

A decorative blue crosshair consisting of a vertical line and a horizontal line intersecting in the upper-left quadrant of the slide.

CS 228 : Logic in Computer Science

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Recap



Signatures, Formulae over signatures, Structure for a signature

Example of Satisfaction

- ▶ $\mathcal{G} = (\{1, 2, 3\}, E^{\mathcal{G}} = \{(1, 2), (2, 1), (2, 3), (3, 2)\})$
 - ▶ For any assignment α , $\mathcal{G} \models_{\alpha} \forall x \forall y (E(x, y) \rightarrow E(y, x))$ iff for every $a, b \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a, y \mapsto b]} (E(x, y) \rightarrow E(y, x))$

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 - ▶ There is an assignment α which satisfies $\mathcal{G} \models_{\alpha} \exists x (E(x, y) \wedge E(x, z) \wedge y \neq z)$

What about Y and Z, they are not bound.

Assignment alpha is taking care of that.

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 - ▶ There is no assignment α which satisfies $\exists x \forall y (E(x, y))$
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 - ▶ There is an assignment α for which $\mathcal{W} \models_{\alpha} (Q_a(x) \wedge Q_a(y) \wedge S(x, y))$
consecutive a's

$Q_a(X)$ means a at position X, and same for Y.

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 - ▶ There is an assignment α for which $\mathcal{W} \models_{\alpha} (Q_a(x) \wedge Q_a(y) \wedge S(x, y))$
 - ▶ There is no assignment α which satisfies $\exists x \exists y (Q_b(x) \wedge Q_b(y) \wedge x \neq y)$
no two b's are there.

Satisfiability, Validity and Equivalence

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- ▶ A formula φ over a signature τ is said to be **valid** iff for every τ -structure \mathcal{A} and assignment α , $\mathcal{A} \models_{\alpha} \varphi$
- ▶ Formulae $\varphi(x_1, \dots, x_n)$ and $\psi(x_1, \dots, x_n)$ are **equivalent** denoted $\varphi \equiv \psi$ iff **for every \mathcal{A} and α** , $\mathcal{A} \models_{\alpha} \varphi$ iff $\mathcal{A} \models_{\alpha} \psi$

Equisatisfiability

Let $\varphi_1(x) = \forall y R(x, y)$ and $\varphi_2 = \exists x \forall y R(x, y)$.

- ▶ It is clear that whenever $\mathcal{A} \models \varphi_2$, one can find an assignment α such that $\mathcal{A} \models_{\alpha} \varphi_1(x)$.
- ▶ Likewise, if $\mathcal{A} \models_{\alpha} \varphi_1(x)$, then $\mathcal{A} \models \varphi_2$.
- ▶ Thus, $\varphi_1(x), \varphi_2$ are **equisatisfiable**.

True or False?

For a formula φ and assignments α_1 and α_2 such that for every $x \in \text{free}(\varphi)$, $\alpha_1(x) = \alpha_2(x)$, $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

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- ▶ Consider two assignments α_1, α_2 such that $\alpha_1(y) = \alpha_2(y) = \alpha(\text{say})$
- ▶ Evaluate for all $a, b \in u(\mathcal{A})$, $R(a, \alpha) \rightarrow P(b)$
- ▶ $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

True or False?

For a sentence φ , and any two assignments α_1 and α_2 , $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

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No free variables!

Check Satisfiability

Let τ be a signature with a single unary relation P . Consider the structure $\mathcal{A} = (U_{\mathcal{A}} = \{0, 1\}, P^{\mathcal{A}} = \{1\})$.

Let $\varphi = \forall x_1 \forall x_2 \dots \forall x_n (P(x_1) \rightarrow (P(x_2) \rightarrow (P(x_3) \dots \rightarrow (P(x_n) \rightarrow P(x_1))) \dots)))$.

Does $\mathcal{A} \models \varphi$?

Check Satisfiability

Let $\varphi(y) = \exists x(E(x, y) \wedge \neg(y = x) \wedge \forall z[E(z, y) \rightarrow z = x])$ over the signature τ containing a binary relation E . Is $\varphi(y)$ satisfiable under some graph structure?

No it is not satisfiable.