CS 228 : Logic in Computer Science

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Normal Forms in FOL

Translation Lemma

Translation Lemma

If t is a term and F is a formula such that no variable in t occurs bound in F, then $\mathcal{A} \models_\alpha F[t/x]$ iff $\mathcal{A} \models_{\alpha[x \mapsto \alpha(t)]} F$.

$F[t/x]$ denotes substituting t for x in F, where x is free in F

- What if t contains a variable bound in F?
- Results in *Variable Capture*

Translation Lemma Proof : Optional

Proof by Induction on formulae.

- Base case. Atomic formulae $P(t_1, \ldots, t_k)$.
- $\mathcal{A} \models_\alpha P(t_1, \ldots, t_k)[t/x]$ iff $\mathcal{A} \models_\alpha P(t_1[t/x], \ldots, t_k[t/x])$.
- Show that $\mathcal{A} \models_{\alpha[x \mapsto \alpha(t)]} P(t_1, \ldots, t_k)$.
  - Base Cases within : $t_i = c$, $t_i = y$ for $y \neq x$, $t_i = x$ for each t_i.
  - Case $t_i = f(s_1, \ldots, s_j)$ for a function f.
  - $f(s_1, \ldots, s_j)[t/x] = f(s_1[t/x], \ldots, s_j[t/x])$
- $\mathcal{A} \models_\alpha P(t_1[t/x], \ldots, t_k[t/x])$ iff $(\ \alpha(t_1[t/x]), \ldots, \alpha(t_k[t/x])\ ) \in P^{\mathcal{A}}$
- iff $(\ \alpha([x \mapsto \alpha(t)](t_1), \ldots, \alpha([x \mapsto \alpha(t)](t_k)\ ) \in P^{\mathcal{A}}$
- iff $\mathcal{A} \models_{\alpha[x \mapsto \alpha(t)]} P(t_1, \ldots, t_k)$
- Cases for formulae with propositional connectives is routine.
- Case with quantifier, $\forall y F[t/x]$, $\exists y F[t/x]$ where $y \neq x$.

Renaming

$\int_0^\infty f(s)ds$ has the same value as $\int_0^\infty f(t)dt$

Renaming Lemma

Let $F = Qx[G]$ be a formula with $Q \in \{\exists, \forall\}$. Let y be a variable which does not appear in G. Then $\mathcal{A} \models_\alpha F$ iff $\mathcal{A} \models_\alpha Qy(G[y/x])$.

Assume $Q = \forall$.
$\mathcal{A} \models_\alpha \forall y G[y/x]$ iff $\mathcal{A} \models_{\alpha[y \mapsto a]} G[y/x]$ for all $a \in U^{\mathcal{A}}$

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iff $\mathcal{A} \models_{\alpha[y \mapsto a, x \mapsto \alpha[y \mapsto a](y)]} G$ for all $a \in U^{\mathcal{A}}$

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\mathcal{A} \models_\alpha \forall y G[y/x] & \text{ iff } \mathcal{A} \models_{\alpha[y \mapsto a]} G[y/x] \text{ for all } a \in U^{\mathcal{A}} \\
& \text{ iff } \mathcal{A} \models_{\alpha[y \mapsto a, x \mapsto \alpha[y \mapsto a](y)]} G \text{ for all } a \in U^{\mathcal{A}} \\
& \qquad\qquad\qquad\qquad \text{(Translation Lemma)} \\
& \text{ iff } \mathcal{A} \models_{\alpha[y \mapsto a, x \mapsto a]} G \text{ for all } a \in U^{\mathcal{A}} \\
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\end{aligned}$$

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\begin{aligned}
\mathcal{A} \models_\alpha \forall y G[y/x] &\text{ iff } \mathcal{A} \models_{\alpha[y\mapsto a]} G[y/x] \text{ for all } a \in U^{\mathcal{A}}\\
&\text{ iff } \mathcal{A} \models_{\alpha[y\mapsto a, x\mapsto \alpha[y\mapsto a](y)]} G \text{ for all } a \in U^{\mathcal{A}}\\
&\qquad\qquad\qquad\qquad\text{(Translation Lemma)}\\
&\text{ iff } \mathcal{A} \models_{\alpha[y\mapsto a, x\mapsto a]} G \text{ for all } a \in U^{\mathcal{A}}\\
&\text{ iff } \mathcal{A} \models_{\alpha[x\mapsto a]} G \text{ for all } a \in U^{\mathcal{A}}\\
&\text{ iff } \mathcal{A} \models_\alpha \forall x G
\end{aligned}
$$

Rectified Formulae

A FOL formula is *rectified* if no variable occurs both free and bound, and if all quantifiers in the formula refer to different variables.

$$\forall x \exists y P(x, f(y)) \land \forall y (Q(x, y) \lor R(x))$$

is not rectified. By renaming we obtain an equivalent rectified formula

$$\forall u \exists v P(u, f(v)) \land \forall y (Q(x, y) \lor R(x))$$

By Renaming Lemma, we can always obtain an equivalent rectified formula by renaming bound variables.

Rename the bound variables and get the rectified variables.

Prenex Normal Form

A formula is in prenex normal form if it can be written as

$$Q_1 x_1 Q_2 x_2 \dots Q_n x_n F$$

where $Q_i \in \{\forall, \exists\}$, $n \geqslant 0$ and F has no quantifiers. F is called the matrix of the formula.

Prenex Normal Form : Example

Convert the rectified formula $\neg(\exists x P(x, y) \vee \forall z Q(z)) \wedge \exists w Q(w)$ to Prenex Normal Form

- $(\neg\exists x P(x, y) \wedge \neg\forall z Q(z)) \wedge \exists w Q(w)$
- $(\forall x \neg P(x, y) \wedge \exists z \neg Q(z)) \wedge \exists w Q(w)$
- $\forall x \exists z (\neg P(x, y) \wedge \neg Q(z)) \wedge \exists w Q(w)$
- $\forall x \exists z \exists w ((\neg P(x, y) \wedge \neg Q(z)) \wedge Q(w))$
- Note that we have used the equivalences from the last lecture

Rectified, Prenex normal form (RPF)

- Given a rectified formula, we can use the equivalences from the last lecture to convert F into rectified, prenex normal form, by "pushing all quantifiers up front".
- Otherwise, rectify the formula first, and then convert to prenex normal form.

Every formula is equivalent to a rectified formula in prenex normal form.

Skolemisation

A formula in RPF is in *Skolem form* if it has no occurrences of the existential quantifier.

We can transform any formula in RPF to an equisatisfiable formula in Skolem form by using extra function symbols.

- $\forall x \exists y P(x, y)$ is equisatisfiable with $\forall x P(x, f(x))$.
- $\forall x \forall z \exists y P(x, y, z)$ is equisatisfiable with $\forall x \forall z P(x, f(x, z), z)$.
- $\exists x \forall y G(x, y)$ is equisatisfiable with $\forall y G(c, y)$ where c is a constant.
- $\exists x \forall y \exists z \exists w G(x, y, z, w)$ is equisatisfiable with $\forall y G(c, y, f(y), g(y))$ where c is a constant.

Skolemisation

Skolem Lemma

Let $F = \forall y_1 \forall y_2 \ldots \forall y_n \exists z G$ be in RPF. Given a function symbol f of arity n which does not appear in F, write

$$F' = \forall y_1 \forall y_2 \ldots \forall y_n G[f(y_1, \ldots, y_n)/z]$$

Then F and F' are equisatisfiable.

Assume F is satisfiable. Let $\mathcal{A} \models_\alpha F$.

- Extend structure $\mathcal{A}$ with an interpretation for a function f such that $\mathcal{A}' \models_{\alpha'} F'$.
- Given $a_1, \ldots, a_n \in U^\mathcal{A}$, choose $a \in U^\mathcal{A}$ such that $\mathcal{A} \models_{\alpha[y_1 \mapsto a_1, \ldots, y_n \mapsto a_n, z \mapsto a]} G$, and define $f^{\mathcal{A}'}(a_1, \ldots, a_n) = a$.
- f does not appear in G, $\mathcal{A}' \models_{\alpha[y_1 \mapsto a_1, \ldots, y_n \mapsto a_n, z \mapsto f^{\mathcal{A}'}(a_1, \ldots, a_n)]} G$,
- By Translation Lemma, $\mathcal{A}' \models_{\alpha[y_1 \mapsto a_1, \ldots, y_n \mapsto a_n]} G[f(y_1, \ldots, y_n)/z]$
- Since this holds for any $a_1, \ldots, a_n \in U^\mathcal{A}$, $\mathcal{A}' \models \forall y_1 \forall y_2 \ldots \forall y_n G[f(y_1, \ldots, y_n)/z]$

Skolemisation : Example

$\forall x \exists y \forall z \exists w (\neg P(a, w) \vee Q(f(x), y))$

- By Skolem Lemma, eliminate $\exists y$ and introduce a new function g, obtaining $\forall x \forall z \exists w (\neg P(a, w) \vee Q(f(x), g(x)))$
- By Skolem Lemma, eliminate $\exists w$ introducing a new function h obtaining $\forall x \forall z (\neg P(a, h(x, z)) \vee Q(f(x), g(x)))$

Conversion to Skolem Form : Summary

Convert an arbitrary FOL formula to an equisatisfiable formula in Skolem formula as follows:

1. Rectify F systematically renaming bound variables, obtaining an equivalent formula F_1
2. Use the equivalences in the beginning and move all quantifiers outside, yielding an equivalent formula F_2 in prenex normal form
3. Repeatedly eliminate the outermost existential quantifier in F_2 until an equisatisfiable formula F_3 is obtained in Skolem form.

Semi Decidability of Satisfiability

- Given a FOL formula in Skolem normal form, if F is unsatisfiable, there is a technique of *Ground Resolution* which gives $\perp$ and terminates.
- However, if F is satisfiable, then this process may go on forever.
- Validity is *semi decidable* : a valid formula F has a finite witness of its validity, namely, a finite resolution refutation for $\neg F$.
- If F is not valid, and satisfiable, then there may not be a finite witness.
- This is for general FOL : however, we can focus on FOL over some special signatures where satisfiability is decidable.