$$\boxed{\text{IVP}: \quad y' = f(x,y) \quad , \quad y(x_0) = y_0}$$

$f : R \to \mathbb{R}$, $(x_0, y_0) \in R$, $|x - x_0| < a$
$|y - y_0| < b$

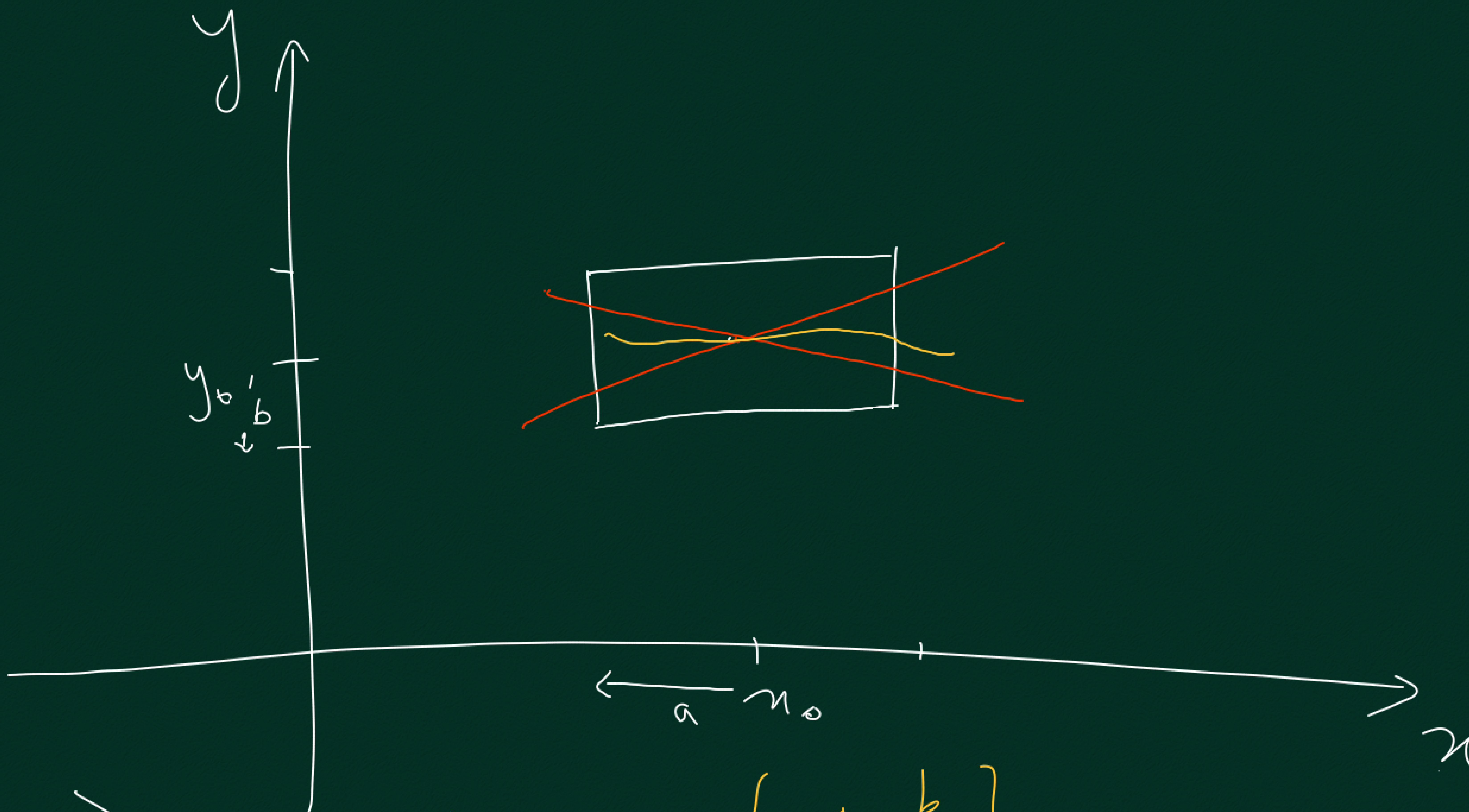
- continuity, bdd $\Rightarrow$ existence

$\exists K$ s.t. $|f(x,y)| < K \quad \forall (x,y) \in R$

With $\alpha = \min\left\{a, \frac{b}{K}\right\}$, $\exists$ sol^n for $|x - x_0| < \alpha$

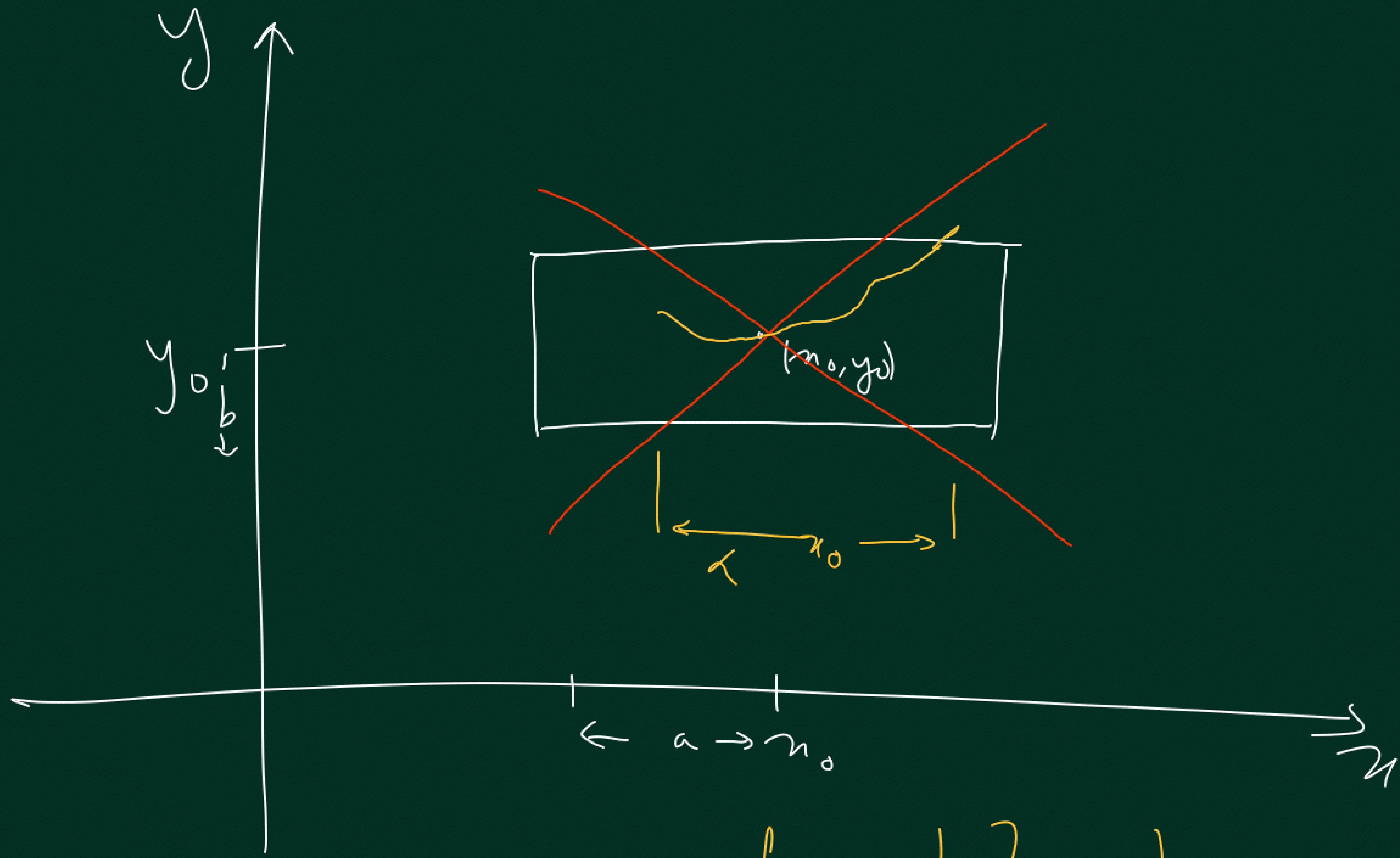
- If f satisfy Lipschitz condition wrt y in R,

then sol^n is unique



K

$\frac{b}{K} \geq a$, $\alpha = \min\left\{ a, \frac{b}{K} \right\} = a$



$$\frac{b}{K} < a$$

$$\alpha = \min\left\{ a, \frac{b}{k} \right\} = \frac{b}{k}$$

- $f(x) = x^2$, $x \in [1, 2]$

$$|x_1^2 - x_2^2| \le \zeta\, |x_1 - x_2|$$

- $f(x) = x^2$ on $\mathbb{R}$

$x_1 = 0$, $x_2 = x > 0$

$$\left|\frac{x^2}{x}\right| = x \to \infty \quad \text{as } x \to \infty$$

$x_1 = t$, $x_2 = 2t$, $t > 0$

$$\left|\frac{x_1^2 - x_2^2}{x_1 - x_2}\right| = x_1 + x_2 = 3t \to \infty \quad \text{as } t \to \infty$$

- $f(x) = \frac{1}{x^2}$, $[\alpha, \infty]$. So, Lipschitz

$$\left|\frac{df}{dx}\right| = \frac{2}{x^3} \leq \frac{2}{\alpha^3}$$

- $f(x) = \frac{1}{x^2}$, $(0, \infty)$

$x_1 = t$, $x_2 = 2t$

$$\left|\frac{\frac{1}{t^2} - \frac{1}{(2t)^2}}{t - 2t}\right| = \left|\frac{3}{-t\,(4t^2)}\right| \to \infty \text{ as } t \to 0$$

Not Lipschitz

$$y' = y^{1/3} \qquad y(0) = 0, \quad |x| \leq a, \quad |y| \leq b$$

$$\frac{dy}{y^{1/3}} = dx \Rightarrow \frac{3y^{2/3}}{2} = x + c \Rightarrow \frac{3y^{2/3}}{2} = x$$

$$\boxed{y \equiv 0} \qquad \Rightarrow \boxed{y = \left(\frac{2}{3}x\right)^{3/2}}$$

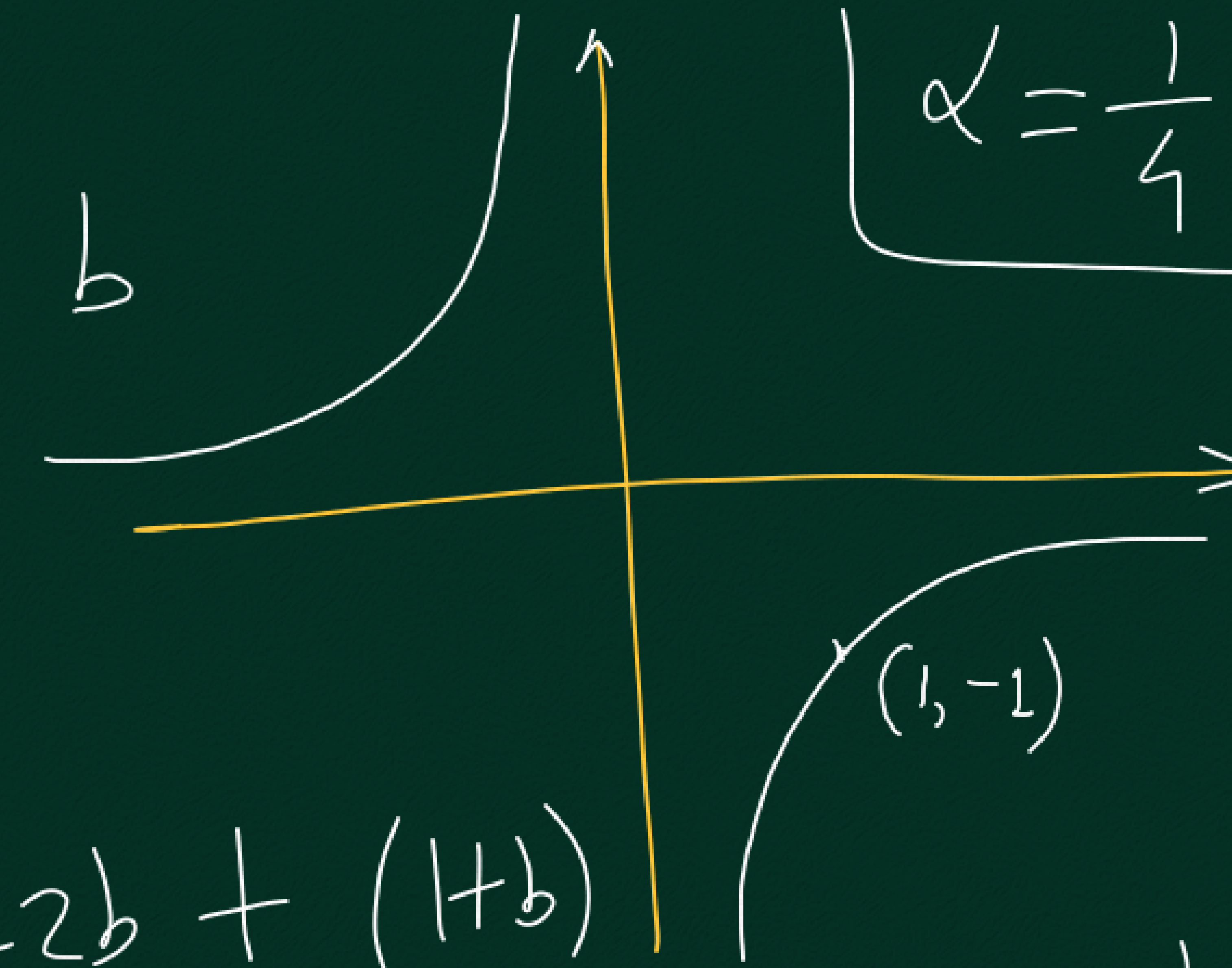
$y' = y^2$, $y(1) = -1$

$\frac{dy}{y^2} = dx \Rightarrow -\frac{1}{y} = x + C$, $1 = 1 + C \Rightarrow C = 0$

$$\boxed{-\frac{1}{y} = x}$$

$-b \le y+1 \le b$

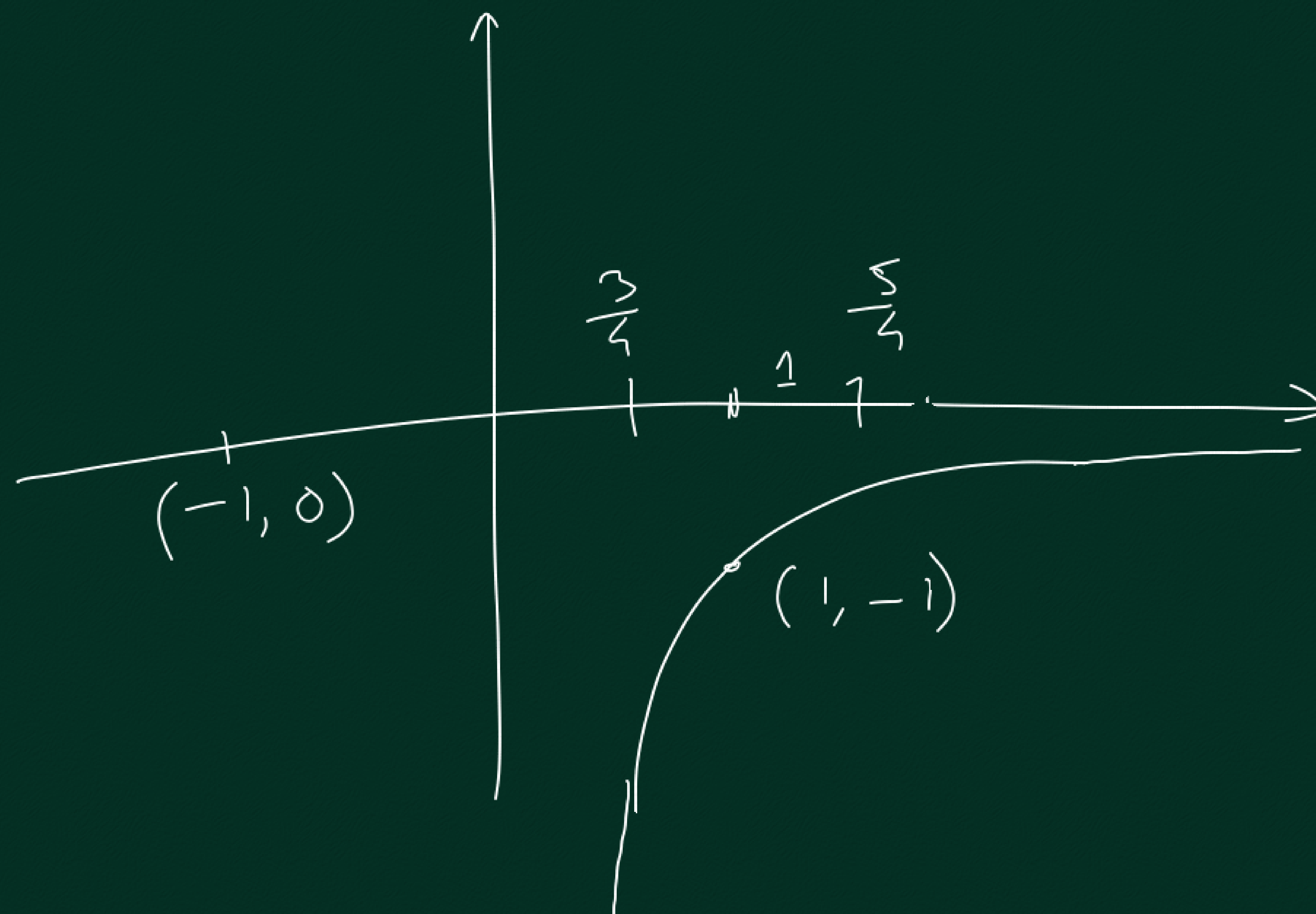
$$\boxed{\alpha = \frac{1}{4}}$$



$\alpha = \min\left\{a, \frac{b}{(1+b)^2}\right\}$

$F(b) = \frac{b}{(1+b)^2}$, $F'(b) = \frac{-2b + (1+b)}{(1+b)^3} = \frac{1-b}{(1+b)^3}$

$F'(b) = 0 \Rightarrow b = 1$, $F(1) = \frac{1}{(1+1)^2} = \frac{1}{4}$



$\left(\frac{3}{4}, \frac{5}{4}\right)$

$(0, \infty)$

$$\boxed{y' = f(x, y), \quad y(x_0) = y_0} \quad \text{IVP}$$

$$y(x) = y(x_0) + \int_{x_0}^{x} f(t, y(t))\,dt$$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\,dt$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\,dt$$

$$\vdots \qquad \vdots$$

$$y_n(x) = y_0 + \int_{x_0} \cdots$$

$$y' = xy \quad , \quad y(0) = 1$$

$$y_1 = 1 + \int_0^x t \, dt = 1 + \frac{x^2}{2}$$

$$y_2 = 1 + \int_0^x t\left(1 + \frac{t^2}{2}\right) dt$$

$$= 1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 4}$$

$$|\sqrt{x} - \sqrt{y}|^2 \le (\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})$$

$$= x - y \le \varepsilon^2$$

$\delta = \varepsilon^2$

unif continuous

$$y = \frac{x}{1 - Kx}, \quad \frac{dy}{dx} = \frac{(1-Kx) + Kx}{(1-kx)^2} = \frac{1}{(1-kx)^2}$$

$$= \left(\frac{x}{1-kx}\right)^2 \frac{1}{x^2} = \frac{y^2}{x^2}$$

OT $\dfrac{dy}{dx} = -\dfrac{x^2}{y^2}$

$$x^3 + y^3 = c$$