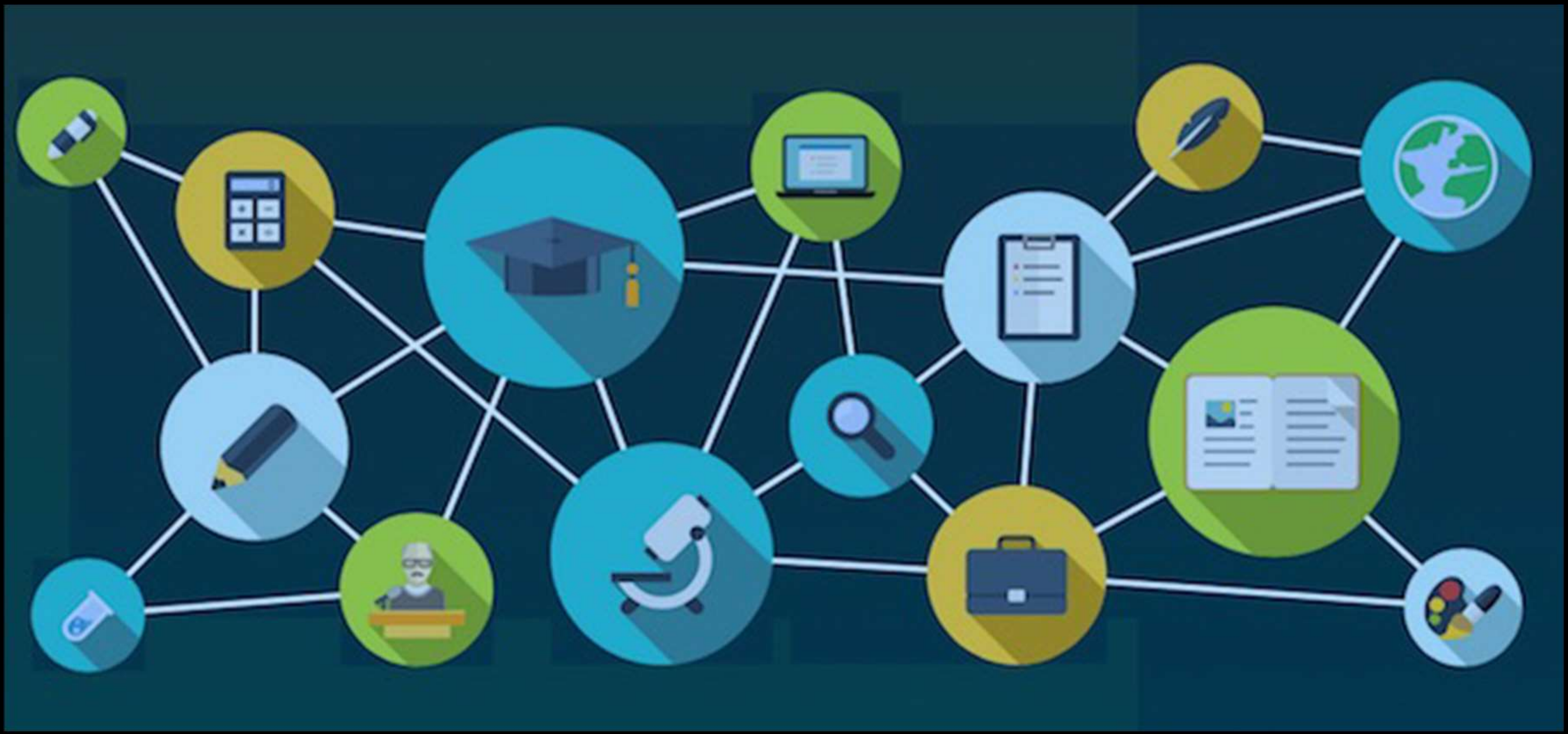
Multiple Random Variables

What Are We Missing?

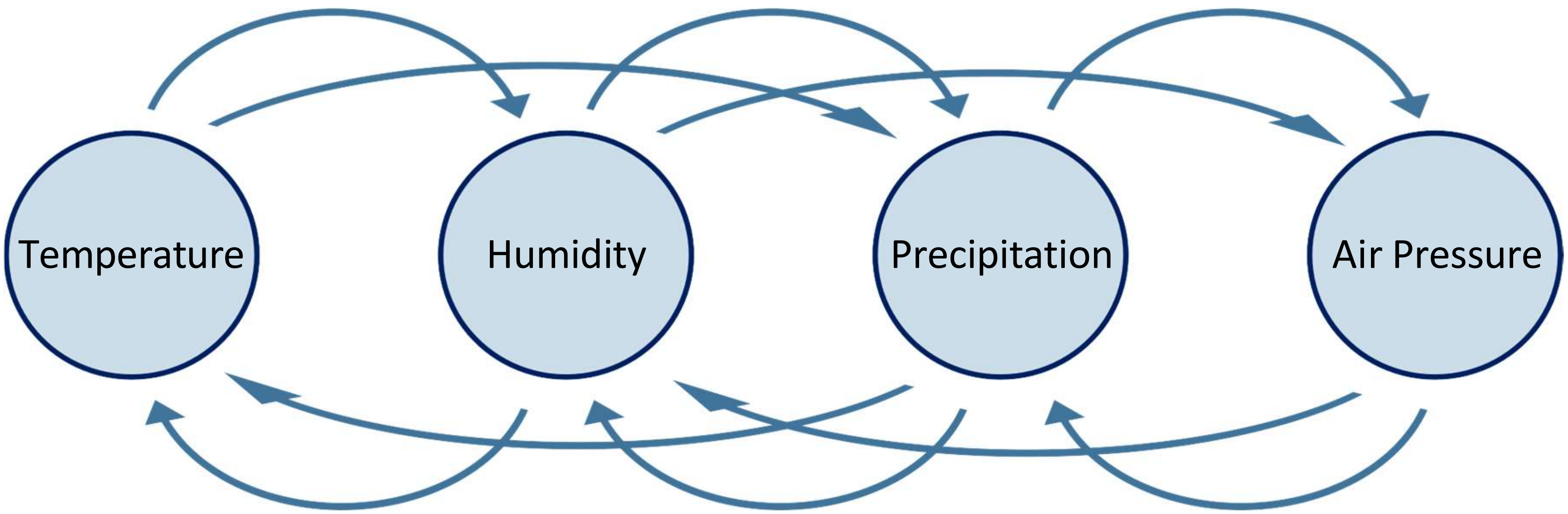


The world is full of interesting probability problems…

…and many of them involve *multiple* random variables, being random *together*

How Do We Model Multiple Random Variables Together?

Often, all the random variables involved are not independent of each other.



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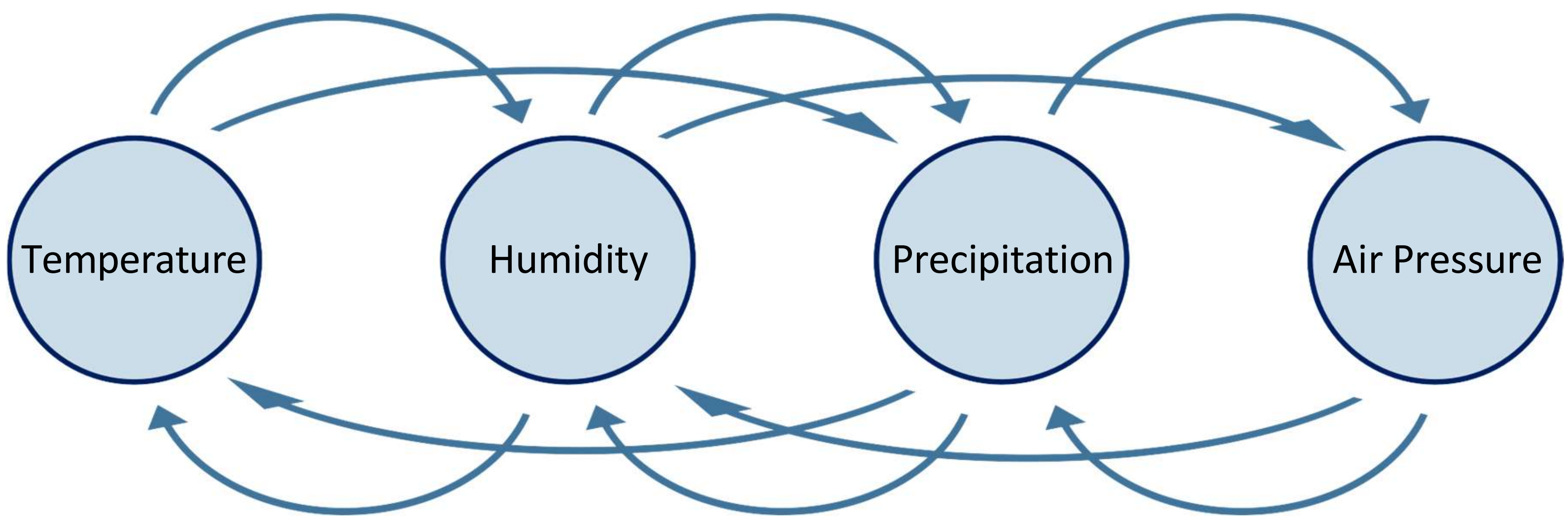

So we can’t just have a single distribution for each random variable —
we need a way to talk about all the random variables at the same time.

The "Joint" Distribution of Multiple Random Variables

For *discrete* random variables X and Y, we have a **joint probability mass function**:

$$P(X = x, Y = y)$$

The joint is the "and" between an assignment to X, and an assignment to Y

The same as P(A and B) for events A and B!

The "Joint" Distribution of Multiple Random Variables

For *discrete* random variables X and Y, we have a **joint probability mass function**:

$0.5134\ldots$

$$P(\underbrace{X = x}, \underbrace{Y = y})$$

$X = 2, Y = 4$

$P(\underset{\text{gender}}{X} = \text{male}, \underset{\text{height}}{Y} = 5.9 \text{ feet})$

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The same as P(A and B) for events A and B!

The "Joint" Distribution of Multiple Random Variables

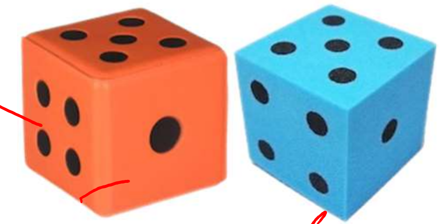
For *discrete* random variables X and Y, we have a **joint probability mass function**:

$$P(X = x, Y = y)$$

For *continuous* random variables, we have a **joint probability density function**:

$$f(X = x, Y = y) \qquad P\{(X, Y) \in C\} = \iint_{(x,y)\in C} f(x, y)\, dx\, dy$$

Example Joint PMF: Two Dice



Roll two 6-sided dice, yielding values X and Y.

$$\boxed{X}$$

random variable

$$P(X = 1)$$

probability of ~~an event~~

$$P(X = k)$$

probability mass function

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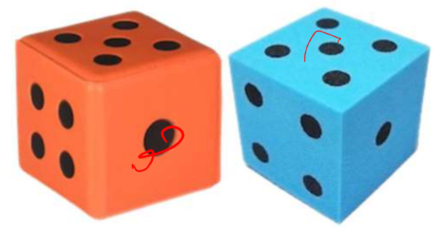

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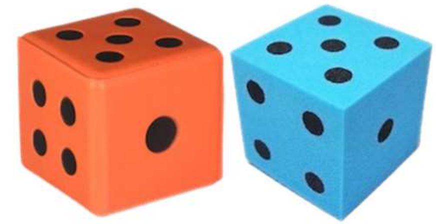
X random variable	$P(X = 1)$ probability of an event	$P(X = k)$ probability mass function
X, Y random variables	$P(X = 1, Y = 6)$ probability of the intersection of two events	$P(X = x, Y = y)$ joint probability mass function

Example Joint PMF: Two Dice



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What is $P(X = x, Y = y)$?

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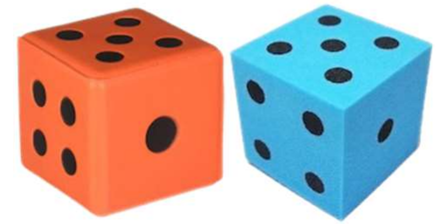


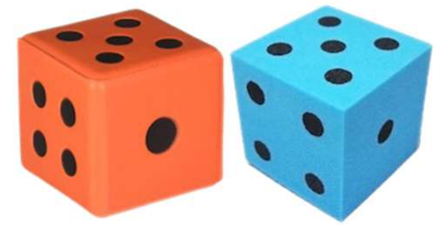
Roll two 6-sided dice, yielding values X and Y.

What is $P(X = x, Y = y)$?

$$P(X = x, Y = y) = \frac{1}{36}$$

$$(x, y) \in \{(1,1), \dots, (6,6)\}$$

Example Joint PMF: Two Dice



Roll two 6-sided dice, yielding values X and Y.

What is $P(X = x, Y = y)$?

		X					
		1	2	3	4	5	6
Y	1	1/36	…	…	…	…	1/36
	2	…	…	…	…	…	…
	3	…	…	…	…	…	…
	4	…	…	…	…	…	…
	5	…	…	…	…	…	…
	6	1/36	…	…	…	…	1/36

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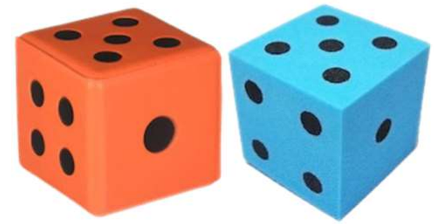
Example Joint PMF: Two Dice



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		X					
		1	2	3	4	5	6
	1	1/36	…	…	…	…	1/36
	2	…	…	…	…	.	
	3	…	…	…	…	…	…
Y	4	…	…	…	…	…	…
	5	…	…	…	…	…	…
	6	1/36	…	…	…	…	1/36

$P(X = 4, Y = 3)$

This is a **joint probability table**: it contains the probabilities of all possible outcomes for a set of discrete random variables

Another Example

Example 4.3.a. Suppose that 3 batteries are randomly chosen from a group of 3 new, 4 used but still working, and 5 defective batteries. If we let X and Y denote, respectively, the number of new and used but still working batteries that are chosen, then the joint probability mass function of X and Y, $p(i, j) = P\{X = i, Y = j\}$, is given by

$$\boxed{p(i, j) = \frac{\binom{3}{i}\binom{4}{j}\binom{5}{3-i-j}}{\binom{12}{3}}}$$

$$p(0,0) = \binom{5}{3} \Big/ \binom{12}{3} = 10/220$$

$$p(0,1) = \binom{4}{1}\binom{5}{2} \Big/ \binom{12}{3} = 40/220$$

$$p(0,2) = \binom{4}{2}\binom{5}{1} \Big/ \binom{12}{3} = 30/220$$

X ~ # of new batteries

Y ~ # of used batteries

new, used, defective

Table 4.1 $P\{X = i, Y = j\}$

i \ j	**0**	**1**	**2**	**3**	**Row Sum $= P\{X = i\}$**
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$\frac{4}{220}$	$\frac{84}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0	$\frac{108}{220}$
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0	$\frac{27}{220}$
3	$\frac{1}{220}$	0	0	0	$\frac{1}{220}$
Column Sums $= P\{Y = j\}$	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$	

Example with continuous density

$C \equiv \{X < Y\}$

Example 4.3.c. The joint density function of X and Y is given by

$$f(x, y) = \begin{cases} 2e^{-x}e^{-2y} & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

$C = \{X > 1, Y < 1\}$

$$P\{X > 1, Y < 1\} = \int_0^1 \int_1^\infty 2e^{-x}e^{-2y}\,dx\,dy$$
$$= \int_0^1 2e^{-2y}(-e^{-x}|_1^\infty)\,dy$$

$$P\{X < Y\} = \iint_{(x,y):x<y} 2e^{-x}e^{-2y}\,dx\,dy$$
$$= \int_0^\infty \int_0^y 2e^{-x}e^{-2y}\,dx\,dy$$
$$= \int_0^\infty 2e^{-2y}(1 - e^{-y})\,dy$$
$$= \int_0^\infty 2e^{-2y}\,dy - \int_0^\infty 2e^{-3y}\,dy$$
$$= 1 - \tfrac{2}{3}$$
$$= \tfrac{1}{3}$$

$$P(X < Y) = \iint_{y\ x} f(x,y)\,dy$$

Marginals

$x = a$

$$P\{X < a\} = \int_0^a \int_0^\infty 2e^{-2y} e^{-x}\, dy\, dx$$

$x = 0$ $\quad y = 0$

$$= \int_0^a e^{-x}\, dx$$

$$= 1 - e^{-a} \qquad \blacksquare$$

$P(X < a) \sim \exp(\lambda = 1)$

Law of total probability

Joint table expresses the complete information about the random variables

$$P(X = x) = \sum_{y} P(X = x, Y = y)$$

$P(X = x)$ is called the marginal of the joint distribution $P(X, Y)$

$$f(X=x) = \int_{Y} f(x, y)\, dy$$

Independent Random Variables

The random variables X and Y are said to be independent if for any two sets of real numbers A and B

$$P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\} \tag{4.3.7}$$

This also implies that

$$P(X \leq a, Y \leq b) = P(X \leq a)P(Y \leq b)$$

Or $F_{X,Y}(a,b) = F_X(a)F_Y(b)$

In the jointly continuous case, the condition of independence is equivalent to

$$f(x, y) = f_X(x)f_Y(y) \qquad \text{for all } x, y$$

Example 4.3.d. Suppose that X and Y are independent random variables having the common density function

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f(y) = \begin{cases} e^{-y} & y > 0 \\ 0 & \text{other.} \end{cases}$$

$e^{-(x+y)}$

$\frac{x}{y} = c$

$x = cy$

$\int_0^{\infty} e^{-(c+1)y} \, dy$

Find the density function of the random variable X/Y.

$$f\left(z = \frac{X}{Y}\right)$$

$$P(z \le a) = P\left(\frac{X}{Y} \le a\right) = P(X \le aY) = \int_0^{\infty}\int_0^{ay} f(x,y)\,dx\,dy$$

$$= \int_0^{\infty} e^{-y} \int_0^{ay} e^{-x}\,dx\,dy = F_z(a) = \frac{a}{a+1}$$

$$f(z) = \frac{\partial}{\partial a} F_z(a)$$

$z = T(x)$ $\quad f_X(x)$

$$f_z(z) = f_X(T^{-1}(z)) \frac{\partial T^{-1}}{\partial z}$$

Conditional Probability

Given two discrete random variables X, Y. The conditional probability of X given a specific value of Y is given as:

$$P(X = x|Y = y) = P(X = x, Y = y)/P(Y = y)$$

For continuous variables with joint density of X,Y as $f(x, y)$:

$$f_{X|Y}(x|y) = f(x, y)/f(y)$$

$$\begin{aligned} f_{X|Y}(x\,|\,y)\,dx &= \frac{\boxed{f(x, y)\,dx\,dy}}{f_Y(y)\,dy} \\ &\approx \frac{\boxed{P\{x \le X \le x + dx, y \le Y \le y + dy\}}}{P\{y \le Y \le y + dy\}} \\ &= P\{x \le X \le x + dy\,|\,y \le Y \le y + dy\} \end{aligned}$$

Example 4.3.h. The joint density of X and Y is given by

$$f(x, y) = \begin{cases} \frac{12}{5}x(2-x-y) & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute the conditional density of X, given that $Y = y$, where $0 < y < 1$.

Solution. For $0 < x < 1$, $0 < y < 1$, we have

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)\,dx} \\ &= \frac{x(2-x-y)}{\int_0^1 x(2-x-y)\,dx} \\ &= \frac{x(2-x-y)}{\frac{2}{3} - y/2} \\ &= \frac{6x(2-x-y)}{4-3y} \quad \blacksquare \end{aligned}$$

$$f_Y(y) = \int_{x=0}^{1} f(x,y)\,dx$$

Joint distribution of n random variables

If $X_1, X_2, \dots, X_n$ are n random variables. Their joint distribution is defined for the discrete case as

$$p(x_1, x_2, \dots, x_n) = P\{X_1 = x_1, X_2 = x_2, \dots, X_n = x_n\}$$

Further, the n random variables are said to be jointly continuous if there exists a function $f(x_1, x_2, \dots, x_n)$, called the joint probability density function, such that for any set C in n-space

$$P\{(X_1, X_2, \dots, X_n) \in C\} = \int\int_{(x_1,\dots,x_n)\in C} \cdots \int f(x_1, \dots, x_n)\, dx_1\, dx_2 \cdots dx_n$$

In particular, for any n sets of real numbers $A_1, A_2, \dots, A_n$

$$P\{X_1 \in A_1, X_2 \in A_2, \dots, X_n \in A_n\}$$
$$= \int_{A_n} \int_{A_{n-1}} \cdots \int_{A_1} f(x_1, \dots, x_n)\, dx_1\, dx_2 \dots dx_n$$

Example 4.3.e. Suppose that the successive daily changes of the price of a given stock are assumed to be independent and identically distributed random variables with probability mass function given by

X_i

$$P\{\text{daily change is } i\} = \begin{cases} -3 & \text{with probability } .05 \\ -2 & \text{with probability } .10 \\ -1 & \text{with probability } .20 \\ 0 & \text{with probability } .30 \\ 1 & \text{with probability } .20 \\ 2 & \text{with probability } .10 \\ 3 & \text{with probability } .05 \end{cases}$$

$P(X_i)$

Then the probability that the stock's price will increase successively by 1, 2, and 0 points in the next three days is

$$P\{X_1 = 1, X_2 = 2, X_3 = 0\} = (.20)(.10)(.30) = .006$$

where we have let X_i denote the change on the ith day. ■