CS 105: Department Introductory Course on Discrete Structures

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Aug 22, 2023

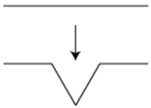
Lecture 07 – Basic Mathematical Structures

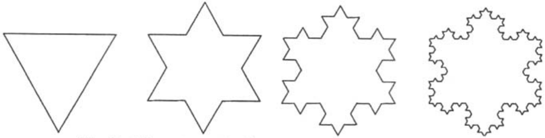
Countable Sets and functions

Pop Quiz

Define a sequence of shapes as follows:

- $K(0)$ is an equilateral triangle.
- For $n > 0$, $K(n)$ is formed by replacing each line segment of $K(n-1)$ by the shape shown in bottom of left fig., such that the central vertex points outwards.





The first four stages in the construction of the Koch snowflake

1. Show by induction that $\forall n \in \mathbb{N}$, the no. of line segments in $K(n)$ is $4^n \cdot 3$.
2. If the original equilateral triangle $K(0)$ has side length 1,
   - 2.1 what is the perimeter of $K(n)$ as a function of n?
   - 2.2 what is the area of $K(n)$? (Prove both by induction)

Difference between finite and infinite sets

Theorem

Let A be a set and $b \notin A$. Then A is infinite iff there is a bijection from A to $A \cup \{b\}$.

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1. Show contrapositive! If A is finite, then there can't be a bijection from A to $A \cup \{b\}$.

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- Even if A, B are infinite, $A \subset B$, there can be a bijection from A to B, i.e., they have the same "cardinality".
- From any set A, there is a surjection from A to $\mathbb{N}$.

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Some questions...

1. Is there a bijection between $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$? yes
2. Is there a bijection between $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$? yes
3. Is there a bijection from $\mathbb{Q}$ to $\mathbb{N}$? yes
4. Is there a bijection from the set of all subsets of $\mathbb{N}$ to $\mathbb{N}$? no
5. Is there a bijection from $\mathbb{R}$ to $\mathbb{N}$? no

Countable and countably infinite sets

Definition

- For a given set C, if there is a bijection from C to $\mathbb{N}$, then C is called countably infinite.
- A set is countable if it is finite or countably infinite.

Examples: even numbers, number of horses,...

By previous corollary ($\exists$ surj from any infinite set to $\mathbb{N}$)

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What are the other properties of countable sets?

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Are the following sets countable?

That is, is there a bijection from these sets to $\mathbb{N}$?

- the set of all integers $\mathbb{Z}$
- $\mathbb{N} \times \mathbb{N}$
- $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$
- the set of rationals $\mathbb{Q}$
- the set of all (finite and infinite) subsets of $\mathbb{N}$
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To show these it suffices to show that why?

- there is an injection from these sets to $\mathbb{N}$

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To show these it suffices to show that

- there is an injection from these sets to $\mathbb{N}$
- or there is a surjection from $\mathbb{N}$ (or any countable set) to these sets.