Linear Algebra (MA106 & MA110 First Half)
Tutorial Problems

Most of these problems are from reference texts for this course. We will add the new tutorial problems to this same file each week. For the latest problems, see the last few pages.

Tutorial 1: Wednesday, 10th Jan 2024

1. Sketch the three lines, and decide if the system is solvable. If yes, find the solution set.

$$x + 2y = 2, \quad x - y = 2, \quad y = 1$$

2. For the equations $x + y = 4$, $2x - 2y = 4$, draw the row picture (two intersecting lines) and the column picture (combination of two columns equal to the column vector $(4, 4)$ on the right side).

3. Describe the intersection of the three hyperplanes in a four dimensional space

$$u + v + w + z = 6, \quad u + w + z = 4, \quad u + w = 2$$

Is it a line or a point or an empty set? What is the intersection if the fourth hyperplane $u = -1$ is included? Find a different fourth equation that leaves us with no solution.

4. Under what conditions on y_1, y_2, y_3, do the points $(0, y_1), (1, y_2), (2, y_3)$ lie on a line?

5. Starting with $x + 4y = 7$, find the equation for the parallel line through $x = 0, y = 0$. Find the equation of another line that meets the first at $x = 3, y = 1$.

6. Starting with a first plane $u + 2v - w = 6$, find the equation for
   (a) the parallel plane through the origin.
   (b) a second plane through origin that also contains the points $(6, 0, 0)$ and $(2, 2, 0)$.
   (c) a third plane that meets the first and second in the point $(4, 1, 0)$.

7. It is impossible for a system of linear equations to have exactly two solutions. Explain why.
   (a) If (x, y, z) and (X, Y, Z) are two solutions of system of linear equations in 3 unknowns, what is another one?
   (b) If 25 planes in $\mathbb{R}^3$ meet at two points, where else do they meet?

8. Show that the set $\left\{ c_1 \begin{pmatrix} -2 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ -15/2 \end{pmatrix} \mid c_1, c_2 \in \mathbb{R} \right\}$ describes a line. Does it describe a line through the origin?

9. Fill in the blanks.
   (a) For four linear equations in two unknowns x and y, the row picture shows four ________. The column picture is in ________ dimensional space. The equations have no solutions unless the vector on the right-hand side is a linear combination of ________.

(b) If a linear system is consistent, then the solution is unique if and only if the following is true about the columns containing pivots: ________.

(c) A 3×4 matrix can have at most ____ pivots.

(d) A 4×3 matrix can have at most ____ pivots.

10. Choose a coefficient a that makes this system singular. Then choose a right-hand side b that makes it solvable. Find two solutions in that singular case.

$$2x + ay = 16, \quad 4x + 8y = b.$$

11. What test on b_1, and b_2 decides whether these two equations allow a solution? How many solutions will they have? Draw the column picture.

$$3x - 2y = b_1, \quad 6x - 4y = b_2$$

12. If the following system is consistent for all values of c and d, what can you say about the coefficients a and b?

$$2x_1 + 4x_2 = d, \quad ax_1 + bx_2 = c$$

13. Find h and k, if they exist, such that the following system $x_1 + hx_2 = 2, \quad 4x_1 + 8x_2 = k$ has (a) no solution, (b) a unique solution, and (c) many solutions.

14. Which number b leads later to a row exchange? Which b leads to a missing pivot? In that singular case find a non-zero solution x, y, z.

$$x + by = 0, \quad x - 2y - z = 0, \quad y + z = 0$$

15. Apply elimination (circle the pivots) and back-substitution to solve

$$2x - 3y = 3, \quad 4x - 5y + z = 7, \quad 2x - y - 3z = 5$$

16. (a) Verify that $(1, 1)$ is a solution to $3x + y = 4$. Find the solution set of this system.

(b) Find two systems of equations such that the solution set is $\{(1, 1)\}$.

17. Use elimination to solve

(a)

$$u + v + w = 6, \quad u + 2v + 2w = 11, \quad 2u + 3v - 4w = 3$$

(b)

$$u + v + w = 7, \quad u + 2v + 2w = 10, \quad 2u + 3v - 4w = 3$$

18. Find a polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ such that $p(1) = 6$, p(2)=15, p(3)=28.

19. Consider a 3×3 system in variables u, v and w, with three (nonzero) pivots. State true or false with explanation:

(a) If the third equation starts with a zero coefficient (it begins with $0u$) then no multiple of equation 1 will be subtracted from equation 3.

(b) If the third equation has zero as its second coefficient (it contains $0v$ then no multiple of equation 2 will be subtracted from equation 3.

(c) If the third equation contains $0u$ and $0v$ then no multiple of equation 1 or no multiple of equation 2 will be subtracted from equation 3.

20. Suppose a 3×5 coefficient matrix for a system has three pivot columns. Is the system consistent? Why or why not?

21. Suppose A is a 3×3 matrix and b is a 3×1 column vector such that $Ax = b$ does not have a solution. Does there exist a 3×1 column vector y such that $Ax = y$ has a unique solution?

22. Suppose A is a 3×4 matrix and b is a 3×1 column vector such that $Ax = b$ does not have a solution. Does there exist a 3×1 column vector y such that $Ax = y$ has a unique solution?

23. Let $A = \begin{pmatrix} 1 & -5 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 7 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$.

    (a) Find all possible solutions to $Ax = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

    State true or false with explanation: $Ax = d$ is consistent for any 4×1 matrix d.

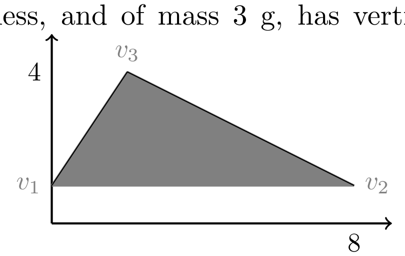
    (b) Find all possible solutions to $Bx = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

    State true or false with explanation: $Bx = b$ is consistent for any 3×1 matrix b.

24. Let $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, B be 2×2 matrices such that $AB = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Show that $BA = I_2$.

25. Write your CPI as a linear combination (or weighted sum) of your grades.

26. A thin triangular plate of uniform density and thickness, and of mass 3 g, has vertices at



$v_1 = (0, 1)$, $v_2 = (8, 1)$, and $v_3 = (2, 4)$, as in the figure.

    (a) Find the (x, y)-coordinates of the centre of mass of the plate. (Hint: Find the centroid).

    (b) The balance point of the plate coincides with the centre of mass of a system consisting of three 1 gram point masses located at the vertices. Determine how to distribute an additional mass of $6g$ at the three vertices of the plate to move the balance point to $(2, 2)$. $\left(\text{Center of mass, of point masses } m_j \text{ located at } v_j, j = 1, \ldots, n, \text{ is given by } \frac{\sum_{j=1}^{n} m_j v_j}{\sum_{j=1}^{n} m_j}\right)$.

27. Consider an economy with three sectors: Fuels and Power, Manufacturing, and Services. Fuels and Power sells 80% of its output to Manufacturing, 10% to services and retains the rest. Manufacturing sells 10% of its out put to Fuels and Power, 80% to Services, and retains the rest. Services sells 20% to Fuels and Power, 40% to Manufacturing, and retains the rest.

Develop a system of equations that leads to prices at which each sector's income matches its expenses. Then write the augmented matrix that can be row reduced to find these prices.

28. Limestone, $CaCO_3$, neutralizes the acid H_3O, in acid rain by the following unbalanced equation.

$$H_3O + CaCO_3 \to H_2O + Ca + CO_2.$$

Balance this equation.

Tutorial 2: Wednesday, 17th Jan 2024

1. Let A and B be $n \times n$ matrices. State true or false with explanation:
   (a) $(AB)^T = B^T A^T$.
   (b) If $AB = 0$ then $A = 0$ or $B = 0$.
   (c) The zero matrix is diagonal.
   (d) If A is upper triangular, then so is A^T.
   (e) The identity matrix I is upper triangular.
   (f) Every lower triangular matrix is symmetric.
   (g) If A is symmetric and skew-symmetric, then $A = 0$
   (h) If A and B are triangular, then so is $A + B$.

2. Prove or disprove.
   (a) If a 2×2 matrix A is such that $AB = BA$ for all 2×2 matrices B, then A is a constant multiple of the identity matrix.
   (b) Let A be a matrix. There does not exist a matrix B such that $BA = 2A$.
   (c) Product of triangular matrices is triangular.
   (d) Inverse of an invertible triangular matrix is triangular.
   (e) Inverse of an invertible symmetric matrix is symmetric.
   (f) If u and v are solutions to $Ax = b$ then so is $(u + v)$.
   (g) Given a square matrix A, if $Ax = b$ has a solution for all b, then the solutions are all unique.
   (h) If $A^2 = A$, then $A = I$ or $A = 0$.

3. By trial and error find examples of 2 by 2 matrices such that
   (a) $A^2 = -I$, A having only real entries.
   (b) $B^2 = 0$, although $B \neq 0$.
   (c) $CD = -DC$, not allowing the case $CD = 0$.
   (d) $EF = 0$, although no entries of E or F are zero.

4. Let $C = \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$.

(a) Find a 2×2 matrix X, if it exists such that $CX = \begin{pmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 3 \end{pmatrix}$.

(b) Find all column vectors X such that $DX = 3X$.

5. What three elementary matrices E_{21}, E_{31}, E_{32} put $A = \begin{pmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{pmatrix}$ into triangular form U? Multiply the E's to get one matrix M that does the elimination to give $MA = U$.

6. Fill in the blanks.

   (a) Let A be a 3×3 matrix, with no row exchanges are needed in elimination to get U. Suppose $a_{33} = 7$ and the third pivot is 5.
   (i) If you change a_{33} to 11, what is the third pivot?
   (ii) What should you change a_{33} to, so that there is a zero in the third pivot position?

   (b) To obtain the entry in row 3, column 4 of AB we need to multiply the ____ row of ____ with the ____ column of ____ .

   (c) If a 5×5 matrix has __ number of pivots, then it is invertible.

7. Find A such that

$$A \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad A \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and } A \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

How is A related to the matrix $B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}$?

8. Let A be $m \times n$, and b be an $m \times 1$ vector. If $Ax = 0$ has a unique solution, what can you say about the number of solutions for $Ax = b$ for some b?

9. Factor A into LU and write down the upper triangular system $Ux = c$ which appears after elimination, for

$$Ax = \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$$

10. How could you factor A into a product UL, upper triangular times lower triangular? Would they be the same factors as in $A = LU$?

11. Solve as two triangular system, without multiplying LU to find A:

$$LUx = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

12. For which numbers c, will A have LU decomposition ?

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

13. Find the inverses of

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 3 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 4 & 5 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

14. If A, B and C are $n \times n$ matrices such that $AB = I_n$, and $CA = I_n$, then show that $B = C$.

15. (a) If P_1 and P_2 are permutation matrices, so is $P_1 P_2$. This still has the rows of I in some order. Give examples with $P_1 P_2 \neq P_2 P_1$ and $P_3 P_4 = P_4 P_3$.

   (b) Find the inverses of the permutation matrices

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

   (c) Explain for permutations why P^{-1} is always the same as P^T. Show that the 1's are in the right place to give $PP^T = I$.

16. Suppose A is invertible and you exchange its first two rows to reach B. Is the new matrix B invertible? How would you find B^{-1} from A^{-1}?

17. Let A and B be $n \times n$. Show that $I - AB$ is invertible if $I - BA$ is invertible. Start from $B(I - AB) = (I - BA)B$.

18. This matrix has a remarkable inverse. Find A^{-1} by elimination on $[A \mid I]$. Extend it to 5×5 "alternating matrix in 1, -1" and guess its inverse.

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

19. (a) There are sixteen 2 by 2 matrices whose entries are 1's and 0's. How many are invertible?

   (b) If you put 1's and 0's at random into the entries of a 10 by 10 matrix, is it more likely to be invertible or singular?

Tutorial 3: Wednesday, 24th Jan 2024

1. If $Ax = b$ has infinitely many solutions, why is it impossible for $Ax = c$ (a new constant vector) to have exactly one solution? Is it possible for $Ax = c$ to be inconsistent?

2. If $Ax = b$ has two solutions x_1 and x_2, find:

   (a) two solutions to $Ax = 0$ and (b) another solution to $Ax = b$.

3. Solve the following system of equations

$$\begin{aligned} 2x_1 + 2x_2 + 4x_3 &= 0 \\ -4x_1 - 4x_2 - 8x_3 &= 0 \\ -3x_2 + 3x_3 &= 0 \end{aligned}$$

   and

$$\begin{aligned} 2x_1 + 2x_2 + 4x_3 &= 8 \\ -4x_1 - 4x_2 - 8x_3 &= -16 \\ -3x_2 + 3x_3 &= 12 \end{aligned}$$

   (a) How are these two solution sets related?

   (b) Give a geometric description of the solution sets.

   (c) Are either of these solutions sets closed under linear combinations?

4. Fill in the blanks.

   (a) Suppose column 4 of a 3×5 matrix is all 0s. Then x_4 is certainly a ____ variable. The special solution corresponding to x_4 is $x =$ ____.

   (b) If A is an invertible 8×8 matrix, then its column space is ____. Why?

   (c) If the 9×12 system $Ax = b$ is solvable for every b, then $C(A) =$ ____.

   (d) Let $\mathbf{L}$ be a line in $\mathbb{R}^3$ through the origin, $\mathbf{P}$ be a plane in $\mathbb{R}^3$ through the origin, and $W \subseteq \mathbb{R}^3$ be a subset which is closed under linear combinations. If W conatins $\mathbf{P}$ and $\mathbf{L}$, then W is either ____ or ____.

   (e) If we add an extra column b to a matrix A, then the column space gets larger unless ____. Give an example in which the column space gets larger and an example in which it does not.

5. Write down all possible 3×4 row reduced forms.

6. If the r pivot variables come first, the reduced R must look like $R = \begin{pmatrix} I & F \\ 0 & 0 \end{pmatrix}$, where I is $r \times r$, and F is $r \times (n - r)$. What is the null space matrix containing the special solutions?

7. Let $A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. Under what conditions on b does $Ax = b$ have a solution? Find two vectors in $N(A)$ and a complete solution to $Ax = b$.

8. Find q (if possible) so that the ranks are (a) 1, (b) 2, (c) 3:

$$A = \begin{pmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 1 & 3 \\ q & 2 & q \end{pmatrix}.$$

9. Let $u = \begin{pmatrix} 7 \\ 2 \\ 5 \end{pmatrix}$, $v = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$ and $w = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}$. Use the fact that $2u - 3v - w = 0$ to solve the system.

$$\begin{pmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}.$$

10. Construct a matrix whose column space contains $(1, 1, 1)$ and whose nullspace is the line of multiples of $(1, 1, 1, 1)$.

11. Reduce A and B to their echelon forms, find their ranks, the free and the dependent variables.

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Find the special solutions to $Ax = 0$ and $Bx = 0$, and their nullspaces.

12. Reduce the matrices A and B to their echelon forms U. Find a special solution for each variable and describe all solutions in the nullspace.

$$A = \begin{pmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{pmatrix}.$$

Reduce the echelon forms U to R, find the rank r and draw a box around the $r \times r$ identity matrix in R.

13. Given a 4×4 matrix A with three pivot positions,

(a) does the equation $Ax = 0$ have a non-trivial solution?

(b) does the equation $Ax = b$ have a least one solution for every possible b?

Repeat the above exercise when A is a 3×2 matrix with two pivot positions.

14. Let $u = \begin{pmatrix} 4 \\ -1 \\ 4 \end{pmatrix}$ and $A = \begin{pmatrix} 2 & 5 & -1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{pmatrix}$. Does u belong to $C(A)$? Why or why not?

15. Mark all the correct options.

(a) The solutions of $Ax = 0$, where $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ form

(i) a plane (ii) a line (iii) a point (iv) a subspace of $\mathbb{R}^2$.
(v) a subspace of $\mathbb{R}^3$ (vi) the nullspace of A (vii) the column space of A.

(b) A is $m \times n$ with row reduced form R. The rank of A is:
(i) The number of nonzero rows in R. (ii) $n - m$.
(iii) $n -$ number of free columns. (iv) The number of 1's in R.
(v) The number of dependent variables. (vi) $\min\{m, n\}$.

16. Prove or disprove.

(a) The set of nonsingular 2×2 matrices is a vector space.

(b) The set of singular 2×2 matrices is not a vector space.

(c) Let $B = [A|b]$. The system $Ax = b$ is solvable exactly when $C(A) = C(B)$.

(d) A system of equations $Ax = 0$ where A is a square matrix has no free variables.

(e) A system of equations $Ax = 0$ where A is an invertible matrix has no free variables.

(f) An $m \times n$ matrix has no more than $\min\{m, n\}$ pivot variables.

(g) Any linear combination of vectors can always be written as Ax for appropriate choices of matrices A and column vector x.

(h) If A is a $m \times n$ matrix such that $C(A) \subsetneq \mathbb{R}^m$, then the equation $Ax = b$ is not consistent for every $b \in \mathbb{R}^m$.

Tutorial 4: Wednesday, 31st Jan 2024

1. Write down 5 different subsets of $\mathbb{R}$. Draw 5 different subsets in $\mathbb{R}^2$. Identify which of them are subspaces of the respective spaces, and if not, identify why it is not so.

2. Which of the following are subspaces of $\mathbb{R}^3$?
(i) The plane of vectors (b_1, b_2, b_3) with (i) $b_1 = 0$. (ii) $b_1 = 1$.
(ii) The set of vectors (b_1, b_2, b_3) with $b_2 b_3 = 0$.
(iii) All linear combinations of the vectors $(1, 1, 0)$ and $(2, 0, 1)$.
(iv) The plane of vectors (b_1, b_2, b_3) satisfying $b_3 - b_2 + 3b_1 = 0$.

3. Consider $\mathcal{M}$, the set 3×3 matrices with standard operations. Which of the following are subspaces of $\mathcal{M}$?
(i) The symmetric matrices in $\mathcal{M}$ (i.e., $A = A^T$) form a subspace.
(ii) The skew symmetric matrices in $\mathcal{M}$ (i.e., $A = -A^T$) form a subspace.
(iii) The non-symmetric matrices in $\mathcal{M}$ (i.e., $A \neq A^T$) form a subspace.
(iv) The set of upper triangular matrices in $\mathcal{M}$ form a subspace.
(v) The matrices that have $(1, 1, 1)$ in their nullspace form a subspace.

4. Let $V = \mathcal{C}[0, 1]$, the vector space of continuous real-valued functions on the closed interval $[0, 1]$. Which of the following are subspaces of V? Justify.
(i) $W_0 = \{f \in V \mid f(0) = 1\}$ (ii) $W_1 = \{f \in V \mid f(1) = 0\}$
(iii) W_2 = set of polynomials of degree 2.
(iv) $W_3 = \{f \mid f$ is a real valued function on $[0, 1]$ such that $\int_0^1 f(x)dx$ is finite.$\}$
(v) $\mathcal{C}^1[0, 1]$ the set of differentiable real-valued functions on $[0, 1]$
(vi) $\mathcal{C}^\infty[0, 1]$ the set of infinitely differentiable real-valued functions on $[0, 1]$.
(vii) $\mathcal{P}_2$ = set of polynomials of degree at most 2.

5. Let V be a vector space, $v_1, v_2, v_3 \in V$, W_1 and W_2 be subspaces of V. Prove or disprove:
(i) $W_1 \cap W_2$ is a subspace of V. (ii) $W_1 \cup W_2$ is a subspace of V.
(iii) $W_1 + W_2 = \{u + v \mid u \in W_1, v \in W_2\}$ is a subspace of V.
(iv) $V \setminus W_1 = \{u \in V \mid u \notin W_1\}$ is a subspace of V.
(v) W = set of all possible linear combinations of v_1, v_2 and v_3.
(vi) $W' = \{a_1 v_1 + a_2 v_2 + a_3 v_3 \mid a_1 \geq 0\}$.

6. Describe the subspace of $\mathbb{R}^3$ spanned by:

(a) $u_1 = (1, 1, -1)^T$ and $u_2 = (-1, -1, 1)^T$.

(b) $v_1 = (0, 1, 1)^T$, $v_2 = (1, 1, 0)^T$ and $v_3 = (0, 0, 0)^T$.

(c) The columns of a 3×5 echelon matrix with 2 pivots.

(d) All vectors with positive components.

7. Is v in $\text{Span}\{v_1, \ldots, v_n\}$? If yes, write v as a combination of the v_i's.

   (a) $v_1 = (1, 1, 0)^T$, $v_2 = (2, 2, 1)^T$, $v_3 = (0, 0, 2)^T$; $v = (3, 4, 5)^T$.

   (b) $v_1 = (1, 2, 0)^T$, $v_2 = (2, 5, 0)^T$, $v_3 = (0, 0, 2)^T$, $v_4 = (0, 0, 0)^T$; $v = (a, b, c)^T$.

   In each case, find a basis of $\text{Span}\{v_1, \ldots, v_n\}$.

8. Construct a 3×3 matrix whose column space contains $(1, 1, 0)$ and $(1, 0, 1)$, but not $(1, 1, 1)$. Construct a 3×3 matrix whose column space is only a line.

9. Suppose A is a 5×4 matrix with $\text{rank}(A) = 4$. Show that $Ax = v$ has no solution if and only if the 5×5 matrix $[A|v]$ is invertible. Show $Ax = v$ is solvable when $[A|v]$ is singular.

10. The matrix $A = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$ is a vector in $\mathcal{M}$, the space of all 2×2 matrices. Write the zero vector in this space, the vectors $\frac{1}{2}A$ and $-A$. What matrices are in the smallest subspace containing A?

11. Describe the column space and null space for:

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

12. $x = v + w$ and $y = v - w$ are combinations of v and w. Show that v and w can be written as combinations of x and y. How are $\text{Span}\{v, w\}$ and $\text{Span}\{x, y\}$ related? When is each pair of vectors a basis for its span?

13. Let $\mathcal{P}$ be the set of polynomials with real coefficients. Show that $\mathcal{P}$ is a real vector space under term-wise addition and scalar multiplication. Can you find a linearly independent set of size 2? 3? 50?

14. How many pivots should a 6×4 matrix have if its columns are linearly independent? Why?

15. Find values of h for which the following set is linearly dependent. $\left\{ \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ -9 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ h \\ -9 \end{pmatrix} \right\}$

16. Determine by inspection whether the following sets of vectors are linearly independent.

    (i) $\left\{ \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 7 \end{pmatrix} \right\}$. (ii) $\left\{ \begin{pmatrix} -8 \\ 12 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \right\}$. (iii) $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

17. Are the following vectors linearly independent?

    (a) $(1, 3, 2)^T$, $(2, 1, 3)^T$, $(3, 2, 1)^T$.

    (b) $(1, -3, 2)^T$, $(2, 1, -3)^T$, $(-3, 2, 1)^T$.

18. Let $v_1 = (1,0,0)^T$, $v_2 = (1,1,0)^T$, $v_3 = (1,1,1)^T$ and $v_4 = (2,3,4)^T$.

   (a) v_1, v_2, v_3, v_4 are linearly dependent because ______.

   (b) Find scalars a_1, a_2, a_3, a_4, not all zero, such that $a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0$.

   (c) Show that v_1, v_2, v_3 are linearly independent.

   (d) Find all combinations of 3 vectors from v_1, v_2, v_3, v_4, which are linearly independent.

   (e) Compute the rank of $A = (v_1 \quad v_2 \quad v_3 \quad v_4)$, and the dimensions of its four fundamental subspaces.

19. Find the largest possible number of independent vectors among: $v_1 = (1,-1,0,0)^T$, $v_2 = (1,0,-1,0)^T$, $v_3 = (1,0,0,-1)^T$, $v_4 = (0,1,-1,0)^T$, $v_5 = (0,1,0,-1)^T$, $v_6 = (0,0,1,-1)^T$. How is this number related to $\text{Span}\{v_1, \ldots, v_6\}$?

20. Fill in the blanks.

   (a) Let $\mathbf{P}$ be the plane in $\mathbb{R}^3$ with the equation $x + 2y + z = 6$. The equation of the parallel plane $\mathbf{P_0}$ through the origin is ______. Of the two, ___ is a subspace of $\mathbb{R}^3$. A basis is $\{$_______$\}$, and its dimension is ___.

   (b) Two vectors v_1 and v_2 in $\mathbb{R}^4$ will be dependent if and only if ______.

   (c) If v is any vector in $\mathbb{R}^4$, v and 0 are dependent because ______.

   (d) Consider $f = 1$, $g = x$, $h = x^2 \in \mathcal{C}[0,1]$. Then $\text{Span}\{f, g, h\} =$ _____. It has a basis ___ and its dimension is ___.

   (e) Let $W = \text{Span}\{\cos(x), \sin(x)\} \subseteq \mathcal{C}[0,1]$. A basis of W is ___, and $\dim(W) =$ ___.

   (f) A basis for the subspace of symmetric 3×3 matrices is ___, and its dimension is ___.
   Do the same for the subspaces of diagonal, skew-symmetric and lower triangular matrices respectively.

21. Are the following true or false? Briefly explain if it is true, give a counter-example if it is false.

   (a) If $v_1, \ldots, v_4$ are in $\mathbb{R}^4$ and $\{v_1, v_3, v_4\}$ is linearly independent then $\{v_1, v_2, v_3\}$ is linearly independent.

   (b) If S_1 and S_2 are subsets of a vector space V, then $\text{Span}(S_1 \cup S_2) = \text{Span}(S_1) \cup \text{Span}(S_2)$.

   (c) If the columns of a matrix are dependent, so are the rows.

   (d) The columns of a matrix are a basis for its column space.

   (e) If the vectors $v_1, \ldots, v_n$ span a subspace V, then $\dim(V) = n$.

   (f) If $v_1, \ldots, v_n$ are linearly independent in a vector space V, then $\dim(V) \geq n$.

   (g) If W is a subspace of V, then $\dim(W) \leq \dim(V)$.

   (h) The intersection of two subspaces of a vector space V cannot be empty.

   (i) Given any non-zero 3×3 matrix A, if $Ax = Ay$, then $x = y$.

   (j) If a square matrix A has independent columns, then so does A^2.

   (k) If $AB = 0$, then $C(B)$ is contained in $N(A)$.

   (l) The union of two linearly independent subsets in a vector space V is linearly independent.

(m) The intersection of two linearly independent subsets in a vector space V is linearly independent.

(n) If $\dim(V) = n$, then any set $S \subseteq V$ with n elements spans V.

(o) If $\dim(V) = n$, then any set $S \subseteq V$ with n elements that spans V is linearly independent.

22. A and B are 3×3 matrices. Mark all the correct options. Justify.

    (a) $C(A) = \{0\} \Rightarrow A = 0$.

    (b) $C(2A) = C(A)$.

    (c) $C(A - I) = C(A)$.

    (d) $C(AB) = C(A)$.

    (e) $C(A) = C(A^T)$.

    (f) $C(A + B) \subseteq C(A)$.

23. Let $\mathbf{P}$ be the plane $x - 2y + 3z = 0$ in $\mathbb{R}^3$.

    (a) Find a basis for $\mathbf{P}$.

    (b) Find a basis for the space of all the vectors perpendicular to $\mathbf{P}$.

    (c) Find a basis for the intersection of $\mathbf{P}$ with the x-y plane.

24. Find a basis for each of the following subspaces of $\mathbb{R}^4$.

    (a) All vectors whose components are equal.

    (b) All vectors whose components add to zero.

    (c) All vectors that are perpendicular to $(1, 1, 0, 0)^T$ and $(1, 0, 1, 1)^T$.

25. Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ be vectors such that:

    (i) $\text{Span}\{v_1, v_2, v_3, v_4\} = \mathbb{R}^3$ and (ii) the vectors $\{v_2, v_3, v_4\}$ are linearly independent.

    For each of the following statements, state if it is true or false. Justify.

    (a) The vectors $\{v_1, v_2, v_3, v_4\}$ are linearly independent.

    (b) The vectors $\{v_1, v_2, v_3\}$ form a spanning set for $\mathbb{R}^3$.

    (c) The vectors $\{v_2, v_3\}$ are linearly independent.

    (d) For any other vector v_5 in $\mathbb{R}^3$, the vectors $\{v_1, v_2, v_3, , v_4, v_5\}$ form a spanning set for $\mathbb{R}^3$.

    (e) The vectors $v_2 + v_3$, $v_2 + v_4$, $v_3 + v_4$ are linearly independent.

26. If A is $m \times n$, the columns of A are n vectors in $\mathbb{R}^m$. If they are linearly independent, what is $\text{rank}(A)$? If they span $\mathbb{R}^m$, what is $\text{rank}(A)$? What happens if they are a basis of $\mathbb{R}^m$?

Tutorial 5: Wednesday, 7th Feb 2024

1. Are the following true or false? Briefly explain if it is true, give a counter-example if it is false.

    (a) The vectors b not in the column space of a matrix A form a subspace.

    (b) For any matrix A, $\dim(C(A)) = \dim(C(A^T))$.

(c) For matrices A and B (of the right size), $\text{rank}(AB) \leq \text{rank}(A)$. (Hint: How are $C(AB)$ and $C(A)$ related?)

(d) There is a 3×3 matrix A whose column space is the same as its null space.

(e) A and A^T have the same number of pivots.

(f) If W is a subspace of V, then $\dim(W) \leq \dim(V)$.

(g) If a square matrix A has independent columns, then so does A^2.

(h) If $AB = 0$, then $C(B)$ is contained in $N(A)$ and the row space of A is contained in the left null space of B) .

(i) If the row space equals the column space, then $A = A^T$.

(j) If $A^T = -A$, then the row space of A equals its column space.

(k) If $\dim(V) = n$, then any set $S \subseteq V$ with n elements spans V.

(l) If $\dim(V) = n$, then any set $S \subseteq V$ with n elements that spans V is linearly independent.

2. Find a basis for the null space and column space of $U = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$.

3. Prove that the following are subspaces of $\mathcal{M}$, the set of 3×3 matrices, and find a basis.

(a) All diagonal matrices.

(b) All symmetric matrices ($A^T = A$).

(c) All skew-symmetric matrices ($A^T = -A$).

(d) All lower triangular matrices.

4. Find an echelon form U of A, and bases and the dimensions the row space, column space and null space of A:

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{pmatrix}.$$

5. Without computing A, find bases of the row space, column space and null space of A:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

6. Let A be $m \times n$ with rank r. Suppose there are right hand sides b for which $Ax = b$ is not solvable.

(a) What inequalities must be true between m, n and r?

(b) Explain why $A^T y = 0$ has non-trivial solutions.

7. Fill in the blanks: Let A be an $m \times n$ matrix, with rank r.

(a) If A has linearly independent columns, then $r =$ _____, the nullspace is _____, and the row space is _____.

(b) If $Ax = b$ always has at least one solution, then the solutions to $A^T y = 0$ is/are ____. (Hint: Find r).

(c) If $m = n = 3$ and A is invertible, then a basis for (i) $N(A)$ is ____, (ii) $C(A)$ is ____, (iii) $N(A^T)$ is ____ and (iv) $C(A^T)$ is ____. Do the same for the 3×6 matrix $B = \begin{pmatrix} A & A \end{pmatrix}$.

(d) If $m = 7$, $n = 9$ and $r = 5$, then the dimension of (i) $N(A)$ is ____, (ii) $C(A)$ is ____., (iii) $N(A^T)$ is ____ and (iv) $C(A^T)$ is ____.

(e) If $m = 3$, $n = 4$ and $r = 3$, then $C(A) =$ ____. and $N(A^T) =$ ____.

(f) Fill in the blanks: A is a 6×4 matrix whose columns are independent. The number of pivots of A are ___. Why?

(g) If B is obtained by exchanging the first two rows of A, then the fundamental subspaces which remain unchanged are ____.

(h) If B is obtained by exchanging the first two columns of A, if $(1, 2, 3, 4)^T$ is in the nullspace of A, a non-zero vector in the nullspace of B is ___.

8. Define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ as $Tu = Au$.
Describe what T does geometrically to a point in $\mathbb{R}^2$ for A =
(i) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (iii) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ (iv) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (v) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

9. Write down as many functions as possible from $\mathbb{R}$ to $\mathbb{R}$. Which of them are linear?

10. Show that $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined as $T(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_2 + x_3, x_3 + x_4)$ is a linear transformation. Find a matrix A such that for every u in $\mathbb{R}^4$, we have $T(u) = Au$.
NOTE: Such a matrix A is called *the standard matrix of* T.

11. Is $\det : \mathcal{M}_{2\times2} \to \mathbb{R}$ a linear transformation? What about the *trace of a matrix* given by tr: $\mathcal{M}_{2\times2} \to \mathbb{R}$ defined as the sum of the diagonal entries?

12. State true or false. If true, prove the statement, and if false, explain why and give a counter-example.

(a) If $T : V \to V$ is a linear transformation of vector space V and $W \subseteq V$ a subspace of V. Then $T(W)$ is a subspace of V.

(b) There is no one-one linear transformation from $\mathbb{R}^3 \to \mathbb{R}^2$.

(c) There is no onto linear transformation from $\mathbb{R}^3 \to \mathbb{R}^2$.

(d) There is no one-one linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$.

(e) There is no onto linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$.

13. Show that the function $T : \mathcal{M}_{2\times3} \to \mathcal{M}_{3\times2}$ defined as $T(A) = A^t$ is a linear transformation, which is a bijection.
NOTE: Such a map is called a *linear isomorphism*, or just *isomorphism*.

14. Show that $S_1, S_2 : \mathcal{M}_{n\times n} \to \mathcal{M}_{n\times n}$ defined as $S_1(A) = A + A^T$ and $S_2(A) = A - A^T$ are linear transformations . Find $N(S_1)$, $N(S_2)$, $C(S_1)$ and $C(S_2)$.

15. Define $D : \mathcal{P}_3 \to \mathcal{P}_2$ as $D(f) = \frac{df}{dx}$ for all polynomials f of degree less than or equal to 3. Show that this is a linear transformation. Find $N(D)$ and $C(D)$.

16. Let $V = \mathbb{R}^\infty$ and W be the subspace of all convergent sequences.
Let $T : W \to \mathbb{R}$ be defined as follows: for $x = (x_1, x_2, \ldots 0$, define $T(x) = \lim_{n\to\infty}(x_n)$.
Show that T is a linear transformation. What is $N(T)$? $C(T)$?

17. Let $V = \mathcal{C}^\infty(\mathbb{R})$ be the vector space of all real-valued functions of a real variable which are differentiable infinitely many times.
Show that $D : V \to V$, defined a $D(f) = f'$ is a linear transformation.
Is $D^2 = D \circ D$ linear? How about $D - 2$, defined by $(D-2)(f) = D(f) - 2f$?
What is $N(D)$? $N(D^2)$? $N(D-2)$? $N(D-2)^2$? $N(D^2 - 3D + 2)$?

18. Is $I : \mathcal{P} \to \mathcal{P}$ defined as $I(f) = \int f \, dx$ a linear transformation? Prove or disprove.

19. Construct a linear map T from $P_2(\mathbb{R}) \to \mathbb{R}$ such that $T(1) = 1$, $T(1-x) = 2$ and $T(x^2) = 3$. What is $N(T)$ and $C(T)$? How many such maps can you construct? Is there one with $T(x) = 0$?

20. Find the standard matrix of $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x_1, x_2) = (x_1 + 2x_2, x_2)$. Can you find a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that $[T]_{\mathcal{B}}^{\mathcal{B}}$ is diagonal?

21. Let Let $\dim(V) = n$ and $\mathcal{B} = \{v_1, v_2, \cdots v_n\}$ be a basis of V.

    (a) Let $T : V \to V$ be the transformation defined by $T(v_i) = v_{i+1}$ for all $i = 1, 2, \cdots, n-1$ and $T(v_n) = 0$. Find the matrix A representing T with respect to the basis $\mathcal{B}$.

    (b) Prove that $T^n = 0$ but $T^{n-1} \neq 0$.

22. Let V be a finite dimensional vector space with a basis $\mathcal{B} = \{v_1, v_2, \ldots v_k\}$. Let $T : V \to V$ be a linear transformation and $A = [T]_{\mathcal{B}}^{\mathcal{B}}$. Show that the following statements are equivalent.

    (a) The linear transformation T is invertible.

    (b) $\text{Span}\{T(v_1), T(v_2), \ldots, T(v_k)\} = V$.

    (c) Dimension of $C(T) = k$.

    (d) Dimension of $N(T) = 0$.

    (e) T is one-one.

    (f) $T(v_1), T(v_2), \ldots, T(v_k)$ are linearly independent.

    (g) $Ax = 0$ has a unique solution.

    (h) The columns of A form al linearly independent set of $\mathbb{R}^k$.

    (i) $Ax = b$ has a solution for all $b \in \mathbb{R}^k$.

    (j) Rank of $A = k$.

    (k) A is invertible.

    (l) A^t is invertible .

    (m) Rows of A form a linearly independent subset of $\mathbb{R}^k$.

Tutorial 6: Wednesday, 14 Feb 2024

1. Find the standard matrix of $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x_1, x_2) = (x_1 + 2x_2, x_2)$. Can you find a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that $[T]_{\mathcal{B}}^{\mathcal{B}}$ is diagonal?

2. Let Let $\dim(V) = n$ and $\mathcal{B} = \{v_1, v_2, \cdots v_n\}$ be a basis of V.

    (a) Let $T : V \to V$ be the transformation defined by $T(v_i) = v_{i+1}$ for all $i = 1, 2, \cdots, n-1$ and $T(v_n) = 0$. Find the matrix A representing T with respect to the basis $\mathcal{B}$.

(b) Prove that $T^n = 0$ but $T^{n-1} \neq 0$.

3. Let V be a finite dimensional vector space with a basis $\mathcal{B} = \{v_1, v_2, \dots v_k\}$. Let $T : V \to V$ be a linear transformation and $A = [T]_{\mathcal{B}}^{\mathcal{B}}$. Show that the following statements are equivalent.

   (a) The linear transformation T is invertible.

   (b) $\text{Span}\{T(v_1), T(v_2), \dots, T(v_k)\} = V$.

   (c) Dimension of $C(T) = k$.

   (d) Dimension of $N(T) = 0$.

   (e) T is one-one.

   (f) $T(v_1), T(v_2), \dots, T(v_k)$ are linearly independent.

   (g) $Ax = 0$ has a unique solution.

   (h) The columns of A form al linearly independent set of $\mathbb{R}^k$.

   (i) $Ax = b$ has a solution for all $b \in \mathbb{R}^k$.

   (j) Rank of $A = k$.

   (k) A is invertible.

   (l) A^t is invertible .

   (m) Rows of A form a linearly independent subset of $\mathbb{R}^k$.

   (n) $\det A \neq 0$.

4. Consider the linear transformation $D^2 : \mathcal{P}_3 \to \mathcal{P}_3$. Let $\mathcal{S} = \{1, x, x^2, x^3\}$ denote the standard basis of $\mathcal{P}_3$. Find $[D^2]_{\mathcal{S}}^{\mathcal{S}}$, $N(D^2)$ and $C(D^2)$. What do they mean in terms of polynomials? If $\mathcal{B} = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$, then find $[D^2]_{\mathcal{B}}^{\mathcal{B}}$? Also, find $[D^2]_{\mathcal{B}}^{\mathcal{S}}$.

5. What 3×3 (standard) matrices represent the transformations that

   (a) project every vector onto the $x - y$ plane?

   (b) reflect every vector through the $x - y$ plane?

   (c) rotate the $x - y$ plane through $90°$, leaving the z-axis alone?

   (d) rotate the $x - y$ plane, then $x - z$, then $y - z$, through $90°$.?

   (e) carry out the same three rotations, but each one through $180°$.?

6. (a) What matrix M transforms $(1, 0)$ and $(0, 1)$ to (r, t) and (s, u)?

   (b) What matrix N transforms (a, c) and (b, d) to $(1, 0)$ and $(0, 1)$?

   (c) What conditions on a, b, c and d will make the previous part impossible?

7. Prove or disprove:

   (a) If A and B are identical except that $b_{11} = 2a_{11}$, then $\det(B) = 2\det(A)$.

   (b) If $T(v)$ is known for n different nonzero vectors in $\mathbb{R}^n$, then we know $T(v)$ for every vector in $\mathbb{R}^n$.

   (c) If A is invertible and B is singular, then $A + B$ is invertible.

   (d) If A is invertible and B is singular, then AB is singular.

(e) The eigenvectors of a 3×3 matrix A will give a basis for $\mathbb{R}^3$.

8. If every row of A adds to zero, prove that $\det(A) = 0$. If every row adds to 1, prove that $\det(A - I) = 0$. Show by an example that this does not imply $\det(A) = 1$.

9. Suppose $CD = -DC$. Find the flaw in the following argument:
Taking determinants gives $\det(C)\det(D) = -\det(D)\det(C) \Rightarrow \det(C) = 0$ or $\det(D) = 0$.
Thus $CD = -DC \Rightarrow C$ is singular or D is singular.

10. Find these determinants by Gaussian elimination:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & t & t^2 & t^3 \\ t & 1 & t & t^2 \\ t^2 & t & 1 & t \\ t^3 & t^2 & t & 1 \end{pmatrix}$$

.

11. (a) If $a_{11} = a_{22} = a_{33} = 0$, how many of the six terms in $\det(A_{3\times 3})$ will be zero?

    (b) If $a_{11} = a_{22} = a_{33} = a_{44} = 0$, how many of the 24 terms in $\det(A_{4\times 4})$ will be zero?

12. Choose the third row of the following matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \_ & \_ & \_ \end{bmatrix}$$

so that its characteristic polynomial is $-\lambda^3 + 4\lambda^2 + 5\lambda + 6$.

13. Find the rank and all four eigenvalues for the matrix of ones and the chess board matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

14. Suppose A is a 3×3 matrix whose eigenvalues are $0, 3, 5$ with independent eigenvectors u, v, w.

    (a) Give a basis for the nullspace and a basis for the column space of A.

    (b) Find a particular solution to $Ax = v + w$. Find all solutions.

    (c) Show that $Ax = u$ has no solution. (Hint: If it had a solution, then $C(A) =$ ______)

15. Let A be a 2×2 matrix with eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 5$. Explain why Trace$(A) = 9$, and $\det(A) = 20$. (Hint: Characteristic polynomial of A is ____.) Find at least three such matrices A.

16. Suppose $u, v \in \mathbb{R}^n$, and $A = uv^T$ i.e., a column times a row.

(a) By multiplying A times u, show that u is an eigenvector. What is λ ?

(b) What are the other eigenvalues of A (and why)? (Hint: rank$(A) = ___$.)

(c) Compute trace(A) from the sum on the diagonal and the sum of λ's.

17. When P exchanges rows 1 and 2 and columns 1 and 2, the eigenvalues do not change. Find eigenvectors of A and PAP for $\lambda = 11$:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix} \text{ and } PAP = \begin{bmatrix} 6 & 3 & 3 \\ 2 & 1 & 1 \\ 8 & 4 & 4 \end{bmatrix}$$

Tutorial 7: Wednesday, 21st Feb 2024

1. Let A be a 2×2 matrix satisfying $A^2 = I$.

   (a) What are the possible eigenvalues of A?

   (b) If $A \neq \pm I$, find its trace and determinant.

   (c) If the first row of A is $(3, -1)$, what is the second row?

2. If $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$. Find A^{100} by diagonalising A.

3. Find all the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

   and write two different diagonalising matrices S.

4. Suppose A has eigenvalues $1, 2, 4$. What is the trace of A^2? What is the determinant of $(A^{-1})^T$?

5. Which of the following cannot be diagonalised?

$$A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

6. Mark all the choices which are correct and explain why.

   (a) If the eigenvalues of A are 1, 1, 2 then,

      i. A is invertible.

      ii. A is diagonalizable.

      iii. A is not diagonalizable.

   (b) If the n columns of P (eigenvectors of A) are independent, then

      i. A is invertible.

      ii. A is diagonalizable.

      iii. P is invertible.

      iv. P is diagonalizable.

7. The matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalisable because the rank of $A - 3I$ is ____. Explain.

   If you are allowed to change one entry to make A diagonalizable, which entries could you change?

8. Find Λ and P to diagonalize $A = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}$.

   (a) What is the limit of Λ^k as $k \to \infty$?

   (b) What is the limit of $P\Lambda^k P^{-1}$?

   (c) $A^k = P\Lambda^k P^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than _____?

9. Let A be a 2×2 matrix. Let $N(A - I) = \text{Span}\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$ and $N(A - 4I) = \text{Span}\left\{ \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right\}$.

   (a) Find A. Is A diagonalizable? Explain why.

   (b) Find a diagonal matrix B such that $B^2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

   (c) Use previous parts to find a matrix X such that $X^2 = A$.

10. Project $b = (0, 3, 0)$ onto each of the orthonormal vectors $a_1 = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ and $a_2 = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ and then find its projection p onto the plane of a_1 and a_2.

11. Project the vector $b = (1, 2)$ onto two vectors that are not orthogonal, $a_1 = (1, 0)$ and $a_2 = (1, 1)$. Show that, unlike the orthogonal case, the sum of the two one- dimensional projections does not equal b.

12. Find a third column so that the matrix

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & \\ 1/\sqrt{3} & 2/\sqrt{14} & \\ 1/\sqrt{3} & -3/\sqrt{14} & \end{bmatrix}$$

   is orthogonal. It must be a unit vector that is orthogonal to the other columns; how much freedom does this leave? Verify that the rows automatically become orthonormal at the same time.

13. If the vectors q_1, q_2, q_3 are orthonormal, what combination of q_1 and q_2 is closest to q_3?

14. Find an orthonormal set q_1, q_2, q_3 for which q_1, q_2, span the column space of

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}.$$

   Which fundamental subspace contains q_3 ? What is the least-squares solution of $Ax = b$ if $b = [1\ 2\ 7]^T$?

15. Apply Gram-Schmidt to $(1,-1,0)$, $(0,1,-1)$, and $(1,0,-1)$, to find an orthonormal basis on the plane $x_1+x_2+x_3=0$. What is the dimension of this subspace, and how many nonzero vectors come out of Gram-Schmidt?

16. Find orthogonal vectors A, B, C by Gram-Schmidt from $a=(1,-1,0,0)$, $b=(0,1,-1,0)$, $c=(0,0,1,-1)$. A, B, C and a, b, c are bases for the vectors perpendicular to $d=(1,1,1,1)$.

17. Construct the projection matrix P onto the space spanned by $(1,1,1)$ and $(0,1,3)$.

18. If Q is orthogonal, is the same true of Q^3?

19. The system $Ax=b$ has a solution if and only if b is orthogonal to which of the four fundamental spaces?

20. Find an orthonormal basis for the plane $x-y+z=0$, and find the matrix P that projects onto the plane. What is the nullspace of P.

21. Find the best straight-line fit (least squares) to the measurements: $b=4$ at $t=-2$, $b=3$ at $t=-1$, $b=1$ at $t=0$ and $b=0$ at $t=2$.

    Find the projection of $b=(4,3,1,0)^t$ onto $\text{Span}\{(1,1,1,1)^t,(-2,-1,0,2)^t\}$.

22. A certain experiment produces the data $(1,7.9)$, $(2,5.4)$and $(3,-0.9)$. Describe the model that produces a least squares fit of these points by a function of the form $y=a\cos(\frac{\pi x}{6})+b\sin(\frac{\pi x}{6})$.

23. CT scanners examine the patient from different directions and produce a matrix giving the densities of bone and tissue at each point. Mathematically, the problem is to recover a matrix from its projections. in the 2 by 2 case, can you recover the matrix A if you know the sum along each row and down each column?

24. Find an orthonormal basis for $\mathbb{R}^3$ starting with the vector $(1,1,1)$.

25. Let W be a subspace of $\mathbb{R}^n$. Define $W^\perp=\{v\in\mathbb{R}^n \mid v^Tw=0 \text{ for alll } w\in W\}$. Show that $W^\perp$ is a subspace of $\mathbb{R}^n$.

26. Let W be a subspace of $\mathbb{R}^n$, $\mathcal{B}_1=\{w_1,\ldots,w_r\}$ and $\mathcal{B}_2$ be ordered bases of W and $W^\perp$ respectively, and $\pi:\mathbb{R}^n\to\mathbb{R}^n$ be defined by $\pi(v)=\text{proj}_W(v)$.

    (a) Show that $\mathcal{B}=\mathcal{B}_1\cup\mathcal{B}_2$ is a basis of $\mathbb{R}^n$.

    (b) Show that π is linear.

    (c) Find $N(\pi)$, and $C(\pi)$.

    (d) Find $P=[\pi]_{\mathcal{B}}^{\mathcal{B}}$.

    (e) If $\mathcal{B}_1$ is an orthogonal basis of W, show that for $v\in\mathbb{R}^n$, $\pi(v)=\text{proj}_{w_1}(v)+\cdots+\text{proj}_{w_r}(v)$.

27. State true or false. If true, explain your answer and if false give a counter-example.

    (a) Any matrix with determinant 1 is a orthogonal matrix.

    (b) An orthogonal matrix cannot have eigenvalue 3.

    (c) Let A be a 2×2 diagonalizable matrix. Applying Gram-Schmidt process to a basis of $\mathbb{R}^2$ consisting of eigenvectors of A will give a orthogonal basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

(d) Product of orthogonal matrices is orthogonal.

(e) Any projection matrix P (that satisfies $P^2 = P$) is invertible.

28. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Find a orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^T$.

29. Hooke's Law states that displacement x of the spring is directly proportional to the load (mass) applied, i.e., $m = kx$. A student performs experiments to calculate spring constant k. The data collected says for loads $4, 7, 11$ kg applied, the displacement is $3, 5, 8$ inches respectively. Hence we have:

$$\begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix} k = \begin{pmatrix} 4 \\ 7 \\ 11 \end{pmatrix} \qquad (ak = b).$$

Show this is an inconsistent system. Estimate k using the method of least squares.

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