Tutorial Sheet 3 (I. Multiple integrals and change of variables)

1. Find the volume that lies under the paraboloid $z = x^2 + y^2$, above the xy plane, and inside the cylinder $x^2 + y^2 = 2x$.

Sol.

Though the question is worded differently, the region is exactly the same as that of Question 9 in Tutorial sheet 2, hence the same solution follows.

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2. Using a suitable change of variables, evaluate the integral $\int\int_D y\,dy\,dx$, where D is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x, y \geq 0$.

Sol.

We can use the following variables:

$$x = 1 + u$$
$$y = \sqrt{v}$$

Since D is the region given by,

$$D = \left\{(x, y) \;\middle|\; \frac{y^2 - 4}{4} \leq x \leq \frac{4 - y^2}{4}, y \geq 0\right\}$$

On changing variables to u and v, the D is transformed to a new region D',

$$D' = \left\{(u, v) \;\middle|\; 0 \leq v \leq 4, \frac{v}{4} - 2 \leq u \leq \frac{-v}{4}\right\}$$

The Jacobian $\mathbb{J}$ is easy enough to find. $|\mathbb{J}| = \frac{1}{2\sqrt{v}}$ (Verify).

Thus the integral $\int\int_D y\,dy\,dx$ can be computed as follows:

$$\begin{aligned}
\iint_D y\,dy\,dx &= \iint_{D'} \sqrt{v} \times \frac{1}{2\sqrt{v}} du\,dv \\
&= \int_0^4 \int_{\frac{v}{4}-2}^{\frac{-v}{4}} \frac{1}{2} du\,dv \\
&= \int_0^4 \frac{1}{2}\left(2 - \frac{v}{2}\right) dv \\
&= \boxed{2}
\end{aligned}$$

■

3. Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

Sol.

Since we are supposed to use spherical coordinates, we can write the change of coordinates as,

$$x = r\cos\phi\sin\theta$$
$$y = r\sin\phi\sin\theta$$
$$z = r\cos\theta$$

We define a region D as the following:

$$D := \left\{ (x, y, z) \;\middle|\; x^2 + y^2 \leq \frac{1}{2}, \sqrt{x^2 + y^2} \leq z \leq \frac{1}{2} + \sqrt{\frac{1}{4} - x^2 - y^2} \right\}$$

It should be clear that D is the region of which the volume we have to compute. We now represent D in spherical coordinates and call this transformed region D'.

$$D' := \left\{ (r, \phi, \theta) \;\middle|\; 0 \leq \theta \leq \frac{\pi}{4}, 0 \leq \phi \leq 2\pi, 0 \leq r \leq \cos\theta \right\}$$

So the volume to be computed is simply,

$$\begin{aligned}
V &= \iiint_{D'} r^2 \sin\theta \, dr d\theta d\phi \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\cos\theta} r^2 \sin\theta \, dr d\theta d\phi \\
&= 2\pi \int_0^{\frac{\pi}{4}} \sin\theta \left(\frac{\cos^3\theta}{3} \right) d\theta \\
&= -\frac{2\pi}{3} \int_0^{\frac{\pi}{4}} \cos^3\theta \, d(\cos\theta) \\
&= \boxed{\frac{\pi}{8}}
\end{aligned}$$

■

4. Use cylindrical coordinates to evaluate $\iiint_W (x^2+y^2)\,dz\,dy\,dx$ where

$$W = \{(x,y,z) \in \mathbb{R}^3 \mid -2 \le x \le 2, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, \sqrt{x^2+y^2} \le z \le 2\}.$$

Sol.
We perform the following substitution to get the cylindrical coordinates,

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

Observe that the region W can be written as,

$$W = \left\{ (r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3 \,\middle|\, 0 \le \theta < 2\pi, 0 \le r \le 2, r \le z \le 2 \right\}$$

The given integral therefore becomes,

$$\begin{aligned} \int_0^{2\pi}\int_0^2\int_r^2 r^3\,dz\,dr\,d\theta &= \int_0^{2\pi}\int_0^2 r^3(2-r)\,dr\,d\theta \\ &= (2\pi) \times \left(8 - \frac{32}{5}\right) \\ &= \boxed{\frac{16\pi}{5}} \end{aligned}$$

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5. Describe the solid whose volume is given by the integral

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta,$$

and evaluate the integral.

Sol.
The limits of integral describe the thick spherical shell S shown in Fig. 1 such that,

$$S := \left\{ (\rho,\theta,\phi) \,\middle|\, 1 \le \rho \le 2, 0 \le \phi \le \frac{\pi}{2}, 0 \le \theta \le \frac{\pi}{2} \right\}$$

The integration is trivial,

$$\begin{aligned} \iiint_S \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta &= \left(\int_0^{\pi/2} \sin\phi\,d\phi\right) \cdot \left(\int_0^{\pi/2} d\theta\right) \cdot \left(\int_1^2 \rho^2\,d\rho\right) \\ &= 1 \times \frac{\pi}{2} \times \frac{7}{3} \\ &= \boxed{\frac{7\pi}{6}} \end{aligned}$$

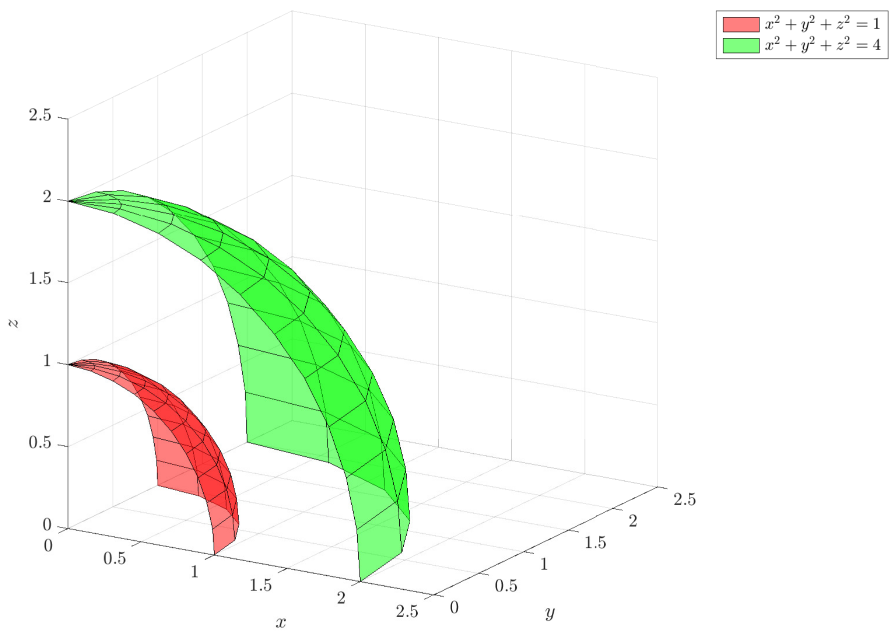

Figure 1: The region between the two spherical boundaries is the volume for question 5

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6. Find $\iiint_F \frac{1}{(x^2+y^2+z^2)^{\frac{n}{2}}} dV$ where F is the region bounded by the spheres with the center the origin and radii r and R with $0 < r < R$

Sol.
We will change the coordinates from Cartesian to spherical:

$$\begin{aligned} x &= r_1 \sin\theta \cos\varphi \\ y &= r_1 \sin\theta \sin\varphi \\ z &= r_1 \cos\theta \end{aligned}$$

where $\theta \in [0, \pi)$, $\varphi \in [0, 2\pi)$ and we are given $r_1 \in (r, R)$.
So the integral after transformation is,

$$\begin{aligned} \iiint_F \frac{1}{(x^2+y^2+z^2)^{\frac{n}{2}}} dV &= \int_0^{\pi}\int_0^{2\pi}\int_r^R \frac{1}{r_1^n} r_1^2 \sin\theta dr_1 d\theta d\varphi \\ &= \left(\int_0^{2\pi} d\varphi\right) \cdot \left(\int_0^{\pi} \sin\theta d\theta\right) \cdot \left(\int_r^R r_1^{2-n} dr_1\right) \\ &= 4\pi\wp(n) \end{aligned}$$

where,

$$\wp(n) = \begin{cases} \ln\left(\frac{R}{r}\right) & n = 3 \\ \frac{1}{3-n}\left(R^{3-n} - r^{3-n}\right) & \text{otherwise} \end{cases}$$

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1. Let f, g be differentiable functions on $\mathbb{R}^2$. Show that

i. $\nabla(fg) = f\nabla g + g\nabla f$;

ii. $\nabla f^n = nf^{n-1}\nabla f$;

iii. $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$ whenever $g \neq 0$.

Sol.

i. We use the definition of the gradient to rewrite the expressions in terms of partial derivatives, and then use the previously established product rule of the partial derivative to prove these identities.

$$\begin{aligned}\nabla(fg) &= \frac{\partial(fg)}{\partial x}\mathbf{i} + \frac{\partial(fg)}{\partial y}\mathbf{j}\\ &= \left(f\frac{\partial g}{\partial x} + \frac{\partial f}{\partial x}g\right)\mathbf{i} + \left(f\frac{\partial g}{\partial y} + \frac{\partial f}{\partial y}g\right)\mathbf{j} && (\because \text{ product rule for partial derivatives})\\ &= f\left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j}\right) + g\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)\\ &= f\nabla g + g\nabla f\end{aligned}$$

□

ii. We use the result obtained in part i. above, and the principle of induction. Let $P(n)$ denote the statement $\nabla f^n = nf^{n-1}\nabla f$. Observe that $P(1)$ is $\nabla f = \nabla f$, which is trivially true.
Now suppose $P(n)$ is true for some $n \geq 1$, i.e. $\nabla f^n = nf^{n-1}\nabla f$ holds. Then we have,

$$\begin{aligned}\nabla f^{n+1} &= \nabla(f^n f) = f\nabla f^n + f^n\nabla f && (\because \text{ i.})\\ &= fnf^{n-1}\nabla f + f^n\nabla f && (\because P(n) \text{ is true})\\ &= (n+1)f^n\nabla f,\end{aligned}$$

which is the same as $P(n+1)$. Hence we have that $P(n+1)$ is true whenever $P(n)$ is, for all $n \geq 1$, and that $P(1)$ is trivially true, which completes the proof for all positive integers n.

By the result in iii. (which will be proved in the next part), we may similarly show that $P(n-1)$ is true whenever $P(n)$ is, for any $n \leq 1$; hence by the principle of induction we have that $P(n)$ holds for non-positive integers as well. This completes the proof for all integers n. □

iii. Taking $h = \frac{1}{g}$ (which is possible when $g \neq 0$), we have

$$\nabla h = \frac{\partial h}{\partial x}\mathbf{i} + \frac{\partial h}{\partial y}\mathbf{j} = -\frac{1}{g^2}\frac{\partial g}{\partial x}\mathbf{i} - \frac{1}{g^2}\frac{\partial g}{\partial y}\mathbf{j} = -\frac{1}{g^2}\nabla g.$$

We now apply i. to the product fh, to obtain

$$\nabla\left(\frac{f}{g}\right) = \nabla(fh) = f\nabla h + h\nabla f = -f\frac{1}{g^2}\nabla g + \frac{1}{g}\nabla f = \frac{g\nabla f - f\nabla g}{g^2}$$

■

2. Let $\mathbf{a}$, $\mathbf{b}$ be two fixed vectors, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r^2 = x^2 + y^2 + z^2$. Prove the following:

1. $\nabla(r^n) = nr^{n-2}\mathbf{r}$ for any integer n
2. $\mathbf{a} \cdot \nabla(\frac{1}{r}) = -\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}$
3. $\mathbf{b} \cdot \nabla \left(\mathbf{a} \cdot \nabla(\frac{1}{r})\right) = \dfrac{3(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})}{r^5} - \dfrac{\mathbf{a}\cdot\mathbf{b}}{r^3}$

Sol.

1. Let $n \in \mathbb{Z}$ be given. Define $f : \mathbb{R}^3 \to \mathbb{R}$ as

$$f(x, y, z) := \left(x^2 + y^2 + z^2\right)^{\frac{n}{2}}$$

Thus,

$$\frac{\partial f}{\partial x} = nx\left(x^2 + y^2 + z^2\right)^{\frac{n-2}{2}}$$

Similar will be the face for the other partial derivatives. Therefore,

$$\begin{aligned}\nabla r^n &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \\ &= nr^{n-2}\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right) \\ &= nr^{n-2}\mathbf{r}\end{aligned}$$

□

2. Let $n = -1$. By the previously proved part, we have,

$$\begin{aligned}\nabla\left(\frac{1}{r}\right) &= -\frac{\mathbf{r}}{r^3} \\ \implies \mathbf{a}\cdot\nabla\left(\frac{1}{r}\right) &= -\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\end{aligned}$$

□

3. Let $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$. Using the result obtained in part 2.,

$$\begin{aligned}\mathbf{b}\cdot\nabla\left(\mathbf{a}\cdot\nabla\left(\frac{1}{r}\right)\right) &= \mathbf{b}\cdot\nabla\left(-\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\right) \\ &= \mathbf{b}\cdot\nabla\left(-\frac{a_x x + a_y y + a_z z}{r^3}\right) \\ &= \frac{3(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})}{r^5} - \frac{\mathbf{a}\cdot\mathbf{b}}{r^3}\end{aligned}$$

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3. Calculate the line integral of the vector field

$$\mathbf{F}(x,y) = \left(x^2 - 2xy\right)\mathbf{i} + \left(y^2 - 2xy\right)\mathbf{j}$$

from $(-1,1)$ to $(1,1)$ along $y = x^2$.

Sol.

We can represent the curve $y = x^2$ (let's call it C) in parametric form as,

$$x = \tau$$
$$y = \tau^2$$
$$\mathbf{c}(\tau) = (x,y) = (\tau, \tau^2)$$

where τ is the parameter, varying from -1 to 1.

The tangent $\mathbf{c}'(\tau)$ at every point (τ, τ^2) on C will be,

$$\mathbf{c}'(\tau) = \mathbf{i} + 2\tau\mathbf{j}$$

Therefore, we can write,

$$\begin{aligned}\int_C \mathbf{F}(\mathbf{r}) \cdot \mathbf{ds} &= \int \mathbf{F}(\mathbf{c}(\tau)) \cdot \mathbf{c}'(\tau) d\tau \\ &= \int_{-1}^{1} \left(\tau^2 - 2\tau^3 + 2\tau^5 - 4\tau^4\right) d\tau \\ &= \boxed{-\frac{14}{15}}\end{aligned}$$

■

4. Calculate the line integral of

$$\mathbf{F}(x,y) = \left(x^2 + y^2\right)\mathbf{i} + (x - y)\mathbf{j}$$

once around the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ in the counter clockwise direction.

Remark Often line integral of a vector field $\mathbf{F}$ along a 'geometric curve' C is represented by $\int_C \mathbf{F.ds}$. A geometric curve C is a set of points in the plane or in the space that can be traversed by a parametrized path in the given direction.

To evaluate $\int_C \mathbf{F.ds}$, choose a convenient parametrization $\mathbf{c}$ of C traversing C in the given direction and then

$$\int_C \mathbf{F.ds} := \int_{\mathbf{c}} \mathbf{F.ds}$$

'$\oint_C$' means the line integral over a closed curve C.

Sol.

Let's call the ellipse E. Every point on the ellipse can be parametrized in the following manner:

$$x(\theta) = a\cos\theta$$
$$y(\theta) = b\sin\theta$$
$$\mathbf{c}(\theta) = (x,y) = (a\cos\theta, b\sin\theta)$$

where $\theta \in [0, 2\pi)$.

We can now calculate the line integral as,

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{ds} &= \int \mathbf{F}(\mathbf{c}(\theta)) \cdot \mathbf{c}'(\theta) d\theta \\ &= \int_0^{2\pi} \left[\left(a^2\cos^2\theta + b^2\sin^2\theta\right)\mathbf{i} + (a\cos\theta - b\sin\theta)\mathbf{j}\right] \cdot (-a\sin\theta\mathbf{i} + b\cos\theta\mathbf{j})\, d\theta \\ &= \int_0^{2\pi} \left(-a^3\cos^2\theta\sin\theta - ab^2\sin^3\theta + ab\cos^2\theta - b^2\sin\theta\cos\theta\right) d\theta \\ &= \boxed{\pi ab}\end{aligned}$$

■

5. Calculate the value of the line integral

$$\oint_C \frac{(x+y)dx - (x-y)dy}{x^2+y^2}.$$

where C is the curve $x^2+y^2=a^2$ traversed once in the counter clockwise direction.

Sol.

The curve C can be parametrised as $\mathbf{c}(t) := (a\cos t, a\sin t)\,, t \in [0, 2\pi]$. Define $\mathbf{F}(x,y) := \left(\frac{x+y}{x^2+y^2}, \frac{y-x}{x^2+y^2}\right)$. The integral can then be written as,

$$\oint_C \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt$$

which is equal to,

$$\begin{aligned}\oint_C \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt &= \int_0^{2\pi} F(a\cos t, a\sin t) \cdot (-a\sin t, a\cos t)dt \\ &= \int_0^{2\pi} -(\cos t + \sin t)\sin t + \cos t(\sin t - \cos t)\,dt \\ &= -\int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right) dt \\ &= \boxed{-2\pi}\end{aligned}$$

■

6. Calculate

$$\oint_C y dx + z dy + x dz$$

where C is the intersection of two surfaces $z = xy$ and $x^2+y^2=1$ traversed once in a direction that appears counter clockwise when viewed from high above the xy-plane.

Sol.

Verify that the given curve can be paramaterized as follows:

$$c(t) := (x(t), y(t), z(t)) = (\cos t, \sin t, \cos t \sin t)\,, t \in [-\pi, \pi]$$

It can be seen that this respects the direction given.
Also, $(x'(t), y'(t), z'(t)) = (-\sin t, \cos t, \cos 2t)$. Hence, we can now evaluate our integral as follows:

$$\begin{aligned}\oint_C y dx + z dy + x dz &= \int_{-\pi}^{\pi} [\sin t(-\sin t) + \cos t \sin t(\cos t) + \cos t(\cos 2t]\,dt \\ &= -\int_{-\pi}^{\pi} \sin^2 t dt \\ &= \boxed{-\pi}\end{aligned}$$

7. Let the curve C be given by $x^2+y^2=1, z=0$. Let $\mathbf{c}_1$ be a parametrization defined by $\mathbf{c}_1(t)=(\cos t, \sin t)$ for $t \in [0, 2\pi]$. Find the line integral of $\mathbf{F}(x,y,z)=-y\mathbf{i}+x\mathbf{j}$ along this curve. Also find the line integral along the curve parametrized by $\mathbf{c}_2(t)=(\cos t, -\sin t)$, for $t \in [0, \pi]$.

Sol.

For the parametrization $\mathbf{c}_1$, we have,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{s} &= \int \mathbf{F}\left(\mathbf{c}_1(t)\right)\cdot \mathbf{c}_1'(t)dt \\ &= \int_0^{2\pi} (-\sin t, \cos t)\cdot(-\sin t, \cos t)dt \\ &= \int_0^{2\pi} \left(\sin^2 t+\cos^2 t\right) dt \\ &= \int_0^{2\pi} dt \\ &= \boxed{2\pi}\end{aligned}$$

For the parametrization $\mathbf{c}_2$, we have,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{s} &= \int \mathbf{F}\left(\mathbf{c}_2(t)\right)\cdot \mathbf{c}_2'(t)dt \\ &= \int_0^{\pi} (\sin t, \cos t)\cdot(-\sin t, -\cos t)dt \\ &= -\int_0^{\pi} \left(\sin^2 t+\cos^2 t\right) dt \\ &= -\int_0^{\pi} dt \\ &= \boxed{-\pi}\end{aligned}$$

8. Show that a constant force field does zero work on a particle that moves once uniformly around the circle: $x^2+y^2=1$. Is this also true for a force field $\mathbf{F}(x,y,z)=\alpha\left(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\right)$, for some constant α.

Sol.

Consider the following constant force field,

$$\mathbf{F}'=F_x\mathbf{i}+F_y\mathbf{j}$$

where F_x, F_y are constants.

We can parametrize the path along the circle easily by substituting,

$$\begin{aligned}x &= \cos\theta \\ y &= \sin\theta \\ \mathbf{c}(\theta) &= (x,y) \\ &= (\cos\theta, \sin\theta)\end{aligned}$$

where $\theta \in [0, 2\pi)$.

We now calculate the line integral for F',

$$\begin{aligned}\int_C \mathbf{F}'\cdot d\mathbf{s} &= \int_0^{2\pi} \mathbf{F}'(\mathbf{c}(\theta))\cdot \mathbf{c}'(t)dt \\ &= \int_0^{2\pi} \left(-F_x\sin\theta+F_y\cos\theta\right)d\theta \\ &= -F_x\int_0^{2\pi}\sin\theta d\theta + F_y\int_0^{2\pi}\cos\theta d\theta \\ &= \boxed{0}\end{aligned}$$

We see that, independent of our choice of F_x and F_y, the integral comes out to be zero.
In the case of F, the line integral comes out to be,

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{ds} &= \int_0^{2\pi}\mathbf{F}(\mathbf{c}(\theta))\cdot\mathbf{c}'(t)dt\\ &= \alpha\int_0^{2\pi}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})\,d\theta\\ &= \alpha\int_0^{2\pi}(-\cos\theta\sin\theta+\cos\theta\sin\theta)\,d\theta\\ &= \boxed{0}\end{aligned}$$

9. Let $C: x^2+y^2=1$. Find

$$\oint_C \text{grad}\left(x^2-y^2\right)\mathbf{.ds}.$$

Sol.
We can parametrize the given C by substituting,

$$\begin{aligned}x &= \cos\theta\\ y &= \sin\theta\\ \mathbf{c}(\theta) &= (x,y)\\ &= (\cos\theta,\sin\theta)\end{aligned}$$

where $\theta\in(0,2\pi)$.

$$\mathbf{c}'(\theta) = (-\sin\theta,\cos\theta)$$

$$\begin{aligned}\text{grad}\left(x^2-y^2\right) &= \left(\frac{\partial\left(x^2-y^2\right)}{\partial x},\frac{\partial\left(x^2-y^2\right)}{\partial y}\right)\\ &= (2x,-2y)\\ &= (2\cos\theta,-2\sin\theta)\end{aligned}$$

The required line integral is,

$$\begin{aligned}\oint_C \text{grad}\left(x^2-y^2\right)\cdot\mathbf{ds} &= \int_0^{2\pi}(2\cos\theta,-2\sin\theta)\,.\,(-\sin\theta,\cos\theta)\,d\theta\\ &= \int_0^{2\pi}(-2\sin\theta\cos\theta-2\sin\theta\cos\theta)\,d\theta\\ &= \int_0^{2\pi}(-2\sin 2\theta)\,d\theta\\ &= 0\end{aligned}$$

This is to be expected as gradient of a scalar field is a conservative vector field and hence the line integral over a continuous closed loop is 0.

■

10. Evaluate

$$\int_C \text{grad}\left(x^2 - y^2\right) \mathbf{.ds}$$

where C is $y = x^3$, joining $(0, 2)$ and $(2, 8)$.

Sol.

We can parametrize C as follows,

$$\begin{aligned} x &= t \\ y &= t^3 \\ \mathbf{c}(t) &= (x, y) \\ &= \left(t, t^3\right) \end{aligned}$$

where $t \in (0, 2)$.

$$\mathbf{c}'(t) = \left(1, 3t^2\right)$$

As shown in the previous question,

$$\begin{aligned} \text{grad}\left(x^2 - y^2\right) &= (2x, -2y) \\ &= \left(2t, -2t^3\right) \\ \implies \int_C \text{grad}\left(x^2 - y^2\right) \mathbf{.ds} &= \int_0^2 \left(2t, -2t^3\right) . \left(1, 3t^2\right) dt \\ &= \int_0^2 \left(2t - 6t^5\right) dt \\ &= \boxed{-60} \end{aligned}$$

■

11. Compute the line integral

$$\oint_C \frac{dx + dy}{|x| + |y|}$$

where C is the square with vertices $(1, 0), (0, 1), (-1, 0)$ and $(0, -1)$ traversed once in the counter clockwise direction.

Sol.

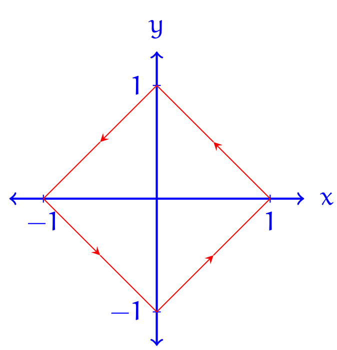

Figure 1: Figure for question 11

The path C is given by the region in Fig. 1. Note that along this path, $|x| + |y| = 1$. Hence the line integral reduces to

$$\oint_C (dx + dy), \text{ which can be rewritten as } \oint_C \nabla(x + y) \cdot \mathbf{ds}.$$

The fundamental theorem of calculus (FTC) for gradient fields can then be applied here to conclude that this integral is 0, but not directly, since C is not a smooth path. We break up C into 4 constituent paths $C_1, \ldots, C_4$, which are cyclically from $(1,0)$ to $(0,1)$, ..., and from $(0,-1)$ to $(1,0)$ respectively. Each of these paths are smooth, and thus we may apply FTC to each of them, giving us

$$\begin{aligned}
\oint_C \nabla(x+y)\cdot \mathbf{ds} &= \oint_{C_1} \nabla(x+y)\cdot \mathbf{ds} + \oint_{C_2} \nabla(x+y)\cdot \mathbf{ds} + \oint_{C_3} \nabla(x+y)\cdot \mathbf{ds} + \oint_{C_4} \nabla(x+y)\cdot \mathbf{ds} \\
&= (x+y)\Big|_{(x,y)=(1,0)}^{(x,y)=(0,1)} + (x+y)\Big|_{(x,y)=(0,1)}^{(x,y)=(-1,0)} + (x+y)\Big|_{(x,y)=(-1,0)}^{(x,y)=(0,-1)} + (x+y)\Big|_{(x,y)=(0,-1)}^{(x,y)=(1,0)} \\
&= \boxed{0}
\end{aligned}$$

■

12. A force $F = xy\mathbf{i} + x^6y^2\mathbf{j}$ moves a particle from $(0,0)$ onto the line $x = 1$ along $y = ax^b$ where $a, b > 0$. If the work done is independent of b find the value of a.

Sol.

Parametrize the curve as,

$$\begin{aligned}
x &= \tau \\
y &= a\tau^b \\
\mathbf{c}(\tau) &= (x,y) \\
&= \left(\tau, a\tau^b\right)
\end{aligned}$$

We can therefore write,

$$\begin{aligned}
\int_C \mathbf{F.ds} &= \int \mathbf{F}(\mathbf{c}(\tau))\cdot \mathbf{c}'(\tau)d\tau \\
&= \int_0^1 \left(a\tau^{b+1}\mathbf{i} + a^2\tau^{2b+6}\mathbf{j}\right)\cdot\left(\mathbf{i} + ab\tau^{b-1}\mathbf{j}\right)d\tau \\
&= \int_0^1 \left(a\tau^{b+1} + a^3b\tau^{3b+5}\right)d\tau \\
&= \frac{3a+ba^3}{3b+6} = I(a,b)
\end{aligned}$$

If we want $I(a,b)$ to be independent of b, then we will have to ensure that $\frac{\partial I}{\partial b} = 0$.

$$\begin{aligned}
\frac{\partial I}{\partial b} &= 0 \\
\implies \frac{a^3(3b+6) - 3\left(3a+ba^3\right)}{(3b+6)^2} &= 0 \\
\implies a &= 0, \pm\sqrt{\frac{3}{2}}
\end{aligned}$$

We will take the value of $a > 0$. So for $I(a,b)$ to be independent of b, we will need $a = \boxed{\sqrt{\frac{3}{2}}}$.

■
