CS 228 : Logic in Computer Science

Krishna. S

Recap

Signatures, Formulae over signatures, Structure for a signature

Example of Satisfaction

- $\mathcal{G} = (\{1, 2, 3\}, E^{\mathcal{G}} = \{(1, 2), (2, 1), (2, 3), (3, 2)\})$
  - For any assignment α, $\mathcal{G} \models_\alpha \forall x \forall y(E(x, y) \to E(y, x))$ iff for every $a, b \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a, y \mapsto b]} (E(x, y) \to E(y, x))$

Example of Satisfaction

- $\mathcal{G} = (\{1,2,3\}, E^{\mathcal{G}} = \{(1,2),(2,1),(2,3),(3,2)\})$
  - For any assignment α, $\mathcal{G} \models_\alpha \forall x \forall y (E(x,y) \to E(y,x))$ iff for every $a, b \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a, y \mapsto b]} (E(x,y) \to E(y,x))$
  - There is an assignment α which satisfies $\mathcal{G} \models_\alpha \exists x (E(x,y) \wedge E(x,z) \wedge y \neq z)$

What about Y and Z, they are not bound.

Assignment alpha is taking care of that.

Example of Satisfaction

- $\mathcal{G} = (\{1,2,3\}, E^{\mathcal{G}} = \{(1,2),(2,1),(2,3),(3,2)\})$
  - For any assignment α, $\mathcal{G} \models_\alpha \forall x \forall y (E(x,y) \rightarrow E(y,x))$ iff for every $a, b \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a, y \mapsto b]} (E(x,y) \rightarrow E(y,x))$
  - There is an assignment α which satisfies $\mathcal{G} \models_\alpha \exists x (E(x,y) \wedge E(x,z) \wedge y \neq z)$
  - There is no assignment α which satisfies $\exists x \forall y (E(x,y))$
- $\mathcal{W} = abaaa$

Example of Satisfaction

- $\mathcal{G} = (\{1,2,3\}, E^{\mathcal{G}} = \{(1,2),(2,1),(2,3),(3,2)\})$
  - For any assignment α, $\mathcal{G} \models_{\alpha} \forall x \forall y (E(x,y) \rightarrow E(y,x))$ iff for every $a, b \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a, y \mapsto b]} (E(x,y) \rightarrow E(y,x))$
  - There is an assignment α which satisfies $\mathcal{G} \models_{\alpha} \exists x (E(x,y) \wedge E(x,z) \wedge y \neq z)$
  - There is no assignment α which satisfies $\exists x \forall y (E(x,y))$
- $\mathcal{W} = abaaa$
  - There is an assignment α for which $\mathcal{W} \models_{\alpha} (Q_a(x) \wedge Q_a(y) \wedge S(x,y))$

consecutive a's

Q_a(X) means a at position X, and same for Y.

Example of Satisfaction

- $\mathcal{G} = (\{1,2,3\}, E^{\mathcal{G}} = \{(1,2),(2,1),(2,3),(3,2)\})$
  - For any assignment α, $\mathcal{G} \models_{\alpha} \forall x \forall y(E(x,y) \rightarrow E(y,x))$ iff for every $a, b \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a, y \mapsto b]} (E(x,y) \rightarrow E(y,x))$
  - There is an assignment α which satisfies $\mathcal{G} \models_{\alpha} \exists x(E(x,y) \wedge E(x,z) \wedge y \neq z)$
  - There is no assignment α which satisfies $\exists x \forall y(E(x,y))$
- $\mathcal{W} = abaaa$
  - There is an assignment α for which $\mathcal{W} \models_{\alpha} (Q_a(x) \wedge Q_a(y) \wedge S(x,y))$
  - There is no assignment α which satisfies $\exists x \exists y(Q_b(x) \wedge Q_b(y) \wedge x \neq y)$

There are two b's in the word.

Satisfiability, Validity and Equivalence

- A formula φ over a signature τ is said to be satisfiable iff for some τ-structure $\mathcal{A}$ and assignment α, $\mathcal{A} \models_\alpha \varphi$

Satisfiability, Validity and Equivalence

- A formula φ over a signature τ is said to be satisfiable iff for some τ-structure $\mathcal{A}$ and assignment α, $\mathcal{A} \models_\alpha \varphi$
- A formula φ over a signature τ is said to be valid iff for every τ-structure $\mathcal{A}$ and assignment α, $\mathcal{A} \models_\alpha \varphi$

Satisfiability, Validity and Equivalence

- A formula φ over a signature τ is said to be satisfiable iff for some τ-structure $\mathcal{A}$ and assignment α, $\mathcal{A} \models_\alpha \varphi$
- A formula φ over a signature τ is said to be valid iff for every τ-structure $\mathcal{A}$ and assignment α, $\mathcal{A} \models_\alpha \varphi$
- Formulae $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are equivalent denoted $\varphi \equiv \psi$ iff for every $\mathcal{A}$ and α, $\mathcal{A} \models_\alpha \varphi$ iff $\mathcal{A} \models_\alpha \psi$

Equisatisfiability

Predicate, requires specific

Closed formula, asserts existence of at least one

Let $\varphi_1(x) = \forall y R(x, y)$ and $\varphi_2 = \exists x \forall y R(x, y)$.

x here is free, can assign value.

x here is not free, can't assign value.

- It is clear that whenever $\mathcal{A} \models \varphi_2$, one can find an assignment α such that $\mathcal{A} \models_\alpha \varphi_1(x)$.
- Likewise, if $\mathcal{A} \models_\alpha \varphi_1(x)$, then $\mathcal{A} \models \varphi_2$.
- Thus, $\varphi_1(x), \varphi_2$ are equisatisfiable.

True or False?

For a formula φ and assignments α_1 and α_2 such that for every $x \in \mathit{free}(\varphi)$, $\alpha_1(x) = \alpha_2(x)$, $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

True or False?

For a formula φ and assignments α_1 and α_2 such that for every $x \in \mathit{free}(\varphi)$, $\alpha_1(x) = \alpha_2(x)$, $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

- For example, $\varphi(y) = \forall x(R(x, y) \rightarrow \forall z P(z))$

True or False?

For a formula φ and assignments α_1 and α_2 such that for every $x \in \mathit{free}(\varphi)$, $\alpha_1(x) = \alpha_2(x)$, $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

- For example, $\varphi(y) = \forall x(R(x, y) \rightarrow \forall z P(z))$
- Consider two assignments α_1, α_2 such that $\alpha_1(y) = \alpha_2(y) = \alpha(\mathit{say})$

True or False?

For a formula φ and assignments α_1 and α_2 such that for every $x \in \mathit{free}(\varphi)$, $\alpha_1(x) = \alpha_2(x)$, $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

- For example, $\varphi(y) = \forall x(R(x, y) \to \forall z P(z))$
- Consider two assignments α_1, α_2 such that $\alpha_1(y) = \alpha_2(y) = \alpha(\mathit{say})$
- Evaluate for all $a, b \in u(\mathcal{A})$, $R(a, \alpha) \to P(b)$

True or False?

For a formula φ and assignments α_1 and α_2 such that for every $x \in \mathit{free}(\varphi)$, $\alpha_1(x) = \alpha_2(x)$, $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

- For example, $\varphi(y) = \forall x(R(x, y) \to \forall z P(z))$
- Consider two assignments α_1, α_2 such that $\alpha_1(y) = \alpha_2(y) = \alpha(\mathit{say})$
- Evaluate for all $a, b \in u(\mathcal{A})$, $R(a, \alpha) \to P(b)$
- $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

True or False?

For a sentence φ, and any two assignments α_1 and α_2, $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

True or False?

For a sentence φ, and any two assignments α_1 and α_2, $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

No free variables!

Sentence asserts no free variable that means, we can't give any assignment to them.

Check Satisfiability

Let τ be a signature with a single unary relation P. Consider the structure $\mathcal{A} = (U_{\mathcal{A}} = \{0, 1\}, P^{\mathcal{A}} = \{1\})$.

Let $\varphi = \forall x_1 \forall x_2 \dots \forall x_n (P(x_1) \rightarrow (P(x_2) \rightarrow (P(x_3) \cdots \rightarrow (P(x_n) \rightarrow P(x_1))) \dots)))$.

assign x1,...,xn-1 = 1 and xn = 0

Does $\mathcal{A} \models \varphi$?

Nope.

Check Satisfiability

for all z, for which the edge relation holds.

Let $\varphi(y) = \exists x(E(x,y) \wedge \neg(y = x) \wedge \forall z[E(z,y) \rightarrow z = x])$ over the signature τ containing a binary relation E. Is $\varphi(y)$ satisfiable under some graph structure?

Satisfiable.