Exponential Random Variable

For any Poisson Process, the **Exponential** RV models *time until an event*:

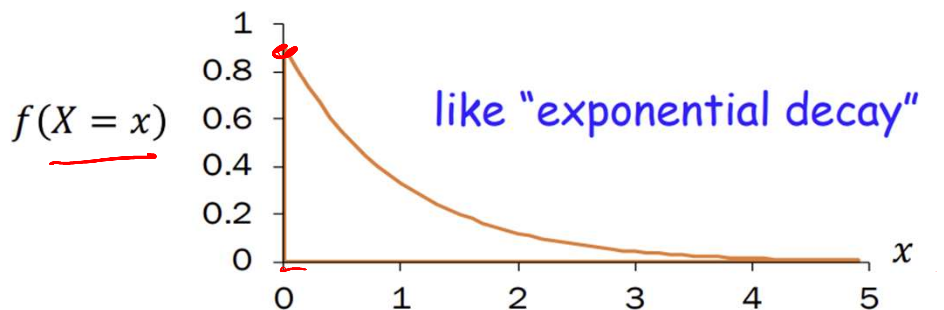
$$X \sim \text{Exp}(\lambda)$$

PDF:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

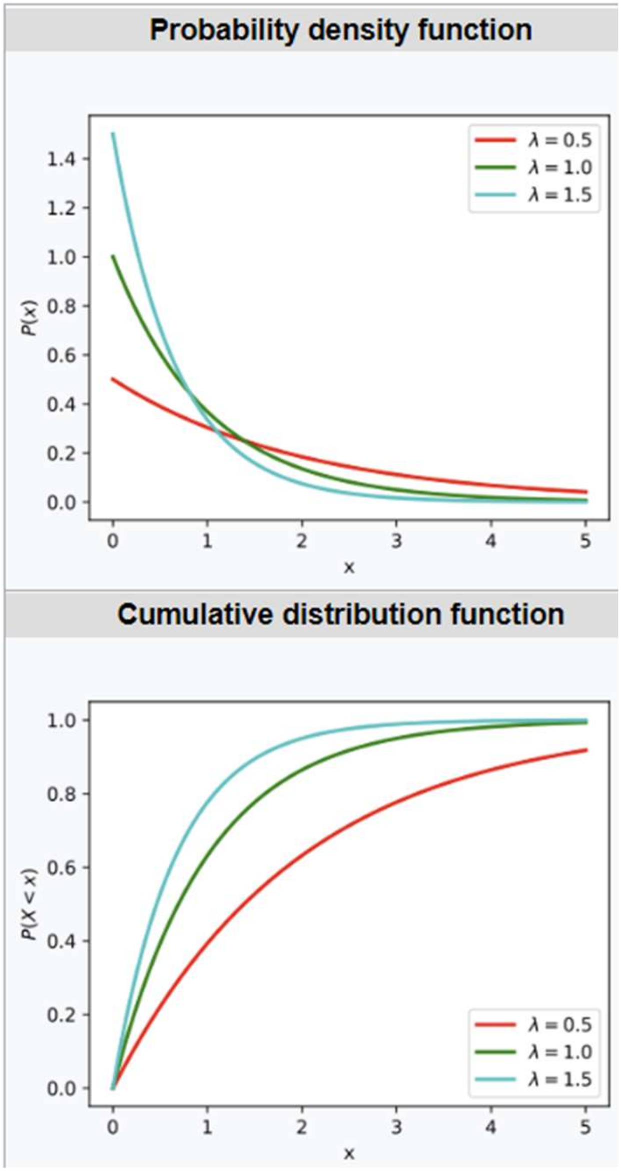
Examples:

- Time until next earthquake
- Time until a ping reaches a web server
- Time until a Uranium atom decays



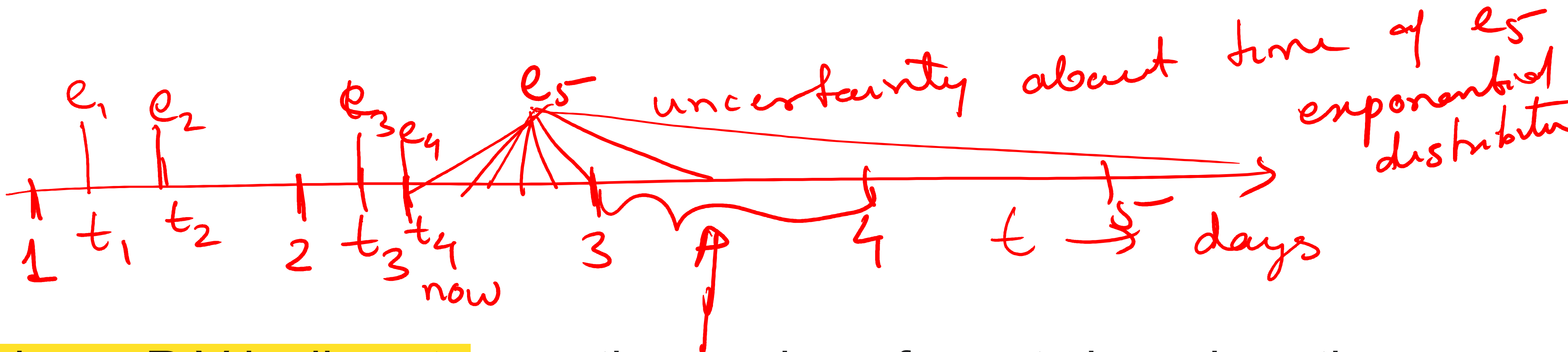
Cumulative Distribution function

$$F(x) = P\{X \leq x\}$$
$$= \int_0^x \lambda e^{-\lambda y}\, dy$$
$$= 1 - e^{-\lambda x}, \qquad x \geq 0$$



Relationship to Poisson distribution

- Both are applicable when events occur continuously and independently at a constant average rate λ



- Poisson R.V is discrete over the number of events in a given time
- Exponential R.V is continuous and is the distance between two events.

Moment Generating Function , Mean, Variance

$$\phi(t) = E[e^{tX}]$$ $X \sim \exp(\lambda)$

$$= \int_0^\infty e^{tx} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - t)x}\, dx$$

$$= \frac{\lambda}{\lambda - t}, \qquad t < \lambda$$

Differentiation yields

$$\phi'(t) = \frac{\lambda}{(\lambda - t)^2}$$

$$\phi''(t) = \frac{2\lambda}{(\lambda - t)^3} \cdot \quad \frac{2\lambda}{\lambda^3}$$

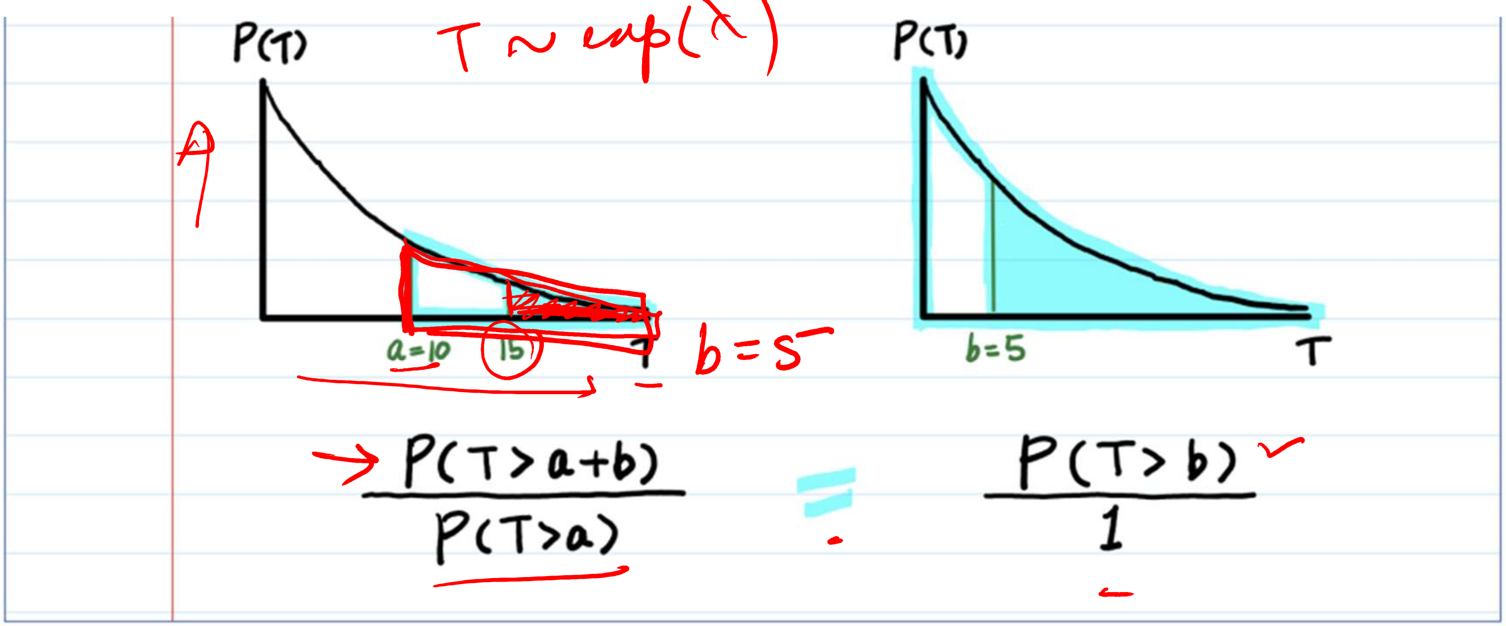
$$E[X] = \phi'(0) = 1/\lambda$$

$$\begin{aligned} \mathrm{Var}(X) &= \phi''(0) - (E[X])^2 \\ &= 2/\lambda^2 - 1/\lambda^2 \\ &= 1/\lambda^2 \end{aligned}$$

Memoryless property of exponential distribution

$$P(X > s + t \mid X > s) = P(X > t)$$

Example: lifetime T of a lamp if exponentially distributed, then remaining lifetime does not depend on how long lamp has been in use!



https://towardsdatascience.com/what-is-exponential-distribution-7bdd08590e2a

Proof of the memory-less property

$$P(A|B) = \frac{P(A,B)}{P(B)}$$

$X \sim \exp(\lambda)$

$$CDF(x) = \underline{1 - e^{-\lambda x}} \qquad P(X \leq x) \qquad t > 0$$

$$P(X > s+t \mid X > s) = \frac{P(X > s+t,\ X > s)}{P(X > s)} = \frac{P(X > s+t)}{P(X > s)}$$

$$= \frac{1 - CDF_X(s+t)}{1 - CDF_X(s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t}$$

$$= 1 - CDF_X(t)$$

$$= \underline{P(X > t)}$$

$\Rightarrow$ X is memoryless

Important.

Memoryless property is unique to exponential!

- If X is a continuous random variable where P(X>s+t|X>s)=P(X>t) then P(X) is an exponential distribution. [Proof not part of the syllabus]

Proof

Let F be the CDF of X, and let $G(x) = P(X > x) = 1 - F(x)$. The memoryless property says $G(s+t) = G(s)G(t)$, we want to show that only the exponential will satisfy this.

Try $s = t$, this gives us $G(2t) = G(t)^2, G(3t) = G(t)^3, \ldots, G(kt) = G(t)^k$.

Similarly, from the above we see that $G(\frac{t}{2}) = G(t)^{\frac{t}{2}}, \ldots, G(\frac{t}{k}) = G(t)^{\frac{1}{k}}$.

Combining the two, we get $G(\frac{m}{n}t) = G(t)^{\frac{m}{n}}$ where $\frac{m}{n}$ is a rational number.

Now, if we take the limit of rational numbers, we get real numbers. Thus, $G(xt) = G(t)^x$ for all real $x > 0$.

If we let $t = 1$, we see that $G(x) = G(1)^x$ and this looks like the exponential. Thus, $G(1)^x = e^{xlnG(1)}$, and since $0 < G(1) \leq 1$, we can let $lnG(1) = -\lambda$.

Therefore $e^{xlnG(1)} = e^{-\lambda x}$ and only exponential can be memoryless.

https://math.stackexchange.com/questions/1801830/on-the-proof-that-every-positive-continuous-random-variable-with-the-memoryless

Example

- Suppose the number of kms that a car can run before the battery wears down is exponentially distributed with average distance as 10000. If the person takes a 5000 km trip, what is the probability that the battery will not run down.

$X \sim \exp(\lambda)$ $\qquad \lambda = \frac{1}{10000}$ $\qquad E(X) = \frac{1}{\lambda}$

$P(X > 5000) = e^{-\lambda 5000} = e^{-\frac{1}{2}}$

Another interesting property of exponential distribution

Proposition 5.6.1. If $X_1, X_2, \ldots, X_n$ are independent exponential random variables having respective parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $\min(X_1, X_2, \ldots, X_n)$ is exponential with parameter $\sum_{i=1}^{n} \lambda_i$.

$$Y = \min(x_1, x_2 \quad x_n) \qquad x_i \sim \exp(\lambda_i)$$

$$P(Y > b) = P(x_1 > b,\ x_2 > b \quad x_n > b)$$

$x_1 - - x_n$ are independent

$$P(\min(x_1 \quad x_n) > b) = \prod_{i=1}^{n} P(x_i > b)$$

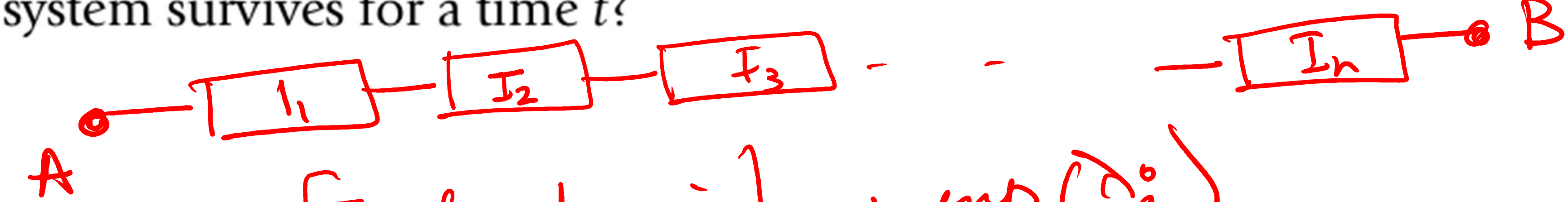
$$= \prod_{i=1}^{n} e^{-\lambda_i b} = e^{-b\left(\sum_{i=1}^{n} \lambda_i\right)}$$

$$\Rightarrow \quad Y \sim \exp\left(\sum_{i=1}^{n} \lambda_i\right)$$

if minimum is bigger than a threshold then all are bigger than the threshold.

Example

Example 5.6.c. A series system is one that needs all of its components to function in order for the system itself to be functional. For an n-component series system in which the component lifetimes are independent exponential random variables with respective parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, what is the probability that the system survives for a time t?



$[I_j \text{ functioning}] \sim \exp(\lambda_j)$

Y = A to B connection is functioning

$P(Y \geq \gamma) = P(\min(I_1, I_2 \ldots I_n) > \gamma) = e^{-\gamma\left[\sum_{i=1}^{n} \lambda_i\right]}$

Another fun property of exponential distribution

Maximum entropy distribution

Among all continuous probability distributions with support $[0, \infty)$ and mean μ, the exponential distribution with $\lambda = 1/\mu$ has the largest differential entropy. In other words, it is the maximum entropy probability distribution for a random variate X which is greater than or equal to zero and for which $E[X]$ is fixed.[2]