

# MA 110 - Ordinary Differential Equations

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# Outline of the lecture

- Integrating factors
- Bernoulli equation

# Warm up!

- 1 The value of  $b$  that makes  $(xy^2 + bx^2y)dx + (x + y)x^2dy = 0$  exact is .....
- 2 The value of  $r$  for which the DE  $y'' + y' - 6y = 0$  has solutions of the form  $y = e^{rt}$  are .....  
(Ans.  $(r^2 + r - 6)e^{rt} = 0$ . So,  $r = 2, -3$ )
- 3 The solution of  $-ydx + (x + \sqrt{xy})dy = 0$  is .....
- 4 The solution of  $y' = e^{3x} - y$  is .....  
(Ans.  $ye^x = \int Q(x)e^x dx + C$  where  $Q(x) = e^{3x}$ )
- 5 Let  $L := x^2 \frac{d^2}{dx^2} + 2x \frac{d}{dx} + 2$  be a differential operator.  $L(x^3)$  is .....  
(Ans.  $L(x^3) = 6x^3 + 6x^3 + 2x^3 = 14x^3$ )

# Integrating Factors

Suppose the first order ODE

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0$$

is **not** exact; that is,  $M_y \neq N_x$ . In this situation, we try to find a function  $\mu(x, y)$  such that

$$\mu \cdot M + \mu \cdot N \frac{dy}{dx} = 0$$

is exact; i.e.,

$$(\mu \cdot M)_y = (\mu \cdot N)_x.$$

Thus,

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x.$$

That is,  $\mu(x, y)$  satisfies the DE

$$\mu_y M - \mu_x N + (M_y - N_x)\mu = 0.$$

Such a function  $\mu(x, y)$  is called an **integrating factor** of the given ODE.

# Integrating Factor - function of $x$ alone

In practice, we start by looking for an IF which depends only on one variable  $x$  or  $y$ , because it may be difficult to solve the PDE  $\mu_y M - \mu_x N + (M_y - N_x)\mu = 0$ .

Case 1 :

Suppose  $\mu$  is a function of  $x$  alone. That is,  $\mu = \mu(x)$ ,  $\mu_y = 0$ .  
Then, the PDE above reduces to

$$\mu_x N = (M_y - N_x) \mu.$$

You have to make a good guess for what the "myu" can be.

Thus,

$$\frac{d\mu}{dx} = \left( \frac{M_y - N_x}{N} \right) \mu.$$

If further,  $\frac{M_y - N_x}{N}$  is a function of  $x$  then the above DE is separable & we try to solve it to find  $\mu(x)$ .

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}.$$

# Integrating Factor - function of $y$ alone

Case 2 :

If we assume  $\mu$  to be a function of  $y$  alone in the PDE

$$\mu_y M - \mu_x N + (M_y - N_x)\mu = 0,$$

then we get an analogous equation:

$$\frac{d\mu}{dy} = \left( \frac{N_x - M_y}{M} \right) \mu.$$

If further,  $\frac{N_x - M_y}{M}$  is a function of  $y$  then the above DE is separable & we try to solve it to find  $\mu(y)$ .

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}.$$

# Example 1

Solve the ODE:

$$(8xy - 9y^2) + (2x^2 - 6xy)\frac{dy}{dx} = 0.$$

Let  $M = 8xy - 9y^2$  and  $N = 2x^2 - 6xy$ .

Thus,  $M_y = 8x - 18y$  and  $N_x = 4x - 6y$ . As  $M_y \neq N_x$ , the given ODE is not exact.

We first try to find an IF depending only upon one variable.

Note that

$$\frac{M_y - N_x}{N} = \frac{4x - 12y}{2x(x - 3y)} = \frac{2}{x}, \text{ a function of } x \text{ alone.}$$

Hence by the earlier discussion, we have:

$$\frac{d\mu}{dx} = \frac{2}{x}\mu.$$

Solving this separable ODE, we get  $\ln |\mu| = \ln x^2$ . Hence,

$\mu(x) = x^2$  can be chosen as an IF for the given ODE.

# Integrating Factors

Multiplying the given ODE by  $\mu(x) = x^2$ , we get:

$$(8x^3y - 9x^2y^2) + (2x^4 - 6x^3y)\frac{dy}{dx} = 0.$$

Check that this is an exact ODE. (How? )

To solve this exact ODE, we need to find  $u(x, y)$  such that

$$8x^3y - 9x^2y^2 = u_x \text{ \& } 2x^4 - 6x^3y = u_y.$$

To find  $u(x, y)$ :

Step I:  $u(x, y) = 2x^4y - 3x^3y^2 + k(y)$ .

Step II:  $2x^4 - 6x^3y = u_y = 2x^4 - 6x^3y + k'(y)$ .

Thus,  $k'(y) = 0$ . Hence,

$$u(x, y) = 2x^4y - 3x^3y^2 = c$$

is a solution of the given ODE.



## Example 2

Solve the DE:  $-y + x \frac{dy}{dx} = 0$ .

Check that this is not an exact DE.

Let  $M(x, y) = -y$  and  $N(x, y) = x$ .

To find a possible IF  $\mu$ : note that  $\frac{N_x - M_y}{M} = -\frac{2}{y}$ , a function of  $y$  alone.

By the earlier discussion, we obtain :

$$\frac{d\mu}{dy} = -\frac{2}{y}\mu.$$

Thus,  $\ln |\mu| = -2 \ln |y|$ .

So we choose

$$\mu(y) = \frac{1}{y^2}$$

as an IF.

Then,  $\frac{-y + x \frac{dy}{dx}}{y^2} = 0$  is exact. Thus,  $d \left( -\frac{x}{y} \right) = 0$ .

Therefore, solution is given by  $\frac{x}{y} = c$ .

# Bernoulli equation - (non-linear reduced to linear)

Consider

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \quad (n = 0, 1 \text{ yields linear equations!})$$

**Claim :** Let  $n \neq 0, 1$ .

Then the transformation  $v = y^{1-n}$  reduces the Bernoulli equation to a linear equation in  $v$ .

**Justification :**

Let  $v = y^{1-n}$ .

$$\frac{dv}{dx} = (1 - n) y^{1-n-1} \frac{dy}{dx}$$

That is,

$$\frac{dy}{dx} = \frac{1}{1 - n} y^n \frac{dv}{dx}.$$

Substituting in the DE,

$$\frac{1}{1-n} y^n \frac{dv}{dx} + P(x)y = Q(x)y^n$$

$$\frac{1}{1-n} \frac{dv}{dx} + P(x)v = Q(x) \quad (\text{assuming } y \neq 0)$$

Hence,

$$\frac{dv}{dx} + (1-n)P(x)v = Q(x)(1-n), \text{ which is a linear DE in } v.$$

## Example - Bernoulli

Solve :  $\frac{dy}{dx} + y = xy^3$ .

Let  $\boxed{v = y^{-2}}$ .

$$\frac{dv}{dx} = -2y^{-3} \frac{dy}{dx} \implies -\frac{1}{2} \frac{dv}{dx} + v = x$$

That is,  $\frac{dv}{dx} - 2v = -2x$  (linear equation in  $v$ )

Integrating factor is  $e^{-2x}$ .

$$\begin{aligned} ve^{-2x} &= - \int 2xe^{-2x} dx + C \\ &= \frac{2xe^{-2x}}{-2} - \int 2 \frac{e^{-2x}}{2} + C \\ &= xe^{-2x} + \frac{e^{-2x}}{2} + C \end{aligned}$$

$$\implies \frac{1}{y^2} = x + \frac{1}{2} + Ce^{2x}.$$

# Equations reducible to linear equations

Consider

$$\frac{d}{dy}(f(y)) \frac{dy}{dx} + P(x)f(y) = Q(x),$$

where  $f$  is an unknown function of  $y$ .

Set  $v = f(y)$ .

Then,

$$\frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = \frac{d}{dy}(f(y)) \frac{dy}{dx}.$$

Hence the given equation is

$$\frac{dv}{dx} + P(x)v = Q(x), \text{ which is linear in } v.$$

Remark : Bernoulli DE is a special case when  $f(y) = y^{1-n}$ .

# Example

Solve :  $\cos y \frac{dy}{dx} + \frac{1}{x} \sin y = 1.$

Set  $v = \sin y$ .

Then,

$$\frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = \cos y \frac{dy}{dx}.$$

Hence the given equation is

$$\frac{dv}{dx} + \frac{1}{x}v = 1, \text{ which is linear in } v.$$

That is,

$$e^{\int \frac{1}{x} dx} v(x) = \int e^{\int \frac{1}{x} dx} dx + C$$

$$\implies x v(x) = \frac{x^2}{2} + C$$

$$\sin y = \frac{x}{2} + \frac{C}{x}.$$