# MA 105 Calculus II

## Week 2


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# Integrals over any bounded region in $\mathbb{R}^2$

So far we have learnt to integrate bounded functions on any rectangle in $\mathbb{R}^2$.

Let $D$ be any bounded subset (not necessarily rectangle) of $\mathbb{R}^2$.

How to define integral of $f : D \to \mathbb{R}$ on $D$?

Remedy: If $D$ is a bounded subset of $\mathbb{R}^2$, then there exists a rectangle $R$ in $\mathbb{R}^2$ containing $D$, i.e., $D \subset R$. Why?
Since $D$ is a bounded subset of $R^2$, there exists $a > 0$ such that any $(x, y) \in D$ satisfies $x^2 + y^2 < a^2$, i.e, $D \subset B_a = \{(x, y) \mid x^2 + y^2 \leq a^2\}$.

Note $B_a \subset [-a, a] \times [-a, a]$

Then the rectangle $R := [-a, a] \times [-a, a]$ contains $D$.

Extend $f$ from $D$ to $R$ by defining

$$f^*(x,y) := \begin{cases} f(x,y), & (x,y) \in D, \\ 0, & (x,y) \notin D. \end{cases}.$$

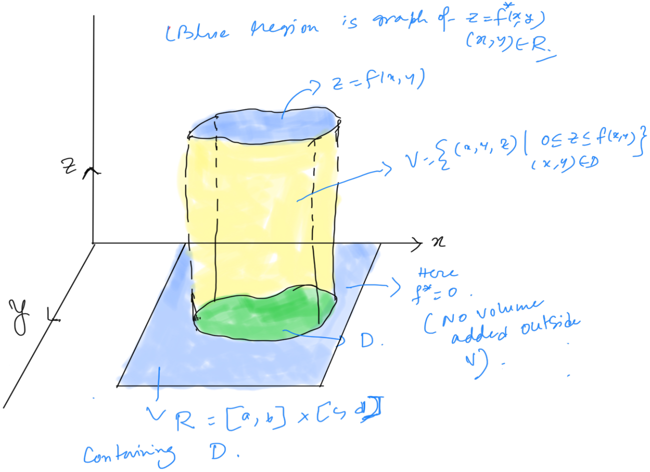
## Definition

The function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be integrable on bounded $D \subset \mathbb{R}^2$, if $f^*$ is integrable on $R$ and the integral of $f$ on $D$ is defined by

$$\int\int_D f(x,y)\,dx\,dy := \int\int_R f^*(x,y)\,dx\,dy.$$

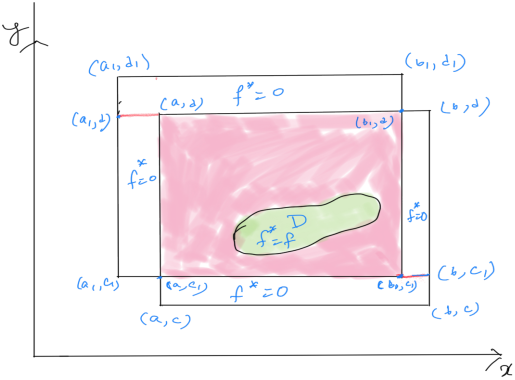
- If $f \geq 0$ on $D \subset \mathbb{R}^2$ and $f$ is integrable on $D$, then the double integral of $f$ on $D$ is the volume of the solid that lies above $D$ in the $x$-$y$ plane and below the graph of the surface $z = f(x,y)$ for all $(x,y) \in D$.

# $\int \int_D f =$ volume of $V$



# Independent of choice of rectangle

- The choice of rectangle $R$ containing $D$ is not unique.
- But the value of the integral of $f$ on $D$ does not depend on the choice of the rectangle $R$ containing $D$.
- Use the additivity property of integrals on rectangle and note that only ‘zero’ is getting added outside $D$.



# Properties of Integrals over bounded sets in $\mathbb{R}^2$

Let $D$ be a bounded subset of $\mathbb{R}^2$. Let $f : D \to \mathbb{R}$ be an integrable function.

- The algebraic properties for integrals on any bounded set $D$ in $\mathbb{R}^2$ hold similarly to those of the case of integrals on rectangle.

Domain additivity property: Let $D \subseteq \mathbb{R}^2$ be a bounded set. Let $D_1, D_2 \subseteq D$ such that $D = D_1 \cup D_2$. Let $f : D \to \mathbb{R}^2$ be a bounded function. If $f$ is integrable over $D_1$ and $D_2$ and $D_1 \cap D_2$ has content zero then $f$ is integrable on $D$ and

$$\int\int_D f = \int\int_{D_1} f + \int\int_{D_2} f.$$

# Boundary of a bounded region

Let $D \subseteq \mathbb{R}^2$ be a bounded set. A point in the boundary of $D$ is one which has a sequence in $D$ and a sequence in $\mathbb{R}^2 - D$ converging to it. The set of boundary points of $D$ is denoted by $\partial D$.

Example. $D = \{(x, y) \mid x^2 + y^2 \leq r^2\}$. The boundary of $D$, $\partial D = \{(x, y) \mid x^2 + y^2 = r^2\}$.

Example. $R = [a, b] \times [c, d]$. The boundary of rectangle $R$, $\partial R = \{(a, y) \in \mathbb{R}^2 \mid c \leq y \leq d\} \cup \{(b, y) \in \mathbb{R}^2 \mid c \leq y \leq d\} \cup \{(x, c) \in \mathbb{R}^2 \mid a \leq x \leq b\} \cup \{(x, d) \in \mathbb{R}^2 \mid a \leq x \leq b\}$.

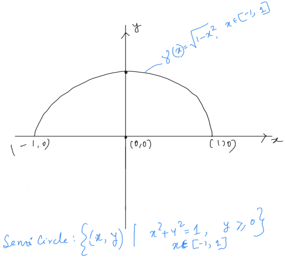
What is the boundary of the set $S = \{(x, y) \mid x, y \in \mathbb{Q}\}$? $\partial S = \mathbb{R}^2$.

Therefore for $f : D \to \mathbb{R}$ to be integrable we need $\partial D$ to be content zero and same should be true for the points of discontinuity of $f$ on $D$.

# Path and Curve

Convention : A *path* $\gamma$ in $\mathbb{R}^2$ (or $\mathbb{R}^3$) will mean a continuous function $\gamma : [a, b] \to \mathbb{R}^2$ (or $\gamma : [a, b] \to \mathbb{R}^3$) for $a, b \in \mathbb{R}$. It is said to be *closed* if $\gamma(a) = \gamma(b)$.

By a *curve* $\gamma$ we mean the image of a path $\gamma$ in $\mathbb{R}^2$ (or $\mathbb{R}^3$).



# Existence of Integrals over bounded sets in $\mathbb{R}^2$

**Theorem**

*Let $D \subset \mathbb{R}^2$ be a bounded set whose boundary $\partial D$ is given by finitely many continous closed curve then any bounded and continuous function $f : D \to \mathbb{R}$ is integrable over $D$.*

Example. Let $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$ and $f(x, y) = x^2 + y^2, \quad \forall (x, y) \in D$. Then $f$ is integrable over $D$.

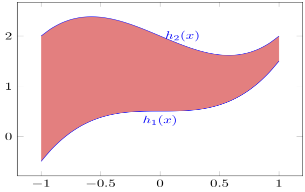
A slightly more general theorem is as follows:
Let $D$ be a bounded set in $\mathbb{R}^2$ such that $\partial D$ is of content zero. Let $f : D \to \mathbb{R}$ be a bounded function whose points of discontinuity have 'content zero'. Then $f$ is integrable over $D$.

# Elementary region: Type 1

Let $h_1, h_2 : [a, b] \to \mathbb{R}$ be two continuous functions such that $h_1(x) \leq h_2(x)$ for all $x \in [a, b]$. Consider the set of points

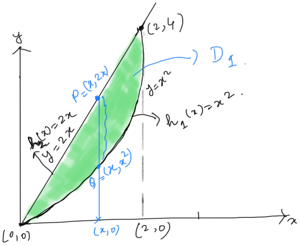
$$D_1 = \{(x, y) \,|\, a \leq x \leq b \text{ and } h_1(x) \leq y \leq h_2(x)\}$$

Such a region is said to be of *Type 1* and for every $x \in \mathbb{R}$ vertical cross-section of $D_1$ is an interval.



# Type 1 contd.

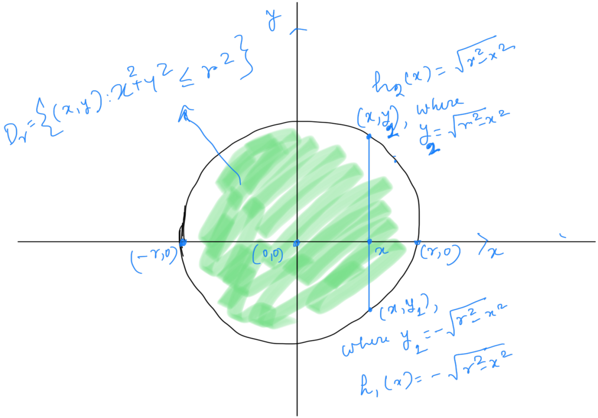
Example I: $D_1 = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, \quad x^2 \leq y \leq 2x\}$. Here for all $x \in [0, 2]$, $h_1(x) = x^2$ and $h_2(x) = 2x$. Note $h_1(x) \leq h_2(x)$ for $x \in [0, 2]$.



Example II: The closed disc $D_r$ of radius $r$ around the origin,

$$D_r := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\}.$$

Take $h_1(x) = -\sqrt{r^2 - x^2}$ and $h_2(x) = \sqrt{r^2 - x^2}$. We see that $D_r$ is of Type 1.



# Integrability on Type 1 region

For Type 1, when $D_1 = \{(x, y) \,|\, a \le x \le b \text{ and } h_1(x) \le y \le h_2(x)\}$, the boundary

$$\begin{aligned}\partial D_1 = \{(a, y) \mid h_1(a) \le y \le h_2(a)\} \cup \{(b, y) \mid h_1(b) \le y \le h_2(b)\} \\ \cup\{(x, h_1(x)) \mid a \le x \le b\} \cup \{(x, h_2(x)) \mid a \le x \le b\}\end{aligned}.$$

The region $D_1$ is bounded by continuous curves (the straight lines $x = a$ and $x = b$ and the graphs of the curves $y = h_1(x)$ and $y = h_2(x)$).

Thus $\partial D_1$ is of 'content zero' in $\mathbb{R}^2$.

Hence any continuous function defined on $D_1$ is integrable over the elementary region $D_1$.

# Evaluating integrals on regions of Type 1

Let $D$ be a region of Type 1 and assume that $f : D \to \mathbb{R}$ is continuous. Let $D \subset R = [\alpha, \beta] \times [\gamma, \delta]$ and let $f^*$ be the corresponding function on $R$ (obtained by extending $f$ by zero).

The region $D$ is bounded by continuous curves (the straight lines $x = a$ and $x = b$ and the graphs of the curves $y = h_1(x)$ and $y = h_2(x)$). Hence we can conclude that $f^*$ is integrable on $R$. Applying Fubini's theorem on $f^*$ we get,

$$\int\int_D f(x,y)dxdy := \int\int_R f^*(x,y)dxdy = \int_\alpha^\beta \left[\int_\gamma^\delta f^*(x,y)dy\right] dx.$$

In turn, this gives

$$\int_\alpha^\beta \left[\int_{h_1(x)}^{h_2(x)} f^*(x,y)dy\right] dx = \int_a^b \left[\int_{h_1(x)}^{h_2(x)} f(x,y)dy\right] dx,$$

since $f^*(x,y) = 0$ if $y < h_1(x)$ or $y > h_2(x)$.

# Examples

Example Let $D = \{(x, y) \mid 0 \leq x \leq 2, \quad x^2 \leq y \leq 2x\}$ and $f(x, y) = x + y$. Find $\int\int_D f(x, y)dxdy$.

Ans Since $D$ is a Type 1 region and $f$ is continuous over $D$, $f$ is integrable over $D$.

$$\int\int_D f(x,y)\,dxdy = \int_0^2 \Big(\int_{x^2}^{2x} (x+y)\,dy\Big)dx = \int_0^2 [xy + \frac{y^2}{2}]_{y=x^2}^{y=2x}\,dx$$
$$= \int_0^2 [2x^2 + 4\frac{x^2}{2} - x^3 - \frac{x^4}{2}]\,dx$$

Example Let $D = \{(x, y) \mid x^2 + y^2 \leq 1, \quad x \geq 0, \quad y \geq 0\}$ and $f(x, y) = \sqrt{1 - y^2}$. Find $\int\int_D f(x, y)dxdy$.

Ans Type 1, i.e, $D = \{(x, y) \mid 0 \leq x \leq 1, \quad 0 \leq y \leq \sqrt{1 - x^2}\}$. Then

$$\int\int_D f(x,y)\,dxdy = \int_0^1 \Big(\int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\,dy\Big)\,dx.$$

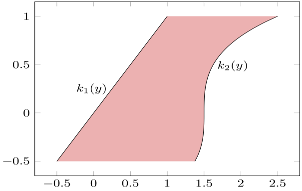
Not easy to compute!

# Elementary region: Type 2

Similarly, if $k_1, k_2 : [c, d] \to \mathbb{R}$ are two continuous functions such that $k_1(y) \le k_2(y)$, for all $y \in [c, d]$. The set of points

$$D_2 = \{(x, y) \mid c \le y \le d \text{ and } k_1(y) \le x \le k_2(y)\}$$

is called a region of Type 2 and for every $y \in \mathbb{R}$ horizontal cross-section of $D_2$ is an interval.




Example $D_2 = \{(x, y) \mid x^2 + y^2 \le 1\}$. If we take $k_1(y) = -\sqrt{1 - y^2}$ and $k_2(y) = \sqrt{1 - y^2}$, we see that $D_2$ is of Type 2.

# Evaluating integrals on regions of type 2

Note that the boundary of $D_2$ is of content zero in $\mathbb{R}^2$. Hence any continuous function defined on $D_2$ is integrable over the elementary region.

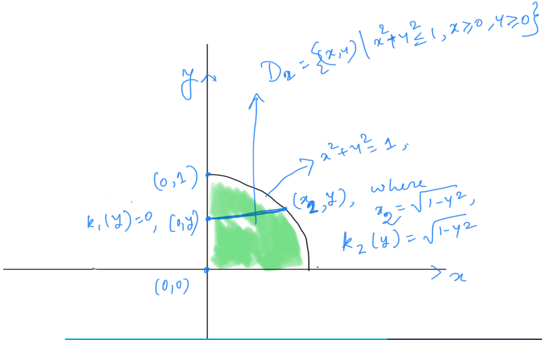
Using exactly the same reasoning as in the previous case (basically, interchanging the roles of $x$ and $y$) we can obtain a formula for regions of Type 2.

Let $D$ be a bounded set of Type 2 in $\mathbb{R}^2$. Let $f : D \to \mathbb{R}$ be a continuous function on $D$. We get

$$\int\int_D f(x,y)dxdy = \int_c^d \left[\int_{k_1(y)}^{k_2(y)} f(x,y)dx\right] dy.$$

Example: Let $D = \{(x,y) \mid x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}$. Evaluate $\int\int_D \sqrt{1-y^2}dxdy$.

Ans.
$$\int\int_D \sqrt{1-y^2}dxdy = \int_0^1 \left( \int_0^{\sqrt{1-y^2}} \sqrt{1-y^2}\, dx \right) dy$$
$$= \int_0^1 [x\sqrt{1-y^2}]_{x=0}^{\sqrt{1-y^2}}\, dy = \int_0^1 (1-y^2)\, dy = \frac{2}{3}.$$



# Remark

Both of these formulæ can be viewed as special cases of Cavalieri's principle when $f(x, y) \geq 0$. In the first case we are slicing by planes perpendicular to the $x$-axis, while in the second case, we are slicing by planes perpendicular to the $y$-axis.

Caution! There exist bounded subsets of $\mathbb{R}^2$ which are not elementary regions; for example, *star-shaped subset* of $\mathbb{R}^2$ or an *annulus*.

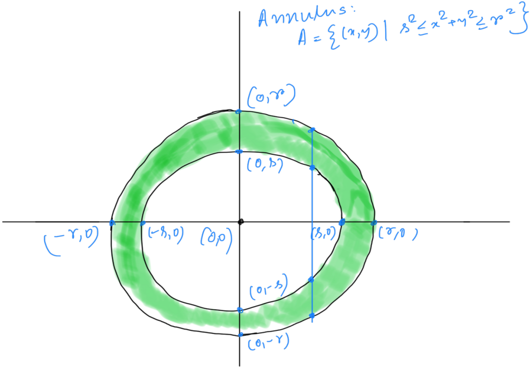
Often we can write $D$ as a union of regions of Types 1 and 2 and then we call it a region of type 3.

We could also view the disc as a region of type 3, by dividing it into four quadrants.

# Remark contd.

What about the *annulus* $A = \{(x,y) \in \mathbb{R}^2 \mid s^2 \le x^2 + y^2 \le r^2\}$?

Is it a type 3 region? yes



# The mean value theorem for double integrals

**Theorem**

*If $D$ is an elementary region in $\mathbb{R}^2$, and $f : D \to \mathbb{R}$ is continuous. There exists $(x_0, y_0)$ in $D$ such that*

$$f(x_0, y_0) = \frac{1}{A(D)} \int\int_D f(x, y) dA.$$

The proof follows using the boundedness of $f(x, y)$ and mean value theorem for continuous functions . How does one interpret the above statement geometrically?

In summary,

- If $D$ is a bounded region in $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ is bounded, then consider any rectangle $R$ containing the region $D$ in $\mathbb{R}^2$ and extend $f$ to the rectangle by 0 outside $D$ and denote it by $f^*$. The integral of $f$ over $D$ is defined by the integral of $f^*$ on the rectangle $R$.
- The above definition is consistent because the definition of integral of $f$ on $D$ is independent of the choice of rectangle $R$.
- To determine the integrability of $f$ over region $D$, conditions on $f$ and $D$? The boundary of $D$ should be 'well-behaved'. The set containing points of discontinuity of $f$ is of 'content zero'.
- Algebraic properties of integrals on $D$ are similar to that of the integrals on rectangle.
- To evaluate the value, use Fubini's theorem.
- To apply Fubini's theorem, hardest part is to determine the lower limit and upper limit of the integration: elementary regions Type 1 and Type 2 or combination of both.

wherever you find x^2+y^2 then substitute by rcostheta and rsintheta.

Example 1: Compute the integral of $f(x, y) = x^2 + y^2$ on $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

Can we compute this integral using iterated integrals ?

Example 2: Compute the integral of $g(x, y) = e^{x^2+y^2}$ on $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

Can we use substitution like we did in one variable?

Let us see what happens when we use polar coordinates.

# Polar Coordinates

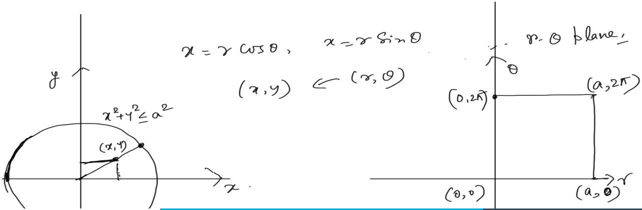
Change of variables from Cartesian coordinate system to polar coordinate system, any $(x, y) \in \mathbb{R}^2$ in Cartesian coordinate can be written as

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad r > 0, \theta \in [0, 2\pi].$$

Transformation of region under change of variables:

Ex. $D := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq a^2\}$ is transformed in polar coordinate system as a rectangle

$$D^* = \{(r, \theta) \mid 0 \leq r \leq a, \quad \theta \in [0, 2\pi]\}.$$



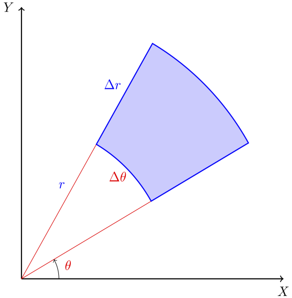
# The integral in polar coordinates

Let $D^*$ be a subset of $\mathbb{R}^2$ in polar coordinate system, such that for all $(r, \theta) \in D^*$, $(r\cos(\theta), r\sin(\theta)) \in D$, for $0 \le r \le 1$, and

$$g(r, \theta) := f(r\cos(\theta), r\sin(\theta)), \quad (r, \theta) \in D^*.$$

To integrate the function $g$ on a domain $D^*$ we need to cut up $D^*$ into small rectangles, but these will be rectangles in the $r$-$\theta$ coordinate system.

What shape does a rectangle $[r, r + \Delta r] \times [\theta, \theta + \Delta\theta]$ represent in the $x$-$y$ plane? A part of a sector of a circle.




Then we will be integrating over this sector instead of rectangle.

What is the area of this part of a sector?

Ans: It is $\frac{1}{2} \cdot [(r + \Delta r)^2 \Delta\theta - r^2 \Delta\theta] \sim r^* \Delta r \Delta\theta$, $r \leq r^* \leq r + \Delta r$.

Partitioning the region into subrectangles is equivalent to partitioning the region into parts of sectors as shown earlier.

It follows that the integral we want is approximated by a sum of the form

$$\sum_i \sum_j g(r_i^*, \theta_j^*) r_i^* \Delta r_i \Delta\theta_j,$$

where $\{(r_i^*, \theta_j^*\}$ is a tag for the partition of the "rectangle" in polar coordinates and

$$\int\int_D f(x,y)dxdy = \int\int_{D^*} f(r\cos\theta, r\sin\theta)rdrd\theta,$$

where $D$ is the image of the region $D^*$.

This is the change of variable formula for polar coordinates.

# Examples

Example1: Integrate $f(x,y) = x^2 + y^2$ on $D = \{(x,y) \mid x^2 + y^2 \leq 1\}$.

Solution: Let us use polar coordinates. Let

$$D^* = \{(r,\theta) \mid 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi\}.$$

Denoting $x = r\cos\theta$ and $y = r\sin\theta$, the polar coordinates will transform $D^*$ to $D$ and

$$g(r,\theta) = f(r\cos\theta, r\sin\theta) = r^2.$$

$$\begin{aligned} \int\int_D f(x,y)\ dxdy &= \int\int_{D^*} g(r,\theta)\ r\ drd\theta = \int\int_{[0,1]\times[0,2\pi]} r^2.r\ drd\theta \\ &= \int_0^{2\pi}\int_0^1 r^3\ drd\theta = \int_0^{2\pi} \frac{r^4}{4}\Big|_0^1\ d\theta = \frac{\pi}{2} \end{aligned}$$

# Examples contd.

Example 2: Integrate $f(x,y) = e^{x^2+y^2}$ on $D = \{(x,y) \mid x^2+y^2 \leq 1\}$.
Solution: Using the same transformation as above

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we get

$$\begin{aligned}
\int\int_D f(x,y)\ dxdy &= \int\int_{D^*} g(r,\theta)\ r\ drd\theta = \int\int_{[0,1]\times[0,2\pi]} e^{r^2} r\ drd\theta \\
&= \int_0^{2\pi}\int_0^1 e^{r^2} r\ drd\theta = \int_0^{2\pi} \frac{e^{r^2}}{2}\Big|_0^1\ d\theta = \pi(e-1)
\end{aligned}$$

# An Applicaton:The integral of the Gaussian

We would like to evaluate the following integral:

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx.$$

What does this integral mean? - so far we have only looked at Riemann integrals inside closed bounded intervals, so the end points were always finite numbers $a$ and $b$.

An integral like the one above is called an improper integral. We can assign it a meaning as follows. It is defined as

$$\lim_{T\to\infty} \int_{-T}^{T} e^{-x^2}\,dx,$$

provided, of course, this limit exists. We will see how to evaluate this.

# The most amazing trick ever

Consider

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \cdot \int_{-\infty}^{\infty} e^{-y^2} dy.$$

We view this product as an iterated integral!

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dxdy.$$

Now under polar coordinates, the plane is sent to the plane. Hence, we can write this as

$$\int_0^{2\pi} \left[ \int_0^{\infty} e^{-r^2} rdr \right] d\theta.$$

But we can now evaluate the inner integral. Hence, we get

$$\int_0^{2\pi} \left[ -\frac{1}{2} e^{-r^2} \Big|_0^{\infty} \right] d\theta = \int_0^{2\pi} \frac{1}{2} d\theta = \pi$$

# The answer

Since $I^2 = \pi$, we see that $I = \sqrt{\pi}$.

Using the above result you can easily conclude that

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}.$$

The integral above arises in a number of places in mathematics - in probability, the study of the heat equation, the study of the Gamma function and in many other contexts.

There are many other ways of evaluating the integral $I$, but the method above is easily the cleverest.

Example: Evaluate $\int\int_D (3x + 4y^2)\, dxdy$, where $D$ is the region in the upper half-plane bounded by the circled $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
Ans: The region

$$D = \{(x, y) \mid y \geq 0, \quad 1 \leq x^2 + y^2 \leq 4\}.$$

In polar coordinate, after using change of variables $x = r\cos\theta$ and $y = r\sin\theta$, in $r - \theta$ plane, $D$ becomes

$$D^* = \{(r, \theta) \mid 1 \leq r \leq 2, \quad 0 \leq \theta \leq \pi\}.$$

$$\int\int_D (3x + 4y^2)\, dxdy = \int_{\theta=0}^{\pi}\int_{r=1}^{2} (3r\cos\theta + 4r^2\sin^2\theta) r\, drd\theta$$
$$= \int_0^{\pi} [r^3\cos\theta + r^4\sin^2\theta]_{r=1}^{2}\, d\theta = \int_0^{\pi} [7\cos\theta + 15\sin^2\theta]\, d\theta = \frac{15\pi}{2}.$$

# The mean value theorem for double integrals

## Theorem

*If $D$ is an elementary region in $\mathbb{R}^2$, and $f : D \to \mathbb{R}$ is continuous. There exists $(x', y')$ in $D$ such that*

$$f(x', y') = \frac{1}{A(D)} \int\int_D f(x, y) dA.$$

The proof follows using the boundedness of $f(x, y)$ and mean value theorem for continuous functions .

Sketch of Proof Since $D$ is closed and bounded and $f$ is continuous, the function attains its maximum and minimum at some points $(x_0, y_0) \in D$ and $(x_1, y_1) \in D$ respectively. Since $D$ is an elementary region, there exists a path $\gamma : [0, 1] \to \mathbb{R}^2$ such that $\gamma(0) = (x_0, y_0) \in D$ and $\gamma(1) = (x_1, y_1)$.

Now apply the intermediate value theorem function $f \circ \gamma : [0, 1] \to \mathbb{R}$.

# Average value contd.

How does one interpret the above statement geometrically?
If $f(x,y) \geq 0$, $f(x_0,y_0)$, the solid region under the graph of $f$ and over the region $D$ is same as the volume of the region over $D$ whose height is the average value or mean value of $f$ defined above.i.e.,

$$f(x_0,y_0) \times A(D) = \int\int_D f(x,y)dxdy.$$

Application: Center of Mass of a thin plate: (Weighted average): Let a plate occupies a region $D$ of the $x-y$ plane and $\rho(x,y)$ be its density at a point $(x,y)$ in $D$. Let $\rho$ be a positive continuous function on $D$. The the coordinate of the center of mass $(\bar{x},\bar{y})$ is given by

$$\bar{x} = \frac{\int\int_D x\rho(x,y)\,dxdy}{\int\int_D \rho(x,y)\,dxdy}, \quad \bar{y} = \frac{\int\int_D y\rho(x,y)\,dxdy}{\int\int_D \rho(x,y)\,dxdy}.$$

Note that for $\rho \equiv 1$, $\bar{x}$ is the average of $f(x,y) = x$ over the region $D$ and $\bar{y}$ is the average of $g(x,y) = y$ over the region $D$.

# Generalising integration for $n \geq 3$

Recall our definition of Darboux integrals and Riemann integrals. Both these definitions have an analogue in dimensions $n \geq 3$.

In this course, we only extend these ideas to functions on 3 variables. Note we already cannot imagine the graph of a function of 3 variables and much of the geometry is lost.

As an exercise you can think about which of the following definitions are specific to $n = 3$ and which can be generalized further.

If we have a bounded function $f : B = [a, b] \times [c, d] \times [e, f] \to \mathbb{R}$, we can integrate it over this rectangular cuboid (which we often refer to as a cuboid.) We divide the rectangular cuboid into smaller ones $B_{ijk}$, making sure that the length, breadth and height of the subcuboids are all small.

# Integrals over rectangular cuboids

In particular, we can use the regular partition of order $n$ to obtain the Riemann sum

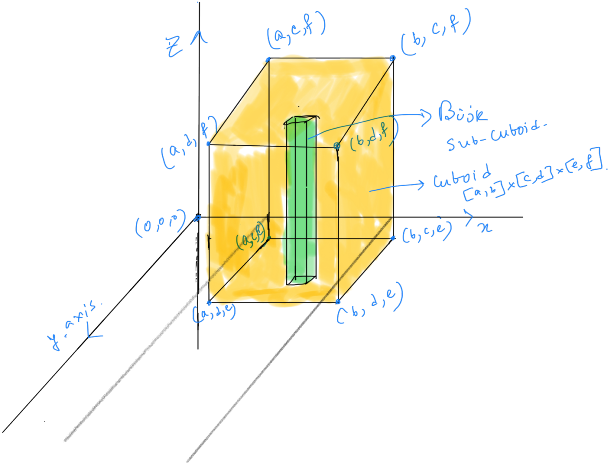
$$S(f, P_n, t) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} f(t_{ijk})\Delta B_{ijk},$$

where $\Delta B_{ijk}$ is the volume of $B_{ijk}$, and $t = \{t_{ijk} \in B_{ijk}\}$ is an arbitrary tag.

As before we say that $f$ is integrable if $\lim_{n\to\infty} S(f, P_n, t)$ converges to some fixed $S \in \mathbb{R}$ for any choice of tag $t$. The value of this limit is denoted by

$$\iiint_B f dV, \iiint_B f(x, y, z) dV \quad \text{or} \quad \iiint_B f(x, y, z) dxdydz.$$

All the theorems for integrals over rectangles go through for integrals over rectangular cuboids.



Z
(a,c,f)
(b,c,f)
(a,d,f)
(b,d,f)
Box
sub-cuboid.
cuboid
[a,b]×[c,d]×[e,f].
(0,0,0)
(a,c,e)
(b,c,e)
x
(a,d,e)
(b,d,e)
y-axis.

# Integrating over bounded regions $B$ in $\mathbb{R}^3$

First, if $f : B \subset \mathbb{R}^3 \to \mathbb{R}$ is bounded and continuous in $B$, except possibly on (a finite union of) graphs of continuous functions of the form $z = a(x, y)$, $y = b(x, z)$ and $x = c(y, z)$, then it is integrable.

This allows us to define the integral of (say) a continuous function on any bounded region $B$ whose boundary is a set of content zero in $\mathbb{R}^3$. Let $B^*$ be a cuboid enclosing the bounded region and $f^* : B^* \to \mathbb{R}$ be defined as $f$ on $B$ and 0 elsewhere.

Then integral of $f$ over $B$ exists if integral of $f^*$ over $B^*$ exists and

$$\iiint_{B^*} f^* = \iiint_B f.$$

Once we have defined the triple integral in this way, it remains to evaluate it.

# Evaluating triple integrals: Fubini's Theorem

Fubini's Theorem can be generalized - that is, triple integrals can usually be expressed as iterated integrals, this time by integrating functions of a single variable three times.

Let $f$ be integrable on the cuboid $B$. Then any iterated integral that exists is equal to the triple integral; i.e.,

$$\iiint_B f(x, y, z)dxdydz = \int_a^b \int_c^d \int_e^f f(x, y, z)dzdydx,$$

provided the right hand side iterated integral exists.
There are, in fact, five other possibilities for the iterated integrals.

We have a theorem saying if $f$ is integrable, whenever any of these iterated integral exists, it is equal to the value of the integral of $f$ over $B$. If $f$ is continuous on $B$, then $f$ is integrable on $B$ and all iterated integrals exist and their values are equal to the integral of $f$ on $B$.

# Elementary regions in $\mathbb{R}^3$

The triple integrals that are easiest to evaluate are those for which the region $W$ in space can be described by bounding $z$ between the graphs of two functions in $x$ and $y$ with the domain of these functions being an elementary region in two variables.

For example,

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid \gamma_1(x, y) \leq z \leq \gamma_2(x, y), (x, y) \in D\},$$

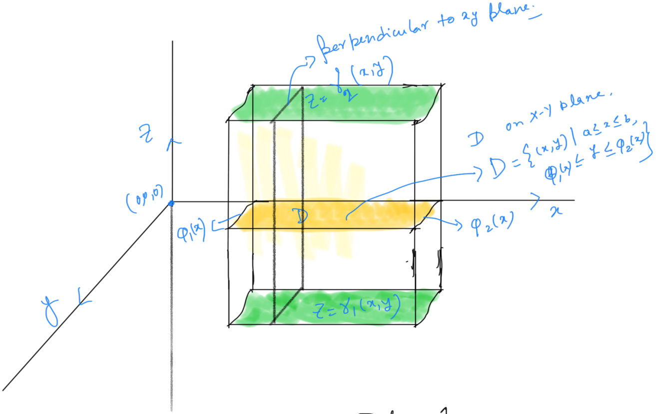
where $\gamma_1$ and $\gamma_2$ are continuous on $D \subset \mathbb{R}^2$ and $D$ is an elementary region in $\mathbb{R}^2$. For rxample, if $D$ is Type 1, then

$$D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \quad \phi_1(x) \leq y \leq \phi_2(x)\},$$

where $\phi_1 : [a, b] \to \mathbb{R}$ and $\phi_2 : [a, b] \to \mathbb{R}$ are continuous functions. The region $D$ can be Type 2 also.

## Example:

- The region $W$ between the paraboloid $z = x^2 + y^2$ and the plane $z = 2$.
- The region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 4$ and $x = z - y - 1$.




When D is Type 1.

# Elementary regions (Example)

Suppose that the region $W$ lies between $z = \gamma_1(x, y)$ and $z = \gamma_2(x, y)$. Suppose that the projection of $W$ on the $xy$ plane is bounded by the curves $y = \phi_1(x)$ and $y = \phi_2(x)$ and the straight lines $x = a$ and $x = b$, then for a continuous function $f$ defined over $W$, we have

$$\iiint_W f(x, y, z)dxdydz = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x, y, z)dzdydx.$$

Example: Let us find the volume of the sphere using the above formula. In other words, let us integrate the function 1 on the region $W$, where $W$ is the unit sphere, i.e.,

$$\int\int\int_W 1dxdydz = ?, \quad \text{where} \quad W = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}.$$

# The volume of the unit sphere

The sphere can be described as the region lying between $z = -\sqrt{1-x^2-y^2}$ and $z = \sqrt{1-x^2-y^2}$.

The projection of the sphere onto the $xy$ plane gives a disc of unit radius. This can be described as the set of points lying between the curves $-\sqrt{1-x^2}$ and $\sqrt{1-x^2}$ and the lines $x = \pm 1$. Thus our triple integral reduces to the iterated integral

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dzdydx.$$

This yields

$$2\int_{-1}^{1}\left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(1-x^2-y^2)^{1/2}dy\right]dx.$$

After evaluating the inner integral we obtain

$$2\pi\int_{-1}^{1}\frac{1-x^2}{2}dx = \frac{4}{3}\pi.$$