CS 105: DIC on Discrete Structures

Graph theory

Characterizing maximum matchings via augmenting paths

Lecture 31

Oct 31 2023

Topic 3: Graph theory

Basic concepts

- Basics: graphs, paths, cycles, walks, trails, ...
- Cliques and independent sets.
- Graph representations, isomorphisms and automorphisms.
- Matchings: perfect, maximal and maximum.

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4. Maximum matchings: Using augmenting paths.

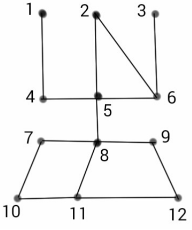
Matchings

Important slide.

Definitions

- A matching in a graph G is a set of (non-loop) edges with no shared end-points. The vertices incident to edges in a matching are called matched or saturated. Others are unsaturated.
- A perfect matching in a graph is a matching that saturates every vertex.
- A maximal matching in a graph is a matching that cannot be enlarged by adding an edge.
- A maximum matching is a matching of maximum size (# edges) among all matchings in a graph.

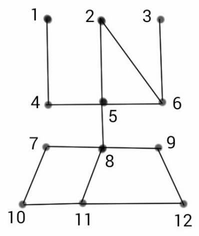
Matchings: Pop Quiz



Give an example of the following, if possible:

1. A maximal matching in G which is not a maximum matching.
2. A maximum matching in G. How do you know it is maximum?
   No augmenting path.
3. Can there be more than one maximum matching in a graph?
4. A graph which has no perfect matching but has a maximum matching. Is G such a graph? Odd no of vertices.

Matchings: Pop Quiz



- Perfect matching $\implies$ maximum matching $\implies$ maximal matching
- The reverse directions in the above implications do not hold.

Alternating and Augmenting paths

Definition

- Given a matching M, an M-alternating path is a path that alternates between edges in M and edges not in M.
- An M-alternating path whose endpoints are unmatched by M is an M-augmenting path.

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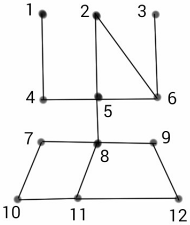

- Ex 1: Give an example of a matching M in G and
  1. a M-alternating path which is an M-augmenting path and
  2. a M-alternating path which is not an M-augmenting path

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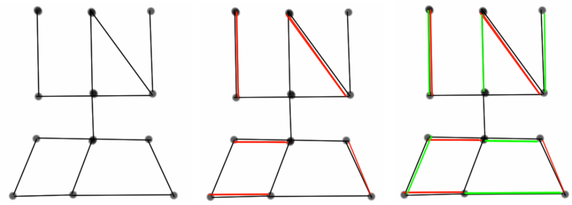

Characterization of Maximum matchings

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Theorem

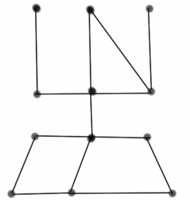
A matching M in G is a maximum matching iff G has no M-augmenting path.

Characterizing maximum matchings

We need a definition and a lemma.

Definition

If M, M' are matchings in a graph H, the symmetric difference $M \triangle M'$ is the set of edges which are either in M or in M' but not both, i.e., $M \triangle M' = (M \setminus M') \cup (M' \setminus M)$.Not in both.



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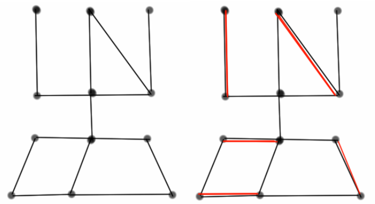

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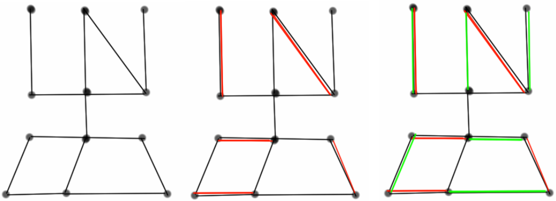

Ex 2: What is the symmetric difference of M (red) and M' (green) in the above graph?

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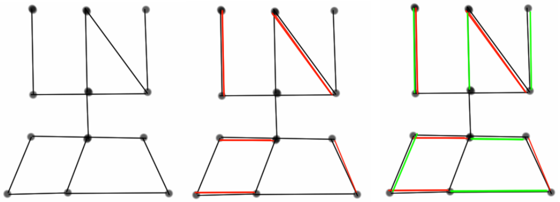
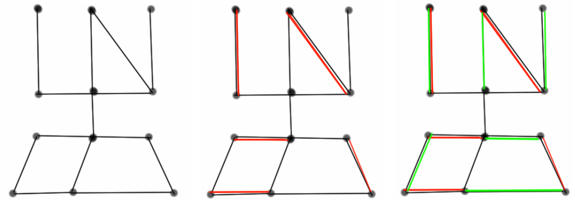


Ex 2: What is the symmetric difference of M (red) and M' (green) in the above graph? Can you generalize this?

Characterizing maximum matchings

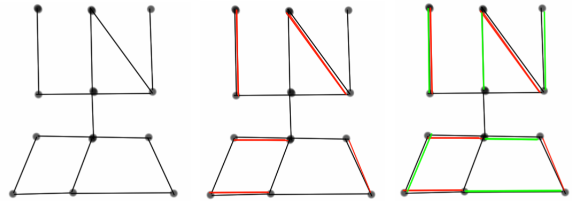


Lemma

Every component of the symmetric difference of two matchings is either a path or an even cycle.

Important lemma.

Characterizing maximum matchings



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- Let $F = M \triangle M'$. F has at most 2 edges at each vertex, hence every component is a path or a cycle.

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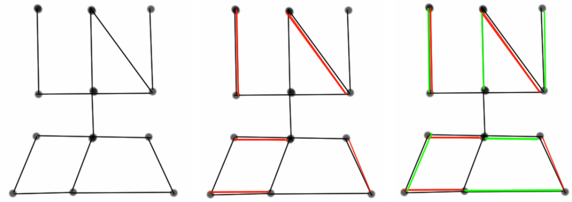

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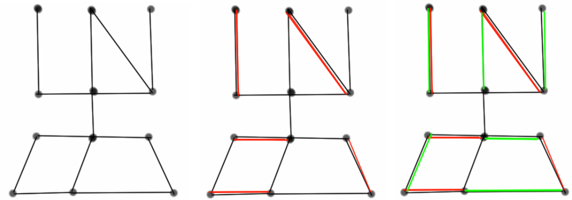

Lemma

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- Let $F = M \triangle M'$. F has at most 2 edges at each vertex, hence every component is a path or a cycle.
- Further every path/cycle alternates between edges of $M \setminus M'$ and $M' \setminus M$.
- Thus, each cycle has even length with equal edges from M and M'. $\square$

Characterizing maximum matchings

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A matching M in G is a maximum matching iff G has no M-augmenting path.

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- But then since $|M'| > |M|$ it must have a component with more edges in M' than M.
- This component can only be a path that starts and ends with an edge of M'; i.e., it is an M-augmenting path in G. $\square$

Perfect matchings in bipartite graphs

- If there are n women and n men, and each woman is compatible with exactly k men and each man compatible with exactly k women, can they be perfectly matched?

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- If there are n women and n men, and each woman is compatible with exactly k men and each man compatible with exactly k women, can they be perfectly matched?
- If there are m jobs and n applicants, when can we find a perfect matching where all m jobs are saturated?

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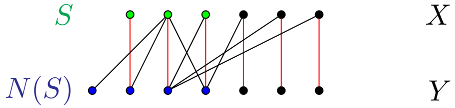
Perfect matchings in bipartite graphs

- Consider a bipartite graph with X, Y as partitions.
- If a matching M saturates X, then for every $S \subseteq X$, what can we say?



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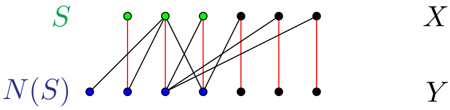
- If a matching M saturates X, then for every $S \subseteq X$, there must exist at least $|S|$ vertices that have neighbours in S.



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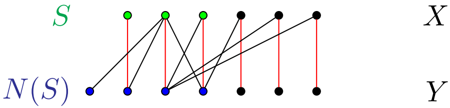

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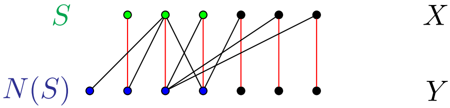

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Theorem (Hall'35)

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Proof: ($\implies$) is straightforward:

- Let M be a matching.
- Then for any $S \subseteq X$, each vertex of S is matched to a distinct vertex in $N(S)$
- So $|N(S)| \geq |S|$.