# MA 110 - Ordinary Differential Equations

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# Outline of the lecture

- Recall : Definition & Properties of Laplace transforms

# Examples

We know that

$$L(\cos \beta t) = \frac{s}{s^2 + \beta^2}, \;\; L(\sin \beta t) = \frac{\beta}{s^2 + \beta^2}.$$

Therefore, using

$$F'(s) = -L(tf(t)),$$

$$L(t \cos \beta t) = -\frac{d}{ds}\left(\frac{s}{s^2 + \beta^2}\right) = \frac{s^2 - \beta^2}{(s^2 + \beta^2)^2},$$

and

$$L(t \sin \beta t) = -\frac{d}{ds}\left(\frac{\beta}{s^2 + \beta^2}\right) = \frac{2s\beta}{(s^2 + \beta^2)^2}.$$

# Inverse Laplace Transforms - Examples

$$L(t\sin\beta t) = -\frac{d}{ds}\left(\frac{\beta}{s^2+\beta^2}\right) = \frac{2s\beta}{(s^2+\beta^2)^2}.$$

Thus,

$$L^{-1}\left(\frac{s}{(s^2+\beta^2)^2}\right) = \frac{t\sin\beta t}{2\beta}.$$

Thus, from Property 5 $\left(L\left(\int_0^t f(\tau)\,d\tau\right) = \frac{F(s)}{s},\ \text{for } s>\alpha\right)$

$$L^{-1}\left(\frac{1}{(s^2+\beta^2)^2}\right) = \int_0^t \frac{\tau\sin\beta\tau}{2\beta}d\tau,$$

and from Property 4 $(L(f') = sL(f) - f(0))$,

$$\frac{d}{dt}\left(\frac{t\sin\beta t}{2\beta}\right) = L^{-1}\left(\frac{s^2}{(s^2+\beta^2)^2}\right).$$

Find the inverse transform of

$$G(s) = \frac{1}{s^2 - 4s + 5}.$$

Note that

$$G(s) = \frac{1}{(s-2)^2 + 1}.$$

Using I shift theorem, $L(e^{at}f(t)) = F(s-a)$.
Here,

$$F(s) = \frac{1}{s^2+1} = L(\sin t).$$

Hence,

$$L^{-1}(G(s)) = L^{-1}(F(s-2)) = e^{2t}\sin t.$$

# Property 7 : Integration of Laplace transforms

Suppose $f : [0, \infty) \to \mathbb{R}$ is piecewise continuous of exponential order. Suppose further that $\lim_{t\to 0^+} \frac{f(t)}{t}$ exists. Then,

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\tilde{s})\, d\tilde{s}, \quad s > \alpha.$$

Proof:

$$\begin{aligned}
\int_s^\infty F(\tilde{s})\, d\tilde{s} &= \int_{\tilde{s}=s}^\infty \left(\int_{t=0}^\infty e^{-\tilde{s}t} f(t) dt\right) d\tilde{s} \\
&= \int_{t=0}^\infty \int_{\tilde{s}=s}^\infty e^{-\tilde{s}t} f(t) d\tilde{s} dt \\
&= \int_0^\infty f(t) \left[\frac{e^{-\tilde{s}t}}{-t}\right]_s^\infty dt \\
&= \int_0^\infty e^{-st} \frac{f(t)}{t} dt = L\left(\frac{f(t)}{t}\right).
\end{aligned}$$

# Example

Find $L^{-1}$ of $\ln(1+\frac{w^2}{s^2})$.

We have:

$$\ln\left(1+\frac{\omega^2}{s^2}\right) = -\int_s^\infty \frac{d}{ds}\left(\ln\left(1+\frac{\omega^2}{s^2}\right)\right)\,ds.$$

Hence,

$$L^{-1}\left(\ln\left(1+\frac{\omega^2}{s^2}\right)\right) = L^{-1}\left(-\int_s^\infty \frac{d}{ds}\left(\ln\left(1+\frac{\omega^2}{s^2}\right)\right)\,ds\right)$$

From Property 7,

$$L^{-1}\left(\int_s^\infty F(\tilde{s})\,d\tilde{s}\right) = \frac{f(t)}{t}, \quad s > \alpha.$$

$$\begin{aligned} -\frac{d}{ds}\left(\ln(1+\frac{w^2}{s^2})\right) &= \frac{2w^2}{s(s^2+w^2)} \\ &= \frac{2}{s} - \frac{2s}{s^2+w^2} \end{aligned}$$

Let $f(t) = 2 - 2\cos wt$. Hence,

$$L^{-1}\left(\int_s^\infty F(\tilde{s})\, d\tilde{s}\right) = L^{-1}\left(\int_s^\infty \left(\frac{2}{\tilde{s}} - \frac{2\tilde{s}}{\tilde{s}^2+w^2}\right)\, d\tilde{s}\right)$$

$$= \left(\frac{2-2\cos wt}{t}\right).$$

# Exercises

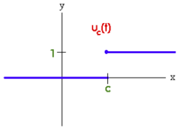
➪ $L^{-1}\left(ln\,(1-\dfrac{a^2}{s^2})\right)$

➪ $L^{-1}(\tan^{-1}(\dfrac{1}{s}))$.

For $c \geq 0$, the function

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$$

is called the Heaviside function.



# Example

Write the following piecewise continuous function in terms of Heaviside functions:

$$f(t) = \begin{cases} 2 & t \in [0, 4) \\ 5 & t \in [4, 7) \\ -1 & t \in [7, 9) \\ 1 & t \geq 9. \end{cases}$$

Note that $u_c - u_d$ takes 1 on $[c, d)$ and 0 everywhere else. Thus,

$$\begin{aligned} f(t) &= 2(u_0 - u_4) + 5(u_4 - u_7) - (u_7 - u_9) + u_9 \\ &= 2u_0 + 3u_4 - 6u_7 + 2u_9. \end{aligned}$$

# Laplace Transform of Heaviside function

$$L(u_c(t))(s) = \frac{e^{-cs}}{s}.$$

$$\begin{aligned} L(u_c(t))(s) &= \int_0^\infty e^{-st} u_c(t) dt \\ &= \int_c^\infty e^{-st} dt \\ &= \frac{e^{-cs}}{s}, \end{aligned}$$

for $s > 0$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Consider the new function

$$g(t) = \begin{cases} 0 & \text{if } t < c \\ f(t-c) & \text{if } t \geq c. \end{cases}$$

Note that

$$g(t) = u_c(t) f(t-c).$$

Suppose $L(f(t)) = F(s)$ for $s > a \geq 0$. If $c > 0$, then for $s > a$,

$$L(u_c(t)f(t-c)) = e^{-cs}F(s).$$

Proof:

$$\begin{aligned} L(u_c(t)f(t-c)) &= \int_0^\infty e^{-st}u_c(t)f(t-c)dt \\ &= \int_c^\infty e^{-st}f(t-c)dt \\ &= \int_0^\infty e^{-s(u+c)}f(u)du \\ &= e^{-cs}F(s). \end{aligned}$$

# Example

Find the Laplace transform of

$$f(t) = \begin{cases} \sin t & 0 \le t < \frac{\pi}{4} \\ \sin t + \cos(t - \frac{\pi}{4}) & t \ge \frac{\pi}{4}. \end{cases}$$

Write

$$f(t) = \sin t + u_{\frac{\pi}{4}}(t) \cdot \cos(t - \frac{\pi}{4}).$$

Hence,

$$\begin{aligned} L(f(t)) &= L(\sin t) + L(u_{\frac{\pi}{4}}(t) \cdot \cos(t - \frac{\pi}{4})) \\ &= \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s} \cdot \frac{s}{s^2+1} \\ &= \frac{1 + e^{-\frac{\pi}{4}s}s}{s^2+1}. \end{aligned}$$

(use $L(u_c(t)f(t-c)) = e^{-cs}F(s)$).

# Convolution of functions

The convolution of $f(t)$ and $g(t)$ is defined as:

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\, d\tau.$$

Check:

1. $f * g = g * f$ (Put $y = t - \tau$.)
2. $f * (g_1 + g_2) = f * g_1 + f * g_2$
3. $(f * g) * h = f * (g * h)$
4. $f * 0 = 0 * f = 0.$

Remark: $f * 1$ need not be $f$.
Check that $\sin t * 1 = 1 - \cos t$.

# Property 9 : Laplace transform of convolution

Suppose $L(f)$ and $L(g)$ exist for all $s > a \geq 0$. Then,

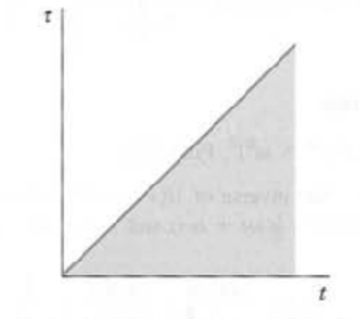
$$L(f * g) = L(f) \cdot L(g),$$

for $s > a$.

Proof: Let $L(f) = F(s)$ and $L(g) = G(s)$. Fix $\tau \geq 0$.

$$\begin{aligned}
e^{-s\tau} G(s) &= L\left(u_\tau(t) g(t-\tau)\right) \text{ using II shifting theorem} \\
&= \int_0^\infty e^{-st} u_\tau(t) g(t-\tau)\, dt \\
&= \int_\tau^\infty e^{-st} g(t-\tau)\, dt.
\end{aligned}$$

$$\begin{aligned}
L(f) \cdot L(g) &= F(s)G(s) = \left(\int_0^\infty e^{-s\tau} f(\tau)\, d\tau\right) G(s) \\
&= \int_0^\infty e^{-s\tau} G(s) f(\tau)\, d\tau \\
&= \int_{\tau=0}^\infty f(\tau) \left(\int_{t=\tau}^\infty e^{-st} g(t-\tau) dt\right)\, d\tau
\end{aligned}$$




That is,

$$\begin{aligned} L(f)\cdot L(g) &= \int_{\tau=0}^{\infty} f(\tau)\left(\int_{t=\tau}^{\infty} e^{-st} g(t-\tau)dt\right)\, d\tau \\ &= \int_{t=0}^{\infty} e^{-st}\left(\int_{\tau=0}^{t} f(\tau)g(t-\tau)d\tau\right)\, dt \\ &= \int_{0}^{\infty} e^{-st}(f*g)(t)dt = L(f*g). \end{aligned}$$

# Example

Find $L^{-1}$ of

$$H(s) = \frac{a}{s^2(s^2+a^2)}.$$

Recall

$$L(t) = \frac{1}{s^2},$$

and

$$L(\sin at) = \frac{a}{s^2+a^2}.$$

Thus,

$$L(t * \sin at) = H(s).$$

Now,

$$t * \sin at = \int_0^t (t-\tau)\sin a\tau \, d\tau = \frac{at - \sin at}{a^2}.$$

# Example

Solve the IVP:

$$y'' + 4y = g(t),\ \ y(0) = 3, y'(0) = -1.$$

Taking Laplace transforms:

$$L(y'') + 4L(y) = L(g) = G(s).$$

Thus,

$$s^2 L(y) - sy(0) - y'(0) + 4L(y) = G(s).$$

Therefore,

$$\begin{aligned}
L(y) &= \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4} \\
&= 3 \cdot \frac{s}{s^2+4} - \frac{1}{2} \cdot \frac{2}{s^2+4} + \frac{1}{2} \cdot \frac{2}{s^2+4} \cdot G(s) \\
&= 3L(\cos 2t) - \frac{1}{2} L(\sin 2t) + \frac{1}{2} L(\sin 2t) \cdot L(g) \\
&= 3L(\cos 2t) - \frac{1}{2} L(\sin 2t) + \frac{1}{2} L(\sin 2t * g).
\end{aligned}$$

Hence,

$$y = 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t \sin 2(t-x) g(x) dx.$$

## Property 10. Laplace transform of periodic functions

Let $f$ be a periodic function with period $p$ whose Laplace transform exists. Then,

$$L(f) = \frac{1}{1 - e^{-sp}} \int_0^p e^{-st} f(t)dt.$$

$$\begin{aligned}
L(f)(s) &= \int_0^\infty e^{-st} f(t)dt \\
&= \int_0^p e^{-st} f(t)dt + \int_p^{2p} e^{-st} f(t)dt + \int_{2p}^{3p} e^{-st} f(t)dt + \dots \\
&\quad \text{setting } t - (n-1)p \text{ as } t \text{ in the } n^{th} \text{ integral} \\
&= \int_0^p e^{-st} f(t)dt + \int_0^p e^{-s(t+p)} f(t)dt + \int_0^p e^{-s(t+2p)} f(t)dt \\
&\quad + \dots \\
&= \frac{1}{1 - e^{-sp}} \int_0^p e^{-st} f(t)dt.
\end{aligned}$$

Tut 5, Q 6

# Example

Solve

$$2y_1' - y_2' - y_3' = 0, y_1' + y_2' = 4t + 2, y_2' + y_3 = t^2 + 2;$$
$$y_1(0) = 0, y_2(0) = 0, y_3(0) = 0.$$

Taking Laplace transforms and denoting Laplace transforms of $y_1, y_2, y_3$ by $Y_1$, $Y_2$, $Y_3$, we have

$$\begin{aligned} 2sY_1 - sY_2 - sY_3 &= 0 \\ sY_1 + sY_2 &= \frac{4}{s^2} + \frac{2}{s} \\ sY_2 + Y_3 &= \frac{2}{s^3} + \frac{2}{s}. \end{aligned}$$

Solving:

$$Y_1 = \frac{2}{s^3}, Y_2 = \frac{2}{s^3} + \frac{2}{s^2}, Y_3 = \frac{2}{s^3} - \frac{2}{s^2}.$$

Thus,

$$y_1(t) = t^2, y_2(t) = t^2 + 2t, y_3(t) = t^2 - 2t.$$

# Solution of a system of DE using LT

Solve $x' = x + y, y' = 4x + y$.
Denoting $X(s)$ and $Y(s)$ as the LT's of $x$ and $y$ respectively.
Taking Laplace transforms,

$$\begin{aligned} sX - x(0) &= X + Y \\ sY - y(0) &= 4X + Y. \end{aligned}$$

Solving:

$$X(s) = \frac{(s-1)x(0) + y(0)}{s^2 - 2s - 3}, Y(s) = \frac{4x(0) + (s-1)y(0)}{s^2 - 2s - 3}.$$

Take $L^{-1}$ to get $x(t)$ and $y(t)$.
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