CS 228 : Logic in Computer Science

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Resolution

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- How does a solver do it?
- Assume it is in CNF

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- Let C_1, C_2 be two clauses. Assume $p \in C_1$ and $\neg p \in C_2$ for some literal p. Then the clause $R = (C_1 - \{p\}) \cup (C_2 - \{\neg p\})$ is a resolvent of C_1 and C_2.
- Let $C_1 = \{p_1, \neg p_2, p_3\}$ and $C_2 = \{p_2, \neg p_3, p_4\}$. As $p_3 \in C_1$ and $\neg p_3 \in C_2$, we can find the resolvent. The resolvent is $\{p_1, p_2, \neg p_2, p_4\}$.
- Resolvent not unique : $\{p_1, p_3, \neg p_3, p_4\}$ is also a resolvent.

3 rules in Resolution

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- Let F be a formula in CNF, and let C be a clause in F. Then $F \vdash C$ (Prove!)
- Let F be a formula in CNF. Let R be a resolvent of two clauses of F. Then $F \vdash R$ (Prove!)

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Show that resolution can be used to determine whether any given formula is unsatisfiable.

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- $Res^0(F) = F$, there are finitely many clauses that can be derived from F.
- There is some $m \geqslant 0$ such that $Res^m(F) = Res^{m+1}(F)$. Denote it by $Res^*(F)$.

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Let $F = \{\{p_1, p_2, \neg p_3\}, \{\neg p_2, p_3\}\}$.

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- $Res^2(F) = Res^1(F) \cup \{p_1, p_2, \neg p_3\} \cup \{p_1, p_3, \neg p_2\}$

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- Since $\emptyset \notin Res^0(F)$ ($\emptyset$ is not a clause), there is an $m > 0$ such that $\emptyset \notin Res^m(F)$ and $\emptyset \in Res^{m+1}(F)$.
- Then $\{p\}, \{\neg p\} \in Res^m(F)$. By the rules of resolution, we have $F \vdash p, \neg p$, and thus $F \vdash \bot$. Hence, F is unsatisfiable.

Resolution

Prove the converse: F is unsatisfiable implies $\emptyset \in \mathit{Res}^*(F)$.

(Discuss substitution before the proof)

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- If $F = \{\{p\}\}$ or $F = \{\{\neg p\}\}$, F is satisfiable.
- Hence, $F = \{\{p\}, \{\neg p\}\}$. Clearly, $\emptyset \in Res(F)$.

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- Clauses in F= Clauses in $G_0 \cup$ Clauses in G_1

- Let $F_0 = \{C_i - \{p_{n+1}\} \mid C_i \in G_0\}$
- Let $F_1 = \{C_i - \{\neg p_{n+1}\} \mid C_i \in G_1\}$

Resolution

Let $F = \{\{p_1, p_3\}, \{p_2\}, \{\neg p_1, \neg p_2, p_3\}, \{\neg p_2, \neg p_3\}\}$ and $n = 2$.

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- $F_0 = \{\{p_1\}, \{p_2\}, \{\neg p_1, \neg p_2\}\}$ and $F_1 = \{\{p_2\}, \{\neg p_2\}\}$
- If $p_{n+1} = \mathit{false}$ in F, then F is equisatisfiable with F_0

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- If $p_{n+1} = \mathit{false}$ in F, then F is equisatisfiable with F_0
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- Hence F is satisfiable iff $F_0 \vee F_1$ is.

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- $F_0 = \{\{p_1\}, \{p_2\}, \{\neg p_1, \neg p_2\}\}$ and $F_1 = \{\{p_2\}, \{\neg p_2\}\}$

- If $p_{n+1} = \mathit{false}$ in F, then F is equisatisfiable with F_0
- If $p_{n+1} = \mathit{true}$ in F, then F is equisatisfiable with F_1
- Hence F is satisfiable iff $F_0 \vee F_1$ is.
- As F is unsatisfiable, F_0 and F_1 are both unsatisfiable.

Resolution

- By induction hypothesis, $\emptyset \in \mathit{Res}^*(F_0)$ and $\emptyset \in \mathit{Res}^*(F_1)$.

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- By induction hypothesis, $\emptyset \in \mathit{Res}^*(F_0)$ and $\emptyset \in \mathit{Res}^*(F_1)$.
- Hence, $\emptyset \in \mathit{Res}^*(G_0)$ or $\{p_{n+1}\} \in \mathit{Res}^*(G_0)$, and $\emptyset \in \mathit{Res}^*(G_1)$ or $\{\neg p_{n+1}\} \in \mathit{Res}^*(G_1)$.

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- Hence, $\emptyset \in Res^*(G_0)$ or $\{p_{n+1}\} \in Res^*(G_0)$, and $\emptyset \in Res^*(G_1)$ or $\{\neg p_{n+1}\} \in Res^*(G_1)$.
- If $\emptyset \in Res^*(G_0)$ or $\emptyset \in Res^*(G_1)$, then $\emptyset \in Res^*(F)$.

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- Hence, $\emptyset \in Res^*(G_0)$ or $\{p_{n+1}\} \in Res^*(G_0)$, and $\emptyset \in Res^*(G_1)$ or $\{\neg p_{n+1}\} \in Res^*(G_1)$.
- If $\emptyset \in Res^*(G_0)$ or $\emptyset \in Res^*(G_1)$, then $\emptyset \in Res^*(F)$.
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- Hence, $\emptyset \in Res^*(G_0)$ or $\{p_{n+1}\} \in Res^*(G_0)$, and $\emptyset \in Res^*(G_1)$ or $\{\neg p_{n+1}\} \in Res^*(G_1)$.
- If $\emptyset \in Res^*(G_0)$ or $\emptyset \in Res^*(G_1)$, then $\emptyset \in Res^*(F)$.
- Else, $\{p_{n+1}\} \in Res^*(G_0)$ and $\{\neg p_{n+1}\} \in Res^*(G_1)$.
- Hence $\emptyset \in Res^*(F)$.

This is the completeness of Resolution theorem.

Resolution Summary

Given a formula ψ, convert it into CNF, say ζ. ψ is satisfiable iff $\emptyset \notin \textit{Res}^*(\zeta)$.

- If ψ is unsat, we might get $\emptyset$ before reaching $\textit{Res}^*(\zeta)$.
- If ψ is sat, then truth tables are faster : stop when some row evaluates to 1.