MA-110 Linear Algebra and Differential Equations

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Lecture 10 D3

Linear Span: Definition

Given a collection $S = \{v_1, v_2, \ldots, v_n\}$ in a vector space V, the *linear span* of S, denoted $\text{Span}(S)$ or $\text{Span}\{v_1, \ldots, v_n\}$, is the set of all linear combinations of v_1, $v_2, \ldots, v_n$, i.e.,

$$\boxed{\text{Span}(S) = \{v = a_1 v_1 + \cdots + a_n v_n, \text{ for scalars } a_1, \ldots, a_n\}.}$$

Let $\{v_1, \ldots, v_n\}$ be n vectors in $\mathbb{R}^n$, $A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$.

Note:

1. If $v_1, \ldots, v_n$ are in $\mathbb{R}^m$, $\text{Span}\{v_1, \ldots, v_n\} = C(A)$. Thus v is in $\text{Span}\{v_1, \ldots, v_n\} \Leftrightarrow Ax = v$ is consistent.
2. $\text{Span}\{v_1, \ldots, v_n\} = \mathbb{R}^m \Leftrightarrow Ax = v$ is consistent for all $v \in \mathbb{R}^m$ $\Leftrightarrow$ A has m pivots. This implies, $m \leq n$.
3. Let $m = n$. Then A is invertible $\Leftrightarrow$ A has n pivots $\Leftrightarrow$ $Ax = v$ is consistent for every v in $\mathbb{R}^n$ $\Leftrightarrow$ $\text{Span}\{v_1, \ldots, v_n\} = \mathbb{R}^n$.

Example: $\text{Span}\{e_1, \ldots, e_n\} = \mathbb{R}^n$.

Linear Span: Examples

Examples:

1. $\text{Span}\{0\} = \{0\}$.
2. If $v \neq 0$ is a vector, $\text{Span}\{v\} = \{av, \text{ for scalars } a\}$.

   Geometrically (in $\mathbb{R}^m$): $\text{Span}\{v\}$ = the line in the direction of v passing through the origin.
3. $\text{Span}\left\{\begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} = \mathbb{R}^2$.
4. If A is $m \times n$, then $\text{Span}\{A_1, \ldots, A_n\} = C(A)$.
5. If $v_1, \ldots, v_k$ are the special solutions of A, then $\text{Span}\ \{v_1, \ldots, v_k\} = N(A)$.

Remark: All of the above are subspaces.

Exercise: $\text{Span}(S)$ is a subspace of V. Why?

6. Let $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$, $v_3 = \begin{pmatrix} 3 \\ 8 \\ 7 \end{pmatrix}$ and $v_4 = \begin{pmatrix} 5 \\ 12 \\ 13 \end{pmatrix}$. Is $v = \begin{pmatrix} 1 & 0 & 4 \end{pmatrix}^T$ in $\mathrm{Span}\{v_1, v_2, v_3, v_4\}$?

Set $A = \begin{pmatrix} v_1 & \cdots & v_4 \end{pmatrix}$, and $b = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}$.

Recall $Ax = b$ is solvable $\Leftrightarrow 5b_1 - b_2 - b_3 = 0$.

$\Rightarrow v$ is not in $\mathrm{Span}\{v_1, v_2, v_3, v_4\}$,

and $w = \begin{pmatrix} 1 & 0 & 5 \end{pmatrix}^T = 4v_1 + (-1)v_3$ is in it.

Observe: $v_2 = 2v_1$ and $v_4 = 2v_1 + v_3$. Hence v_2, v_4 are in $\mathrm{Span}\{v_1, v_3\} \Rightarrow \mathrm{Span}\{v_1, v_2, v_3, v_4\} = \mathrm{Span}\{v_1, v_3, \}$.

Thus, $C(A) =$ the plane $P : (5x - y - z = 0) = \mathrm{Span}\{v_1, v_3\}$.

Question:

Is the span of two vectors in $\mathbb{R}^3$ always a plane?

7. Let $v_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$, $v_3 = \begin{pmatrix} 6 \\ 7 \\ 5 \end{pmatrix}$ and $v_4 = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}$? .
   Is $v = \begin{pmatrix} 4 & 3 & 5 \end{pmatrix}^T$ in Span$\{v_1, v_2, v_3, v_4\}$? If yes, write v as a linear combination of v_1, v_2, v_3 and v_4.

Let $A = \begin{pmatrix} v_1 & \cdots & v_4 \end{pmatrix}$. The question can be rephrased as:

Question: Is v in $C(A)$, i.e., is $Ax = v$ solvable? If yes, find a solution.

Exercise: $Ax = \begin{pmatrix} a & b & c \end{pmatrix}^T$ is consistent $\Leftrightarrow 2a - b - c = 0$.

Observe and prove:

(i) that Span$\{v_1, v_2, v_3, v_4\} = C(A)$ is a plane! **(ii)** that v is in Span$\{v_1, v_2, v_3, v_4\}$ (and $w = \begin{pmatrix} 4 & 3 & 4 \end{pmatrix}^T$ is not).

Solve $Ax = v$ using the row reduced form of A to get particular solution: $\begin{pmatrix} 4 & -1 & 0 & 0 \end{pmatrix}^T$ and $v = 4v_1 + (-1)v_2$.

Linear Independence: Example

With $v_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$, $v_3 = \begin{pmatrix} 6 \\ 7 \\ 5 \end{pmatrix}$ and $v_4 = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}$

Observe: $v_3 = v_1 + v_2$ and $v_4 = -2v_1 + 2v_2$.

Hence v_3 and v_4 are in $\mathrm{Span}\{v_1, v_2\}$.

Therefore, $\mathrm{Span}\{v_1, v_2\} = \mathrm{Span}\{v_1, v_2, v_3, v_4\}$
$= C(A) =$ the plane $P : (2x - y - z = 0)$.

Question: Is the span of two vectors in $\mathbb{R}^3$ always a plane?
A: Not always. If v is a multiple of w, then $\mathrm{Span}\{v, w\} = \mathrm{Span}\{w\}$, which is a line through the origin or zero.

Question: If v and w are not on the same line through the origin? **A:** Yes. v, w are examples of *linearly independent vectors*.

Linear Independence: Definition

The vectors $v_1, v_2, \ldots, v_n$ in a vector space V, are *linearly independent* if $\boxed{a_1 v_1 + \cdots + a_n v_n = 0 \Rightarrow a_1 = 0, \ \ldots, \ a_n = 0}$.

Equivalently, for every nonezero $(a_1, \ldots, a_n)^T$ in $\mathbb{R}^n$,

we have $a_1 v_1 + \cdots + a_n v_n \neq 0$ in V.

The vectors $v_1, \ldots, v_n$ are *linearly dependent* if they are not linearly independent. i.e., we can find $(a_1, \ldots, a_n)^T \neq 0$ in $\mathbb{R}^n$, such that $a_1 v_1 + \cdots + a_n v_n = 0$ in V.

Observe: When $V = \mathbb{R}^m$, if $A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$, then

$\boxed{Ax = x_1 v_1 + \cdots + x_n v_n = 0 \text{ has a } \textbf{non-trivial} \text{ solution,}}$

$\Leftrightarrow N(A) \neq 0 \Leftrightarrow v_1, \ldots, v_n$ are linearly **dependent** and

$\boxed{Ax = x_1 v_1 + \cdots + x_n v_n = 0 \text{ has only the } \textbf{trivial} \text{ solution}}$

$\Leftrightarrow N(A) = 0 \Leftrightarrow v_1, \ldots, v_n$ are linearly **independent**.