# CS 105: Department Introductory Course on Discrete Structures

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Aug 29, 2023

Lecture 10 – Basic Mathematical Structures

Equivalence relations and partitions

# Recap: Proofs and Structures

## Chapter 1: Proofs

1. Propositions, predicates
2. Types of proofs, axioms
3. Mathematical Induction, Well-ordering principle
4. Strong Induction

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1. Finite and infinite sets.
2. Using functions to compare sets: focus on bijections.
3. Countable, countably infinite and uncountable sets.
4. Cantor's diagonalization (New/powerful proof technique!).

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## Chapter 3: Relations

# Relations

**Definition: Relation**

- A relation $R$ from $A$ to $B$ is a subset of $A \times B$. If $(a, b) \in R$, we also write this as $a \ R \ b$.

We write $R(A, B)$ for a relation from $A$ to $B$ and just $R(A)$ if $A = B$. Also if $A$ is clear from context, we just write $R$.

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## Examples of relations

- All functions are relations.
- $R_1(\mathbb{Z}) = \{(a, b) \mid a, b \in \mathbb{Z}, a - b \text{ is even } \}$.
- $R_2(\mathbb{Z}) = \{(a, b) \mid a, b \in \mathbb{Z}, a \leq b\}$.
- Let $S$ be a set, $R_3(\mathcal{P}(S)) = \{(A, B) \mid A, B \subseteq S, A \subseteq B\}$.

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### Representations of a relation from $A$ to $B$.

As a set of ordered pairs of elements, i.e., subset of $A \times B$; As a directed graph; As a (database) table.

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### Examples

- Natural numbers are partitioned into even and odd.
- Class is partitioned into sets of students from same hostel.

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What properties does this relation have?

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Thus, from any partition, we get an equivalence relation. Is the converse true?

# Examples

- Reflexive: $\forall a \in S$, $aRa$.
- Symmetric: $\forall a, b \in S$, $aRb$ implies $bRa$.
- Transitive: $\forall a, b, c \in S$, $aRb, bRc$ implies $aRc$.
- Equivalence: Reflexive, Symmetric and Transitive.

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- Reflexive: $\forall a \in S,\ aRa$.
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- Equivalence: Reflexive, Symmetric and Transitive.

| Relation | Refl. | Sym. | Trans. | Equiv. |
|---|---|---|---|---|
| $aR_4b$ if students $a$ and $b$ take same set of courses | ✓ | ✓ | ✓ | ✓ |
| $aR_5b$ if student $a$ takes course $b$ | | | | |
| $\{(a,b) \mid a,b \in \mathbb{Z}, (a-b) \mod 2 = 0\}$ | | | | |
| $\{(a,b) \mid a,b \in \mathbb{Z}, a \leq b\}$ | | | | |
| $\{(a,b) \mid a,b \in \mathbb{Z}, a < b\}$ | | | | |
| $\{(a,b) \mid a,b \in \mathbb{Z}, a \mid b\}$ | | | | |
| $\{(a,b) \mid a,b \in \mathbb{R}, \lvert a-b \rvert < 1\}$ | | | | |
| $\{((a,b),(c,d)) \mid (a,b),(c,d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), (ad = bc)\}$ | | | | |

# Partitions of a set – grouping "like" elements

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Example: natural numbers partitioned into even and odd...

## Theorem

Every partition of set $S$ gives rise to a canonical equivalence relation $R$ on $S$, namely,

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Is the converse true? Can we generate a partition from every equivalence relation?

# Equivalence classes

## Definition

- Let $R$ be an equivalence relation on set $S$, and let $a \in S$.
- Then the equivalence class of $a$, denoted $[a]$, is the set of all elements related to it, i.e., $[a] = \{b \in S \mid (a, b) \in R\}$.

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**Lemma**

Let $R$ be an equivalence relation on $S$. Let $a, b \in S$. Then, the following statements are equivalent:

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Proof Sketch: (1) to (2) symm and trans, (2) to (3) refl, (3) to (1) symm and trans. (H.W.: Redo the proof formally.)

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Proof sketch of (1): Union, non-emptiness follows from reflexivity. The rest (pairwise disjointness) follows from the previous lemma.
(H.W.): Write the formal proofs of (1) and (2).

# More "applications" of equivalence relations

## Defining new objects using equivalence relations

Consider

$R = \{((a, b), (c, d)) \mid (a, b), (c, d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), (ad = bc)\}.$

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- Then the equivalence classes of $R$ define the rational numbers.
- e.g., $\left[\frac{1}{2}\right] = \left[\frac{2}{4}\right]$ are two names for the same rational number.
- Indeed, when we write $\frac{p}{q}$ we implicitly mean $\left[\frac{p}{q}\right]$.

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Can we define integers and real numbers starting from naturals by using equivalence classes?