MA 110 - Ordinary Differential Equations

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Outline of the lecture

- Second order linear equations
- Method of reduction of order

Test for l.i.

Suppose that

$$\boxed{y'' + p(x)y' + q(x)y = 0}$$

has continuous coefficients on an open interval I. Then

1. two solutions y_1 and y_2 of the DE on I are linearly dependent iff their Wronskian is 0 at some $x_0 \in I$.
2. Wronskian $=0$ for some $x = x_0 \implies W \equiv 0$ on I.
3. if there exists an $x_1 \in I$ at which $W \neq 0$, then y_1 and y_2 are linearly independent on I.

Proof contd..

$\boxed{2.\ \text{Wronskian } = 0 \text{ for some } x = x_0 \implies W \equiv 0 \text{ on } I.}$

If Wronskian $=0$ for some $x = x_0$, then by the first part of the result, y_1 & y_2 are linearly dependent
$\implies W(y_1, y_2)(x) = 0 \ \ \forall x \in I$.

$\boxed{3.\ \text{if there exists an } x_1 \in I \text{ at which } W \neq 0, \text{ then } y_1 \text{ and } y_2 \text{ are l.i. on } I.}$

$W(y_1, y_2)(x_1) \neq 0 \implies y_1$ & y_2 can't be linearly dependent
$\implies y_1$ & y_2 are l.i.

Definition

A basis or fundamental set of solutions of $y'' + p(x)y' + q(x)y = 0$ on an interval I is a pair y_1, y_2 of linearly independent solutions of $y'' + p(x)y' + q(x)y = 0$ on I.

Examples

1. The continuity of $p(x)$ and $q(x)$ is required in the results of the previous slide. Consider the DE

$$x^2y'' - 4xy' + 6y = 0.$$

Then, x^2 and x^3 are linearly independent solutions, but
$W(x^2, x^3) = x^4$ and so $W(x^2, x^3)(0) = 0$.
Note that $p(x) = -\frac{4}{x}$ and $q(x) = \frac{6}{x^2}$
2. Consider $y_1(x) = x^2$ and

$$y_2(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0, \end{cases}$$

Then, $W(y_1, y_2)(x) = 0$ for all $x \in \mathbb{R}$, but y_1 and y_2 are linearly independent.
Does it contradict the result in the previous slide? No.

Result : If $p(x)$ and $q(x)$ are continuous on an open interval I, then $y'' + p(x)y' + q(x)y = 0$ has a basis of solutions on I.

Proof : Consider the IVP's

$$\begin{aligned} y'' + p(x)y' + q(x)y = 0, \;\; y(x_0) = 1, \; y'(x_0) = 0 \\ y'' + p(x)y' + q(x)y = 0, \;\; y(x_0) = 0, \; y'(x_0) = 1 \end{aligned}$$

By existence-uniqueness theorem of IVP, the above problems have unique solutions $y_1(x)$ and $y_2(x)$ respectively on I.

Now, $W(y_1, y_2)(x_0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \Longrightarrow y_1 \,\& y_2$ are l.i. Why?

Hence, they form a basis of solutions of $y'' + p(x)y' + q(x)y = 0$.

General solution

Let y_1 & y_2 be a basis of solutions of the homogeneous second order linear DE $y'' + p(x)y' + q(x)y = 0$ on I, where $p(x)$ and $q(x)$ are continuous on I. Then,

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

is a general solution of $y'' + p(x)y' + q(x)y = 0$.
Every solution $y = Y(x)$ of the DE has the form

$$Y(x) = C_1 y_1(x) + C_2 y_2(x),$$

where C_1 and C_2 are arbitrary constants.

Proof

Let $Y(x)$ be a solution of the given ODE. We want to find C_1 and C_2 such that

$$Y(x) = C_1 y_1(x) + C_2 y_2(x).$$

This implies for $x_0 \in I$,

$$\begin{aligned} Y(x_0) &= C_1 y_1(x_0) + C_2 y_2(x_0) \\ Y'(x_0) &= C_1 y_1'(x_0) + C_2 y_2'(x_0). \end{aligned}$$

Thus,

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} Y(x_0) \\ Y'(x_0) \end{bmatrix}.$$

As y_1 and y_2 form a basis of solutions of the DE,

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix}$$

is invertible (Justify!), i.e., $W(x_0)$ is not zero.

Therefore,

$$C_1 = \frac{\begin{vmatrix} Y(x_0) & y_2(x_0) \\ Y'(x_0) & y_2'(x_0) \end{vmatrix}}{W(x_0)},$$

and

$$C_2 = \frac{\begin{vmatrix} y_1(x_0) & Y(x_0) \\ y_1'(x_0) & Y'(x_0) \end{vmatrix}}{W(x_0)}.$$

(Is the representation of Y in terms of C_1 and C_2 unique?)

Now,

$$u(x) = Y(x) - C_1 y_1(x) - C_2 y_2(x)$$

satisfies the given DE, and

$$u(x_0) = 0 = u'(x_0).$$

But the constant function $u(x) \equiv 0$ also satisfies the IVP. Thus,

$$Y(x) = C_1 y_1(x) + C_2 y_2(x) \text{ by the uniqueness theorem.}$$

Method of reduction of order

We've been looking at the second order linear homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0.$$

As we remarked earlier, there is no general method to find a basis of solutions. However, if we know one non-zero solution $y_1(x)$ then we have a method to find $y_2(x)$ such that $y_1(x)$ and $y_2(x)$ are linearly independent.

To find such a $y_2(x)$, set

$$\boxed{y_2(x) = v(x)y_1(x)}$$

We'll choose v such that y_1 and y_2 are linearly independent.

Can v be a constant? No.

Now for y_2 to be a solution of the given ODE

$$y_2'' + p(x)y_2' + q(x)y_2 = 0.$$

that is,

$$(vy_1)'' + p(x)(vy_1)' + q(x)(vy_1) = 0.$$

Second solution

Thus,

$$
\begin{aligned}
0 &= (v'y_1 + vy_1')' + p(v'y_1 + vy_1') + qvy_1 \\
&= v''y_1 + 2v'y_1' + vy_1'' + p(v'y_1 + vy_1') + qvy_1 \\
&= v(y_1'' + py_1' + qy_1) + v'(2y_1' + py_1) + v''y_1 \\
&= 0 + v'(2y_1' + py_1) + v''y_1.
\end{aligned}
$$

Thus, $\dfrac{v''}{v'} = -\dfrac{(2y_1' + py_1)}{y_1} = -\dfrac{2y_1'}{y_1} - p$. Therefore,

$$\ln|v'| = \ln\left(\frac{1}{y_1^2}\right) - \int p\,dx;$$

That is,

$$v' = \frac{e^{-\int p\,dx}}{y_1^2}, \text{ or } \boxed{v = \int \frac{e^{-\int p\,dx}}{y_1^2}\,dx.}$$

Second solution

Claim: y_1 and vy_1 are linearly independent.
Proof. Enough to check Wronskian!

$$
\begin{aligned}
W(y_1, vy_1) &= \begin{vmatrix} y_1 & vy_1 \\ y_1' & (vy_1)' \end{vmatrix} \\
&= y_1(v'y_1 + y_1'v) - y_1'vy_1 \\
&= y_1^2 v' \\
&= y_1^2 \frac{e^{-\int p dx}}{y_1^2} \\
&= e^{-\int p dx} \neq 0.
\end{aligned}
$$

Example

Given that $y = x$ is a solution, find a l.i. solution of

$$(x^2 + 1)y'' - 2xy' + 2y = 0$$

by reducing the order.

$y_2 = vy_1 = vx$. Then,

$$v(x) = \int \frac{e^{-\int p dx}}{y_1^2} dx = \int \frac{e^{-\int \frac{-2x}{x^2+1} dx}}{x^2} dx = \int \frac{x^2 + 1}{x^2} dx = \int (1 + \frac{1}{x^2}) dx$$

Hence, $v(x) = x - \dfrac{1}{x}$ and $y_2 = x\left(x - \dfrac{1}{x}\right) = x^2 - 1.$

Are y_1 & y_2 l.i.? What is the general solution?