Logic in CS Autumn 2024

Problem Sheet 6

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1. Translate the following formula to rectified form, then to prenex form, and finally to Skolem form:
$$\forall z\, \exists y\, (Q(x, g(y), z) \vee \neg \forall x\, P(x)) \wedge \neg \forall z\, \exists x\, \neg R(f(x, z), z)\,.$$

2. Are the following claims correct? Justify your answers.
   (a) For any formula F and term t, if F is valid then $F[t/x]$ is valid.
   (b) $\exists x\, (P(x) \to \forall y\, P(y))$ is valid.
   (c) For any formula F and constant symbol c, if $F[c/x]$ is valid and c does not appear in F then $\forall x\, F$ is valid.

3. Which of the following languages are regular? Which are FO definable?
   (a) The set of words over $\{a, b\}$ which has equal number of occurrences of ab and ba. For example, aba is in the language, while $abab$ is not.
   (b) The set of words over $\{a, b, \#\}$ with a single occurrence of $\#$, and every symbol before the $\#$ is an a, and all symbols after the $\#$ are b's.
   (c) The set of strings over $\{a, b\}$ which does not contain any occurrence of ba.
   (d) The set of strings over $\{0, 1\}$ such that the second symbol from both ends is 0.
   (e) Let $\Sigma = \{\begin{pmatrix} a \\ b \end{pmatrix} \mid a, b \in \{0, 1\}\}$. A string over Σ gives two rows of 0's and 1's. Treat each row as a binary number. The set of words
$$\{w \in \Sigma^* \mid \text{ the top row is larger than the bottom row } \}$$

4. Consider the following FO formulae. In each case, answer the following questions:
   - What is $L(\varphi)$?
   - What is $\overline{L(\varphi)}$?
   - Is $L(\varphi)$ regular?
   - Is $\overline{L(\varphi)}$ regular?

(a) $\forall x(x \neq x)$

(b) $\exists x \exists y[x < y \wedge Q_b(x) \wedge Q_a(y) \wedge \forall z[(x < z < y) \rightarrow Q_a(z)]]$

(c) $\exists x[Q_a(x) \wedge \exists y[S(x,y) \wedge \forall z[z \leq y]]]$

(d) $\exists x \forall y[x \leq y \wedge Q_a(x)] \wedge \exists x \forall y[y \leq x \wedge Q_b(x)] \wedge$
$\forall x \forall y[Q_a(x) \wedge S(x,y) \rightarrow Q_b(y)] \wedge \forall x \forall y[Q_b(x) \wedge S(x,y) \rightarrow Q_a(y)]$

5. Consider the following automaton. What is the language L accepted? Can you write an FO formula φ such that $L = L(\varphi)$?