

MA-110 Linear Algebra and Differential Equations

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Lecture 3 D3

Recap Choosing pivots: Two examples

Example 2: 3 equations in 3 unknowns (u, v, w)

$$0u + v + 2w = 1, \quad 0u + 6v + 4w = -2, \quad 0u + 7v - 2w = -9.$$

$$[A|b] = \left(\begin{array}{ccc|c} 0 & 1 & 2 & 1 \\ 0 & 6 & 4 & -2 \\ 0 & 7 & -2 & -9 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 0 & 1 & 2 & 1 \\ 0 & 0 & -8 & -8 \\ 0 & 0 & -16 & -16 \end{array} \right)$$

Coefficient of u is 0 in every equation. The first pivot is 1 and we eliminate v from the second and third equations. Solve for w and v to get $w = 1$, and $v = -1$.

Note: $(0, -1, 1)$ is a solution of the system. So is $(1, -1, 1)$. In general, $(*, -1, 1)$ is a solution, for any real number $*$.

Observe: Unique solution is not an option. **Why?** This system has infinitely many solutions.

Q: Does such a system always have infinitely many solutions?

A: Depends on the constant vector b .

Exercise: Find 3 vectors b for which the above system has (i) no solutions (ii) infinitely many solutions.

Summary: Pivots

- Can a pivot be zero? No (since we need to divide by it).
- If the first pivot (coefficient of 1st variable in 1st equation) is zero, then interchange it with next equation so that you get a non-zero first pivot. Do the same for other pivots.
- If the coefficient of the 1st variable is zero in every equation, consider the 2nd variable as 1st and repeat the previous step.
- Consider system of n equations in n variables.

The non-singular case, i.e. the system has **exactly** n pivots:

The system has a unique solution.

The singular case, i.e., the system has **atmost** $n - 1$ pivots:

The system has no solutions, i.e., it is **inconsistent**, or it will have infinitely many solutions, provided it is **consistent**.

What is a matrix?

A *matrix* is a collection of numbers arranged into a fixed number of rows and columns.

If a matrix A has m rows and n columns, the size of A is $m \times n$.

The rows of A are denoted $A_{1*}, A_{2*}, \dots, A_{m*}$, i.e., $A = \begin{pmatrix} A_{1*} \\ A_{2*} \\ \vdots \\ A_{m*} \end{pmatrix}$,

the columns are denoted $A_{*1}, A_{*2}, \dots, A_{*n}$, i.e.,

$A = (A_{*1} \ A_{*2} \ \cdots \ A_{*n})$, and the (i,j) th entry is A_{ij} (or a_{ij}).

Operations on Matrices: Matrix Addition

Example 1. We know how to add two row or column vectors.

$$(1 \ 2 \ 3) + (-3 \ -2 \ -1) = (-2 \ 0 \ 2) \text{ (component-wise)}$$

We can add matrices if and only if they have the same size,

and the addition is **component-wise**.

Example 2.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix} + \begin{pmatrix} -1 & -4 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 1 \\ 2 & 2 & 5 \end{pmatrix}$$

Thus

$$(A + B)_{i*} = A_{i*} + B_{i*} \text{ and } (A + B)_{*j} = A_{*j} + B_{*j}$$

Linear Systems: Multiplying a Matrix and a Vector

One row at a time (dot product): The system

$2u + v + w = 5$, $4u - 6v = -2$, $-2u + 7v + 2w = 9$
can be rewritten using **dot product** as follows:

$$\begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 5, \quad \begin{pmatrix} 4 & -6 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = -2 \quad \text{and}$$

$$\begin{pmatrix} -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 9.$$

Write the system in the $Ax = b$ form:

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2u + v + w \\ 4u - 6v \\ -2u + 7v + 2w \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}$$

Note: No. of columns of A = length of the vector x .

Multiplication of a Matrix and a Vector

Dot Product (row method): Ax is obtained by taking dot product of each row of A with x .

$$\text{If } A = \begin{pmatrix} A_{1*} \\ A_{2*} \\ A_{3*} \end{pmatrix}, \text{ then } Ax = \begin{pmatrix} A_{1*} \cdot x \\ A_{2*} \cdot x \\ A_{3*} \cdot x \end{pmatrix}$$

Linear Combinations (column method):

The column form of the system

$2u + v + w = 5$, $4u - 6v = -2$, $-2u + 7v + 2w = 9$ is:

$$u \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + v \begin{pmatrix} 1 \\ -6 \\ 7 \end{pmatrix} + w \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Thus Ax is a linear combination of columns of A , with the coordinates of x as weights, i.e., $Ax = uA_{*1} + vA_{*2} + wA_{*3}$.

An Example

$$\text{Let } A = \begin{pmatrix} 1 & 3 & -3 & -1 \\ 1 & 2 & 0 & -2 \\ 1 & 0 & -2 & 0 \end{pmatrix}, x = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \text{ and } e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$A_{1*} = (1 \ 3 \ -3 \ -1), \ A_{2*} = (1 \ 2 \ 0 \ -2) \ A_{3*} = ?.$$

$$\text{Then } A_{1*} \cdot x = ?, \ A_{2*} \cdot x = 0, \ A_{3*} \cdot x = 0, \text{ hence } Ax = \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}.$$

Q: What is Ae_1 ? **A:** The first column A_{*1} of A .

Exercise:

What should x be so that $Ax = A_{*j}$, the j th column of A ?

Observe: No. of rows of Ax = No. of rows of A ,
and No. of columns of Ax = No. of columns of x .

Question: What can you say about the solutions of $Ax = 0$?

Operations on Matrices: Matrix Multiplication

Two matrices A and B can be multiplied if and only if

no. of columns of A = no. of rows of B .

If A is $m \times \underline{n}$ and B is $\underline{n} \times r$, then AB is $m \times r$.

Key Idea: We know how to multiply a matrix and a vector.

Column wise: Write B column-wise, i.e., let

$B = (B_{*1} \ B_{*2} \ \cdots \ B_{*r})$. Then

$$AB = (AB_{*1} \ AB_{*2} \ \cdots \ AB_{*r})$$

Note: Each B_{*j} is a column vector of length n . Hence, AB_{*j} is a column vector of length m . So, the size of AB is $m \times r$.

Operations on Matrices: Matrix Multiplication

Row wise: Write A row-wise, i.e., let A_{1*}, \dots, A_{m*} be the rows of A . Then

$$AB = \begin{pmatrix} A_{1*} \\ \vdots \\ A_{m*} \end{pmatrix} B = \begin{pmatrix} A_{1*}B \\ \vdots \\ A_{m*}B \end{pmatrix}$$

Note: Each A_{i*} is a row vector of size $1 \times n$. Hence, $A_{i*}B$ is a row vector of size $1 \times r$. So, the size of AB is $m \times r$.

Working Rule:

The entry in the i th row and j th column of AB is the dot product of the i th row of A with the j th column of B , i.e.,

$$(AB)_{ij} = A_{i*} \cdot B_{*j}.$$

Properties of Matrix Multiplication

If A is $m \times n$ and B is $n \times r$.

a) $(AB)_{ij} = A_{i*} \cdot B_{*j} = (\text{ith row of } A) \cdot (\text{jth column of } B)$

b) $\text{jth column of } AB = A \cdot (\text{jth column of } B)$, i.e.,
 $(AB)_{*j} = AB_{*j}$.

c) $\text{ith row of } AB = (\text{ith row of } A) \cdot B$, i.e., $(AB)_{i*} = A_{i*}B$.

Properties of Matrix Multiplication:

- (associativity) $(AB)C = A(BC)$. Why?
- (distributivity) $A(B + C) = AB + AC$. How to verify?
 $(B + C)D = BD + CD$. Why?
- (non-commutativity) $AB \neq BA$, in general. Why?
Find examples.