MA 110 - Ordinary Differential Equations

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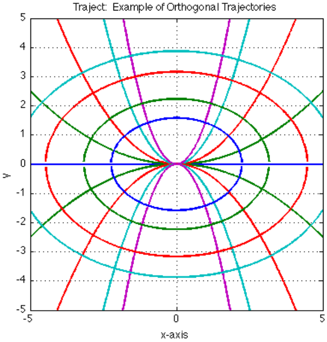
Outline of the lecture

- Orthogonal Trajectories
- Lipschitz continuity
- Existence & uniqueness

Orthogonal Trajectories

If two families of curves always intersect each other at right angles, then they are said to be orthogonal trajectories of each other.



Working Rule

To find the OT of a family of curves

$$F(x, y, c) = 0.$$

- Find the DE $\dfrac{dy}{dx} = f(x, y)$.
- Slopes of the OT's are given by

$$\frac{dy}{dx} = -\frac{1}{f(x, y)}.$$

- Obtain a one parameter family of curves $G(x, y, c) = 0$ as solutions of the above DE.

(Leaving a part certain trajectories that are vertical lines!)

Example

Find the set of OT's of the family of circles $x^2 + y^2 = c^2$.

$$x + y\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = -\frac{x}{y}$$

The slope of OT's are $\dfrac{dy}{dx} = \dfrac{y}{x} \Longrightarrow y = kx$.

Hence the orthogonal trajectories are given by $\boxed{y = kx}$.

1. Let f be a real function defined on D, where D is either a domain or a closed domain of the xy plane. The function f is said to be bounded in D if there exists a positive number M such that
$$|f(x,y)| \leq M$$
for all (x,y) in D.
2. Let f be defined and continuous on a closed rectangle $R : a \leq x \leq b,\ c \leq y \leq d$. Then, f is bounded in R.
3. Let f be defined on D, where D is either a domain or a closed domain of the xy- plane. The function f is said to satisfy Lipschitz condition (with respect to y) in D if $\exists$ a constant $M > 0$ such that
$$|f(x,y_1) - f(x,y_2)| \leq M|y_1 - y_2|$$
for every $(x,y_1),(x,y_2)$ which belong to D. The constant M is called the Lipschitz constant.

Understanding the Lipschitz condition - $y = g(x)$

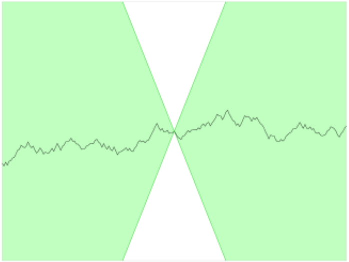
Consider

$$|g(x_2) - g(x_1)| \leq M|x_2 - x_1| \ \ \forall x_1, \ x_2 \text{ in the domain of } g.$$

This condition in the form $\dfrac{|g(x_2) - g(x_1)|}{|x_2 - x_1|} \leq M$ can be interpreted as follows :

At each point $(a, g(a))$, the entire graph of g lies between the lines

$$y = g(a) - M(x - a) \ \& y = g(a) + M(x - a).$$



Example : x^2 is Lipschitz in $[1, 2]$.

Understanding Lipschitz condition - $z = f(x, y)$

- Let (x, y_1) and (x, y_2) be any two points in D having the same abscissa x.
- Consider the corresponding points

$$P_1(x, y_1, f(x, y_1)) \;\&\; P_2(x, y_2, f(x, y_2))$$

  on the surface $z = f(x, y)$, and let α $(0 \leq \alpha \leq \pi/2)$ denote the angle that the chord joining P_1 and P_2 makes with the xy- plane.
- Then if the condition

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|$$

  holds in D, then $\tan \alpha$ is bounded in absolute value.
- That is, the chord joining P_1 and P_2 is bounded away from being perpendicular to the xy- plane.
- Further, this bound is independent of the points (x, y_1) and (x, y_2) belonging to D.

Lipschitz condition $\Longrightarrow$ Continuity ?

If f satisfies Lipschitz condition with respect to y in D, then for each fixed x, the resulting function of y is a continuous function of y, for all (x, y) in D.

Example : Let $f(x, y) = y + [x]$ where $g(x) = [x]$ is the greatest integer function. For fixed x,

$$\begin{aligned} f(x, y_1) - f(x, y_2) &= y_1 + [x] - y_2 - [x] \\ &= y_1 - y_2 \end{aligned}$$

That is, $|f(x, y_1) - f(x, y_2)| = |y_1 - y_2| \leq 1 \cdot |y_1 - y_2|$
But we know that f is discontinuous w.r.t. x for every integral value of x.

Note that the condition of Lipschitz continuity implies nothing concerning the continuity of f with respect to x.

Does Continuity w.r.t. second variable $\implies$ Lipschitz condtn. w.r.t. second variable?

Continuity w.r.t. second variable DOES NOT imply Lipschitz condtn. w.r.t. second variable.

Example : Consider $f(x, y) = \sqrt{|y|}$.

f is continuous for all y.
Note that f doesn't satisfy Lipschitz condition in any region that includes $y = 0$ as for $y_1 = 0,\ y_2 > 0$, we have

$$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} = \frac{\sqrt{y_2}}{|y_2|} = \frac{1}{\sqrt{y_2}}$$

which can be made as large as we want by making y_2 smaller.
The Lipschitz condition requires that the quotient should be bounded by a fixed constant K.

Does Continuity w.r.t. second variable $\Longrightarrow$ Lipschitz condtn. w.r.t. second variable?

Lipschitz condition implies the continuity but continuity does not imply the Lipschitz condition.

Continuity w.r.t. second variable $\not\Rightarrow$ Lipschitz condtn. w.r.t. second variable.

Example : Consider $f(x, y) = \sqrt{|y|}$.

f is continuous for all y.
Note that f doesn't satisfy Lipschitz condition in any region that includes $y = 0$ as for $y_1 = 0, \; y_2 > 0$, we have

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which can be made as large as we want by making y_2 smaller.
The Lipschitz condition requires that the quotient should be bounded by a fixed constant K.

Sufficiency

Result : If f is such that $\dfrac{\partial f}{\partial y}$ exists and is bounded for all $(x, y) \in D$, then f satisfies Lipschitz condition w.r.t. y in D, where the Lipschitz constant

$$M = l.u.b._{(x,y)\in D} \left|\frac{\partial f}{\partial y}(x, y)\right|.$$

Proof : Mean value theorem

$$\Longrightarrow f(x, y_1) - f(x, y_2) = (y_1 - y_2)\frac{\partial f}{\partial y}(x, \xi),\ \xi \in (y_1, y_2).$$

$$\begin{aligned} |f(x, y_1) - f(x, y_2)| &= |y_1 - y_2|\left|\frac{\partial f}{\partial y}(x, \xi)\right| \\ &\leq |y_1 - y_2|\ \ l.u.b._{(x,y)\in D}\left|\frac{\partial f}{\partial y}(x, y)\right|. \end{aligned}$$

That is, f satisfies Lipschitz condition.

Example

Consider

$$f(x,y) = y^2 \text{ defined in } D : |x| \leq a, \ |y| \leq b.$$

$f_y = 2y$ is bounded in D. The Lipschitz contant is

$$M = \ l.u.b._{(x,y)\in D} |\frac{\partial f}{\partial y}(x,y)| = \ l.u.b._{(x,y)\in D} |2y| = 2b.$$

(Verify Lipschitz condition directly!)

To get the Lipchitz constant, we just find the derivative with respect to the second variable and then the maximum value of that derivative function in the given interval.

Bounded derivative - sufficient condition

Consider

$$f(x,y) = x|y| \text{ defined in } D : |x| \le a,\ |y| \le b.$$

$\dfrac{\partial f}{\partial y}$ doesn't exist for any point $(x, 0) \in D$. (Why?)

Now f satisfies Lipschitz condition :

$$\begin{aligned} |f(x,y_1) - f(x,y_2)| &= \big|x|y_1| - x|y_2|\big| \\ &= |x|\,\big||y_1| - |y_2|\big| \\ &\le |x|\,|y_1 - y_2| \\ &\le a|y_1 - y_2| \end{aligned}$$

Existence of bounded derivative f_y is a sufficient condition for Lipschitz condition to hold true (not necessary).

Existence - Uniqueness Theorem

Let R be a rectangle containing (x_0, y_0) in the domain D,

- $f(x, y)$ be continuous at all points (x, y) in $R : |x - x_0| < a, \ |y - y_0| < b$ and
- bounded in R, that is, $|f(x, y)| \leq K \ \ \forall (x, y) \in R$.

Then, the IVP $\boxed{y' = f(x, y), \ y(x_0) = y_0}$ has at least one solution $y(x)$ defined for all x in the interval $|x - x_0| < \alpha$, where

$$\alpha = \min \left\{ a, \frac{b}{K} \right\}.$$

In addition to the above conditions, if f satisfies the Lipschitz condition with respect to y in R, that is,

$$|f(x, y_1) - f(x, y_2)| \leq M |y_1 - y_2| \ \ \forall (x, y_1), (x, y_2) \text{ in } R,$$

then, the solution $y(x)$ defined at least for all x in the interval $|x - x_0| < \alpha$, with α defined above is unique [1].

[1] Existence - Peano, Existence & uniqueness -Picard