Random Variables: Continued

Types of Random Variables

- Discrete Vs Continuous

Specifying probability of discrete R.V.

- Probability Mass Function P(X=k)

$X \in \{1, 2, 3, 4, 5, 6\} \leftarrow$ outcome of dice roll.

$$
\left.
\begin{aligned}
P(X=1) &= p_1 \\
P(X=2) &= p_2 \\
&\vdots \\
P(X=k) &= p_k
\end{aligned}
\right\}
$$

Enumeration representation of PMF.

Specifying probability of continuous R.Vs

- If X is continuous, it can take an infinite number of values.
- Probability mass function: $P(X = k)$ cannot be defined.
- Instead we ask for probability that x lies in an interval B of non-zero size: $P(X \in B)$

Probability density function: $f(X) \geq 0$

$$P(X \in B) = \int_{x \in B} f(x)\,dx \qquad \int_{-\infty}^{\infty} f(x)\,dx = 1$$

X

$$f(x) = 2 \quad \text{if} \quad 0 \le x \le \frac{1}{2}$$
$$= 0 \quad \text{otherwise}$$

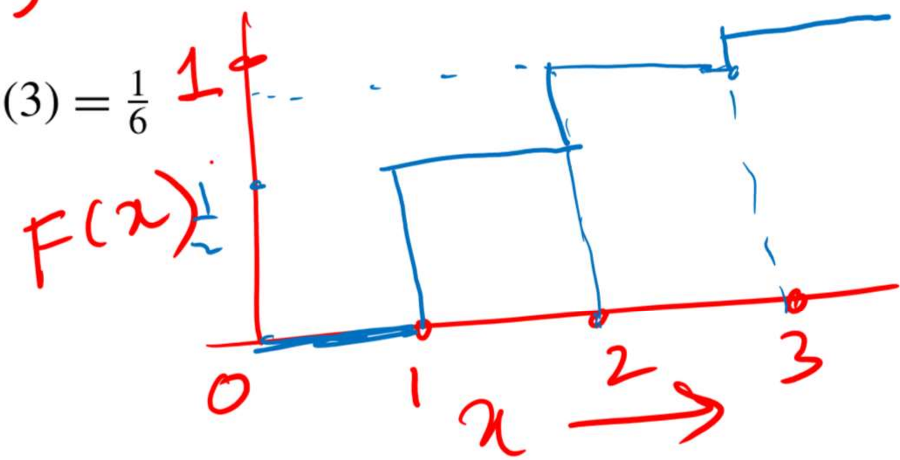
Cumulative distribution function (CDF)

- Assume R.V. X is ordered.
- CDF of X is a function F(a) that takes a value a and return $P(X \le a)$
- CDF of a discrete distribution. X is discrete and ordered: $x_1, x_2 \cdots x_k$

$$F(x) = \sum_{x_i \le x} P(X = x_i)$$

- Example: $p(1) = \frac{1}{2}, \quad p(2) = \frac{1}{3}, \quad p(3) = \frac{1}{6}$

$x_1 = 1, \; x_2 = 2, \; x_3 = 3$



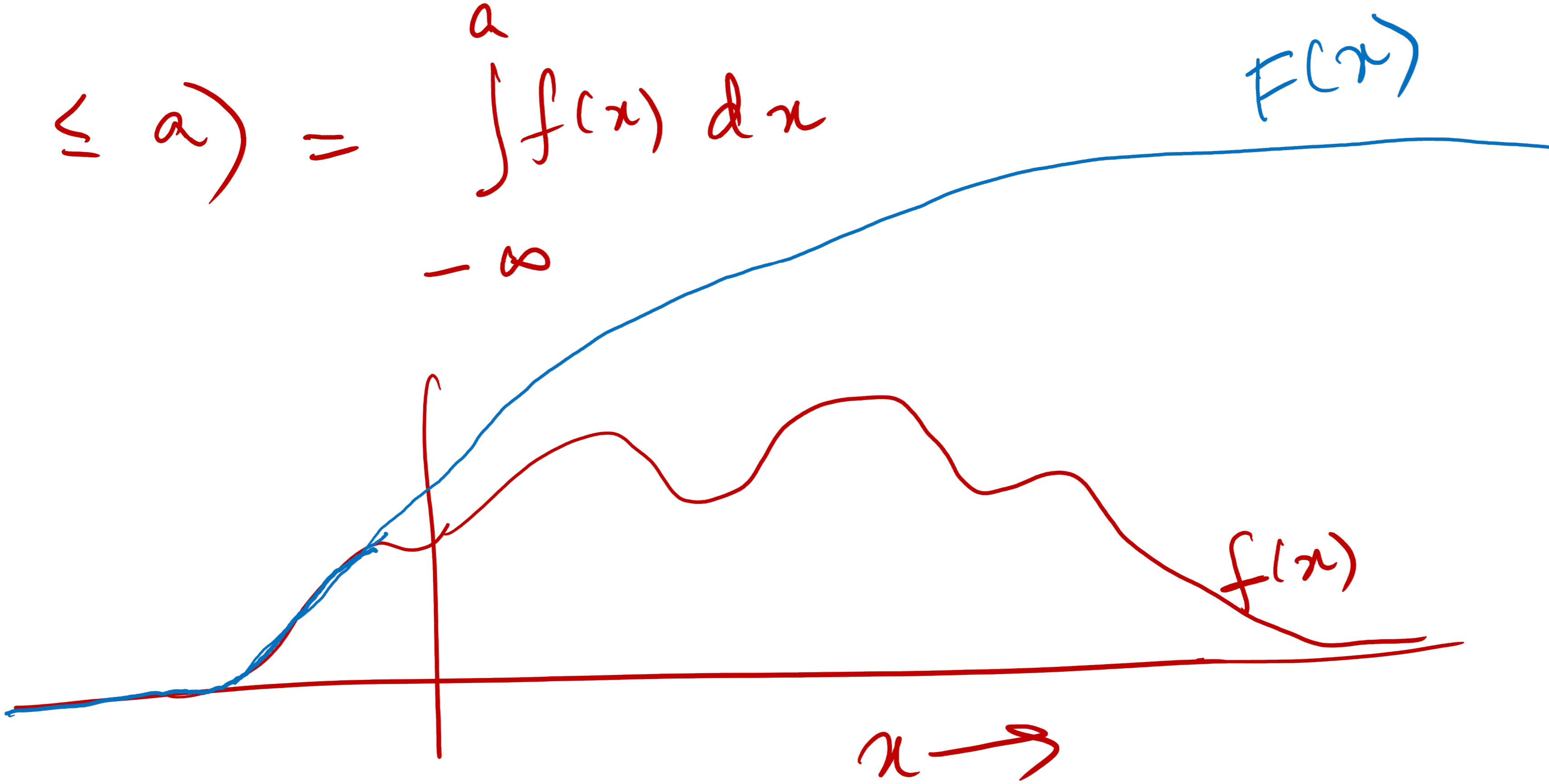
$F(1) = \frac{1}{2}$

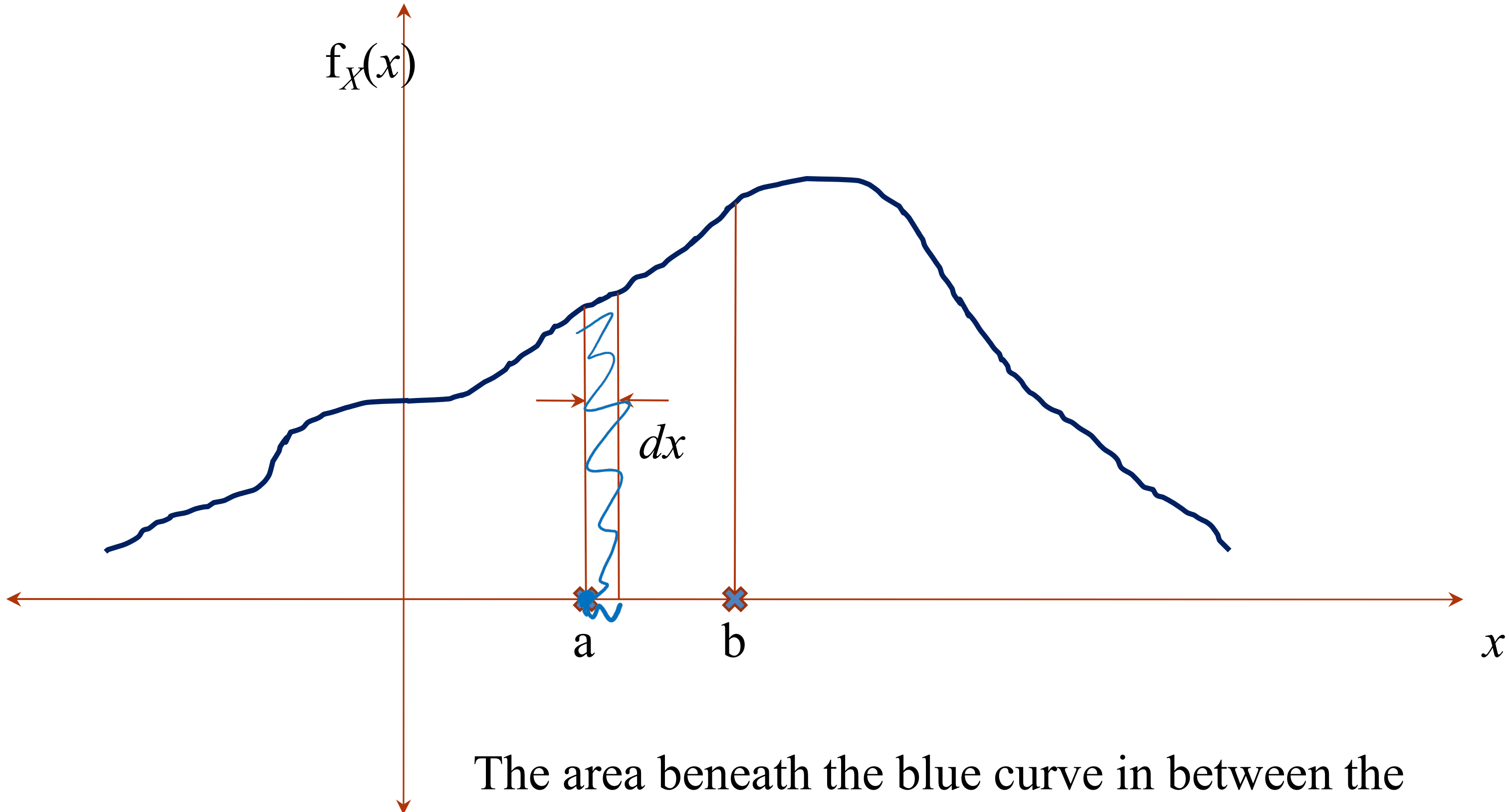
$F(2) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

$F(3) = \frac{5}{6} + \frac{1}{6} = 1$

CDF of a continuous distribution

$$P(X \le a) = \int_{-\infty}^{a} f(x)\, dx$$





The area beneath the blue curve in between the lines $x = a$ and $x = b$ is the **cumulative interval measure** $P(a < X \le b) = F_X(b) - F_X(a)$.

$f_X(a)dx$ = probability that the random variable X takes on values between a and $a+dx$.

Random variable: continuous - example

Consider a CDF of the form:

$F_X(x) = 0$ for $x \leq 0$, and

$F_X(x) = 1 - \exp(-x^2)$ otherwise

To find: probability that X exceeds 1

$$P(X > 1) = 1 - P(X \leq 1)$$

$$= 1 - F(1)$$

$$= 1 - \left(1 - e^{-1^2}\right)$$

$$= e^{-1}$$

Expected Value of a random variable

For a discrete random variable *X*, is defined as:

$$E(X) = \sum_i x_i P(X = x_i)$$

The expected value that shows up when you throw a die is 1/6(1+2+3+4+5+6) = 3.5.

For continuous random variable *X*, is defined as:

$$E(X) = \int_{-\infty}^{+\infty} x f_X(x) dx$$

Expected Value: examples

The game of roulette consists of a ball and wheel with 38 numbered pockets on its side. The ball rolls and settles on one of the pockets. If the number in the pocket is the same as the one you guessed, you win \$35 (probability 1/38), otherwise you lose \$1 (probability 37/38). The expected value of the amount you earn after one trial is: (-1)37/38 +(35)1/38 = \$-0.0526

A Game of Roulette

https://en.wikipedia.org/wiki/Roulette#/media/File:Roulette_casino.JPG

Expected value of a function of random variable

Consider a function $g(X)$.
The expected value of $g(X)$:
For discreet R.V. (provided the summation is well-defined):

$$E(g(X)) = \sum_i g(x_i) P(X = x_i)$$

For a continuous random variable,

$$E(g(X)) = \int_{-\infty}^{+\infty} g(x) f_X(x) dx$$

Properties of expected value

$$E(ag(X)+b)=\int_{-\infty}^{+\infty}(ag(x)+b)f_X(x)dx$$

$$=\int_{-\infty}^{+\infty}ag(x)f_X(x)dx+\int_{-\infty}^{+\infty}bf_X(x)dx$$

$$=\boxed{aE(g(X))+b}---why?$$

This property is called the **linearity** of the expected value. In general, a function $f(x)$ is said to be linear in x is $f(ax+b) = af(x)+f(b)$ where a and b are constants. In this case, the expected value is not a function but an operator (it takes a function as input). An operator E is said to be linear if $E(af(x) + b) = a E(f(x)) + E(b)$. This is equal to $aE(f(x)) + b$ for the expectation operator.

Properties of expected value

Consider a set of random variables $X_1, X_2, \ldots, X_n$; a set of functions $g_1, g_2, \ldots, g_n$. Then we have:

$$E\left(\sum_{i=1}^{n} a_i g_i(X_i) + b_i\right) = \sum_{i=1}^{n}\left(a_i E[g_i(X_i)] + b_i\right)$$

a_i, b_i are scalars

$$= \qquad E(g(X_1)^2) \neq E(g(X))^2 .$$

- Note: for a general nonlinear function g, we have:

$$E(g(X)) \neq g(E(X))$$

What if you have to guess the value of a R.V.?

Suppose you want to predict the value of a random variable with a known mean. On an average, what value will yield the least squared error?

$X \sim P(X=k)$

Goal: guess a value c s.t.

$\min_c E[(X-c)^2]$

To prove that at $c = \underbrace{E[X]}_{\mu}$ the above error is minimized.

$E[(X-c)^2] = E[(X - c + \mu - \mu)^2]$

$= E[(X-\mu)^2 + (c-\mu)^2 - 2(X-\mu)(c-\mu)]$

$= E[(x-\mu)^2] + E[(c-\mu)^2] - 2(c-\mu)\underbrace{[E(X) - \mu]}_{=0}$

$= E(x-\mu)^2 + (c-\mu)^2$

Variance

- The **variance** of a random variable *X* tells you how much its values deviate from the mean – on an average.
- The definition of variance for a continuous r.v. with mean μ is:

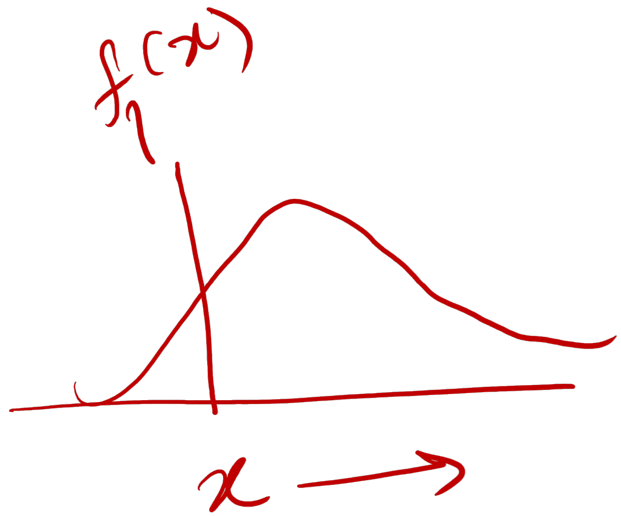
$$Var(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)dx$$

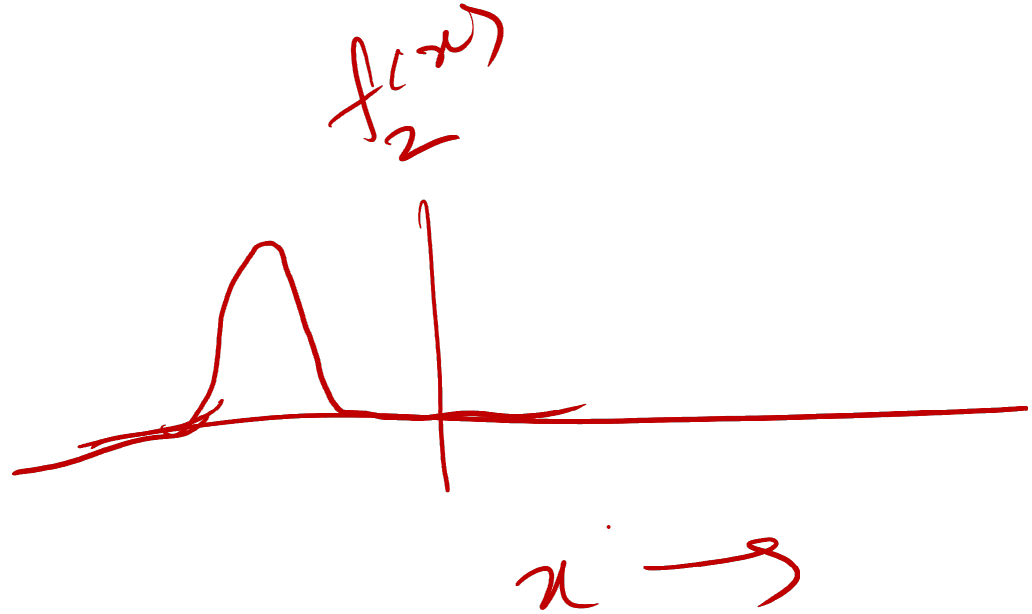
- For a discrete r.v., the integration is replaced by a summation:

$$Var(X) = E[(X-\mu)^2] = \sum_i (x_i-\mu)^2 P(X=x_i)$$

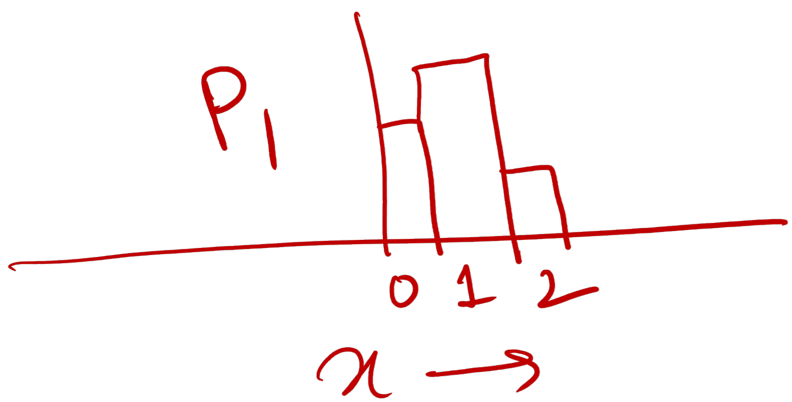
- The positive square-root of the variance is called the **standard deviation**.
- Low-variance probability mass functions or probability densities tend to be concentrated around one point. High variance densities are spread out.

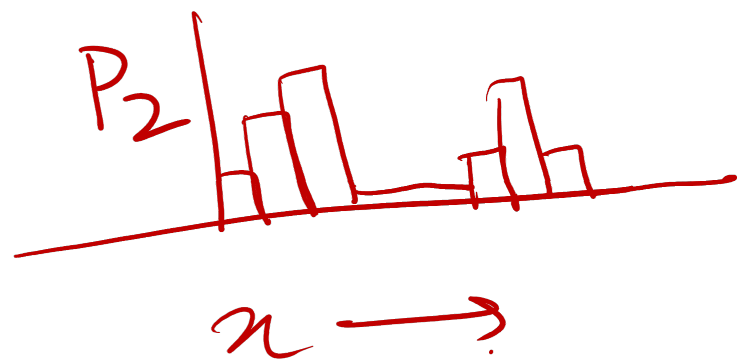
Variance: examples





$\mathrm{Var}(f_1) > \mathrm{Var}(f_2)$





$\mathrm{Var}(P_2) > \mathrm{Var}(P_1)$

The Simplest Random Variable

- Bernoulli Random Variable

$X \in \{0, 1\}$

PMF of X $\qquad P(X=1) = \theta$

$$P(X=x) = \theta^{x}(1-\theta)^{1-x}$$

$$E[X] = 0 \cdot (1-\theta) + 1 \cdot \theta = \theta$$

$$V(X) = (0-\theta)^2 (1-\theta) + (1-\theta)^2 \cdot \theta$$

$$= \theta \cdot (1-\theta)$$