MA-110 Linear Algebra and Differential Equations

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Echelon Form: Recap

Recall: If A is $n \times n$, then $PA = LU$, where P is a product of permutation matrices, L is lower triangular, U is upper triangular, and all of size $n \times n$.

Q: What happens when A is not a square matrix?

Let $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$. By elimination, we see:

$$A \to \begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U.$$

Thus $A = LU$, where $L = E_{21}(2)E_{31}(3)E_{32}(-1) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix}$.

Echelon Form

If A is $m \times n$, we can find P, L and U as before. In this case, L and P will be $m \times m$ and U will be $m \times n$, $PA = LU$.

U has the following properties:

1. Pivots are the 1st nonzero entries in their rows.
2. Entries below pivots are zero, by elimination.
3. Each pivot lies to the right of the pivot in the row above.
4. Zero rows are at the bottom of the matrix.

U is called an *echelon form* of A.
What are all possible 2×2 echelon forms: Let $\bullet$ = pivot entry.
$\begin{pmatrix} \bullet & * \\ 0 & \bullet \end{pmatrix}$, $\begin{pmatrix} \bullet & * \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & \bullet \\ 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Row Reduced Form

To obtain the row reduced form R of a matrix A:
1) Get the echelon form U. 2) Make the pivots 1.
3) Make the entries above the pivots 0.

Ex: Find all possible 2×2 row reduced forms.

Eg. Let $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$. Then $U = \begin{pmatrix} \mathbf{1} & 2 & 3 & 5 \\ 0 & 0 & \mathbf{2} & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Divide by pivots: $R_2/2$ gives $\begin{pmatrix} \mathbf{1} & 2 & 3 & 5 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

By $R_1 = R_1 - 3R_2$, Row reduced form of A: $R = \begin{pmatrix} \mathbf{1} & 2 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

U and R are used to solve $Ax = 0$ and $Ax = b$.

Null Space: Solution of $Ax = 0$

Let A be $m \times n$. **Q:** For which $x \in \mathbb{R}^n$, is $Ax = 0$?

The Null Space of A, denoted by $N(A)$, is the set of all vectors x in $\mathbb{R}^n$ such that $Ax = 0$.

Example 1: $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$. Are the following in $N(A)$?

$$x = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}? \quad y = \begin{pmatrix} -5 \\ 0 \\ 0 \\ 1 \end{pmatrix}? \quad z = \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}?$$

Note: x is in $N(A) \iff A_{1*} \cdot x = 0$, $A_{2*} \cdot x = 0$, and $A_{3*} \cdot x = 0$, i.e., x is perpendicular to every row of A.

Linear Combinations in $N(A)$

Example 1 (contd.): If $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$, then

$x = \begin{pmatrix} -2 & 1 & 0 & 0 \end{pmatrix}^T$ and $y = \begin{pmatrix} -2 & 0 & -1 & 1 \end{pmatrix}^T$ are in $N(A)$.

Q: What about $x + y = \begin{pmatrix} -4 & 1 & -1 & 1 \end{pmatrix}^T$,

$-3 \cdot x = \begin{pmatrix} 6 & -3 & 0 & 0 \end{pmatrix}^T$?

Remark: Let A be an $m \times n$ matrix, u, v be real numbers.

- The null space of A, $\boxed{N(A) \text{ contains vectors from } \mathbb{R}^n}$,
- If x, y are in $N(A)$, i.e., $Ax = 0$ and $Ay = 0$, then

$A(ux + vy) = u(Ax) + v(Ay) = 0$, i.e., $ux + vy$ is in $N(A)$.

i.e., a linear combination of vectors in $N(A)$ is also in $N(A)$.

$\boxed{\text{Thus } N(A) \text{ is } \textit{closed under} \text{ linear combinations}}$.

Key Point: $Ax = 0$ has the same solutions as $Ux = 0$,

which has the same solutions as $Rx = 0$, i.e.,

$$\boxed{N(A) = N(U) = N(R)}.$$

Reason: If A is $m \times n$, and Q is an invertible $m \times m$ matrix, then $N(A) = N(QA)$. (Verify this)!

Example 2:

For $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$, we have $Rx = \begin{pmatrix} \mathbf{1} & 2 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix}$.

$R\,x = 0$ gives $t + 2u + 2w = 0$ and $v + w = 0$.

i.e., $t = -2u - 2w$ and $v = -w$.

Null Space: Solution of $Ax = 0$

$Rx = 0$ gives $t = -2u - 2w$ and $v = -w$,

t and v are *dependent* on the values of u and w.

u and w are *free* and *independent*, i.e., we can choose any value for these two variables.

Special solutions:

$u = 1$ and $w = 0$, gives $x = \begin{pmatrix} -2 & 1 & 0 & 0 \end{pmatrix}^T$.

$u = 0$ and $w = 1$, gives $x = \begin{pmatrix} -2 & 0 & -1 & 1 \end{pmatrix}^T$.

The null space contains:

$$x = \begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -2u - 2w \\ u \\ -w \\ w \end{pmatrix} = u \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix},$$

i.e., all possible linear combinations of the special solutions.

Rank of A

$Ax = 0$ always has a solution: the trivial one, i.e., $x = 0$.

Main Q1: When does $Ax = 0$ have a non-zero solution?

A: When there is at least one free variable,
i.e., not every column of R contains a pivot.

To keep track of this, we define:

$\boxed{\text{rank}(A) = \text{number of columns containing pivots in } R}$.

If A is $m \times n$ and $\text{rank}(A) = r$, then

- $\text{rank}(A) \leq \min\{m, n\}$.
- no. of dependent variables $= r$.
- no. of free variables $= n - r$.
- $Ax = 0$ has only the 0 solution $\Leftrightarrow$ $r = n$.
- $m < n \Rightarrow Ax = 0$ has non-zero solutions.

True/False: If $m \geq n$, then $Ax = 0$ has only the 0 solution.

Rank of A

$\boxed{\text{rank}(A) = \text{number of columns containing pivots in } R}$.

$= $ number of dependent variables in the system $Ax = 0$.

Example: $R = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ when $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$.

The no.of columns containing pivots in R is 2, $\Rightarrow$ $\text{rank}(A) = 2$.

R contains a 2×2 identity matrix, namely the rows and columns corresponding to the pivots.

This is the row reduced form of the corresponding submatrix $\begin{pmatrix} 1 & 3 \\ 2 & 8 \end{pmatrix}$ of A, which is invertible, since it has 2 pivots.

Thus, $\boxed{\text{rank}(A) = r \Rightarrow A \text{ has an } r \times r \text{ invertible submatrix.}}$

State the converse. The converse is also true. Why?

Summary: Finding $N(A) = N(U) = N(R)$

Let A be $m \times n$. To solve $Ax = 0$, find R and solve $Rx = 0$.

1. Find free (independent) and pivot (dependent) variables:
   pivot variables: columns in R with pivots ($\longleftrightarrow t$ and v).
   free variables: columns in R without pivots ($\longleftrightarrow u$ and w).

2. No free variables, i.e., $\text{rank}(A) = n \Rightarrow N(A) = 0$.

3. (a) If $\text{rank}(A) < n$, obtain a special solution:
   Set one free variable $= 1$, the other free variables $= 0$.
   Solve $Rx = 0$ to obtain values of pivot variables.

   (b) Find special solutions for each free variable.
   $N(A)$ = space of linear combinations of special solutions.

- This information is stored in a compact form in:

  Null Space Matrix: Special solutions as columns.