MA-110 Linear Algebra and Differential Equations

Rekha Santhanam

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

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Recap

- If a $n \times n$ matrix is invertible it has n-pivots.
- Elementary matrix $E_{ij}(\lambda)$ is a matrix corresponding to adding λ multiple of the j^{th} row to the i^{th} row. Its inverse corresponds to adding λ multiple of the j^{th} row to the i^{th} row, $E_{ij}(-\lambda)$.
- Permutation matrices P_{ij} are matrices which correspond to row exchanges. Product of any matrices of this form is also called a permutation matrix . The inverse of the matrix P_{ij} is P_{ij}.
- Note that P_{ij} is a symmetric matrix.
- Any square matrix without row exchanges can be written as a product of a lower triangular and upper triangular matrix.

Triangular Factorization

Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}$. Note that each $E_{ij}(a)$ is a *lower triangular*. Product of lower triangular matrices is lower triangular. In particular L is lower triangular, where

$$L = E_{21}(2)\ E_{31}(-1)\ E_{32}(-1) =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{2} & 1 & 0 \\ \mathbf{-1} & \mathbf{-1} & 1 \end{pmatrix}$$

Observe: L is lower triangular with diagonal entries 1 and *below the diagonals are* the multipliers.
$(2, -1, -1$ in the earlier example).

LU Decomposition

If A is an $n \times n$ matrix, *with no row interchanges needed* in the Gaussian elimination of A, then $\boxed{A = L\,U}$, where

• U is an upper triangular matrix, which is obtained by forward elimination, with non-zero diagonal entries as pivots.

• L is a lower triangular with diagonal entries 1 and with the multipliers needed in the elimination algorithm below the diagonals.

Q: What happens if row exchanges are required?

LU Decomposition: with Row Exchanges

Example: $A = \begin{pmatrix} 0 & 2 \\ 3 & 4 \end{pmatrix}$. A can not be factored as LU. (Why?)

How to verify?

The 1st step in the Gaussian elimination of A is a row exchange.

$$P_{12}\ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix}$$

Now elimination can be carried out without row exchanges.

• If A is an $n \times n$ non-singular matrix, then there is a matrix P which is a permutation matrix (needed to take care of row exchanges in the elimination process) such that $\boxed{PA = LU}$, where L and U are as defined earlier. Why?

Q: What happens when A is an $m \times n$ matrix? **A:** Coming Soon!

Application 1: Solving systems of equations

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -12 & -5 \\ 1 & -6 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -8 & 1 \\ 0 & 0 & -1 \end{pmatrix}$

To solve $Ax = b$, we can solve two triangular systems $Lc = b$ and $Ux = c$. Then $Ax = LUx = Lc = b$.

Take $b = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$. First solve $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$.

We get $c_1 = 1$, $-2c_1 + c_2 = 2 \Rightarrow c_2 = 4$, and similarly $c_3 = 0$.

Now solve $\begin{pmatrix} 1 & 2 & 3 \\ 0 & -8 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$.

We get $w = 0$, $v = -1/2$, $u = 2$.

Applications: 2. Invertibility of a Matrix

Let A be $n \times n$, P, L and U as before be such that $PA = LU$.

- P is invertible and $P^{-1} = P^T \Rightarrow A = P^{-1}LU$.
- L is lower triangular, with diagonal entries $1 \Rightarrow L$ is invertible.

Q: What is L^{-1}? e.g., Try $L = E_{21}(2)E_{31}(-1)E_{32}(-1)$ first.

- The non-zero diagonal entries of U are the pivots of A.

Thus, A invertible $\Rightarrow A$ has n pivots
$\Rightarrow$ all diagonal entries of U are non-zero $\Rightarrow U$ is invertible.

Why? Hint: U^T is invertible.

Conversely, suppose U is invertible. Then A is invertible and has n pivots. Why? Moreover, $A^{-1} =$ _______.

We have proved:

$$\boxed{A \text{ is invertible} \Leftrightarrow U \text{ is invertible} \Leftrightarrow A \text{ has } n \text{ pivots.}}$$

Observe: $A = L\,U \Rightarrow A^{-1} = U^{-1}\,L^{-1}$.

Example: $A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}$ is invertible. Find A^{-1}.

If $A^{-1} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$, where x_i is the i-th column of A^{-1}, then $AA^{-1} = I$ gives three systems of linear equations

$$Ax_1 = e_1, \quad Ax_2 = e_2, \quad Ax_3 = e_3$$

where e_i is the i-th column of I. Since the coefficient matrix A is same in three systems, we can solve them simultaneously as follows:

Calculation of A^{-1} : Gauss-Jordan Method

Steps: $(A|I) \longrightarrow (U|L^{-1}) \longrightarrow (I|U^{-1}L^{-1})$.

$$
\begin{aligned}
(A \mid e_1 \quad e_2 \quad e_3) &= \begin{pmatrix} 2 & 1 & 1 & | & 1 & 0 & 0 \\ 4 & -6 & 0 & | & 0 & 1 & 0 \\ -2 & 7 & 2 & | & 0 & 0 & 1 \end{pmatrix} \\
&\xrightarrow[R_3+R_1]{R_2-2R_1} \begin{pmatrix} \mathbf{2} & 1 & 1 & | & 1 & 0 & 0 \\ 0 & -8 & -2 & | & -2 & 1 & 0 \\ 0 & 8 & 3 & | & 1 & 0 & 1 \end{pmatrix} \\
&\xrightarrow{R_3+R_2} \begin{pmatrix} \mathbf{2} & 1 & 1 & | & 1 & 0 & 0 \\ 0 & \mathbf{-8} & -2 & | & -2 & 1 & 0 \\ 0 & 0 & \mathbf{1} & | & -1 & 1 & 1 \end{pmatrix} \\
&= (U \mid L^{-1}).
\end{aligned}
$$

Steps: $(A|I) \longrightarrow (U|L^{-1}) \longrightarrow (I|U^{-1}L^{-1})$.

$$(U\,|\,L^{-1}) = \begin{pmatrix} \mathbf{2} & 1 & 1 & | & 1 & 0 & 0 \\ 0 & \mathbf{-8} & -2 & | & -2 & 1 & 0 \\ 0 & 0 & \mathbf{1} & | & -1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow[R_1-R_3]{R_2+2R_3} \begin{pmatrix} \mathbf{2} & 1 & 0 & | & 2 & -1 & -1 \\ 0 & \mathbf{-8} & 0 & | & -4 & 3 & 2 \\ 0 & 0 & \mathbf{1} & | & -1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1+\frac{1}{8}R_2} \begin{pmatrix} \mathbf{2} & 0 & 0 & | & 12/8 & -5/8 & -6/8 \\ 0 & \mathbf{-8} & 0 & | & -4 & 3 & 2 \\ 0 & 0 & \mathbf{1} & | & -1 & 1 & 1 \end{pmatrix}$$

$$\text{Divide by pivots} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & \mathbf{12/16} & \mathbf{-5/16} & \mathbf{-6/16} \\ 0 & 1 & 0 & | & \mathbf{4/8} & \mathbf{-3/8} & \mathbf{-2/8} \\ 0 & 0 & 1 & | & \mathbf{-1} & \mathbf{1} & \mathbf{1} \end{pmatrix}$$

$$= (I \mid U^{-1}L^{-1}) \quad = \quad (I \mid \mathbf{A}^{-1})$$

Echelon Form

Recall: If A is $n \times n$, then $PA = LU$, where P is a product of permutation matrices, L is lower triangular, U is upper triangular, and all of size $n \times n$.

Q: What happens when A is not a square matrix?

Let $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$. By elimination, we see:

$$A \to \begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U.$$

Thus $A = LU$, where $L = E_{21}(2)E_{31}(3)E_{32}(-1) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix}$.

Echelon Form

If A is $m \times n$, we can find P, L and U as before. In this case, L and P will be $m \times m$ and U will be $m \times n$.

U has the following properties:

1. Pivots are the 1st nonzero entries in their rows.
2. Entries below pivots are zero, by elimination.
3. Each pivot lies to the right of the pivot in the row above.
4. Zero rows are at the bottom of the matrix.

U is called an *echelon form* of A.

Find all possible 2×2 echelon forms: Let $\bullet$ = pivot entry.