MA-110 Linear Algebra and Differential Equations

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Lecture 4 D3

Recap

- We discussed how number of pivots and solution set is related.
- Last class we discussed various matrix operations.
- We can add any two matrices of same size.
- We can multiply two matrices only if the number of columns in first matrix is same as the number of rows in the second matrix.
- Matrix multiplication is associative. It is distributive with matrix addition.
- Matrix multiplication is not commutative.

Examples:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ **(Identity)**

- $AB =$ **??**
- **size of** BA **is** $\_\_ \times \_\_$
- $BA = \begin{pmatrix} 4 & 10 & 7 \\ 4 & 18 & 10 \end{pmatrix}$,
- **and** $IA = A = AI$.

Matrix Multiplication: Examples

Examples:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(Permutation) \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e_2 & e_1 & e_3 \end{pmatrix}$$

Then $AP = \begin{pmatrix} Ae_2 & Ae_1 & Ae_3 \end{pmatrix} = \begin{pmatrix} A_{*2} & A_{*1} & A_{*3} \end{pmatrix}$

Exercise: Find EA and PA.

Question: Can you obtain EA and PA directly from A? How?

Transpose A^T of a Matrix A

Defn. The i-th row of A is the i-th column of A^T, the transpose of A and vice-versa.
Hence if $A_{ij} = a$, then $(A^T)_{ji} = a$.

Example: If $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & 1 \end{pmatrix}$, then $A^T = \begin{pmatrix} 1 & 0 \\ 2 & -2 \\ 3 & 1 \end{pmatrix}$.

- If A is $m \times n$, then A^T is $n \times m$.
- If A is upper triangular, then A^T is lower triangular.
- $(A^T)^T = A$, $\boxed{(A+B)^T = A^T + B^T}$.
- $\boxed{(AB)^T = B^T A^T}$. *Proof.* Exercise.

Symmetric Matrix

Defn. If $A^T = A$, then A is called a *symmetric* matrix.

Note: A symmetric matrix is always $n \times n$.

Examples: $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are symmetric.

- If A, B are symmetric, then AB may **NOT be symmetric.**

In the above case, $AB = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$.

- If A and B are symmetric, then $A + B$is symmetric. Why?
- If A is a $n \times n$ matrix , $A + A^T$ is symmetric.Why?
- For any $m \times n$ matrix B, BB^T and $B^T B$ are symmetric.Why?

Exercise: If $A^T = -A$, we say that A is *skew-symmetric*.
Verify if similar observations are true for skew-symmetric matrices.

Inverse of a Matrix

Defn. Given A of size $n \times n$, we say B is an inverse of A if $AB = I = BA$. If this happens, we say A is *invertible*.

- What would be the inverse of $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$?
- An inverse may not exist. Find an example. *Hint:* $n = 1$.
- An inverse of A, if it exists, has size $n \times n$.
- If the inverse of A exists, it is unique, and is denoted A^{-1}. Why unique?

Proof. Let B and C be inverses of A.

$$\begin{array}{ll} \Rightarrow BA = I & \text{by definition of inverse.} \\ \Rightarrow (BA)C = IC & \text{multiply both sides on the right by } C. \\ \Rightarrow B(AC) = IC & \text{by associativity.} \\ \Rightarrow BI = IC & \text{since } C \text{ is an inverse of } A. \\ \Rightarrow B = C & \text{by property of the identity matrix } I. \end{array}$$

Inverse of a Matrix

- If A and B are invertible, what about AB? AB is invertible, with inverse $(AB)^{-1} = B^{-1}A^{-1}$.

Proof. Exercise.

- If A, B are invertible, what about $A + B$? $A + B$ may not be invertible.

Example: $I + (-I) = (0)$.

- If A is invertible, what about A^T? A^T is invertible with inverse $(A^T)^{-1} = (A^{-1})^T$.

Proof. Use $AA^{-1} = I$. Take transpose.

- If A is symmetric and invertible then, is A^{-1} symmetric?

Yes. *Proof.* Exercise!

- (Identity) $I^{-1} = I$.

Inverses and Linear Systems

- If A is invertible then the system $Ax = b$ has a solution, for every constant vector b, namely $x = A^{-1}b$. Is this unique?
- Since $x = 0$ is always a solution of $Ax = 0$, if $Ax = 0$ has a non-zero solution, then A is not invertible by the last remark.
- If A is invertible, then the Gaussian elimination of A produces n pivots.

Exercise:

1. A diagonal matrix A is invertible if and only if ____.
(Hint: When are the diagonal entries pivots?)
2. When is an upper triangular matrix invertible?

- Since $AB = \begin{pmatrix} AB_{*1} & AB_{*2} \cdots AB_{*n} \end{pmatrix}$ and $I = \begin{pmatrix} e_1 & e_2 \cdots e_n \end{pmatrix}$, if $B = A^{-1}$, then B_{*j} is a solution of $Ax = e_j$ for all j.
- Strategy to find A^{-1}: Let A be an $n \times n$ invertible matrix. Solve $Ax = e_1$, $Ax = e_2$, $\ldots$, $Ax = e_n$.

Things to think about

- Complete the proofs left as exercise.

- Currently we can are unable to show that if $AB = I$ then $BA = I$ for square matrices A and B. Why so?

- Can you rephrase what we proved about transposes as a property of the transpose function from the set of $m \times n$ matrices to $n \times m$ matrices?