MA 109 D3 Lecture 11

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Recap: Darboux and Riemann integration

The fundamental theorem of calculus

The Riemann integral

Definition 1: A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|R(f, P, t) - R| < \epsilon,$$

whenever $\|P\| < \delta$. In this case R is called the Riemann integral of the function f on the interval $[a, b]$.

Definition 2: A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ and a partition P, such that for every tagged refinement (P', t') of P with $\|P'\| \leq \delta$,

$$|R(f, P', t') - R| < \epsilon.$$

The nice thing about the above definition is that one only has to check that $|R(f, P', t') - R|$ is small for for refinements of a fixed partition, and not for all partitions.

Comparison with the Darboux integral

Theorem 20: The Riemann integral (using either definition) exists if and only if the Darboux integral exists and in this case the two integrals are equal.

Theorem 21: Let $f : [a, b] \to \mathbb{R}$ be a function that is bounded, and continuous at all but finitely many points of $[a, b]$. Then f is Riemann integrable on $[a, b]$.

Because of Theorem 20, we are now free to use Darboux integrability and Riemann integrability interchangably.

An example of a function that is not Darboux/Riemann integrable

Here is a function that is not Darboux integrable of $[0, 1]$. Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

It should be clear that no matter what partition one takes the infimum on any sub-interval in the partition will be 0 and the supremum will be 1.

From this one can see immediately that

$$L(f, P) = 0 \neq 1 = U(f, P),$$

for every P, and hence that $L(f) = 0 \neq 1 = U(f)$.

Another property of the Riemann Integral

Theorem 23: Suppose f is Riemann integrable on $[a, b]$ and $c \in [a, b]$. Then

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$$

Proof: First we note that if $c = a$ or $c = b$, there is nothing to prove.

Next, if $c \in (a, b)$ we proceed as follows. If P_1 is a partition of $[a, c]$ and P_2 is a partition of $[c, b]$, then $P_1 \cup P_2 = P'$ is obviously a partition of $[a, b]$. Thus, partitions of the form $P_1 \cup P_2$ constitute a subset of the set of all partitions of $[a, b]$. For such partitions P' we have

$$L(f, P') = L(f, P_1) + L(f, P_2).$$

Let us denote by $L(f)_{[a,c]}$ (resp. $L(f)_{[c,b]}$) the Darboux lower integral of f on the interval $[a, c]$ (resp. $[c, b]$).

If we take the supremum over all partitions P_1 of $[a, c]$ and P_2 of $[c, b]$ we get

$$\sup_{P'} L(f, P') = L(f)_{[a,c]} + L(f)_{[c,b]}.$$

Now the supremum on the left hand side is taken only over partitions P' having the special form $P_1 \cup P_2$. Hence it is less than or equal to $\sup_P L(f, P)$ where this supremum is taken over all partitions P. We thus obtain

$$L(f)_{[a,c]} + L(f)_{[c,b]} \leq L(f).$$

On the other hand, for any partition $P = \{a < x_1 < \ldots x_{n-1} < b\}$ we can consider the partition $P' = P \cup \{c\}$. This will be a refinement of the partition P and can be written as a union of two partitions P_1 of $[a, c]$ and P_2 of $[c, b]$.

Remarks not made in class

Note that if f is Riemann integrable on $[a, b]$ it is automatically Riemann integrable on $[a, c]$ and $[c, b]$. With notation as above, note that

$$0 \leq U(f, P_1) - L(f, P_1) \leq U(f, P') - L(f, P'),$$

so if the second expression is small, the first one will be too. Thus f will be integrable on $[a, c]$, and similarly for $[c, b]$.
Two students (including Arrol - unfortunately I did not get the other student's name) have pointed this out and have also pointed out that one can then be more efficient with the proof. My thanks to them. However, I prefer to not to revisit the proof on these slides.

By the property for refinements for Darboux sums we know that $L(f, P) \leq L(f, P')$.

Thus, given any partition P of $[a, b]$, there is a refinement P' which can be written as the union of two partitions P_1 and P_2 of $[a, c]$ and $[c, b]$ respectively, and by the above inequality,

$$\sup_P L(f, P) \leq \sup_{P'} L(f, P'),$$

where the first supremum is taken over all partitions of $[a, b]$ and the second only over those partitions P' which can be written as a union of two partitions as above. This shows that

$$L(f) \leq L(f)_{[a,c]} + L(f)_{[c,b]},$$

so, together with the previous inequality, we get

$$L(f) = L(f)_{[a,c]} + L(f)_{[c,b]}.$$

The same kind of reasoning applies to the upper sums which allows us to prove the required property.

Motivation

The Fundamental Theorem of calculus allows us to relate the process of Riemann integration to the process of differentiation. Essentially, it tells us that integrating and differentiating are inverse processes. This is a tremendously useful theorem for several reasons.

It turns out that (Riemann) integrating even simple functions is much harder than differentiating them (if you don't believe me, try integrating $(\tan x)^3$ via Riemann sums!). In practice, however, integration is what we need to do to solve physical problems. Usually, when we are studying the motion of a particle or a planet what we find is that the position of a particle, which is a function of time, satisfies some differential equation. Solving the differential equation involves performing the inverse operation of taking some combination of derivatives. The simplest such inverse operation is taking the inverse of the first derivative, which the Fundamental Theorem says, is the same as integrating.

Calculating Integrals

Thus, calculating integrals is one of the basic things one needs to do for solving even the simplest physics and engineering problems. The problem is that this is quite difficult to do.

Once we know the derivatives of some basic functions (polynomials, trigonometric functions, exponentials, logarithms) we can differentiate a wide class of functions using the rules for differentiation , especially the product and chain rules. By contrast, the only rule for Riemann integration that can be proved from the basic definitions is the sum rule.

The Fundamental Theorem solves this problem (partially) because it allows us to deduce formalæ for the integrals of the products and the composition of functions from the corresponding rules for derivatives.

The Fundamental Theorem - Part I

Theorem 24 (Part I): Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let

$$F(x) = \int_a^x f(t)dt$$

for any $x \in [a, b]$. Then $F(x)$ is continuous on $[a, b]$, differentiable on (a, b) and

$$F'(x) = f(x),$$

for all $x \in (a, b)$.

Proof: We know that $f(t)$ is Riemann integrable for any $x \in [a, b]$ because of Theorem 21 (every continuous function is Riemann integrable).

The proof of Part I continued

By Theorem 23 we know that

$$\int_a^{x+h} f(t)dt = \int_a^x f(t)dt + \int_x^{x+h} f(t)dt,$$

for $x + h \in [a, b]$. Hence

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \cdot \int_x^{x+h} f(t)dt.$$

We know that if $f(t) \leq g(t)$ on $[a, b]$, then $\int f(t)dt \leq \int g(t)dt$. We apply this to the three functions $m(h)$, f and $M(h)$, where $m(h)$ and $M(h)$ are the constant functions given by the minimum and maximum of the function f on $[x, x + h]$ to get:

$$m(h) \cdot h \leq \int_x^{x+h} f(t)dt \leq M(h) \cdot h.$$

Dividing by h and taking the limit gives

$$\lim_{h \to 0} m(h) \leq \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \leq \lim_{h \to 0} M(h).$$

But f is a continuous function, so $\lim_{h\to 0} m(h) = \lim_{h\to 0} M(h) = f(x)$. By the Sandwich theorem for limits (use version 2), we see that limit in the middle exists and is equal to $f(x)$, that is $F'(x) = f(x)$. This proves the first part of the Fundamental Theorem of Calculus. $\square$

This first form of the Fundamental Theorem allows us to compute definite integrals. Keeping the notation as in the Theorem we obtain

Corollary:

$$\int_c^d f(t)dt = F(d) - F(c),$$

for any two points $c, d \in [a, b]$.

The Fundamental Theorem of Calculus Part 2

Theorem 24 (Part II): Let $f : [a, b] \to \mathbb{R}$ be given and suppose there exists a continuous function $g : [a, b] \to \mathbb{R}$ which is differentiable on (a, b) and which satisfies $g'(x) = f(x)$. Then, if f is Riemann integrable on $[a, b]$,

$$\int_a^b f(t)dt = g(b) - g(a).$$

Note that this statement does not assume that the function $f(t)$ is continuous, and is thus stronger than the corollary just stated.

Proof: We can write:

$$g(b) - g(a) = \sum_{i=1}^{n} [g(x_i) - g(x_{i-1})],$$

where $\{a = x_0, x_1, \ldots, x_n = b\}$ is an arbitrary partition of $[a, b]$. Using the mean value theorem for each of the intervals $I_j = [x_j, x_{j-1}]$, we can write

The proof of the Fundamental Theorem part II continued

$$g(x_i) - g(x_{i-1}) = g'(c_i)(x_i - x_{i-1}).$$

where $c_i \in (x_{i-1}, x_i)$.

Substituting this in the previous expression and using the fact that $g'(c_i) = f(c_i)$, we get

$$g(b) - g(a) = \sum_{i=1}^{n} [f(c_i)(x_i - x_{i-1})].$$

The calculation above is valid for any partition The right hand side obviously represents a Riemann sum. By hypothesis f is Riemann integrable. It follows (using Definition 1, for example) that as $\|P\| \to 0$, the right hand side goes to the Riemann integral. $\square$