MA-110 Linear Algebra and Differential Equations

Rekha Santhanam

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

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Matrix Associated to a Linear Map

Example: The matrix of $S(a_0 + a_1x + a_2x^2) = a_1 + 4a_2x$, w.r.t. the bases $\mathscr{B} = \{1, x, x^2\}$ of $\mathscr{P}_2$ and $\mathscr{C} = \{1, x\}$ of $\mathscr{P}_1$ is $A =$
$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ and $\boxed{A_{*1} = [S(1)]_{\mathscr{C}}, A_{*2} = [S(x)]_{\mathscr{C}}, A_{*3} = [S(x^2)]_{\mathscr{C}}.}$

General Case: If $T : V \to W$ is linear, then the matrix of T w.r.t. the ordered bases $\mathscr{B} = \{v_1, \ldots, v_n\}$ of V, and $\mathscr{C} = \{w_1, \ldots, w_m\}$ of W, denoted $[T]_{\mathscr{C}}^{\mathscr{B}}$, is

$$A = \begin{pmatrix} [T(v_1)]_{\mathscr{C}} & \cdots & [T(v_n)]_{\mathscr{C}} \end{pmatrix} \in \mathscr{M}_{m \times n}.$$

Example: Projection onto the line $x_1 = x_2$

$P\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1+x_2}{2} \\ \frac{x_1+x_2}{2} \end{pmatrix}$ has standard matrix $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$.
This is the matrix of P w.r.t. the standard basis.

Question: What is $[P]_{\mathscr{B}}^{\mathscr{B}}$ where $\mathscr{B} = \{(1,1)^T, (-1,1)^T\}$?

Conclusion: The matrix of a transformation depends on the chosen basis. Some are better than others!

• Solve the differential equation for u: $du/dt = 3u$.

The solution is $u(t) = c\,e^{3t}$, $c \in \mathbb{R}$. With initial condition $u(0) = 2$, the solution is $u(t) = 2e^{3t}$.

• Consider the system of linear 1^{st} order differential equations (ODE) with constant coefficients:

$$du_1/dt = 4u_1 - 5u_2, \qquad du_2/dt = 2u_1 - 3u_2,$$

How does one find the solution?

• Write the system in matrix form $\boxed{du/dt = Au,}$

where $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$, $A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$.

• Assuming the solution is $u(t) = e^{\lambda t}\, v$, where $v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, we need to find λ and v.

Eigenvalues and Eigenvectors: Definition

We have $u_1' = 4u_1 - 5u_2,\ u_2' = 2u_1 - 3u_2$, where $u_1(t) = e^{\lambda t}x, u_2(t) = e^{\lambda t}y$

$$\lambda e^{\lambda t}x = 4e^{\lambda t}x - 5e^{\lambda t}y,$$
$$\lambda e^{\lambda t}y = 2e^{\lambda t}x - 3e^{\lambda t}y.$$

Cancelling $e^{\lambda t}$, we get

Eigenvalue problem: Find λ and $v = (x, y)^T$ satisfying

$$4x - 5y = \lambda x, \qquad 2x - 3y = \lambda y.$$

In the matrix form, it is $\boxed{Av = \lambda v}$. This equation has two unknowns, λ and v.

If there exists a λ such that $Av = \lambda v$ has a non-zero solution v, then λ is called an eigenvalue of A and all *nonzero* v satisfying $Av = \lambda v$ are called eigenvectors of A associated to λ.

Question: How many eignevalues can A have? How do we find them & the associated eigenvectors? Reduce the number of unknowns!

Eigenvalues and Eigenvectors: Solving $Ax = \lambda x$

- Rewrite $Av = \lambda v$ as $\boxed{(A - \lambda I)v = 0.}$

- λ is an eigenvalue of A

  $\iff$ there is a nonzero v in the nullspace of $A - \lambda I$

  $\iff N(A - \lambda I) \neq 0$, i.e., $\dim(N(A - \lambda I)) \geq 1$,

  $\iff A - \lambda I$ is not invertible

  $\iff \det(A - \lambda I) = 0.$

- $\det(A - \lambda I)$ is a polynomial in the variable λ of degree n. Hence it has *at most* n roots $\Rightarrow$ A has atmost n eigenvalues.

- $\det(A - \lambda I)$ is called the **characteristic polynomial** of A.

- If λ is an eigenvalue of A, then the nullspace of $A - \lambda I$ is called the **eigenspace** of A associated to eigenvalue λ.

Question: When is 0 an eigenvalue of A? What are the corresponding eigenvectors?

To summarise: An eigenvalue of A is a root (in $\mathbb{R}$) of its characteristic polynomial. Any non-zero vector in the corresponding eigenspace is an associated eigenvector.

Recall: The ODE system we want to solve is

$u_1' = 4u_1 - 5u_2, \qquad u_2' = 2u_1 - 3u_2,$

The solutions are $u_1(t) = e^{\lambda t} x, u_2(t) = e^{\lambda t} y$, where $(x, y)^T$ is a solution of:

$$\begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \qquad (Av = \lambda v)$$

The characteristic polynomial of A is $\det(A - \lambda I)$

$$= \det \begin{pmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{pmatrix} = (4-\lambda)(-3-\lambda) + 10 = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2)$$

The eigenvalues of A are $\boxed{\lambda_1 = -1,\ \lambda_2 = 2.}$

Eigenvalues and Eigenvectors: Example

Eigenvectors v_1 and v_2 associated to $\lambda_1 = -1$ and $\lambda_2 = 2$ respectively are in: $N(A - \lambda_1 I) = N(A + I)$, and $N(A - \lambda_2 I) = N(A - 2I)$.

Solving $(A+I)v = 0$, i.e., $\begin{pmatrix} 5 & -5 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$, we get $N(A+I) = \left\{ \begin{pmatrix} y \\ y \end{pmatrix} \mid y \in \mathbb{R} \right\}$ and hence $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector associated to $\lambda_1 = -1$.

Similarly, solving $(A-2I)v = 0$ gives $N(A-2I) = \left\{ \begin{pmatrix} \frac{5y}{2} \\ y \end{pmatrix} \mid y \in \mathbb{R} \right\}$. In particular, $v_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ is an eigenvector associated to $\lambda_2 = 2$.

Thus, the system $du/dt = Au$ has two special solutions $e^{-t} v_1$ and $e^{2t} v_2$.

Reading Slide - Complete Solution to ODE

Note: When two functions satisfy $du/dt = Au$, then so do their linear combinations.

Complete solution: $u(t) = c_1 e^{-t} v_1 + c_2 e^{2t} v_2$, i.e.,

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

$$\text{i.e. } u_1(t) = c_1 e^{-t} + 5c_2 e^{2t}, \qquad u_2(t) = c_1 e^{-t} + 2c_2 e^{2t}.$$

If we put initial conditions (IC) $u_1(0) = 8$ and $u_2(0) = 5$, then

$$c_1 + 5c_2 = 8, \quad c_1 + 2c_2 = 5 \Rightarrow c_1 = 3, \; c_2 = 1.$$

Hence the solution of the original ODE system with the given IC is

$$u_1(t) = 3e^{-t} + 5e^{2t}, \quad u_2(t) = 3e^{-t} + 2e^{2t}.$$

Finding Eigenvalues: Examples

In some cases it is easy to find the eigenvalues.

Example: $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ is diagonal. Characteristic polynomial $(3-\lambda)(2-\lambda)$.

Eigenvalues: $\lambda_1 = 3$, $\lambda_2 = 2$.
Eigenvectors: $(A - 3I)v_1 = 0 \Rightarrow Av_1 = 3v_1$.

Can take $v_1 = e_1$
Similarly, an eigenvector associated to λ_2 is $v_2 = e_2$
Further, $\mathbb{R}^2$ has a basis consisting of eigenvectors of A: $\{e_1, e_2\}$.

Special case: If A is a diagonal matrix with diagonal entries $\lambda_1, \cdots, \lambda_n$, then

Eigenvalues: $\lambda_1, \cdots, \lambda_n$
Eigenvectors: $e_1, \cdots, e_n$, which form a basis for $\mathbb{R}^n$.

Finding Eigenvalues: Examples

Example: Projection onto the line $x = y$: $P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. $v_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$ projects onto itself $\Rightarrow \lambda_1 = 1$ with eigenvector v_1. $v_2 = \begin{pmatrix} 1 & -1 \end{pmatrix}^T \mapsto 0$ $\Rightarrow \lambda_2 = 0$ with eigenvector v_2. Further, $\{v_1, v_2\}$ is a basis of $\mathbb{R}^2$.

Question: Do a collection of eigenvectors always form a basis of $\mathbb{R}^n$?

A: No! **Example:** For $c \in \mathbb{R}$, let $A = \begin{pmatrix} c & 1 \\ 0 & c \end{pmatrix}$.

Characteristic Polynomial: $\det(A - \lambda I) = (c - \lambda)^2$.

Eigenvalues: $\lambda = c$.
Eigenvectors: $(A - I)v = 0 \Rightarrow v = (1\ 0)^T$

Question: Is it unique? Eigenspace of A is 1 dimensional $\Rightarrow$ $\mathbb{R}^2$ has no basis of eigenvectors of A.

Think: What is the advantage of a basis of eigenvectors?

Defn. The $n \times n$ matrices A and B are *similar*, if there exists an invertible matrix P such that $P^{-1}AP = B$.

Observe: If $B = P^{-1}AP$, then (i) $\det(A) = \det(B)$, and (ii) $B^n = P^{-1}A^nP$ for each n.

Theorem: If A and B are similar, then they have the same characteristic polynomial. In particular, they have the same eigenvalues, $\det(A) = \det(B)$ and $\text{Trace}(A) = \text{Trace}(B)$.

Proof. Given: $B = P^{-1}AP$. prove: $\det(A - \lambda I) = \det(B - \lambda I)$.
Note: It is enough to prove that $A - \lambda I$ and $B - \lambda I$ are similar!
Indeed,

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(A - \lambda I)P. \qquad \square$$

Ques: Why care?
Write $\det(A - \lambda I) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$. Compare constant coeff.: $\det(A) = \lambda_1 \cdots \lambda_n = \det(B)$; Compare coeff. of λ^{n-1}: Sum of diagonal entries $= a_{11} + \cdots + a_{nn} =$ Trace of $A = \lambda_1 + \ldots + \lambda_n =$ Trace of B.
Ques: How are eigenvalues of A and B related?