

# CS213/293 Data Structure and Algorithms 2024

## Lecture 3: Stack and queue

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# Topic 3.1

## Stack

# Stack

## Definition 3.1

*Stack* is a container where elements are added and deleted according to the last-in-first-out (LIFO) order.

- ▶ Addition is called **pushing**
- ▶ Deleting is called **popping**

## Example 3.1

- ▶ *Stack of papers in a copier*
- ▶ *Undo-redo features in editors*
- ▶ *Back button on Browser*

# Interface of stack

Reference: <https://en.cppreference.com/w/cpp/container/stack>

Stack supports four interface methods

- ▶ `stack<T> s` : allocates new stack `s`
- ▶ `s.push(e)` : Pushes the given element `e` to the top of the stack.
- ▶ `s.pop()` : Removes the top element from the stack.
- ▶ `s.top()` : accesses the top element of the stack.

Some support functions

- ▶ `s.empty()` : checks whether the stack is empty
- ▶ `s.size()` : returns the number of elements

## Exercise 3.1

*Why define stack when we can use vector for the same effect?*

**Commentary:** Answer: vector in C++ promises to provide efficient random access but stack does not make such a promise. Therefore, the implementations of stack may make implementation choices that may result in inefficient random access.

# Axioms of stack

Let  $s1$  and  $s$  be stacks.

- ▶ `Assume(s1 == s); s.push(e); s.pop(); Assert(s1==s);`
- ▶ `s.push(e); Assert(s.top()==e);`

`Assume(s1 == s)` means that we **assume** that the content of  $s1$  and  $s$  are the same.  
`Assert(s1 == s)` means that we **check** that the content of  $s1$  and  $s$  are the same.

## Exercise: action on the empty stack

### Exercise 3.2

Let `s` be an empty stack in C++.

- ▶ What happens when we run `s.top()`?
- ▶ What happens when we run `s.pop()`?

Ask ChatGPT.

**Commentary:** Answer: `s.top()` will cause a segmentation fault. `s.pop()` will not cause any error and exit without any effect.

## Topic 3.2

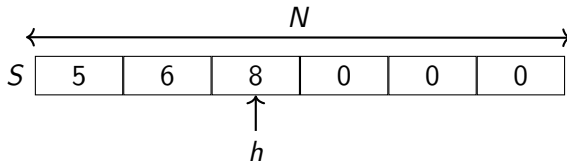
### Implementing stack

## Array-based stack

Let us look at a simplified array-based implementation of an array of integers.

The stack consists of three variables.

- ▶  $N$  specifies the currently available space in the stack
- ▶  $S$  is the integer array of size  $N$
- ▶  $h$  is the position of the head of the stack





## Implementing stack

```
class arrayStack {
    int    N = 2;        // Capacity
    int*   S = NULL;     // pointer to array
    int    h = -1;       // Current head of the stack
public:
    arrayStack() { S = (int*)malloc(sizeof(int)*N ); }
    int    size() { return h+1; }
    bool   empty() { return h<0; }
    int    top() { return S[h]; } // On empty stack what happens?
    void   push(int e) {
        if( size() == N ) expand(); // Expand capacity of the stack
        S[++h] = e;
    }
    void   pop() { if( !empty() ) h--; }
```

**Commentary:** The behavior of the above implementation may not match the behavior of the C++ stack library. To ensure segmentation fault in top() when the stack is empty one may use the following code. `if( empty() ) return *(int*)0; else return S[t];`

## Implementing stack (expanding when full)

```
private:
    void expand() {
        int new_size = N*2; // We observed the growth in our lab!!
        int* tmp = (int*) malloc( sizeof(int)*new_size ); //New array
        for( unsigned i =0; i < N; i++ ) { // copy from the old array
            tmp[i] = S[i];
        }
        free(S);          // Release old memory
        S = tmp;          // Update local fields
        N = new_size; //
    }
};
```

Allocating memory in heap

# Efficiency

All operations are performed in  $O(1)$  if there is no expansion to stack capacity.

What is the cost of expansion?

## Topic 3.3

Why exponential growth strategy?

# Growth strategy

Let us consider two possible choices for growth.

- ▶ Constant growth:  $\text{new\_size} = N + c$  (for some fixed constant  $c$ )
- ▶ Exponential growth:  $\text{new\_size} = 2*N$

Which of the above two is better?

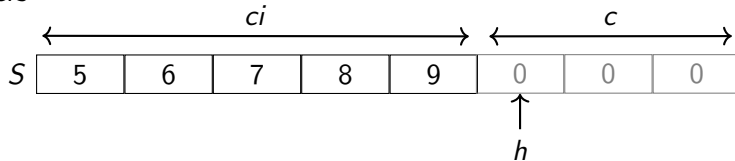
# Analysis of constant growth

Let us suppose initially  $N = 0$  and there are  $n$  consecutive pushes.

After every  $c$ th push, there will be an expansion operation.

Therefore, the expansion operation at  $(ci + 1)$ th push will

- ▶ allocate memory of size  $c(i + 1)$
- ▶ copy  $ci$  integers



Cost of  $i$ th expansion:  $c(2i + 1)$ .

**Commentary:** We are assuming that allocating memory of size  $k$  costs  $k$  time, which may be more efficient in practice. Bulk memory copy can also be sped up by vector instructions.

## Analysis of constant growth(2)

For  $n$  pushes, there will be  $n/c$  expansions.

The total cost of expansions:

$$c(1 + 3 + \dots + (2^{\frac{n}{c}} + 1)) = c(n/c)^2 \in O(n^2)$$

Non-linear cost!

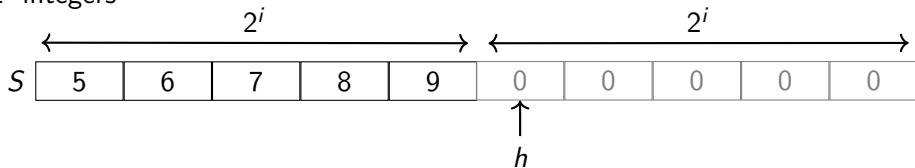
## Analysis of exponential growth

Let us suppose initially  $N = 1$  and there are  $n = 2^r$  consecutive pushes.

The expansion operations will only occur at  $2^i + 1$ th push, where  $i \in [0, r - 1]$ .

The expansion operation at  $2^i + 1$ th push will

- ▶ allocate memory of size  $2^{i+1}$
- ▶ copy  $2^i$  integers



Cost of the expansion:  $3 * 2^i$ .



## Analysis of exponential growth(2)

For  $2^r$  pushes, the last expansion would be at  $2^{r-1} + 1$ .

The total cost of expansions:

$$3(2^0 + \dots + 2^{r-1}) = 3 * (2^r - 1) = 3 * (n - 1)$$

Linear cost! The average cost of push remains  $O(1)$ .

### Exercise 3.3

*Why double? Why not triple? Why not 1.5 times? Is there a trade-off?*

## Topic 3.4

### Applications of stack

# Stacks are everywhere

Stack is a foundational data structure.

It shows up in a vast range of algorithms.

## Example: matching parentheses

```
bool parenMatch(string text ) {  
    std::stack<char> s;  
    for(char c : text ) {  
        if( c == '{' or c == '[' ) s.push(c);  
        if( c == '}' or c == ']' ) {  
            if( s.empty() ) return false;  
            if( c-s.top() != 2 ) return false;  
            s.pop();  
        }  
    }  
    if( s.empty() ) return true;  
    return false;  
}
```

Problem:

Given an input text check if it has matching parentheses.

Examples:

▶ "{a[sic]tik}" ✓

▶ "{a[sic}tik}" ✗

## Topic 3.5

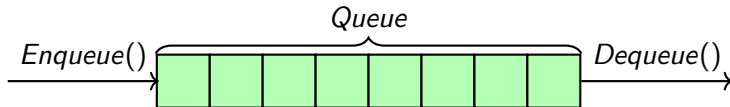
### Queue

# Queue

## Definition 3.2

*Queue* is a container where elements are added and deleted according to the first-in-first-out (FIFO) order.

- ▶ Addition is called **enqueue**
- ▶ Deleting is called **dequeue**



## Example 3.2

- ▶ *Entry into an airport*
- ▶ *Calling lift in a building (priority queue)*

# Interface of queue

Reference: <https://en.cppreference.com/w/cpp/container/queue>

Queue supports four main interface methods

- ▶ `queue<T> q` : allocates new queue `q`
- ▶ `q.enqueue(e)` : Adds the given element `e` to the end of the queue. (push)
- ▶ `q.dequeue()` : Removes the first element from the queue. (pop)
- ▶ `q.front()` : access the first element .

Some support functions

- ▶ `q.empty()` : checks whether the queue is empty
- ▶ `q.size()` : returns the number of elements

**Commentary:** All literature uses the terms enqueue and dequeue, but unfortunately C++ library uses push for enqueue and pop uses for dequeue. Other languages such as Java uses the terms enqueue and dequeue.

## Axioms of queue

1. `queue<T> q; Assert(q.empty() == true);`
2. `q.enqueue(e); Assert(q.empty() == false);`
3. `Assume(q.empty() == true);`  
`q.enqueue(e); Assert(q.front() == e);`
4. `Assume(q.empty() == false && old_q == q);`  
`q.enqueue(e); Assert(old_q.front() == q.front());`
5. `Assume(q.empty() == true && old_q == q);`  
`q.enqueue(e); q.dequeue(); Assert(old_q == q);`
6. `Assume(q.empty() == false && q == q1);`  
`q.enqueue(e); q.dequeue(); q1.dequeue(); q1.enqueue(e); Assert(q == q1);`

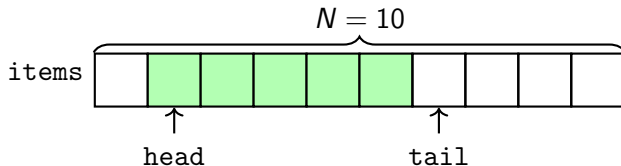


## Topic 3.6

### Array implementation of queue

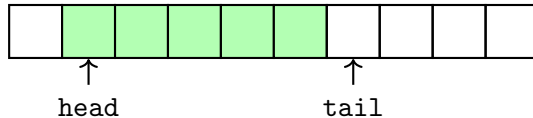
# Array-based implementation

- ▶ Queue is stored in an array `items` in a circular fashion
- ▶ Three integers record the state of the queue
  1. `N` indicates the available capacity ( $N-1$ ) of the queue
  2. `head` indicates the position of the front of the queue
  3. `tail` indicates position one after the rear of the queue

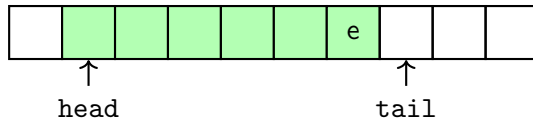


## Enqueue operation on array

Consider the state of the queue

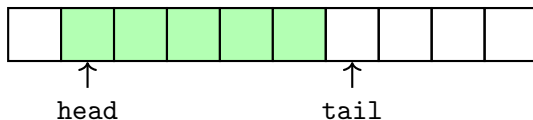


After the enqueue(e) operation:

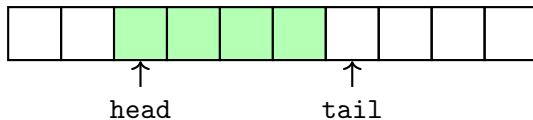


## Deque operation on array

Consider the state of the queue



After dequeue() operation:

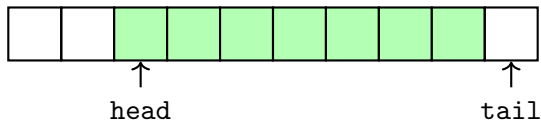


### Exercise 3.4

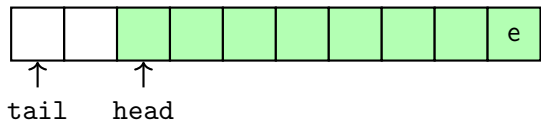
1. *Where will front() read from?*
2. *What is the size of the queue?*

## Wrap around to utilize most of the array

Consider the state of the queue

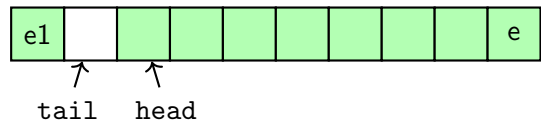


After enqueue(e) operation, we move the tail to 0.



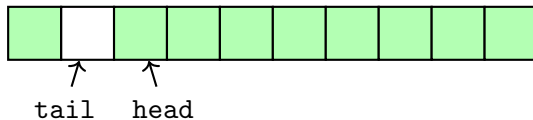
Wrap-around allows us to use the array repeatedly.

After another enqueue(e1) operation:

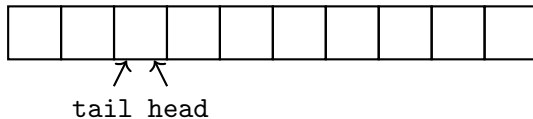


## Full and empty queue

Full queue:



Empty queue:



### Exercise 3.5

*Can we use all  $N$  cells for storing elements?*

## Array implementation

The code is not written in C++; We will slowly move towards pseudo code to avoid clutter on slides.

```
int head = 0, tail=0, N = INITIAL_CAPACITY;
```

```
Object items[N];           //Some initial size
```

```
bool empty()    { return (head == tail); }
```

```
bool size()     { return (N+tail-head)%N; }
```

```
Object front() { return items[head]; }
```

# Array implementation

```
void dequeue() {
    if( empty() ) throw Empty;           // Queue is empty
    free(items[head]); items[head] = NULL; // Clear memory
    head = (head+1)%N;                   // Remove an element
}

void enqueue( Object x ) {
    if ( size() == N-1 ) expand(); // Queue is full; expand
    items[tail] = x;
    tail = (tail+1)%N; // insert element
}
```

## Exercise 3.6

*In our stack implementation, we did not invoke free in pop, but we invoke free in dequeue. Why?*

**Commentary:** In the stack implementation, we were only handling stack of int. Here, we are handling stack of arbitrary objects. If the object has dynamic size then it cannot live on the stack itself. We will store a reference to the object on stack and the object will be allocated somewhere else. Therefore, the objects must be freed on dequeue (the above syntax of free is not in C++; It will only work when items[head] is a reference). However, there is no need to free an int because it has fixed size and it was stored on the array of the stack itself.



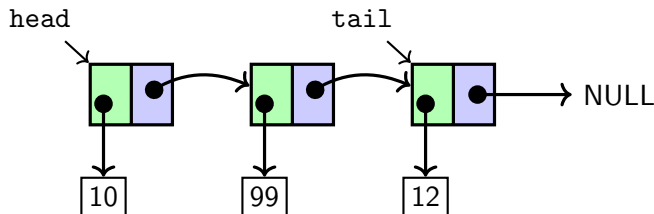
## Topic 3.7

### Queue via linked list

# Linked lists

## Definition 3.3

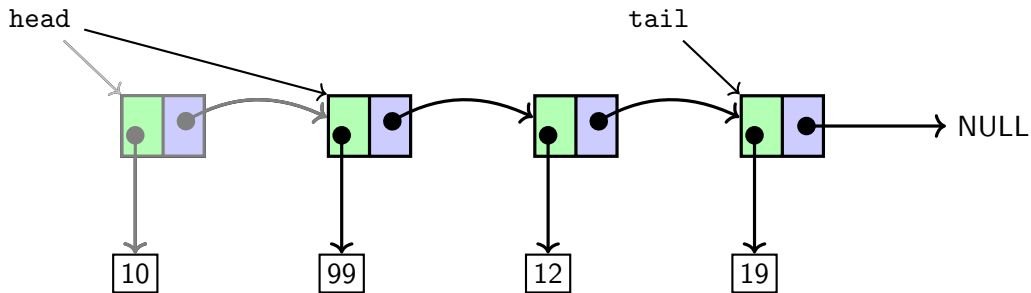
A linked list consists of nodes with two fields *data* and *next* pointer. The nodes form a chain via the *next* pointer. The data pointers point to the objects that are stored on the linked list.



## Exercise 3.7

If we use a linked list for implementing a queue, which side should be the front of the queue?

## Deque in linked lists

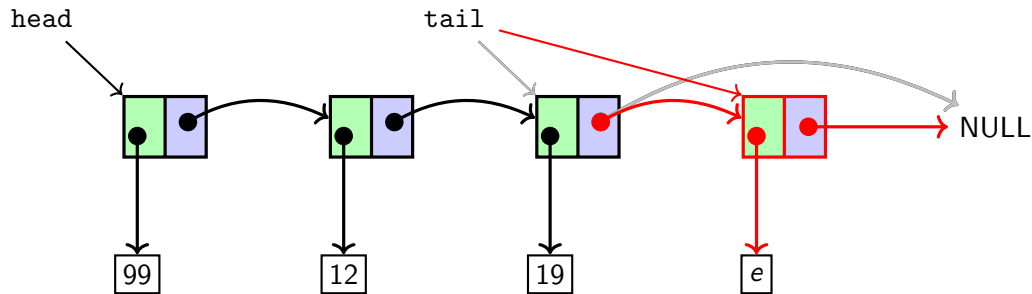


### Exercise 3.8

*What happens to the object containing 10?*

**Commentary:** Answer: There are several choices. The object is deallocated, the reference to the object is returned to the caller, or the copy of the the object is returned to the caller. Different implementations may do it differently. This behavior is not part of the specification of queue.

## Enqueue(e) in linked lists



### Exercise 3.9

- Which one is better: array or linked list?
- Do we need the tail pointer?

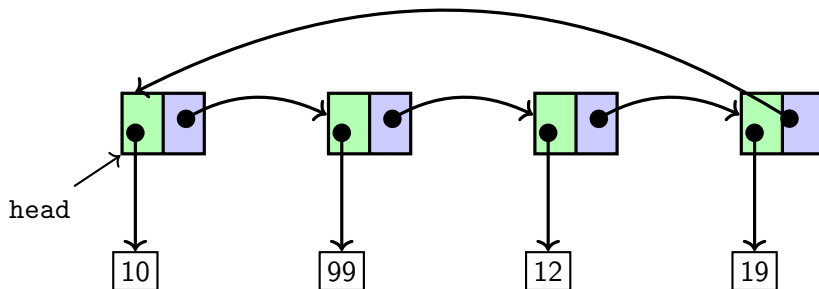
## Topic 3.8

### Circular linked list

# Circular linked lists

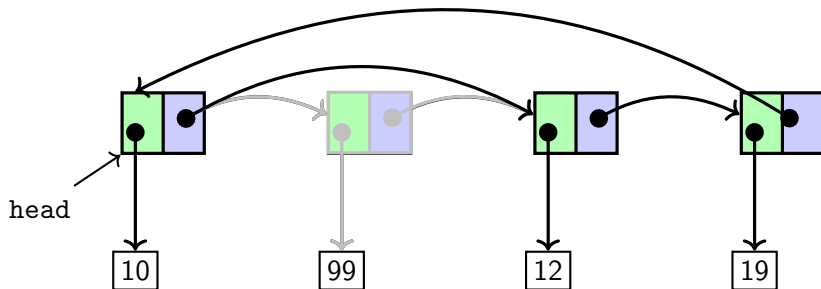
## Definition 3.4

*In a circular linked list, the nodes form a circular chain via the next pointer.*

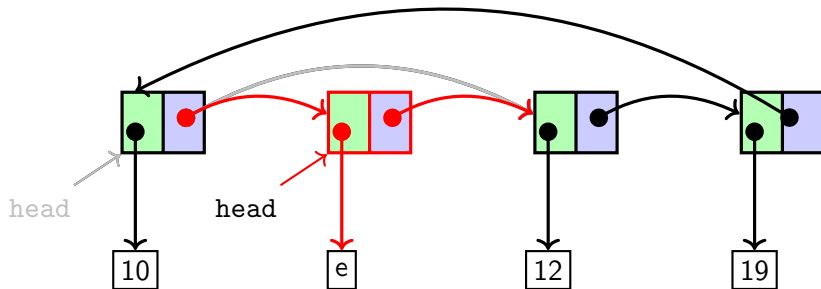


A head pointer points at some node of the circular list. A single pointer can do the job of the head and tail.

## Dequeue in circular linked lists



## enqueue(e) in circular linked lists



### Exercise 3.10

- Which element should be returned by `front()`? **Head**
- Give pseudo code of the implementation of queue using circular linked list. (Midsem 2023)



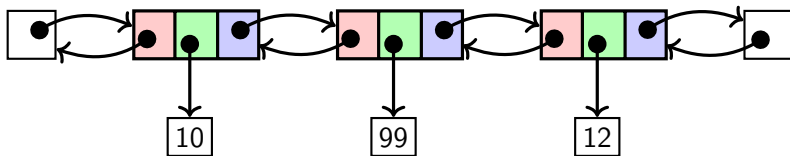
## Topic 3.9

### Deque via a doubly linked list

# Doubly linked lists

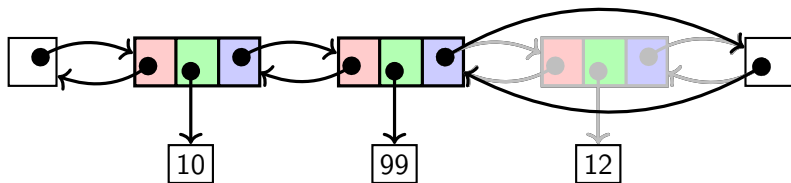
## Definition 3.5

A doubly linked list consists of nodes with three fields *prev*, *data*, and *next* pointer. The nodes form a bidirectional chain via the *prev* and *next* pointer. The data pointers point to the objects that are stored on the linked list.



At both ends, two *dummy or sentinel* nodes do not store any data and are used to store the start and end points of the list.

## Deleting a node in a doubly linked list



# Deque (Double-ended queue)

## Definition 3.6

*Deque* is a container where elements are added and deleted according to both last-in-first-out (LIFO) and first-in-first-out (FIFO) order.

# Interface of Deque

Reference: <https://en.cppreference.com/w/cpp/container/deque>

Queue supports four main interface methods

- ▶ `deque<T> q` : allocates new queue `q`
- ▶ `q.push_back(e)` : Adds the given element `e` to the back.
- ▶ `q.push_front(e)` : Adds the given element `e` to the front.
- ▶ `q.pop_front()` : Removes the first element from the queue.
- ▶ `q.pop_back()` : Removes the last element from the queue.
- ▶ `q.front()` : access the first element .
- ▶ `q.back()` : access the first element .

Some support functions

- ▶ `q.empty()` : checks whether the stack is empty
- ▶ `q.size()` : returns the number of elements  
Gives the number of elements in the deque.

We can implement the Deque data structure using the doubly linked lists.

# Stack and queue via Deque

We can implement both stack and queue using the interface of deque.

## Exercise 3.11

- ▶ *Which functions of deque implement stack?*
- ▶ *Which functions of deque implement queue?*

All modification operations are implemented in  $O(1)$ .

## Exercise 3.12

Can we implement `size` in  $O(1)$  in a doubly linked list?

## Topic 3.10

### Tutorial problems

# Use of stack

## Exercise 3.13

*The span of a stock's price on  $i$ th day is the maximum number of consecutive days (up to  $i$ th day) the price of the stock has been less than or equal to its price on day  $i$ .*

*Example: for the price sequence 2 4 6 3 5 7 of a stock, the span of prices is 1 2 3 1 2 6.*

*Give a linear-time algorithm that computes  $s_i$  for a given price series.*



# Flipping Dosa

## Exercise 3.14

*There is a stack of dosas on a tava, of distinct radii. We want to serve the dosas of increasing radii. Only two operations are allowed: (i) serve the top dosa, (ii) insert a spatula (flat spoon) in the middle, say after the first  $k$ , hold up this partial stack, flip it upside-down, and put it back. Design a data structure to represent the tava, input a given tava, and produce an output in sorted order. What is the time complexity of your algorithm?*

*This is also related to the train-shunting problem.*

# Exponential growth

## Exercise 3.15

- a. Analyze the performance of exponential growth if the growth factor is three instead of two. Does it give us better or worse performance than doubling policy?*
- b. Can we do a similar analysis for growth factor 1.5?*

## Problem: reversing a linked list

### Exercise 3.16

*Give an algorithm to reverse a linked list. You must use only three extra pointers.*

## Problem: middle element

### Exercise 3.17

*Give an algorithm to find the middle element of a singly linked list.*

# Stack and queue (Endsem 2023)

## Exercise 3.18

*Given two stacks  $S1$  and  $S2$  (working in the LIFO method) as black boxes, with the regular methods: “Push”, “Pop”, and “isEmpty”, you need to implement a Queue (specifically : Enqueue and Dequeue working in the FIFO method). Assume there are  $n$  Enqueue/ Dequeue operations on your queue. The time complexity of a single method Enqueue or Dequeue may be linear in  $n$ , however the total time complexity of the  $n$  operations should also be  $\Theta(n)$ .*

# Topic 3.11

## Problems

## Problem: messy queue

### Exercise 3.19

*The mess table queue problem: There is a common mess for  $k$  hostels. Each hostel has some  $N_1, \dots, N_k$  students. These students line up to pick up their trays in the common mess. However, the queue is implemented as follows: If a student sees a person from his/her hostel, she/he joins the queue behind this person. This is the "enqueue" operation. The "dequeue" operation is as usual, at the front. Think about how you would implement such a queue. What would be the time complexity of enqueue and dequeue? Do you think the average waiting time in this queue would be higher or lower than a normal queue? Would there be any difference in any statistic? If so, what?*

# Merge sorted queues (Quiz 2023)

## Exercise 3.20

*Write a time and space efficient algorithm to merge  $k$  sorted-linked list in sorted order, each containing the same no of elements?*



## Exercise: axioms of queue\*\*

### Exercise 3.21

*Using axioms of queue show that the assert in the following does not fail.*

```
queue<int> q, q1;  
q.enqueue(2);  
q.enqueue(0);  
q.enqueue(7);  
q.dequeue();  
q.dequeue();  
q.enqueue(3);  
  
q1.enqueue(7);  
q1.enqueue(3);  
Assert(q == q1);
```

# End of Lecture 3