CS 228: Logic in CS Autumn 2024

Mid Semester Examination

Total Marks: 60 marks *19 September 2024*

Instructions.

- Write your ROLL NUMBER on your answer sheet
- Unless asked for explicitly, you may cite results/proofs covered in class without reproducing them.
- When asked to prove something, give a formal proof with suitable explanation. Similarly, if asked to disprove give a counter-example and argue why it is in-fact a counter-example.
- If you need to make any assumptions, state them clearly.
- **DO NOT COPY SOLUTIONS FROM OTHERS**

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1. **[10 marks]** State TRUE or FALSE for each of the below. **1 mark for correct answer, -1 for incorrect answer, 0 marks if not attempted**. However, the negative marks won't overflow to other questions. So the minimum marks that you can score in this question is 0. **No justifications are expected**.

   Make sure that you write TRUE *or* FALSE *legibly. In case of any ambiguous answers, your answer will be treated as incorrect.*

   (a) Let φ be any FOL sentence and $\mathsf{free}(\varphi)$ and $\mathsf{bound}(\varphi)$ be the set of free and bound variables in φ, respectively. Then $\mathsf{free}(\varphi) \cap \mathsf{bound}(\varphi)$ is always empty.

   (b) In a proof system where $p \wedge \neg p$ is an axiom, the system is complete but not sound.

   (c) Let C_1 and C_2 be any two clauses. Then, C_1 and C_2 have a unique resolvent.

   (d) Tseitin encoding preserves the validity of the input CNF.

   (e) The FOL formula $\forall x\ (P(x) \vee Q(x))$ is true if $P(x)$ is true for all x.

   (f) Over the usual signature $\tau = (S, <, Q_a)$ for finite words discussed in class, consider the FOL formula
   $$\varphi = \forall x \exists y\, (x < y \wedge Q_a(x) \wedge Q_a(y))$$
   $L(\varphi)$ is the set of all words w such that for every position x in w holding an a, there is a later position y where a holds.

   (g) If a FOL formula F in Skolem Normal Form is valid, there is a finite witness of its validity. But if it is not valid but satisfiable, there may not be a finite witness.

   (h) Every DFA has at least one accepting state.

(i) Let A be a DFA accepting a language L, having only one final state. Consider the operation called "reversal of A" defined as follows, which gives a new automaton $\mathsf{rev}(A)$. To obtain $\mathsf{rev}(A)$ from A, (i) each transition (q, a, q') in A is reversed to (q', a, q), (ii) the initial state of A is made a final state, and (iii) the final state of A is made the initial state. Then $\mathsf{rev}(A)$ is a DFA which accepts the reverses of all words in $L(A)$.

(*The reverse of a word w is the word w' obtained by reading it back to front, for example, if $w = aab$, then its reverse is baa.*)

(j) Given a fixed natural number n, one can write a FOL formula that will evaluate to false under a structure whose universe has exactly n elements.

2. **[7 marks]** Define *Positive resolution* as a restriction of ordinary resolution as follows: derive a resolvent from clauses C_1 and C_2 only if C_1 is a positive clause, i.e., it consists only of positive literals. Prove or disprove : If F is an unsatisfiable CNF formula then one can derive the empty clause from F using only positive resolution.

3. **[5 marks]** Define a clause of a CNF formula to be *syntactically valid* if it contains both a literal and its complement. Give an efficient algorithm to check the *semantic validity* of a CNF formula and give its time complexity. Argue for the correctness of your algorithm.

   A formula F is semantically valid *if F evaluates to true in all rows of its truth table.*

4. **[7 marks]** A *renamable Horn formula* is a CNF formula that can be turned into a Horn formula by negating (all occurrences of) some of its variables. For example,

$$(p_1 \vee \neg p_2 \vee \neg p_3) \wedge (p_2 \vee p_3) \wedge (\neg p_1)$$

   can be turned into a Horn formula by negating p_1 and p_2.

   Given a CNF-formula F, show how to derive a 2-CNF formula G such that G is satisfiable if and only if F is a renamable Horn formula. Show moreover that one can derive a renaming that turns F into a Horn formula from a satisfying assignment for G.

5. **[2 + 5 + 8 = 15 marks]** In this question we work with first-order logic without equality; that is, we do not use the atomic formula $x = y$ where x, y are terms. *So, you are bound to lose marks if you use the "=" predicate unless you properly define it using the signature given.*

   (a) Consider a signature τ containing only a binary relation symbol R. For each positive integer n show that there is a satisfiable τ-formula F_n such that every structure $\mathcal{A}$ that satisfies F_n has at least n elements.

   (b) Fix a signature τ and τ-structures $\mathcal{A}, \mathcal{B}$ and assignments α, β. Consider a relation $\sim$ on pairs $(\mathcal{A}, \alpha)$, $(\mathcal{B}, \beta)$ that satisfies the following two properties:

   (P1) If $(\mathcal{A}, \alpha) \sim (\mathcal{B}, \beta)$ then for every atomic formula F we have $\mathcal{A} \models_\alpha F$ iff $\mathcal{B} \models_\beta F$.

   (P2) If $(\mathcal{A}, \alpha) \sim (\mathcal{B}, \beta)$ then for each variable x we have (i) for each $a \in U^{\mathcal{A}}$ there exists $b \in U^{\mathcal{B}}$ such that $(\mathcal{A}, \alpha[x \mapsto a]) \sim (\mathcal{B}, \beta[x \mapsto b])$, and (ii) for all $b \in U^{\mathcal{B}}$ there exists $a \in U^{\mathcal{A}}$ such that $(\mathcal{A}, \alpha[x \mapsto a]) \sim (\mathcal{B}, \beta[x \mapsto b])$.

If $(\mathcal{A}, \alpha) \sim (\mathcal{B}, \beta)$, then show that for any formula F built from atomic formulae using the connectives $\neg, \wedge, \exists$, we have $\mathcal{A} \models_\alpha F$ iff $\mathcal{B} \models_\beta F$.

(c) Consider a signature τ containing only unary predicate symbols $P_1, \ldots, P_k$. Using (b), show that any satisfiable τ-formula has a structure with at most 2^k elements in its universe satisfying it.

6. **[5 marks]** Assume that a 2-CNF formula is input to DPLL. Prove that every clause that is learnt is either empty or a singleton.

7. **[4 marks]** Consider the language $L = \{w \in \{a, b\}^* \mid w$ has equal number of occurrences of patterns ab and $ba\}$. For example, $aab \notin L$ since it has one occurrence of ab and zero occurrences of ba while $aba \in L$ since it has one occurrence each of both ab and ba. Show that L is FO-definable and regular.

   In order to show that L is FO-definable you need to exhibit a FOL formula φ and argue why $L(\varphi) = L$. Similarly to show that L is regular, you should draw a DFA and argue that the language accepted by the DFA is L.

8. **[7 marks]** Consider the formula in 3-CNF,

$$\varphi = (x_1 \vee x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee \neg x_4) \wedge (x_1 \vee \neg x_2 \vee x_4)$$

   Show that one can construct a graph G_φ from formula φ such that

   - G_φ has an independent set of size k iff φ is satisfiable, where k is the number of literals assigned true in the satisfying assignment.
   - The size of G_φ is polynomial in the size of φ.
   - G_φ must be inspired from φ; that is, you must have an "encoding scheme" to construct a graph G_φ from any formula φ.

   An independent set in a graph is a set of vertices such that no two of them are adjacent.

   Give proper explanation to validate that your construction of G_φ is correct.