MA-110 Linear Algebra and Differential Equations

Rekha Santhanam

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

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Summary: Finding $N(A) = N(U) = N(R)$

Let A be $m \times n$. To solve $Ax = 0$, find R and solve $Rx = 0$.

1. Find free (independent) and pivot (dependent) variables:
   pivot variables: columns in R with pivots ($\longleftrightarrow$ t and v).
   free variables: columns in R without pivots ($\longleftrightarrow$ u and w).

2. No free variables, i.e., $\operatorname{rank}(A) = n \Rightarrow N(A) = 0$.

3. (a) If $\operatorname{rank}(A) < n$, obtain a special solution:
   Set one free variable $= 1$, the other free variables $= 0$.
   Solve $Rx = 0$ to obtain values of pivot variables.

   (b) Find special solutions for each free variable.
   $N(A) =$ space of linear combinations of special solutions.

- This information is stored in a compact form in:

  Null Space Matrix: Special solutions as columns.

Solving $Ax = b$

Caution: If $b \neq 0$, solving $Ax = b$ may not be the same as solving $Ux = b$ or $Rx = b$.

Example: $Ax = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix} \begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = b.$

Convert to $Ux = c$ and then $Rx = d$.

$$\begin{pmatrix} 1 & 2 & 3 & 5 & | & b_1 \\ 2 & 4 & 8 & 12 & | & b_2 \\ 3 & 6 & 7 & 13 & | & b_3 \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{1} & 2 & 3 & 5 & | & b_1 \\ 0 & 0 & 2 & 2 & | & b_2 - 2b_1 \\ 0 & 0 & -2 & -2 & | & b_3 - 3b_1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \mathbf{1} & 2 & 3 & 5 & | & b_1 \\ 0 & 0 & \mathbf{2} & 2 & | & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & | & b_3 + b_2 - 5b_1 \end{pmatrix}$$

System is consistent $\Leftrightarrow b_3 + b_2 - 5b_1 = 0$, i.e., $b_3 = 5b_1 - b_2$

Solving $Ax = b$ or $Ux = c$ or $Rx = d$

$Ax = b$ has a solution $\iff b_3 = 5b_1 - b_2$.

for example, there is no solution when $b = \begin{pmatrix} 1 & 0 & 4 \end{pmatrix}^T$.

Suppose $b = \begin{pmatrix} 1 & 0 & 5 \end{pmatrix}^T$. Then $[A|b] \rightarrow$

$$\left(\begin{array}{cccc|c} \mathbf{1} & 2 & 3 & 5 & b_1 \\ 0 & 0 & \mathbf{2} & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{array}\right) = \left(\begin{array}{cccc|c} \mathbf{1} & 2 & 3 & 5 & 1 \\ 0 & 0 & \mathbf{2} & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} \mathbf{1} & 2 & 3 & 5 & 1 \\ 0 & 0 & \mathbf{1} & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} \mathbf{1} & 2 & 0 & 2 & 4 \\ 0 & 0 & \mathbf{1} & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$Ax = b$ is reduced to solving $Ux = c = \begin{pmatrix} 1 & -2 & 0 \end{pmatrix}^T$,

which is further reduced to solving $Rx = d = \begin{pmatrix} 4 & -1 & 0 \end{pmatrix}^T$.

Solving $Ax = b$ or $Ux = c$ or $Rx = d$

Solving $\boxed{Ax = b}$ is reduced to solving $\boxed{Rx = d}$, that is., we want to solve

$$\begin{pmatrix} \mathbf{1} & 2 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}$$

that is., $t = 4 - 2u - 2w$ and $v = -1 - w$

Set the free variables u and $w = 0$ to get $t = 4$ and $v = -1$
A particular solution: $\mathbf{x} = \begin{pmatrix} 4 & 0 & -1 & 0 \end{pmatrix}^T$.

Exercise: Check it is a solution i.e., check $Ax = b$.

Observe: In $Rx = d$, the vector d gives values for the pivot variables, when the free variables are 0.

From $Rx = d$, we get $t = 4 - 2u - 2w$ and $v = -1 - w$, where u and w are free. Complete set of solutions to $Ax = b$:

$$\begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 4 - 2u - 2w \\ u \\ -1 - w \\ w \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -1 \\ 0 \end{pmatrix} + u \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

To solve $Ax = b$ completely, reduce to $Rx = d$. Then:

1. Find $x_{\text{NullSpace}}$, i.e., $N(A)$, by solving $Rx = 0$.
2. Set free variables $= 0$, solve $Rx = d$ for pivot variables.
This is a particular solution: $x_{\text{particular}}$.

3. Complete solutions: $x_{\text{complete}} = x_{\text{particular}} + x_{\text{NullSpace}}$

Exercise: Verify geometrically for a 1×2 matrix, say $A = \begin{pmatrix} 1 & 2 \end{pmatrix}$.
Exercise: Prove statement 3 for solutions of any $Ax = b$.

The Column Space of A

Q: Does $Ax = b$ have a solution? **A:** Not always.

Main Q2: When does $Ax = b$ have a solution?

If $Ax = b$ has a solution, then we can find numbers $x_1, \ldots, x_n$

such that $\begin{pmatrix} A_{*1} & \cdots & A_{*n} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 A_{*1} + \cdots + x_n A_{*n} = b$,

that is, b can be written as a linear combination of columns of A.

The *column space* of A, denoted $C(A)$;

is the set of all linear combinations of the columns of A

$= \{b \text{ in } \mathbb{R}^m \text{ such that } Ax = b \text{ is } \textbf{consistent}\}$.

Finding $C(A)$: Consistency of $Ax = b$

Example: Let $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$. Then $Ax = b$, where

$b = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}^T$, has a solution whenever $-5b_1 + b_2 + b_3 = 0$.

- $C(A)$ is a plane in $\mathbb{R}^3$ passing through the origin with normal vector $\begin{pmatrix} -5 & 1 & 1 \end{pmatrix}^T$.
- $c = \begin{pmatrix} 1 & 0 & 4 \end{pmatrix}^T$ is not in $C(A)$ as $Ax = c$ is inconsistent.
- $d = \begin{pmatrix} 1 & 0 & 5 \end{pmatrix}^T$ is in $C(A)$ as $Ax = d$ is consistent.

Exercise: Write b as a linear combination of the columns of A. (A different way of saying: Solve $Ax = b$).

$x = \begin{pmatrix} 4 & 0 & -1 & 0 \end{pmatrix}^T$ is a solution of $Ax = b$, and

$$\boxed{\begin{pmatrix} 1 & 0 & 5 \end{pmatrix}^T = 4A_{*1} + (-1)A_{*3}.}$$

Q: Can you write b as a different combination of $A_{*1}, \ldots, A_{*4}$?

Linear Combinations in $C(A)$

Let A be an $m \times n$ matrix, u and v be real numbers.

- The column space of A, $\boxed{C(A) \text{ contains vectors from } \mathbb{R}^m}$.
- If a, b are in $C(A)$, i.e., $Ax = a$ and $Ay = b$ for some x, y in $\mathbb{R}^n$, then $ua + vb = u(Ax) + v(Ay) = A(ux + vy) = Aw$, where $w = ux + vy$. Hence, if $w = \begin{pmatrix} w_1 & \cdots & w_n \end{pmatrix}^T$, then
$$ua + vb = w_1 A_{*1} + \cdots w_n A_{*n},$$
i.e., a linear combination of vectors in $C(A)$ is also in $C(A)$.

  Thus, $C(A)$ is *closed under* linear combinations.

- If b is in $C(A)$, then b can be written as a linear combination of the columns of A in *as many ways* as the solutions of $Ax = b$.

Summary: $N(A)$ and $C(A)$

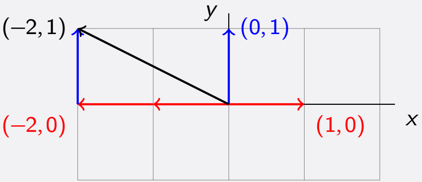
Remark: Let A be an $m \times n$ matrix.

- The null space of A, $N(A)$ contains vectors from $\mathbb{R}^n$.
- $Ax = 0 \Leftrightarrow x$ is in $N(A)$.
- The column space of A, $C(A)$ contains vectors from $\mathbb{R}^m$.
- If B is the nullspace matrix of A, then $C(B) = N(A)$.
- $Ax = b$ is consistent $\Leftrightarrow$ b is in $C(A)$ $\Leftrightarrow$
  b can be written as a linear combination of the columns of A. This can be done in as many ways as the solutions of $Ax = b$.
- Let A be $n \times n$.
  A is *invertible* $\Leftrightarrow N(A) = \{0\} \Leftrightarrow C(A) = \mathbb{R}^n$. Why?
- $N(A)$ and $C(A)$ are closed under linear combinations.

Vector Spaces: $\mathbb{R}^n$

We begin with $\mathbb{R}^1$, $\mathbb{R}^2, \ldots, \mathbb{R}^n$, etc., where $\mathbb{R}^n$ consists of all column vectors of length n, i.e.,
$\mathbb{R}^n = \{x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^T$, where $x_1, \ldots, x_n$ are in $\mathbb{R}\}$.
We can add two vectors, and we can multiply vectors by scalars, (i.e., real numbers). Thus, we can take linear combinations in $\mathbb{R}^n$.

Examples:
$\mathbb{R}^1$ is the real line, $\mathbb{R}^3$ is the usual 3-dimensional space, and $\mathbb{R}^2$ is represented by the x-y plane; the x and y co-ordinates are given by the two components of the vector.



Vector Spaces: Definition

Defn. A non-empty set V is a vector space if it is *closed under* vector addition (i.e., if x, y are in V, then $x + y$ must be in V) and scalar multiplication, (i.e., if x is in V, a is in $\mathbb{R}$, then $a * x$ must be in V) satisfying a few axioms.

Equivalently, $\boxed{x, y \text{ in } V,\ a, b \text{ in } \mathbb{R} \Rightarrow a * x + b * y \text{ must be in } V.}$

- A vector space is a triple $(V, +, *)$ with vector addition $+$ and scalar multiplication $*$(see next reading slide).
- The elements of V are called vectors and the scalars are chosen to be real numbers (for now).
- If the scalars are allowed to be complex numbers, then V is a *complex* vector space.
- **Primary Example:** $\mathbb{R}^n$. Under which operations.

Reading Slide: Vector Spaces definition continued

Let x, y and z be vectors, a and b be scalars The vector addition and scalar multiplication are required to satisfy the following axioms:

- $x + y = y + x$ Commutativity of addition
- $(x + y) + z = x + (y + z)$ Associativity of addition
- There is a unique vector 0, such that $x + 0 = x$
  Existence of additive identity
- For each x, there is a unique $-x$ such that $x + (-x) = 0$
  Existence of additive inverse
- $1 * x = x$ Unit property
- $(a + b) * x = a * x + b * x, \quad a * (x + y) = a * x + a * y$
  $(ab) * x = a * (b * x)$ Compatibility

Notation: For a scalar a, and a vector x, we denote $a * x$ by ax.

Subspaces: Definition and Examples

If V is a vector space, and W is a non-empty subset, then W is a *subspace* of V if:

$$\boxed{x,\ y \text{ in } W, \quad a,\ b \text{ in } \mathbb{R} \Rightarrow a * x + b * y \text{ are in } W.}$$

i.e., linear combinations stay in the subspace.

Examples:

1. $\{0\}$: The zero subspace and $\mathbb{R}^n$ itself.
2. $\{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$ is not a subspace of $\mathbb{R}^2$. Why?
3. The line $x - y = 1$ is not a subspace of $\mathbb{R}^2$. Why?
   Exercise: A line not passing through the origin is not a subspace of $\mathbb{R}^2$.
4. The line $x - y = 0$ is a subspace of $\mathbb{R}^2$. Why?
   Exercise: Any line passing through the origin is a subspace of $\mathbb{R}^2$.