CS 228 : Logic in Computer Science

Krishna. S

First-Order Logic : Semantics

Structures

- A structure $\mathcal{A}$ of signature τ consists of

Structures

- A structure $\mathcal{A}$ of signature τ consists of
  - A non-empty set A or $u(\mathcal{A})$ called the universe

Structures

- A structure $\mathcal{A}$ of signature τ consists of
  - A non-empty set A or $u(\mathcal{A})$ called the universe
  - For each constant c in the signature τ, a fixed element $c_{\mathcal{A}}$ is assigned from the universe A

Structures

- A structure $\mathcal{A}$ of signature τ consists of
  - A non-empty set A or $u(\mathcal{A})$ called the universe
  - For each constant c in the signature τ, a fixed element $c_{\mathcal{A}}$ is assigned from the universe A
  - For each k-ary relation R^k in the signature τ, a set of k-tuples from A^k is assigned to $R^{\mathcal{A}}$

Structures

- A structure $\mathcal{A}$ of signature τ consists of
  - A non-empty set A or $u(\mathcal{A})$ called the universe
  - For each constant c in the signature τ, a fixed element $c_{\mathcal{A}}$ is assigned from the universe A
  - For each k-ary relation R^k in the signature τ, a set of k-tuples from A^k is assigned to $R^{\mathcal{A}}$
  - The structure $\mathcal{A}$ is finite if A (or $u(\mathcal{A})$) is finite

Examples of Structures : A Graph

- $\tau = \{E\}$, with E binary.

Examples of Structures : A Graph

- $\tau = \{E\}$, with E binary.
  - A graph structure over τ is $\mathcal{G} = (V, E^{\mathcal{G}})$,
  - The universe $u(\mathcal{G})$ is the set of vertices V
  - The relation E is the edge relation

Examples of Structures : A Graph

- $\tau = \{E\}$, with E binary.
  - A graph structure over τ is $\mathcal{G} = (V, E^{\mathcal{G}})$,
  - The universe $u(\mathcal{G})$ is the set of vertices V
  - The relation E is the edge relation
  - $\mathcal{G} = (V = \{1, 2, 3, 4\}, E^{\mathcal{G}} = \{(1, 2), (2, 3), (3, 4), (1, 1)\})$. We could just as well draw the graph for convenience.

Examples of Structures : An Order

- $\tau = \{<, S\}$ with $<, S$ binary.

Examples of Structures : An Order

- $\tau = \{<, S\}$ with $<, S$ binary.
  - A finite order structure over τ is $\mathcal{O} = (O, <^{\mathcal{O}}, S^{\mathcal{O}})$
  - The universe $u(\mathcal{O})$ is the finite ordered set O
  - $<^{\mathcal{O}}$ is the ordering on O and $S^{\mathcal{O}}$ is the successor on O

Examples of Structures : An Order

- $\tau = \{<, S\}$ with $<, S$ binary.
  - A finite order structure over τ is $\mathcal{O} = (O, <^{\mathcal{O}}, S^{\mathcal{O}})$
  - The universe $u(\mathcal{O})$ is the finite ordered set O
  - $<^{\mathcal{O}}$ is the ordering on O and $S^{\mathcal{O}}$ is the successor on O
  - $\mathcal{O} = (O = \{1, 2, 4\}, <^{\mathcal{O}} = \{(1, 2), (1, 4), (2, 4)\}, S^{\mathcal{O}} = \{(1, 2)\})$

Examples of Structures : A Word

- $\tau = \{<, S, Q_a, Q_b\}$, where $<$, S are binary, Q_a, Q_b are unary relations.

Examples of Structures : A Word

- $\tau = \{<, S, Q_a, Q_b\}$, where $<, S$ are binary, Q_a, Q_b are unary relations.
  - A word structure $\mathcal{W} = (u(\mathcal{W}), <^{\mathcal{W}}, S^{\mathcal{W}}, {Q_a}^{\mathcal{W}}, {Q_b}^{\mathcal{W}})$

Examples of Structures : A Word

- $\tau = \{<, S, Q_a, Q_b\}$, where $<, S$ are binary, Q_a, Q_b are unary relations.
  - A word structure $\mathcal{W} = (u(\mathcal{W}), <^{\mathcal{W}}, S^{\mathcal{W}}, {Q_a}^{\mathcal{W}}, {Q_b}^{\mathcal{W}})$
  - The universe $u(\mathcal{W})$ consists of the positions in a word W over symbols a, b

Examples of Structures : A Word

- $\tau = \{<, S, Q_a, Q_b\}$, where $<, S$ are binary, Q_a, Q_b are unary relations.
  - A word structure $\mathcal{W} = (u(\mathcal{W}), <^{\mathcal{W}}, S^{\mathcal{W}}, {Q_a}^{\mathcal{W}}, {Q_b}^{\mathcal{W}})$
  - The universe $u(\mathcal{W})$ consists of the positions in a word W over symbols a, b
  - $<^{\mathcal{W}}$ is the ordering relation on the positions of W

Examples of Structures : A Word

- $\tau = \{<, S, Q_a, Q_b\}$, where $<, S$ are binary, Q_a, Q_b are unary relations.
  - A word structure $\mathcal{W} = (u(\mathcal{W}), <^{\mathcal{W}}, S^{\mathcal{W}}, {Q_a}^{\mathcal{W}}, {Q_b}^{\mathcal{W}})$
  - The universe $u(\mathcal{W})$ consists of the positions in a word W over symbols a, b
  - $<^{\mathcal{W}}$ is the ordering relation on the positions of W
  - $S^{\mathcal{W}}$ is the successor relation on the positions of W

Examples of Structures : A Word

- $\tau = \{<, S, Q_a, Q_b\}$, where $<, S$ are binary, Q_a, Q_b are unary relations.
  - A word structure $\mathcal{W} = (u(\mathcal{W}), <^{\mathcal{W}}, S^{\mathcal{W}}, {Q_a}^{\mathcal{W}}, {Q_b}^{\mathcal{W}})$
  - The universe $u(\mathcal{W})$ consists of the positions in a word W over symbols a, b
  - $<^{\mathcal{W}}$ is the ordering relation on the positions of W
  - $S^{\mathcal{W}}$ is the successor relation on the positions of W
  - ${Q_a}^{\mathcal{W}}$ is the set of positions labeled a in W

Examples of Structures : A Word

- $\tau = \{<, S, Q_a, Q_b\}$, where $<, S$ are binary, Q_a, Q_b are unary relations.
  - A word structure $\mathcal{W} = (u(\mathcal{W}), <^{\mathcal{W}}, S^{\mathcal{W}}, {Q_a}^{\mathcal{W}}, {Q_b}^{\mathcal{W}})$
  - The universe $u(\mathcal{W})$ consists of the positions in a word W over symbols a, b
  - $<^{\mathcal{W}}$ is the ordering relation on the positions of W
  - $S^{\mathcal{W}}$ is the successor relation on the positions of W
  - ${Q_a}^{\mathcal{W}}$ is the set of positions labeled a in W
  - ${Q_b}^{\mathcal{W}}$ is the set of positions labeled b in W

Examples of Structures : A Word

- $\tau = \{<, S, Q_a, Q_b\}$, where $<, S$ are binary, Q_a, Q_b are unary relations.
  - A word structure $\mathcal{W} = (u(\mathcal{W}), <^{\mathcal{W}}, S^{\mathcal{W}}, {Q_a}^{\mathcal{W}}, {Q_b}^{\mathcal{W}})$
  - The universe $u(\mathcal{W})$ consists of the positions in a word W over symbols a, b
  - $<^{\mathcal{W}}$ is the ordering relation on the positions of W
  - $S^{\mathcal{W}}$ is the successor relation on the positions of W
  - ${Q_a}^{\mathcal{W}}$ is the set of positions labeled a in W
  - ${Q_b}^{\mathcal{W}}$ is the set of positions labeled b in W
  - The structure with $u(\mathcal{W}) = \{0, 1, 2, \dots, 8\}$, ${Q_a}^{\mathcal{W}} = \{0, 1, 4, 6, 8\}$, ${Q_b}^{\mathcal{W}} = \{2, 3, 5, 7\}$,
  - $<^{\mathcal{W}} = \{(0, 1), (0, 2), \dots, (7, 8)\}$, $S^{\mathcal{W}} = \{(0, 1), (1, 2), \dots, (7, 8)\}$ uniquely defines the word $W = aabbababa$.
  - For convenience, we can just write the word instead of the structure.

Free and Bound Variables

- For a wff $\varphi = \forall x \psi$ or $\exists x \psi$, ψ is said to be the scope of the quantifier x

Free and Bound Variables

- For a wff $\varphi = \forall x \psi$ or $\exists x \psi$, ψ is said to be the scope of the quantifier x
- Every occurrence of x in $\forall x \psi$ or $\exists x \psi$ is bound
- Any occurrence of x which is not bound is called free

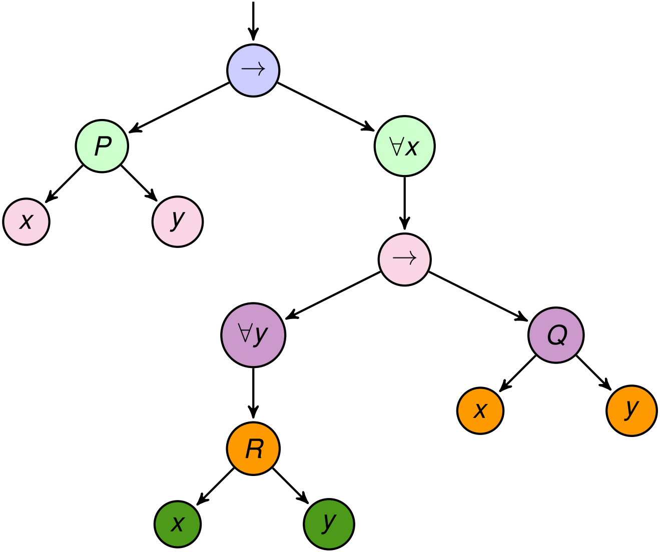
Free and Bound Variables

- For a wff $\varphi = \forall x \psi$ or $\exists x \psi$, ψ is said to be the scope of the quantifier x
- Every occurrence of x in $\forall x \psi$ or $\exists x \psi$ is bound
- Any occurrence of x which is not bound is called free
- $\varphi = P(x, y) \rightarrow \forall x((\forall y R(x, y)) \rightarrow Q(x, y))$
  - y is free in $Q(x, y)$ and bound in $R(x, y)$,
  - x is free in $P(x, y)$, and bound in $Q(x, y)$, $R(x, y)$

Free and Bound Variables

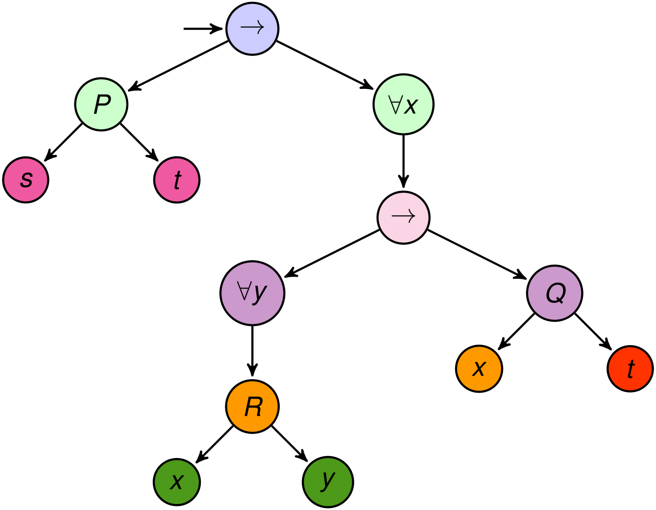
- For a wff $\varphi = \forall x \psi$ or $\exists x \psi$, ψ is said to be the scope of the quantifier x
- Every occurrence of x in $\forall x \psi$ or $\exists x \psi$ is bound
- Any occurrence of x which is not bound is called free
- $\varphi = P(x, y) \rightarrow \forall x((\forall y R(x, y)) \rightarrow Q(x, y))$
  - y is free in $Q(x, y)$ and bound in $R(x, y)$,
  - x is free in $P(x, y)$, and bound in $Q(x, y)$, $R(x, y)$
- Given φ, denote by $\varphi(x_1, \ldots, x_n)$, that $x_1, \ldots, x_n$ are the free variables of φ, also $free(\varphi)$
- A sentence is a formula φ none of whose variables are free

$P(x, y) \to \forall x((\forall y R(x, y)) \to Q(x, y))$

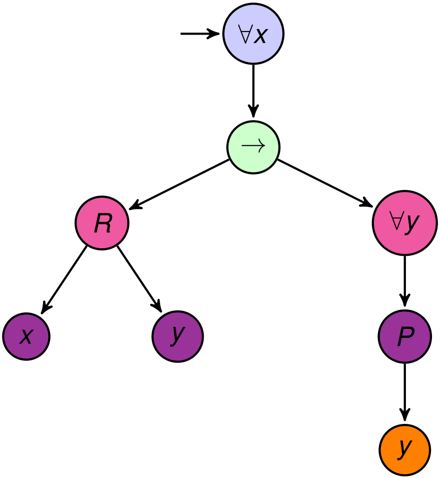


$P(x, y) \to \forall x((\forall y R(x, y)) \to Q(x, y))$

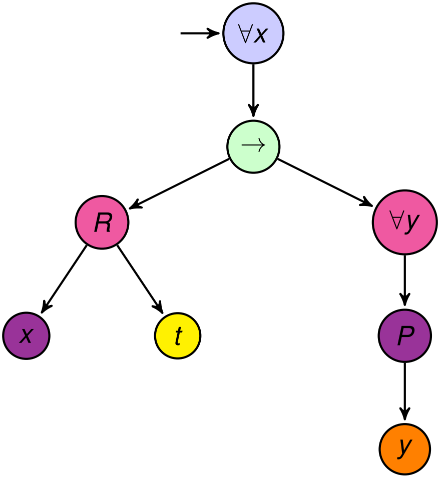
$\varphi(s, t) = P(s, t) \to \forall x((\forall y R(x, y)) \to Q(x, t))$



$\forall x(R(x, y) \to \forall y P(y))$



$\forall x(R(x,y) \to \forall y P(y))$



$\varphi(t) = \forall x(R(x,t) \to \forall y P(y))$

Assignments on τ-structures

Assignments

For a τ-structure $\mathcal{A}$, an assignment over $\mathcal{A}$ is a function $\alpha : \mathcal{V} \to u(\mathcal{A})$ that assigns every variable $x \in \mathcal{V}$ a value $\alpha(x) \in u(\mathcal{A})$. If t is a constant symbol c, then $\alpha(t)$ is $c^{\mathcal{A}}$

Assignments on τ-structures

Assignments

For a τ-structure $\mathcal{A}$, an assignment over $\mathcal{A}$ is a function $\alpha : \mathcal{V} \to u(\mathcal{A})$ that assigns every variable $x \in \mathcal{V}$ a value $\alpha(x) \in u(\mathcal{A})$. If t is a constant symbol c, then $\alpha(t)$ is $c^{\mathcal{A}}$

Binding on a Variable

For an assignment α over $\mathcal{A}$, $\alpha[x \mapsto a]$ is the assignment

$$\alpha[x \mapsto a](y) = \begin{cases} \alpha(y), y \neq x, \\ a, y = x \end{cases}$$

Satisfaction

We define the relation $\mathcal{A} \models_\alpha \varphi$ (read as φ is true in $\mathcal{A}$ under the assignment α) inductively:

Satisfaction

We define the relation $\mathcal{A} \models_\alpha \varphi$ (read as φ is true in $\mathcal{A}$ under the assignment α) inductively:

- $\mathcal{A} \not\models_\alpha \bot$

Satisfaction

We define the relation $\mathcal{A} \models_\alpha \varphi$ (read as φ is true in $\mathcal{A}$ under the assignment α) inductively:

- $\mathcal{A} \not\models_\alpha \perp$
- $\mathcal{A} \models_\alpha t_1 = t_2$ iff $\alpha(t_1) = \alpha(t_2)$

Satisfaction

We define the relation $\mathcal{A} \models_\alpha \varphi$ (read as φ is true in $\mathcal{A}$ under the assignment α) inductively:

- $\mathcal{A} \not\models_\alpha \bot$
- $\mathcal{A} \models_\alpha t_1 = t_2$ iff $\alpha(t_1) = \alpha(t_2)$
- $\mathcal{A} \models_\alpha R(t_1, \ldots, t_k)$ iff $(\alpha(t_1), \ldots, \alpha(t_k)) \in R^{\mathcal{A}}$

Satisfaction

We define the relation $\mathcal{A} \models_\alpha \varphi$ (read as φ is true in $\mathcal{A}$ under the assignment α) inductively:

- $\mathcal{A} \not\models_\alpha \bot$
- $\mathcal{A} \models_\alpha t_1 = t_2$ iff $\alpha(t_1) = \alpha(t_2)$
- $\mathcal{A} \models_\alpha R(t_1, \ldots, t_k)$ iff $(\alpha(t_1), \ldots, \alpha(t_k)) \in R^{\mathcal{A}}$
- $\mathcal{A} \models_\alpha (\varphi \to \psi)$ iff $\mathcal{A} \not\models_\alpha \varphi$ or $\mathcal{A} \models_\alpha \psi$

Satisfaction

We define the relation $\mathcal{A} \models_\alpha \varphi$ (read as φ is true in $\mathcal{A}$ under the assignment α) inductively:

- $\mathcal{A} \not\models_\alpha \bot$
- $\mathcal{A} \models_\alpha t_1 = t_2$ iff $\alpha(t_1) = \alpha(t_2)$
- $\mathcal{A} \models_\alpha R(t_1, \ldots, t_k)$ iff $(\alpha(t_1), \ldots, \alpha(t_k)) \in R^{\mathcal{A}}$
- $\mathcal{A} \models_\alpha (\varphi \to \psi)$ iff $\mathcal{A} \not\models_\alpha \varphi$ or $\mathcal{A} \models_\alpha \psi$
- $\mathcal{A} \models_\alpha (\forall x)\varphi$ iff for every $a \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a]} \varphi$

Satisfaction

We define the relation $\mathcal{A} \models_\alpha \varphi$ (read as φ is true in $\mathcal{A}$ under the assignment α) inductively:

- $\mathcal{A} \not\models_\alpha \bot$
- $\mathcal{A} \models_\alpha t_1 = t_2$ iff $\alpha(t_1) = \alpha(t_2)$
- $\mathcal{A} \models_\alpha R(t_1, \ldots, t_k)$ iff $(\alpha(t_1), \ldots, \alpha(t_k)) \in R^{\mathcal{A}}$
- $\mathcal{A} \models_\alpha (\varphi \to \psi)$ iff $\mathcal{A} \not\models_\alpha \varphi$ or $\mathcal{A} \models_\alpha \psi$
- $\mathcal{A} \models_\alpha (\forall x)\varphi$ iff for every $a \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a]} \varphi$
- $\mathcal{A} \models_\alpha (\exists x)\varphi$ iff there is some $a \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a]} \varphi$

Last two cases, α has no effect on the value of x. Thus, assignments matter only to free variables.