# CS 228 : Logic in Computer Science

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# The DPLL Algorithm

Input : CNF formula $F$.

1. Initialise $\alpha$ as the empty assignment
2. While there is a unit clause $L$ in $F|\alpha$, add $L = 1$ to $\alpha$ (unit propagation)
3. If $F|\alpha$ contains no clauses, then stop and output $\alpha$
4. If $F|\alpha$ contains the empty clause, then apply the learning procedure to add a new clause $C$ to $F$. If it is the empty clause, output UNSAT. Otherwise, backtrack to the highest level at which $C$ is a unit clause, go to line 2.
5. Decide on a new assignment $p = b$ to be added to $\alpha$, goto line 2.

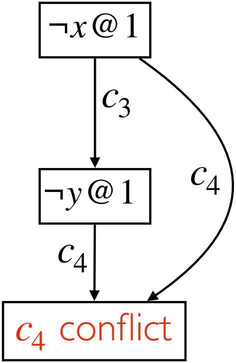
# DPLL Example

$c_1 = \neg x \vee \neg y$

$c_2 = \neg x \vee y$

$c_3 = x \vee \neg y$

$c_4 = x \vee y$




Clause learnt : $x$ (Resolve $c_4$ with $c_3$)

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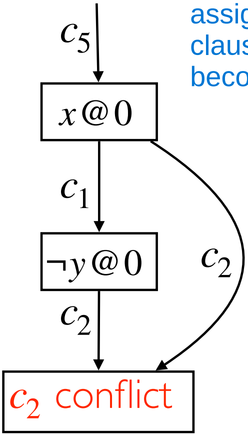
$c_4 = x \vee y$

$c_5 = x$




If all the literals in a clause are assigned values that make the clause false, then that clause becomes an empty clause.

Clause learnt : Resolve $c_2$ with $c_1, c_5$. Empty clause.

# DPLL Correctness

### Termination

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## Correctness

Correctness is straightforward : $F \vdash$ the learned clause. Thus, if the empty clause is learnt, then $F$ is unsat. Otherwise, if DPLL terminates with a satisfying assignment $\alpha$, then the input formula is also satisfied by $\alpha$.

# Modern SAT Solvers

Numerous enhancements/heuristics

- Decision heuristics to choose decision variables
- Random restarts

First Order Logic

# FOL

Extends propositional logic

- Propositional logic : atomic formulas have no internal structure
- FOL : atomic formulas are predicates that assert a relationship between certain elements
- Quantification in FOL : ability to assert that a certain property holds for all elements or only for some element.
- Formulae in FOL are over some signature.

# Signatures

- A vocabulary or signature $\tau$ is a set consisting of
  - constants $c_1, c_2, \ldots$
  - Relation symbols $R_1, R_2 \ldots$, each with some arity $k$, denoted $R_i^k$
  - Function symbols $f_1, \ldots$ each with some arity $k$, denoted $f_i^k$
- We look at finite signatures
- $\tau = (E^2, F^3, f^1)$ is a finite signature with two relations, $E$ with arity 2 and $F$ with arity 3, and a function $f$ with arity 1

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- The symbols ( and ) called paranthesis

# Terms

Given a signature $\tau$, the set of $\tau$-terms are defined inductively as follows.

- Each variable is a term
- Each constant symbol is a term
- If $t_1, \ldots, t_k$ are terms and $f$ is a $k$-ary function, then $f(t_1, \ldots, t_k)$ is a term
- Ground Terms : Terms without variables. For instance $f(c_1, \ldots, c_k)$ for constants $c_1, \ldots, c_k$.

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- If $\varphi$ is a wff and $x$ is a variable, then $(\forall x)\varphi$ and $(\exists x)\varphi$ are wff
- The second and third are atomic formulae.
- If a formula $F$ occurs as part of another formula $G$, then $F$ is called a sub formula of $G$.

# Logical Abbreviations : Boolean Connectives

- $\neg\varphi = \varphi \to \bot$
- $\top = \neg\bot$
- $\varphi \vee \psi = \neg\varphi \to \psi$
- $\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$
- $\exists x.\varphi = \neg(\forall x.\neg\varphi)$
- Precedence of operators : Quantifiers and negation highest, followed by $\vee$, $\wedge$, followed by $\to$.
  - $\forall x\ P(x) \wedge R(x)$ is $[\forall x.[P(x)]] \wedge R(x)$

# An Example

Consider the signature $\tau = \{R\}$ where $R$ is a binary relation. The following are FO formulae over this signature.

- $\forall x R(x, x)$ Reflexivity
- $\forall x (R(x, x) \to \bot)$ Irreflexivity
- $\forall x \forall y (R(x, y) \to R(y, x))$ Symmetry
- $\forall x \forall y \forall z (R(x, y) \to (R(y, z) \to R(x, z)))$ Transitivity