MA-110 Linear Algebra and Differential Equations

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Lecture 2 D3

Recap

- The solution to a system of equations can be thought as points of intersection of lines, planes, hyperplanes. This is the row method.
- The solution could also be thought of as coefficients required to write a vector as a linear combination of some vectors. This is the column method.
- We observed that the solution set could be empty, have only one point,or have infinitely many points.
- We discussed Cramer's rule and the elimination method .
- We noted that the elimination method generalizes to systems of equations with more than 3 variables in 3 unknowns.
- We make this formal using the idea of pivots.

Gaussian Elimination

Example: $2u + v + w = 5, \quad 4u - 6v = -2, \quad -2u + 7v + 2w = 9.$

Algorithm: Eliminate u from last 2 equations by $(2) - \frac{4}{\mathbf{2}} \times (1)$, and $(3) - \frac{-2}{\mathbf{2}} \times (1)$ to get the *equivalent system:*

$$\mathbf{2}u + v + w = 5, \qquad \text{-}\mathbf{8}v - 2w = -12, \qquad \mathbf{8}v + 3w = 14$$

The coefficient used for eliminating a variable is called a *pivot*.
The first pivot is 2. The second pivot is -8. The third pivot is 1.
Eliminate v from the last equation to get an equivalent *triangular system*:

$$\mathbf{2}u + v + w = 5, \qquad \text{-}\mathbf{8}v - 2w = -12, \qquad \mathbf{1} \cdot w = 2$$

Solve this triangular system by *back substitution*, to get the *unique solution*

$$w = 2, \qquad v = 1, \quad u = 1.$$

Matrix notation ($A\vec{x} = \vec{b}$) for linear systems

Consider the system
$2u + v + w = 5, \quad 4u - 6v = -2, \quad -2u + 7v + 2w = 9.$

Let $\vec{x} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ be the unknown vector, and $\vec{b} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}$.

The coefficient matrix is $A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}$.

If we have m equations in n variables, then A has m rows and n columns, the column vector $\vec{b}$ has size m, and the unknown vector $\vec{x}$ has size n.

Notation: From now on, we will write $\vec{x}$ as x and $\vec{b}$ as b.

Elimination: Matrix form

Example: $2u + v + w = 5, \quad 4u - 6v = -2, \quad -2u + 7v + 2w = 9.$

Forward elimination in the *augmented* matrix form $[A|b]$:

(Note: The last column is the constant vector b).

$$\begin{pmatrix} 2 & 1 & 1 & | & 5 \\ 4 & -6 & 0 & | & -2 \\ -2 & 7 & 2 & | & 9 \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{2} & 1 & 1 & | & 5 \\ 0 & -8 & -2 & | & -12 \\ 0 & 8 & 3 & | & 14 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \mathbf{2} & 1 & 1 & | & 5 \\ 0 & \mathbf{-8} & -2 & | & -12 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}. \quad \text{Solution is: } x = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Q: Is there a relation between 'pivots' and 'unique solution'?

Singular case: No solution

Example: $\mathbf{2}u + v + w = 5, \quad 4u - 6v = -2, \quad -2u + 7v + w = 9.$

$\boxed{\text{Step 1}}$ Eliminate u (using the 1st pivot 2) to get:

$\mathbf{2}u + v + w = 5, \quad \text{-}\mathbf{8}v - 2w = -12, \quad 8v + 2w = 14$

$\boxed{\text{Step 2}}$: Eliminate v (using the 2nd pivot -8) to get:

$\mathbf{2}u + v + w = 5, \quad -\mathbf{8}v - 2w = -12, \quad 0 = 2.$

The last equation shows that there is no solution,
i.e., the system is *inconsistent*.

Geometric reasoning: In Step 1, notice we get two distinct parallel planes $8v + 2w = 12$ and $8v + 2w = 14$.
They have no point in common.

Note: The planes in the original system were not parallel, but in an equivalent system, we get two distinct parallel planes!

Singular Case: Infinitely many solutions

Example: $\mathbf{2}u + v + w = 5, \quad 4u - 6v = -2, \quad -2u + 7v + w = 7.$

$\boxed{\text{Step 1}}$ Eliminate u (using the 1st pivot 2) to get:

$\mathbf{2}u + v + w = 5, \qquad \text{-}\mathbf{8}v - 2w = -12, \quad 8v + 2w = 12$

$\boxed{\text{Step 2}}$: Eliminate y (using the 2nd pivot -8) to get:

$\mathbf{2}u + v + w = 5, \quad -\mathbf{8}v - 2w = -12, \qquad 0 = 0.$

There are only two equations. For every value of w, values for u and v are obtained by back-substitution, e.g, $(1, 1, 2)$ or $\left(\frac{7}{4}, \frac{3}{2}, 0\right)$. Hence the system has infinitely many solutions.

Geometric reasoning: In Step 1, notice we get two parallel planes $-8v - 2w = 12$ and $8v + 2w = 12$.
They give the same plane. Hence we are looking at the intersection of the two planes, $2u + v + w = 5$ and $8u + 2v = 12$, which is a line.

Singular Cases: Matrix Form

Eg. 1 $2u+v+w=5, \quad 4u-6v=-2, \quad -2u+7v+w=9.$

$$\begin{pmatrix} \mathbf{2} & 1 & 1 & | & 5 \\ 4 & -6 & 0 & | & -2 \\ -2 & 7 & 1 & | & 9 \end{pmatrix} \to \begin{pmatrix} \mathbf{2} & 1 & 1 & | & 5 \\ 0 & \mathbf{-8} & -2 & | & -12 \\ 0 & 0 & 0 & | & 2 \end{pmatrix}.$$

No Solution! Why?

Eg 2. $2u+v+w=5, \quad 4u-6v=-2, \quad -2u+7v+w=7.$

$$\begin{pmatrix} \mathbf{2} & 1 & 1 & | & 5 \\ 4 & -6 & 0 & | & -2 \\ -2 & 7 & 1 & | & 7 \end{pmatrix} \to \begin{pmatrix} \mathbf{2} & 1 & 1 & | & 5 \\ 0 & \mathbf{-8} & -2 & | & -12 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

Infinitely many solutions! Why?

Q: Is there a relation between pivots and number of solutions? Think!

Choosing pivots: Two examples

Example 1:

$-6v + 4w = -2, \quad u + v + 2w = 5, \quad 2u + 7v - 2w = 9.$

Forward elimination in the *augmented* matrix form $[A|b]$:

$$\begin{pmatrix} \mathbf{0} & -6 & 4 & | & -2 \\ 1 & 1 & 2 & | & 5 \\ 2 & 7 & -2 & | & 9 \end{pmatrix}$$

Coefficient of u in the first equation is 0. To get a non-zero coefficient we exchange the first two equations , i.e, interchange the first two rows of the matrix and get

$$\begin{pmatrix} \mathbf{1} & 1 & 2 & | & 5 \\ 0 & -6 & 4 & | & -2 \\ 2 & 7 & -2 & | & 9 \end{pmatrix}$$

Exercise: Continue using elimination method; find all solutions.

Choosing pivots: Two examples

Example 2: 3 equations in 3 unknowns (u, v, w)

$0u + 6v + 4w = -2, \quad 0u + v + 2w = 1, \quad 0u + 7v - 2w = -9.$

$$[A|b] = \begin{pmatrix} \mathbf{0} & \mathbf{1} & 2 & | & 1 \\ \mathbf{0} & 6 & 4 & | & -2 \\ \mathbf{0} & 7 & -2 & | & -9 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & \mathbf{1} & 2 & | & 1 \\ 0 & 0 & \mathbf{-8} & | & -8 \\ 0 & 0 & -16 & | & -16 \end{pmatrix}$$

Coefficient of u is 0 in every equation.The first pivot is 1 and we eliminate v from the second and third equations. Solve for w and v to get $w = 1$, and $v = -1$.

Note: $(0, -1, 1)$ is a solution of the system. So is $(1, -1, 1)$. In general, $(*, -1, 1)$ is a solution, for any real number $*$.

Observe: Unique solution is not an option. Why? This system has infinitely many solutions.

Q: Does such a system always have infinitely many solutions?

A: Depends on the constant vector b.

Exercise: Find 3 vectors b for which the above system has (i) no solutions (ii) infinitely many solutions.

Questions to think about

- How does the process of Gaussian elimination change the line or plane geometrically?
- Draw three planes which are non parallel but do not have common points of intersection.
- Draw three planes which are non parallel but intersect in a line.
- Are the pivots related to getting a unique solution or infinite solutions or having no solution? How so?
- What does having a column of zeros in the augmented system signify for the solution of the corresponding system of linear equations?