CS 105: Department Introductory Course on Discrete Structures

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Lecture 08 – Basic Mathematical Structures

Countable and Uncountable Sets

Countable and countably infinite sets

Definition

- Set C is called countably infinite, if there is a bijection from set C to $\mathbb{N}$.
- A set is countable if it is finite or countably infinite.

Examples: even numbers, number of horses,...

By previous corollary ($\exists$ surj from any infinite set to $\mathbb{N}$)

Countably infinite sets are the "smallest" infinite sets.

Some questions...

Are the following sets countable?

That is, is there a bijection from these sets to $\mathbb{N}$?

1. the set of all integers $\mathbb{Z}$
2. $\mathbb{N} \times \mathbb{N}$
3. $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$
4. the set of rationals $\mathbb{Q}$
5. the set of all (finite and infinite) subsets of $\mathbb{N}$
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To show these it suffices to show that

- there is an injection from these sets to $\mathbb{N}$
- or there is a surjection from $\mathbb{N}$ (or any countable set) to these sets.

Unions of countable sets is countable

Let $A = \{a_0, \ldots, \}$ be a countably infinite set and B be a set. Then, is $A \cup B$ countable, under the following conditions?

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- Is this correct?

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- $\mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{Z} \times \mathbb{N}$ are countable.

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Hint: Show that $f(a, b) = \begin{cases} a/b \text{ if } b \neq 0 \\ 0 \text{ if } b = 0 \end{cases}$, is a surjection. How does the result follow?

Countable sets and functions

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Proof by contradiction: Suppose there is such a bijection, say f. This would imply that each $i \in \mathbb{N}$ maps to some set $f(i) \subseteq \mathbb{N}$.

	0	1	2	3	$\dots$
$f(0)$	✓	$\times$	$\times$	$\times$	$\dots$
$f(1)$	✓	$\times$	✓	✓	$\dots$
$f(2)$	$\times$	$\times$	$\times$	$\times$	$\dots$
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- Consider the set $S \subseteq \mathbb{N}$ obtained by switching the diagonal elements, i.e., $S = \{i \in \mathbb{N} \mid i \notin f(i)\}$.

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- As f is bij, $\exists j \in \mathbb{N}, f(j) = S$.
- S and $f(j)$ differ at position j, for any j.
- Thus, $S \neq f(j)$ for all $j \in \mathbb{N}$, which is a contradiction! $\square$

Cantor's diagonalization

Does this proof look familiar??

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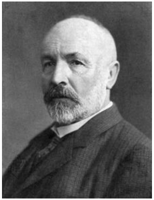 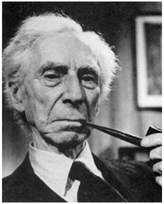

Figure: Cantor and Russell

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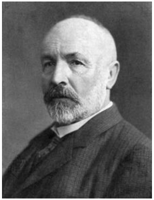 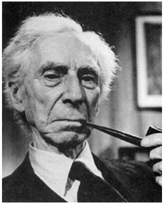

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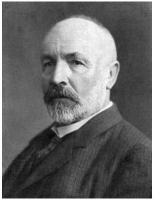 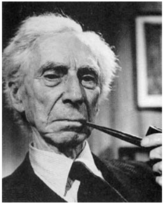

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  - If $j \in S$, then $j \notin f(j) = S$.
  - If $j \notin S$, then $j \notin f(j)$, which implies $j \in S$.

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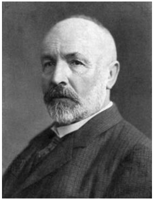 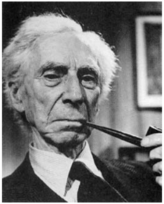

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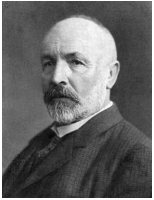

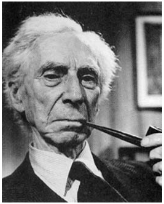

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- So there is an infinite hierarchy of "larger" infinities...
- There is no bijection from $\mathbb{R}$ to $\mathbb{N}$ (H.W). Moreover, there is a bijection from $\mathbb{R}$ to set of subsets of $\mathbb{N}$.