# CS 228 : Logic in Computer Science

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# Rules for Natural Deduction

**Another implies elimination rule or Modus Tollens MT**

$$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi}$$

# A Proof

- Show that $p \to \neg q, q \vdash \neg p$

| | | |
|---|---|---|
| 1. | $p \to \neg q$ | premise |
| 2. | | |

# A Proof

- Show that $p \rightarrow \neg q, q \vdash \neg p$

| | | |
|---|---|---|
| 1. | $p \rightarrow \neg q$ | premise |
| 2. | $q$ | premise |
| 3. | | |

# A Proof

- Show that $p \to \neg q, q \vdash \neg p$

| | | |
|---|---|---|
| 1. | $p \to \neg q$ | premise |
| 2. | $q$ | premise |
| 3. | $\neg\neg q$ | $\neg\neg i$ 2 |
| 4. | | |

# A Proof

- Show that $p \to \neg q, q \vdash \neg p$

| | | |
|---|---|---|
| 1. | $p \to \neg q$ | premise |
| 2. | $q$ | premise |
| 3. | $\neg\neg q$ | $\neg\neg i$ 2 |
| 4. | $\neg p$ | MT 1,3 |

# More Rules

- Thanks to MT, we have $p \to q, \neg q \vdash \neg p$.

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- So far, no proof rule that can do this.

# More Rules

- Thanks to MT, we have $p \to q, \neg q \vdash \neg p$.
- Can we prove $p \to q \vdash \neg q \to \neg p$?
- So far, no proof rule that can do this.
- Given $p \to q$, let us assume $\neg q$. Can we then prove $\neg p$?

# More Rules

- Thanks to MT, we have $p \to q, \neg q \vdash \neg p$.
- Can we prove $p \to q \vdash \neg q \to \neg p$?
- So far, no proof rule that can do this.
- Given $p \to q$, let us assume $\neg q$. Can we then prove $\neg p$?
- Yes, using MT.

# The implies introduction rule $\rightarrow i$

- $p \rightarrow q \vdash \neg q \rightarrow \neg p$

| | | |
|---|---|---|
| 1. | $p \rightarrow q$ | premise |
| 2. | $\neg q$ | assumption |
| 3. | $\neg p$ | MT 1,2 |
| 4. | $\neg q \rightarrow \neg p$ | $\rightarrow i$ 2-3 |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | | |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | | |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | $true$ | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | $\neg q \rightarrow \neg p$ | assumption |
| 4. | | |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | $true$ | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | $\neg q \rightarrow \neg p$ | assumption |
| 4. | $p$ | assumption |
| 5. | | |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | $\neg q \rightarrow \neg p$ | assumption |
| 4. | $p$ | assumption |
| 5. | $\neg\neg p$ | $\neg\neg i$ 4 |
| 6. | | |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | $true$ | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | $\neg q \rightarrow \neg p$ | assumption |
| 4. | $p$ | assumption |
| 5. | $\neg\neg p$ | $\neg\neg i$ 4 |
| 6. | $\neg\neg q$ | MT 3,5 |
| 7. | | |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | $true$ | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | $\neg q \rightarrow \neg p$ | assumption |
| 4. | $p$ | assumption |
| 5. | $\neg\neg p$ | $\neg\neg i$ 4 |
| 6. | $\neg\neg q$ | MT 3,5 |
| 7. | $q$ | $\neg\neg e$ 6 |
| 8. | | |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | $true$ | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | $\neg q \rightarrow \neg p$ | assumption |
| 4. | $p$ | assumption |
| 5. | $\neg\neg p$ | $\neg\neg i$ 4 |
| 6. | $\neg\neg q$ | MT 3,5 |
| 7. | $q$ | $\neg\neg e$ 6 |
| 8. | $r$ | MP 2,7 |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | $true$ | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | $\neg q \rightarrow \neg p$ | assumption |
| 4. | $p$ | assumption |
| 5. | $\neg\neg p$ | $\neg\neg i$ 4 |
| 6. | $\neg\neg q$ | MT 3,5 |
| 7. | $q$ | $\neg\neg e$ 6 |
| 8. | $r$ | MP 2,7 |
| 9. | $p \rightarrow r$ | $\rightarrow i$ 4-8 |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | $true$ | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | $\neg q \rightarrow \neg p$ | assumption |
| 4. | $p$ | assumption |
| 5. | $\neg\neg p$ | $\neg\neg i$ 4 |
| 6. | $\neg\neg q$ | MT 3,5 |
| 7. | $q$ | $\neg\neg e$ 6 |
| 8. | $r$ | MP 2,7 |
| 9. | $p \rightarrow r$ | $\rightarrow i$ 4-8 |
| 10. | $(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$ | $\rightarrow i$ 3-9 |
| 11. | | |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | $\neg q \rightarrow \neg p$ | assumption |
| 4. | $p$ | assumption |
| 5. | $\neg\neg p$ | $\neg\neg i$ 4 |
| 6. | $\neg\neg q$ | MT 3,5 |
| 7. | $q$ | $\neg\neg e$ 6 |
| 8. | $r$ | MP 2,7 |
| 9. | $p \rightarrow r$ | $\rightarrow i$ 4-8 |
| 10. | $(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$ | $\rightarrow i$ 3-9 |
| 11. | $(q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$ | $\rightarrow i$ 2-10 |

# Transforming Proofs

- $(q \rightarrow r), (\neg q \rightarrow \neg p), p \vdash r$
- Transform any proof $\varphi_1, \ldots, \varphi_n \vdash \psi$ to $\vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_n \rightarrow \psi) \ldots))$ by adding $n$ lines of the rule $\rightarrow i$

# More Examples

- $p \rightarrow (q \rightarrow r) \vdash (p \wedge q) \rightarrow r$

| | | |
|---|---|---|
| 1. | $p \rightarrow (q \rightarrow r)$ | premise |
| 2. | | |

# More Examples

- $p \to (q \to r) \vdash (p \wedge q) \to r$

| | | |
|---|---|---|
| 1. | $p \to (q \to r)$ | premise |
| 2. | $p \wedge q$ | assumption |
| 3. | | |

# More Examples

- $p \to (q \to r) \vdash (p \wedge q) \to r$

| | | |
|---|---|---|
| 1. | $p \to (q \to r)$ | premise |
| 2. | $p \wedge q$ | assumption |
| 3. | $p$ | $\wedge e_1$ 2 |
| 4. | | |

# More Examples

- $p \to (q \to r) \vdash (p \land q) \to r$

| | | |
|---|---|---|
| 1. | $p \to (q \to r)$ | premise |
| 2. | $p \land q$ | assumption |
| 3. | $p$ | $\land e_1$ 2 |
| 4. | $q$ | $\land e_2$ 2 |
| 5. | | |

# More Examples

- $p \to (q \to r) \vdash (p \wedge q) \to r$

| | | |
|---|---|---|
| 1. | $p \to (q \to r)$ | premise |
| 2. | $p \wedge q$ | assumption |
| 3. | $p$ | $\wedge e_1$ 2 |
| 4. | $q$ | $\wedge e_2$ 2 |
| 5. | $q \to r$ | MP 1,3 |
| 6. | | |

# More Examples

- $p \to (q \to r) \vdash (p \wedge q) \to r$

| | | |
|---|---|---|
| 1. | $p \to (q \to r)$ | premise |
| 2. | $p \wedge q$ | assumption |
| 3. | $p$ | $\wedge e_1$ 2 |
| 4. | $q$ | $\wedge e_2$ 2 |
| 5. | $q \to r$ | MP 1,3 |
| 6. | $r$ | MP 4,5 |
| 7. | | |

# More Examples

- $p \to (q \to r) \vdash (p \wedge q) \to r$

| | | |
|---|---|---|
| 1. | $p \to (q \to r)$ | premise |
| 2. | $p \wedge q$ | assumption |
| 3. | $p$ | $\wedge e_1$ 2 |
| 4. | $q$ | $\wedge e_2$ 2 |
| 5. | $q \to r$ | MP 1,3 |
| 6. | $r$ | MP 4,5 |
| 7. | $p \wedge q \to r$ | $\to i$ 2-6 |

# More Rules

**The or introduction rule $\vee i_1$**

$$\frac{\varphi}{\varphi \vee \psi}$$

**The or introduction rule $\vee i_2$**

$$\frac{\psi}{\varphi \vee \psi}$$

# More Rules

**The or elimination rule $\vee e$**

$$\frac{\varphi \vee \psi \qquad \varphi \vdash \chi \qquad \psi \vdash \chi}{\chi}$$

# Or Elimination Example

- $q \to r \vdash (p \lor q) \to (p \lor r)$

| | | |
|---|---|---|
| 1. | $q \to r$ | premise |
| 2. | | |

# Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

| | | |
|---|---|---|
| 1. | $q \to r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | | |

# Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

| | | |
|---|---|---|
| 1. | $q \to r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | $p$ | $\vee e$ (1) |
| 4. | | |

# Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

| | | |
|---|---|---|
| 1. | $q \to r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | $p$ | $\vee e$ (1) |
| 4. | $p \vee r$ | $\vee i_1$ 3 |
| 5. | | |

# Or Elimination Example

- $q \rightarrow r \vdash (p \vee q) \rightarrow (p \vee r)$

| | | |
|---|---|---|
| 1. | $q \rightarrow r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | $p$ | $\vee e$ (1) |
| 4. | $p \vee r$ | $\vee i_1$ 3 |
| 5. | $q$ | $\vee e$ (2) |
| 6. | | |

# Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

| | | |
|---|---|---|
| 1. | $q \to r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | $p$ | $\vee\ e\ (1)$ |
| 4. | $p \vee r$ | $\vee i_1\ 3$ |
| 5. | $q$ | $\vee\ e\ (2)$ |
| 6. | $r$ | MP 1,5 |
| 7. | | |

# Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

| | | |
|---|---|---|
| 1. | $q \to r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | $p$ | $\vee e$ (1) |
| 4. | $p \vee r$ | $\vee i_1$ 3 |
| 5. | $q$ | $\vee e$ (2) |
| 6. | $r$ | MP 1,5 |
| 7. | $p \vee r$ | $\vee i_2$ 6 |

# Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

| | | |
|---|---|---|
| 1. | $q \to r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | $p$ | $\vee\ e\ (1)$ |
| 4. | $p \vee r$ | $\vee i_1$ 3 |
| 5. | $q$ | $\vee\ e\ (2)$ |
| 6. | $r$ | MP 1,5 |
| 7. | $p \vee r$ | $\vee i_2$ 6 |
| 8. | $p \vee r$ | $\vee e$ 2, 3-4, 5-7 |

# Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

| | | |
|---|---|---|
| 1. | $q \to r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | $p$ | $\vee\ e\ (1)$ |
| 4. | $p \vee r$ | $\vee i_1$ 3 |
| 5. | $q$ | $\vee\ e\ (2)$ |
| 6. | $r$ | MP 1,5 |
| 7. | $p \vee r$ | $\vee i_2$ 6 |
| 8. | $p \vee r$ | $\vee e$ 2, 3-4, 5-7 |
| 9. | | |

# Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

| | | |
|---|---|---|
| 1. | $q \to r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | $p$ | $\vee\ e\ (1)$ |
| 4. | $p \vee r$ | $\vee i_1\ 3$ |
| 5. | $q$ | $\vee\ e\ (2)$ |
| 6. | $r$ | MP 1,5 |
| 7. | $p \vee r$ | $\vee i_2\ 6$ |
| 8. | $p \vee r$ | $\vee e$ 2, 3-4, 5-7 |
| 9. | $(p \vee q) \to (p \vee r)$ | $\to i$ 2-8 |

# Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

| | | |
|---|---|---|
| 1. | $(p \vee q) \vee r$ | premise |
| 2. | | |

# Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

| | | |
|---|---|---|
| 1. | $(p \vee q) \vee r$ | premise |
| 2. | $p \vee q$ | $\vee e\,(1)$ |
| 3. | | |

# Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

| | | |
|---|---|---|
| 1. | $(p \vee q) \vee r$ | premise |
| 2. | $p \vee q$ | $\vee e$ (1) |
| 3. | $p$ | $\vee e$ (1.1) |
| 4. | | |

# Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

| | | |
|---|---|---|
| 1. | $(p \vee q) \vee r$ | premise |
| 2. | $p \vee q$ | $\vee e$ (1) |
| 3. | $p$ | $\vee e$ (1.1) |
| 4. | $p \vee (q \vee r)$ | $\vee i_1$ 3 |
| 5. | | |

# Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

| | | |
|---|---|---|
| 1. | $(p \vee q) \vee r$ | premise |
| 2. | $p \vee q$ | $\vee e\,(1)$ |
| 3. | $p$ | $\vee e\,(1.1)$ |
| 4. | $p \vee (q \vee r)$ | $\vee i_1\ 3$ |
| 5. | $q$ | $\vee e\,(1.2)$ |
| 6. | | |

# Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

| | | |
|---|---|---|
| 1. | $(p \vee q) \vee r$ | premise |
| 2. | $p \vee q$ | $\vee e\,(1)$ |
| 3. | $p$ | $\vee e\,(1.1)$ |
| 4. | $p \vee (q \vee r)$ | $\vee i_1$ 3 |
| 5. | $q$ | $\vee e\,(1.2)$ |
| 6. | $q \vee r$ | $\vee i_1$ 5 |
| 7. | | |

# Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

| | | |
|---|---|---|
| 1. | $(p \vee q) \vee r$ | premise |
| 2. | $p \vee q$ | $\vee e\,(1)$ |
| 3. | $p$ | $\vee e\,(1.1)$ |
| 4. | $p \vee (q \vee r)$ | $\vee i_1\ 3$ |
| 5. | $q$ | $\vee e\,(1.2)$ |
| 6. | $q \vee r$ | $\vee i_1\ 5$ |
| 7. | $p \vee (q \vee r)$ | $\vee i_2\ 6$ |
| 8. | | |

# Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

| | | |
|---|---|---|
| 1. | $(p \vee q) \vee r$ | premise |
| 2. | $p \vee q$ | $\vee\, e\,(1)$ |
| 3. | $p$ | $\vee\, e\,(1.1)$ |
| 4. | $p \vee (q \vee r)$ | $\vee i_1$ 3 |
| 5. | $q$ | $\vee\, e\,(1.2)$ |
| 6. | $q \vee r$ | $\vee i_1$ 5 |
| 7. | $p \vee (q \vee r)$ | $\vee i_2$ 6 |
| 8. | $p \vee (q \vee r)$ | $\vee e$ 2, 3-4, 5-7 |
| 9. | | |

# Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

| | | |
|---|---|---|
| 1. | $(p \vee q) \vee r$ | premise |
| 2. | $p \vee q$ | $\vee\ e\ (1)$ |
| 3. | $p$ | $\vee\ e\ (1.1)$ |
| 4. | $p \vee (q \vee r)$ | $\vee i_1\ 3$ |
| 5. | $q$ | $\vee\ e\ (1.2)$ |
| 6. | $q \vee r$ | $\vee i_1\ 5$ |
| 7. | $p \vee (q \vee r)$ | $\vee i_2\ 6$ |
| 8. | $p \vee (q \vee r)$ | $\vee e$ 2, 3-4, 5-7 |
| 9. | $r$ | $\vee\ e\ (2)$ |
| 10. | | |

# Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

| | | |
|---|---|---|
| 1. | $(p \vee q) \vee r$ | premise |
| 2. | $p \vee q$ | $\vee\, e$ (1) |
| 3. | $p$ | $\vee\, e$ (1.1) |
| 4. | $p \vee (q \vee r)$ | $\vee i_1$ 3 |
| 5. | $q$ | $\vee\, e$ (1.2) |
| 6. | $q \vee r$ | $\vee i_1$ 5 |
| 7. | $p \vee (q \vee r)$ | $\vee i_2$ 6 |
| 8. | $p \vee (q \vee r)$ | $\vee e$ 2, 3-4, 5-7 |
| 9. | $r$ | $\vee\, e$ (2) |
| 10. | $q \vee r$ | $\vee i_2$ 9 |
| 11. | | |

# Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

| | | |
|---|---|---|
| 1. | $(p \vee q) \vee r$ | premise |
| 2. | $p \vee q$ | $\vee\, e\,(1)$ |
| 3. | $p$ | $\vee\, e\,(1.1)$ |
| 4. | $p \vee (q \vee r)$ | $\vee i_1$ 3 |
| 5. | $q$ | $\vee\, e\,(1.2)$ |
| 6. | $q \vee r$ | $\vee i_1$ 5 |
| 7. | $p \vee (q \vee r)$ | $\vee i_2$ 6 |
| 8. | $p \vee (q \vee r)$ | $\vee e$ 2, 3-4, 5-7 |
| 9. | $r$ | $\vee\, e\,(2)$ |
| 10. | $q \vee r$ | $\vee i_2$ 9 |
| 11. | $p \vee (q \vee r)$ | $\vee i_2$ 10 |

# Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

| | | |
|---|---|---|
| 1. | $(p \vee q) \vee r$ | premise |
| 2. | $p \vee q$ | $\vee\ e\ (1)$ |
| 3. | $p$ | $\vee\ e\ (1.1)$ |
| 4. | $p \vee (q \vee r)$ | $\vee i_1$ 3 |
| 5. | $q$ | $\vee\ e\ (1.2)$ |
| 6. | $q \vee r$ | $\vee i_1$ 5 |
| 7. | $p \vee (q \vee r)$ | $\vee i_2$ 6 |
| 8. | $p \vee (q \vee r)$ | $\vee e$ 2, 3-4, 5-7 |
| 9. | $r$ | $\vee\ e\ (2)$ |
| 10. | $q \vee r$ | $\vee i_2$ 9 |
| 11. | $p \vee (q \vee r)$ | $\vee i_2$ 10 |
| 12. | $p \vee (q \vee r)$ | $\vee e$ 1, 2-8, 9-11 |

# Basic Rules So Far

- $\wedge$i, $\wedge e_1$, $\wedge e_2$ (and introduction and elimination)
- $\neg\neg e$, $\neg\neg i$ (double negation elimination and introduction)
- MP (Modus Ponens)
- $\rightarrow i$ (Implies Introduction : remember opening boxes)
- $\vee i_1, \vee i_2, \vee e$ (Or introduction and elimination)

# The Copy Rule

- $\vdash p \rightarrow (q \rightarrow p)$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | | |

# The Copy Rule

- $\vdash p \to (q \to p)$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | $p$ | assumption |
| 3. | | |

# The Copy Rule

- $\vdash p \to (q \to p)$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | $p$ | assumption |
| 3. | $q$ | assumption |
| 4. | | |

# The Copy Rule

- $\vdash p \to (q \to p)$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | $p$ | assumption |
| 3. | $q$ | assumption |
| 4. | $p$ | copy 2 |
| 5. | | |

# The Copy Rule

- $\vdash p \rightarrow (q \rightarrow p)$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | $p$ | assumption |
| 3. | $q$ | assumption |
| 4. | $p$ | copy 2 |
| 5. | $q \rightarrow p$ | $\rightarrow i$ 3-4 |
| 6. | | |

# The Copy Rule

- $\vdash p \to (q \to p)$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | $p$ | assumption |
| 3. | $q$ | assumption |
| 4. | $p$ | copy 2 |
| 5. | $q \to p$ | $\to i$ 3-4 |
| 6. | $p \to (q \to p)$ | $\to i$ 2-5 |

# The Rules of Single Negation

- We have seen $\neg\neg e$ and $\neg\neg i$, the elimination and introduction of double negation.

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- $\perp \rightarrow \varphi$ for any formula $\varphi$.

# Rules with $\perp$

### The $\perp$ elimination rule $\perp e$

$$\frac{\perp}{\psi}$$

### The $\perp$ introduction rule $\perp i$

$$\frac{\varphi \qquad \neg\varphi}{\perp}$$

# An Example

- $\neg p \vee q \vdash p \rightarrow q$

| | | |
|---|---|---|
| 1. | $\neg p \vee q$ | premise |
| 2. | | |

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| 2. | $\neg p$ | $\vee e$ (1) |
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| 2. | $\neg p$ | $\vee e$ (1) |
| 3. | $p$ | assumption |
| 4. | | |

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- $\neg p \vee q \vdash p \rightarrow q$

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| 1. | $\neg p \vee q$ | premise |
| 2. | $\neg p$ | $\vee e$ (1) |
| 3. | $p$ | assumption |
| 4. | $\bot$ | $\bot i$ 2,3 |
| 5. | | |

# An Example

- $\neg p \lor q \vdash p \to q$

| | | |
|---|---|---|
| 1. | $\neg p \lor q$ | premise |
| 2. | $\neg p$ | $\lor e$ (1) |
| 3. | $p$ | assumption |
| 4. | $\bot$ | $\bot i$ 2,3 |
| 5. | $q$ | $\bot e$ 4 |
| 6. | | |

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| 1. | $\neg p \lor q$ | premise |
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| 4. | $\bot$ | $\bot i$ 2,3 |
| 5. | $q$ | $\bot e$ 4 |
| 6. | $p \to q$ | $\to i$ 3-5 |
| 7. | $q$ | $\lor e$ (2) |
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| 9. | | |

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| 1. | $\neg p \vee q$ | premise |
| 2. | $\neg p$ | $\vee e$ (1) |
| 3. | $p$ | assumption |
| 4. | $\bot$ | $\bot i$ 2,3 |
| 5. | $q$ | $\bot e$ 4 |
| 6. | $p \rightarrow q$ | $\rightarrow i$ 3-5 |
| 7. | $q$ | $\vee e$ (2) |
| 8. | $p$ | assumption |
| 9. | $q$ | copy 7 |
| 10. | $p \rightarrow q$ | $\rightarrow i$ 8-9 |
| 11. | $p \rightarrow q$ | $\vee e$ 1, 2-6, 7-10 |

# Introducing Negations (PBC)

- In the course of a proof, if you assume $\varphi$ (by opening a box) and obtain $\perp$ in the box, then we conclude $\neg\varphi$
- This rule is denoted $\neg i$ and is read as $\neg$ introduction.
- Also known as Proof By Contradiction

# An Example

- $p \to \neg p \vdash \neg p$

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|---|---|---|
| 1. | $p \to \neg p$ | premise |
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| 1. | $p \to \neg p$ | premise |
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| 1. | $p \to \neg p$ | premise |
| 2. | $p$ | assumption |
| 3. | $\neg p$ | MP 1,2 |
| 4. | | |

# An Example

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| | | |
|---|---|---|
| 1. | $p \to \neg p$ | premise |
| 2. | $p$ | assumption |
| 3. | $\neg p$ | MP 1,2 |
| 4. | $\perp$ | $\perp i$ 2,3 |
| 5. | $\neg p$ | $\neg i$ 2-4 |

# The Last One

**Law of the Excluded Middle (LEM)**

$$\frac{}{\varphi \vee \neg\varphi}$$

# Summary of Basic Rules

- $\wedge i$, $\wedge e_1$, $\wedge e_2$,
- $\neg\neg e$
- MP
- $\rightarrow i$
- $\vee i_1, \vee i_2, \vee e$
- Copy, $\neg i$ or PBC
- $\bot e, \bot i$

# Derived Rules

- MT (derive using MP, $\bot i$ and $\neg i$)
- $\neg\neg i$ (derive using $\bot i$ and $\neg i$)
- LEM (derive using $\vee i_1, \bot i, \neg i, \vee i_2, \neg\neg e$)

# The Proofs So Far

- So far, the "proof" we have seen is purely syntactic, no true/false meanings were attached

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- So far, the "proof" we have seen is purely syntactic, no true/false meanings were attached
- Intuitively, $p \rightarrow q \vdash \neg p \vee q$ makes sense because you think semantically. However, we never used any semantics so far.
- Now we show that whatever can be proved makes sense semantically too.

# Semantics

- Each propositional variable is assigned values true/false. Truth tables for each of the operators $\vee, \wedge, \neg, \rightarrow$ to determine truth values of complex formulae.

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  - Recall $\vdash$, and compare with $\models$
- Formulae $\varphi$ and $\psi$ are provably equivalent iff $\varphi \vdash \psi$ and $\psi \vdash \varphi$
- Formulae $\varphi$ and $\psi$ are semantically equivalent iff $\varphi \models \psi$ and $\psi \models \varphi$