MA 110 - Ordinary Differential Equations

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Outline of the lecture

- Annihilator Method
- Laplace Transforms
- Examples
- Existence
- Properties

Non-homogeneous Cauchy-Euler ODE

Solve $x^2y'' + 2xy' - 6y = 10x^2$.
$0 = m(m-1) + 2m - 6 = m^2 + m - 6 = (m+3)(m-2)$.
So, the solution of the homogeneous equation is

$$c_1x^{-3} + c_2x^2. \tag{7}$$

To find the particular solution, we notice that D^3 is an annihilator for $r(x) = 10x^2$. But if we apply D^3 to the given ODE, we do not get a Cauchy-Euler equation. Note that $(xD - 2)$ annihilates $10x^2$. Applying $(xD - 2)$ to both the sides of ODE we get

$$(xD - 2)(x^2D^2 + 2xD - 6)y_p = 0$$

Note that $x^2D^2 = xD(xD - 1)$.
$x^3D^3 = xD(xD - 1)(xD - 2)$, etc

Thus,

$$(xD-2)(xD-2)(xD+3)y_p = 0$$

Hence, $y_p = c_1x^{-3} + c_2x^2 + Ax^2 \ln x$. But in view of (7), we need to choose $y_p = Ax^2 \ln x$. Substituting y_p into the ODE we get

$$(xD-2)(xD+3)Ax^2 \ln x = 10x^2.$$

Therefore $5Ax^2 = 10x^2$, i. e., $A = 2$. Hence $y_p = 2x^2 \ln x$. The general solution is

$$y = c_1x^{-3} + c_2x^2 + 2x^2 \ln x.$$

Let $f : (0,\infty) \to \mathbb{R}$ be a function. The Laplace transform $L(f)$ of f is the function defined by

$$\boxed{L(f)(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{a\to\infty} \int_0^a e^{-st} f(t)dt, \qquad s > 0,}$$

for all values of s for which the integral exists. Sometimes we denote $F(s) = L(f)(s)$.
The integral above may not converge for every s. We may impose suitable restrictions on f later under which the intergral exists.

Find the Laplace transforms of the functions in the examples.

- $\boxed{f(t) = c \text{ for all } t \geq 0}$.

$$L(c)(s) = \int_0^\infty ce^{-st}dt = c\left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{c}{s}.$$

- $\boxed{f(t) = e^{at}, t \geq 0,\ a \text{ being a constant.}}$

$$L(e^{at})(s) = \int_0^\infty e^{-st}e^{at}dt = \int_0^\infty e^{(a-s)t}dt = \frac{1}{s-a},$$

for $s > a$.

➩ $\boxed{f(t) = \sin at,\, t \geq 0.}$

$$\begin{aligned}
L(\sin at)(s) &= \int_0^\infty e^{-st} \sin at\, dt \\
&= \lim_{b\to\infty} \int_0^b e^{-st} \sin at\, dt \\
&= \lim_{b\to\infty} \left[-\frac{e^{-st}\cos at}{a}\right]_0^b - \frac{s}{a}\int_0^\infty e^{-st}\cos at\, dt \\
&= \frac{1}{a} - \frac{s^2}{a^2}\int_0^\infty e^{-st}\sin at\, dt \\
&= \frac{1}{a} - \frac{s^2}{a^2} L(\sin at)(s)
\end{aligned}$$

Therefore,

$$L(\sin at)(s) = \frac{a}{s^2 + a^2}, \text{ for } s > 0.$$

- $\boxed{f(t) = t^2, t \geq 0.}$

$$L(t^2)(s) = \frac{2}{s^3}, \text{ for } s > 0.$$

- $\boxed{L(\cos at)(s) = \frac{s}{s^2 + a^2}, s > 0.}$

Example

Prove that $L(t^n) = \dfrac{n!}{s^{n+1}}, \; n \in \mathbb{N}.$

We show this by induction.

Show that $L(t) = \dfrac{1}{s^2}$.

$$
\begin{aligned}
L(t^{n+1}) &= \int_0^\infty e^{-st} t^{n+1}\, dt \\
&= \cancel{t^{n+1} \frac{e^{-st}}{-s} \bigg|_0^\infty} - \int_0^\infty (n+1) t^n \frac{e^{-st}}{-s}\, dt \\
&= \frac{n+1}{s} L(t^n) = \frac{n+1}{s} \frac{n!}{s^{n+1}} \\
&= \frac{(n+1)!}{s^{n+2}}.
\end{aligned}
$$

Hence, $\boxed{L(t^n) = \dfrac{n!}{s^{n+1}}.}$

Recap

Function	Laplace transform
c	$\frac{c}{s}$
e^{at}	$\frac{1}{s-a}, \quad s > a$
t^n	$\frac{n!}{s^{n+1}}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$

- For a given f, $L(f)$ may or may not exist.
- Sufficient conditions under which convergence is guaranteed for the integral in the definition of the Laplace transform is that f is piecewise continuous and is of exponential order.
- Piecewise continuity - The function is continuous except possibly for finitely many jump discontinuities.
- A function f is said to be of exponential order if there exists $a \in \mathbb{R}$ and positive constants t_0 and K such that

$$|f(t)| \leq Ke^{at},$$

for all $t \geq t_0 > 0$.