MA 110 - Ordinary Differential Equations

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Outline of the lecture

- Laplace Transforms-Recap
- Properties

I.

- ⇨ Suppose f is continuous,
- ⇨ f' is piecewise continuous on $[0, a]$ for all $a > 0$,
- ⇨ $|f(t)| \leq Ke^{\alpha t}$, for $t \geq t_0 > 0$, where $K > 0$, t_0, $\alpha \in \mathbb{R}$.

Then, $L(f')(s)$ exists for $s > \alpha$ and

$$L(f') = sL(f) - f(0).$$

II.

- ⇨ Suppose $f, f', \ldots, f^{(n-1)}$ are continuous
- ⇨ $f^{(n)}$ is piecewise continuous on $[0, a]$, for all $a > 0$,
- ⇨ For all $t \geq t_0 > 0$, $|f^{(i)}(t)| \leq Ke^{\alpha t}$, $0 \leq i \leq n-1$, where $K > 0$, t_0, $\alpha \in \mathbb{R}$.

Then, $L(f^{(n)})(s)$ exists for all $s > \alpha$ and

$$L(f^{(n)}) = s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Consider the interval $[0, a]$. Let f' be discontinuous at $t_1, t_2, \ldots, t_n$, where $0 = t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1} = a$. Then,

$$\int_0^a e^{-st} f'(t)dt = \int_0^{t_1} e^{-st} f'(t)dt + \int_{t_1}^{t_2} e^{-st} f'(t)dt + \ldots + \int_{t_n}^{a} e^{-st} f'(t)dt$$

Integrating by parts, we get:

$$\begin{aligned} \int_0^a e^{-st} f'(t)dt &= \sum_{i=1}^{n+1} [e^{-st} f(t)]_{t_{i-1}}^{t_i} + s \sum_{i=1}^{n+1} \int_{t_{i-1}}^{t_i} e^{-st} f(t)dt \\ &= e^{-sa} f(a) - f(0) + s \int_0^a e^{-st} f(t)dt. \end{aligned}$$

Taking limit as $a \to \infty$, we get:

$$L(f')(s) = sL(f) - f(0),$$

for $s > \alpha$.

Proof of Corollary

Induction. $n = 1$ is already done. Assume that the result is true up to $n-1$. Then,

$$\begin{aligned} L(f^{(n)}) &= L((f^{(n-1)})') \\ &= sL(f^{(n-1)}) - f^{n-1}(0) \\ &= s(s^{n-1}L(f) - s^{n-2}f(0) - \ldots - f^{(n-2)}(0)) - f^{(n-1)}(0) \\ &= s^n L(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0). \end{aligned}$$

✔ First derivative :

$$L(f') = sL(f) - f(0).$$

✔ Second derivative :

$$L(f'') = s^2 L(f) - sf(0) - f'(0).$$

Lerch's theorem

Lerch's theorem Suppose $f, g : [0, \infty) \to \mathbb{R}$ be continuous functions such that $L(f) = L(g)$. Then $f = g$.

Infact it can be shown that if two functions have the same Laplace transform, these functions cannot differ over an interval of positive length.

This gives essentially the uniqueness of inverse of Laplace Transforms. If $h = L(f)$, then we write $f = L^{-1}(h)$ and f is said to be the inverse Laplace transform of h.

Examples

Let's now see how to solve IVP's using Laplace transforms.

Solve

$$y'' - y' - 2y = 0, y(0) = 1, y'(0) = 0.$$

Apply Laplace transform through out to obtain :

$$L(y'') - L(y') - 2L(y) = 0.$$

Assume that there is a solution y such that

(i) y'' exists for all $x \geq 0$

(ii) y, y' are of exponential order.

Thus,

$$(s^2L(y) - sy(0) - y'(0)) - (sL(y) - y(0)) - 2L(y) = 0.$$

Thus,

$$(s^2 - s - 2)L(y) - s + 1 = 0.$$

So,

$$L(y)(s) = \frac{s-1}{(s-2)(s+1)} = \frac{1/3}{s-2} + \frac{2/3}{s+1}.$$

The rhs is

$$\frac{1}{3}L(e^{2t}) + \frac{2}{3}L(e^{-t}).$$

Thus,

$$y = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.$$

Remark: If you had done this via characteristic polynomial, first you would have got

$$y(t) = c_1 e^{2t} + c_2 e^{-t},$$

and you would evaluate c_1, c_2 from the constraints.

Example

Solve $y'' - 2y' + 5y = 8\sin t - 4\cos t,\ y(0) = 1,\ y'(0) = 3.$
Taking Laplace transforms,

$$L(y'') - 2L(y') + 5L(y) = \frac{8}{s^2+1} - \frac{4s}{s^2+1}$$

$$(s^2L(y) - sy(0) - y'(0)) - 2(sL(y) - y(0)) + 5L(y) = \frac{8-4s}{s^2+1}.$$

Using the initial conditions,

$$L(y)(s^2-2s+5)-s-3+2 = \frac{8-4s}{s^2+1} \implies L(y)(s^2-2s+5) = \frac{4(2-s)}{s^2+1}+s+1$$

This yields

$$L(y) = \frac{s^3+s^2-3s+9}{(s^2-2s+5)(s^2+1)}.$$

That is, $y = L^{-1}\left(\dfrac{s^3+s^2-3s+9}{(s^2-2s+5)(s^2+1)}\right).$

$$\frac{s^3+s^2-3s+9}{(s^2-2s+5)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{C(s-1)+D}{(s-1)^2+2^2}$$

That is,

$$(As+B)(s^2-2s+5)+(C(s-1)+D)(s^2+1) = s^3+s^2-3s+9.$$

This yields, $A=0,\ B=2,\ C=1,\ D=0.$
Hence,

$$y = L^{-1}\left(\frac{2}{s^2+1}\right) + L^{-1}\left(\frac{s-1}{(s-1)^2+2^2}\right) = 2\sin t + e^t\cos 2t.$$

Property 5: Integration

Let f be piecewise continuous and suppose there exist K, $t_0 > 0$ and $\alpha \in \mathbb{R}$ such that

$$|f(t)| \leq Ke^{\alpha t},$$

for $t \geq t_0 > 0$. Also, let $L(f) = F(s)$. Then,

$$L\left(\int_0^t f(\tau)\, d\tau\right) = \frac{F(s)}{s}, \qquad \text{for } s > \alpha.$$

Proof: We need to show that

$$L\left(\int_0^t f(\tau)\, d\tau\right) = \frac{1}{s}L(f),$$

for $s > \alpha$, where $|f(t)| \leq Ke^{\alpha t}$ for $t \geq t_0 > 0$. We may assume that $\alpha > 0$. Set

$$g(t) = \int_0^t f(\tau) d\tau.$$

Then,

$$g'(t) = f(t)$$

except at the points of discontinuities of $f(t)$.

Hence $g'(t)$ is piecewise continuous. Also $g(t)$ is of exponential order. Hence,

$$L(f) = L(g') = sL(g) - g(0) = sL(g),$$

for $s > \alpha$. Thus,

$$L\left(\int_0^t f(\tau)\, d\tau\right) = \frac{1}{s}L(f).$$

If $L(f) = \dfrac{1}{s(s^2+\omega^2)}$, find $f(t)$.

Solution:

$$L\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}L(f).$$

We know,

$$L\left(\frac{\sin\omega\tau}{\omega}\right) = \frac{1}{s^2+\omega^2}.$$

$$\begin{aligned} f(t) &= L^{-1}\left(\frac{1}{s(s^2+\omega^2)}\right) = L^{-1}\left(\frac{1}{s}\,\frac{1}{s^2+\omega^2}\right) \\ &= \int_0^t \frac{\sin\omega\tau}{\omega}\,d\tau = -\frac{\cos\omega\tau}{\omega^2}\bigg|_0^t = \frac{1}{\omega^2}(1-\cos\omega t). \end{aligned}$$

$$L^{-1}\left(\frac{1}{s^2(s^2+\omega^2)}\right) = L^{-1}\left(\frac{1}{s}\,\frac{1}{s(s^2+\omega^2)}\right)$$

$$= \int_0^t \frac{1-\cos\omega\tau}{\omega^2}\,d\tau = \frac{1}{\omega^2}\left(\tau - \frac{\sin\omega\tau}{\omega}\right)_0^t = \frac{1}{\omega^2}\left(t - \frac{\sin\omega t}{\omega}\right).$$

Property 6 : Differentiation of Laplace transforms

Suppose $f : [0, \infty) \to \mathbb{R}$ is piecewise continuous and of exponential order. Then,

$$F'(s) = -L(tf(t)).$$

Also,

$$L(t^n f(t)) = (-1)^n F^{(n)}(s).$$

Proof.

$$F(s) = \int_0^\infty e^{-st} f(t) dt.$$

Then,

$$\begin{aligned} \frac{dF(s)}{ds} &= \int_0^\infty \frac{\partial}{\partial s}(e^{-st}) f(t) dt \\ &= \int_0^\infty -te^{-st} f(t) dt \\ &= -L(tf(t)). \end{aligned}$$

Exercise : Prove for nth derivative.

Examples

We know that

$$L(\cos\beta t) = \frac{s}{s^2+\beta^2}, \; L(\sin\beta t) = \frac{\beta}{s^2+\beta^2}.$$

Therefore, using

$$F'(s) = -L(tf(t)),$$

$$L(t\cos\beta t) = -\frac{d}{ds}\left(\frac{s}{s^2+\beta^2}\right) = \frac{s^2-\beta^2}{(s^2+\beta^2)^2},$$

and

$$L(t\sin\beta t) = -\frac{d}{ds}\left(\frac{\beta}{s^2+\beta^2}\right) = \frac{2s\beta}{(s^2+\beta^2)^2}.$$