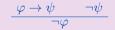
CS 228 : Logic in Computer Science

Krishna, S

Rules for Natural Deduction

Another implies elimination rule or Modus Tollens MT



▶ Show that $p \rightarrow \neg q, q \vdash \neg p$

1. $p \rightarrow \neg q$ premise

2.

▶ Show that $p \rightarrow \neg q, q \vdash \neg p$

- 1. $p \rightarrow \neg q$ premise
- 2. q premise
- 3.

▶ Show that $p \rightarrow \neg q, q \vdash \neg p$

1.	p ightarrow eg q	premise
2.	q	premise
3.	$\neg \neg q$	¬¬ <i>i</i> 2
4.		

▶ Show that $p \rightarrow \neg q, q \vdash \neg p$

1.	$oldsymbol{ ho} ightarrow eg oldsymbol{q}$	premise
2.	q	premise
3.	$\neg \neg q$	¬¬ <i>i</i> 2
4.	$\neg p$	MT 1,3

▶ Thanks to MT, we have $p \rightarrow q, \neg q \vdash \neg p$.

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- ▶ Can we prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$?

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- ▶ So far, no proof rule that can do this.

- ▶ Thanks to MT, we have $p \rightarrow q, \neg q \vdash \neg p$.
- ▶ Can we prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$?
- So far, no proof rule that can do this.
- ▶ Given $p \rightarrow q$, let us assume $\neg q$. Can we then prove $\neg p$?

- ▶ Thanks to MT, we have $p \rightarrow q, \neg q \vdash \neg p$.
- ▶ Can we prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$?
- So far, no proof rule that can do this.
- ▶ Given $p \rightarrow q$, let us assume $\neg q$. Can we then prove $\neg p$?
- ► Yes, using MT.

The implies introduction rule $\rightarrow i$

1.	p o q	premise
2.	$\neg q$	assumption

4.
$$\neg q \rightarrow \neg p \rightarrow i \ 2-3$$

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

premise

- true
- 2.

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

- 1. true premise 2. $q \rightarrow r$ assumption
- 3.

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	$q \rightarrow r$	assumption
3.	eg q o eg p	assumption
4.		

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	q o r	assumption
3.	eg q ightarrow eg p	assumption
4.	p	assumption
5.		

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	$q \rightarrow r$	assumption
3.	eg q o eg p	assumption
4.	p	assumption
5.		¬¬ <i>i</i> 4
6.		

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	q o r	assumption
3.	eg q o eg p	assumption
4.	p	assumption
5.	$ \ \ \ \neg \neg p$	¬¬ <i>i</i> 4
6.	$ \ \ \ \neg \neg q$	MT 3,5
7.		

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	$q \rightarrow r$	assumption
3.	eg q ightarrow eg p	assumption
4.	p	assumption
5.		¬¬ <i>i</i> 4
6.		MT 3,5
7.	q	¬¬ <i>e</i> 6
8.		

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	q o r	assumption
3.	eg q ightarrow eg p	assumption
4.	р	assumption
5.	$ \ \ \ \neg \neg \rho$	¬¬ <i>i</i> 4
6.	$ \ \ \ \neg \neg q$	MT 3,5
7.		¬¬ <i>e</i> 6
8.		MP 2,7

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	q o r	assumption
3.	eg q ightarrow eg p	assumption
4.	p	assumption
5.		¬¬ <i>i</i> 4
6.	$ \neg \neg q$	MT 3,5
7.	q	¬¬ <i>e</i> 6
8.	r	MP 2,7
9	$p \rightarrow r$	→ <i>i</i> 4-8

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	q o r	assumption
3.	eg q o eg p	assumption
4.	P	assumption
5.	$ \cdot \cdot \neg \neg p$	¬¬ <i>i</i> 4
6.	$ \cdot \cdot \neg \neg q$	MT 3,5
7.	q	¬¬ <i>e</i> 6
8.	r	MP 2,7
9.	ho ightarrow r	→ <i>i</i> 4-8
10.	(eg q ightarrow eg p) ightarrow (p ightarrow r)	→ <i>i</i> 3-9

6/24

11.

 $(q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)] \rightarrow i \text{ 2-10}$

 \rightarrow *i* 4-8

 \rightarrow *i* 3-9

6/24

9.

10.

11.

 $p \rightarrow r$

 $(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$

Transforming Proofs

- $ightharpoonup (q
 ightarrow r), (\neg q
 ightarrow \neg p), p \vdash r$
- ► Transform any proof $\varphi_1, \ldots, \varphi_n \vdash \psi$ to $\vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_n \rightarrow \psi) \ldots))$ by adding n lines of the rule $\rightarrow i$

▶
$$p \to (q \to r) \vdash (p \land q) \to r$$

1. $p \to (q \to r)$ premise 2.

$$\begin{array}{c|cccc} \blacktriangleright & p \rightarrow (q \rightarrow r) \vdash (p \land q) \rightarrow r \\ & 1. & p \rightarrow (q \rightarrow r) & \text{premise} \\ & 2. & p \land q & \text{assumption} \\ & 3. & p & \land e_1 \ 2 \\ & 4. & q & \land e_2 \ 2 \\ & 5. & q \rightarrow r & \text{MP 1,3} \\ & 6. & \end{array}$$

▶
$$p \rightarrow (q \rightarrow r) \vdash (p \land q) \rightarrow r$$

1. $p \rightarrow (q \rightarrow r)$ premise

2. $p \land q$ assumption

3. $p \land e_1 2$

4. $q \land e_2 2$

5. $q \rightarrow r \land P 1,3$

6. $r \land P 4,5$

7.

$$\begin{array}{c|cccc} \blacktriangleright & p \rightarrow (q \rightarrow r) \vdash (p \land q) \rightarrow r \\ & 1. & p \rightarrow (q \rightarrow r) & \text{premise} \\ & 2. & p \land q & \text{assumption} \\ & 3. & p & \land e_1 \ 2 \\ & 4. & q & \land e_2 \ 2 \\ & 5. & q \rightarrow r & \text{MP 1,3} \\ & 6. & r & \text{MP 4,5} \\ & 7. & p \land q \rightarrow r & \rightarrow i \ 2\text{-}6 \end{array}$$

The or introduction rule $\vee i_1$

$$\frac{\varphi}{\varphi\vee\psi}$$

The or introduction rule $\vee i_2$

$$\frac{\psi}{\varphi \vee \psi}$$

The or elimination rule $\vee e$

$$\begin{array}{ccc} \varphi \lor \psi & \varphi \vdash \chi & \psi \vdash \chi \\ \hline \chi & \end{array}$$

Or Elimination Example

$$P q \to r \vdash (p \lor q) \to (p \lor r)$$

- 1. $q \rightarrow r$
- 2

premise

Or Elimination Example

$$P q \rightarrow r \vdash (p \lor q) \rightarrow (p \lor r)$$

1.	$q \rightarrow r$	premise
2.	$p \lor q$	assumpt
3.		

Or Elimination Example

$$P q \to r \vdash (p \lor q) \to (p \lor r)$$

1.	$oldsymbol{q} ightarrow oldsymbol{r}$	premise
2.	$p \lor q$	assumption
3.	p	∨ <i>e</i> (1)
4.		

$$P q \to r \vdash (p \lor q) \to (p \lor r)$$

1.	$q \rightarrow r$	premise
2.	$p \lor q$	assumption
3.	p	∨ <i>e</i> (1)
4.	$p \lor r$	√ <i>i</i> ₁ 3
5.		

$$P q \rightarrow r \vdash (p \lor q) \rightarrow (p \lor r)$$

	q o r	premise
	$p \lor q$	assumption
3.	p	∨ <i>e</i> (1)
	p∨r	∨ <i>i</i> ₁ 3
) .	q	∨ e (2)
.		

$$P q \rightarrow r \vdash (p \lor q) \rightarrow (p \lor r)$$

1.	$oldsymbol{q} ightarrow oldsymbol{r}$	premise
2.	$p \lor q$	assumption
3.	p	∨ <i>e</i> (1)
4.	$p \lor r$	∨ <i>i</i> ₁ 3
5.	q	∨ e (2)
6.	r	MP 1,5
7.		

$$P q \to r \vdash (p \lor q) \to (p \lor r)$$

1.	$oldsymbol{q} ightarrow oldsymbol{r}$	premise
2.	$p \lor q$	assumption
3.	p	∨ <i>e</i> (1)
4.	$p \lor r$	√ <i>i</i> ₁ 3
5.	q	∨ e (2)
6.	r	MP 1,5
7.	p∨r	∨ <i>i</i> ₂ 6

$$P q \to r \vdash (p \lor q) \to (p \lor r)$$

1.	q o r	premise
2.	$p \lor q$	assumption
3.	р	∨ <i>e</i> (1)
4.	p∨r	∨ <i>i</i> ₁ 3
5.	q	∨ <i>e</i> (2)
6.	r	MP 1,5
7.	p∨r	∨ <i>i</i> ₂ 6
8.	p∨r	∨ <i>e</i> 2, 3-4, 5-7

$$P q \to r \vdash (p \lor q) \to (p \lor r)$$

1.	$q \rightarrow r$	premise
2.	$p \lor q$	assumption
3.	р	∨ <i>e</i> (1)
4.	<i>p</i> ∨ <i>r</i>	∨ <i>i</i> ₁ 3
5.	q	∨ e (2)
6.	r	MP 1,5
7.	p∨r	∨ <i>i</i> ₂ 6
8.	p∨r	∨ <i>e</i> 2, 3-4, 5-7
9.		

$$P q \rightarrow r \vdash (p \lor q) \rightarrow (p \lor r)$$

1.	$q \rightarrow r$	premise
2.	$p \lor q$	assumption
3.	р	∨ <i>e</i> (1)
4.	p∨r	∨ <i>i</i> ₁ 3
5.	q	∨ e (2)
6.	r	MP 1,5
7.	p∨r	∨ <i>i</i> ₂ 6
8.	p∨r	∨ <i>e</i> 2, 3-4, 5-7
9.	$(p \lor q) \rightarrow (p \lor r)$	→ <i>i</i> 2-8

▶
$$(p \lor q) \lor r \vdash p \lor (q \lor r)$$

1. $(p \lor q) \lor r$ premise

 $(p \lor q) \lor r \vdash p \lor (q \lor r)$

```
1. (p \lor q) \lor r premise
2. p \lor q \lor e(1)
3.
```

 $(p \lor q) \lor r \vdash p \lor (q \lor r)$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	∨ <i>e</i> (1)
3.	р	∨ <i>e</i> (1.1)

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	∨ <i>e</i> (1)
3.	p	∨ e (1.1)
4.	$p \lor (q \lor r)$	∨ <i>i</i> ₁ 3
5		

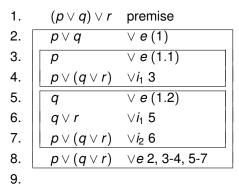
1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	∨ <i>e</i> (1)
3.	p	∨ <i>e</i> (1.1)
4.	$p \lor (q \lor r)$	∨ <i>i</i> ₁ 3
5.	q	∨ <i>e</i> (1.2)
6.		

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	∨ <i>e</i> (1)
3.	p	∨ <i>e</i> (1.1)
4.	$p \lor (q \lor r)$	∨ <i>i</i> ₁ 3
5.	q	∨ <i>e</i> (1.2)
6.	$q \vee r$	∨ <i>i</i> ₁ 5
7.		

 $(p \lor q) \lor r \vdash p \lor (q \lor r)$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	∨ <i>e</i> (1)
3.	p	∨ <i>e</i> (1.1)
4.	$p \lor (q \lor r)$	√ <i>i</i> ₁ 3
5.	q	∨ e (1.2)
6.	$ q \lor r$	∨ <i>i</i> ₁ 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> ₂ 6
8.		

 $(p \lor q) \lor r \vdash p \lor (q \lor r)$



1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	∨ <i>e</i> (1)
3.	p	∨ <i>e</i> (1.1)
4.	$p \lor (q \lor r)$	∨ <i>i</i> ₁ 3
5.	q	∨ e (1.2)
6.	$q \vee r$	∨ <i>i</i> ₁ 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> ₂ 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.	r	∨ e (2)
0.		

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	∨ <i>e</i> (1)
3.	p	∨ e (1.1)
4.	$p \lor (q \lor r)$	∨ <i>i</i> ₁ 3
5.	q	∨ <i>e</i> (1.2)
6.	$ q \lor r$	∨ <i>i</i> ₁ 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> ₂ 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.	r	∨ e (2)
0.	$q \lor r$	∨ <i>i</i> ₂ 9
1.		

$$(p \lor q) \lor r \vdash p \lor (q \lor r)$$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	∨ <i>e</i> (1)
3.	p	∨ e (1.1)
4.	$p \lor (q \lor r)$	∨ <i>i</i> ₁ 3
5.	q	∨ <i>e</i> (1.2)
6.	$q \lor r$	∨ <i>i</i> ₁ 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> ₂ 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.	r	∨ <i>e</i> (2)
10.	$q \vee r$	√ <i>i</i> ₂ 9
11.	$p \lor (q \lor r)$	√ <i>i</i> ₂ 10

$$\blacktriangleright (p \lor q) \lor r \vdash p \lor (q \lor r)$$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	∨ <i>e</i> (1)
3.	p	∨ <i>e</i> (1.1)
4.	$p \lor (q \lor r)$	∨ <i>i</i> ₁ 3
5.	q	∨ <i>e</i> (1.2)
6.	$q \lor r$	∨ <i>i</i> ₁ 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> ₂ 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.	r	∨ <i>e</i> (2)
10.	$q \vee r$	∨ <i>i</i> ₂ 9
11.	$p \lor (q \lor r)$	∨ <i>i</i> ₂ 10
12.	$p \lor (q \lor r)$	∨ <i>e</i> 1, 2-8, 9-11

Basic Rules So Far

- $ightharpoonup \land i, \land e_1, \land e_2$ (and introduction and elimination)
- $\rightarrow \neg \neg e, \neg \neg i$ (double negation elimination and introduction)
- ► MP (Modus Ponens)
- $ightharpoonup \rightarrow i$ (Implies Introduction : remember opening boxes)
- $\vee i_1, \forall i_2, \forall e$ (Or introduction and elimination)

▶
$$\vdash p \rightarrow (q \rightarrow p)$$

1. true

premise

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
2		

▶
$$\vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
3.	q	assumption

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

	true	premise
2.	р	assumption
3.	q	assumption
ŀ.	р	copy 2

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
3.	q	assumption
4.	p	copy 2
5.	$oldsymbol{q} ightarrow oldsymbol{p}$	<i>→ i</i> 3-4

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
3.	q	assumption
4.	р	copy 2
5.	$oldsymbol{q} ightarrow oldsymbol{p}$	→ <i>i</i> 3-4
6.	$p \rightarrow (q \rightarrow p)$	\rightarrow i 2-5

▶ We have seen $\neg \neg e$ and $\neg \neg i$, the elimination and introduction of double negation.

- ▶ We have seen $\neg \neg e$ and $\neg \neg i$, the elimination and introduction of double negation.
- ▶ How about introducing and eliminating single negations?

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- How about introducing and eliminating single negations?
- ▶ We use the notion of contradictions, an expression of the form $\varphi \land \neg \varphi$, where φ is any propositional logic formula.

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- How about introducing and eliminating single negations?
- ▶ We use the notion of contradictions, an expression of the form $\varphi \land \neg \varphi$, where φ is any propositional logic formula.
- ▶ Any two contradictions are equivalent : $p \land \neg p$ is equivalent to $\neg r \land r$. Contradictions denoted by \bot .

- We have seen ¬¬e and ¬¬i, the elimination and introduction of double negation.
- How about introducing and eliminating single negations?
- ▶ We use the notion of contradictions, an expression of the form $\varphi \land \neg \varphi$, where φ is any propositional logic formula.
- ▶ Any two contradictions are equivalent : $p \land \neg p$ is equivalent to $\neg r \land r$. Contradictions denoted by \bot .
- $ightharpoonup \perp \to \varphi$ for any formula φ .

Rules with \bot

The \perp elimination rule $\perp e$

$$\frac{\perp}{\psi}$$

The \perp introduction rule $\perp i$

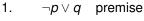
$$\frac{\varphi \qquad \neg \varphi}{\bot}$$

▶
$$\neg p \lor q \vdash p \rightarrow q$$

- 1. $\neg p \lor q$ premise
- 2.

▶
$$\neg p \lor q \vdash p \rightarrow q$$

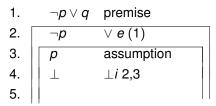
- 1. $\neg p \lor q$ premise
- 2. $\neg p \lor e(1)$
- 3.



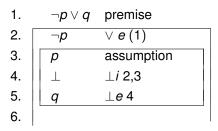
2.
$$\neg p \lor e(1)$$

4.

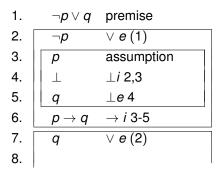
▶
$$\neg p \lor q \vdash p \rightarrow q$$



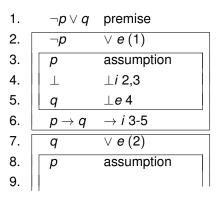
▶
$$\neg p \lor q \vdash p \rightarrow q$$



▶
$$\neg p \lor q \vdash p \rightarrow q$$



▶
$$\neg p \lor q \vdash p \rightarrow q$$



▶
$$\neg p \lor q \vdash p \rightarrow q$$

1.	$\neg p \lor q$	premise
2.	$\neg p$	∨ <i>e</i> (1)
3.	р	assumption
4.		<i>⊥i</i> 2,3
5.	q	⊥ <i>e</i> 4
6.	p o q	→ <i>i</i> 3-5
7.	q	∨ e (2)
8.	р	assumption
9.	q	copy 7
0.	p o q	→ <i>i</i> 8-9
1.	p o q	∨ <i>e</i> 1, 2-6, 7-10

Introducing Negations (PBC)

- In the course of a proof, if you assume φ (by opening a box) and obtain \bot in the box, then we conclude $\neg \varphi$
- ▶ This rule is denoted $\neg i$ and is read as \neg introduction.
- ► Also known as Proof By Contradiction

- 1. $p \rightarrow \neg p$ premise
- 2.

۱.	p ightarrow eg p	premise

2. p assumption 3.

$$\blacktriangleright \ p \to \neg p \vdash \neg p$$

1.	p ightarrow eg p	premise
2.	р	assumption
3.	$\neg p$	MP 1,2
4.		

1.	$oldsymbol{ ho} ightarrow eg eta$	premise
2.	р	assumption
3.	$\neg p$	MP 1,2
4.		<i>⊥i</i> 2,3
5.	$\neg p$	<i>¬i</i> 2-4

The Last One

Law of the Excluded Middle (LEM)



Summary of Basic Rules

- $\rightarrow \land i, \land e_1, \land e_2,$
- ¬¬e
- ► MP
- $\rightarrow i$
- $\triangleright \forall i_1, \forall i_2, \forall e$
- ▶ Copy, $\neg i$ or PBC
- **▶** ⊥*e*, ⊥*i*

Derived Rules

- ▶ MT (derive using MP, $\perp i$ and $\neg i$)
- $ightharpoonup \neg \neg i$ (derive using $\bot i$ and $\neg i$)
- ▶ LEM (derive using $\forall i_1, \bot i, \neg i, \forall i_2, \neg \neg e$)

The Proofs So Far

► So far, the "proof" we have seen is purely syntactic, no true/false meanings were attached

The Proofs So Far

- So far, the "proof" we have seen is purely syntactic, no true/false meanings were attached
- ▶ Intuitively, $p \rightarrow q \vdash \neg p \lor q$ makes sense because you think semantically. However, we never used any semantics so far.

The Proofs So Far

- So far, the "proof" we have seen is purely syntactic, no true/false meanings were attached
- ▶ Intuitively, $p \rightarrow q \vdash \neg p \lor q$ makes sense because you think semantically. However, we never used any semantics so far.
- Now we show that whatever can be proved makes sense semantically too.

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