Well-known discrete Random Variables.

Many slides from Chris Piech. CS109 in Stanford Univ

The Simplest Random Variable

- Bernoulli Random Variable   Boolean R.V.

$X \in \{0, 1\}$

Examples: ◦ coin-toss
◦ equipment will fail or not
◦ whether your crush will show up or not.

PMF $P(X=1) = p$

$P(X=0) = 1-p$

$E(X) = \sum_{x \in X} x \cdot P(X=x) = 0 \cdot (1-p) + 1 \cdot p = p$

$V(X) = \sum_{x \in X} (x - E(X))^2 P(X=x) = (0-p)^2 (1-p) + (1-p)^2 \cdot p$

$= p(1-p)$

Binomial Random Variable

$X \in \{0, 1, 2, \ldots n\}$

Imagine flipping a coin *n* times and counting the number of heads.

1. We will flip a coin ***n*** times: ***n* independent trials** of the same experiment
2. Each coin flip has a **probability *p*** of being heads
3. What we want to model: what is the probability of **exactly *k* heads**?

(This isn't really about flipping coins, though.)

Lots of scenarios fit the same description:

- # of 1's in randomly generated in length *n* bit string
- # of servers working in a large computer cluster
- # of people who vote for one of two candidates in an election
- # of jury members selected from a particular demographic

Binomial Random Variable

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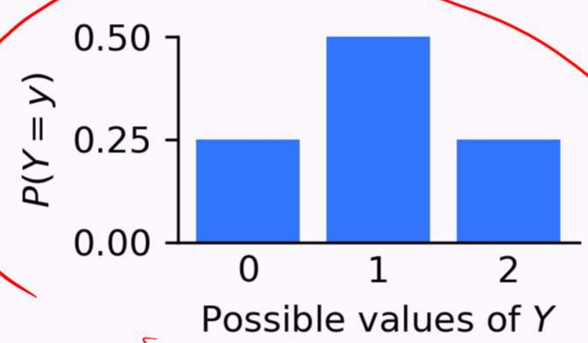
"Let Y be the # of heads in 2 coin flips."

- $P(Y = 0) = 1/4$ (T, T)
- $P(Y = 1) = 1/2$ (H, T), (T, H)
- $P(Y = 2) = 1/4$ (H, H)



This is the binomial for n = 2. Can we generalize from this?

Probability of Exactly k Heads in n Coin Flips

To start:

- Let's say we flip the coin 10 times. Probability of heads is p.
- For now, focus on the probability of 4 heads.

What is the probability of the outcome below?

(H, H, H, H, T, T, T, T, T, T)

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$$p^4(1-p)^6$$

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(H, H, H, T, H, T, T, T, T, T) $$p^4(1-p)^6$$

All of the outcomes with exactly 4 heads have the same probability

Probability of Exactly *k* Heads in *n* Coin Flips

```
(H, H, H, H, T, T, T, T, T, T)
(H, H, H, T, H, T, T, T, T, T)
(H, H, H, T, T, H, T, T, T, T)
(H, H, H, T, T, T, H, T, T, T)
(H, H, H, T, T, T, T, H, T, T)
(H, H, H, T, T, T, T, T, H, T)
(H, H, H, T, T, T, T, T, T, H)
(H, H, T, H, H, T, T, T, T, T)
(H, H, T, H, T, H, T, T, T, T)
(H, H, T, H, T, T, H, T, T, T)
(H, H, T, H, T, T, T, H, T, T)
(H, H, T, H, T, T, T, T, H, T)
(H, H, T, H, T, T, T, T, T, H)
(H, H, T, T, H, H, T, T, T, T)
(H, H, T, T, H, T, H, T, T, T)
(H, H, T, T, H, T, T, H, T, T)
(H, H, T, T, H, T, T, T, H, T)
(H, H, T, T, H, T, T, T, T, H)
```

Then, the probability of getting k heads in any ordering is the "**or**" of all of these **mutually exclusive** cases

How many cases are there?

Each outcome has probability $p^k(1-p)^{10-k}$

Probability of Exactly *k* Heads in *n* Coin Flips

(H, H, H, H, T, T, T, T, T, T)
(H, H, H, T, H, T, T, T, T, T)
(H, H, H, T, T, H, T, T, T, T)
(H, H, H, T, T, T, H, T, T, T)
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(H, H, T, T, H, T, T, T, T, H)

Then, the probability of getting k heads in any ordering is the "**or**" of all of these **mutually exclusive** cases

How many cases are there? $\binom{10}{k}$

Each outcome has probability $p^k(1-p)^{10-k}$

Probability of Exactly *k* Heads in *n* Coin Flips

(H, H, H, H, T, T, T, T, T, T)
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Then, the probability of getting k heads in any ordering is the "**or**" of all of these **mutually exclusive** cases

How many cases are there? $\binom{10}{k}$

Each outcome has probability $p^k(1-p)^{10-k}$

$$P(k \text{ heads}) = \binom{10}{k} p^k (1-p)^{10-k}$$

We Have Invented The Binomial

This type of random variable is so common it needs a name so that I can talk about it generally.

I shall call it: the ***Binomial*** *Random Variable. Huzzah.*

Jacob "James" Bernoulli (1654-1705): Swiss mathematician

One of many mathematicians in the Bernoulli family

Declaring a Random Variable to be Binomial

Our random variable

Num trials

Probability of success on each trial

$$X \sim \text{Bin}(n, p)$$

Is distributed as a

Binomial

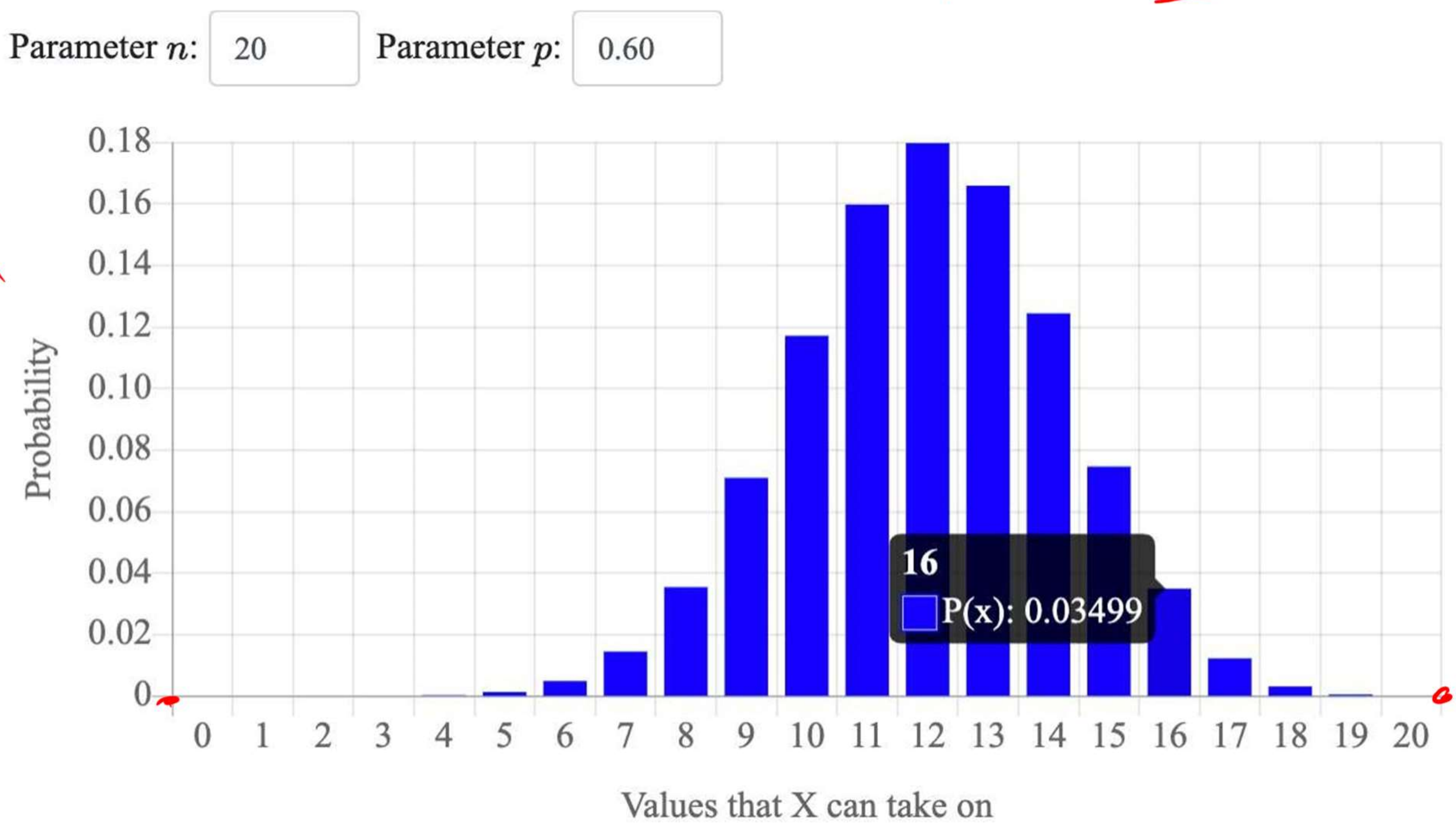
With these parameters

Then We Automatically Know the PMF!

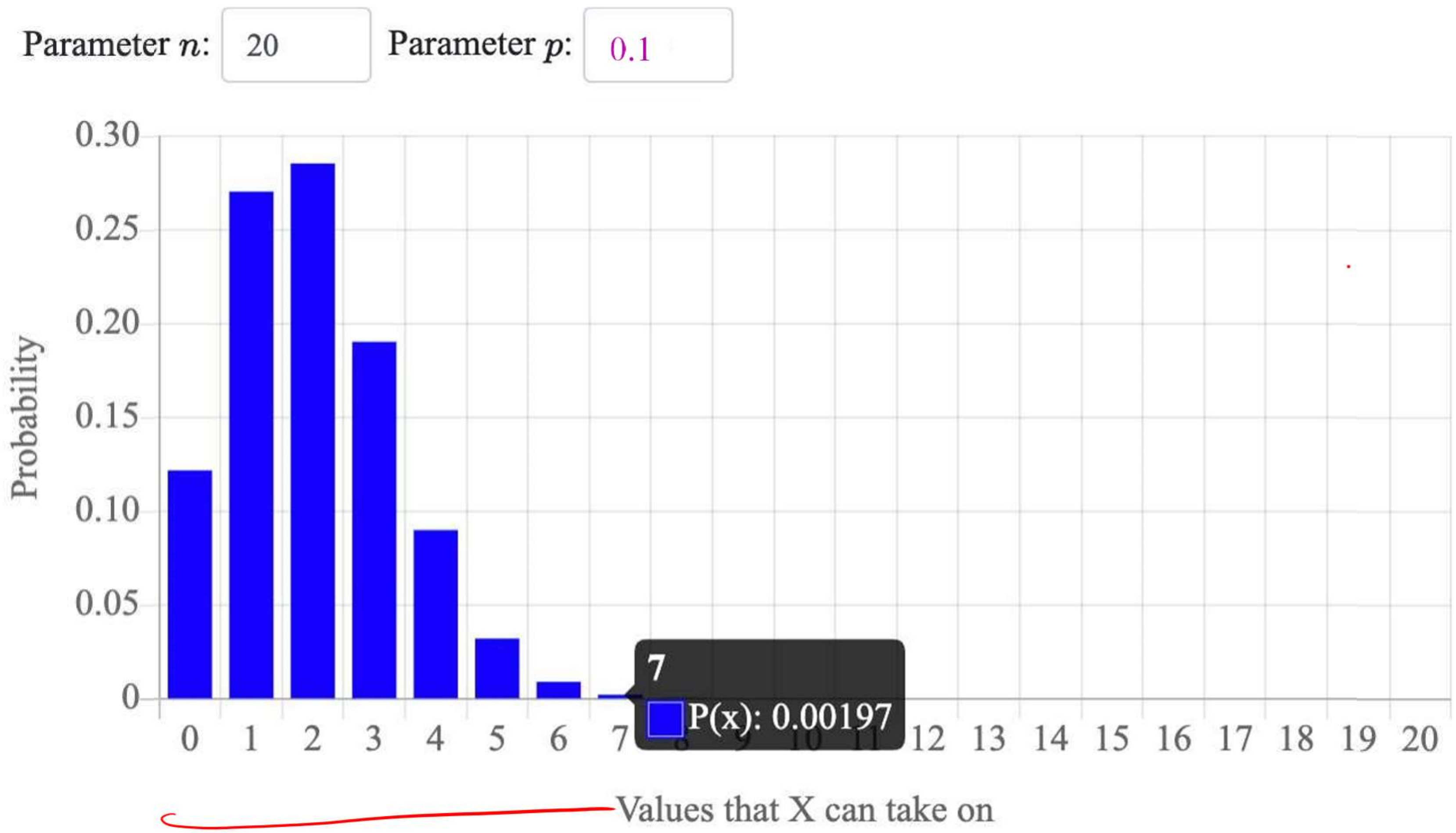
$$P(X = k) = \overbrace{\binom{n}{k} p^k (1-p)^{n-k}}^{\text{Probability Mass Function for a Binomial}}$$

Probability that our variable takes on the value k

The PMF as a Graph: $X \sim \text{Bin}(n = 20, p = 0.6)$

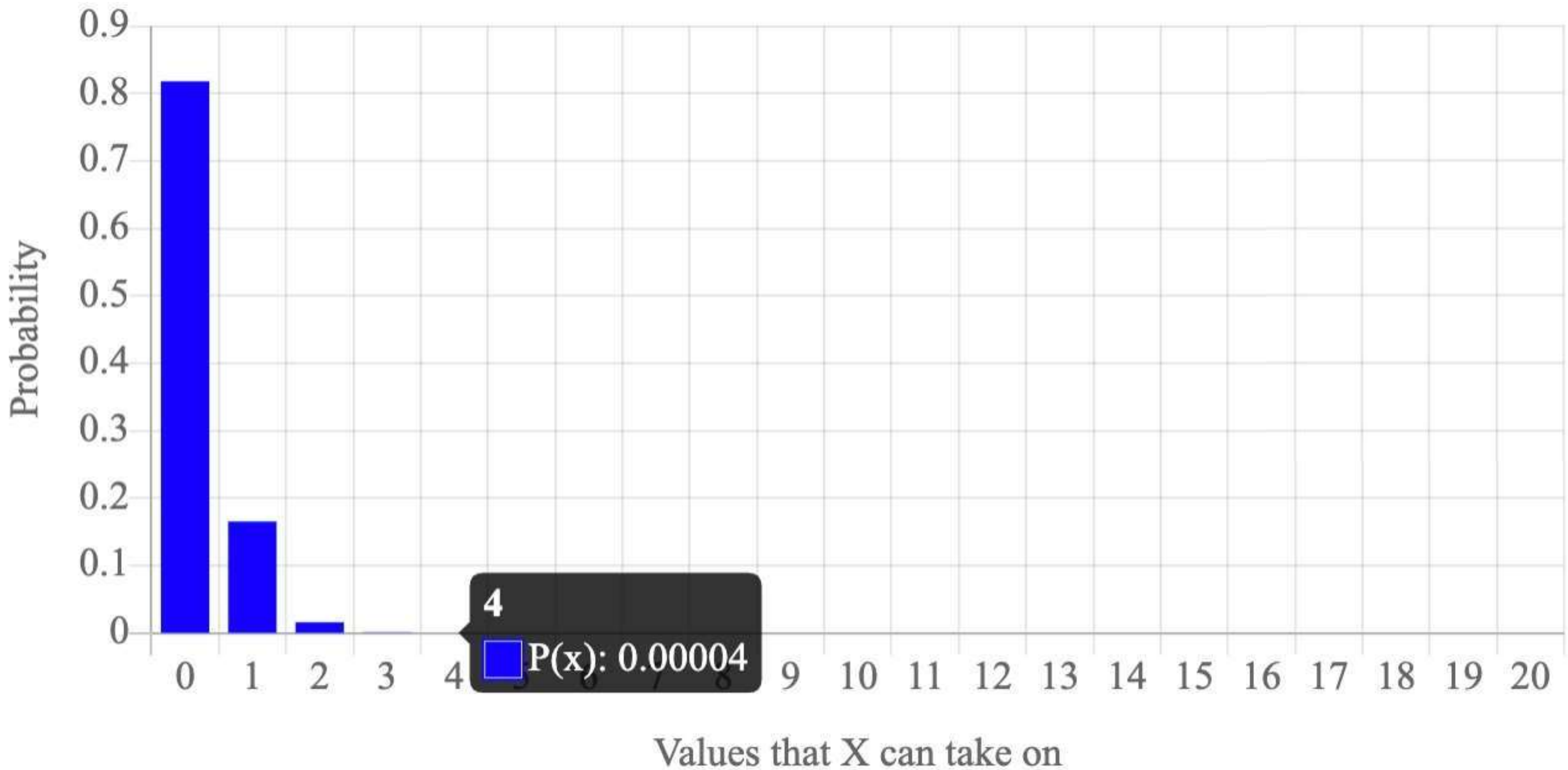


The PMF as a Graph: $X \sim \text{Bin}(n = 20, p = 0.1)$



The PMF as a Graph: $X \sim \text{Bin}(n = 20, p = 0.01)$

Parameter n: 20 Parameter p: 0.01



Probability of k Heads In n Flips: Now With Binomial

Three fair (p = 0.5 of heads) coins are flipped.

Let X be the number of heads.

$$X \sim \text{Bin}(n = 3, p = 0.5)$$

Probability of k Heads In n Flips: Now With Binomial

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What is the probability of…

… 0 heads?

… 1 heads?

… 2 heads?

… 3 heads?

Probability of k Heads In n Flips: Now With Binomial

Three fair (p = 0.5 of heads) coins are flipped.

Let X be the number of heads.

$$X \sim \text{Bin}(n = 3, p = 0.5)$$

What is the probability of…

… 0 heads?

$$P(X = 0) = \binom{3}{0} p^0 (1-p)^3 = \frac{1}{8}$$

… 1 heads?

$$P(X = 1) = \binom{3}{1} p^1 (1-p)^2 = \frac{3}{8}$$

… 2 heads?

$$P(X = 2) = \binom{3}{2} p^2 (1-p)^1 = \frac{3}{8}$$

… 3 heads?

$$P(X = 3) = \binom{3}{3} p^3 (1-p)^0 = \frac{1}{8}$$

Random Variable Sums

$n = 9$

The Binomial ...is a sum of Bernoulli random variables

Heads Tails Heads Tails Heads Heads Tails Tails Heads

1 2 3 4 5 6 7 8 9

Random Variable Sums

The Binomial ...is a sum of Bernoulli random variables

Let $X_1 \sim \text{Bern}(p = 1/2)$ and $X_2 \sim \text{Bern}(p = 1/2)$.

$$Y \sim \text{Bin}(n = 2, p = 1/2)$$

$$Y = X_1 + X_2$$

We Can Now Calculate Expectation of Binomial

$X \sim \text{Bin}(n, p)$

Let Y_i be 1 if trial i was a success, otherwise 0, with i from 1 to n. $Y_i \sim \text{Bern}(p)$.

The Binomial ...is a sum of Bernoulli random variables

Heads Tails Heads Tails Heads Heads Tails Tails Heads

Y_1 Y_2 Y_3 Y_9

We Can Now Calculate Expectation of Binomial

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Let Y_i be 1 if trial i was a success, otherwise 0, with i from 1 to n. $Y_i \sim \text{Bern}(p)$.

$$\mathrm{E}[X] = \mathrm{E}\left[\sum_{i=1}^{n} Y_i\right] \qquad \text{Since } X = \sum_{i=1}^{n} Y_i$$

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$$= \sum_{i=1}^{n} \mathrm{E}[Y_i] \qquad \text{Expectation of sum}$$

Expectation of a sum is the sum of expectations: $E[X + Y] = E[X] + E[Y]$

We Can Now Calculate Expectation of Binomial

$$X \sim \text{Bin}(n, p)$$

Let Y_i be 1 if trial i was a success, otherwise 0, with i from 1 to n. $Y_i \sim \text{Bern}(p)$.

$$\begin{aligned}
\mathrm{E}[X] &= \mathrm{E}\left[\sum_{i=1}^{n} Y_i\right] && \text{Since } X = \sum_{i=1}^{n} Y_i \\
&= \sum_{i=1}^{n} \mathrm{E}[Y_i] && \text{Expectation of sum} \\
&= \sum_{i=1}^{n} p && \text{Expectation of Bernoulli} \\
&= n \cdot p && \text{Sum } n \text{ times}
\end{aligned}$$

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&= \sum_{i=1}^{n} p && \text{Expectation of Bernoulli} \\
&= n \cdot p && \text{Sum } n \text{ times}
\end{aligned}$$

True for every binomial ever

Independent R.V.'s thus variance distributes over without any difficulty.

Variance of Binomial R.V.s

$$X = \sum_{i=1}^{n} Y_i \qquad Y_i \sim Bern(p)$$

$$Var(Y_i) = p(1-p)$$

$$Var(X) = Var\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} Var(Y_i)$$

If $Y_i \perp\!\!\!\perp Y_j \quad \forall\, i, j$

$$Var(X) = np(1-p)$$

Practice: Ad Clicks

Every day, Youtube shows a particular ad 1000 times.

Each ad served is clicked with p = 0.01 (otherwise it's ignored).

What is the probability of this ad getting 10 clicks?

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What is the probability of this ad getting 10 clicks?

Let $\boldsymbol{X}$ be the number of ad clicks. $\boldsymbol{X} \sim \mathrm{Bin}(n = 1000, p = 0.01)$.

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What is the probability of this ad getting 10 clicks?

Let X be the number of ad clicks. $X \sim \mathrm{Bin}(n = 1000, p = 0.01)$.

$$\boldsymbol{P}(X = k) = \binom{1000}{k}(0.01)^k(0.99)^{1000-k}$$

$$\boldsymbol{P}(X = 10) = \binom{1000}{10}(0.01)^{10}(0.99)^{990} \approx 0.125$$

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What is the probability of this ad getting **20** clicks?

Let X be the number of ad clicks. $X \sim \text{Bin}(n = 1000, p = 0.01)$.

$$P(X = k) = \binom{1000}{k}(0.01)^k(0.99)^{1000-k}$$

$$P(X = 20) = \binom{1000}{20}(0.01)^{20}(0.99)^{980} \approx 0.0018$$

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```
>>> from scipy import stats
>>> stats.binom.pmf(10, 1000, 0.01)
0.125740211262075
>>> stats.binom.pmf(20, 1000, 0.01)
0.0017918782400182195
```

k n p

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What is the probability of this ad getting **20** clicks?

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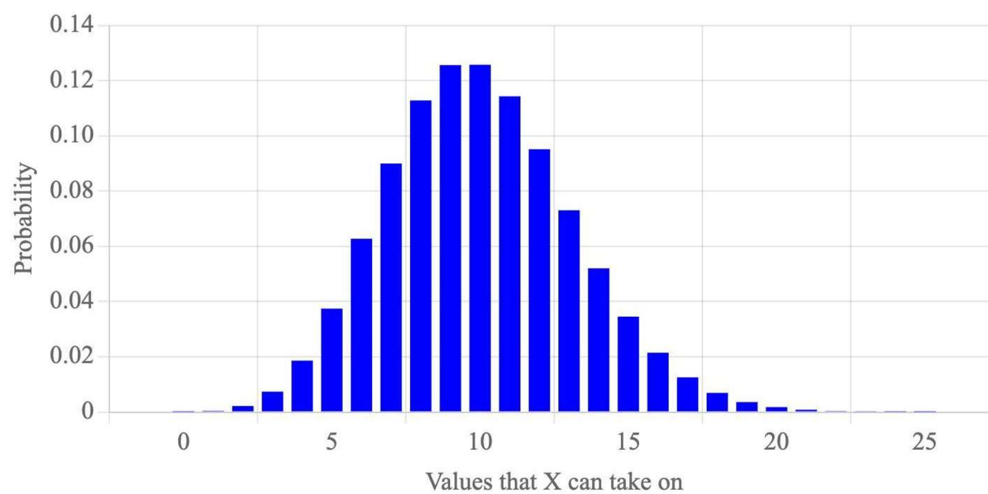
PMF graph:

Parameter n: 1000 Parameter p: 0.01



Server Redundancy

A network can remain functional as long as at least 2 out of 7 servers are alive.

The probability of any server working is 0.8.

What is the probability that less than 2 servers are alive?

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Server Redundancy

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Let $\boldsymbol{X}$ be the number of servers alive. $\boldsymbol{X} \sim \mathrm{Bin}(n = 7, p = 0.8)$.

$$P(X = k) = \binom{7}{k}(0.8)^k(0.2)^{7-k}$$

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$$P(X < 2) = P(X = 0) + P(X = 1)$$

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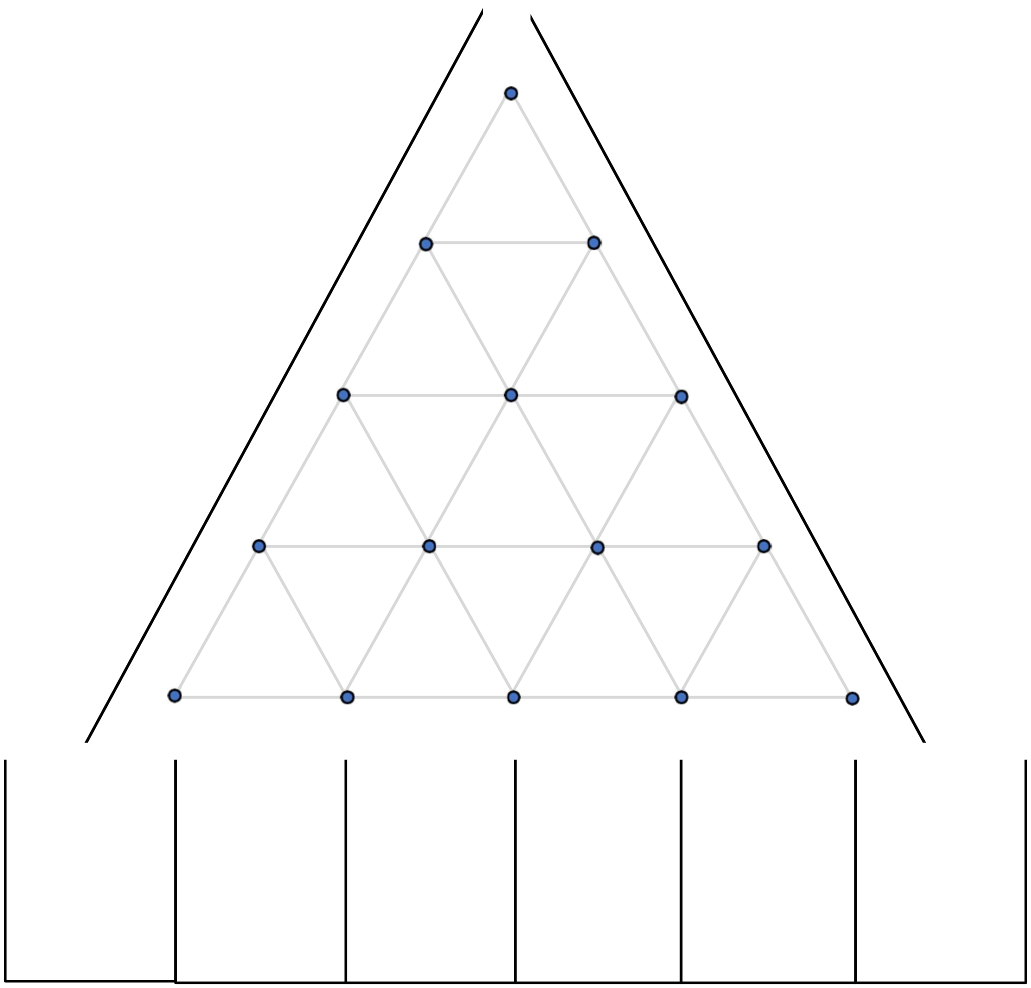
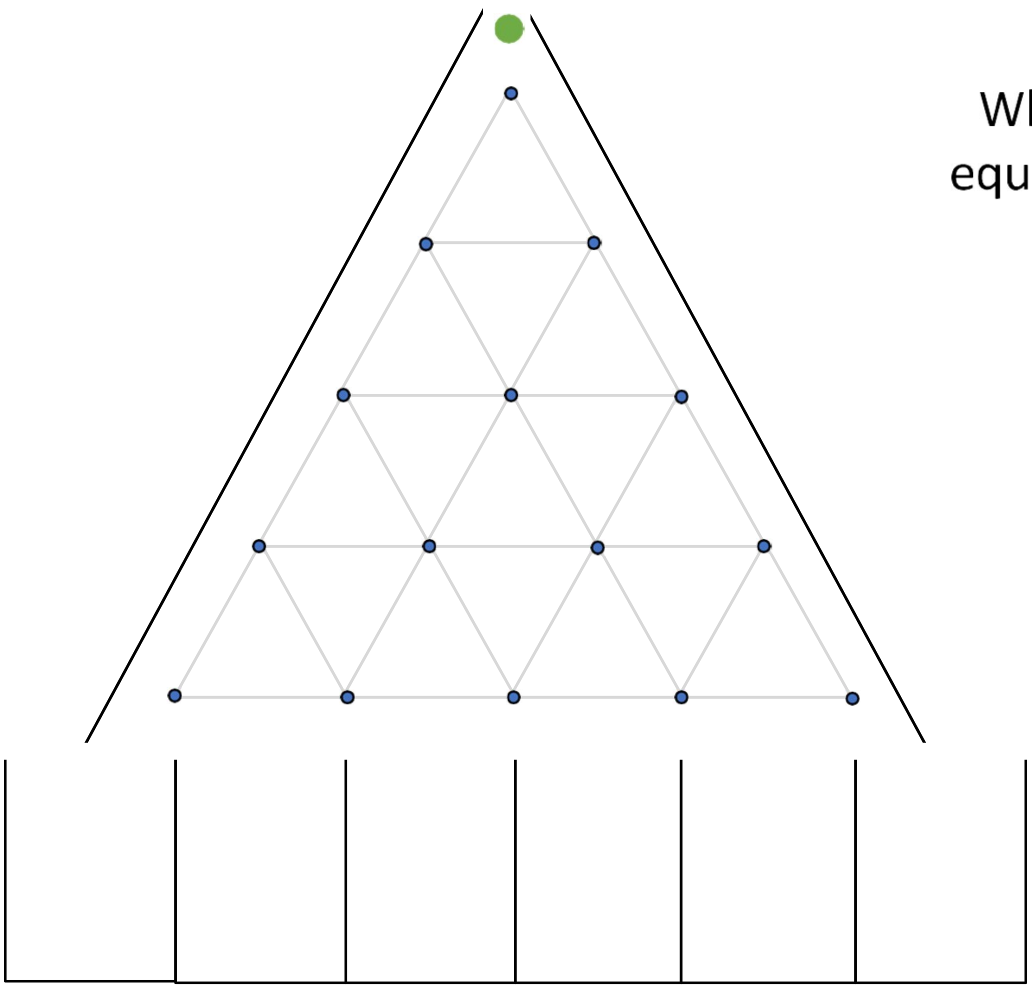
$$P(X = k) = \binom{7}{k}(0.8)^k(0.2)^{7-k}$$

$$P(X < 2) = P(X = 0) + P(X = 1) = \binom{7}{0}(0.8)^0(0.2)^{7-0} + \binom{7}{1}(0.8)^1(0.2)^{7-1} \approx \underline{0.0004}$$

Galton Board Fun

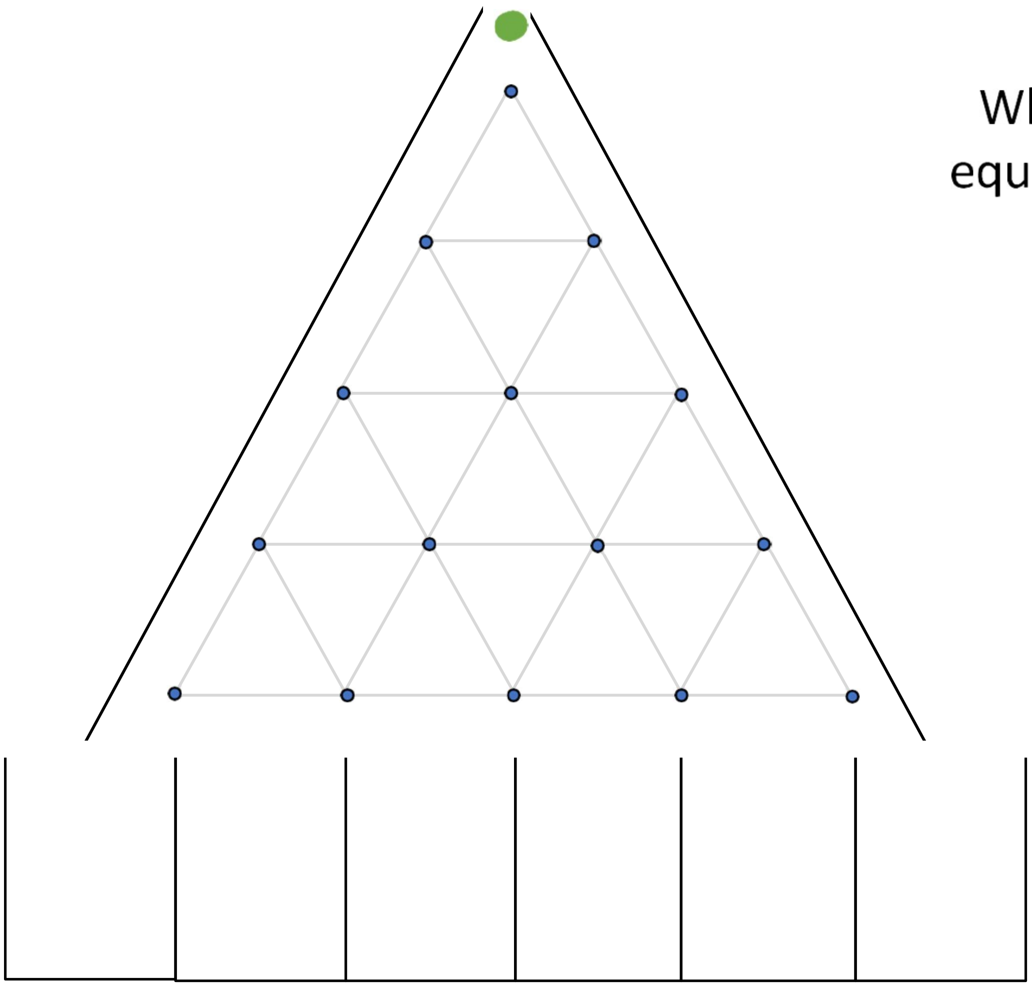


Galton Board Fun



When a marble hits a pin, it has equal chance of going left or right.

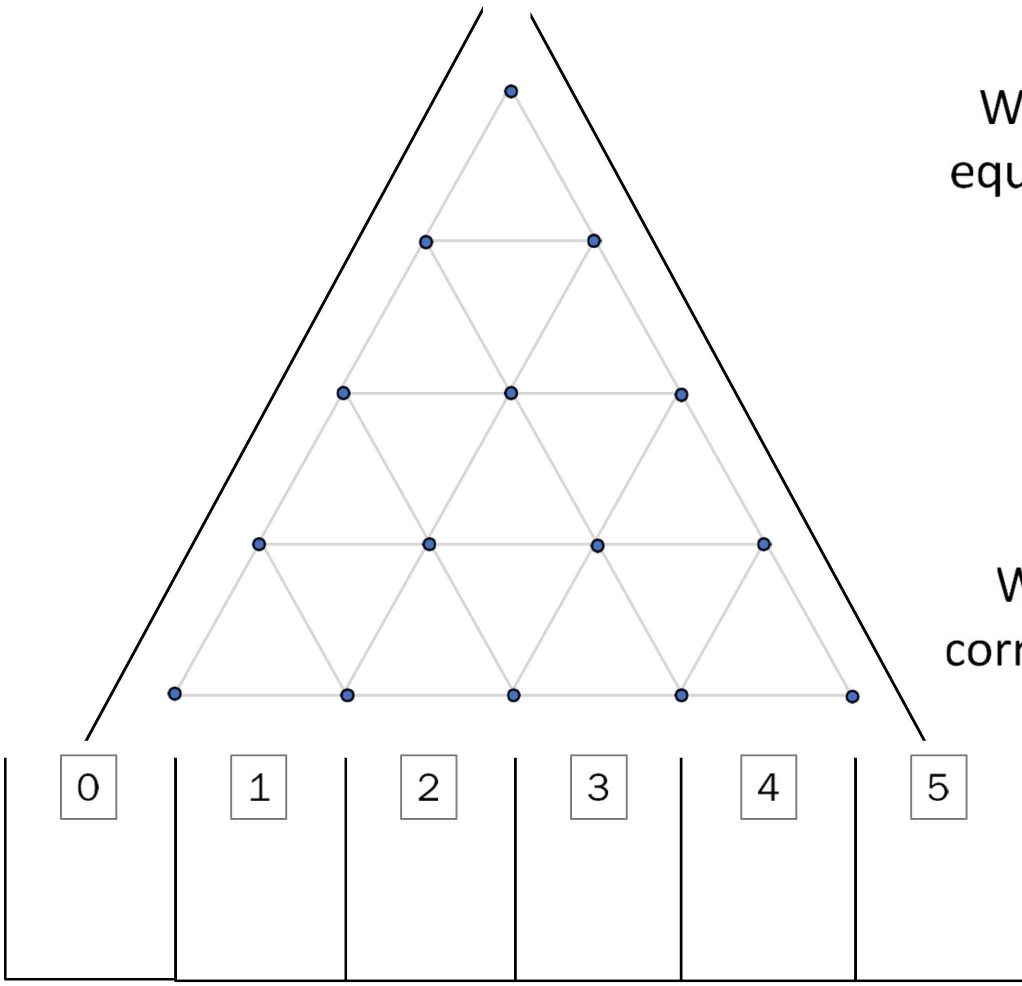
Galton Board Fun



When a marble hits a pin, it has equal chance of going left or right.

Each pin represents an independent event.

Galton Board Fun

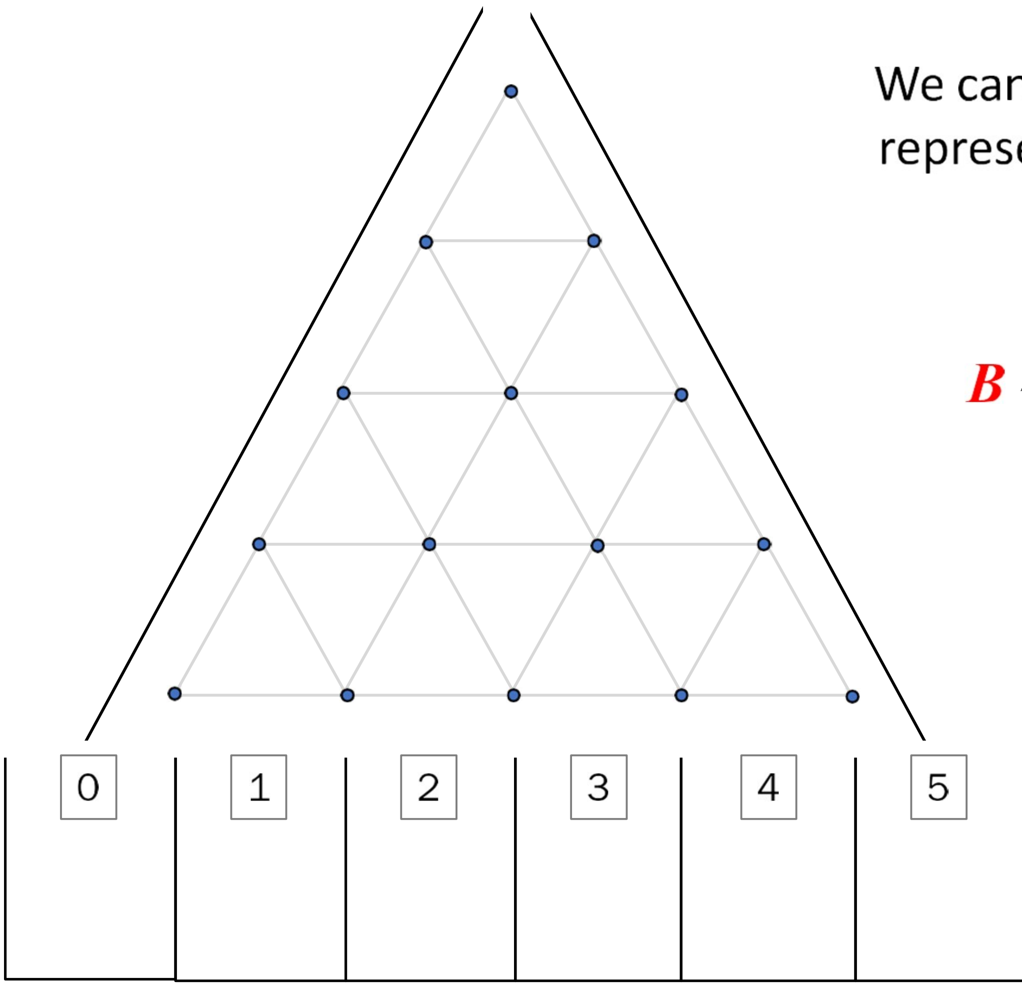


When a marble hits a pin, it has equal chance of going left or right.

Each pin represents an independent event.

Which bucket a marble lands in corresponds to the number of times the marble went right.

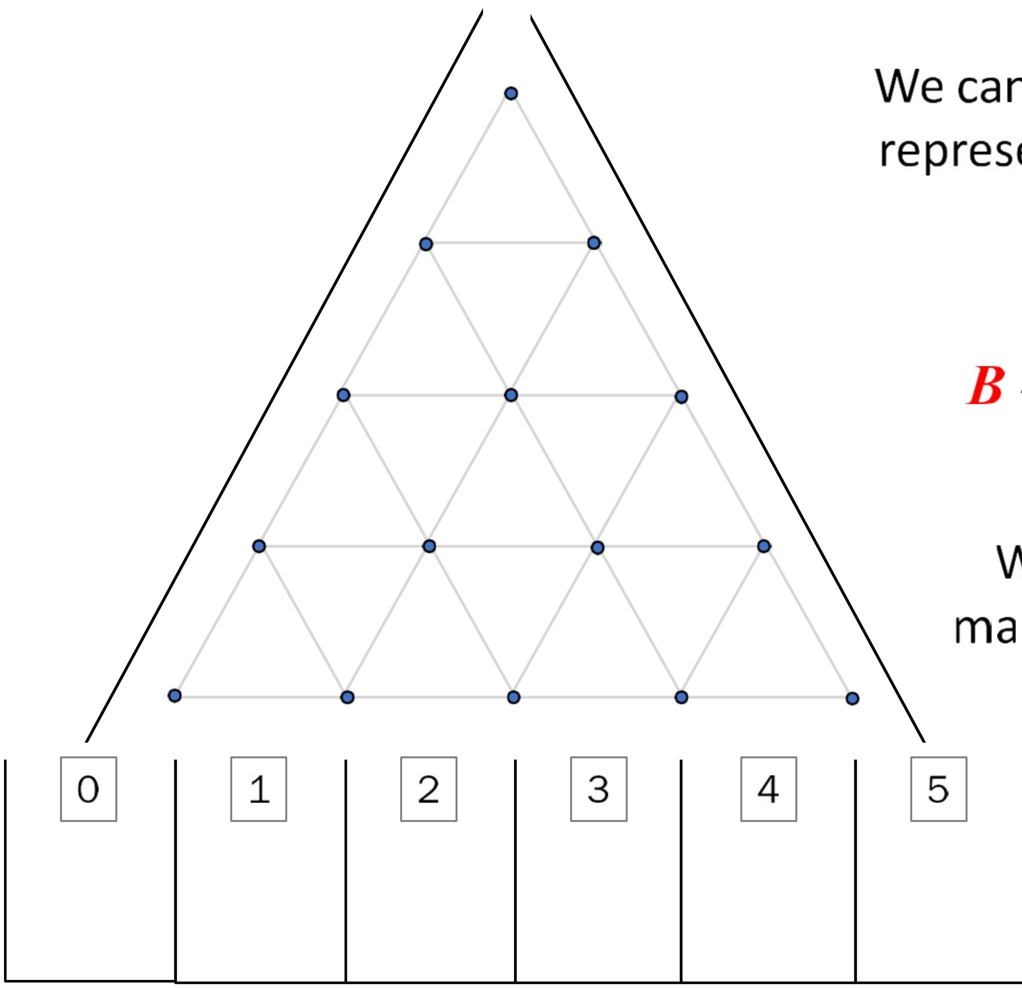
Galton Board Fun



We can define a random variable (***B***) representing which bucket a marble lands in.

$$B \sim \text{Bin}(n = \text{levels}, p = 0.5)$$

Galton Board Fun



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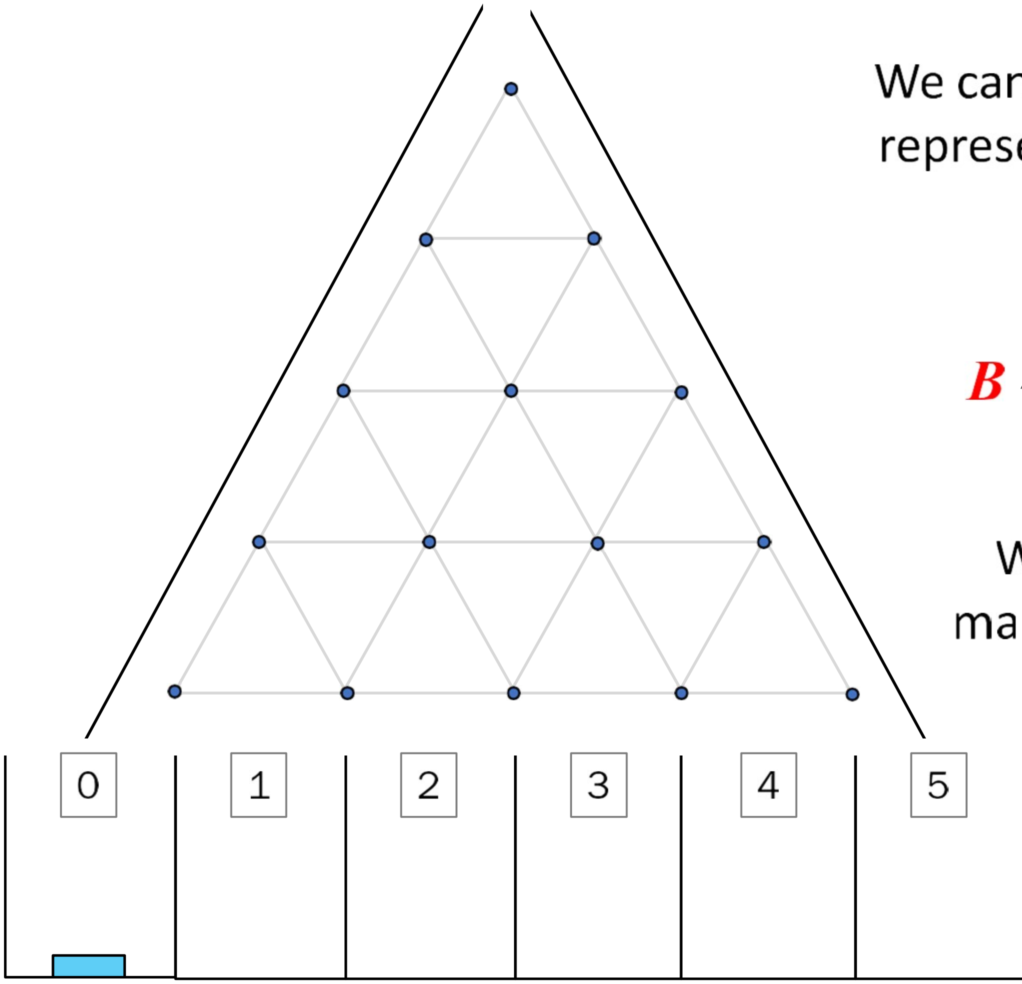
What is the probability of a marble landing in each bucket?

Galton Board Fun



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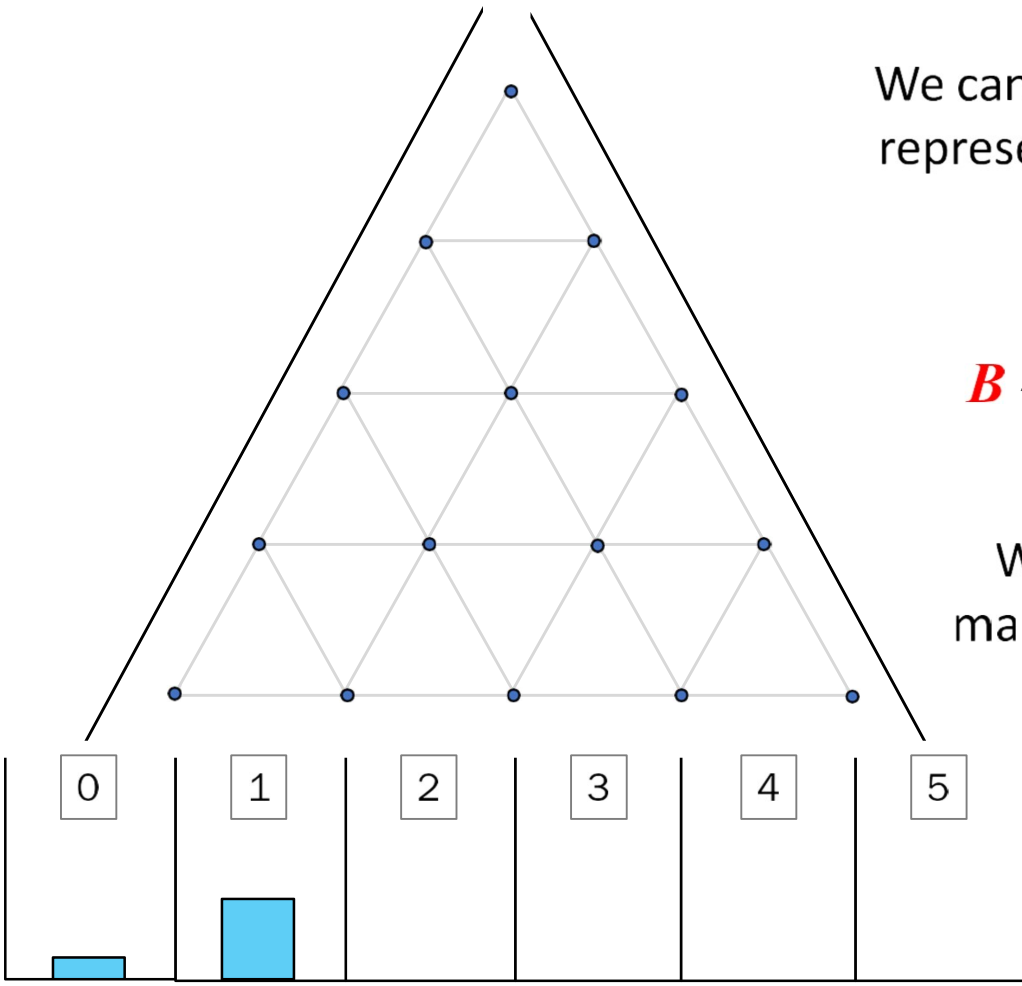
What is the probability of a marble landing in each bucket?

$$P(B=0) = \binom{5}{0}\frac{1}{2}^{5} \approx 0.03$$

Galton Board Fun



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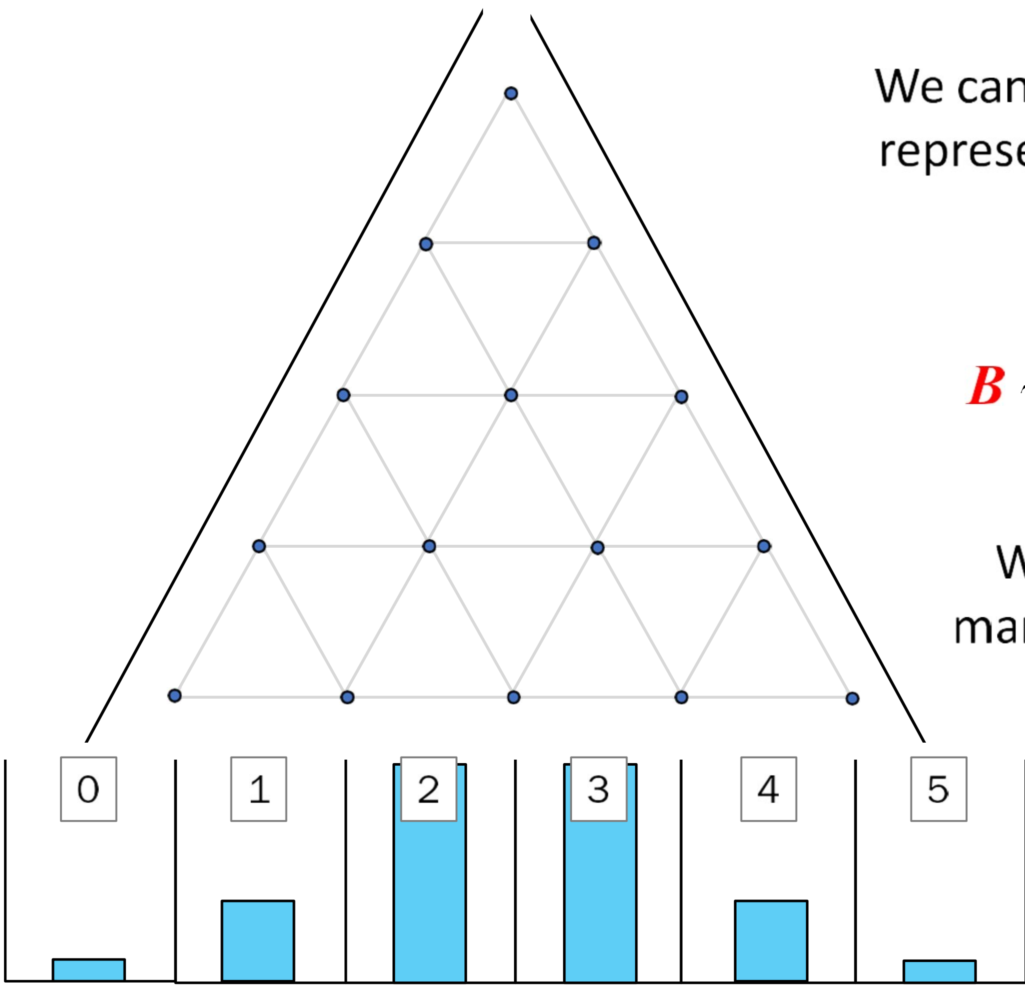
$$B \sim \text{Bin}(\text{n} = \text{levels}, p = 0.5)$$

What is the probability of a marble landing in each bucket?

$$P(B=0) = \binom{5}{0} \frac{1}{2}^5 \approx 0.03$$

$$P(B=1) = \binom{5}{1} \frac{1}{2}^5 \approx 0.16$$

Galton Board Fun



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$$\boldsymbol{B} \sim \text{Bin}(\text{n} = \text{levels}, p = 0.5)$$

What is the probability of a marble landing in each bucket?

This is the PMF of the binomial

The Geometric Random Variable

Imagine flipping a coin *until you see your first heads*.

Each coin flip is an independent trial, with probability *p* of getting heads.

Want to model: how many coin flips until the first heads?

$$X \sim \mathrm{Geo}(p)$$

$X \in \{1, 2, 3, \ldots \infty\}$

The Geometric Random Variable

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$$X \sim \mathrm{Geo}(p)$$

Deriving the PMF:

P(heads on first flip) = p

P(tails, then heads) = $(1 - p) * p$

P(tails, tails, heads) = $(1 - p)^2 * p$

...

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P(heads on first flip) = p

P(tails, then heads) = $(1 - p) * p$

P(tails, tails, heads) = $(1 - p)^2 * p$

…

$$P(X = \underline{n}) = \underline{(1 - p)}^{n-1} p$$

The Negative Binomial Random Variable

Imagine flipping a coin *until you see* ***r*** *heads*.

Each coin flip is an independent trial, with probability *p* of getting heads.

Want to model: how many coin flips until ***r*** heads?

$$X \in \{r, r+1, \ldots \infty\}$$

$$P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}$$

We need success at n-1'th position and that is fixed.

The Negative Binomial Random Variable

Imagine flipping a coin *until you see **r** heads*.

Each coin flip is an independent trial, with probability *p* of getting heads.

Want to model: how many coin flips until ***r*** heads?

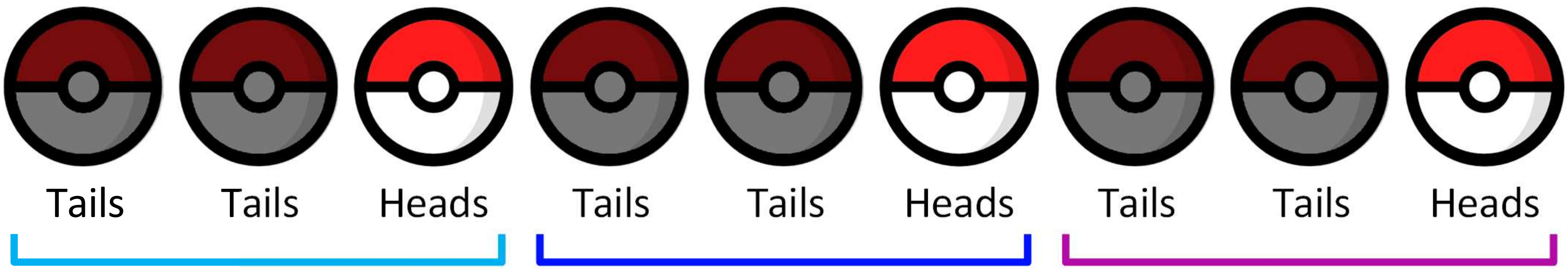
$$X \sim \mathrm{NegBin}(r, p)$$

$$P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}$$

Random Variable Sums

The Negative Binomial X

$r=3$

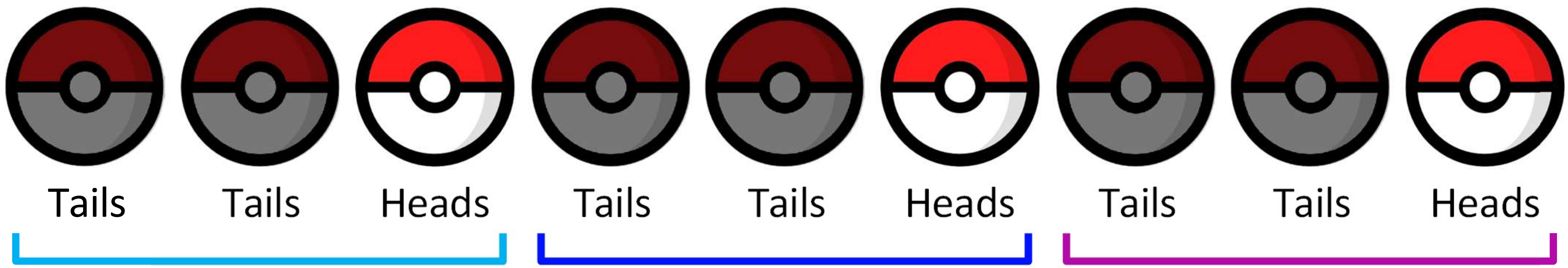


Y_1 Y_2 Y_3

$$X = Y_1 + Y_2 + Y_3$$

Random Variable Sums

The Negative Binomial ...is a sum of r Geometric random variables



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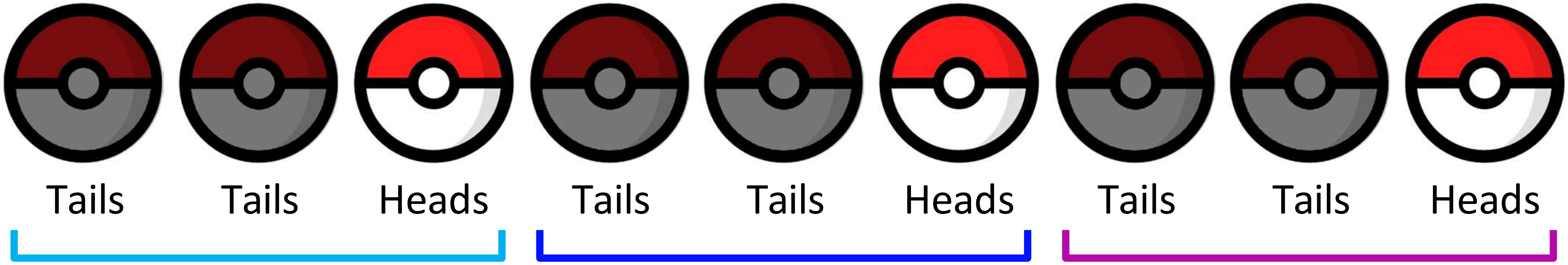

Let $X_1 \sim \text{Geo}(p = 1/3)$, $X_2 \sim \text{Geo}(p = 1/3)$, and $X_3 \sim \text{Geo}(p = 1/3)$.

Random Variable Sums

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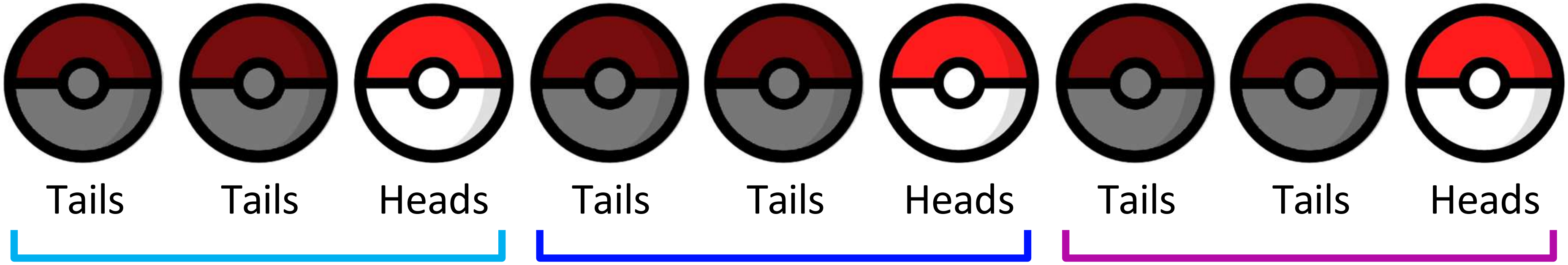

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$$Y \sim \text{NegBin}(r = 3, p = 1/3)$$

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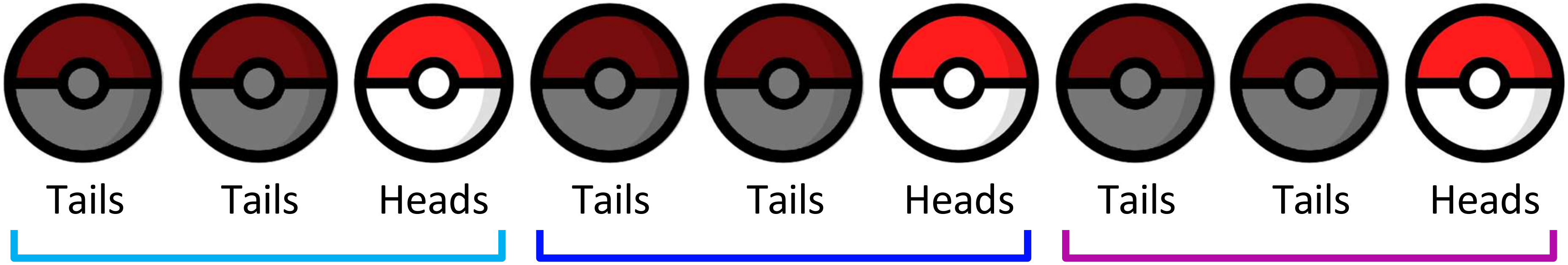

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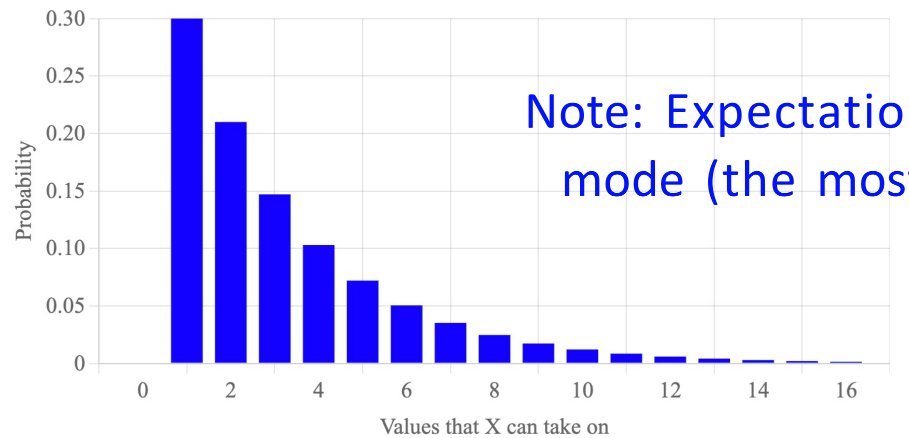
$$Y = X_1 + X_2 + X_3$$

Expected Value of The Geometric

If $X \sim \text{Geo}(p)$, then $E[X] = \frac{1}{p}$

This definition has intuition built in:

- If a coin has probability ½ of a head, then on average, it will take him two tosses to get a head. E[X] = $(1/2)^{-1}$ = 2.



Note: Expectation is often **not** the mode (the most likely outcome)

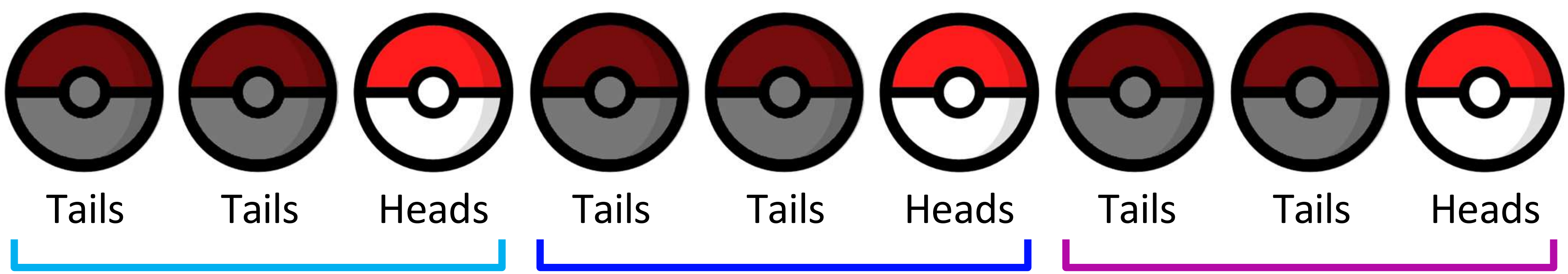
Expected Value of The Geometric

$$E[Y] = \sum_{i=1}^{\infty} \underline{n} \cdot \underbrace{(1-p)^{n-1} \cdot p} = \frac{1}{p}$$

Expected Value of The Negative Binomial

We can derive using the **sum of expectations** property, similar to binomials.

The Negative Binomial ...is a sum of Geometric random variables



Expected Value of The Negative Binomial

We can derive using the **sum of expectations** property, similar to binomials.

Let $X_i \sim \text{Geo}(p)$, for each i from 1 to r.

$$E[X_i] = \frac{1}{p}$$

Let $Y \sim \text{NegBin}(r, p)$.

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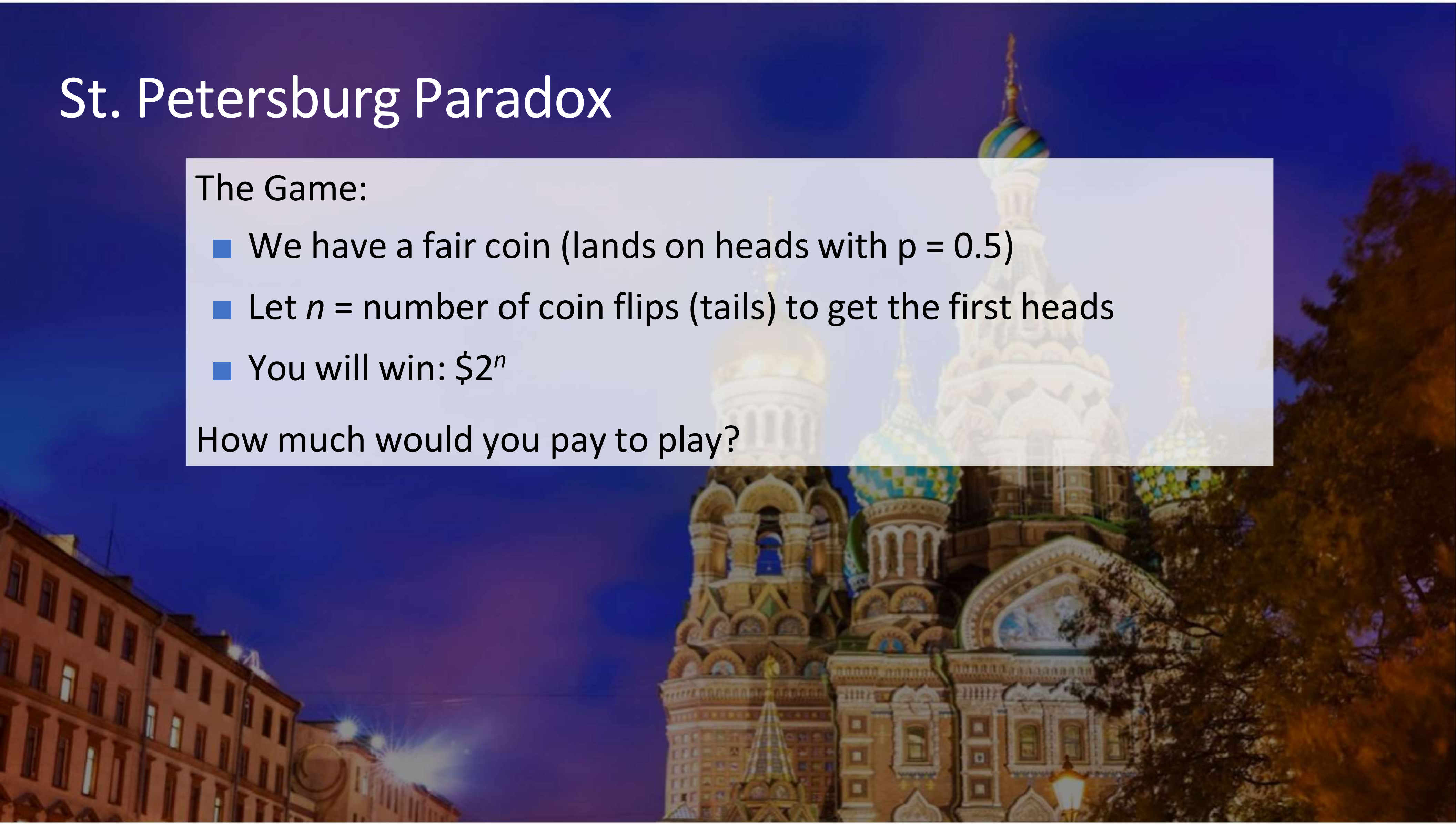
St. Petersburg Paradox



The Game:

- We have a fair coin (lands on heads with p = 0.5)
- Let n = number of coin flips (tails) to get the first heads
- You will win: $\$2^n$

How much would you pay to play?

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Let X be your winnings.

$$E[X] = \left(\frac{1}{2}\right)^1 2^1 + \left(\frac{1}{2}\right)^2 2^2 + \left(\frac{1}{2}\right)^3 2^3 + \ldots = \sum_{i=0}^{\infty} 1 = \infty$$

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What if you could play this game for only $1000...but just once?

Expectations of Classic Random Variables

$X \sim \mathrm{Geo}(p)$

$$E[X] = \frac{1}{p}$$

$X \sim \mathrm{Bern}(p)$

$$E[X] = p$$

$Y \sim \mathrm{NegBin}(r, p)$

$$E[Y] = \frac{r}{p}$$

$Y \sim \mathrm{Bin}(n, p)$

$$E[Y] = n \cdot p$$

Variance of Classic Random Variables

$X \sim \text{Geo}(p)$

$$Var(X) = \frac{1-p}{p^2}$$

$X \sim \text{Bern}(p)$

$$Var(X) = p(1-p)$$

$Y \sim \text{NegBin}(r, p)$

$$Var(X) = \frac{r \cdot (1-p)}{p^2}$$

$Y \sim \text{Bin}(n, p)$

$$Var(Y) = n \cdot p(1-p)$$