

MA 110 - Ordinary Differential Equations

Santanu Dey

Department of Mathematics,
Indian Institute of Technology Bombay,
Powai, Mumbai 76
santanudey@iitb.ac.in

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Outline of the lecture

- Laplace Transforms
- Examples
- Existence
- Properties

Existence of Laplace transforms

- For a given f , $L(f)$ **may or may not exist**.
- **Sufficient conditions** under which **convergence** is guaranteed for the integral in the definition of the Laplace transform is that f is piecewise continuous and is of exponential order.
- **Piecewise continuity** - The function is continuous except possibly for finitely many **jump** discontinuities.



A function f is said to be of **exponential order** if there exists $a \in \mathbb{R}$ and positive constants t_0 and K such that

$$|f(t)| \leq Ke^{at},$$

for all $t \geq t_0 > 0$.

Exponential Order

- ⇒ In other words, we say f is of **exponential order** if there exists a constant a such that $e^{-at}|f(t)|$ is bounded for all sufficiently large values of t .
- ⇒ That is, if f is of **exponential order** and the values $f(t)$ of f become infinite as $t \rightarrow \infty$, these values cannot become infinite more rapidly than a multiple of K of the corresponding e^{at} values of some constant a .

Examples

- 1 Every bounded function is of exponential order with the constant $a = 0$. Further, $\sin bt$ and $\cos bt$ are of exponential order. Also, if $|f(t)| \leq K$ for $t \geq t_0 > 0$, then f is of exponential order.
- 2 $e^{\alpha t} \sin bt$ is of exponential order, with constant $a = \alpha$.
- 3 t^n for $n > 0$ is of exponential order, since for $a > 0$,
 $\lim_{t \rightarrow \infty} e^{-at} t^n = 0$ and thus, there exists $K > 0$ and $t_0 > 0$ such that

$$e^{-at}|f(t)| = e^{-at} t^n < K, \text{ for } t > t_0.$$

- 4 e^{t^2} is not of exponential order, for in this case,

$$e^{-at}|f(t)| = e^{t^2-at}$$

and this becomes unbounded as $t \rightarrow \infty$, no matter what is value of a .

- 5 Sum of functions of exponential order is also of exponential order.

Theorem

Suppose $f(t)$ is piecewise continuous on $[0, \alpha]$ for all $\alpha > 0$.
Further suppose

$$|f(t)| \leq Ke^{at},$$

for $t \geq t_0 > 0$, where $K > 0$, $a, t_0 \in \mathbb{R}$. Then

$$L(f)(s) = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

exists for $s > a$.

Proof: We have:

$$L(f)(s) = \int_0^{t_0} e^{-st} f(t) dt + \int_{t_0}^{\infty} e^{-st} f(t) dt.$$

As f is piecewise continuous on $[0, t_0]$, $\int_0^{t_0} e^{-st} f(t) dt$ exists.

We need to show that $\int_{t_0}^{\infty} e^{-st} f(t) dt$ converges.

For $t \geq t_0$, we have:

$$|e^{-st} f(t)| \leq e^{-st} K e^{at} = K e^{-(s-a)t}.$$

For $s > a$, $\int_{t_0}^{\infty} e^{-(s-a)t} dt$ converges. Hence,

$$\int_{t_0}^{\infty} |e^{-st} f(t)| dt,$$

and thus

$$\int_{t_0}^{\infty} e^{-st} f(t) dt$$

converges.

The conditions in the theorem are sufficient but not necessary for the existence of $L(f)$.

Example:

$$\frac{1}{\sqrt{t}} \rightarrow \infty$$

as $t \rightarrow 0$. Hence it is not piecewise continuous on $[0, b]$ for any

$b > 0$. But **we will prove later** that $L\left(\frac{1}{\sqrt{t}}\right)(s) = \sqrt{\frac{\pi}{s}}$.

Property 1 : Linearity

Let $f, g : [0, \infty) \rightarrow \mathbb{R}$ be functions such that $L(f)$ and $L(g)$ exist.
Let $a, b \in \mathbb{R}$. Then,

$$L(af(t) + bg(t)) = aL(f(t)) + bL(g(t)).$$

Proof:

$$\begin{aligned} L(af + bg) &= \int_0^{\infty} e^{-st}(af(t) + bg(t))dt \\ &= \int_0^{\infty} e^{-st}af(t)dt + \int_0^{\infty} e^{-st}bg(t)dt \\ &= aL(f) + bL(g). \end{aligned}$$

Laplace of whole is equal to Laplace of individual.

Example 1

$$\begin{aligned}L(e^{i\omega t}) &= \int_0^{\infty} e^{-st} e^{i\omega t} dt \\&= \int_0^{\infty} e^{-(s-i\omega)t} dt \\&= \left. \frac{e^{-(s-i\omega)t}}{-(s-i\omega)} \right|_{t=0}^{\infty} = \frac{1}{s-i\omega} \\&= \frac{s+i\omega}{s^2+\omega^2}.\end{aligned}$$

Hence,

$$L(\cos \omega t + i \sin \omega t) = \frac{s+i\omega}{s^2+\omega^2}.$$

Using linearity,

$$L(\cos \omega t) = \frac{s}{s^2+\omega^2}, \quad L(\sin \omega t) = \frac{\omega}{s^2+\omega^2}.$$

Example 2

$$\begin{aligned}L(\cosh at) &= L\left(\frac{e^{at} + e^{-at}}{2}\right) \\&= \frac{1}{2}L(e^{at}) + \frac{1}{2}L(e^{-at}) \\&= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2 - a^2}.\end{aligned}$$

Hence, $L(\cosh at) = \frac{s}{s^2 - a^2}.$

Similarly, $L(\sinh at) = \frac{a}{s^2 - a^2}.$

Example 3

$$\begin{aligned}L(te^{i\omega t}) &= \int_0^{\infty} e^{-st} te^{i\omega t} dt = \int_0^{\infty} te^{-(s-i\omega)t} dt \\&= \left. \frac{te^{-(s-i\omega)t}}{-(s-i\omega)} \right|_{t=0}^{\infty} + \int_0^{\infty} \frac{e^{-(s-i\omega)t}}{s-i\omega} dt \\&= \left. \frac{1}{(s-i\omega)} \frac{e^{-(s-i\omega)t}}{-(s-i\omega)} \right|_{t=0}^{\infty} \\&= \frac{1}{(s-i\omega)^2} = \frac{1}{s^2 - \omega^2 - 2is\omega} \frac{s^2 - \omega^2 + 2is\omega}{s^2 - \omega^2 + 2is\omega} \\&= \frac{s^2 - \omega^2}{(s^2 - \omega^2)^2 + 4s^2\omega^2} + i \frac{2s\omega}{(s^2 - \omega^2)^2 + 4s^2\omega^2} \\&= \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2} + i \frac{2s\omega}{(s^2 + \omega^2)^2}.\end{aligned}$$

Hence,

$$L(t \cos \omega t) = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}, \quad L(t \sin \omega t) = \frac{2s\omega}{(s^2 + \omega^2)^2}.$$

Property 2 : I Shifting theorem (s shifting)

If $L(f(t)) = F(s)$, then $L(e^{at}f(t)) = F(s - a)$.

Proof :

$$\begin{aligned}L(e^{at}f(t)) &= \int_0^{\infty} e^{-st} e^{at} f(t) dt \\&= \int_0^{\infty} e^{-(s-a)t} f(t) dt \\&= F(s - a).\end{aligned}$$

Examples :

$$1. \quad L(t^2) = \frac{2}{s^3} \implies L(e^{-t}t^2) = \frac{2}{(s+1)^3}.$$

$$2. \quad L(\cos \omega t) = \frac{s}{s^2 + \omega^2} \implies L(e^{at} \cos \omega t) = \frac{s - a}{(s - a)^2 + \omega^2}.$$

Find the Laplace transforms of

- ① $t^n e^{at}$
- ② $\cosh at \cos at$
- ③ $e^{-t} \sin^2 t$

Property 3 : Scaling

If $L(f) = F(s)$, then $L(f(ct)) = \frac{1}{c} F\left(\frac{s}{c}\right)$, $c > 0$.

Proof: Let $\xi = ct$. Then, $d\xi = c dt$.

$$\begin{aligned} L(f(ct)) &= \int_0^{\infty} e^{-st} f(ct) dt \\ &= \int_0^{\infty} e^{-\left(\frac{s\xi}{c}\right)} \frac{1}{c} f(\xi) d\xi \\ &= \frac{1}{c} \int_0^{\infty} e^{-\left(\frac{s\xi}{c}\right)} f(\xi) d\xi \\ &= \frac{1}{c} F\left(\frac{s}{c}\right). \end{aligned}$$

Example :

$$L(e^t) = \frac{1}{s-1} \implies L(e^{at}) = \frac{1}{a} \frac{1}{\left(\frac{s}{a}-1\right)} = \frac{1}{s-a}.$$

Property 4 : Differentiation

I.

- ⇒ Suppose f is continuous,
- ⇒ f' is piecewise continuous on $[0, a]$ for all $a > 0$,
- ⇒ $|f(t)| \leq Ke^{\alpha t}$, for $t \geq t_0 > 0$, where $K > 0$, $t_0, \alpha \in \mathbb{R}$.

Then, $L(f')(s)$ exists for $s > \alpha$ and

$$L(f') = sL(f) - f(0).$$

II.

- ⇒ Suppose $f, f', \dots, f^{(n-1)}$ are continuous
- ⇒ $f^{(n)}$ is piecewise continuous on $[0, a]$, for all $a > 0$,
- ⇒ For all $t \geq t_0 > 0$, $|f^{(i)}(t)| \leq Ke^{\alpha t}$, $0 \leq i \leq n-1$, where $K > 0$, $t_0, \alpha \in \mathbb{R}$.

Then, $L(f^{(n)})(s)$ exists for all $s > \alpha$ and

$$L(f^{(n)}) = s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$