}^3$ is given by $\mathbf{c}(t) = (t, t^2, 1)$.

Solution: Let $\mathbf{c}(t) = (t, t^2, 1)$.
Let $\mathbf{F}(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)) = (x^2, xy, 1)$.

Thus $F_1(t, t^2, 1) = t^2$, $F_2(t, t^2, 1) = t^3$ and $F_3(t, t^2, 1) = 1$.

We have $\mathbf{c}'(t) = (1, 2t, 0)$, hence

$$(F_1(t, t^2, 1), F_2(t, t^2, 1), F_3(t, t^2, 1)) \cdot \mathbf{c}'(t) = t^2 + 2t^4 + 0.$$

$$\int_{\mathbf{c}} x^2 dx + xydy + dz = \int_0^1 (t^2 + 2t^4) dt = 11/15.$$

Examples

Example 2 (Marsden, Tromba, Weinstein): Find the work done by the force field $\mathbf{F} = (x^2 + y^2)(\mathbf{i} + \mathbf{j})$ around the loop $\mathbf{c}(t) = (\cos t, \sin t)$, $0 \leq t \leq 2\pi$.
Solution: The work done is given by

$$
\begin{aligned}
W &= \int_{\mathbf{c}} \mathbf{F} \cdot ds \\
&= \int_0^{2\pi} \mathbf{F}(\mathbf{c}(t)).\mathbf{c}'(t)\ dt \\
&= \int_0^{2\pi} (\cos t + \sin t) dt \\
&= (\sin t - \cos t)\big|_0^{2\pi} = 0
\end{aligned}
$$

Integrating along successive paths

It is easy to see that if $\mathbf{c}_1$ is a path joining two points P_0 and P_1 and $\mathbf{c}_2$ is a path joining P_1 and P_2 and $\mathbf{c}$ is the union of these paths (that is, it is a path from P_0 to P_2 passing through P_1), which is C^1 then

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}.$$

This property follows directly from the corresponding property for Riemann integrals:

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt,$$

where c is a point between a and b.

This allows us to define integration over peicewise differentiable curves for example the perimeter of a square.

Let the curve $\mathbf{c}$ be a union of curves $\mathbf{c}_1, \ldots, \mathbf{c}_n$. We often write this as $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2 + \ldots \mathbf{c}_n$, where end point of $\mathbf{c}_i$ is the starting point of $\mathbf{c}_{i+1}$ for all $i = 1, \ldots, n-1$.

Then we can define

$$\int_{\mathbf{c}} \mathbf{F} \cdot ds := \int_{\mathbf{c}_1} \mathbf{F} \cdot ds + \ldots + \int_{\mathbf{c}_n} \mathbf{F} \cdot ds.$$

Divide the curve $\mathbf{c}$ at a point p into two curves $\mathbf{c}_1$ and c_2. Then there it is easy to verify that $\int_{\mathbf{c}} \mathbf{F} \cdot = \int_{\mathbf{c}_1} \mathbf{F} \cdot ds + \int_{\mathbf{c}_2} \mathbf{F} \cdot ds$.

Let $\mathbf{c}$ be a curve on $[a, b]$ and $\widetilde{\mathbf{c}}(t) = \mathbf{c}(a + b - t)$, that is the curve $\widetilde{\mathbf{c}}$ traversed in the reverse direction and is denoted by $-\mathbf{c}$. What is $\int_{\mathbf{c}} \mathbf{F} \cdot ds + \int_{-\mathbf{c}} \mathbf{F} \cdot ds$?

$\int_{c} \mathbf{F} \cdot ds = -\int_{-c} \mathbf{F} \cdot ds$ (use change of variables formula).