

MA-110 Linear Algebra and Differential Equations

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Lecture 9 D3

Reading Slide: Vector Spaces definition continued

Let x , y and z be **vectors**, a and b be **scalars**. The vector addition and scalar multiplication are required to satisfy the following axioms:

- $x + y = y + x$ Commutativity of addition
- $(x + y) + z = x + (y + z)$ Associativity of addition
- There is a unique vector 0 , such that $x + 0 = x$
Existence of additive identity
- For each x , there is a unique $-x$ such that $x + (-x) = 0$
Existence of additive inverse
- $1 * x = x$ Unit property
- $(a + b) * x = a * x + b * x$, $a * (x + y) = a * x + a * y$
 $(ab) * x = a * (b * x)$ Compatibility

Notation: For a **scalar** a , and a **vector** x , we denote $a * x$ by ax .

Subspaces: Definition and Examples

If V is a vector space, and W is a non-empty subset, then W is a *subspace* of V if:

$$x, y \text{ in } W, \quad a, b \text{ in } \mathbb{R} \Rightarrow a * x + b * y \text{ are in } W.$$

i.e., linear combinations stay in the subspace.

Examples:

- ① $\{0\}$: The zero subspace and \mathbb{R}^n itself.
- ② $\{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$ is not a subspace of \mathbb{R}^2 . Why?
- ③ The line $x - y = 1$ is not a subspace of \mathbb{R}^2 . Why?

Exercise: A line not passing through the origin is not a subspace of \mathbb{R}^2 .

- ④ The line $x - y = 0$ is a subspace of \mathbb{R}^2 . Why?

Exercise: Any line passing through the origin is a subspace of \mathbb{R}^2 .

Vector Spaces: Examples

- 1 $V = \{0\}$, the space consisting of only the zero vector.
- 2 $V = \mathbb{R}^n$, the n -dimensional space.
- 3 $V = \mathbb{R}^\infty =$ sequences of real numbers, e.g.,
 $x = (0, 1, 0, 2, 0, 3, 0, 4, \dots)$, with component-wise addition and scalar multiplication.
- 4 $V = \mathcal{M}_{m \times n}$, the set of $m \times n$ matrices, with entry-wise $+$ and $*$.
- 5 $V = \mathcal{P}$, the set of polynomials, e.g.
 $1 + 2x + 3x^2 + \dots + 2023x^{2022}$, with term-wise $+$ and $*$.
- 6 $V = \mathcal{C}[0, 1]$, the set of continuous real-valued functions on the closed interval $[0, 1]$. e.g., x^2 , e^x are vectors in V . How about $1/x$ and $1/(x-5)$? Are they vectors in V ?
Vector addition and scalar multiplication are pointwise:
 $(f + g)(x) = f(x) + g(x)$ and $(a * f)(x) = af(x)$.

Subspaces: More Examples

- 5 Let A be an $m \times n$ matrix.
The null space of A , $N(A)$, is a subspace of \mathbb{R}^n .
The column space of A , $C(A)$, is a subspace of \mathbb{R}^m .
Recall: They are both closed under linear combinations.
- 6 The set of 2×2 symmetric matrices is a subspace of \mathcal{M} .
The set of 2×2 lower triangular matrices is also a subspace of \mathcal{M} .
Q. Is the set of invertible 2×2 matrices a subspace of \mathcal{M} ?
- 7 The set of convergent sequences is a subspace of \mathbb{R}^∞ .
What about the set of sequences convergent to 1?
- 8 The set of differentiable functions is a subspace of $\mathcal{C}[0, 1]$.
Is the same true for the set of functions integrable on $[0, 1]$? Create your own examples.
- 9 See the tutorial sheet for many more examples!

Exercise: (i) A subspace must contain the 0 vector!

(ii) Show that a subspace of a vector space is a vector space.

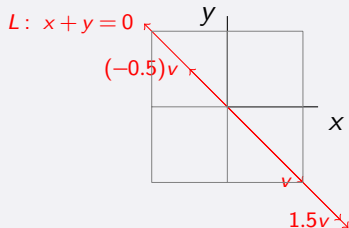
Examples: Subspaces of \mathbb{R}^2

What are all the subspaces of \mathbb{R}^2 ?

- $V = \{(0 \ 0)^T\}$.
- $V = \mathbb{R}^2$.
- What if V is neither of the above?

Example:

Suppose V contains a non-zero vector, say $v = (-1 \ 1)^T$.



V must contain the entire line $L: x + y = 0$, i.e., all multiples of v .

Examples: Subspaces of \mathbb{R}^2

Let V be a subspace of \mathbb{R}^2 containing $v_1 = (-1 \ 1)^T$. Then V must contain the entire line $L: x + y = 0$.

If $V \neq L$, it contains a vector v_2 , which is not a multiple of v_1 , say $v_2 = (0 \ 1)^T$.

Observe: $A = (v_1 \ v_2) = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ has two pivots,

$\Leftrightarrow A$ is invertible.

\Leftrightarrow for any v in \mathbb{R}^2 , $Ax = v$ is solvable,

$\Leftrightarrow v$ is in $C(A)$,

$\Leftrightarrow v$ can be written as a linear combination of v_1 and v_2 .

$\Rightarrow v$ is in V , i.e., $V = \mathbb{R}^2$

To summarise: A subspace of \mathbb{R}^2 , which is non-zero, and not \mathbb{R}^2 , is a line passing through the origin.

Linear Span: Definition

Given a collection $S = \{v_1, v_2, \dots, v_n\}$ in a vector space V , the *linear span* of S , denoted $\text{Span}(S)$ or $\text{Span}\{v_1, \dots, v_n\}$, is the set of all linear combinations of v_1, v_2, \dots, v_n , i.e.,

$$\text{Span}(S) = \{v = a_1 v_1 + \dots + a_n v_n, \text{ for scalars } a_1, \dots, a_n\}.$$

Let $\{v_1, \dots, v_n\}$ be n vectors in \mathbb{R}^n , $A = \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix}$.

Note:

- 1 If v_1, \dots, v_n are in \mathbb{R}^m , $\text{Span}\{v_1, \dots, v_n\} = C(A)$. Thus v is in $\text{Span}\{v_1, \dots, v_n\} \Leftrightarrow Ax = v$ is consistent.
- 2 $\text{Span}\{v_1, \dots, v_n\} = \mathbb{R}^m \Leftrightarrow Ax = v$ is consistent for all $v \in \mathbb{R}^m \Leftrightarrow A$ has m pivots. This implies, $m \leq n$.
- 3 Let $m = n$. Then A is invertible $\Leftrightarrow A$ has n pivots $\Leftrightarrow Ax = v$ is consistent for every v in $\mathbb{R}^n \Leftrightarrow \text{Span}\{v_1, \dots, v_n\} = \mathbb{R}^n$.

Example: $\text{Span}\{e_1, \dots, e_n\} = \mathbb{R}^n$.