# MA-110 Linear Algebra and Differential Equations

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# Basis: Definition

**Defn.** A subset $\mathscr{B}$ of a vector space $V$, is said to be a *basis* of $V$, if it is linearly independent and $\text{Span}(\mathscr{B}) = V$.

**Theorem:** For any subset $S$ of a vector space $V$, the following are equivalent:

(i) $S$ is a maximal linearly independent set in $V$
(ii) $S$ is linearly independent and $\text{Span}(S) = V$.
(iii) $S$ is a minimal spanning set of $V$.

**Remark/Examples:**

- Every vector space $V$ has a basis.
- By convention, the empty set is a basis for $V = \{0\}$.
- $\{e_1, \ldots, e_n\}$ is a basis for $\mathbb{R}^n$, called the *standard basis*.
- A basis of $\mathbb{R}$ is just $\{1\}$. Is this unique?
- $\left\{\begin{pmatrix}-1 & 1\end{pmatrix}^T, \begin{pmatrix}0 & 1\end{pmatrix}^T\right\}$ is a basis for $\mathbb{R}^2$. So is $\{e_1, e_2\}$, as is the set consisting of columns of a $2 \times 2$ invertible matrix.
- Find a basis in all the examples seen so far.

# Coordinate Vector: Definition

• Let $\mathscr{B} = \{v_1, \ldots, v_n\}$ be a basis for $V$ and $v$ a vector in $V$. $\text{Span}(\mathscr{B}) = V \Rightarrow v = a_1 v_1 + \cdots + a_n v_n$ for scalars $a_1, \ldots, a_n$. Linear independence $\Rightarrow$ this expression for $v$ is unique. Thus

$$\boxed{\text{Every } v \in V \text{ can be } \textit{uniquely} \text{ written}}$$

$$\boxed{\text{as a linear combination of } \{v_1, \ldots, v_n\}.}$$

**Exercise:** Prove this!

**Definition:** If $v = a_1 v_1 + \cdots + a_n v_n$, then $(a_1, \ldots, a_n)^T \in \mathbb{R}^n$ is called the *coordinate vector* of $v$ w.r.t. $\mathscr{B}$, denoted $[v]_{\mathscr{B}}$.

**Note:** $[v]_{\mathscr{B}}$ depends not only on the basis $\mathscr{B}$, but also the order of the elements in $\mathscr{B}$.

**Question:**
How does $[v]_{\mathscr{B}}$ change, if $\mathscr{B}$ is rewritten as $\{v_2, v_1, v_3, \ldots, v_n\}$?

# Dimension of a Vector Space

**Question:** The number of vectors in each basis of $\mathbb{R}^3$ is 3. Why?
**Recall:** If $v_1, \dots, v_n$ span $\mathbb{R}^m$, then $m \leq n$, and
if they are linear independent, then $n \leq m$.

**Defn.:** More generally, if $v_1, \dots v_m$ and $w_1, \dots, w_n$ are both basis of $V$, then $m = n$. This is called the *dimension* of $V$. Thus

$$\boxed{\dim(V) = \text{number of elements in a basis of } V.}$$

**Examples:** • $\dim(\{0\}) = 0$. • $\dim(\mathbb{R}^n) = n$.

• A line through origin in $\mathbb{R}^3$ is of the form $\mathbf{L} = \{tu \mid t \in \mathbb{R}\}$ for some $u$ in $\mathbb{R}^3 \setminus \{0\}$. A basis for $\mathbf{L}$ is $\{\_\_\_\}$, and $\dim(\mathbf{L}) = \_\_\_$.

• The dimension of a plane ($\mathbf{P}$) in $\mathbb{R}^3$ is 2. Why?

• A basis for $\mathbb{C}$ as a vector space over $\mathbb{R}$ is $\{1, i\}$.
A basis for $\mathbb{C}$ as a *complex* vector space is $\{1\}$.
i.e., $\dim(\mathbb{C}) = 2$ as a $\mathbb{R}$-vector space and 1 as a $\mathbb{C}$-vector space.
Thus, dimension depends on the choice of scalars!

# Basis: Remarks

Let $\dim(V) = n$, $S = \{v_1, \ldots, v_k\} \subseteq V$.

**Recall:** A basis is a minimal spanning set.

In particular, if $\text{Span}(S) = V$, then $k \geq n$, and $S$ contains a basis of $V$, i.e., there exist $\{v_{i_1}, \ldots, v_{i_n}\} \subseteq S$ which is a basis of $V$.

**Example:** The columns of a $3 \times 4$ matrix $A$ with 3 pivots span $\mathbb{R}^3$. Hence the columns contain a basis of $\mathbb{R}^3$.

**Recall:** A basis is a maximal linearly independent set.

In particular, if $S$ is linear independent, then $k \leq n$, and $S$ can be extended to a basis of $V$, i.e., there exist $w_1, \ldots, w_{n-k}$ in $V$ such that $\{v_1, \ldots, v_k, w_1, \ldots, w_{n-k}\}$ is a basis of $V$.

**Example:** The columns of a $3 \times 2$ matrix $A$ with 2 pivots has linearly independent columns, and hence can be extended to a basis of $\mathbb{R}^3$.

# Summary: Basis and Dimension

- A basis of a vector space $V$ is a linearly independent subset $\mathscr{B}$ which spans $V$.
- A basis is a maximal linearly independent subset of $V$

  $\Rightarrow$ any linearly independent subset in $V$
  can be extended to a basis of $V$.
- A basis is a minimal spanning set of $V$

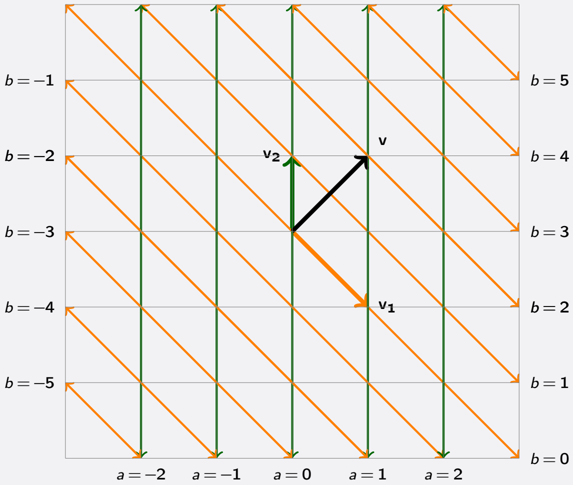
  $\Rightarrow$ every spanning set of $V$ contains a basis.
- The number of elements in each basis is the same, and the dimension of $V$,

  $\dim(V) =$ number of elements in a basis of $V$.
- $\mathscr{B} = \{v_1, \ldots, v_n\}$ is a basis for $V \Longleftrightarrow$ every $v \in V$ can be uniquely written as a linear combination of $\{v_1, \ldots, v_n\}$.
- $\dim\ (\mathbb{R}^n) = n$, and the set $\mathscr{B} = \{v_1, \ldots, v_n\}$ is a basis of $\mathbb{R}^n$

  $\Longleftrightarrow A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ is invertible.

# Example: A basis for $\mathbb{R}^2$

Pick $\mathbf{v}_1 \neq 0$. Choose $\mathbf{v}_2$, not a multiple of $\mathbf{v}_1$. For any $\mathbf{v}$ in $\mathbb{R}^2$, there are unique scalars $a$ and $b$ such that $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$.
e.g., pick $\mathbf{v}_1 = (1, -1)^T$, $\mathbf{v}_2 = (0, 1)^T$, and let $\mathbf{v} = (1, 1)^T$.




Thus the lines $a = 0$ and $b = 0$ give a set of axes for $\mathbb{R}^2$, and $\mathbf{v} = \mathbf{v}_1 + 2\mathbf{v}_2$.

With this basis $\mathscr{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$, the coordinates of $\mathbf{v}$ will be $[\mathbf{v}]_{\mathscr{B}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

A basis for $\mathscr{M}_{2\times 2}$, the vector space of $2 \times 2$ matrices , (called *standard the basis of* $\mathscr{M}$), is $\mathscr{B} = \{e_{11}, e_{12}, e_{21}, e_{22}\}$, where

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(Verify this!) Hence $\dim(\mathscr{M}_{2\times 2}) = 4$.

Every $2 \times 2$ matrix $A = (a_{ij})$ can be written uniquely as

$$A = a_{11}e_{11} + a_{12}e_{12} + a_{21}e_{21} + a_{22}e_{22}.$$

Thus, the coordinate vector of $A$ with respect to $\mathscr{B}$ is

$$[A]_{\mathscr{B}} = (a_{11}, a_{12}, a_{21}, a_{22})^T$$

**Note:** $[A]_{\mathscr{B}}$ completely determines $A$, once we fix $\mathscr{B}$, and order the elements in $\mathscr{B}$.

Since dim $(\mathscr{M}_{2\times 2}) = 4$, once we fix a basis, we will need 4 coordinates to describe each matrix.

**Exercise:** Find two bases (other than the standard one) and the dimension of $\mathscr{M}_{m\times n}$. Find $[e_{11}]_{\mathscr{B}}$ in both cases.

# Coordinate Vectors: Examples

1. Consider the basis $\mathscr{B} = \{v_1 = (1,-1)^T, v_2 = (0,1)^T\}$ of $\mathbb{R}^2$, and $v = (1,1)^T$. Note that $v = 1v_1 + 2v_2$. Hence, the coordinate vector of $v$ w.r.t. $\mathscr{B}$ is $[v]_{\mathscr{B}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

2. **Exercise:** Show that $\mathscr{B} = \{1, x, x^2\}$ is a basis of $\mathscr{P}_2$ (called the *standard basis* of $\mathscr{P}_2$).
   The coordinate vector of $v = 2x^2 - 3x + 1$ w.r.t. $\mathscr{B}$ is $[v]_{\mathscr{B}} = (1, -3, 2)^T$.

3. **Exercise:** Show that $\mathscr{B}' = \{1, (x-1), (x-1)^2, (x-1)^3\}$ is a basis of $\mathscr{P}_3$. Hint: Taylor expansion.
   Let $\mathscr{B}$ be the standard basis of $\mathscr{P}_3$. Then
   $[x^3]_{\mathscr{B}} = (\_\_,\ \_\_,\ \_\_,\ \_\_)^T$, and
   $[x^3]_{\mathscr{B}'} = (\_\_,\ \_\_,\ \_\_,\ \_\_)^T$.

**Recall:** To write the coordinates, we have a to fix a basis $\mathscr{B}$, and fix the order of elements in it!

# Subspaces Associated to a Matrix

Associated to an $m \times n$ matrix $A$, we have four subspaces:

- The **column space** of $A$: $C(A) = \text{Span}\{A_{*1}, \dots A_{*n}\}$ $= \{v : Ax = v \text{ is consistent}\} \subseteq \mathbb{R}^m$.
- The **null space** of $A$: $N(A) = \{x : \ Ax = 0\} \subseteq \mathbb{R}^n$.
- The **row space** of $A = \text{Span}\{A_{1*}, \dots, A_{m*}\} = C(A^T) \subseteq \mathbb{R}^n$.
- The **left null space** of $A = \{x : x^T A = 0\} = N(A^T) \subseteq \mathbb{R}^m$.

**Question:** Why are the row space and the left null space subspaces?

**Recall:** Let $U$ be the echelon form of $A$, and $R$ its reduced form.

Then $\boxed{N(A) = N(U) = N(R).}$

**Observe:** The rows of $U$(and $R$) are linear combinations of the rows of $A$, and vice versa $\Rightarrow$ their row spaces are same, i.e.,

$$\boxed{C(A^T) = C(U^T) = C(R^T).}$$

We compute bases and dimensions of these special subspaces.

# An Example

We illustrate how to find a basis and the dimension of the Null Space $N(A)$, the Column Space $C(A)$, and the Row Space $C(A^T)$ by using the following example.

Let $A = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix}$.

**Recall:**

- The reduced form of $A$ is $R = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.
- The 1st and 2nd are pivot columns $\Rightarrow \text{rank}(A) = 2$.
- $v = \begin{pmatrix} a & b & c \end{pmatrix}^T$ is in $C(A) \Leftrightarrow Ax = v$ is solvable $\Leftrightarrow$ $2a - b - c = 0$.
- We can compute special solutions to $Ax = 0$. The number of special solutions to $Ax = 0$ is the number of free variables.

# The Null Space: $N(A)$

For $A = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix}$, reduced form $R = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

$$N(A) = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -c+2d \\ -c-2d \\ c \\ d \end{pmatrix} = c \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$= \text{Span}\left\{ w_1 = \begin{pmatrix} -1 & -1 & 1 & 0 \end{pmatrix}^T, w_2 = \begin{pmatrix} 2 & -2 & 0 & 1 \end{pmatrix}^T \right\}$.

$w_1$, $w_2$ are linearly independent (Why?)

$\Rightarrow \mathscr{B} = \{w_1, w_2\}$ forms a basis for $N(A) \Rightarrow \dim(N(A)) = 2$.

A basis for $N(A)$ is the set of special solutions.

$\dim(N(A)) =$ no. of free variables $=$ no. of variables - rank$(A)$

$\dim(N(A))$ is called nullity$(A)$.

**Show:** $w = (-3, -7, 5, 1)^T$ is in $N(A)$. Find $[w]_{\mathscr{B}}$.