MA 110 - Ordinary Differential Equations

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Outline of the lecture

- Method of variation of parameters
- Method of undetermined coefficients

Method of Variation of Parameters - a method to obtain $y_p(x)$

A method to find a particular solution of a non-homogeneous ODE is the method of variation of parameters.
Here, we vary the constants c_1, c_2 in the general solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

of the associated homogeneous equation

$$\mathcal{L}y \equiv y'' + p(x)y' + q(x)y = 0.$$

That is, we replace the constants c_1, c_2 by functions $v_1(x), v_2(x)$, so that

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

is a solution of

$$\mathcal{L}y \equiv y'' + p(x)y' + q(x)y = r(x).$$

Now,

$$y_p' = v_1 y_1' + v_2 y_2' + v_1' y_1 + v_2' y_2.$$

Let's also demand

$$v_1' y_1 + v_2' y_2 = 0.$$

Thus $y_p' = v_1 y_1' + v_2 y_2'$ and so

$$y_p'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2'.$$

Substituting y_p, y_p', y_p'' in the given non-homogeneous ODE and rearranging the terms, we get

$$v_1(y_1'' + py_1' + qy_1) + v_2(y_2'' + py_2' + qy_2) + v_1' y_1' + v_2' y_2' = r(x).$$

Thus,

$$v_1'y_1' + v_2'y_2' = r(x).$$

Recall that we also have

$$v_1'y_1 + v_2'y_2 = 0.$$

Thus, we have:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r(x) \end{bmatrix}.$$

Method of Variation of Parameters

Therefore,

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ r(x) & y_2' \end{vmatrix}}{W(y_1, y_2)}, \; v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r(x) \end{vmatrix}}{W(y_1, y_2)}.$$

Thus,

$$v_1 = -\int \frac{y_2 r(x)}{W(y_1, y_2)} dx, \; v_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)} dx.$$

Hence,

$$\boxed{y_p = v_1 y_1 + v_2 y_2 = y_2 \int \frac{y_1 r(x)}{W(y_1, y_2)} dx - y_1 \int \frac{y_2 r(x)}{W(y_1, y_2)} dx.}$$

Hence, the general solution of the non-homogeneous equation is

$$\boxed{y = c_1 y_1 + c_2 y_2 + y_p}$$

Example 1

Find a particular solution of

$$y'' + y = \csc x.$$

$\boxed{\text{Step I :}}$ Find a basis of solutions for the associated homogeneous equation

$$y'' + y = 0.$$

Following is a basis of solution:

$$y_1(x) = \sin x, \qquad y_2(x) = \cos x.$$

The general solution of this is $y(x) = c_1 y_1 + c_2 y_2$.
$\boxed{\text{Step II :}}$ Calculate the Wronskian $W(y_1, y_2)$:

$$W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1.$$

Now,

$$v_1 = -\int \frac{y_2 r(x)}{W(y_1, y_2)} dx = -\int \frac{\cos x \csc x}{-1} dx = \ln|\sin x|,$$

and

$$v_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)} dx = \int \frac{\sin x \csc x}{-1} dx = -x.$$

Hence, a particular solution is given by

$$y_p(x) = \sin x \ln|\sin x| - x \cos x.$$

What about the general solution ?

Example 2

Find the general solution of

$$y'' - y' - 2y = e^{-x}.$$

A basis of solutions of the corresponding homogeneous equation is

$$y_1 = e^{2x},\ y_2 = e^{-x}.$$

Now,

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{vmatrix} = -3e^{x}.$$

$$v_1 = -\int \frac{y_2 r(x)}{W(y_1, y_2)} dx = -\int \frac{e^{-x} e^{-x}}{-3e^{x}} dx = -\frac{1}{9} e^{-3x},$$

and

$$v_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)} dx = \int \frac{e^{2x} e^{-x}}{-3e^{x}} dx = -\frac{1}{3} x.$$

Hence,

$$y_p = -\frac{1}{9}e^{-3x}e^{2x} - \frac{1}{3}xe^{-x} = -\frac{1}{9}e^{-x} - \frac{1}{3}xe^{-x}.$$

The general solution is

$$y = C_1 e^{2x} + C_2 e^{-x} - \frac{1}{9}e^{-x} - \frac{1}{3}xe^{-x} = C_1 e^{2x} + C_2 e^{-x} - \frac{1}{3}xe^{-x}.$$

Example 3

Find a particular solution of

$$y'' + 4y = 3\cos 2t.$$

A basis of solutions of the corresponding homogeneous equation is

$$y_1 = \cos 2t, \; y_2 = \sin 2t,$$

and

$$v_1 = -\int \frac{\sin 2t \cdot 3\cos 2t}{2} dt = \frac{3}{16}\cos 4t,$$

$$v_2 = \int \frac{\cos 2t \cdot 3\cos 2t}{2} dt = \frac{3}{16}\sin 4t + \frac{3}{4}t.$$

Thus, a particular solution is

$$y_p = v_1 y_1 + v_2 y_2$$

Complete it!

Method of Undetermined Coefficients

- A method to find particular solution of non-homogeneous equations.
- Applicable when the right hand side has specific forms like ke^{rx}, kx^n, $k\cos\omega x$, $k\sin\omega x$, $ke^{rx}\cos\omega x$, $ke^{rx}\sin\omega x$, $ke^{rx}x^n$ and so on.
- May fail when if a term of choice of y_p happens to be the solution of the corresponding homogeneous equation!

Example 1

Find the general and a particular solution of the DE:

$$y'' - 3y' - 4y = 3e^{2x}.$$

We'll search for a solution of the form ke^{2x}, where k is a constant. So put $y = ke^{2x}$. We get:

$$(ke^{2x})'' - 3(ke^{2x})' - 4ke^{2x} = 3e^{2x}.$$

Thus,

$$4ke^{2x} - 6ke^{2x} - 4ke^{2x} = 3e^{2x}.$$

Thus, $k = -\frac{1}{2}$. Hence, $y_p(x) = -\frac{1}{2}e^{2x}$ is a particular solution of the DE.

How do you get the general solution? Analyse roots of $m^2 - 3m - 4 = 0$. So general solution is

$$y = c_1 e^{4x} + c_2 e^{-x} - \frac{1}{2}e^{2x}.$$

Example 2

Find the general and a particular solution of the DE:

$$y'' + 5y' + 6y = e^{-3x}.$$

We'll search for a solution of the form ke^{-3x}, where k is a constant. So put $y = ke^{-3x}$. We get:

$$(ke^{-3x})'' + 5(ke^{-3x})' + 6ke^{-3x} = e^{-3x}.$$

This leads to $0 = e^{-3x}$!
This is because $y_p = ke^{-3x}$ satisfies the homogeneous equation $y'' + 5y' + 6y = 0$ and not the equation itself!
Hence, we choose $y_p(x) = kxe^{-3x}$ is a particular solution of the DE.
In case -3 is a double root of the auxiliary equation, we know that xe^{-3x} is also a solution of the homogeneous equation. Then we choose $y_p = kx^2e^{-3x}$ as a particular solution.
In this example, -3 is not the double root of the auxiliary equation and hence $y_p(x) = kxe^{-3x}$ would work.
Check: $k = -1$. Write the general solution.

- If $r(x) = x^d e^{ax}$ or even $p(x)e^{ax}$ with $p(x)$ being polynomial of degree d, then the candidate solution is
  $y(x) = x^{\mu(a)} q(x) e^{ax}$ where
  $\mu(a)$=number of times a occurs as a characteristic root,
  $$q(x) = a_d + a_{d-1}x + \ldots + a_0 x^d.$$
- If $r(x) = x^d e^{ax} \cos bx$ or even $p(x)e^{ax} \cos bx$ with $p(x)$ being polynomial of degree d, then the candidate soln is
  $y(x) = x^{\mu(a+ib)} e^{ax}(q_1(x) \cos bx + q_2(x) \sin bx)$ where
  $\mu(a+ib)$=number of times $(a+ib)$ occurs as a characteristic root,
  $$q_1(x) = a_d + a_{d-1}x + \ldots + a_0 x^d,$$
  $$q_2(x) = b_d + b_{d-1}x + \ldots + b_0 x^d.$$

Example 3

Find a particular solution of

$$y'' - 3y' - 4y = 2\sin t.$$

Make a guess as to functions of which form we'll search for as a solution. $a\sin t$? No. $a\sin t + b\cos t$? Yes. So set

$$y(t) = a\sin t + b\cos t.$$

Thus,

$$y' = a\cos t - b\sin t; \;\; y'' = -a\sin t - b\cos t.$$

Substituting, we get:

$$(-5a + 3b - 2)\sin t + (-3a - 5b)\cos t = 0.$$

Thus,

$$-5a + 3b = 2; \;\; 3a + 5b = 0$$

(Why?). Thus, $a = -\frac{5}{17}$, $b = \frac{3}{17}$, and a particular solution is

$$y(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t.$$

Example 4

Find a particular solution of

$$y'' - 3y' - 4y = 4t^2 - 1.$$

Set

$$y(t) = at^2 + bt + c.$$

Substituting, we get:

$$-4at^2 + (-6a - 4b)t + (2a - 3b - 4c) = 4t^2 - 1.$$

Thus,

$$-4a = 4, \ -6a - 4b = 0, \ 2a - 3b - 4c = -1.$$

Thus,

$$a = -1, \ b = \frac{3}{2}, \ c = -\frac{11}{8}.$$

Thus, a particular solution is

$$y(t) = -t^2 + \frac{3}{2}t - \frac{11}{8}.$$

Example 5

Find a particular solution of

$$y'' - 3y' - 4y = -8e^t \cos 2t.$$

We should search for a solution of the form

$$y(t) = ae^t \cos 2t + be^t \sin 2t.$$

Then,

$$y'(t) = (a + 2b)e^t \cos 2t + (-2a + b)e^t \sin 2t,$$

and

$$y''(t) = (-3a + 4b)e^t \cos 2t + (-4a - 3b)e^t \sin 2t.$$

Substituting, we get:

$$-10a - 2b = -8, \; 2a - 10b = 0.$$

Thus, a particular solution is

$$y(t) = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t.$$