CS 105: DIC on Discrete Structures

Graph theory

Basic terminology, Bipartite graphs and a characterization

Lecture 27
Oct 19 2023

Some simple types of Graphs

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- paths, cycles.

Some simple types of Graphs

- We have already seen some: connected graphs.
- paths, cycles.
- Are there other interesting classes of graphs?

Bipartite graphs

Definition

A graph is called bipartite, if the vertices of the graph can be partitioned into $V = X \cup Y$, $X \cap Y = \emptyset$ s.t., $\forall e = (u, v) \in E$,

- either $u \in X$ and $v \in Y$
- or $v \in X$ and $u \in Y$

Example: m jobs and n people, k courses and ℓ students.

- How can we check if a graph is bipartite?
- Can we characterize bipartite graphs?

Characterizing bipartite graphs using cycles.

- Recall: A path or a cycle has length n if the number of edges in it is n.
- A path (or cycle) is call odd (or even) if its length is odd (or even, respectively).

Exercise: Prove or Disprove:

Every closed odd walk contains an odd cycle.

Characterizing bipartite graphs using cycles.

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Every closed odd walk contains an odd cycle.

Proof: By induction on the length of the given closed odd walk.
Exercise!

Characterizing bipartite graphs using cycles.

Lemma

Every closed odd walk contains an odd cycle.

Theorem, Konig, 1936

A graph is bipartite iff it has no odd cycle.

Proof:

- ($\Longrightarrow$) direction is easy.

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- ($\implies$) direction is easy.
- Let G be bipartite with $(V = X \cup Y)$. Then, every walk in G alternates between X, Y.

$\implies$ if we start from X, each return to X can only happen after an even number of steps.

$\implies$ G has no odd cycles.

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- ($\Longleftarrow$) Suppose G has no odd cycle, then let us construct the bipartition. Wlog assume G is connected.

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- ($\Longleftarrow$) Suppose G has no odd cycle, then let us construct the bipartition. Wlog assume G is connected.
- Let $u \in V$. Break V into

$$X = \{v \in V \mid \text{length of shortest path } P_{uv} \text{ from } u \text{ to } v \text{ is even}\},$$
$$Y = \{v \in V \mid \text{length of shortest path } P_{uv} \text{ from } u \text{ to } v \text{ is odd}\},$$

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- If there is an edge vv' between two vertices of X or two vertices of Y, this creates a closed odd walk: $uP_{uv}vv'P_{v'u}u$.

Characterizing bipartite graphs using cycles.

Lemma DOUBT

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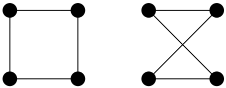
- If there is an edge vv' between two vertices of X or two vertices of Y, this creates a closed odd walk: $uP_{uv}vv'P_{v'u}u$.
- By Lemma, it must contain an odd cycle: contradiction.
- This along with $X \cap Y = \emptyset$ and $X \cup Y = V$, implies X, Y is a bipartition. $\square$

Are these graphs the same?



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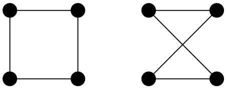

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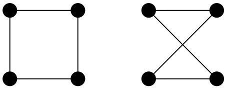

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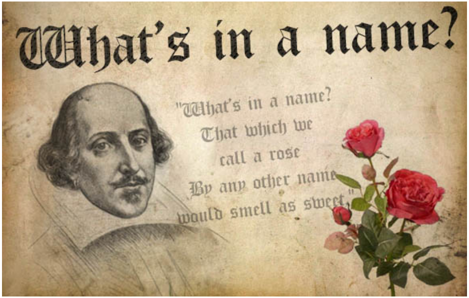
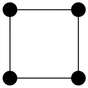


Representing and comparing graphs

We start with simple graphs...



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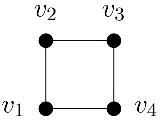

To represent it, we need to name the vertices...

- As an adjacency list:

v_1	v_2, v_4
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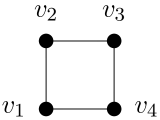

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- As an adjacency matrix:

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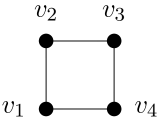

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- Are two given graphs the "same", wrt these properties?

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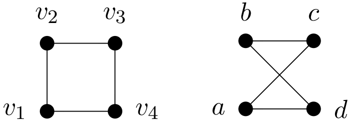

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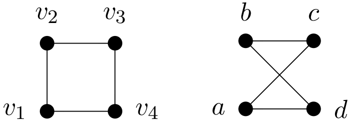

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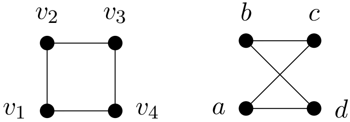

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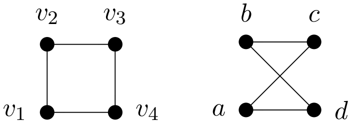

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- Reordering of vertices is same as applying a permutation to rows and colums of $A(G)$.
- So, it seems two graphs are “same” if by reordering and renaming the vertices we get the same graph/matrix.

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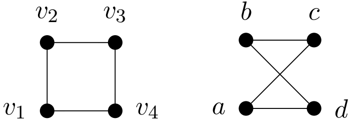

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- So, it seems two graphs are "same" if by reordering and renaming the vertices we get the same graph/matrix.
- How do we formalize this?

Isomorphism

Definition

An isomorphism from simple graph G to H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ iff $f(u)f(v) \in E(H)$.

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- Can be checked using adjacency matrix by reordering/renaming.
- What are the properties of this function/relation: $R = \{(G, H) \mid \exists \text{ an isomorphism from } G \text{ to } H\}$.

Isomorphism There is isomorphism iff there is bijection.

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The isomorphism relation is an equivalence relation.

- The equivalence classes are called isomorphism classes.
- When we talked about an "unlabeled" graph till now, we actually meant the isomorphism class of that graph!