# MA 105 Calculus II

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October 30 - November 03, 2023

# Parametrisation and coordinate change

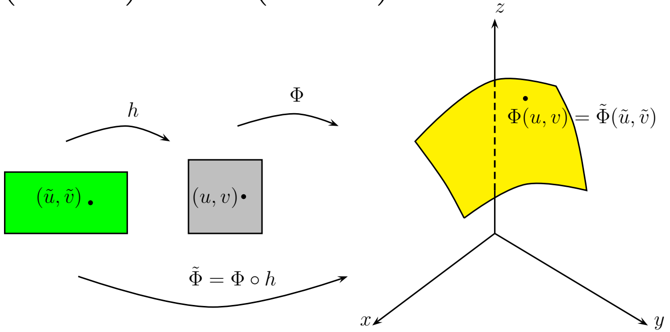
As before, let $S$ be the geometric surface corresponding to a parametrised surface $\mathbf{\Phi}(u, v)$. Given a rectangle $R$ with corners $(u, v)$, $(u + \Delta u, v)$, $(u + \Delta u, v + \Delta v)$ and $(u, v + \Delta v)$, we would like to compute the area of the "area element" on $S$ bounded by the four points $\mathbf{\Phi}(u, v)$, $\mathbf{\Phi}(u + \Delta u, v)$, $\mathbf{\Phi}(u + \Delta u, v + \Delta v)$ and $\mathbf{\Phi}(u, v + \Delta v)$.

Actually, we have already done this! This is exactly what we did when computing the formula for the change of variables, except that in that case, the coordinate change took an area in $\mathbb{R}^2$ back to an area in $\mathbb{R}^2$ (recall the situation for polar coordinates). The only difference now is that $\mathbf{\Phi}(R)$ no longer lies in the plane. This doesn't really change anything.

We do have to be a little careful: we must make sure that $\mathbf{\Phi}$ is bijective and that it is non-singular. In fact, the inverse function theorem guarantees us that if $\mathbf{\Phi}$ is non-singular it is automatically bijective in a small enough neighbourhood on the surface.

Let $E$ be a path-connected subset of $\mathbb{R}^2$ having an area, and let $\mathbf{\Phi} : E \to \mathbb{R}^3$ be a smooth parametrised surface.
Let $\widetilde{E}$ be a path-connected subset of $\mathbb{R}^2$ having an area, and let $h : \widetilde{E} \to \mathbb{R}^2$ be a continuously differentiable and one-one function such that $h(\widetilde{E}) = E$ and its Jacobian $J(h)$ does not vanish on $\widetilde{E}$. Then the smooth surface $\tilde{\mathbf{\Phi}} := \mathbf{\Phi} \circ h$ is called a **reparametrisation** of $\mathbf{\Phi}$. Note that

$$\left(\tilde{\mathbf{\Phi}}_{\tilde{u}} \times \tilde{\mathbf{\Phi}}_{\tilde{v}}\right)(\tilde{u}, \tilde{v}) = \left(\mathbf{\Phi}_u \times \mathbf{\Phi}_v\right)(h(\tilde{u}, \tilde{v}))J(h)(h(\tilde{u}, \tilde{v})).$$



Examples: Let $E := (0, \pi) \times [-\pi, \pi]$, and define $\mathbf{\Phi}(\varphi, \theta) := (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ for $(\varphi, \theta) \in E$.

If $\widetilde{E} := [-\pi, \pi] \times (0, \pi)$, and we define $\widetilde{\mathbf{\Phi}}(\theta, \varphi) := (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ for $(\theta, \varphi) \in \widetilde{E}$, then $\widetilde{\mathbf{\Phi}}$ is a reparametrisation of $\mathbf{\Phi}$ since $\widetilde{\mathbf{\Phi}}(\theta, \varphi) = \mathbf{\Phi}(h(\theta, \varphi))$, where $h : \widetilde{E} \to E$ is given by $h(\theta, \varphi) := (\varphi, \theta)$ with $J(h) = -1$.

Similarly, if $\widetilde{E} := (0, \pi/2) \times [-\pi/2, \pi/2]$, and we define $\widetilde{\mathbf{\Phi}}(\varphi, \theta) := (\sin 2\varphi\cos 2\theta, \sin 2\varphi\sin 2\theta, \cos 2\varphi)$ for $(\varphi, \theta) \in \widetilde{E}$, then $\widetilde{\mathbf{\Phi}}$ is a reparametrisation of $\mathbf{\Phi}$ since $\widetilde{\Phi}(\varphi, \theta) = \Phi(h(\varphi, \theta))$, where $h : \widetilde{E} \to E$ is given by $h(\varphi, \theta) := (2\varphi, 2\theta)$ with $J(h) = 4$.

**Theorem**

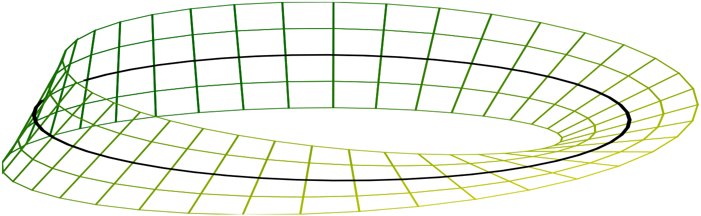
*The integral of a continuous function over a smooth surface is invariant under reparametrisation. In particular, the area of a smooth surface is invariant under reparametrisation.*

# Oriented surfaces

In what follows we will assume that any parametrised surface $\mathbf{\Phi}$ is $\mathcal{C}^1$ and non-singular. We now are setting up to state Stokes' theorem.
We need to define orientation on a surface. One plausible way is to imitate the idea of interior and exterior for a surface. An oriented surface $S$ could be defined as two-sided surface with one side specified as the outside (or positive side) and the other side as the inside (or negative side).

What is the problem with this definition?

There are surfaces which have only one side! The simplest such surface is the Möbius strip, named after its discoverer, a famous Swedish mathematician of the eighteenth century.



# Orientable surfaces -definition

This tells us that we need a better way to make sense of terms like "inside" and "outside".

Recall that for us a vector field on a surface $S$ is a vector field is a function $\mathbf{F} : U \to \mathbb{R}^3$ defined on a open set containing $S \subseteq U$. We say $\mathbf{F}$ is continuous (or $C^1$) if it is continuous on $U$.

Definition: A surface $S$ is said to be orientable if there exists a continuous vector field $\mathbf{F} : S \to \mathbb{R}^3$ such that for each point $P$ in $S$, $\mathbf{F}(P)$ is a unit vector normal to the surface $S$ at $P$.

At each point of $S$ there are two possible directions for the normal vector to $S$. The question is whether the normal vector field be can be chosen so that the resulting vector field is continuous.
Note that this definition is independent of the choice of parametrisation and only dependent on the geometric surface.

# Examples of orientable surfaces

Example: For the unit sphere in $\mathbb{R}^3$ we can choose an orientation by selecting the unit vector $\hat{\mathbf{n}}(x, y, z) = \hat{\mathbf{r}}$, where $\mathbf{r}$ points outwards from the surface of the sphere.

More explicitly, we define

$$\mathbf{F}(x, y, z) = (x, y, z).$$

This obviously defines a continuous vector field on $S$. Hence, we see that the unit sphere in $\mathbb{R}^3$ is orientable.

Notice, that we can also define a vector field $\mathbf{G}(x, y, z) = -(x, y, z)$. The vector field $\mathbf{G} = -\mathbf{F}$ is also obviously continuous. There are two possible choices of orientation.

# Non-orientable surfaces

Definition: A surface on which there exists no continuous vector field consisting of unit normal vectors is called non-orientable.

Exercise 1: Make a Möbius strip out of a piece of paper. Starting at the top draw a series of stick figures, head to toe, and label their left and right hands. When the stick figure comes back to the top (on the underside) compare the left and right hands of the two stick figures at the top.

Exercise 2: Give a parametrisation for the Möbius strip. Here is nice reference to this.

Exercise 3: The Möbius strip is an example of a non-orientable surface. Can you prove this? Answer: This exercise is beyond the scope of this class. But a good advertisement to learn more differential geometry/topology.

# Choosing an orientation

As we have just seen in the preceding example, if $S$ is an orientable surface and $\mathbf{F}$ is a continuous vector field of unit normal vectors, so is $-\mathbf{F}$.

An orientable surface together with a specific choice of continuous vector field $\mathbf{F}$ of unit normal vectors is called an oriented surface. The choice of vector field is called an orientation.

Once one has chosen a particular vector field of normal vectors it makes sense to talk about the "outside" or "positive side" of the surface: usually, it is the side given by the direction of the unit normal vector. The other side is then called the "inside" or "negative side". However, which side one calls "positive" or "negative" is a matter of choice.

# The orientation of parametrised surfaces

Let us suppose that we are given an oriented geometric surface $S$ that is described as a $\mathcal{C}^1$ non-singular parametrised surface $\mathbf{\Phi}(u, v)$.

Notice that a oriented parametrised surface $\mathbf{\Phi}$ comes equipped with a natural vector field of unit normal vectors:

$$\hat{\mathbf{n}} = \frac{\mathbf{\Phi}_u \times \mathbf{\Phi}_v}{\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|}.$$

Definition: If the unit normal vector $\hat{\mathbf{n}}$ agrees with the given orientation of $S$ we say that the parametrisation $\mathbf{\Phi}$ is orientation preserving. Otherwise we say that $\mathbf{\Phi}$ is orientation reversing.
Note this gives a orientation only when we already know that the surface is oriented.

# Examples

(i) Let $E \subset \mathbb{R}^2$ have an area, and $f : E \to \mathbb{R}$ be a smooth scalar field.
Consider the graph $S := \{(x, y, f(x, y)) : (x, y) \in E\}$ of $f$.
For $P := (x, y, z) \in S$, define

$$\hat{\mathbf{n}}(P) := (-f_x(P), -f_y(P), 1)/\|(-f_x(P), -f_y(P), 1)\|.$$

This continuous assignment of upward unit normal vectors gives an orientation of $S$. Hence $S$ is orientable.

Clearly, the parametrisation of $S$ given by $\mathbf{\Phi}(x, y) := (x, y, f(x, y))$ for $(x, y) \in E$, is orientation-preserving.

(ii) Let $S := \{(x, y, z) :\in \mathbb{R}^3 : x^2 + y^2 = a^2 \text{ and } 0 \le z \le h\}$. For $P := (x, y, z) \in S$, define $\hat{\mathbf{n}}\,(P) := (x/a, y/a, 0)$.
This continuous assignment of outward unit normal vectors gives an orientation of $S$. Hence the cylinder $S$ is orientable.

Let $E := [0, 2\pi] \times [0, h]$ and $\mathbf{\Phi}(u, v) := (a \cos u, a \sin u, v)$ for $(u, v) \in E$.

# Examples contd.

If $P := \mathbf{\Phi}(u,v) = (x,y,z) \in S$, then

$$\frac{\mathbf{\Phi}_u \times \mathbf{\Phi}_v}{\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|}(u,v) = \frac{(a\cos u, a\sin u, 0)}{a} = \left(\frac{x}{a}, \frac{y}{a}, 0\right) = \hat{\mathbf{n}}(P).$$

Hence $\mathbf{\Phi}$ is an orientation-preserving parametrisation.

(iii) Let $S := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = a^2\}$. For $P := (x,y,z) \in S$, define $\hat{\mathbf{n}}(P) := (x/a, y/a, z/a)$.
This continuous assignment of outward unit normal vectors gives an orientation of the sphere $S$. Hence $S$ is orientable.
Let $E := [0, 2\pi] \times (0, \pi)$ and $\mathbf{\Phi}(u,v) := (a\cos u \sin v, , a\sin u \sin v, a\cos v)$ for $(u,v) \in E$.
If $P := \mathbf{\Phi}(u,v) = (x,y,z) \in S$, then

$$\frac{\mathbf{\Phi}_u \times \mathbf{\Phi}_v}{\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|}(u,v) = -\frac{(a\sin v)\mathbf{\Phi}(u,v)}{a^2 \sin v} = -\left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right) = -\hat{\mathbf{n}}(P).$$

Hence $\mathbf{\Phi}$ is an orientation-reversing parametrisation of $S \setminus \{(0,0,\pm a)\}$.

# Independence of parametrisation

Let $S$ be an oriented surface. Let $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ be two $\mathcal{C}^1$ non-singular parametrisations of $S$ and let $\mathbf{F}$ be a continuous vector field on $S$.

- If $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ are orientation preserving, then

$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}.$$

- If $\mathbf{\Phi}_1$ is orientation preserving and $\mathbf{\Phi}_2$ is orientation reversing, then

$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}.$$

For an oriented surface, the notation

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS,$$

is unambiguous.

# Homeomorphism

continuous bijective fn are Homeomorphism

We now introduce the notion of 'Homeomorphism'.

Let $\psi$ be function from $U_1 \subset \mathbb{R}^n$ to $U_2 \subset \mathbb{R}^m$.
We call the mapping $\psi : U_1 \longrightarrow U_2$ is a homeomorphism if $\psi$ is continuous, bijective map such that $\psi^{-1}$ is also continuous.

Example. $\psi : (-a, a) \to (-a^3, a^3)$, defined by $\psi(x) = x^3$ is a homeomorphism.

Example. $U_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, \quad z > 0\}$. Consider $U_1 = \{(x, y) \mid x^2 + y^2 < 1\} \subseteq \mathbb{R}^2$ and the mapping

$$\psi(x, y) = (x, y, \sqrt{1 - x^2 - y^2}), \quad \forall (x, y) \in U_1.$$

Then this is a homeomorphism. Check.

Example. The spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic unless $n = m$.

Example: The function $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a homeomorphism.

Under homeomorphism many properties of a domain, are preserved.

# Boundary of a surface

Let us now redefine a geometric surface $S$ as a subset of $\mathbb{R}^3$ such that for every point $P \in S$ there is a open subset $U \subseteq \mathbb{R}^3$ such that $P \in U \cap S$ is homeomorphic to an open disk in $\mathbb{R}^2$.

DEFINITION:[Boundary of a surface] A surface with a boundary is a surface along with boundary points. A boundary point $P$ is a point such that there exists an open set $U$ in $\mathbb{R}^3$ so that $P \in U \cap S$ is homeomorphic to a half disk in $\mathbb{R}^2$. The set of all boundary points is called the boundary of $S$ and is denoted by $\partial S$.
Caution: Boundary of a subset of $\mathbb{R}^3$ is not the same as the boundary of a surface in $\mathbb{R}^3$. The idea however is that we are heuristically looking at points which are neither completely inside not completely outside.

Example: $S = \{x^2 + y^2 = a^2, 0 \leq z \leq h\}$ is a cylinder of height $h$. Then $\partial S$ is the union of the sets $\{x^2 + y^2 = a^2, z = 0\}$ and $\{x^2 + y^2 = a^2, z = h\}$.
Example: Let $S = \{x^2 + y^2 + z^2 = a^2\}$. Then $\partial S = \varnothing$. Why? A sphere, and a torus have no boundary. What about an upper hemisphere?

# Orientation of the boundary of a surface

Let $S$ be an oriented surface with a boundary that is a simple, closed, non-singular parametrised curve (more generally, a disjoint union of simple, closed, piecewise non-singular parametrised curves). For instance, the cylinder of height $h$ or the upper hemisphere.

Let an orientation of $S$ be prescribed. How is the boundary of $S$ oriented?

Suppose $S$ is an oriented surface and let $\mathbf{n}(P)$ be the prescribed unit outward normal vector at a point $P \in S$. We choose the induced orientation of $\partial S$ such that the surface lies to the left of an observer walking along the boundary $\partial S$ with his head in the direction $\mathbf{n}(P)$.
The boundary of an oriented surface automatically acquires an orientation.

Example. Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, \quad z \geq 0\}$, the unit upper hemisphere. Let $S$ be oriented by

$$\mathbf{n}(P) := (x, y, z), \quad \text{for} \quad P := (x, y, z) \in S.$$

Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and $\mathbf{\Phi}(x, y) := (x, y, \sqrt{1 - x^2 - y^2})$ for all $(x, y) \in D$. Note the boundary of the hemisphere $S$ is the circle in $x$-$y$ plane

$$\partial S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, \quad z = 0\}.$$

The induced orientation $\partial S$ by the oriented-parametrisation $\mathbf{\Phi}$ corresponds to the counter clock-wise orientation of

$$\partial D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

# Stokes theorem

A surface $S$ is called piecewise smooth if it is a finite union of smooth surfaces joining along smooth curves.

## Theorem

1. *Let $S$ be a bounded piecewise smooth oriented surface with non-empty boundary $\partial S$.*
2. *Let $\partial S$, the boundary of $S$, be the disjoint union of simple closed curves each of which is a piecewise non-singular parametrised curve with the induced orientation.*
3. *Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ be a $C^1$ vector field defined on an open set containing $S$.*

*Then*

$$\int_{\partial S} \mathbf{F}.d\mathbf{s} = \int\int_S (\nabla \times \mathbf{F}).d\mathbf{S}.$$

Remark: It is sufficient to assume the surface is $C^2$ for this theorem.

# Remarks

- If two different oriented surfaces $S_1$ and $S_2$ have the same boundary $C$, then it follows from Stokes theorem that

$$\int\int_{S_1} (\nabla \times \mathbf{F}).d\mathbf{S} = \int\int_{S_2} (\nabla \times \mathbf{F}).d\mathbf{S},$$

  where the surfaces are correctly oriented.

- Green's theorem is the analogous version of Stokes theorem for the planar regions.

# Stokes theorem for closed surface

## Corollary

*Let $S$ be a closed oriented smooth surface in $\mathbb{R}^3$ (and so $\partial S = \emptyset$). Suppose* **F** *is a smooth vector field on an open subset containing $S$. Then*

$$\iint_S (curl\boldsymbol{F})\cdot d\boldsymbol{S} = 0.$$

Proof: Introduce a hole in $S$ by cutting out a small piece along a smooth simple closed curve $C$ on $S$. Let $S_1$ denote the part of $S$ cut out, and let $S_2$ denote the remaining part of $S$. Then

$$\iint_S (\text{curl}\boldsymbol{F})\cdot d\boldsymbol{S} = \iint_{S_1} (\text{curl}\boldsymbol{F})\cdot d\boldsymbol{S} + \iint_{S_2} (\text{curl}\boldsymbol{F})\cdot d\boldsymbol{S}.$$

by the domain additivity.

Now the Stokes theorem shows that

$$\iint_{S_1} (\text{curl}\boldsymbol{F})\cdot d\boldsymbol{S} = \int_{\partial S_1} \boldsymbol{F}\cdot d\boldsymbol{s} \ \text{ and } \ \iint_{S_2} (\text{curl}\boldsymbol{F})\cdot d\boldsymbol{S} = \int_{\partial S_2} \boldsymbol{F}\cdot d\boldsymbol{s}.$$

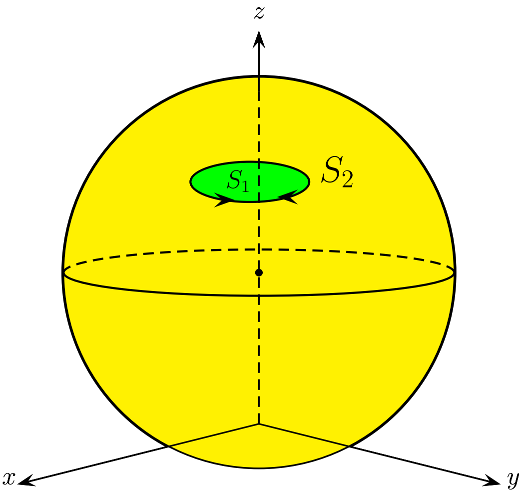



Figure: The Stokes theorem for $S$ with $\partial S = \emptyset$.

We observe that the boundary $\partial S_1$ of $S_1$ is the closed curve $C$ with the orientation induced by the orientation on $S_1$.

Since $\partial S = \emptyset$, the boundary $\partial S_2$ of $S_2$ is also $C$. But the orientations induced on $C$ by the orientations on $S_1$ and on $S_2$ are opposite.

Hence

$$\iint_S (\mathrm{curl}\boldsymbol{F}) \cdot d\boldsymbol{S} = \int_{\partial S_1} \boldsymbol{F} \cdot d\boldsymbol{s} + \int_{\partial S_2} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_C \boldsymbol{F} \cdot d\boldsymbol{s} - \int_C \boldsymbol{F} \cdot d\boldsymbol{s} = 0.$$

□

Example Calculate

$$\oint_C ydx + zdy + xdz,$$

where $C$ is the intersection of the surface $bz = xy$ and the cylinder $x^2 + y^2 = a^2$, for $b \neq 0, a \neq 0$, oriented counter clockwise as viewed from a point high upon the positive $z$-axis.

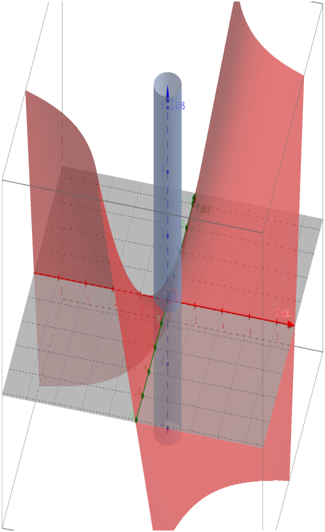
Ans:

$$\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k} \quad \text{and} \quad \mathrm{curl}\ \mathbf{F} = -(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Parametrise the surface lying on the hyperbolic paraboloid $z = xy/b$ and bounded by the curve $C$ as $S = \{x^2 + y^2 \leq a^2 \quad z = \frac{xy}{b}\}$. Then $\mathbf{n}\, dS = (-\frac{y}{b}, -\frac{x}{b}, 1)dxdy$ and

$$\iint_S \mathrm{curl}\ \mathbf{F} \cdot \mathbf{n}\, dS = \frac{1}{b} \iint_{x^2+y^2 \leq a^2} (y + x - b)dxdy$$

$$= \frac{1}{b} \int_0^{2\pi} \int_0^a (r \sin\theta + r\cos\theta - b) rdrd\theta = -\pi a^2.$$


Z-axis
Y-axis

# Examples: Homework

Example Let $C$ be the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$. Let $C$ be oriented so that when it is projected onto the $xy$-plane the resulting curve is traversed counterclockwise. Evaluate

$$\int_C -y^3 dx + x^3 dy - z^3 dz.$$

Ans: Use Stokes theorem.

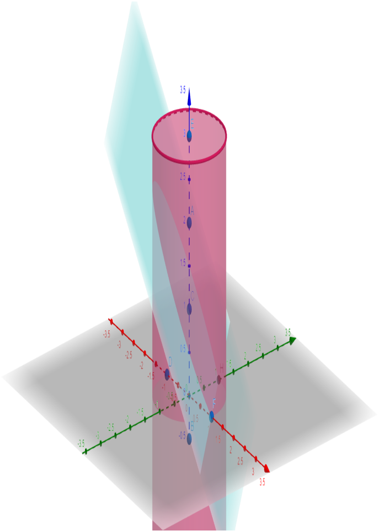
Consider the surface give by the graph of $z = 1 - x - y$ over $x^2 + y^2 \leq 1$,

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1 \quad z = 1 - x - y\}.$$

$S$ is enclosed by the curve $C$.

Check The unit normals to $S$ is given by $\pm\frac{1}{\sqrt{3}}(1, 1, 1)$. To orient $S$ positively so that we traverse $C$ in the counterclockwise direction, we must choose

$$\mathbf{n} = \frac{1}{\sqrt{3}}(1, 1, 1).$$



# Tutorial Problems

- Using Stokes Theorem, evaluate the line integral
$$\oint_C yz\,dx + xz\,dy + xy\,dz$$
where $C$ is the curve of intersection of $x^2 + 9y^2 = 9$ and $z = y^2 + 1$ with clockwise orientation when viewed from the origin.

- Find the integral of $\mathbf{F}(x, y, z) = z\mathbf{i} - x\mathbf{j} - y\mathbf{k}$ around the triangle with vertices $(0, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$.
- Let $C$ be the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$. Let $C$ be oriented so that when it is projected onto the $xy$-plane the resulting curve is traversed counterclockwise. Evaluate
$$\int_C -y^3dx + x^3dy - z^3dz.$$

- Let $\boldsymbol{F}(x, y, z) := (y, -x, e^{xz})$ for $(x, y, z) \in \mathbb{R}^3$, and let $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z - \sqrt{3})^2 = 4 \text{ and } z \geq 0\}$, be oriented by the outward unit normal vectors. Find $\iint_S (\mathrm{curl}\boldsymbol{F})\cdot d\boldsymbol{S}$.

# A more involved example

Example Evaluate the surface integral

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

where $S$ is the portion of the surface of a sphere defined by $x^2 + y^2 + z^2 = 1$ and $x + y + z \geq 1$, $\mathbf{F} = \mathbf{r} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. The outward normal is used to orient $S$.

How does one proceed? One can do this directly as a surface integral or use Stokes' theorem but in either case the evaluation is quite tedious.

Idea: Change the surface, keeping the boundary (and its orientation) unchanged!

After all, Stokes' theorem does not care what surface is being bounded by the curve. The surface integral (no matter what the surface is) is equal to the line integral on the boundary.

# Example contd.

With this idea in mind, we let $C$ be the curve of intersection of the sphere and the plane $x + y + z = 1$, and we let $S_1$ be the region of this plane enclosed by $C$ which is just a disc. We have to make sure that we orient $S_1$ so that $C$ has the same orientation as in the given problem. The normals to $S_1$ are given by

$$\mathbf{n}_1 = \pm\frac{1}{\sqrt{3}}(1,1,1).$$

Which normal should we take for orienting $S_1$? Clearly $\frac{1}{\sqrt{3}}(1,1,1)$. Now $\nabla \times \mathbf{F} = -2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $(\nabla \times \mathbf{F}) \cdot \mathbf{n}_1 = -2\sqrt{3}$. Hence

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_{S_1} -2\sqrt{3}dS = -2\sqrt{3}A(S_1)$$

where $A(S_1)$ is the surface area of the surface $S_1$ which we can easily compute!

# Consequences of Stokes theorem

## Proposition

*Let $\boldsymbol{F}$ be a smooth vector field on an open subset $D$ of $\mathbb{R}^3$ such that* $\mathrm{curl}\boldsymbol{F} = \mathbf{0}$ *on $D$.*

1. *Suppose $S$ is a bounded oriented piecewise $C^2$ surface in $D$, and let $\partial S$ denote its boundary with the induced orientation, as in the Stokes theorem. Then $\int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} = 0$.*

   *In particular, if $\partial S = C_1 \cup (-C_2)$, so that $C_1$ and $-C_2$ have the induced orientation, then $\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{s}$.*

2. *If $F$ is a vector field defined on $\mathbb{R}^3$, then $\boldsymbol{F}$ is a gradient field on $D$.*

# Closed surface

Can we have a generalised version of the divergence for of Green's theorem?

Yes! Gauss's divergence theorem under suitable hypothesis on $W$, a region in $\mathbb{R}^3$.

We define a closed surface $S$ in $\mathbb{R}^3$ to be a surface which is bounded, whose complement is open and boundary of $S$ is empty. This is analogous to the closed curve.

If $S$ is a closed surface, for example like sphere, then it encloses a 3-dimensional region. Call it $W$, and then $S$ will be its boundary, $\partial W$.

This is analogous to a simple closed curve being boundary of $D$ in $\mathbb{R}^2$.

Let us consider a region $W$ in $\mathbb{R}^3$ which is simultaneously Type 1, Type 2, Type 3 and the boundary of the region as a subset of $\mathbb{R}^3$ is a closed surface. We call such region in $\mathbb{R}^3$ as simple solid region.

For example, regions bounded by ellipsoids, spheres, or rectangular boxes are simple solid regions.

# Gauss's divergence theorem

If $W$ is a simple solid region, $W$ is a closed and bounded region in $\mathbb{R}^3$.

## Theorem (Gauss's Divergence Theorem)

1. *Let $W$ be a simple solid region of $\mathbb{R}^3$ whose boundary $S = \partial W$ is a closed surface.*
2. *$\partial W$ is positively oriented (outward pointing normal unit vector field).*
3. *Let $\boldsymbol{F}$ be a smooth vector field on an open subset of $\mathbb{R}^3$ containing $W$.*

*Then*

$$\iint_{\partial W} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_W (div\boldsymbol{F}) dxdydz.$$

Clearly, the importance of Gauss's theorem is that it converts surface integrals to volume integrals and vice-versa. Depending on the context one may be easier to evaluate than the other.

# Consequences of Gauss' theorem

## Theorem

*Let $W$ be a simple solid region in $\mathbb{R}^3$ with positively oriented boundary $\partial W$ and $\mathbf{F}$ be a smooth vector field on an open set in $\mathbb{R}^3$ containing $W$ satisfying div $\mathbf{F} = 0$ on $W$.*
*Then, the following holds:* $\displaystyle\int\int_{\partial W} \mathbf{F}.\mathbf{dS} = 0.$

## Corollary

*Let $\mathbf{F}$ be a vector field defined on $\mathbb{R}^3$. If div $\mathbf{F} = 0$, then*

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{dS} = \iint_{S_2} \mathbf{F} \cdot \mathbf{dS}$$

*whenever $S_1, S_2$ are oriented surfaces in $\mathbb{R}^3$ with $\partial S_1 = \partial S_2$ and $S_1$ and $S_2$ does not intersect each other except along their common boundary and there exists a region $W$ in $\mathbb{R}^3$ with boundary $S_1 \cup S_2$ satisfying the hypothesis in Gauss divergence theorem.*

## Physical Interpretation of the Gauss Divergence Theorem:

Suppose a solid body $W$ in $\mathbb{R}^3$ is enclosed by a closed geometric surface $S$, oriented in the direction of the outward normals. Let $\boldsymbol{F}$ be a vector field on $D$. The Gauss divergence theorem says that the flux of $\boldsymbol{F}$ across $S$ is equal to the triple integral of the divergence of the vector field $\boldsymbol{F}$ over $W$.

Example 1 Calculate the flux of $\mathbf{F}(x,y,z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through the unit sphere.

Solution: Using Gauss's theorem, we see that we need only evaluate

$$\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \iiint_W (\nabla \cdot \mathbf{F}) dV = \iiint_W 3(x^2 + y^2 + z^2) dxdydz,$$

where $W$ is the unit ball.

This problem is clearly ideally suited to the use of spherical coordinates. Making a change of variables, we get

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 3\rho^4 \sin\phi \, d\rho d\phi d\theta = \frac{12\pi}{5}$$

# Examples

Example 2: Let $\mathbf{F} = 2x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, and let $S$ be the unit sphere. Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

Solution: Using Gauss' theorem we see that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W (\nabla \cdot \mathbf{F}) dV,$$

where $W$ is the unit ball bounded by the sphere. Since $\nabla \cdot \mathbf{F} = 2(1 + y + z)$ we get

$$2 \iiint_W (1 + y + z) dV = 2 \iiint_W dV + 2 \iiint_W y dV + 2 \iiint_W z dV.$$

Notice that the last two integrals above are 0, by symmetry. Hence, the flux is simply

$$2 \iiint_W dV = \frac{8\pi}{3}.$$

# Curl and divergence

## Theorem

1. *If* $\mathbf{F} = \nabla \times \mathbf{G}$, *where* $\mathbf{G}$ *is a* $C^2$ *vector field defined on an open set* $W$ *in* $\mathbb{R}^3$, *then*
$$div\,\mathbf{F} = 0 \quad \text{on} \quad W.$$
2. *If* $\mathbf{F}$ *is a* $C^1$ *vector field defined on* $\mathbb{R}^3$ *satisfying* $div\,\mathbf{F} = 0$ *on* $\mathbb{R}^3$, *then there exists a* $C^2$ *vector field* $\mathbf{G}$ *defined on* $\mathbb{R}^3$ *such that*
$$\mathbf{F} = curl\,\mathbf{G}, \quad \text{on} \quad \mathbb{R}^3.$$

If $\operatorname{div}\mathbf{F} = 0$ in $\mathbb{R}^3$, how to find $\mathbf{G}$ such that $\mathbf{F} = \operatorname{curl}\mathbf{G}$?

Example Is $\mathbf{F}(x, y, z) = x\mathbf{i} - 2y\mathbf{j} + z\mathbf{k}$ defined in $\mathbb{R}^3$ the curl of a vector field?

Check $\mathbf{F}$ is smooth vector field satisfying $\operatorname{div}\mathbf{F} = 0$ in $\mathbb{R}^3$. So there exists a smooth vector field $\mathbf{G}$ such that $\mathbf{F} = \mathbf{G}$ in $\mathbb{R}^3$.

# Example contd.

To find $\mathbf{G}$: Let us assume $\mathbf{G}(x,y,z) = G_1(x,y,z)\mathbf{i} + G_2(x,y,z)\mathbf{j}$ for all $(x,y,z) \in \mathbb{R}^3$. Then solve $G_1$ and $G_2$ in such a way that curl $\mathbf{G} = \mathbf{F}$, i.e.,

$$\frac{\partial G_2}{\partial z}(x,y,z) = -F_1(x,y,z) = -x, \quad \frac{\partial G_1}{\partial z}(x,y,z) = F_2(x,y,z) = -2y,$$

$$\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)(x,y,z) = F_3(x,y,z) = z.$$

Now solving the equations, $G_2(x,y,z) = -xz + g(x,y)$ and $G_1(x,y) = -2yz + h(x,y)$. Using the 3rd equation,

$$-z + \partial_x g(x,y) + 2z - \partial_y h(x,y) = z.$$

It yields $\partial_x g(x,y) - \partial_y h(x,y) = 0$. Choosing, $g \equiv 0 \equiv h$, we get

$$\mathbf{G}(x,y,z) = -2yz\mathbf{i} - xz\mathbf{j}, \quad \text{in}\, \mathbb{R}^3.$$