Tutorial Sheet 01

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Questions to be discussed in order: 1 (iii), 2(i), 2(iv), 3(ii), 6, 5(ii).

1. (iii) Using the $(\epsilon - N)$ definition of a limit, prove the following:

$$\lim_{n\to\infty} \frac{n^{2/3}\sin(n!)}{n+1}$$

Solution: For a given $\epsilon > 0$, we have to find $n_0 \in \mathbb{N}$ such that $|a_n| < \epsilon$ for all $n > n_0$. Thus, we need to select a suitable $n_0 \in \mathbb{N}$ (This is possible by the archimedean property of $\mathbb{R}$)

$$\begin{aligned} |a_n| &= \left|\frac{n^{2/3}\sin(n!)}{n+1}\right| \\ &\le \frac{n^{2/3}}{n+1} \\ &\le \frac{1}{n^{1/3}} \end{aligned} \tag{1}$$

Remark: $|\sin x| \le 1\ \forall x$. Also, since n is always positive, we omit the modulus.

Since we need $|a_n|$ to be less than some ϵ,

$$\frac{1}{n^{1/3}} \le \epsilon \Rightarrow n \ge \frac{1}{\epsilon^3}$$

Hence, we can choose a n_0 such that $n_0 \ge \frac{1}{\epsilon^3}$ say $n_0 = \lfloor \frac{1}{\epsilon^3} \rfloor + 1$

2. Show that the following limits exist and find them:

$$(i)\ \lim_{n\to\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}\right)$$

Observe: For any $i \in \mathbb{N}$ such that $1 \le i \le n$, we have $\frac{n}{n^2+n} \le \frac{n}{n^2+i} \le \frac{n}{n^2+1}$
Hence,

$$\left(\frac{n}{n^2+n}+\cdots+\frac{n}{n^2+n}\right) \le \left(\frac{n}{n^2+1}+\frac{n}{n^2+2}+\cdots+\frac{n}{n^2+n}\right) \le \left(\frac{n}{n^2+1}+\cdots+\frac{n}{n^2+1}\right)$$

$$\frac{n^2}{n^2+n} \leq \frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n} \leq \frac{n^2}{n^2+1}$$

Observe:

- $\lim_{n\to\infty} \frac{n^2}{n^2+n} = 1$
- $\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$

Thus, from *Sandwich Theorem*, it follows that

$$\lim_{n\to\infty} (\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n})$$

exists and equals to 1

$$(iv) \lim_{n\to\infty} (n)^{1/n}$$

Observe: $(n)^{1/n} \geq n^0 \Rightarrow (n)^{1/n} \geq 1$

Let $(n)^{1/n} = 1 + h_n$, for some positive sequence $\{h_n\}$. Then for $n \geq 2$, using Binomial Expansion we have,

$$n = (1+h_n)^n = \sum_{k=0}^{n} \binom{n}{k} h_n^k \geq 1 + nh_n + \binom{n}{2} h_n^2 > \binom{n}{2} h_n^2$$

Thus we have,

$$0 \leq h_n^2 \leq \frac{2}{n-1}, (n \geq 2)$$

$$\Rightarrow 0 \leq h_n \leq \sqrt{\frac{2}{n-1}}, \quad (\because h_n \geq 0)$$

Observe: $\lim_{n\to\infty} \sqrt{\frac{2}{n-1}} = 0$

Thus, from *Sandwich Theorem*, it follows that $\lim_{n\to\infty} h_n = 0$. Hence, $\lim_{n\to\infty}(n)^{1/n} = \lim_{n\to\infty} 1 + h_n = 1$

NOTE: In both questions, existence need not be proved separately, Sandwich Theorem guarantees existence as well.

3. Show that the following sequences are not convergent:

$$(ii)\{(-1)^n(\frac{1}{2} - \frac{1}{n})\}_{n\geq 1}$$

The sequence will be convergent iff there exist an L which satisfies the definition of limit.

Here we have,

$$\lim_{n\to\infty} (-1)^n(\frac{1}{2} - \frac{1}{n}) = \lim_{n\to\infty} \frac{(-1)^n}{2} - \lim_{n\to\infty} \frac{(-1)^n}{n}$$

Observe: $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

Hence, the given limit will exist iff $\lim_{n\to\infty} \frac{(-1)^n}{2}$ exists.

Let us assume that $\lim_{n\to\infty} \frac{(-1)^n}{2}$ exists and let it be L'. Hence, from the definition of limit,

$\forall \epsilon > 0$, $\exists n_0 \in \mathbb{N}$, such that $|\frac{(-1)^n}{2} - L'| < \epsilon$ whenever $n > n_0$

Choose n_1 and n_2 such that $n_1, n_2 > n_0$ and n_1 is odd while n_2 is even. Then we have

$$|\frac{(-1)^{n_1}}{2} - L'| < \epsilon \& |\frac{(-1)^{n_2}}{2} - L'| < \epsilon$$

$$|\frac{-1}{2} - L'| < \epsilon \& |\frac{1}{2} - L'| < \epsilon$$

Adding both we get, $|\frac{-1}{2} - L'| + |\frac{1}{2} - L'| < 2\epsilon$. But, from Triangle Inequality it follows that, $|(\frac{-1}{2} - L') - (\frac{1}{2} - L')| < 2\epsilon$, or $\epsilon > \frac{1}{2}$.

But, this leads to contradiction as our argument has to hold true for all $\epsilon > 0$.

Hence $\frac{(-1)^n}{2}$ and thus, $\{(-1)^n(\frac{1}{2} - \frac{1}{n})\}_{n\geq 1}$ are both not convergent.

Aliter: Let $a_n = \{(-1)^n(\frac{1}{2} - \frac{1}{n})\}_{n\geq 1}$

Consider,

$$\begin{aligned}
\lim_{n\to\infty} a_{n+1} - a_n &= \lim_{n\to\infty} (-1)^{n+1}(\frac{1}{2} - \frac{1}{n+1}) - (-1)^n(\frac{1}{2} - \frac{1}{n}) \\
&= \lim_{n\to\infty} (-1)^n(-\frac{1}{2} + \frac{1}{n+1} - \frac{1}{2} + \frac{1}{n}) \\
&= \lim_{n\to\infty} (-1)^n(-1 + \frac{1}{n+1} + \frac{1}{n}) \\
&= \lim_{n\to\infty} (-1)^{n+1} + \lim_{n\to\infty} (-1)^n(\frac{1}{n+1} + \frac{1}{n})
\end{aligned}$$

Observe: $(-1)^n$ is bounded and $\lim_{n\to\infty}(\frac{1}{n+1} + \frac{1}{n}) = 0$.

Hence,

$$\lim_{n\to\infty} a_{n+1} - a_n = \lim_{n\to\infty} (-1)^{n+1} \neq 0$$

Thus, $a_n = \{(-1)^n(\frac{1}{2} - \frac{1}{n})\}_{n\geq 1}$ is not convergent.

6. If $\lim_{n\to\infty} a_n = L$; find the following: $\lim_{n\to\infty} a_{n+1}$, $\lim_{n\to\infty} |a_n|$

   We have, $\lim_{n\to\infty} a_n = L \Rightarrow$ for every $\epsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$, whenever $n > n_0$.

   Now, $n + 1 > n > n_0$. Thus, whenever $n + 1 > n_0$ we can say that $|a_{n+1} - L| < \epsilon$ for every $\epsilon > 0$.

$$\Rightarrow \lim_{n\to\infty} a_{n+1} = L$$

Also, from Triangle inequality we get, $||a_n| - |L|| < |a_n - L| < \epsilon$.

Thus, $||a_n| - |L|| < \epsilon$ for every $\epsilon > 0$, whenever $n > n_0$.

$$\Rightarrow \lim_{n\to\infty} |a_n| = |L|$$

5. (ii) Prove that the following sequences are convergent by showing that they are monotone and bounded. Also find their limits:

$$a_1 = \sqrt{2}, a_{n+1} = \sqrt{2 + a_n} \forall n \geq 1$$

Observe: All elements of the sequence are positive.

Claim 1: $a_n < 2$ for every $n \geq 1$

Proof: Use Mathematical Induction. So define $P(n) := a_n < 2$.

$P(1) := a_1 = \sqrt{2} < 2$ is true.

Assume $P(k) := a_k < 2$ is true. Now consider $P(k+1) := a_{k+1}$.

$a_{k+1} = \sqrt{2 + a_k} < \sqrt{2 + 2} = 2$

Thus, $a_n < 2$ for every $n \geq 1$

Claim 2: $a_{n+1} > a_n$ for every $n \geq 1$.

Proof: Consider

$${a_{n+1}}^2 - {a_n}^2 = 2 + a_n - a_n^2 = (2 - a_n)(a_n + 1) > 0$$

Since $a_n < 2$.

Thus, ${a_{n+1}}^2 > {a_n}^2 \Rightarrow a_{n+1} > a_n$ for every $n \geq 1$.

From above results we see that a_n is monotonically increasing and has an upper bound. Hence, the given sequence is convergent.

Now consider, $a_{n+1} = \sqrt{2 + a_n}$.

Applying limits on both sides, $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2 + a_n}$. Let $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L$. Thus, L = 2.

Hence, $\lim_{n\to\infty} a_n = 2$