

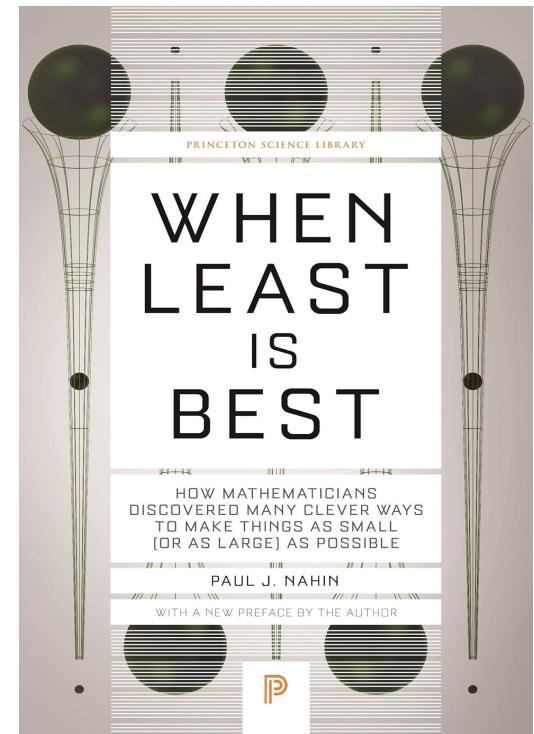
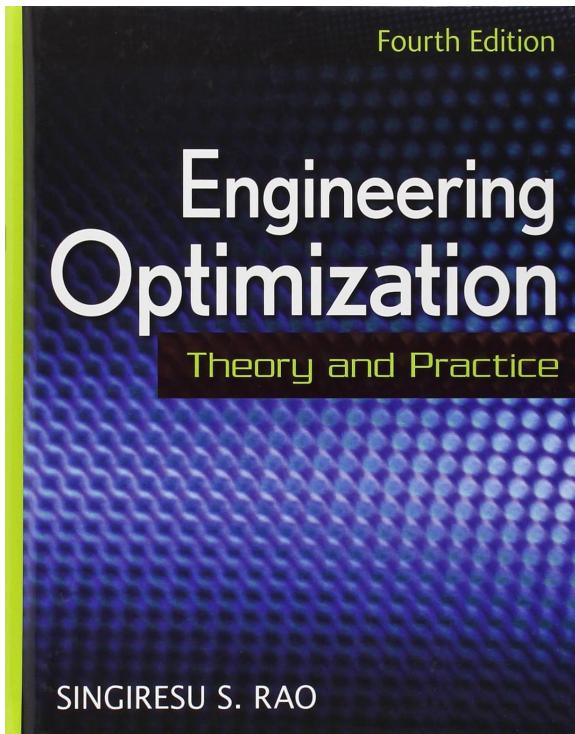
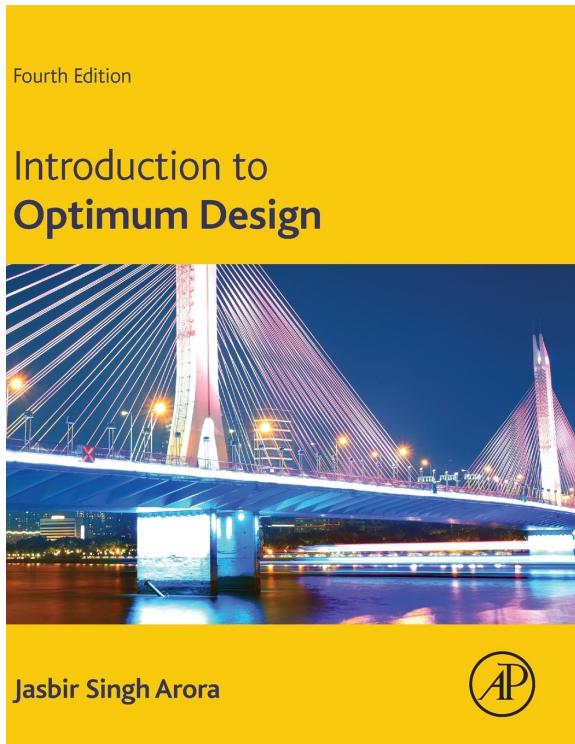


MS 101 Spring 2024

Engineering Optimization

Dnyanesh Pawaskar

References



Engineering Questions

Example: Nose cone of a rocket

- Analysis: Given a shape of the nose cone, what is the drag?
- Design: Given the drag, what is the shape of the nose cone?
- Manufacturing: How do we manufacture the shape?
- Design Optimization: What is the shape of the nose cone that will minimize the drag?



Engineering Design

- Design is a decision-making process. All engineering analysis should lead to design which should eventually lead to manufacturing.
- Engineering is essentially about improving our lives by building better, safer, & cheaper products and services.
- Often must choose between several competing or conflicting alternatives
- Optimization is a formal (mathematical) way to carry out this decision making
- Finds the “best possible” solution to a problem. “Best” + “possible”.
- Often, have to choose the “least bad option”.
- “There are no solutions, only trade-offs.”
- Maximize/minimize requirement(s): safety, cost, reliability, stiffness, utility, strength, profit, etc.

Engineering Optimization for different purposes

Optimization is dependent on the end use of the product.



Boeing 747-400 passenger aircraft



SU-27 multirole fighter aircraft

Some Examples of Design Optimization

- Path optimization
- Shape optimization
- Structural optimization
- And many more

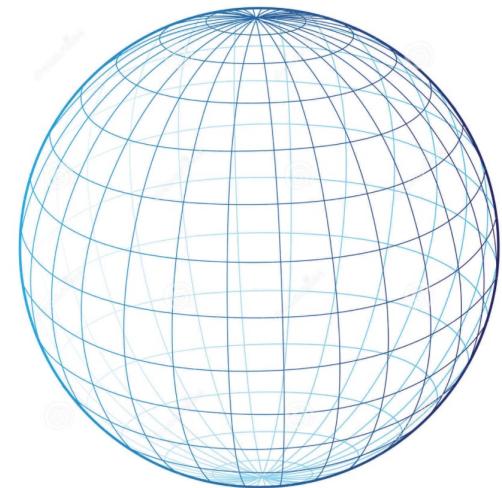
Path Optimization



Debris chute

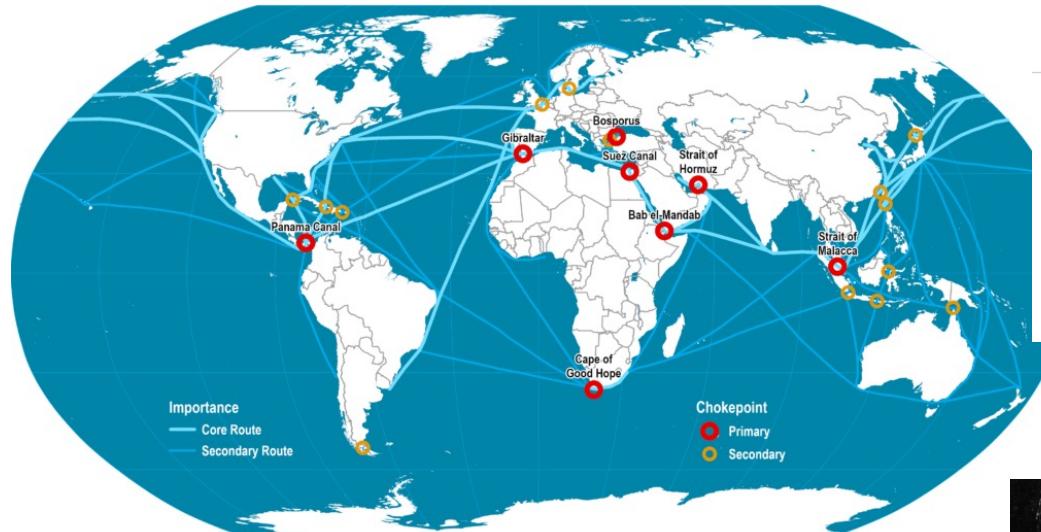


Rollercoaster



Hyperloop

Path Optimization



Shipping routes

REUTERS® World ▾ Business ▾ Markets ▾ Sustainability ▾ Legal ▾ More ▾

World

India deploys unprecedented naval might near Red Sea to rein in piracy

By Krishn Kaushik

January 31, 2024 7:10 PM GMT+5:30 · Updated 7 days ago



Rescue operations



Slow steaming

Optimization?



Path Optimization



- “Understanding the process of fluid mixing is a long-standing challenge in fluid dynamics research. One of the problems that lies at the heart of the study of mixing is the broad concept of ‘efficiency’. For example, what is an optimal mixing strategy that produces, at a given time, the best possible mixture, subject perhaps to some constraints?”

[Journal of Fluid Mechanics](#) , [Volume 850](#) , 10
September 2018 , pp. 875 - 923

Structural Optimization

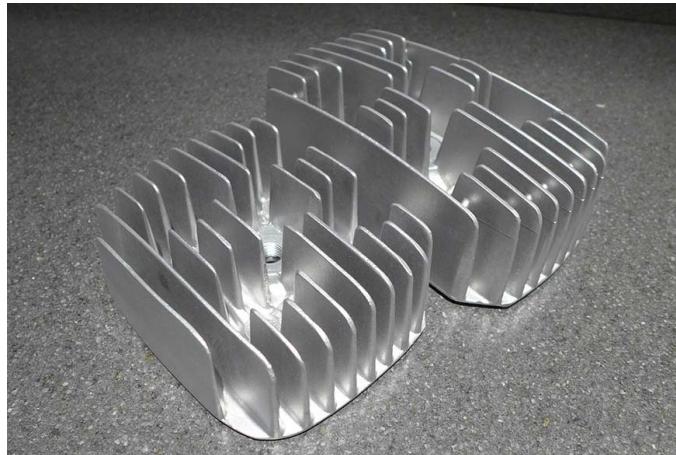
- Maximize buckling load, resistance to wind/wave loads, earthquake loads, stiffness, reliability, fire resistance, visual appeal, etc.
- Minimize weight, cost, construction time, environmental damage, etc.



Shape Optimization



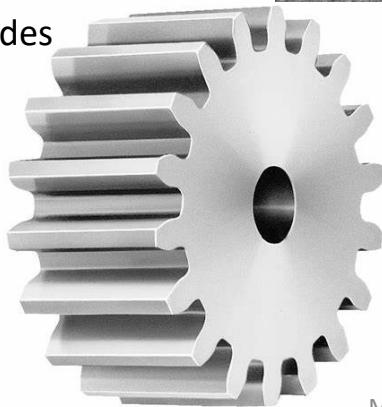
Gas turbine blades



Heat fins



Cooling towers



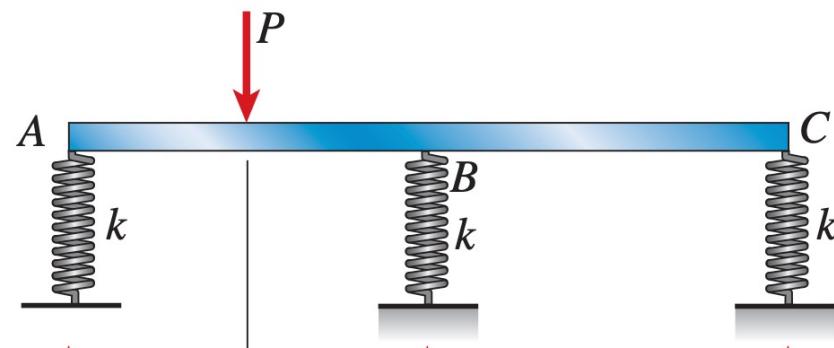
Gear teeth

Optimization Principles in Nature

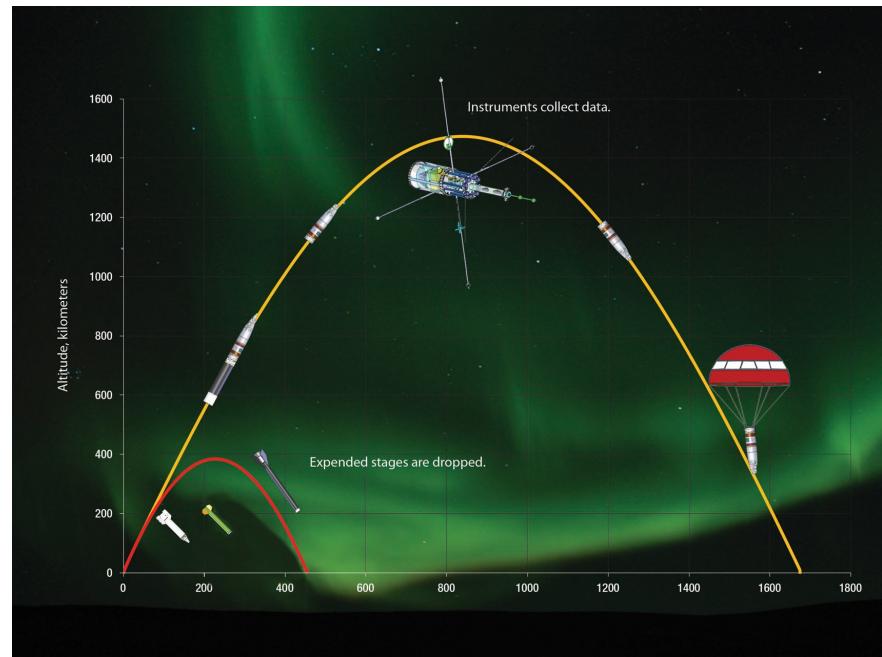
- Statics: Principle of Minimum Potential Energy
- Dynamics: Principle of Least Action
- Ray Optics: Principle of Least Time
- Thermodynamics: Maximum entropy
- Biology: Survival of the fittest

Many scientific/engineering problems can be posed as optimization problems. Many computational methods (like FEM finite element methods) are essentially optimization problems.

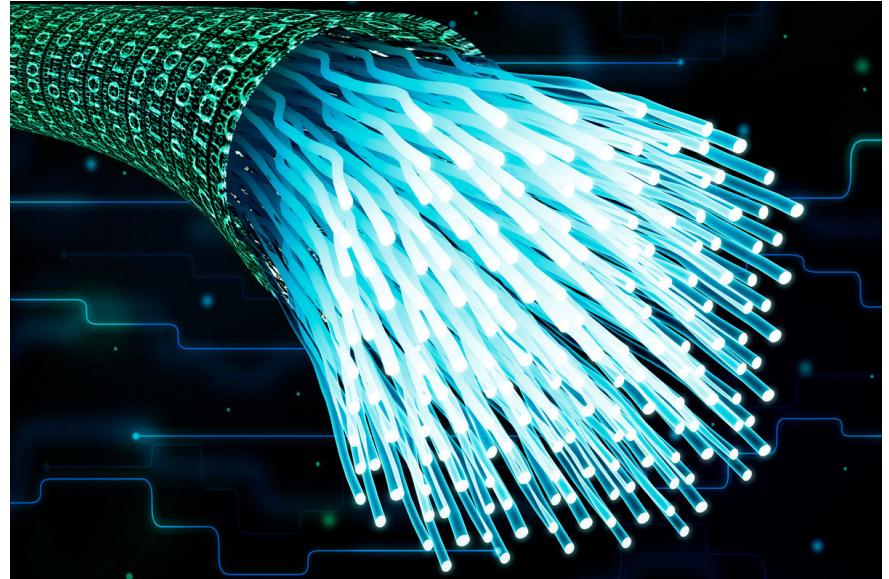
Statics: Principle of Minimum Potential Energy



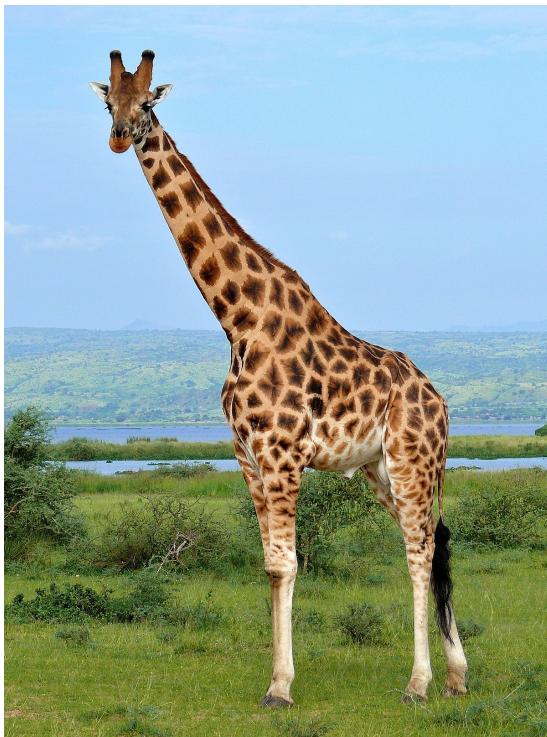
Dynamics: Principle of Least Action



Optics: Fermat Principle of Least Time



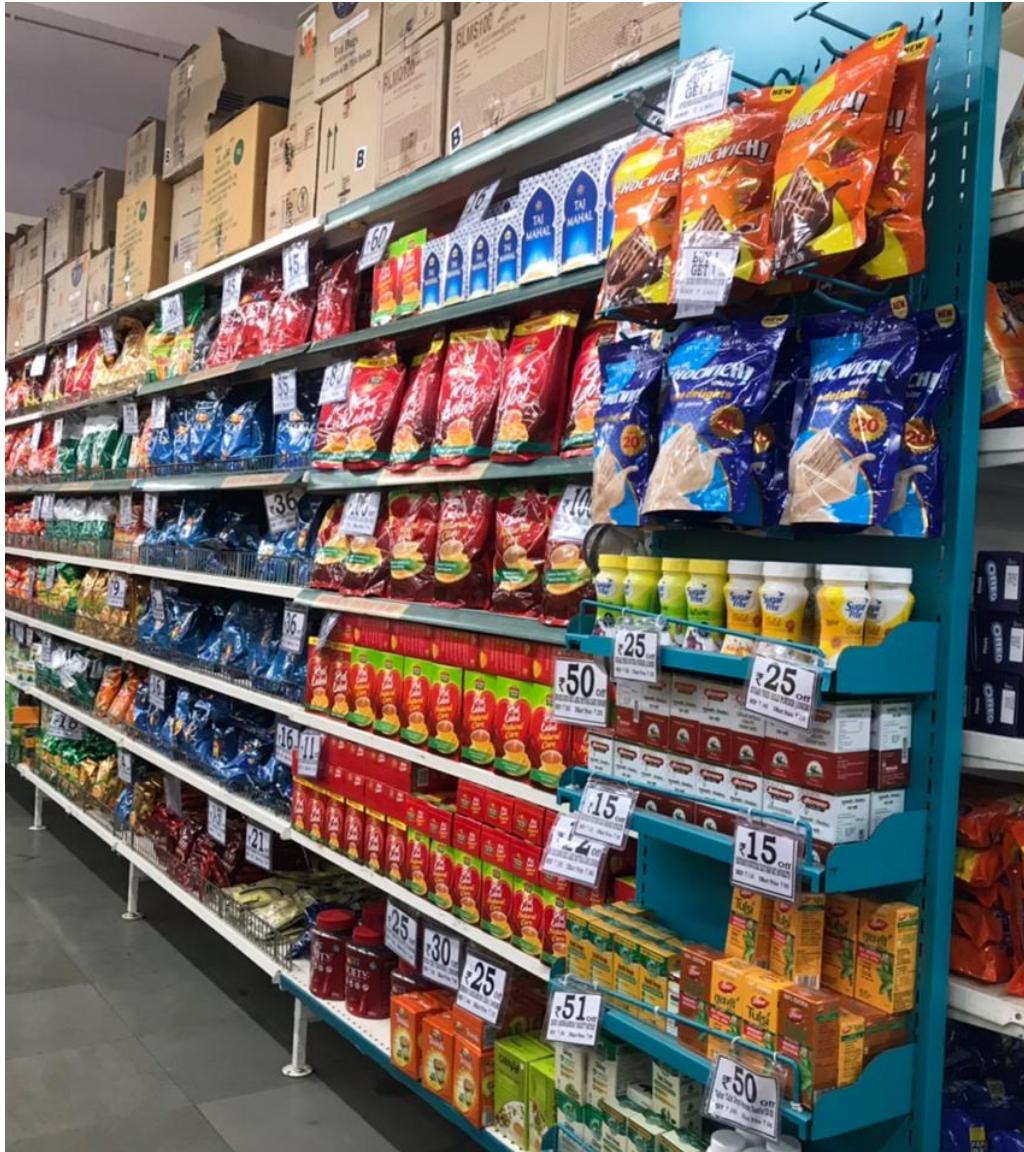
Biology: “Nature is the best teacher”
Can we copy/use ideas from the natural world to develop optimization algos?



Genetic Algorithms
Survival of the fittest



Particle Swarm Optimization
Flock/herd behavior



Economic Optimization

- Optimal production problem
- Optimal consumption problem
- Optimal investment/capital allocation problem
- Optimal tax minimization problem

Objectives of Engineering Optimization

- Given an engineering problem
- Some objective function F (in reality, more than one)
- Dependency on certain underlying variables $x_1, x_2, x_3, \dots, x_N$
- This dependency exists through the underlying mathematical model
- How do we choose $x_1, x_2, x_3, \dots, x_N$ such that F is maximized/minimized?
- Are there any inter-dependencies (underlying relationships) between variables x_1, x_2, \dots, x_N that must be obeyed? Constraints
- Note that the variables x_1, x_2, \dots, x_N must be experimentally measurable, quantifiable, or implementable in practice.

Problem Statement of Engineering Optimization

- Given an objective function $F(x_1, x_2, \dots, x_N)$ of N variables
- (x_1, x_2, \dots, x_N) are called design/decision variables (can be functions also but more on this later).
- Find (x_1, x_2, \dots, x_N) such that F is a minimum
- It is possible that x_1, x_2, \dots, x_N may be inter-related through equations/inequalities called constraints of the sort:
- $G(x_1, x_2, \dots, x_N) = 0$ equality constraint
- $H(x_1, x_2, \dots, x_N) \leq 0$ inequality constraint
- In real life, many objective functions, many constraints.
- F is called objective function / cost function.
- We will talk about minimization. If maximize F , then minimize?

Mathematical Modeling

- Universe is governed by rules/laws and that these rules can be figured out by humans. The way to figure out these rules is to:
- Observe a phenomenon in nature or in a controlled setting aka laboratory
- Propose a mathematical model based on physical laws and reasonable assumptions. Models are typically ODEs/PDEs in engineering. Note that we analyze only the mathematical abstraction of the real system
- Analytically solve / computer simulate the model
- Compare model predictions with experimental results or observations.
- If agreement, keep the model and use it to improve our lives by building better, safer, and cheaper products, and services.
- If no agreement, improve model or discard it and start afresh.
- "All models are wrong, but some are useful."

Optimization Methods

- *Trial-and-error through experimentation.* Learn from your own/others' mistakes.
- *Graphical methods.* Based on visualization. Good for up to 2D problems.
- *Analytical methods.* Calculus based approach i.e. first derivative (gradient) test, second derivative (Hessian) test. Elegant but not suitable for real world applications with complex geometries, loading, boundary conditions, and multi-physics scenarios (say thermal + mechanical + electrical). But physically insightful and gives scaling laws.
- *Computational methods.* Most useful but not as straightforward as thought to be. Output needs to be properly interpreted. Which method best for your problem? Schemes/parameters/startling points in the method need to be chosen with care.
- Even if analytical/computational methods used, experimental validation is always necessary. Build prototype and show in tests that it performs better than other competing designs.

Plan

- We will look at the mathematical formulation of several optimization problems.
- Important to understand where engineering optimization problems come from.
- Try to identify:
 - 1. The objective function
 - 2. The design variables
 - 3. Constraints on the design variables

Plan – Formulation of “Simple” Optimization Problems in 1D, 2D, InfiniteD

- 1D: Rescue problem, Speed limit on highway, Structural optimization
- 2D: Rescue problem, Least squares, Shape optimization
- InfiniteD: Rescue problem, Path optimization, Shape optimization

Note: Keep an eye on the statement of the problem and the underlying assumptions of the mathematical model

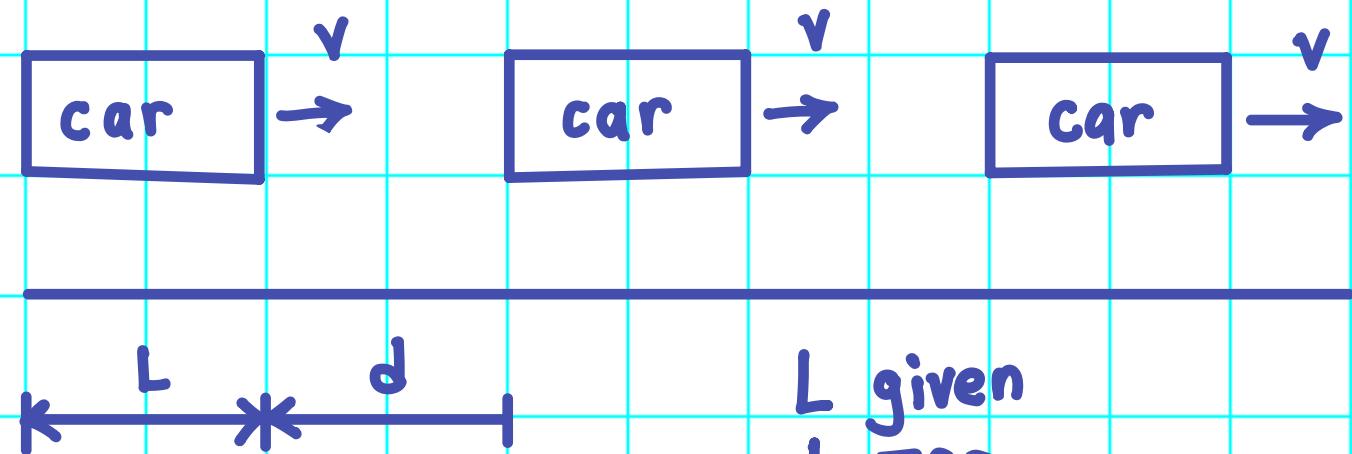
Problem 1 Traffic Congestion (1D)

Find the optimum safe speed v at which cars should travel on a single lane highway to minimize traffic congestion i.e. to maximize the flux of cars.



1D OPTIMIZATION TRAFFIC PROBLEM

Optimum
speed v
single-lane
highway



Maximize flux / throughput $\varphi(v)$

φ Flux = cars passing a point in unit time

If v too high \Rightarrow accidents

v too low \Rightarrow delays / frustration

Let t_r = driver reaction time ≈ 0.5 s , given
 μ = friction coefficient bt tires and
road surface , given

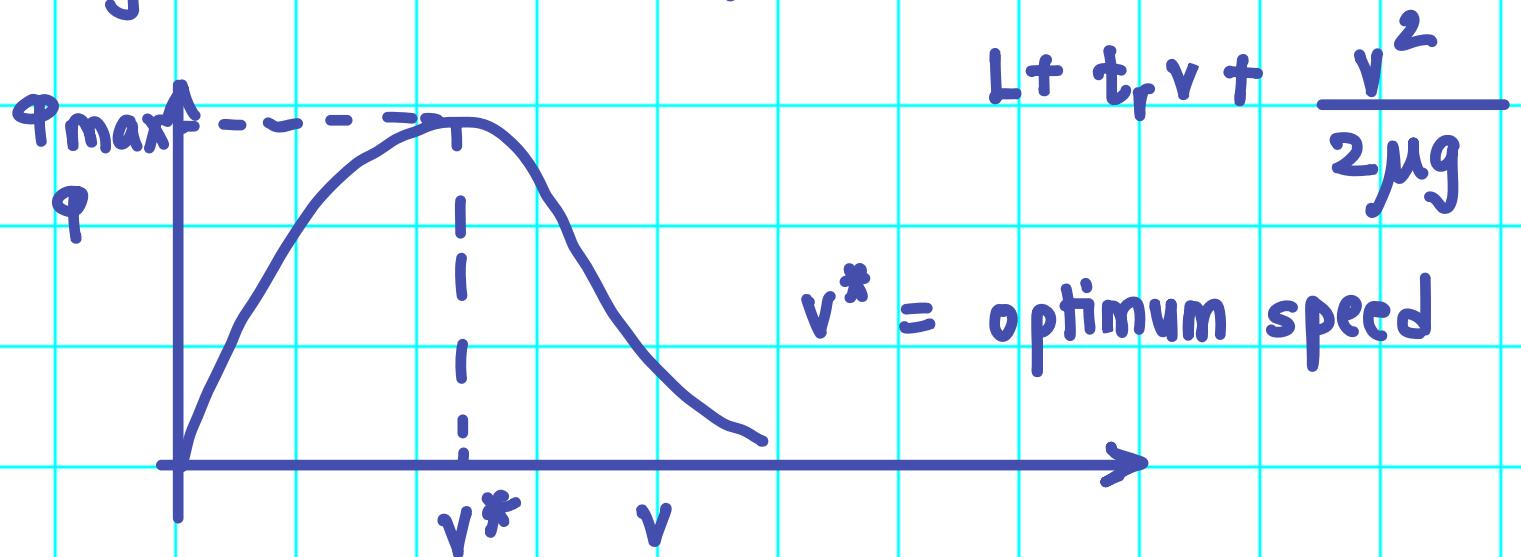
Safe stopping distance $d = t_r v + \frac{v^2}{2\mu g}$

1 car takes $\frac{L+d}{v}$ seconds to pass a point

N cars take $N \left(\frac{L+d}{v} \right)$ seconds to pass a point
 $= t$

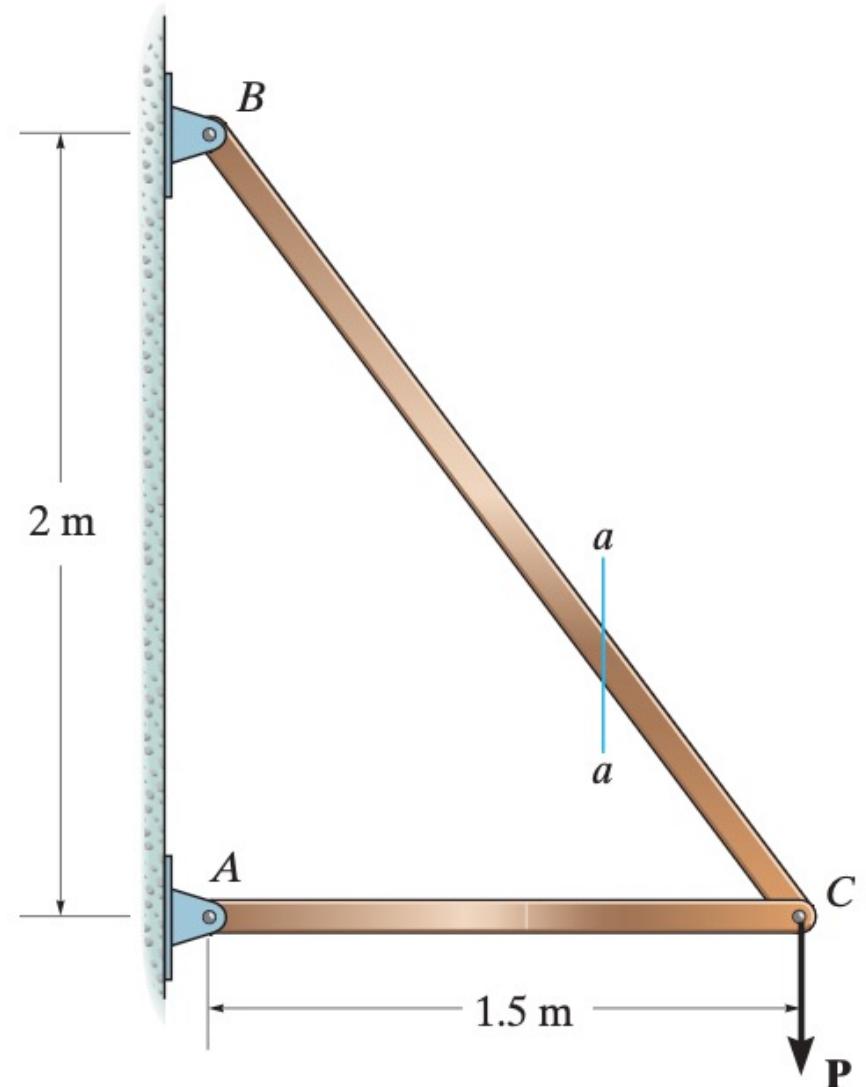
Flux $\varphi = \frac{N}{t} = \frac{v}{L + t_f}$, v = design variable

Objective function $\varphi(v) =$



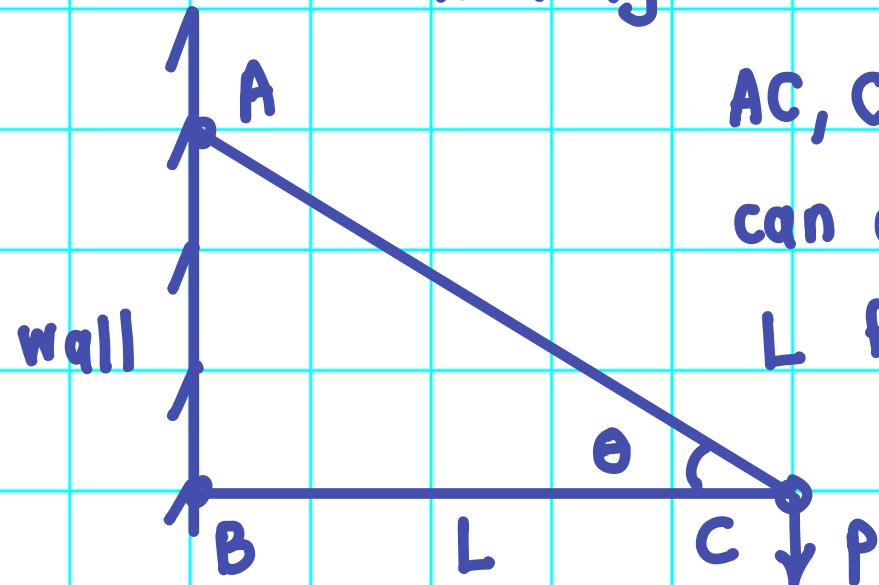
Problem 2 Structure (1D)

For the truss shown, find the angle of the inclined structural member to minimize the total weight of the structure while ensuring that the stress in each member does not exceed the allowable stress.



1D STRUCTURAL OPTIMIZATION

Minimum weight design



AC, CB bars / two force members
can carry only tensile/compressive
force
 $L_{fixed} = L_{BC}$ given
Find θ for min weight

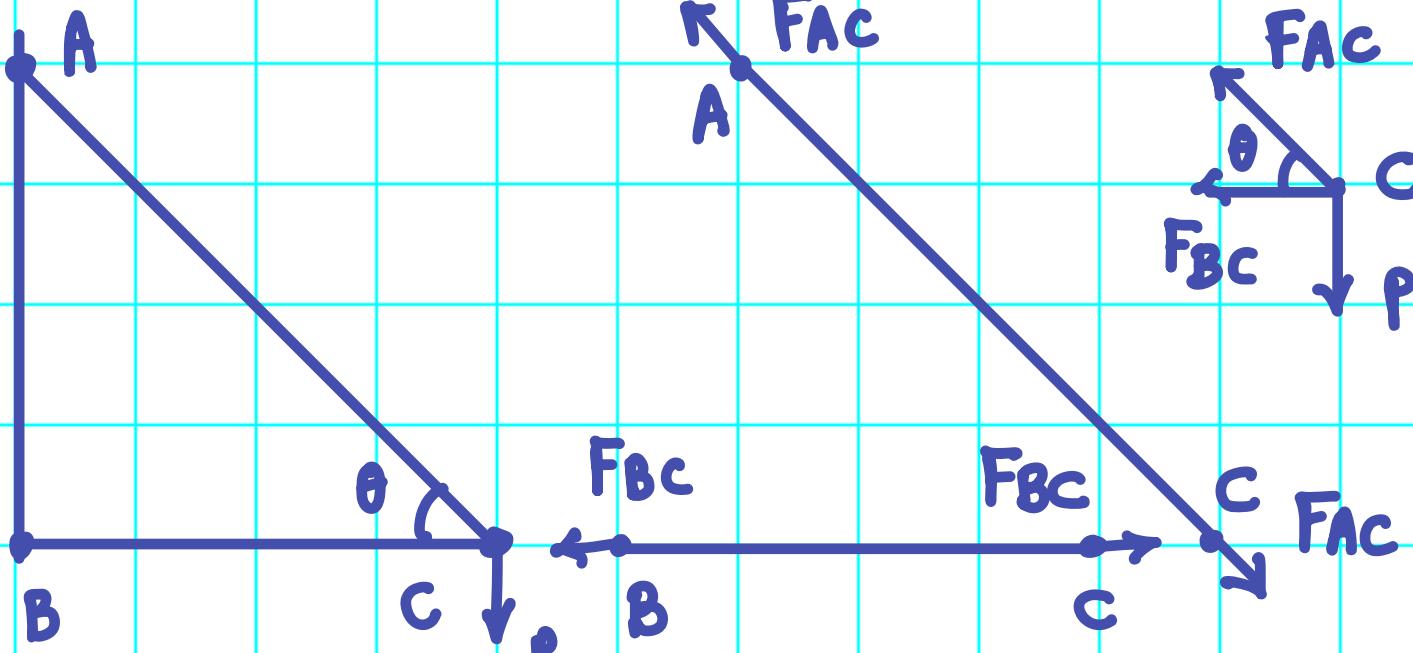
$$\text{Weight of structure } W = \gamma g (L_{AC} A_{AC} + L_{BC} A_{BC})$$

γ mass density of bar material kg/m^3 , given
 g gravitational acceleration N/kg , given

$$A_{AC} = \frac{F_{AC}}{\sigma}, \quad A_{BC} = \frac{F_{BC}}{\sigma}, \quad L_{AC} = \frac{L}{\cos \theta}$$

σ max allowable stress in each bar N/m^2
given

F_{AC} , F_{BC} , P related through static equilibrium



static equilibrium at C $\Rightarrow \sum F_x = 0, \sum F_y = 0$

$$-P + F_{AC} \sin \theta = 0, \quad F_{BC} + F_{AC} \cos \theta = 0$$

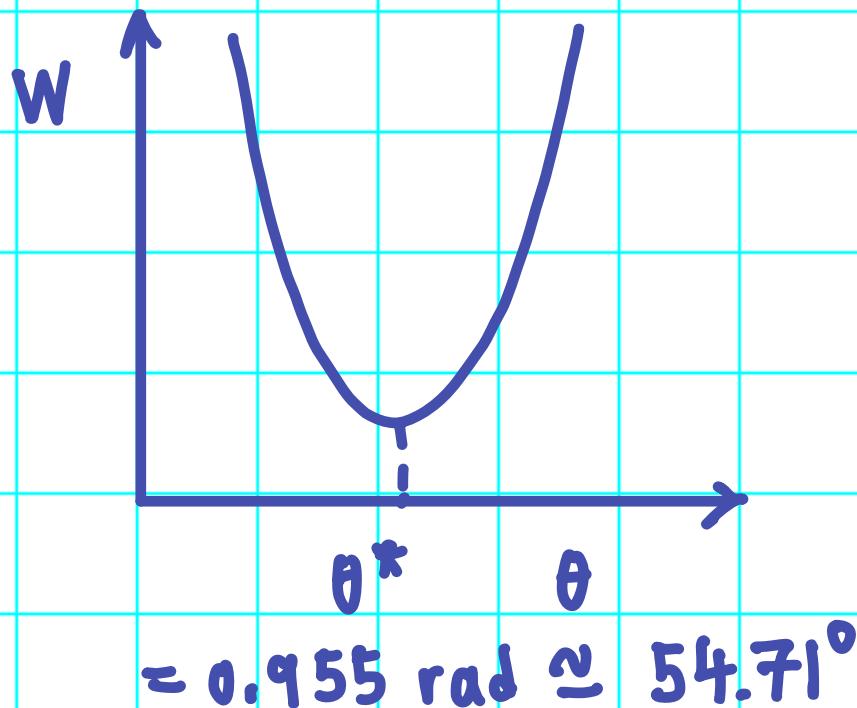
$$F_{AC} = \frac{P}{\sin \theta},$$

$$F_{BC} = \frac{P}{\tan \theta}$$

plug back in expression for W

$$W = \frac{sgL}{r} \left(\frac{1 + \cos^2 \theta}{\sin \theta \cos \theta} \right)$$

objective function



or $\frac{dW}{d\theta} = 0$, θ = design variable

$$\Rightarrow 3 \sin^2 \theta = 2$$

$$\Rightarrow \theta = 0.955 \text{ rad}$$

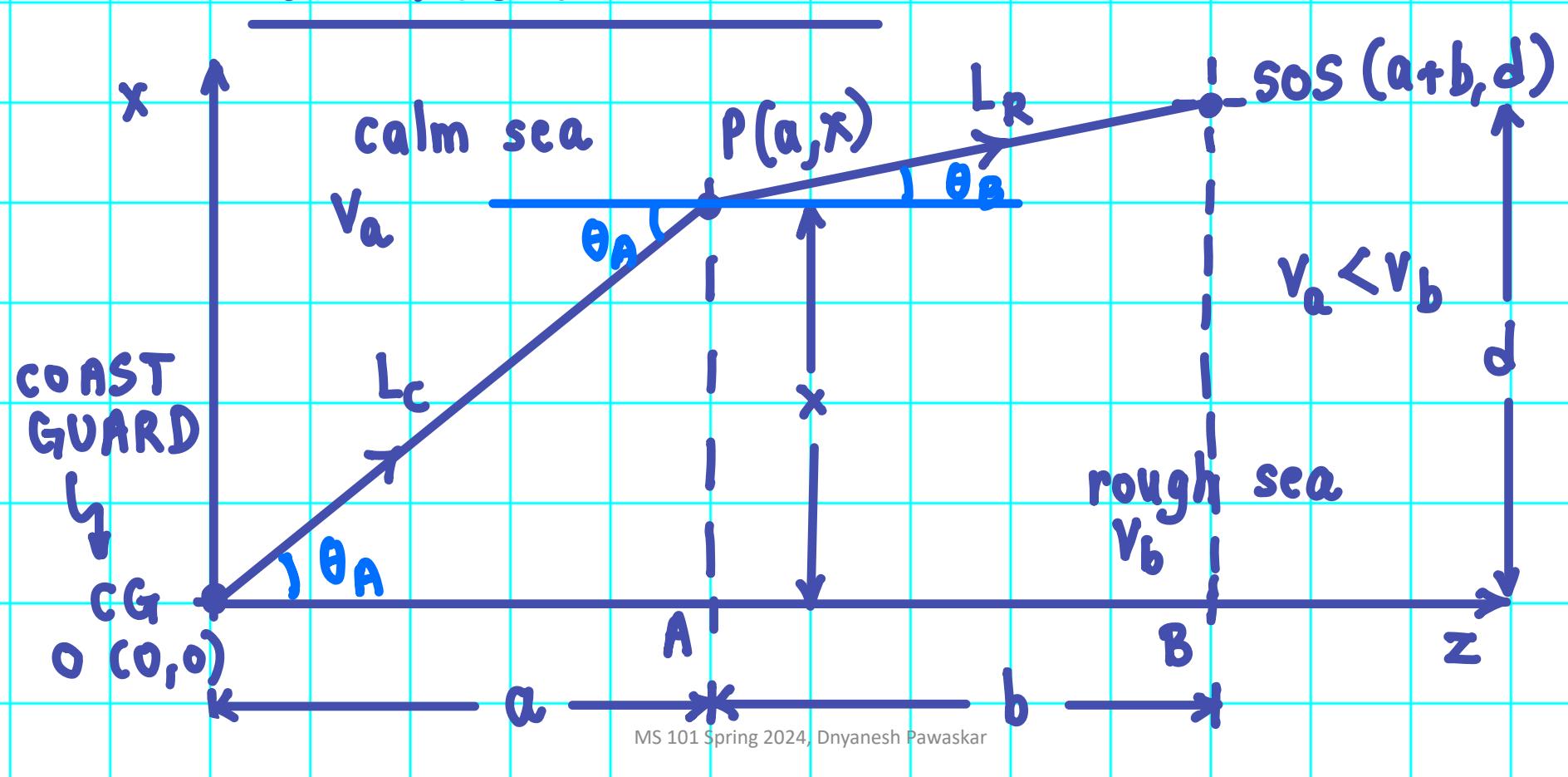
$$\approx 54.71^\circ$$

Problem 3 Rescue (1D)

A merchant ship is under pirate attack and sends out an SOS to a coast guard vessel. Find the best piecewise linear route for the coast guard vessel to reach the merchant ship in minimum time.



1D RESCUE PROBLEM



Find x such that

Rescue (coast guard) ship reaches SOS in
minimum time

$$\text{Total time} = \frac{L_a}{v_a} + \frac{L_b}{v_b}$$

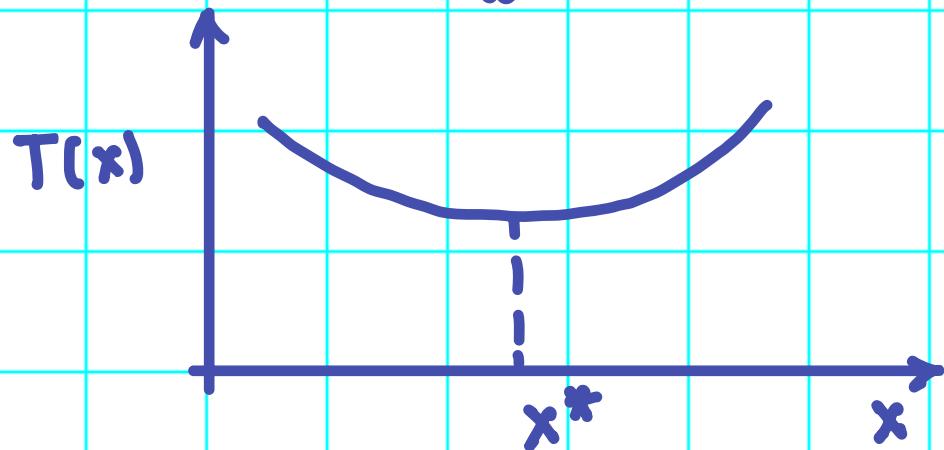
objective function

v_a max ship speed in calm sea , given

v_b max ship speed in rough sea , given

$$L_a = \sqrt{a^2 + x^2}, \quad L_b = \sqrt{b^2 + (d-x)^2}$$

$$T(x) = \frac{\sqrt{a^2 + x^2}}{v_a} + \frac{\sqrt{b^2 + (d-x)^2}}{v_b}$$



x = design variable

$$\frac{dT}{dx} = \frac{1}{v_a} \frac{1x}{\sqrt{a^2 + x^2}} + \frac{1}{v_b} \frac{1(x-d)}{\sqrt{b^2 + (x-d)^2}} = 0$$

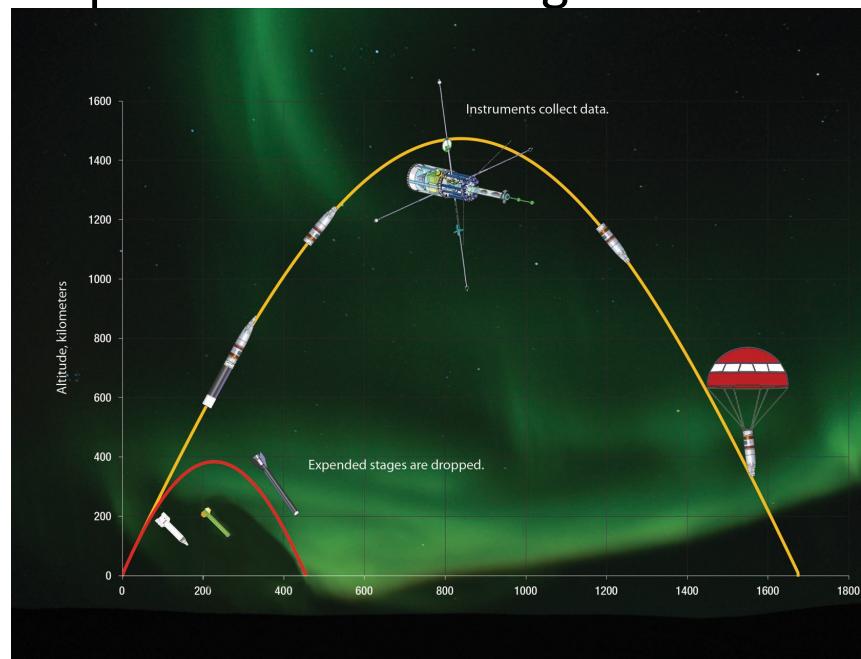
$$\Rightarrow \frac{1}{v_a} \sin \theta_A - \frac{1}{v_b} \sin \theta_B = 0$$

$$\Rightarrow \frac{\sin \theta_A}{v_a} = \frac{\sin \theta_B}{v_b} \quad \text{Snell's Law}$$

Fermat's Principle of Least Time

Problem 4 Projectile Range (1D)

Find the optimum launch angle for a projectile to maximize the horizontal range in the presence of air drag.



1D OPTIMIZATION MAX RANGE OF PROJECTILE

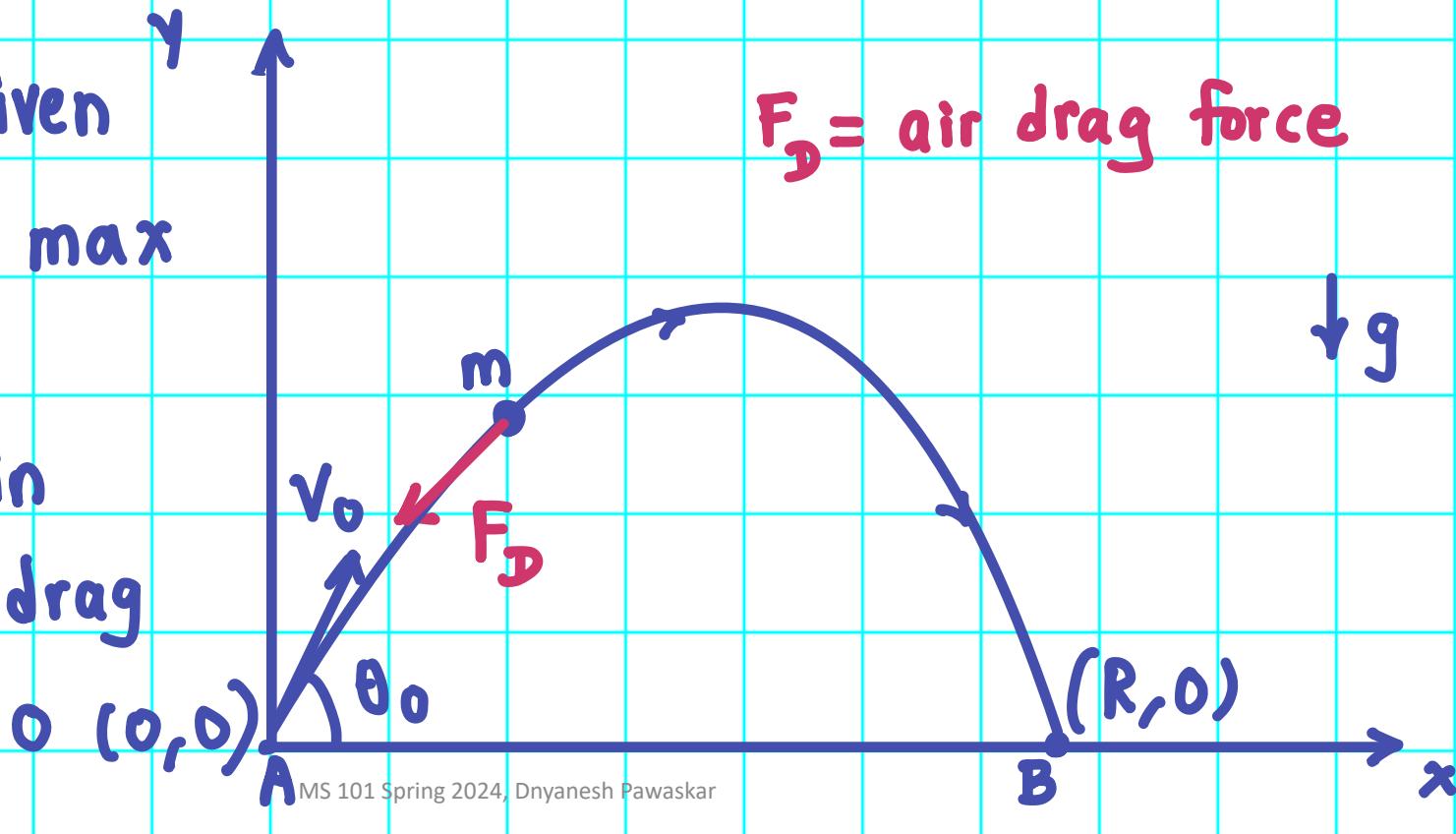
v_0 fixed/given

find θ_0 for max range R

$\theta_0 = 45^\circ$ in absence of drag

F_D = air drag force

$\downarrow g$



Recap, $\underline{F} = m \underline{a} \Rightarrow m \ddot{x} = 0, m \ddot{y} = -mg$
4 initial conditions

$$x(0) = 0, y(0) = 0, \dot{x}(0) = v_0 \cos \theta_0, \dot{y}(0) = v_0 \sin \theta_0$$

$$x(t) = (v_0 \cos \theta_0)t = \dot{x}_0$$

$$y(t) = (v_0 \sin \theta_0)t - \frac{1}{2}gt^2 = \dot{y}_0$$

t_{\max} = total flight time $\Rightarrow y(t_{\max}) = 0$

$$\Rightarrow v_0 \sin \theta_0 = \frac{1}{2} g t_{\max} \Rightarrow t_{\max} = \frac{2v_0 \sin \theta_0}{g}$$

$$R = \text{range} = x(t_{\max}) = \frac{2v_0^2 \cos \theta_0 \sin \theta_0}{g}$$

$R = R_{\max}$ when $\theta_0 = 45^\circ$

$$\Rightarrow R_{\max} = \frac{v_0^2}{g}$$

Linear Air Resistance

$$F_D = m\beta v, \text{ drag force}$$

$$m\ddot{x} = -m\beta\dot{x}, \quad m\ddot{y} = -m\beta\dot{y} - mg$$

β = from expt measurements, given

$$\dot{x} = \dot{x}_0 e^{-\beta t}, \quad \dot{y} = \dot{y}_0 e^{-\beta t} - \frac{g}{\beta} (1 - e^{-\beta t})$$

by integrating once

$$\text{where } \dot{x}_0 = v_0 \cos \theta_0, \quad \dot{y}_0 = v_0 \sin \theta_0$$

Integrate once more, with $x(0)=0, y(0)=0$

$$x(t) = \frac{\dot{x}_0}{\beta} (1 - e^{-\beta t}),$$

$$y(t) = \left(\frac{\dot{y}_0}{\beta} + \frac{g}{\beta^2} \right) (1 - e^{-\beta t}) - \frac{g}{\beta} t$$

For horizontal range $R = x_{max}$, set $y(t) = 0$

and plug in $R = \frac{\dot{x}_0}{\beta} (1 - e^{-\beta t})$, $t = -\frac{1}{\beta} \ln \left(1 - \frac{\beta R}{\dot{x}_0} \right)$

$$\Rightarrow \left(\frac{\dot{y}_0}{\beta} + \frac{g}{\beta^2} \right) \frac{\beta R}{\dot{x}_0} + \frac{g}{\beta^2} \ln \left(1 - \frac{\beta R}{\dot{x}_0} \right) = 0$$

which is an implicit expression for R .

Approximation $\ln(1-u) = -u - \frac{u^2}{2} - \frac{u^3}{3} - \dots$

$$|u| < 1$$

$$u = \frac{\beta R}{\dot{x}_0}, \quad \text{small } \beta \\ \text{i.e. } \beta \ll \frac{g}{v_0}$$

small wrt ?
 $m \beta v_0 \ll m g$

$$R = \frac{2 \dot{x}_0 \dot{y}_0}{g} - \frac{8 \dot{x}_0 \dot{y}_0^2}{3g^2} \beta + \dots$$

$$\dot{x}_0 = v_0 \cos \theta_0, \quad \dot{y}_0 = v_0 \sin \theta_0$$

$$R = \frac{v_0^2 \sin 2\theta_0}{g} - \frac{4v_0^3 \sin 2\theta_0 \sin \theta_0}{3g^2} \beta + \dots$$

without air resistance

effect of linear, β small
air resistance

Quadratic Drag / Air Resistance

$$F_D = m \beta v^2$$

more realistic, agrees
better with experiments

$$F_D = \frac{1}{2} C_D \rho A v^2 \text{ for ball}$$

air density

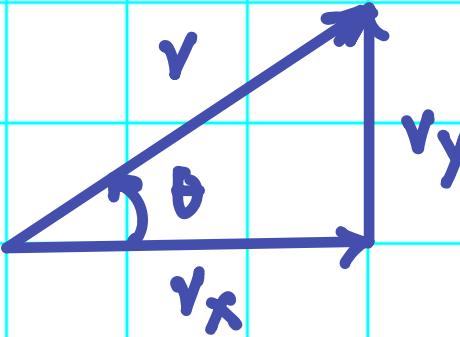
$$C_D \approx 0.45, \rho = 1.23 \text{ kg/m}^3 \text{ at sea-level}$$

$$A = \frac{\pi}{4} d^2, d = \text{diameter of ball}$$

$$v^2 = v_x^2 + v_y^2$$

$$v_x = \dot{x}, v_y = \dot{y}$$

$$\sin \theta = \frac{v_y}{v}, \cos \theta = \frac{v_x}{v}, F = m \underline{a} \Rightarrow$$



$$m\ddot{x} = -F_D \cos \theta = -m\beta (\dot{x}^2 + \dot{y}^2) \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

$$m\ddot{y} = -F_D \sin \theta - mg = -mg - m\beta (\dot{x}^2 + \dot{y}^2) \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

Get two 2nd order nonlinear coupled ODEs,

$$\ddot{x} = -\beta \dot{x} \sqrt{\dot{x}^2 + \dot{y}^2}$$

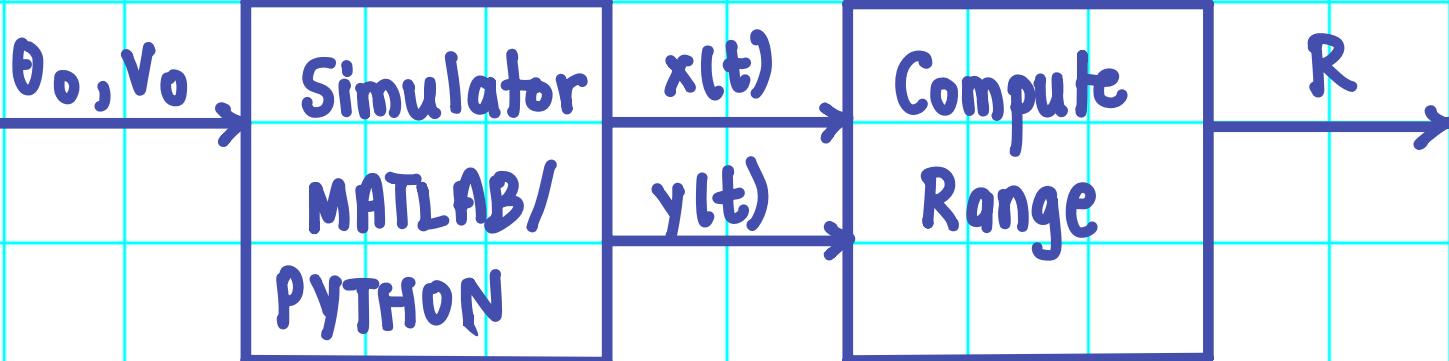
$$\ddot{y} = -g - \beta \dot{y} \sqrt{\dot{x}^2 + \dot{y}^2}$$

Simulate using
computer.
MATLAB, Python, etc

with initial conditions

$$x(0) = 0, y(0) = 0, \dot{x}(0) = v_0 \cos \theta_0, \dot{y}(0) = v_0 \sin \theta_0$$

θ_0 design variable



numerical
integrator like Runge-Kutta 4 (RK4)
to generate time series $x(t), y(t)$

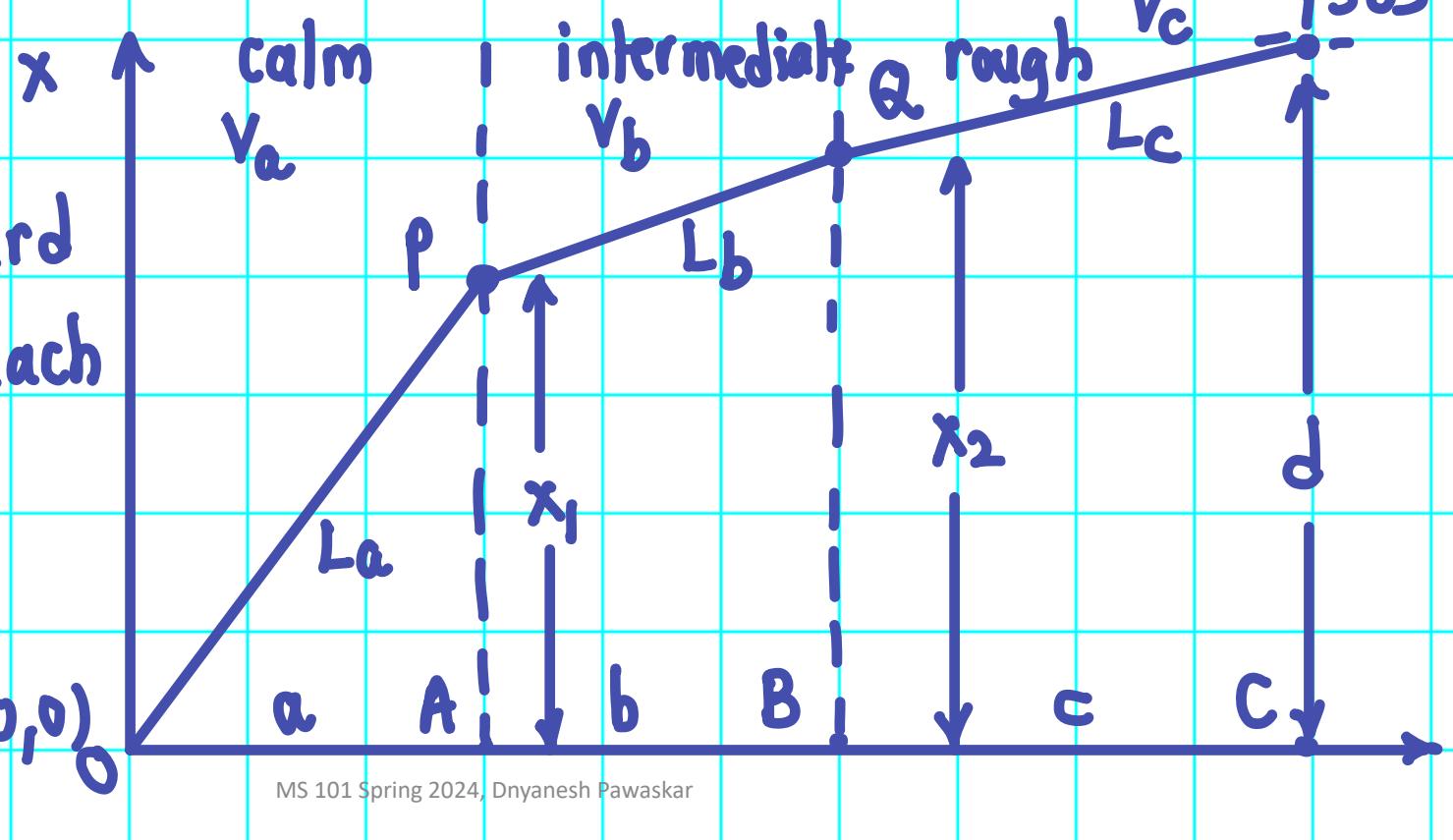
Problem 5 Rescue (2D)

A merchant ship is under pirate attack and sends out an SOS to a coast guard vessel. Find the best piecewise linear route for the coast guard vessel to reach the merchant ship in minimum time.



2 D OPTIMIZATION RESCUE PROBLEM

Find x_1, x_2
for coast guard
at $(0, 0)$ to reach
rescue
 $v_a > v_b > v_c$
COAST GUARD $(0, 0)$



$$T(x_1, x_2) = \frac{L_a}{v_a} + \frac{L_b}{v_b} + \frac{L_c}{v_c}, \text{ objective function}$$

$$L_a = \sqrt{a^2 + x_1^2}, \quad L_b = \sqrt{b^2 + (x_2 - x_1)^2}$$

$$L_c = \sqrt{c^2 + (d - x_2)^2}$$

$$T(x_1, x_2) = \frac{\sqrt{a^2 + x_1^2}}{v_a} + \frac{\sqrt{b^2 + (x_2 - x_1)^2}}{v_b} + \frac{\sqrt{c^2 + (d - x_2)^2}}{v_c}$$

x_1, x_2 design variables

Locate optimum using $\frac{\partial T}{\partial x_1} = 0, \frac{\partial T}{\partial x_2} = 0$

to solve for x_1, x_2

$$\frac{\partial T}{\partial x_1} = 0, \frac{\partial T}{\partial x_2} = 0$$

$$\underline{\nabla T = 0}$$



Problem 6 Least Squares (2D)

N experiments are carried out with

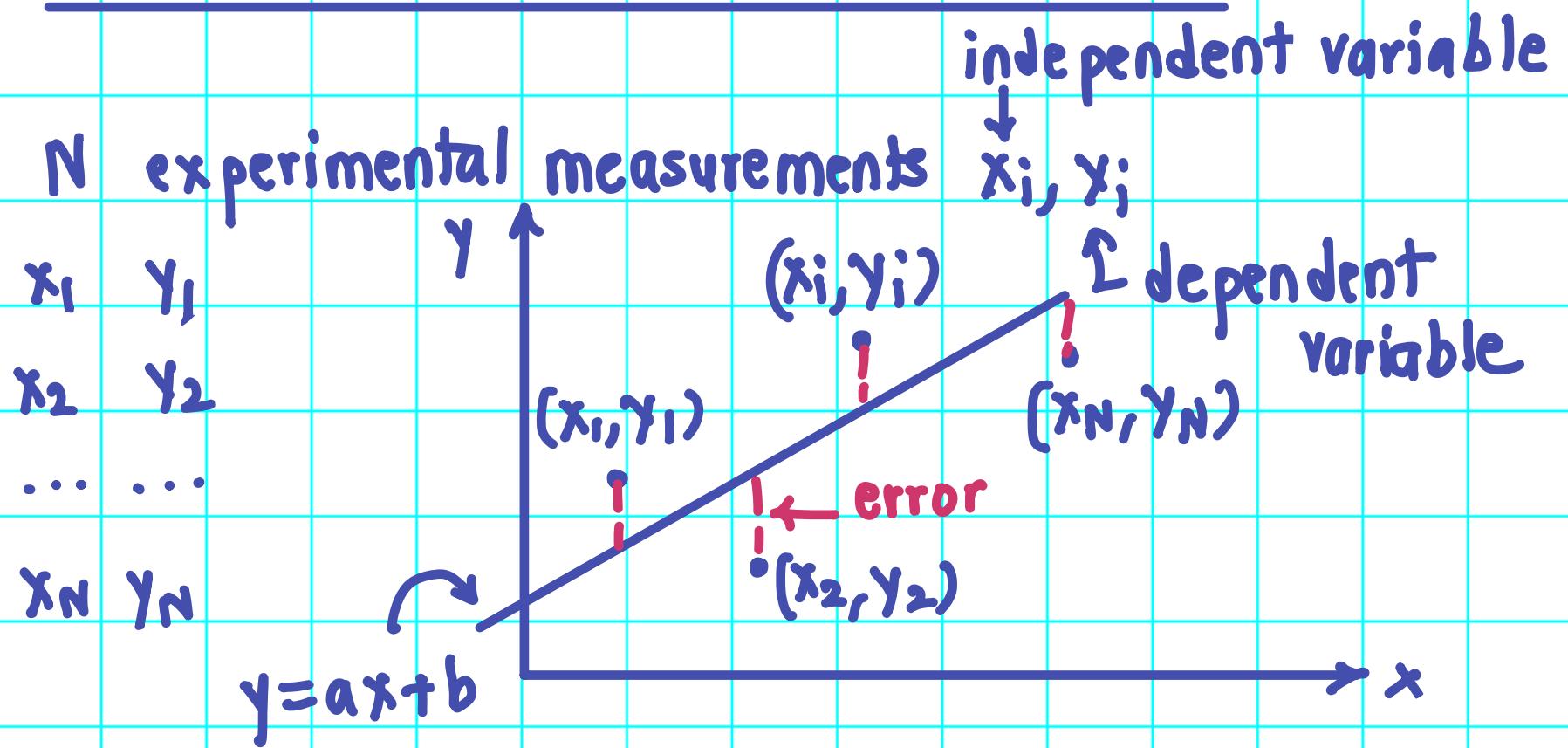
N inputs: x_i
independent variables, $i = 1, 2, 3, \dots, N$

N outputs:

y_i dependent variables,
 $i = 1, 2, 3, \dots, N$

Find the best fit line
between y and x

2D OPTIMIZATION LEAST SQUARES



Expect linear relationship $y = ax + b$ from physical/ logical arguments

Find (a, b) for "best fit" line,

error $e_i = ax_i + b - y_i$

$\sum \text{error}^2 = e_1^2 + e_2^2 + \dots + e_N^2$ = objective function

$f(a, b) = (ax_1 + b - y_1)^2 + (ax_2 + b - y_2)^2 + \dots + (ax_N + b - y_N)^2$

a, b design variables.

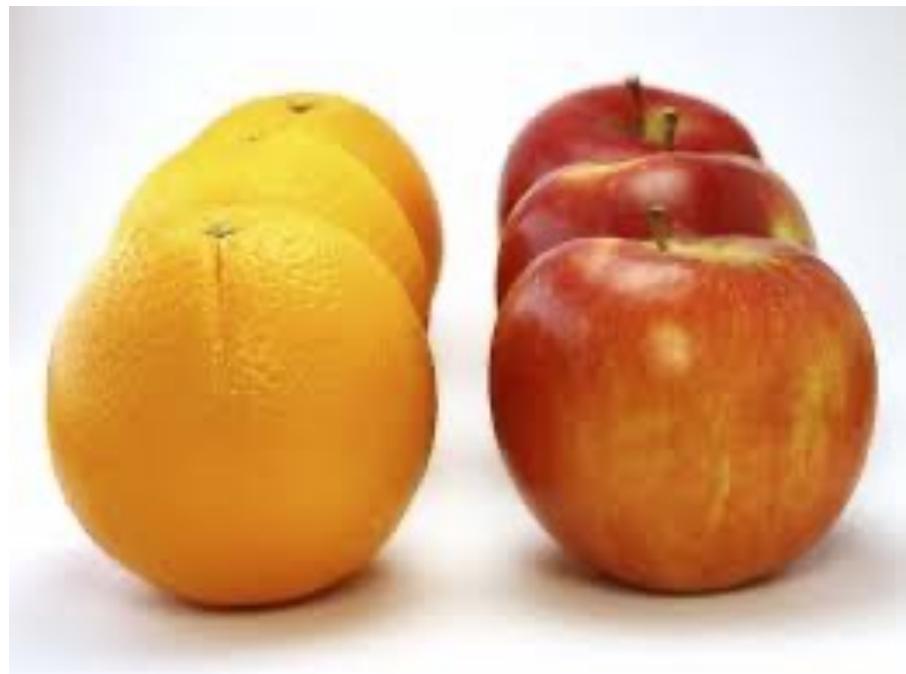
Linear regression as $y = ax + b$ linear in a, b

Locate optimum using $\frac{\partial f}{\partial a} = 0, \frac{\partial f}{\partial b} = 0$
to solve for a and b

$$\nabla f = 0$$

Problem 7 Utility Max (2D)

Find the optimum quantities of apples (x_1) and oranges (x_2) that a person should eat in a month to maximize his/her satisfaction/utility, while operating on a fixed monthly budget.



2D OPTIMIZATION UTILITY MAXIMIZATION

x_1 apples/month, x_2 oranges/month in kg
2 design variables

Utility / satisfaction / benefit / pleasure, objective

$$f = x_1^2 + 8x_2^2 \leftarrow \text{say} \quad \text{function}$$

Apples cost Rs p_1 /kg, Oranges Rs p_2 /kg

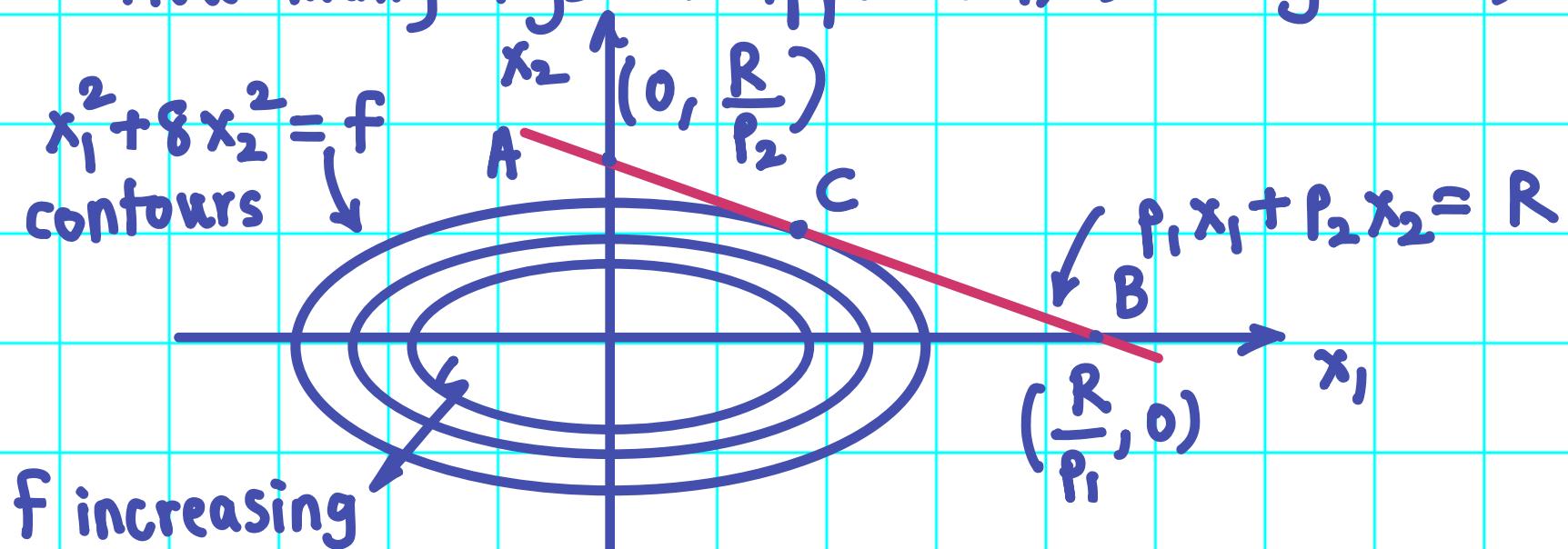
$$p_1 = 100$$

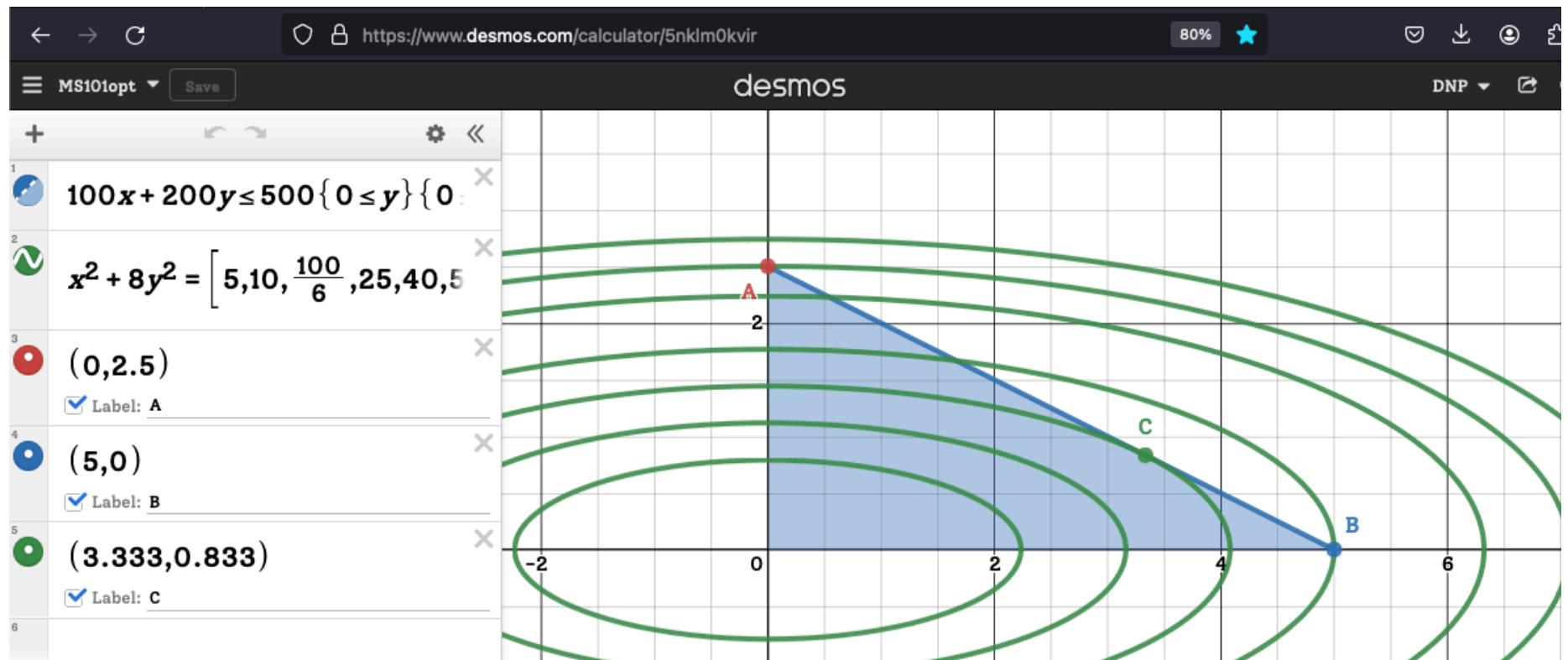
$$p_2 = 200$$

Monthly fruit budget R fixed, say $R = 500$ Rs

$$R = p_1 x_1 + p_2 x_2 \Rightarrow 500 = 100x_1 + 200x_2$$

How many kgs of apples (x_1) & oranges (x_2)?





$$f_A = f\left(0, \frac{R}{P_2}\right) = 0^2 + 8 \frac{R^2}{P_2^2} = 8 \frac{500^2}{200^2} = 50$$

$$f_B = f\left(\frac{R}{P_1}, 0\right) = \frac{R^2}{P_1^2} + 0^2 = \frac{500^2}{100^2} = 25$$

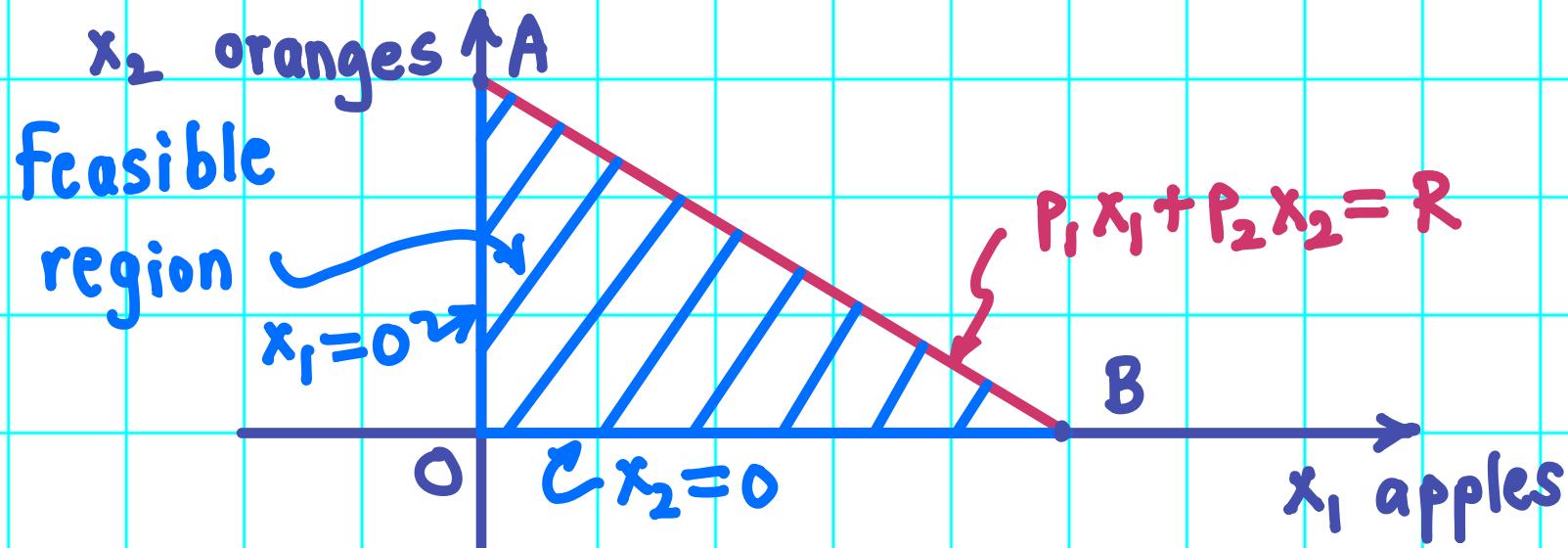
$x_1 = 0, x_2 = 2.5 \text{ kg}$ $\Rightarrow 50 \text{ satisfaction units}$

$x_1 = 5 \text{ kg}, x_2 = 0$ $\Rightarrow 25 \text{ satisfaction units}$

This is a 2D constrained optimization problem.

constraints:

$$x_1 \geq 0, x_2 \geq 0, P_1 x_1 + P_2 x_2 \leq R$$



If additional constraints, they need to be considered

INFINITE DIMENSIONAL PROBLEMS

- "A function is a vector with infinite components"
- Formulation only
- Calculus of Variations / Optimal Control / Dynamic Programming

Problem 8 Rescue (Function, Inf D)

A merchant ship is under pirate attack and sends out an SOS to a coast guard vessel. Find the best route for the coast guard vessel to reach the merchant ship in minimum time.



Infinite Dimensional Optimization

Rescue Problem

Minimize transit time

$$T = \int_0^T dt = \int_A^B \frac{dt}{ds} ds$$

Coast Guard (0,0)

y

x

Merchant
Ship SOS B(a,b)

v(x,y) given

Γ

$\frac{ds}{dx}$

$\frac{dy}{dx}$

$\frac{dt}{ds} = \frac{1}{\dot{s}}$, $\dot{s} = v(x, y)$ tangential speed of boat,
given

Note $y = y(x)$ unknown

$$(ds)^2 = (dx)^2 + (dy)^2 \Rightarrow ds = dx \sqrt{1+y'^2}, y' = \frac{dy}{dx}$$

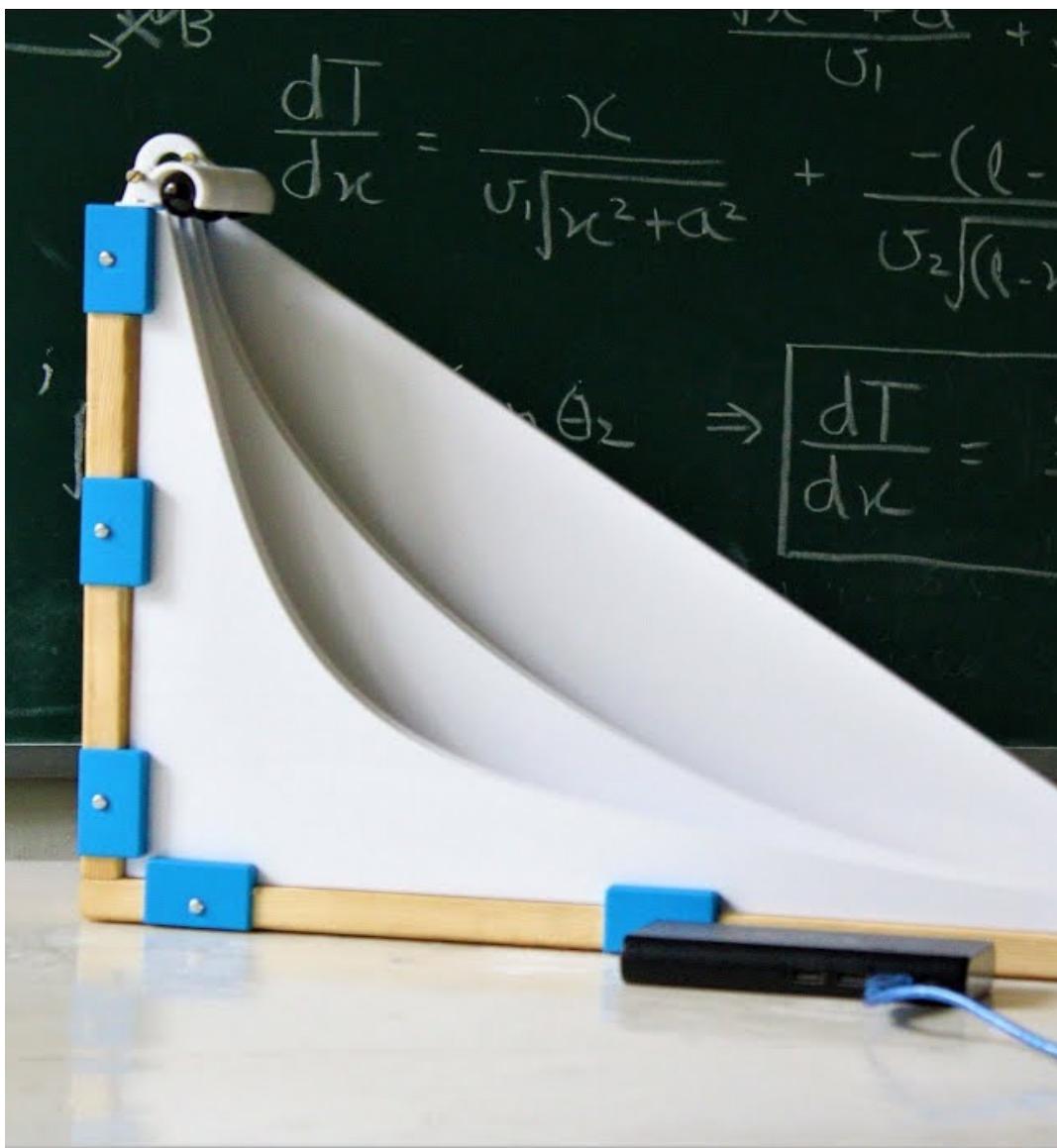
objective

$$a$$

$$\hookrightarrow T = \int_0^a \frac{dt}{ds} \frac{ds}{dx} dx = \int_0^a \frac{1}{v(x, y)} \sqrt{1+y'^2} dx$$

Find $y(x)$ s.t. T is a min \uparrow known dependency

and $y(0) = 0, y(a) = b, y(x)$ design function



Problem 9 Descent Time/Brachistochrone (Function, Inf D)

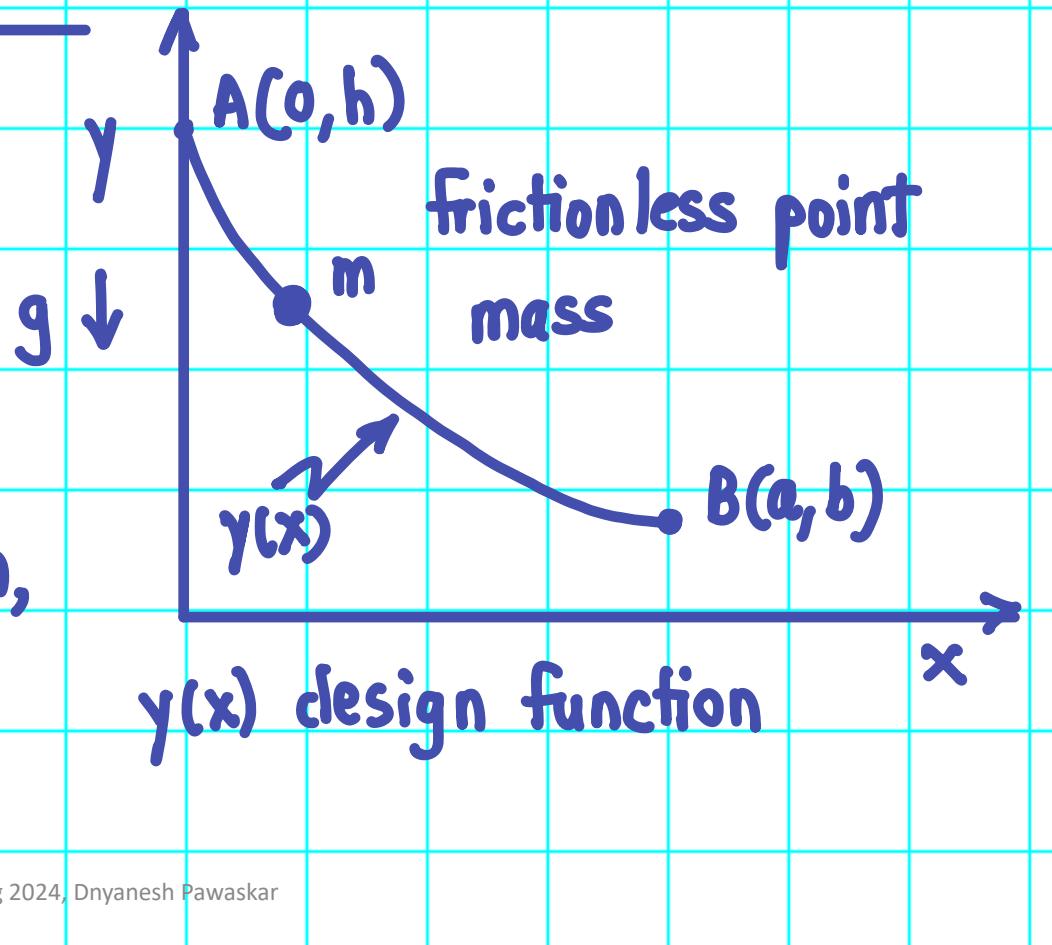
Find the curve that minimizes the descent time for a point mass sliding down under gravity.

Minimum Descent Time

Find $y(x)$ s.t. descent time is minimized

Use energy conservation,

$$\frac{1}{2}mv^2 + mgy = mgh$$



$$\Rightarrow v = \sqrt{2g(h-y)}$$

$$T = \int_0^T dt = \int_A^B \frac{dt}{ds} ds = \int_A^B \frac{1}{v} \sqrt{1+y'^2} dx$$

$\rightarrow T =$
objective
function

$$\int_0^B \sqrt{\frac{1+y'^2}{h-y}} dx$$

Find $y(x)$ s.t.
T is minimized
Brachistochrone

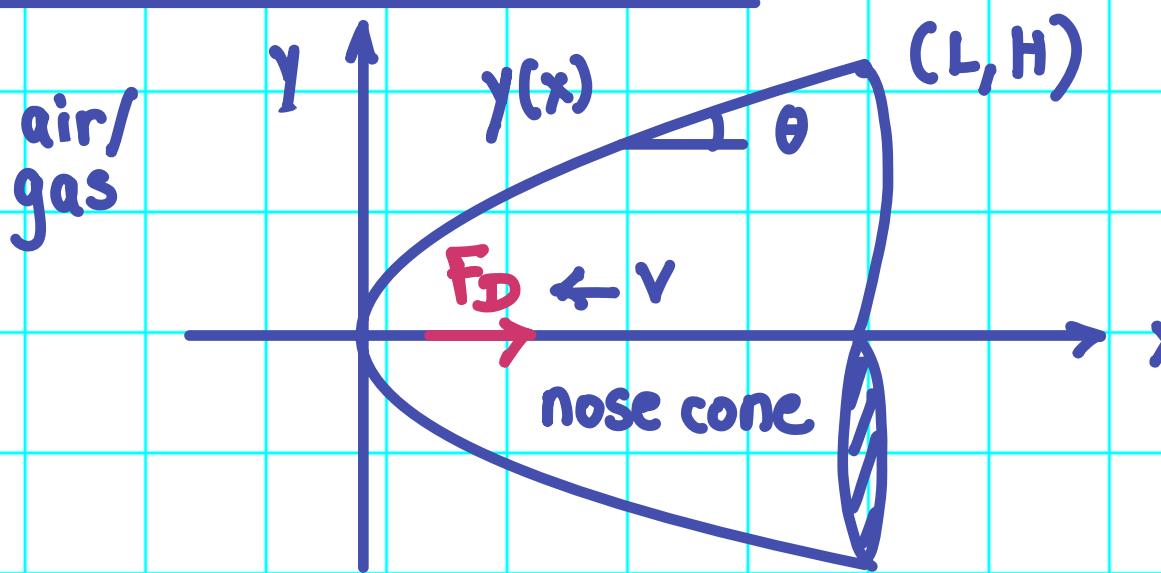
Problem 10 Minimum Drag (Function, Inf D)

Find the profile of a nose cone of an aircraft traveling through air at velocity v that minimizes the total air drag.



MINIMUM DRAG NOSE CONE

Find $y(x)$
s.t. drag
 F_D minimum



Nose cone is volume of revolution of $y(x)$ about x -axis

Consider surface area made of infinite flat plates

jet stream

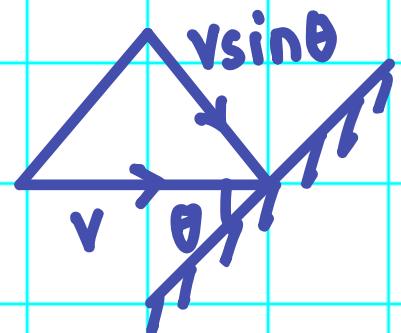
A

ρ, v

wall/plate

$$\text{Normal force on plate} = \rho A v^2$$

Use impulse-momentum principle



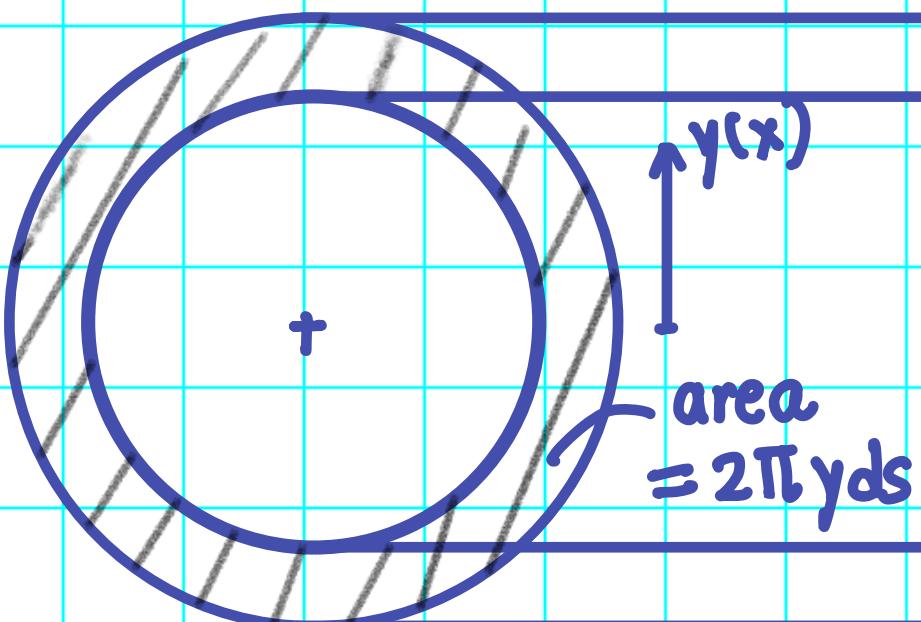
Normal force on inclined plate
 $= \rho A (v \sin \theta)^2$

$$\text{Pressure drag} = \frac{\text{force}}{\text{area}} = \rho v^2 \sin^2 \theta$$

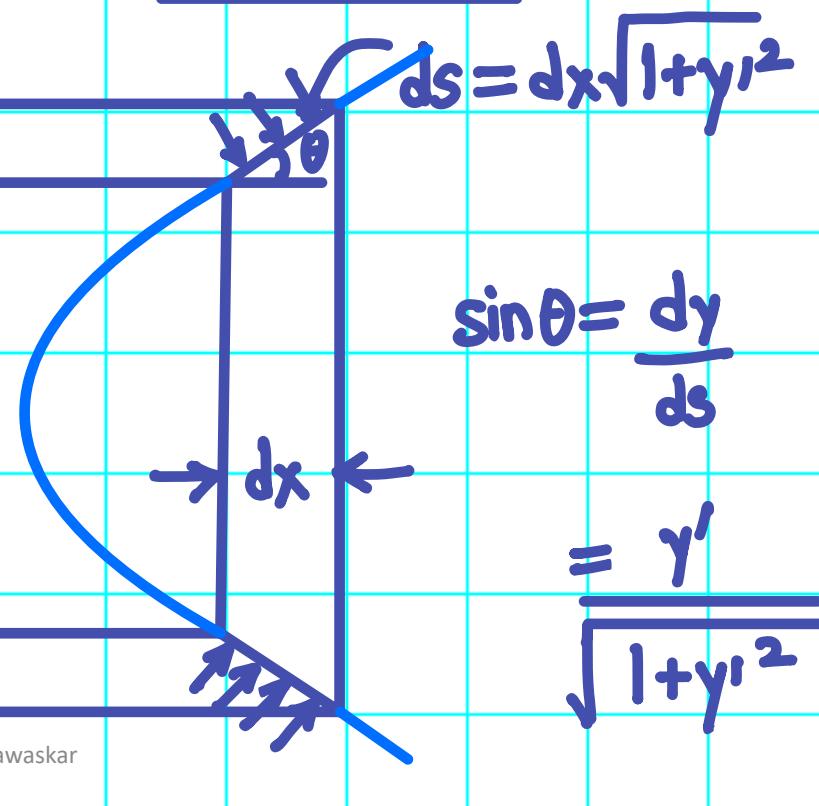
$$\begin{aligned}\text{Horizontal component of pressure drag} &= \rho \sin \theta \\ &= \rho v^2 \sin^3 \theta\end{aligned}$$

Consider an infinitesimal strip of width dx

Front View

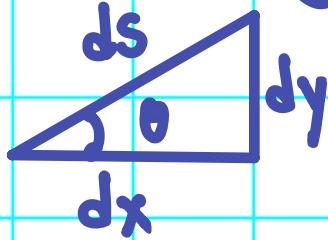


Side View



$$\text{Horizontal force on strip} = \rho 2\pi y ds = \rho v^2 \sin^3 \theta 2\pi y ds$$

$$\text{Total drag on nose cone} = 2\pi \rho v^2 \int_0^L \sin^3 \theta y \sqrt{1+y'^2} dx$$



$$F_D = 2\pi \rho v^2 \int_0^L \frac{y y'^3}{\sqrt{1+y'^2}} dx$$

↑
objective

Find $y(x)$ st. integral minimum