CS 228 : Logic in Computer Science

Krishna. S

Check Satisfiability

Let $\psi(z) = \exists x[Q_a(x) \wedge \forall y[(y \leqslant x \wedge Q_b(y)) \rightarrow (z < x \wedge y < z \wedge Q_c(z))]]$ over the signature τ having the relational symbols $<, Q_a, Q_b, Q_c$ and unary function S. Does $\psi(z)$ evaluate to true under some word structure?

No, because when S(y,x) holds there is no position z in bw them for "c".

Check Satisfiability

Not able to come up with an example which satisfies the given relation.

Let $\zeta = P(0) \wedge \forall x(P(x) \to P(S(x))) \wedge \exists x \neg P(x)$ over a signature τ containing the constant 0, unary function S and unary relation P.

Is ζ satisfiable?

Normal Forms in FOL

Recap : Satisfaction

We define the relation $\mathcal{A} \models_\alpha \varphi$ (read as φ is true in $\mathcal{A}$ under the assignment α) inductively:

- $\mathcal{A} \not\models_\alpha \bot$
- $\mathcal{A} \models_\alpha t_1 = t_2$ iff $\alpha(t_1) = \alpha(t_2)$
- $\mathcal{A} \models_\alpha R(t_1, \ldots, t_k)$ iff $(\alpha(t_1), \ldots, \alpha(t_k)) \in R^{\mathcal{A}}$
- $\mathcal{A} \models_\alpha (\varphi \to \psi)$ iff $\mathcal{A} \not\models_\alpha \varphi$ or $\mathcal{A} \models_\alpha \psi$
- $\mathcal{A} \models_\alpha (\forall x)\varphi$ iff for every $a \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a]} \varphi$
- $\mathcal{A} \models_\alpha (\exists x)\varphi$ iff there is some $a \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a]} \varphi$

Last two cases, α has no effect on the value of x. Thus, assignments matter only to free variables.

Equivalences

See it carefully that the initial negation is over quantifiers, and according to which try to interpret the formula.

Let F, G be arbitrary FOL formulae.

1. $\neg\forall x F \equiv \exists x \neg F$
2. $\neg\exists x F \equiv \forall x \neg F$

saying, there do not exists any x such that, F(x) holds, which is equivalent to for all x, f(x) do not holds.

$\mathcal{A} \models_\alpha \neg\forall x F$ iff $\mathcal{A} \not\models_\alpha \forall x F$ Under some assignment alpha of structure A, F does not holds

iff $\mathcal{A} \not\models_{\alpha[x\mapsto a]} F$ for some $a \in U^{\mathcal{A}}$

iff $\mathcal{A} \models_{\alpha[x\mapsto a]} \neg F$ for some $a \in U^{\mathcal{A}}$

iff $\mathcal{A} \models_\alpha \exists x \neg F$

Renaming is done for bound variables and Substitution is done for free variables.

Equivalences

Even if it occurs with bound, the quantifier that bounds should be same.

If x does not occur free in G then

1. $(\forall x F \wedge G) \equiv \forall x(F \wedge G)$
2. $(\forall x F \vee G) \equiv \forall x(F \vee G)$
3. $(\exists x F \wedge G) \equiv \exists x(F \wedge G)$
4. $(\exists x F \vee G) \equiv \exists x(F \vee G)$

$$
\begin{aligned}
\mathcal{A} \models_\alpha \forall x F \wedge G \text{ iff } & \mathcal{A} \models_\alpha \forall x F \text{ and } \mathcal{A} \models_\alpha G \\
\text{iff } & \text{for all } a \in U^{\mathcal{A}}, \mathcal{A} \models_{\alpha[x \mapsto a]} F \text{ and } \mathcal{A} \models_\alpha G \\
\text{iff } & \text{for all } a \in U^{\mathcal{A}}, \mathcal{A} \models_{\alpha[x \mapsto a]} F \text{ and } \mathcal{A} \models_{\alpha[x \mapsto a]} G \\
\text{iff } & \text{for all } a \in U^{\mathcal{A}}, \mathcal{A} \models_{\alpha[x \mapsto a]} (F \wedge G) \\
\text{iff } & \mathcal{A} \models \forall x(F \wedge G)
\end{aligned}
$$

Here alpha is the original assignment function and if we write alpha[x->a], by that we mean we are putting all the other assignments as it is but checking the x=a in the formula.

Equivalences

Let F, G be arbitrary FOL formulae.

1. $(\forall xF \land \forall xG) \equiv \forall x(F \land G)$
2. $(\exists xF \lor \exists xG) \equiv \exists x(F \lor G)$

Order of quantifier doesn't matter if the quantifier's are same.

1. $\forall x \forall yF \equiv \forall y \forall xF$
2. $\exists x \exists yF \equiv \exists y \exists xF$

Recap : Terms

Given a signature τ, the set of τ-terms are defined inductively as follows.

- Each variable is a term
- Each constant symbol is a term
- If $t_1, \ldots, t_k$ are terms and f is a k-ary function, then $f(t_1, \ldots, t_k)$ is a term
- Ground Terms : Terms without variables. For instance $f(c_1, \ldots, c_k)$ for constants $c_1, \ldots, c_k$.

Translation Lemma

Translation Lemma

If t is a term and F is a formula such that no variable in t occurs bound in F, then $\mathcal{A} \models_{\alpha} F[t/x]$ iff $\mathcal{A} \models_{\alpha[x \mapsto \alpha(t)]} F$.

$F[t/x]$ denotes substituting t for x in F, where x is free in F

- What if t contains a variable bound in F? False results/behaviors because the we are restricting term "t" to some values.
- Results in *Variable Capture*

Translation Lemma Proof : Optional

Proof by Induction on formulae.

- Base case. Atomic formulae $P(t_1, \ldots, t_k)$.
- $\mathcal{A} \models_\alpha P(t_1, \ldots, t_k)[t/x]$ iff $\mathcal{A} \models_\alpha P(t_1[t/x], \ldots, t_k[t/x])$.
- Show that $\mathcal{A} \models_{\alpha[x \mapsto \alpha(t)]} P(t_1, \ldots, t_k)$.
  - Base Cases within : $t_i = c$, $t_i = y$ for $y \neq x$, $t_i = x$ for each t_i.
  - Case $t_i = f(s_1, \ldots, s_j)$ for a function f.
  - $f(s_1, \ldots, s_j)[t/x] = f(s_1[t/x], \ldots, s_j[t/x])$
- $\mathcal{A} \models_\alpha P(t_1[t/x], \ldots, t_k[t/x])$ iff ($\alpha(t_1[t/x]), \ldots, \alpha(t_k[t/x])$) $\in P^{\mathcal{A}}$
- iff ($\alpha([x \mapsto \alpha(t)](t_1), \ldots, \alpha([x \mapsto \alpha(t)](t_k)$) $\in P^{\mathcal{A}}$
- iff $\mathcal{A} \models_{\alpha[x \mapsto \alpha(t)]} P(t_1, \ldots, t_k)$
- Cases for formulae with propositional connectives is routine.
- Case with quantifier, $\forall y F[t/x]$, $\exists y F[t/x]$ where $y \neq x$.

Renaming

$\int_0^\infty f(s)ds$ has the same value as $\int_0^\infty f(t)dt$

Renaming Lemma

Let $F = Qx[G]$ be a formula with $Q \in \{\exists, \forall\}$. Let y be a variable which does not appear in G. Then $\mathcal{A} \models_\alpha F$ iff $\mathcal{A} \models_\alpha Qy(G[y/x])$.

Assume $Q = \forall$.
$\mathcal{A} \models_\alpha \forall y G[y/x]$ iff $\mathcal{A} \models_{\alpha[y \mapsto a]} G[y/x]$ for all $a \in U^{\mathcal{A}}$

Renaming

$\int_0^\infty f(s)ds$ has the same value as $\int_0^\infty f(t)dt$

Renaming Lemma

Let $F = Qx[G]$ be a formula with $Q \in \{\exists, \forall\}$. Let y be a variable which does not appear in G. Then $\mathcal{A} \models_\alpha F$ iff $\mathcal{A} \models_\alpha Qy(G[y/x])$.

Assume $Q = \forall$.

$\mathcal{A} \models_\alpha \forall y G[y/x]$ iff $\mathcal{A} \models_{\alpha[y \mapsto a]} G[y/x]$ for all $a \in U^{\mathcal{A}}$

iff $\mathcal{A} \models_{\alpha[y \mapsto a, x \mapsto \alpha[y \mapsto a](y)]} G$ for all $a \in U^{\mathcal{A}}$

(Translation Lemma)

Renaming

$\int_0^\infty f(s)ds$ has the same value as $\int_0^\infty f(t)dt$

Renaming Lemma

Let $F = Qx[G]$ be a formula with $Q \in \{\exists, \forall\}$. Let y be a variable which does not appear in G. Then $\mathcal{A} \models_\alpha F$ iff $\mathcal{A} \models_\alpha Qy(G[y/x])$.

Assume $Q = \forall$.

$\mathcal{A} \models_\alpha \forall y G[y/x]$ iff $\mathcal{A} \models_{\alpha[y \mapsto a]} G[y/x]$ for all $a \in U^{\mathcal{A}}$

iff $\mathcal{A} \models_{\alpha[y \mapsto a, x \mapsto \alpha[y \mapsto a](y)]} G$ for all $a \in U^{\mathcal{A}}$

(Translation Lemma)

iff $\mathcal{A} \models_{\alpha[y \mapsto a, x \mapsto a]} G$ for all $a \in U^{\mathcal{A}}$

Renaming

$\int_0^\infty f(s)ds$ has the same value as $\int_0^\infty f(t)dt$

Renaming Lemma

Let $F = Qx[G]$ be a formula with $Q \in \{\exists, \forall\}$. Let y be a variable which does not appear in G. Then $\mathcal{A} \models_\alpha F$ iff $\mathcal{A} \models_\alpha Qy(G[y/x])$.

Assume $Q = \forall$.

$\mathcal{A} \models_\alpha \forall y G[y/x]$ iff $\mathcal{A} \models_{\alpha[y \mapsto a]} G[y/x]$ for all $a \in U^{\mathcal{A}}$

iff $\mathcal{A} \models_{\alpha[y \mapsto a, x \mapsto \alpha[y \mapsto a](y)]} G$ for all $a \in U^{\mathcal{A}}$

(Translation Lemma)

iff $\mathcal{A} \models_{\alpha[y \mapsto a, x \mapsto a]} G$ for all $a \in U^{\mathcal{A}}$

iff $\mathcal{A} \models_{\alpha[x \mapsto a]} G$ for all $a \in U^{\mathcal{A}}$

Renaming

$\int_0^\infty f(s)ds$ has the same value as $\int_0^\infty f(t)dt$

Renaming Lemma

Let $F = Qx[G]$ be a formula with $Q \in \{\exists, \forall\}$. Let y be a variable which does not appear in G. Then $\mathcal{A} \models_\alpha F$ iff $\mathcal{A} \models_\alpha Qy(G[y/x])$.

Assume $Q = \forall$.

$$
\begin{aligned}
\mathcal{A} \models_\alpha \forall y G[y/x] &\text{ iff } \mathcal{A} \models_{\alpha[y\mapsto a]} G[y/x] \text{ for all } a \in U^{\mathcal{A}} \\
&\text{ iff } \mathcal{A} \models_{\alpha[y\mapsto a, x\mapsto \alpha[y\mapsto a](y)]} G \text{ for all } a \in U^{\mathcal{A}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(Translation Lemma)} \\
&\text{ iff } \mathcal{A} \models_{\alpha[y\mapsto a, x\mapsto a]} G \text{ for all } a \in U^{\mathcal{A}} \\
&\text{ iff } \mathcal{A} \models_{\alpha[x\mapsto a]} G \text{ for all } a \in U^{\mathcal{A}} \\
&\text{ iff } \mathcal{A} \models_\alpha \forall x G
\end{aligned}
$$