CS 228 : Logic in Computer Science

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- Intuitively, $p \rightarrow q \vdash \neg p \vee q$ makes sense because you think semantically. However, we never used any semantics so far.
- Now we show that whatever can be proved makes sense semantically too.

Semantics

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Soundness of Propositional Logic

$\varphi_1, \ldots, \varphi_n \vdash \psi \Rightarrow \varphi_1, \ldots, \varphi_n \models \psi$

Whenever ψ can be proved from $\varphi_1, \ldots, \varphi_n$, then ψ evaluates to true whenever $\varphi_1, \ldots, \varphi_n$ evaluate to true

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- Consider now a proof with k lines.

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- By inductive hypothesis, we have $\varphi_1, \ldots, \varphi_n \models \psi_1$ and $\varphi_1, \ldots, \varphi_n \models \psi_2$. By semantics, we have $\varphi_1, \ldots, \varphi_n \models \psi_1 \wedge \psi_2$.

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We are inducting on the structure of the proof.

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- Assume ψ was obtained using $\rightarrow i$. Then ψ is of the form $\psi_1 \rightarrow \psi_2$.
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- The line just after the box was ψ.
- Consider adding ψ_1 in the premises along with $\varphi_1, \ldots, \varphi_n$. Then we will get a proof $\varphi_1, \ldots, \varphi_n, \psi_1 \vdash \psi_2$, of length $k - 1$. By inductive hypothesis, $\varphi_1, \ldots, \varphi_n, \psi_1 \models \psi_2$. By semantics, this is same as $\varphi_1, \ldots, \varphi_n \models \psi_1 \rightarrow \psi_2$

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- The equivalence of $\varphi_1, \ldots, \varphi_n \vdash \psi_1 \rightarrow \psi_2$ and $\varphi_1, \ldots, \varphi_n, \psi_1 \vdash \psi_2$ gives the proof.

Soundness : Other cases

Completeness

$\varphi_1, \ldots, \varphi_n \models \psi \Rightarrow \varphi_1, \ldots, \varphi_n \vdash \psi$

Whenever $\varphi_1, \ldots, \varphi_n$ semantically entail ψ, then ψ can be proved from $\varphi_1, \ldots, \varphi_n$. The proof rules are complete

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- Step 2: Show that $\vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow (\ldots(\varphi_n \rightarrow \psi)\ldots))$
- Step 3: Show that $\varphi_1, \ldots, \varphi_n \vdash \psi$

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- If $\not\models \varphi_1 \rightarrow (\varphi_2 \rightarrow (\ldots(\varphi_n \rightarrow \psi)\ldots))$, then ψ evaluates to false when all of $\varphi_1, \ldots, \varphi_n$ evaluate to true, a contradiction.

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- Hence, $\models \varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \psi)\ldots))$.

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- Using this insight, we have to give a proof of ψ.

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Truth Table to Proof

Let φ be a formula with variables $p_1, \ldots, p_n$. Let $\mathcal{T}$ be the truth table of φ, and let l be a line number in $\mathcal{T}$. Let $\hat{p}_i$ represent p_i if p_i is assigned true in line l, and let it denote $\neg p_i$ if p_i is assigned false in line l. Then

1. $\hat{p}_1, \ldots, \hat{p}_n \vdash \varphi$ if φ evaluates to true in line l
2. $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg\varphi$ if φ evaluates to false in line l