MA 105 D3 Lecture 5

Ravi Raghunathan

Department of Mathematics

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Continuity - the definition

Definition: If $f : [a, b] \to \mathbb{R}$ is a function and $c \in [a, b]$, then f is said to be continuous at the point c if and only if

$$\lim_{x \to c} f(x) = f(c).$$

Thus, if c is one of the end points we require only the left or right hand limit to exist.

A function f on (a, b) (resp. $[a, b]$) is said to be continuous if and only if it is continuous at every point c in (a, b) (resp. $[a, b]$).

If f is not continuous at a point c we say that it is dicontinuous at c, or that c is a point of discontinuity for f.

Intuitively, continuous functions are functions whose graphs can be drawn on a sheet of paper without lifting the pencil of the sheet of paper. That is, there should be no "jumps" in the graph of the function.

The basic properties of continuous functions

The sums, differences, products and quotients of continuous functions are continuous (in the last case, the value of the continuous function in the denominator should be non-zero).

The composition of continuous functions is continuous.
Theorem 9: Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function. For every u between $f(a)$ and $f(b)$ there exists $c \in [a, b]$ there such that $f(c) = u$.

Functions which have this property are said to have the Intermediate Value Property (IVP).

We will not be proving this property - it is a consequence of the completeness of the real numbers. Intuitively, this is clear. Since one can draw the graph of the function without lifting one's pencil off the sheet of paper, the pencil must cut every line $y = e$ with e between $f(a)$ and $f(b)$.

The IVT in a picture

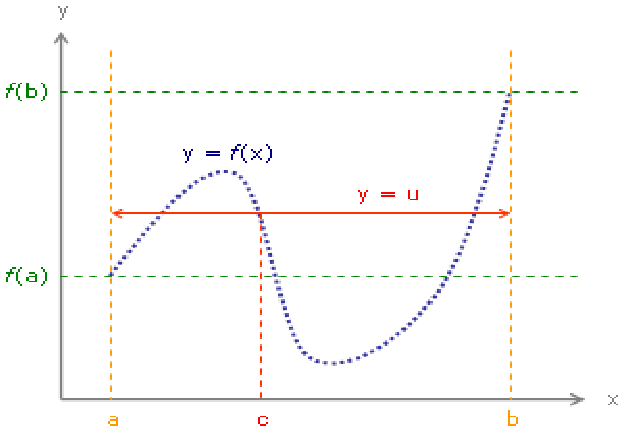


Image created by Enoch Lau see
http://en.wikipedia.org/wiki/File:Intermediatevaluetheorem.png

Continuous functions on closed, bounded intervals

The other major result on continuous functions that we need is the following. A closed bounded interval is one of the form $[a, b]$, where $-\infty < a$ and $b < \infty$.

Theorem 11: A continuous function on a closed bounded interval $[a, b]$ is bounded and attains its infimum and supremum, that is, there are points x_1 and x_2 in $[a, b]$ such that $f(x_1) = m$ and $f(x_2) = M$, where m and M denote the infimum and supremum respectively.

We defined infimum and supremum for sequences previously. The definition for functions of a real variable is the same: Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function. A real number M is called the supremum of $f(x)$ (on X) if
1. If $f(x) \leq M$ for all $x \in X$.
2. If for some real number M_1 $f(x) \leq M_1$ for all $x \in X$, then $M \leq M_1$.

Relaxing the conditions

Again, we will not prove Theorem 11, but will use it quite often. Note the contrast with open intervals. The function $1/x$ on $(0,1)$ does not attain a maximum - in fact it is unbounded. Similarly the function $1/x$ on $(1,\infty)$ does not attain a minimum, although, it is bounded below and the infimum is 0.

Exercise 5: In light of the above theorem, can you find a continuous function $g : (a,b) \to \mathbb{R}$ for part (i) of Exercise 1.11, with $c \in (a,b)$ such that $\lim_{x\to c} f(x)g(x) \neq 0$?
Let us look at Exercise 1.13 part (i).
Consider the function defined as $f(x) = \sin\frac{1}{x}$ when $x \neq 0$, and $f(0) = 0$. Is this function is continuous at $x = 0$.

Let us look at the sequence of points $x_n = 2/(2n+1)\pi$. Clearly $x_n \to 0$ as $n \to \infty$ and $|f(x_n)| = 1$

You can easily check that $f(x)$ has the IVP.

Sequential continuity

The preceding example showed that in order to demonstrate that a function $f(x)$ is not continuous at a point x_0 it is enough to find a sequence x_n tending to x_0 such that the value of the function $|f(x_n) - f(x_0)|$ remains large. Suppose it is not possible to find such a sequence. Does that mean the function is continuous at x_0? Yes.

Theorem 12: A function $f(x)$ is continuous at a point a if and only if for every sequence $x_n \to a$, $\lim_{x_n \to a} f(x_n) = f(a)$.

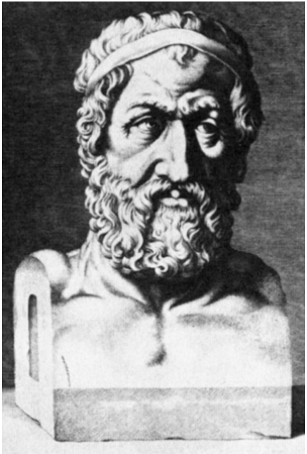
A function that satisfies the property that for every sequence $x_n \to a$, $\lim_{x_n \to a} f(x_n) = f(a)$ is said to be sequentially continuous. The theorem says that sequential continuity and continuity are the same thing. Indeed, it is clear that a continuous function is necessarily sequentially continuous. It is the reverse that is slightly harder to prove.

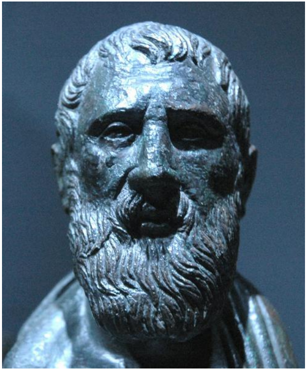
The first man to think about limits?

Zeno of Elea (490 - 460 BCE)
was a famous Greek philosopher
(source: Wikipedia)

Zeno of Elea

First let us record that we have no idea what Zeno looked like. The picture above was painted in the period 1588 - 1594 CE in Spain, about two thousand years after Zeno's time. Here are two more images of Zeno (also from Wikipedia)





Zeno's Paradoxes

I couldn't find out where the first statue came from and when it was made. The second seems to have come from Herculaneum in Italy (incidentally, Elea (modern Vilia) is a town in Italy). Now Herculaneum was destroyed by a volcanic erruption from the nearby volcano Vesuvius in 79 CE, so it looks like the bust was created within 500 years of Zeno's death. Maybe it was even made during his lifetime. Unfortunately, it is not clear whether this statue is one of Zeno of Elea or of another Zeno (of Citium) who lived about 150 years later.

The important about Zeno is that it would appear that he was the first human to think about limits and limiting processes, at least in recorded history. Most of what we know about him is through his paradoxes, nine of which survive in the works of another famous Greek philosopher Aristotle (384 - 322 BCE) , the official guru of Alexander the Great (aka Sikander in India).

Achilles and the tortoise

One of Zeno's motivations for stating his paradoxes seems to have been to defend his own guru Parmenides' philosophy (whatever that was). Anyway here is his most famous paradox as recorded by Aristotle.

Achilles and the tortoise:
In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead.

Aristotle, Physics VI:9, 239b15

General knowledge question: Who was Achilles?

Zeno's paradox animated

Achilles and the Tortoise

A gateway to infinite series

Nowadays, this line of argument does not really bother us, since we understand that an infinite number of terms (in this case consisting of the time travelled in each segment or the distance travelled in each segment) can add up to something finite. Nevertheless there are other philosophical issues that continued to bother mathematicians and physicists for a long time. After all, this kind of discussion does lead us to question whether intervals of time and space can be infinitely subdivided, or if "instantaneous motion" makes sense.

Since we are learning mathematics, we won't speculate on physics or philosophy, but we note that Zeno's argument gives a good way to derive the sum of an infinite geometric series. The geometric series is one of the simplest examples of infinite series, so let us see how this is done.

Geometric series - the formula

Let us suppose that the speed of achilles is v and that the speed of the tortoise is rv for some $0 < r < 1$. We will assume that the tortoise was given a headstart of distance "a".

- The distance covered by Achilles in time t is vt.
- The distance covered by the tortoise in time t is rvt.
- Achilles catches up with the tortoise when $vt = a + rvt$, that is, at time $t = a/(v - rv)$ and when the total distance covered by Achilles is $vt = a/(1 - r)$.

On the other hand,

- Distance covered by the tortoise by the time Achilles has covered distance a is ar.
- Distance covered by the tortoise by the time Achilles has covered distance ar is ar^2 ....
- Total distance covered by Achilles when he has caught up with the tortoise is $a + ar + ar^2 + \dots$.
- Thus we get $a + ar + ar^2 + \cdots = a/(1 - r)$.

The defintion

For now, if you did not understand the rigourous definition of the limit, forget about it. You will be able to understand what follows as long as you remember your 11th standard treatment of limits. Recall that $f : (a, b) \to \mathbb{R}$ is said to be differentiable at a point $c \in (a, b)$ if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists. In this case the value of the limit is denoted $f'(c)$ and is called the derivative of f at c. The derivative may also be denoted by $\frac{df}{dx}(c)$ or by $\frac{dy}{dx}|_c$, where $y = f(x)$.

In general, the derivative measures the rate of change of a function at a given point. Thus, if the function we are studying is the position of a particle on the x-coordinate, then $x'(t)$ is the velocity of the particle. If the function we are studying is the velocity $v(t)$ of the particle, then the derivative $v'(t)$ is the acceleration of the particle. If the function we are studying is the population of India, then the derivative measures the rate of change of the population.

The slope of the tangent

From the point of view of geometry, the derivative $f'(c)$ gives us the slope of the curve, that is, the slope of the tangent to the curve $y = f(x)$ at $(c, f(c))$. This becomes particularly clear if we rewrite the derivative as the following limit:

$$\lim_{y \to c} \frac{f(y) - f(c)}{y - c}.$$

The expression inside the limit obviously represents the slope of a line passing through $(c, f(c))$ and $(y, f(y))$, and as y approaches c this line obviously becomes tangent to $y = f(x)$ at the point $(c, f(c))$.

Another way of thinking of the derivative

Another way of thinking of the derivative of the function f at the point x_0 is as follows. If f is differentiable at x_0 we know that

$$\frac{f(x_0+h)-f(x_0)}{h} - f'(x_0) \to 0$$

as $h \to 0$. Since we are keeping x_0 fixed, we can treat the above quantity as a function of h. Thus we can write

$$\frac{f(x_0+h)-f(x)}{h} - f'(x_0) = o_1(h)$$

for some function $o_1(h)$ with the property that $o(h) \to 0$ as $h \to 0$. Taking a common denominator,

$$\frac{f(x_0+h)-f(x_0)-f'(x_0)h}{h} = o_1(h) \tag{1}$$

We can use the above equality to give an equivalent definition for the derivative. A function f is said to be differentiable at the point x_0 if there exists a real number (denoted $f'(x_0)$) such that (1) holds for some function $o_1(h)$ such that $o_1(h) \to 0$ as $h \to 0$.

The derivative as a linear map

We can rewrite equation (1) as

$$f(x_0 + h) = f(x_0) + f'(x_0)h + o_1(h)h$$

Thus, the derivative of $f(x)$ at a point x_0 can be viewed as that real number (if it exists) by which you have to multiply h so that the resulting remainder goes to 0 faster than h (that is, the remainder divided by h goes to 0 as h goes to 0).

A function $f : \mathbb{R} \to \mathbb{R}$ which has the property that $f(x + y) = f(x) + f(y)$ is called a linear function (or linear map). All such functions are given by multiplication by a real number, that is, every linear function has the form $f(x) = \lambda x$, for some real number λ. Thus the derivative may be regarded as a linear function (in the variable h). This point of view will be particularly useful in multivariable calculus.

Examples

Exercise 1.16: Let $f : (a, b) \to \mathbb{R}$ be a function such that

it must be a constant function
because for any x its
derivative is zero

$$|f(x+h) - f(x)| \leq C|h|^{\alpha}$$

for all $x, x+h \in (a, b)$, where C is a constant and $\alpha > 1$. Show that f is differentiable on (a, b) and compute $f^{'}(x)$ for $x \in (a, b)$.

Solution:

$$\left| \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right| \leq C \lim_{h \to 0} |h|^{\alpha - 1} = 0.$$

Note: Functions that satisfy the property above for some α (not necessarily greater than 1) are said to be Lipschitz continuous with exponent α.

Calculating derivatives

As with limits all of you are already familiar with the rule for calculating the sums, differences, products and quotients of derivatives. You should try and remember how to prove these.

You should also recall the chain rule for calculating the derivative of the composition of functions and try to prove it as an exercise using the $\epsilon - \delta$ definition of a limit.

Note that the proof of the chain rule given in some books involves writing

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \times \frac{\Delta u}{\Delta x}.$$

and then taking limits as $\Delta x \to 0$. This is not quite correct since Δu could be 0 even for infinitely many values of u.

Maxima and minima

Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function (you can think of X as an open, closed or half-open interval, for instance).

Definition: The function f is said to attain a maximum (resp. minimum) at a point $x_0 \in X$ if $f(x) \leq f(x_0)$ (resp. $f(x) \geq f(x_0)$) for all $x \in X$.

Once again, I remind you that, in general, f may not attain a maximum or minimum at all on the set X. The standard example being $X = (0, 1)$ and $f(x) = 1/x$ (can you find an example on the closed interval $[0, 1]$?). However, if X is a closed bounded interval and f is a continuous function, Theorem 11 tells us that the maximum and minimum are actually attained. Theorem 11 is sometimes called the Extreme Value Theorem.

Maxima and minima and the derivative

If f has a maximum at the point x_0 and if it also differentiable at x_0, we can reason as follows. We know that $f(x_0 + h) - f(x_0) \leq 0$ for every $h > 0$ such that $x + h \in X$. Hence, we see that (one half of the Sandwich Theorem!)

$$\lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0.$$

On the other hand, when $h < 0$, we get

$$\lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0.$$

Because f is assumed to be differentiable at x_0 we know that left and right hand limits must be equal. It follows that we must have $f'(x_0) = 0$. A similar argument shows that $f'(x_0) = 0$ if f has a minimum at the point x_0.

Local maxima and minima

The preceeding argument is purely local. Before explaining what this means, we give the following defintion.

Definition: Let $f : X \to \mathbb{R}$ be a function and x_0 be in X. Suppose there is an sub-interval $x_0 \in (c, d) \subset X$ such that $f(x_0) \geq f(x)$ (resp. $f(x_0) \leq f(x)$) for all $x \in (c, d)$, then f is said to have a local maximum (resp. local minimum) at x_0.

Sometimes we use the terms global maximum or global minimum instead of just maximum or minimum in order to emphasize the points are not just local maxima or minima. The argument of the previous slide actually proves the following

Theorem 13: If $f : X \to \mathbb{R}$ is differentiable and has a local minimum or maximum at a point $x_0 \in X$, $f'(x_0) = 0$.