Key Symbols and Abbreviations

$(\mathbf{a} \cdot \mathbf{b})$	Dot product of vectors **a** and **b**; $\mathbf{a}^T\mathbf{b}$
$\mathbf{c}(\mathbf{x})$	Gradient of cost function, $\nabla f(\mathbf{x})$
$f(\mathbf{x})$	Cost function to be minimized
$g_j(\mathbf{x})$	jth inequality constraint
$h_i(\mathbf{x})$	ith equality constraint
m	Number of inequality constraints
n	Number of design variables
p	Number of equality constraints
$\mathbf{x}$	Design variable vector of dimension n
x_i	ith component of design variable vector $\mathbf{x}$
$\mathbf{x}^{(k)}$	kth design variable vector

Note: A superscript (i) indicates optimum value for a variable, (ii) indicates advanced material section, and (iii) indicates a project-type exercise.

ACO	Ant colony optimization
BBM	Branch-and-bound method
CDF	Cumulative distribution function
CSD	Constrained steepest descent
DE	Differential evolution; Domain elimination
GA	Genetic algorithm
ILP	Integer linear programming
KKT	Karush-Kuhn-Tucker
LP	Linear programming
MV-OPT	Mixed variable optimization problem
NLP	Nonlinear programming
PSO	Particle swarm optimization
QP	Quadratic programming
RBDO	Reliability-based design optimization
SA	Simulated annealing
SLP	Sequential linear programming
SQP	Sequential quadratic programming
TS	Traveling salesman (salesperson)

CHAPTER

1

Introduction to Design Optimization

Upon completion of this chapter, you will be able to

- Describe the overall process of designing systems
- Distinguish between engineering design and engineering analysis activities
- Distinguish between the conventional design process and the optimum design process
- Distinguish between optimum design and optimal control problems
- Understand the notations used for operations with vectors, matrices, and functions and their derivatives

Engineering consists of a number of well-established activities, including analysis, design, fabrication, sales, research, and development of systems. The subject of this text—the design of systems—is a major field in the engineering profession. The process of designing and fabricating systems has been developed over centuries. The existence of many complex systems, such as buildings, bridges, highways, automobiles, airplanes, space vehicles, and others, is an excellent testimonial to its long history. However, the evolution of such systems has been slow and the entire process is both time-consuming and costly, requiring substantial human and material resources. Therefore, the procedure has been to design, fabricate, and use a system regardless of whether it is the *best one.* Improved systems have been designed only after a substantial investment has been recovered.

The preceding discussion indicates that several systems can usually accomplish the same task, and that some systems are better than others. For example, the purpose of a bridge is to provide continuity in traffic from one side of the river to the other side. Several types of bridges can serve this purpose. However, to analyze and design all possibilities can be time-consuming and costly. Usually one type is selected based on some preliminary analyses and is designed in detail.

The design of a system can be *formulated as problems of optimization* in which a performance measure is optimized while all other requirements are satisfied. Many numerical methods of optimization have been developed and used to design better systems. This text describes the basic concepts of optimization and numerical methods for the design of engineering systems. Design process, rather than optimization theory, is emphasized. Various theorems are stated as results without rigorous proofs; however, their implications from an engineering point of view are discussed.

Any problem in which certain parameters need to be determined to satisfy constraints can be formulated as one optimization problem. Once this has been done, the concepts and methods described in this text can be used to solve it. For this reason, the optimization techniques are quite general, having a wide range of applicability in diverse fields. It is impossible to discuss every application of optimization concepts and techniques in this introductory text. However, using simple applications, we discuss concepts, fundamental principles, and basic techniques that are used in numerous applications. The student should understand them without becoming bogged down with the notation, terminology, and details of the particular area of application.

1.1 THE DESIGN PROCESS

How Do I Begin to Design a System?

The design of many engineering systems can be a complex process. Assumptions must be made to develop realistic models that can be subjected to mathematical analysis by the available methods, and the models must be verified by experiments. Many possibilities and factors must be considered during problem formulation. *Economic considerations* play an important role in designing cost-effective systems. To complete the design of an engineering system, designers from different fields of engineering usually must cooperate. For example, the design of a high-rise building involves designers from architectural, structural, mechanical, electrical, and environmental engineering as well as construction management experts. Design of a passenger car requires cooperation among structural, mechanical, automotive, electrical, chemical, hydraulics design, and human factors engineers. Thus, in an *interdisciplinary environment* considerable interaction is needed among various design teams to complete the project. For most applications the entire design project must be broken down into several subproblems, which are then treated somewhat independently. Each of the subproblems can be posed as a problem of optimum design.

The design of a system begins with the analysis of various options. Subsystems and their components are identified, designed, and tested. This process results in a set of drawings, calculations, and reports by which the system can be fabricated. We use a systems engineering model to describe the *design process*. Although a complete discussion of this subject is beyond the scope of this text, some basic concepts are discussed using a simple block diagram.

Design is an *iterative process*. *Iterative* implies analyzing several *trial designs* one after another until an acceptable design is obtained. It is important to understand the concept of trial design. In the design process, the designer estimates a trial design of the system based on experience, intuition, or some simple mathematical analyses. The trial design is then analyzed to determine if it is acceptable. If it is, the design process is terminated. In the optimization process, the trial design is analyzed to determine if it is the best. Depending

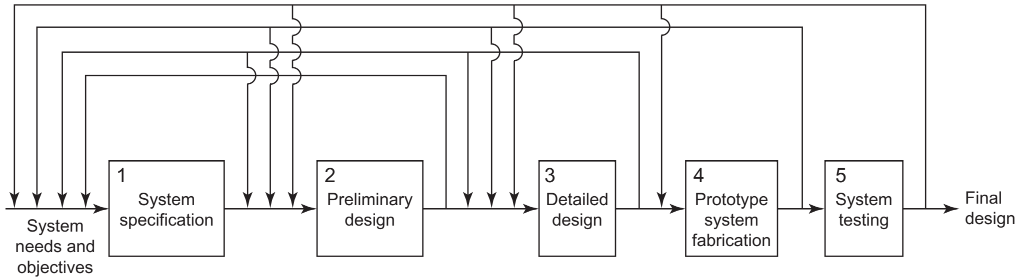

FIGURE 1.1 System evolution model.

on the specifications, "best" can have different connotations for different systems. In general, it implies that a system is cost-effective, efficient, reliable, and durable. The basic concepts are described in this text to aid the engineer in designing systems at the minimum cost and in the shortest amount of time.

The design process should be well organized. To discuss it, we consider a *system evolution model*, shown in Figure 1.1, where the process begins with the identification of a need that may be conceived by engineers or non-engineers. The five steps of the model in the figure are described in the following paragraphs.

The *first step* in the evolutionary process is to precisely define the specifications for the system. Considerable interaction between the engineer and the sponsor of the project is usually necessary to quantify the *system specifications.*

The *second step* in the process is to develop a *preliminary design* of the system. Various system concepts are studied. Since this must be done in a relatively short time, *simplified models* are used at this stage. Various subsystems are identified and their preliminary designs estimated. Decisions made at this stage generally influence the system's final appearance and performance. At the end of the preliminary design phase, a few promising concepts that need further analysis are identified.

The *third step* in the process is a *detailed design* for all subsystems using the iterative process described earlier. To evaluate various possibilities, this must be done for all previously identified promising concepts. The design parameters for the subsystems must be identified. The system performance requirements must be identified and satisfied. The subsystems must be designed to maximize system worth or to minimize a measure of the cost. Systematic optimization methods described in this text aid the designer in accelerating the detailed design process. At the end of the process, a description of the system is available in the form of reports and drawings.

The *fourth and fifth steps* shown in Figure 1.1 may or may not be necessary for all systems. They involve fabrication of a prototype system and testing, and are necessary when the system must be mass-produced or when human lives are involved. These steps may appear to be the final ones in the design process, but they are not because the system may not perform according to specifications during the testing phase. Therefore, the specifications may have to be modified or other concepts may have to be studied. In fact, this reexamination may be necessary at any point during the design process. It is for this reason that *feedback loops* are placed at every stage of the system evolution process, as shown in

Figure 1.1. The iterative process must be continued until the best system evolves. Depending on the complexity of the system, the process may take a few days or several months.

The model described in Figure 1.1 is a simplified block diagram for system evolution. In actual practice, each block may have to be broken down into several sub-blocks to carry out the studies properly and arrive at rational decisions. *The important point is that optimization concepts and methods are helpful at every stage of the process*. Such methods, along with the appropriate software, can be useful in studying various design possibilities rapidly. Therefore, in this text we discuss optimization methods and their use in the design process.

1.2 ENGINEERING DESIGN VERSUS ENGINEERING ANALYSIS

Can I Design without Analysis? No, You Must Analyze!

It is important to recognize the differences between *engineering analysis* and *design activities*. The analysis problem is concerned with determining the behavior of an existing system or a trial system being designed for a given task. Determination of the behavior of the system implies calculation of its response to specified inputs. For this reason, the sizes of various parts and their configurations are given for the analysis problem; that is, the design of the system is known. On the other hand, the design process calculates the sizes and shapes of various parts of the system to meet performance requirements. The design of a system is an iterative process; we estimate a design and analyze it to see if it performs according to given specifications. If it does, we have an *acceptable (feasible) design*, although we may still want to change it to improve its performance. If the trial design does not work, we need to change it to come up with an acceptable system. In both cases, we must be able to *analyze designs* to make further decisions. Thus, analysis capability must be available in the design process.

This book is intended for use in all branches of engineering. It is assumed throughout that students understand the analysis methods covered in undergraduate engineering statics and physics courses. However, *we will not let the lack of analysis capability hinder understanding of the systematic process of optimum design*. Equations for analysis of the system are given wherever feasible.

1.3 CONVENTIONAL VERSUS OPTIMUM DESIGN PROCESS

Why Do I Want to Optimize? Because You Want to Beat the Competition and Improve Your Bottom Line!

It is a challenge for engineers to design efficient and cost-effective systems without compromising their integrity. Figure 1.2(a) presents a self-explanatory flowchart for a conventional design method; Figure 1.2(b) presents a similar flowchart for the optimum design method. It is important to note that both methods are iterative, as indicated by a loop between blocks 6 and 3. Both methods have some blocks that require similar

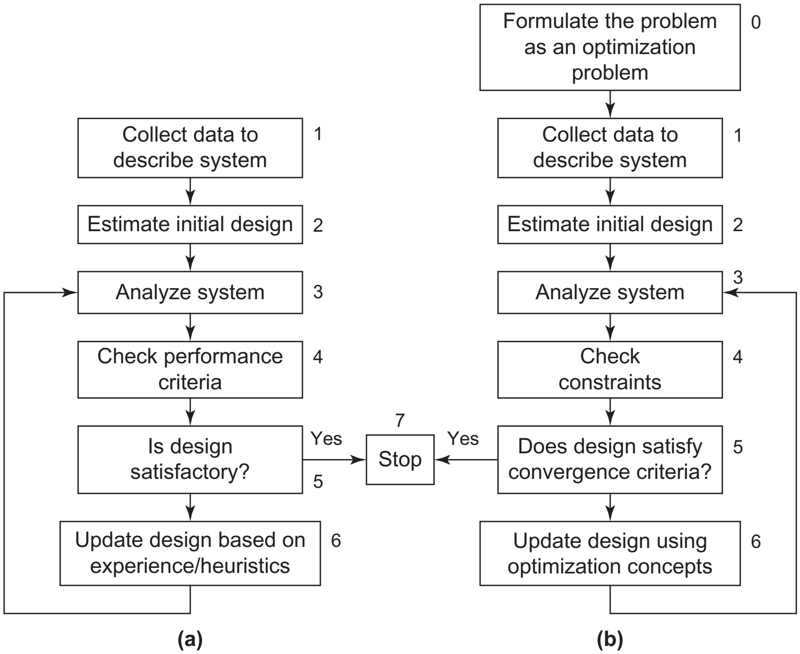

FIGURE 1.2 Comparison of (a) conventional design method and (b) optimum design method.

calculations and others that require different calculations. The key features of the two processes are these:

1. The optimum design method has block 0, where the problem is formulated as one of optimization (discussed in more detail in Chapter 2). An objective function is defined that measures the merits of different designs.
2. Both methods require data to describe the system in block 1.
3. Both methods require an initial design estimate in block 2.
4. Both methods require analysis of the system in block 3.
5. In block 4, the conventional design method checks to ensure that the performance criteria are met, whereas the optimum design method checks for satisfaction of all of the constraints for the problem formulated in block 0.
6. In block 5, stopping criteria for the two methods are checked, and the iteration is stopped if the specified stopping criteria are met.
7. In block 6, the conventional design method updates the design based on the designer's experience and intuition and other information gathered from one or more trial designs; the optimum design method uses optimization concepts and procedures to update the current design.

The foregoing distinction between the two design approaches indicates that the conventional design process is less formal. An objective function that measures a design's merit is not identified. Trend information is usually not calculated; nor is it used in block 6 to make design decisions for system improvement. In contrast, the optimization process is more formal, using trend information to make design changes.

1.4 OPTIMUM DESIGN VERSUS OPTIMAL CONTROL

What Is Optimal Control?

Optimum design and optimal control of systems are separate activities. There are numerous applications in which methods of optimum design are useful in designing systems. There are many other applications where optimal control concepts are needed. In addition, there are some applications in which both optimum design and optimal control concepts must be used. Sample applications of both techniques include *robotics* and *aerospace structures.* In this text, optimal control problems and methods are not described in detail. However, the fundamental differences between the two activities are briefly explained in the sequel. It turns out that optimal control problems can be transformed into optimum design problems and treated by the methods described in this text. Thus, methods of optimum design are very powerful and should be clearly understood. A simple optimal control problem is described in Chapter 14 and is solved by the methods of optimum design.

The optimal control problem consists of finding feedback controllers for a system to produce the desired output. The system has active elements that sense output fluctuations. System controls are automatically adjusted to correct the situation and optimize a measure of performance. Thus, control problems are usually dynamic in nature. In optimum design, on the other hand, we design the system and its elements to optimize an objective function. The system then remains fixed for its entire life.

As an example, consider the cruise control mechanism in passenger cars. The idea behind this feedback system is to control fuel injection to maintain a constant speed. Thus, the system's output (i.e., the vehicle's cruising speed) is known. The job of the control mechanism is to sense fluctuations in speed depending on road conditions and to adjust fuel injection accordingly.

1.5 BASIC TERMINOLOGY AND NOTATION

Which Notation Do I Need to Know?

To understand and to be comfortable with the methods of optimum design, the student must be familiar with linear algebra (vector and matrix operations) and basic calculus. Operations of *linear algebra* are described in Appendix A. Students who are not comfortable with this material need to review it thoroughly. Calculus of functions of single and multiple variables must also be understood. Calculus concepts are reviewed wherever they are needed. In this section, the *standard terminology* and *notations* used throughout the text are defined. It is important to understand and to memorize these notations and operations.

1.5.1 Points and Sets

Because realistic systems generally involve several variables, it is necessary to define and use some convenient and compact notations. *Set* and *vector notations* serve this purpose quite well.

Vectors and Points

A point is an ordered list of numbers. Thus, (x_1, x_2) is a point consisting of two numbers whereas $(x_1, x_2, \ldots, x_n)$ is a point consisting of n numbers. Such a point is often called an n-tuple. The n components $x_1, x_2, \ldots, x_n$ are collected into a column vector as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1 \;\; x_2 \;\; \ldots \;\; x_n]^T \tag{1.1}$$

where the superscript T denotes the *transpose* of a vector or a matrix. This is called an *n-vector*. Each number x_i is called a component of the (point) vector. Thus, x_1 is the first component, x_2 is the second, and so on.

We also use the following notation to represent a point or a vector in the n-dimensional space:

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \tag{1.2}$$

In 3-dimensional space, the vector $\mathbf{x} = [x_1 \;\; x_2 \;\; x_3]^T$ represents a point P, as shown in Figure 1.3. Similarly, when there are n components in a vector, as in Eqs. (1.1) and (1.2), $\mathbf{x}$ is interpreted as a point in the n-dimensional space, denoted as R^n. The space R^n is simply the collection of all n-dimensional vectors (points) of real numbers. For example, the real line is R^1, the plane is R^2, and so on.

The terms *vector* and *point* are used interchangeably, and lowercase letters in roman boldface are used to denote them. Uppercase letters in roman boldface represent matrices.

Sets

Often we deal with *sets* of points satisfying certain conditions. For example, we may consider a set S of all points having three components, with the last having a fixed value of 3, which is written as

$$S = \{\mathbf{x} = (x_1, x_2, x_3) \mid x_3 = 3\} \tag{1.3}$$

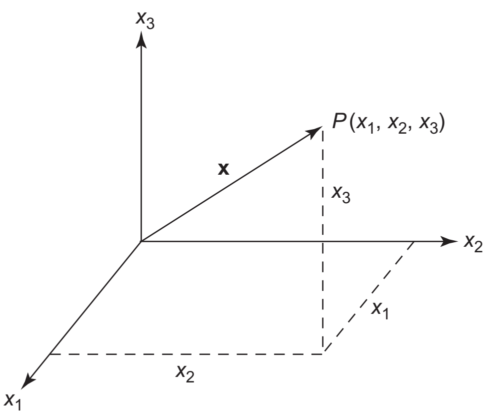

FIGURE 1.3 Vector representation of a point P that is in 3-dimensional space.

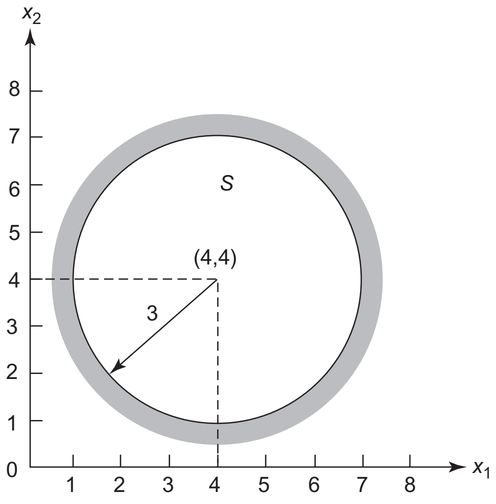

FIGURE 1.4 Image of a geometrical representation for the set $S = \{\mathbf{x} \mid (x_1 - 4)^2 + (x_2 - 4)^2 \le 9\}$.

Information about the set is contained in braces ({}). Equation (1.3) reads as "S equals the set of all points (x_1, x_2, x_3) with $x_3 = 3$." The vertical bar divides information about the set S into two parts: To the left of the bar is the dimension of points in the set; to the right are the properties that distinguish those points from others not in the set (for example, properties a point must possess to be in the set S).

Members of a set are sometimes called *elements*. If a point $\mathbf{x}$ is an element of the set S, then we write $\mathbf{x} \in S$. The expression $\mathbf{x} \in S$ is read as "$\mathbf{x}$ is an element of (belongs to) S." Conversely, the expression "$\mathbf{y} \notin S$" is read as "$\mathbf{y}$ is not an element of (does not belong to) S."

If all the elements of a set S are also elements of another set T, then S is said to be a *subset* of T. Symbolically, we write $S \subset T$, which is read as "S is a subset of T" or "S is contained in T." Alternatively, we say "T is a superset of S," which is written as $T \supset S$.

As an example of a set S, consider a domain of the $x_1 - x_2$ plane enclosed by a circle of radius 3 with the center at the point (4, 4), as shown in Figure 1.4. Mathematically, all points within and on the circle can be expressed as

$$S = \{\mathbf{x} \in R^2 \mid (x_1 - 4)^2 + (x_2 - 4)^2 \le 9\} \tag{1.4}$$

Thus, the center of the circle (4, 4) is in the set S because it satisfies the inequality in Eq. (1.4). We write this as $(4, 4) \in S$. The origin of coordinates (0, 0) does not belong to the set because it does not satisfy the inequality in Eq. (1.4). We write this as $(0, 0) \notin S$. It can be verified that the following points belong to the set: (3, 3), (2, 2), (3, 2), (6, 6). In fact, set S has an infinite number of points. Many other points are not in the set. It can be verified that the following points are not in the set: (1, 1), (8, 8), and (−1, 2).

1.5.2 Notation for Constraints

Constraints arise naturally in optimum design problems. For example, the material of the system must not fail, the demand must be met, resources must not be exceeded, and so on. We shall discuss the constraints in more detail in Chapter 2. Here we discuss the terminology and notations for the constraints.

We encountered a constraint in Figure 1.4 that shows a set S of points within and on the circle of radius 3. The set S is defined by the following constraint:

$$(x_1 - 4)^2 + (x_2 - 4)^2 \leq 9 \tag{1.5}$$

A constraint of this form is a "less than or equal to type" *constraint* and is abbreviated as "≤ type." Similarly, there are *greater than or equal to type constraints*, abbreviated as "≥ type." Both are called *inequality constraints*.

1.5.3 Superscripts/Subscripts and Summation Notation

Later we will discuss a set of vectors, components of vectors, and multiplication of matrices and vectors. To write such quantities in a convenient form, consistent and compact notations must be used. We define these notations here. *Superscripts are used to represent different vectors and matrices.* For example, $\mathbf{x}^{(i)}$ represents the ith vector of a set and $\mathbf{A}^{(k)}$ represents the kth matrix. *Subscripts are used to represent components of vectors and matrices.* For example, x_j is the jth component of $\mathbf{x}$ and a_{ij} is the i–jth element of matrix $\mathbf{A}$. Double subscripts are used to denote elements of a matrix.

To indicate the *range of a subscript or superscript* we use the notation

$$x_i; \quad i = 1 \text{ to } n \tag{1.6}$$

This represents the numbers $x_1, x_2, \ldots, x_n$. Note that "$i = 1$ to n" represents the range for the index i and is read, "i goes from 1 to n." Similarly, a set of k vectors, each having n components, is represented by the *superscript* notation as

$$\mathbf{x}^{(j)}; \quad j = 1 \text{ to } k \tag{1.7}$$

This represents the k vectors $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k)}$. It is important to note that subscript i in Eq. (1.6) and superscript j in Eq. (1.7) are *free indices*; that is, they can be replaced by any other variable. For example, Eq. (1.6) can also be written as x_j, $j = 1$ to n and Eq. (1.7) can be written as $\mathbf{x}^{(i)}$, $i = 1$ to k. Note that the superscript j in Eq. (1.7) does not represent the power of $\mathbf{x}$. It is an index that represents the jth vector of a set of vectors.

We also use the *summation notation* quite frequently. For example,

$$c = x_1y_1 + x_2y_2 + \ldots + x_ny_n \tag{1.8}$$

is written as

$$c = \sum_{i=1}^{n} x_iy_i \tag{1.9}$$

Also, multiplication of an n-dimensional vector $\mathbf{x}$ by an $m \times n$ matrix $\mathbf{A}$ to obtain an m-dimensional vector $\mathbf{y}$ is written as

$$\mathbf{y} = \mathbf{Ax} \tag{1.10}$$

Or, in summation notation, the ith component of $\mathbf{y}$ is

$$y_i = \sum_{j=1}^{n} a_{ij}x_j = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n; \quad i = 1 \text{ to } m \tag{1.11}$$

There is another way of writing the matrix multiplication of Eq. (1.10). Let m-dimensional vectors $\mathbf{a}^{(i)}$; $i = 1$ to n represent columns of the matrix $\mathbf{A}$. Then $\mathbf{y} = \mathbf{Ax}$ is also given as

$$\mathbf{y} = \sum_{j=1}^{n} \mathbf{a}^{(j)}x_j = \mathbf{a}^{(1)}x_1 + \mathbf{a}^{(2)}x_2 + \ldots + \mathbf{a}^{(n)}x_n \tag{1.12}$$

The sum on the right side of Eq. (1.12) is said to be a *linear combination* of columns of matrix $\mathbf{A}$ with x_j, $j = 1$ to n as its multipliers. Or $\mathbf{y}$ is given as a linear combination of columns of $\mathbf{A}$ (refer to Appendix A for further discussion of the linear combination of vectors).

Occasionally, we must use the double summation notation. For example, assuming $m = n$ and substituting y_i from Eq. (1.11) into Eq. (1.9), we obtain the double sum as

$$c = \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} a_{ij}x_j \right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j \tag{1.13}$$

Note that the indices i and j in Eq. (1.13) can be interchanged. This is possible because c is a *scalar quantity*, so its value is not affected by whether we sum first on i or on j. Equation (1.13) can also be written in the matrix form, as we will see later.

1.5.4 Norm/Length of a Vector

If we let $\mathbf{x}$ and $\mathbf{y}$ be two n-dimensional vectors, then their *dot product* is defined as

$$(\mathbf{x} \cdot \mathbf{y}) = \mathbf{x}^T\mathbf{y} = \sum_{i=1}^{n} x_i y_i \tag{1.14}$$

Thus, the dot product is a sum of the product of corresponding elements of the vectors $\mathbf{x}$ and $\mathbf{y}$. Two vectors are said to be *orthogonal* (*normal*) if their dot product is 0; that is, $\mathbf{x}$ and $\mathbf{y}$ are orthogonal if $\mathbf{x} \cdot \mathbf{y} = 0$. If the vectors are not orthogonal, the angle between them can be calculated from the definition of the dot product:

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \; \|\mathbf{y}\| \; \cos\theta \tag{1.15}$$

where θ is the angle between vectors $\mathbf{x}$ and $\mathbf{y}$ and $||\mathbf{x}||$ represents the *length of vector* $\mathbf{x}$. This is also called the *norm of the vector*. The length of vector $\mathbf{x}$ is defined as the square root of the sum of squares of the components:

$$\|\mathbf{x}\| = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{\mathbf{x} \cdot \mathbf{x}} \tag{1.16}$$

The double sum of Eq. (1.13) can be written in the matrix form as follows:

$$c = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j = \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} a_{ij}x_j\right) = \mathbf{x}^T\mathbf{A}\mathbf{x} \tag{1.17}$$

Since **Ax** represents a vector, the triple product of Eq. (1.17) is also written as a dot product:

$$c = \mathbf{x}^T\mathbf{A}\mathbf{x} = (\mathbf{x} \cdot \mathbf{A}\mathbf{x}) \tag{1.18}$$

1.5.5 Functions

Just as a function of a single variable is represented as $f(x)$, a function of n independent variables $x_1, x_2, \ldots, x_n$ is written as

$$f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n) \tag{1.19}$$

We deal with many functions of vector variables. To distinguish between functions, subscripts are used. Thus, the ith function is written as

$$g_i(\mathbf{x}) = g_i(x_1, x_2, \ldots, x_n) \tag{1.20}$$

If there are m functions $g_i(\mathbf{x})$, $i = 1$ to m, these are represented in the vector form

$$\mathbf{g}(\mathbf{x}) = \begin{bmatrix} g_1(\mathbf{x}) \\ g_2(\mathbf{x}) \\ \vdots \\ g_m(\mathbf{x}) \end{bmatrix} = [g_1(\mathbf{x}) \;\; g_2(\mathbf{x}) \;\; \ldots \;\; g_m(\mathbf{x})]^T \tag{1.21}$$

Throughout the text it is *assumed* that all functions are *continuous* and at least *twice continuously differentiable*. A function $f(\mathbf{x})$ of n variables is called *continuous* at a point $\mathbf{x}^*$ if, for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(\mathbf{x}) - f(\mathbf{x}^*)| < \varepsilon \tag{1.22}$$

whenever $||\mathbf{x} - \mathbf{x}^*|| < \delta$. Thus, for all points $\mathbf{x}$ in a small neighborhood of point $\mathbf{x}^*$, a change in the function value from $\mathbf{x}^*$ to $\mathbf{x}$ is small when the function is continuous. A continuous function need not be differentiable. *Twice-continuous differentiability* of a function implies not only that it is differentiable two times, but also that its second derivative is continuous.

Figures 1.5(a) and 1.5(b) show continuous and discontinuous functions. The function in Figure 1.5(a) is differentiable everywhere, whereas the function in Figure 1.5(b) is not differentiable at points x_1, x_2, and x_3. Figure 1.5(c) is an example in which f is not a function because it has infinite values at x_1. Figure 1.5(d) is an example of a discontinuous function. As examples, functions $f(x) = x^3$ and $f(x) = \sin x$ are continuous everywhere and are also continuously differentiable. However, function $f(x) = |x|$ is continuous everywhere but not differentiable at $x = 0$.

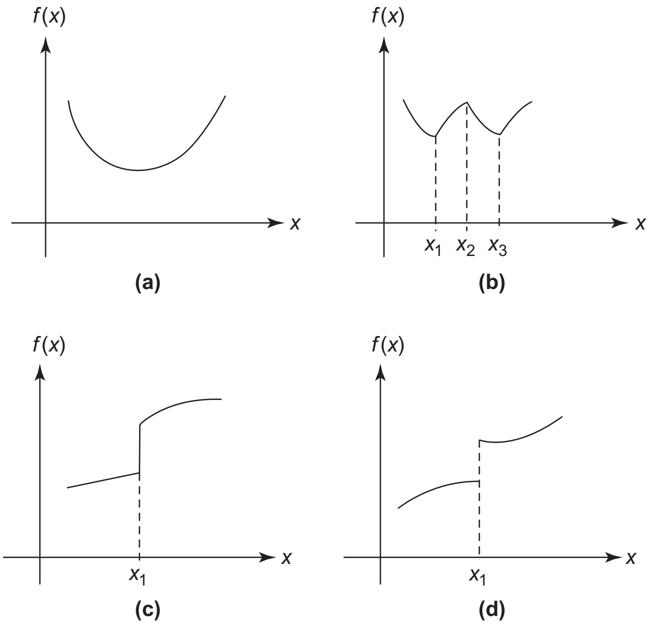

FIGURE 1.5 Continuous and discontinuous functions: (a) and (b) continuous functions; (c) not a function; (d) discontinuous function.

1.5.6 Derivatives of Functions

Often in this text we must calculate derivatives of functions of several variables. Here we introduce some of the basic notations used to represent the partial derivatives of functions of several variables.

First Partial Derivatives

For a function $f(\mathbf{x})$ of n variables, the first partial derivatives are written as

$$\frac{\partial f(\mathbf{x})}{\partial x_i};\quad i = 1 \text{ to } n \tag{1.23}$$

The n partial derivatives in Eq. (1.23) are usually arranged in a column vector known as the *gradient* of the function $f(\mathbf{x})$. The gradient is written as $\partial f/\partial \mathbf{x}$ or $\nabla f(\mathbf{x})$. Therefore,

$$\nabla f(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \dfrac{\partial f(\mathbf{x})}{\partial x_1} \\ \dfrac{\partial f(\mathbf{x})}{\partial x_2} \\ \vdots \\ \dfrac{\partial f(\mathbf{x})}{\partial x_n} \end{bmatrix} \tag{1.24}$$

Note that each component of the gradient in Eq. (1.23) or (1.24) is a function of vector $\mathbf{x}$.

Second Partial Derivatives

Each component of the gradient vector in Eq. (1.24) can be differentiated again with respect to a variable to obtain the second partial derivatives for the function $f(\mathbf{x})$:

$$\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}; \quad i,\ j = 1 \text{ to } n \tag{1.25}$$

We see that there are n^2 partial derivatives in Eq. (1.25). These can be arranged in a matrix known as the *Hessian matrix*, written as $\mathbf{H}(\mathbf{x})$, or simply the matrix of second partial derivatives of $f(\mathbf{x})$, written as $\nabla^2 f(\mathbf{x})$:

$$\mathbf{H}(\mathbf{x}) = \nabla^2 f(\mathbf{x}) = \left[\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}\right]_{n\times n} \tag{1.26}$$

Note that if $f(\mathbf{x})$ is continuously differentiable two times, then Hessian matrix $\mathbf{H}(\mathbf{x})$ in Eq. (1.26) is *symmetric*.

Partial Derivatives of Vector Functions

On several occasions we must differentiate a vector function of n variables, such as the vector $\mathbf{g}(\mathbf{x})$ in Eq. (1.21), with respect to the n variables in vector $\mathbf{x}$. Differentiation of each component of the vector $\mathbf{g}(\mathbf{x})$ results in a gradient vector, such as $\nabla g_i(\mathbf{x})$. Each of these gradients is a*n* n-dimensional vector. They can be arranged as columns of a matrix of dimension $m \times n$, referred to as the gradient matrix of $\mathbf{g}(\mathbf{x})$. This is written as

$$\nabla \mathbf{g}(\mathbf{x}) = \frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}} = [\nabla g_1(\mathbf{x})\ \nabla g_2(\mathbf{x})\ \ldots\ \nabla g_m(\mathbf{x})]_{n\times m} \tag{1.27}$$

This gradient matrix is usually written as matrix $\mathbf{A}$:

$$\mathbf{A} = [a_{ij}]_{n\times m}; \quad a_{ij} = \frac{\partial g_j}{\partial x_i}; \quad i = 1 \text{ to } n; \quad j = 1 \text{ to } m \tag{1.28}$$

1.5.7 U.S.–British versus SI Units

The formulation of the design problem and the methods of optimization do not depend on the units of measure used. Thus, it does not matter which units are used in defining the problem. However, the final form of some of the analytical expressions for the problem does depend on the units used. In the text, we use both U.S.–British and SI units in examples and exercises. Readers unfamiliar with either system should not feel at a disadvantage when reading and understanding the material since it is simple to switch from one system to the other. To facilitate the conversion from U.S.–British to SI units or vice versa, Table 1.1 gives conversion factors for the most commonly used quantities. For a complete list of conversion factors, consult the IEEE ASTM (1997) publication.

TABLE 1.1 Conversion factors for U.S.–British and SI units

To convert from U.S.–British	To SI units	Multiply by
Acceleration		
foot/second^2 (ft/s^2)	meter/second^2 (m/s^2)	0.3048*
inch/second^2 (in/s^2)	meter/second^2 (m/s^2)	0.0254*
Area		
foot^2 (ft^2)	meter^2 (m^2)	0.09290304*
inch^2 (in^2)	meter^2 (m^2)	6.4516E–04*
Bending Moment or Torque		
pound force inch (lbf • in)	Newton meter (N • m)	0.1129848
pound force foot (lbf • ft)	Newton meter (N • m)	1.355818
Density		
pound mass/inch^3 (lbm/in^3)	kilogram/meter^3 (kg/m^3)	27,679.90
pound mass/foot^3 (lbm/ft^3)	kilogram/meter^3 (kg/m^3)	16.01846
Energy or Work		
British thermal unit (BTU)	Joule (J)	1055.056
foot-pound force (ft • lbf)	Joule (J)	1.355818
kilowatt-hour (KWh)	Joule (J)	3,600,000*
Force		
kip (1000 lbf)	Newton (N)	4448.222
pound force (lbf)	Newton (N)	4.448222
Length		
foot (ft)	meter (m)	0.3048*
inch (in)	meter (m)	0.0254*
mile (mi), U.S. statute	meter (m)	1609.347
mile (mi), International, nautical	meter (m)	1852*
Mass		
pound mass (lbm)	kilogram (kg)	0.4535924
slug (lbf • s^{2}ft)	kilogram (kg)	14.5939
ton (short, 2000 lbm)	kilogram (kg)	907.1847
ton (long, 2240 lbm)	kilogram (kg)	1016.047
tonne (t, metric ton)	kilogram (kg)	1000*

TABLE 1.1 (*Continued*)

To convert from U.S.–British	To SI units	Multiply by
Power		
foot-pound/minute (ft • lbf/min)	Watt (W)	0.02259697
horsepower (550 ft • lbf/s)	Watt (W)	745.6999
Pressure or Stress		
atmosphere (std) (14.7 lbf/in^2)	Newton/meter^2 (N/m^2 or Pa)	101,325*
one bar (b)	Newton/meter^2 (N/m^2 or Pa)	100,000*
pound/foot^2 (lbf/ft^2)	Newton/meter^2 (N/m^2 or Pa)	47.88026
pound/inch^2 (lbf/in^2 or psi)	Newton/meter^2 (N/m^2 or Pa)	6894.757
Velocity		
foot/minute (ft/min)	meter/second (m/s)	0.00508*
foot/second (ft/s)	meter/second (m/s)	0.3048*
knot (nautical mi/h), international	meter/second (m/s)	0.5144444
mile/hour (mi/h), international	meter/second (m/s)	0.44704*
mile/hour (mi/h), international	kilometer/hour (km/h)	1.609344*
mile/second (mi/s), international	kilometer/second (km/s)	1.609344*
Volume		
foot^3 (ft^3)	meter^3 (m^3)	0.02831685
inch^3 (in^3)	meter^3 (m^3)	1.638706E–05
gallon (Canadian liquid)	meter^3 (m^3)	0.004546090
gallon (U.K. liquid)	meter^3 (m^3)	0.004546092
gallon (U.S. dry)	meter^3 (m^3)	0.004404884
gallon (U.S. liquid)	meter^3 (m^3)	0.003785412
one liter (L)	meter^3 (m^3)	0.001*
ounce (U.K. fluid)	meter^3 (m^3)	2.841307E–05
ounce (U.S. fluid)	meter^3 (m^3)	2.957353E–05
pint (U.S. dry)	meter^3 (m^3)	5.506105E–04
pint (U.S. liquid)	meter^3 (m^3)	4.731765E–04
quart (U.S. dry)	meter^3 (m^3)	0.001101221
quart (U.S. liquid)	meter^3 (m^3)	9.463529E–04

* *Exact conversion factor.*

CHAPTER

2

Optimum Design Problem Formulation

Upon completion of this chapter, you will be able to

- Translate a descriptive statement of the design problem into a mathematical statement for optimization
- Identify and define the problem's design variables
- Identify and define an optimization criterion for the problem
- Identify and define the design problem's constraints
- Transcribe the problem formulation into a standard model for design optimization

It is generally accepted that the *proper definition and formulation of a problem* take roughly 50 percent of the total effort needed to solve it. Therefore, it is critical to follow well-defined procedures for formulating design optimization problems. In this chapter, we describe the process of transforming the design of a selected system and/or subsystem into an optimum design problem.

Several simple and moderately complex applications are discussed in this chapter to illustrate the problem formulation process. More advanced applications are discussed in Chapters 6 and 7 and 14 through 19.

The *importance of properly formulating* a design optimization problem must be stressed because the optimum solution will be only as good as the formulation. For example, if we forget to include a critical constraint in the formulation, the optimum solution will most likely violate it. Also, if we have too many constraints, or if they are inconsistent, there may be no solution. However, once the problem is properly formulated, good software is usually available to deal with it. For most design optimization problems, we will use the following *five-step* formulation procedure:

Step 1: Project/problem description
Step 2: Data and information collection
Step 3: Definition of design variables
Step 4: Optimization criterion
Step 5: Formulation of constraints

2.1 THE PROBLEM FORMULATION PROCESS

The formulation of an optimum design problem involves translating a descriptive statement of it into a well-defined mathematical statement. We will describe the tasks to be performed in each of the foregoing five steps to develop a mathematical formulation for the design optimization problem. These steps are illustrated with some examples in this section and in later sections.

At this stage, it is also important to understand the solution process for optimization of a design problem. As illustrated earlier in Figure 1.2(b), optimization methods are iterative where the solution process is started by selecting a *trial design* or a *set of trial designs*. The trial designs are analyzed and evaluated, and a new trial design is generated. This iterative process is continued until an optimum solution is reached.

2.1.1 Step 1: Project/Problem Description

Are the Project Goals Clear?

The formulation process begins by developing a descriptive statement for the project/problem, usually by the project's owner/sponsor. The statement describes the overall *objectives* of the project and the *requirements* to be met. This is also called the *statement of work*.

EXAMPLE 2.1 DESIGN OF A CANTILEVER BEAM—PROBLEM DESCRIPTION

Cantilever beams are used in many practical applications in civil, mechanical, and aerospace engineering. To illustrate the step of problem description, we consider the design of a hollow square-cross-section *cantilever beam* to support a load of 20 kN at its end. The beam, made of steel, is 2 m long, as shown in Figure 2.1. The failure conditions for the beam are as follows: (1) the material should not fail under the action of the load, and (2) the deflection of the free end should be no more than 1 cm. The width-to-thickness ratio for the beam should be no more than 8. A minimum-mass beam is desired. The width and thickness of the beam must be within the following limits:

$$60 \leq width \leq 300 \text{ mm} \tag{a}$$

$$10 \leq thickness \leq 40 \text{ mm} \tag{b}$$

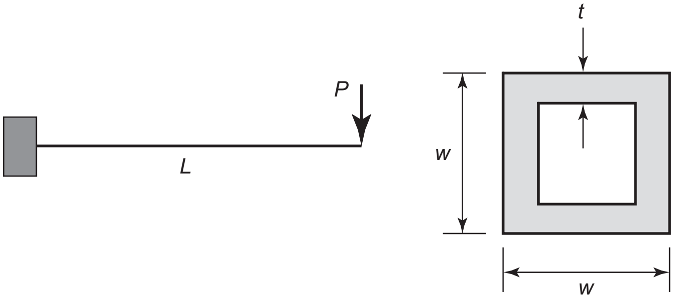

FIGURE 2.1 Cantilever beam of a hollow square cross-section.

2.1.2 Step 2: Data and Information Collection

Is All the Information Available to Solve the Problem?

To develop a mathematical formulation for the problem, we need to gather information on material properties, performance requirements, resource limits, cost of raw materials, and so forth. In addition, most problems require the capability to *analyze trial designs*. Therefore, *analysis procedures* and *analysis tools* must be identified at this stage. For example, the finite-element method is commonly used for analysis of structures, so the software tool available for such an analysis needs to be identified. In many cases, the project statement is vague, and assumptions about modeling of the problem need to be made in order to formulate and solve it.

EXAMPLE 2.2 DATA AND INFORMATION COLLECTION FOR A CANTILEVER BEAM

The information needed for the *cantilever beam design problem* of Example 2.1 includes expressions for bending and shear stresses, and the expression for the deflection of the free end. The notation and data for this purpose are defined in the table that follows.

Useful expressions for the beam are

$$A = w^2 - (w - 2t)^2 = 4t(w - t), \text{ mm}^2 \tag{c}$$

$$I = \frac{8}{3}wt^3 + \frac{2}{3}w^3t - 2w^2t^2 - \frac{4}{3}t^4, \text{ mm}^4 \tag{d}$$

$$Q = \frac{3}{4}w^2t - \frac{3}{2}wt^2 + t^3, \text{ mm}^3 \tag{e}$$

$$M = PL, \text{ N} \cdot \text{mm} \tag{f}$$

$$V = P, \text{ N} \tag{g}$$

$$\sigma = \frac{Mw}{2I}, \text{ N} \cdot \text{mm}^{-2} \tag{h}$$

$$\tau = \frac{VQ}{2It}, \ \mathrm{N \cdot mm^{-2}} \tag{i}$$

$$q = \frac{PL^3}{3EI}, \ \mathrm{mm} \tag{j}$$

Notation	Data
A	cross-sectional area, mm^2
E	modulus of elasticity, 21×10^4 N · mm^{-2}
G	shear modulus, 8×10^4 N · mm^{-2}
I	moment of inertia, mm^4
L	length of the member, 2000 mm
M	bending moment, N · mm
P	load at the free end, 20,000 N
Q	moment about the neutral axis of the area above the neutral axis, mm^3
q	vertical deflection of the free end, mm
q_a	allowable vertical deflection of the free end, 10 mm
V	shear force, N
w	width (depth) of the section, mm
t	wall thickness, mm
σ	bending stress, N · mm^{-2}
σ_a	allowable bending stress, 165 N · mm^{-2}
τ	shear stress, N · mm^{-2}
τ_a	allowable shear stress, 90 N · mm^{-2}

2.1.3 Step 3: Definition of Design Variables

What Are These Variables?
How Do I Identify Them?

The next step in the formulation process is to identify a set of variables that describe the system, called the *design variables*. In general, these are referred to as optimization variables and are regarded as *free* because we should be able to assign any value to them. Different values for the variables produce different designs. The design variables should be independent of each other as far as possible. If they are dependent, their values cannot be specified independently because there are constraints between them. The number of independent design variables gives the *design degrees of freedom* for the problem.

For some problems, different sets of variables can be identified to describe the same system. Problem formulation will depend on the selected set. We will present some examples later in this chapter to elaborate on this point.

Once the design variables are given numerical values, we have a *design of the system*. Whether this design *satisfies all requirements* is another question. We will introduce a number of concepts to investigate such questions in later chapters.

If proper design variables are not selected for a problem, the formulation will be either incorrect or not possible. At the initial stage of problem formulation, all options for specification of design variables should be investigated. Sometimes it may be desirable to designate more design variables than apparent design degrees of freedom. This gives added flexibility to problem formulation. Later, it will be possible to assign a fixed numerical value to any variable and thus eliminate it from the formulation.

At times it is difficult to clearly identify a problem's design variables. In such a case, a complete list of all variables may be prepared. Then, by considering each variable individually, we can decide whether or not to treat it as an optimization variable. If it is a valid design variable, the designer should be able to specify a numerical value for it to select a trial design.

We will use the term "design variables" to indicate all optimization variables for the optimization problem and will represent them in the vector $\mathbf{x}$. To summarize, the following considerations should be given in identifying design variables for a problem:

- Design variables should be independent of each other as far as possible. If they are not, there must be some equality constraints between them (explained later).
- A minimum number of design variables required to properly formulate a design optimization problem must exist.
- As many independent parameters as possible should be designated as design variables at the problem formulation phase. Later on, some of the variables can be assigned fixed values.
- A numerical value should be given to each identified design variable to determine if a trial design of the system is specified.

EXAMPLE 2.3 DESIGN VARIABLES FOR A CANTILEVER BEAM

Only dimensions of the cross-section are identified as design variables for the *cantilever beam design problem* of Example 2.1; all other parameters are specified:

w = width (depth) of the section, mm
t = wall thickness, mm

2.1.4 Step 4: Optimization Criterion

How Do I Know that My Design Is the Best?

There can be many feasible designs for a system, and some are better than others. The question is how we compare designs and designate one as better than another. For this, we must have a criterion that associates a number with each design. Thus, the merit of a given design is specified. The criterion must be a scalar function whose numerical value can be obtained once a design is specified; that is, it must be a *function of the design variable vector* **x**. Such a criterion is usually called an *objective function* for the optimum design problem, and it needs to be *maximized* or *minimized* depending on problem requirements. A criterion that is to be minimized is usually called a *cost function* in engineering literature, which is the term used throughout this text. It is emphasized that a *valid objective function must be influenced directly or indirectly by the variables of the design problem*; otherwise, it is not a meaningful objective function.

The selection of a proper objective function is an important decision in the design process. Some objective functions are cost (to be minimized), profit (to be maximized), weight (to be minimized), energy expenditure (to be minimized), and, for example, ride quality of a vehicle (to be maximized). In many situations an obvious objective function can be identified. For example, we always want to minimize the cost of manufacturing goods or maximize return on investment. In some situations, two or more objective functions may be identified. For example, we may want to minimize the weight of a structure and at the same time minimize the deflection or stress at a certain point. These are called *multiobjective design optimization problems* and are discussed in a later chapter.

For some design problems, it is not obvious what the objective function should be or how it should relate to the design variables. Some insight and experience may be needed to identify a proper objective function. For example, consider the optimization of a passenger car. What are the design variables? What is the objective function, and what is its functional form in terms of the design variables? Although this is a very practical problem, it is quite complex. Usually, such problems are divided into several smaller subproblems and each one is formulated as an optimum design problem. For example, design of a passenger car can be divided into a number of optimization subproblems involving the trunk lid, doors, side panels, roof, seats, suspension system, transmission system, chassis, hood, power plant, bumpers, and so on. Each subproblem is now manageable and can be formulated as an optimum design problem.

EXAMPLE 2.4 OPTIMIZATION CRITERION FOR A CANTILEVER BEAM

For the *design problem in Example 2.1*, the objective is to design a minimum-mass cantilever beam. Since the mass is proportional to the cross-sectional area of the beam, the objective function for the problem is taken as the cross-sectional area:

$$f(w, t) = A = 4t(w - t),\ \text{mm}^2 \tag{k}$$

2.1.5 Step 5: Formulation of Constraints

What Restrictions Do I Have on My Design?

All restrictions placed on the design are collectively called *constraints*. The final step in the formulation process is to identify all constraints and develop expressions for them. Most realistic systems must be designed and fabricated with the given *resources* and must meet *performance requirements*. For example, structural members should not fail under normal operating loads. The vibration frequencies of a structure must be different from the operating frequency of the machine it supports; otherwise, resonance can occur and cause catastrophic failure. Members must fit into the available space.

These constraints, as well as others, must depend on the design variables, since only then do their values change with different trial designs; that is, a meaningful constraint must be a function of at least one design variable. Several concepts and terms related to constraints are explained next.

Linear and Nonlinear Constraints

Many constraint functions have only first-order terms in design variables. These are called *linear constraints*. *Linear programming problems* have only linear constraints and objective functions. More general problems have nonlinear cost and/or constraint functions. These are called *nonlinear programming problems*. Methods to treat both linear and nonlinear constraints and objective functions are presented in this text.

Feasible Design

The design of a system is a set of numerical values assigned to the design variables (i.e., a particular design variable vector **x**). Even if this design is absurd (e.g., negative radius) or inadequate in terms of its function, it can still be called a design. Clearly, some designs are useful and others are not. A design meeting all requirements is called a *feasible design* (*acceptable* or *workable*). An *infeasible design* (*unacceptable*) does not meet one or more of the requirements.

Equality and Inequality Constraints

Design problems may have equality as well as inequality constraints. The problem description should be studied carefully to determine which requirements need to be formulated as equalities and which ones as inequalities. For example, a machine component may be required to move precisely by Δ to perform the desired operation, so we must treat this as an equality constraint. A feasible design must satisfy precisely all equality constraints. Also, most design problems have inequality constraints, sometimes called *unilateral* or *one-sided constraints*. Note that the *feasible region* with respect to an inequality constraint is much larger than that with respect to the same constraint expressed as equality.

To illustrate the difference between equality and inequality constraints, we consider a constraint written in both equality and inequality forms. Figure 2.2(a) shows the equality constraint $x_1 = x_2$. Feasible designs with respect to the constraint must lie on the straight line A−B. However, if the constraint is written as an inequality $x_1 \leq x_2$, the feasible region is much larger, as shown in Figure 2.2(b). Any point on the line A−B or above it gives a feasible design.

Implicit Constraints

Some constraints are quite simple, such as the smallest and largest allowable values for the design variables, whereas more complex ones may be indirectly influenced by the design variables. For example, deflection at a point in a large structure depends on its design. However, it is impossible to express deflection as an explicit function of the design variables except for very simple structures. These are called *implicit constraints*. When there are implicit functions in the problem formulation, it is not possible to formulate the problem functions explicitly in terms of design variables alone. Instead, we must use some intermediate variables in the problem formulation. We will discuss formulations having implicit functions in Chapter 14.

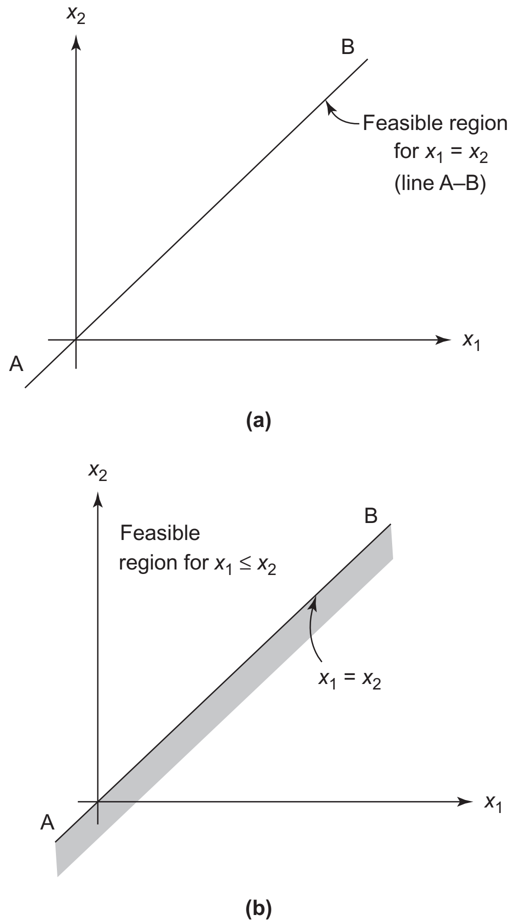

FIGURE 2.2 Shown here is the distinction between equality and inequality constraints: (a) Feasible region for constraint $x_1 = x_2$ (line A−B); (b) feasible region for constraint $x_1 \le x_2$ (line A−B and the region above it).

EXAMPLE 2.5 CONSTRAINTS FOR A CANTILEVER BEAM

Using various expressions given in Eqs. (c) through (j), we formulate the constraints for the *cantilever beam design problem* from Example 2.1 as follows:

Bending stress constraint: $\sigma \le \sigma_a$

$$\frac{PLw}{2I} - \sigma_a \le 0 \tag{1}$$

Shear stress constraint: $\tau \leq \tau_a$

$$\frac{PQ}{2It} - \tau_a \leq 0 \tag{m}$$

Deflection constraint: $q \leq q_a$

$$\frac{PL^3}{3EI} - q_a \leq 0 \tag{n}$$

Width—thickness restriction: $\frac{w}{t} \leq 8$

$$w - 8t \leq 0 \tag{o}$$

Dimension restrictions

$$60 - w \leq 0,\ \text{mm};\quad w - 300 \leq 0,\ \text{mm} \tag{p}$$

$$3 - t \leq 0,\ \text{mm};\quad t - 15 \leq 0,\ \text{mm} \tag{q}$$

Thus the optimization problem is to find w and t to minimize the cost function of Eq. (k) subject to the eight inequality constraints of Eqs. (l) through (q). Note that the constraints of Eqs. (l) through (n) are nonlinear functions and others are linear functions of the design variables. There are eight inequality constraints and no equality constraints for this problem. Substituting various expressions, Eqs. (l) through (n) can be expressed explicitly in terms of the design variables, if desired.

2.2 DESIGN OF A CAN

STEP 1: PROJECT/PROBLEM DESCRIPTION The purpose of this project is to design a can, shown in Figure 2.3, to hold *at least* 400 ml of liquid (1 ml = 1 cm^3), as well as to meet other design requirements. The cans will be produced in the billions, so it is desirable to minimize their manufacturing costs. Since cost can be directly related to the surface area of the sheet metal used, it is reasonable to minimize the amount of sheet metal required. Fabrication, handling, aesthetics, and shipping considerations impose the following restrictions on the size of the can: The diameter should be no more than 8 cm and no less than 3.5 cm, whereas the height should be no more than 18 cm and no less than 8 cm.

STEP 2: DATA AND INFORMATION COLLECTION Data for the problem are given in the project statement.

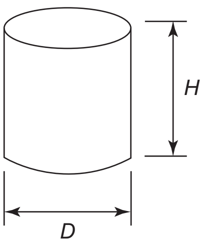

FIGURE 2.3 Can.

STEP 3: DEFINITION OF DESIGN VARIABLES The two design variables are defined as

D = diameter of the can, cm
H = height of the can, cm

STEP 4: OPTIMIZATION CRITERION The design objective is to minimize the total surface area S of the sheet metal for the three parts of the cylindrical can: the surface area of the cylinder (circumference × height) and the surface area of the two ends. Therefore, the optimization criterion, or *cost function* (the total area of sheet metal), is given as

$$S = \pi DH + 2\left(\frac{\pi}{4}D^2\right), \ \text{cm}^2 \tag{a}$$

STEP 5: FORMULATION OF CONSTRAINTS The first constraint is that the can must hold *at least* 400 cm^3 of fluid, which is written as

$$\frac{\pi}{4}D^2H \geq 400, \ \text{cm}^3 \tag{b}$$

If it had been stated that "the can must hold 400 ml of fluid," then the preceding volume constraint would be an equality. The other constraints on the size of the can are

$$\begin{aligned} 3.5 \leq D \leq 8, \ \text{cm} \\ 8 \leq H \leq 18, \ \text{cm} \end{aligned} \tag{c}$$

The explicit constraints on design variables in Eqs. (c) have many different names in the literature, such as *side constraints, technological constraints, simple bounds, sizing constraints*, and *upper and lower limits on the design variables*. Note that for the present problem there are really four constraints in Eqs. (c). Thus, the problem has two design variables and a total of five inequality constraints.

Note also that the cost function and the first constraint are nonlinear in variables; the remaining constraints are linear.

2.3 INSULATED SPHERICAL TANK DESIGN

STEP 1: PROJECT/PROBLEM DESCRIPTION The goal of this project is to choose an insulation thickness t to minimize the life-cycle cooling cost for a spherical tank. The cooling costs include installing and running the refrigeration equipment, and installing the insulation. Assume a 10-year life, a 10 percent annual interest rate, and no salvage value. The tank has already been designed having r (m) as its radius.

STEP 2: DATA AND INFORMATION COLLECTION To formulate this design optimization problem, we need some data and expressions. To calculate the volume of the insulation material, we require the surface area of the spherical tank, which is given as

$$A = 4\pi r^2, \ \text{m}^2 \tag{a}$$

To calculate the capacity of the refrigeration equipment and the cost of its operation, we need to calculate the annual heat gain G, which is given as

$$G = \frac{(365)(24)(\Delta T)A}{c_1 t}, \text{ Watt-hours} \tag{b}$$

where ΔT is the average difference between the internal and external temperatures in Kelvin, c_1 is the thermal resistivity per unit thickness in Kelvin-meter per Watt, and t is the insulation thickness in meters. ΔT can be estimated from the historical data for temperatures in the region in which the tank is to be used. Let c_2 = the insulation cost per cubic meter (\$/m^3), c_3 = the cost of the refrigeration equipment per Watt-hour of capacity (\$/Wh), and c_4 = the annual cost of running the refrigeration equipment per Watt-hour (\$/Wh).

STEP 3: DEFINITION OF DESIGN VARIABLES Only one design variable is identified for this problem:

t = insulation thickness, m

STEP 4: OPTIMIZATION CRITERION The goal is to minimize the life-cycle cooling cost of refrigeration for the spherical tank over 10 years. The life-cycle cost has three components: insulation, refrigeration equipment, and operations for 10 years. Once the annual operations cost has been converted to the present cost, the total cost is given as

$$Cost = c_2 At + c_3 G + c_4 G \tag{c}$$

where *uspwf* (0.1, 10) = 6.14457 is the uniform series present worth factor, calculated using the equation

$$uspwf(i,n) = \frac{1}{i}\left[1 - (1 - i)^{-n}\right] \tag{d}$$

where i is the rate of return per dollar per period and n is the number of periods. Note that to calculate the volume of the insulation as At, it is assumed that the insulation thickness is much smaller than the radius of the spherical tank; that is, $t \ll r$.

STEP 5: FORMULATION OF CONSTRAINTS Although no constraints are indicated in the problem statement, it is important to require that the insulation thickness be non-negative (i.e., $t \geq 0$). Although this may appear obvious, it is important to include the constraint explicitly in the mathematical formulation of the problem. Without its explicit inclusion, the mathematics of optimization may assign negative values to thickness, which is, of course, meaningless. Note also that in reality t cannot be zero because it appears in the denominator of the expression for G. Therefore, the constraint should really be expressed as $t > 0$. However, *strict inequalities* cannot be treated mathematically or numerically in the solution process because they give an open feasible set. We must allow the possibility of satisfying inequalities as equalities; that is, we must allow the possibility that $t = 0$ in the solution process. Therefore, a more realistic constraint is $t \geq t_{min}$, where t_{min} is the smallest insulation thickness available on the market.

EXAMPLE 2.6 FORMULATION OF THE SPHERICAL TANK PROBLEM WITH INTERMEDIATE VARIABLES

A summary of the problem formulation for the design optimization of insulation for a spherical tank with intermediate variables is as follows:

Specified data: r, ΔT, c_1, c_2, c_3, c_4, $t_{\min}$
Design variable: t, m
Intermediate variables:

$$A = 4\pi r^2$$

$$G = \frac{(365)(24)(\Delta T)A}{c_1 t} \tag{e}$$

Cost function: Minimize the life-cycle cooling cost of refrigeration of the spherical tank,

$$Cost = c_2 At + c_3 G + 6.14457 c_4 G \tag{f}$$

Constraint:

$$t \geq t_{\min} \tag{g}$$

Note that A and G may also be treated as design variables in this formulation. In such a case, A must be assigned a fixed numerical value since r has already been determined, and the expression for G must be treated as an equality constraint.

EXAMPLE 2.7 FORMULATION OF THE SPHERICAL TANK PROBLEM WITH THE DESIGN VARIABLE ONLY

Following is a summary of the problem formulation for the design optimization of insulation for a spherical tank in terms of the design variable only:

Specified data: r, ΔT, c_1, c_2, c_3, c_4, $t_{\min}$
Design variable: t, m
Cost function: Minimize the life-cycle cooling cost of refrigeration of the spherical tank,

$$Cost = at + \frac{b}{t}, \quad a = 4c_2\pi r^2, \quad b = \frac{(c_3 + 6.14457c_4)}{c_1}(365)(24)(\Delta T)(4\pi r^2) \tag{h}$$

Constraint:

$$t \geq t_{\min} \tag{i}$$

2.4 SAWMILL OPERATION

STEP 1: PROJECT/PROBLEM DESCRIPTION A company owns two sawmills and two forests. Table 2.1 shows the capacity of each of the mills (logs/day) and the distances

TABLE 2.1 Data for sawmills

Mill	Distance from Mill 1	Distance from Mill 2	Mill capacity per day
A	24.0 km	20.5 km	240 logs
B	17.2 km	18.0 km	300 logs

between the forests and the mills (km). Each forest can yield up to 200 logs/day for the duration of the project, and the cost to transport the logs is estimated at $10/km/log. At least 300 logs are needed daily. The goal is to minimize the total daily cost of transporting the logs.

STEP 2: DATA AND INFORMATION COLLECTION Data are given in Table 2.1 and in the problem statement.

STEP 3: DEFINITION OF DESIGN VARIABLES The design problem is to determine how many logs to ship from Forest i to Mill j, as shown in Figure 2.4. Therefore, the design variables are identified and defined as follows:

x_1 = number of logs shipped from Forest 1 to Mill A
x_2 = number of logs shipped from Forest 2 to Mill A
x_3 = number of logs shipped from Forest 1 to Mill B
x_4 = number of logs shipped from Forest 2 to Mill B

Note that if we assign numerical values to these variables, an operational plan for the project is specified and the cost of daily log transportation can be calculated. The selected design may or may not satisfy all constraints.

STEP 4: OPTIMIZATION CRITERION The design objective is to minimize the daily cost of transporting the logs to the mills. The cost of transportation, which depends on the distance between the forests and the mills, is

$$\begin{aligned} Cost &= 24(10)x_1 + 20.5(10)x_2 + 17.2(10)x_3 + 18(10)x_4 \\ &= 240.0x_1 + 205.0x_2 + 172.0x_3 + 180.0x_4 \end{aligned} \tag{a}$$

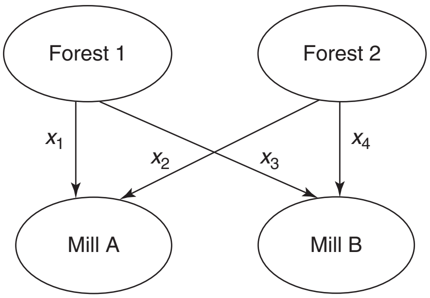

FIGURE 2.4 Sawmill operation.

STEP 5: FORMULATION OF CONSTRAINTS The constraints for the problem are based on mill capacity and forest yield:

$$\begin{aligned} x_1 + x_2 &\le 240 \quad \text{(Mill A capacity)} \\ x_3 + x_4 &\le 300 \quad \text{(Mill B capacity)} \\ x_1 + x_3 &\le 200 \quad \text{(Forest 1 yield)} \\ x_2 + x_4 &\le 200 \quad \text{(Forest 2 yield)} \end{aligned} \tag{b}$$

The constraint on the number of logs needed for each day is expressed as

$$x_1 + x_2 + x_3 + x_4 \ge 300 \quad \text{(demand for logs)} \tag{c}$$

For a realistic problem formulation, all design variables must be non-negative; that is,

$$x_i \ge 0; \quad i = 1 \text{ to } 4 \tag{d}$$

The problem has four design variables, five inequality constraints, and four non-negativity constraints on the design variables. Note that all problem functions are linear in design variables, so this is a *linear programming problem*. Note also that for a meaningful solution, all design variables must have *integer* values. Such problems are called *integer programming problems* and require special solution methods. Some such methods are discussed in Chapter 15.

It is also noted that the problem of sawmill operation falls into a class known as *transportation problems*. For such problems, we would like to ship items from several distribution centers to several retail stores to meet their demand at a minimum cost of transportation.

2.5 DESIGN OF A TWO-BAR BRACKET

STEP 1: PROJECT/PROBLEM DESCRIPTION The objective of this project is to design a two-bar bracket (shown in Figure 2.5) to support a load W without structural failure. The load is applied at an angle θ, which is between 0 and 90°, h is the height, and s is the bracket's base width. The bracket will be produced in large quantities. It has also been determined that its total cost (material, fabrication, maintenance, and so on) is directly related to the size of the two bars. Thus, the design objective is to minimize the total mass of the bracket while satisfying performance, fabrication, and space limitations.

STEP 2: DATA AND INFORMATION COLLECTION First, the load W and its angle of application θ need to be specified. Since the bracket may be used in several applications, it may not be possible to specify just one angle for W. It is possible to formulate the design optimization problem such that a range is specified for angle θ (i.e., load W may be applied at any angle within that specified range). In this case, the formulation will be slightly more complex because performance requirements will need to be satisfied for each angle of application. In the present formulation, it is assumed that angle θ is specified.

Second, the material to be used for the bars must be specified because the material properties are needed to formulate the optimization criterion and performance

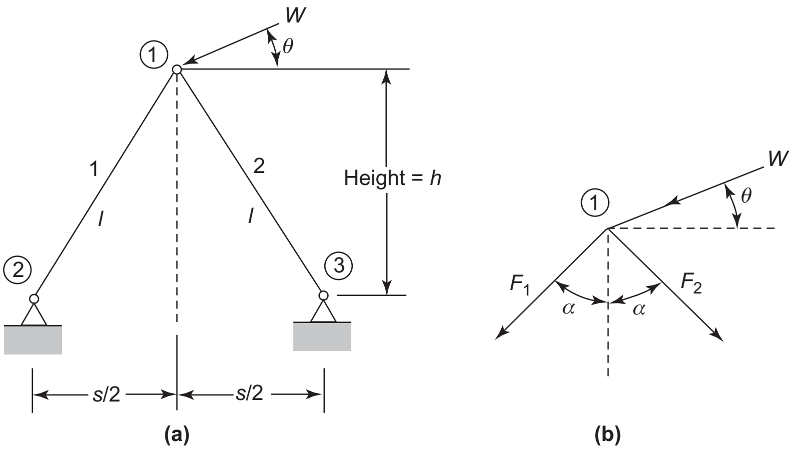

FIGURE 2.5 Two-bar bracket: (a) structure and (b) free-body diagram for node 1.

requirements. Whether the two bars are to be fabricated using the same material also needs to be determined. In the present formulation, it is assumed that they are, although it may be prudent to assume otherwise for some advanced applications. In addition, we need to determine the fabrication and space limitations for the bracket (e.g., on the size of the bars, height, and base width).

In formulating the design problem, we also need to define *structural performance* more precisely. Forces F_1 and F_2 carried by bars 1 and 2, respectively, can be used to define failure conditions for the bars. To compute these forces, we use the principle of *static equilibrium*. Using the *free-body diagram* for node 1 (shown in Figure 2.5(b)), equilibrium of forces in the horizontal and vertical directions gives

$$\begin{aligned} -F_1\sin\alpha + F_2\sin\alpha &= W\ \cos\theta \\ -F_1\cos\alpha - F_2\cos\alpha &= W\ \sin\theta \end{aligned} \tag{a}$$

From the geometry of Figure 2.5, $\sin\alpha = 0.5\ s/l$ and $\cos\alpha = h/l$, where l is the length of members given as $l = \sqrt{h^2 + (0.5s)^2}$. Note that F_1 and F_2 are shown as tensile forces in the free-body diagram. The solution to Eqs. (a) will determine the magnitude and direction of the forces. In addition, the *tensile force will be taken as positive*. Thus, the bar will be in compression if the force carried by it has negative value. By solving the two equations simultaneously for the unknowns F_1 and F_2, we obtain

$$\begin{aligned} F_1 &= -0.5Wl\left[\frac{\sin\theta}{h} + \frac{2\cos\theta}{s}\right] \\ F_2 &= -0.5Wl\left[\frac{\sin\theta}{h} - \frac{2\cos\theta}{s}\right] \end{aligned} \tag{b}$$

To avoid bar failure due to overstressing, we need to calculate bar stress. If we know the force carried by a bar, then the stress σ can be calculated as the force divided by the bar's cross-sectional area (stress = force/area). The SI unit for stress is Newton/meter^2 (N/m^2), also called Pascal (Pa), whereas the U.S.–British unit is pound/in^2 (written as psi). The expression for the cross-sectional area depends on the cross-sectional shape used for the bars and selected design variables. Therefore, a structural shape for the bars and associated design variables must be selected. This is illustrated later in the formulation process.

In addition to analysis equations, we need to define the properties of the selected material. Several formulations for optimum design of the bracket are possible depending on the application's requirements. To illustrate, a material with known properties is assumed for the bracket. However, the structure can be optimized using other materials along with their associated fabrication costs. Solutions can then be compared to select the best possible one for the structure.

For the selected material, let ρ be the mass density and $\sigma_a > 0$ be the allowable design stress. As a performance requirement, it is assumed that if the stress exceeds this allowable value, the bar is considered to have failed. The *allowable stress* is defined as the material failure stress (a property of the material) divided by a factor of safety greater than one. In addition, it is assumed that the allowable stress is calculated in such a way that the buckling failure of a bar in compression is avoided.

STEP 3: DEFINITION OF DESIGN VARIABLES Several sets of design variables may be identified for the two-bar structure. The height h and span s can be treated as design variables in the initial formulation. Later, they may be assigned numerical values, if desired, to eliminate them from the formulation. Other design variables will depend on the cross-sectional shape of bars 1 and 2. Several cross-sectional shapes are possible, as shown in Figure 2.6, where design variables for each shape are also identified.

Note that for many cross-sectional shapes, different design variables can be selected. For example, in the case of the circular tube in Figure 2.6(a), the outer diameter d_o and the ratio between the inner and outer diameters $r = d_i/d_o$ may be selected as the design variables. Or d_o and d_i may be selected. However, it is not desirable to designate d_o, d_i, and r as the design variables because they are not independent of each other. If they are selected, then a relationship between them must be specified as an equality constraint. Similar remarks can be made for the design variables associated with other cross-sections, also shown in Figure 2.6.

As an example of problem formulation, consider the design of a bracket with hollow circular tubes, as shown in Figure 2.6(a). The inner and outer diameters d_i and d_o and wall thickness t may be identified as the design variables, although they are not all independent of each other. For example, we cannot specify $d_i = 10$, $d_o = 12$, and $t = 2$ because it violates the physical condition $t = 0.5(d_o - d_i)$. Therefore, if we formulate the problem with d_i, d_o, and t as design variables, we must also impose the constraint $t = 0.5(d_o - d_i)$. To illustrate a formulation of the problem, let the design variables be defined as

x_1 = height h of the bracket
x_2 = span s of the bracket
x_3 = outer diameter of bar 1
x_4 = inner diameter of bar 1
x_5 = outer diameter of bar 2
x_6 = inner diameter of bar 2

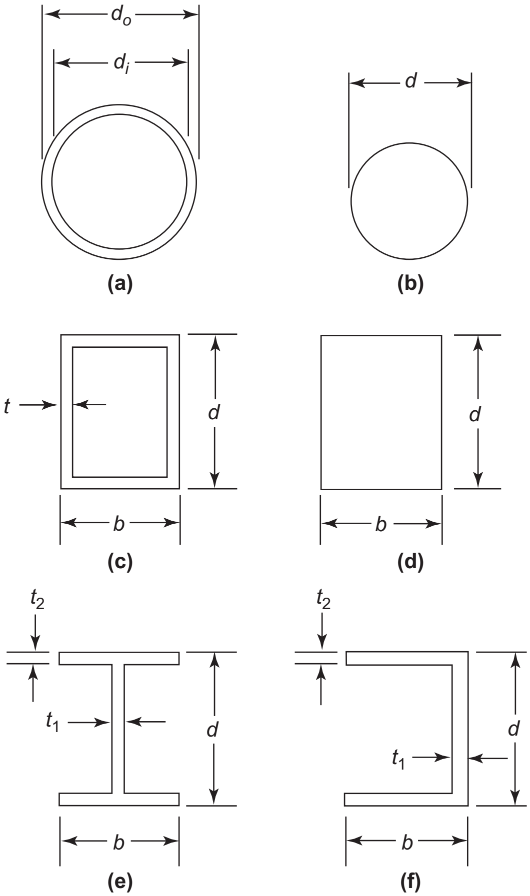

FIGURE 2.6 Bar cross-sectional shapes: (a) circular tube; (b) solid circular; (c) rectangular tube; (d) solid rectangular; (e) I-section; (f) channel section.

In terms of these variables, the cross-sectional areas A_1 and A_2 of bars 1 and 2 are given as

$$A_1 = \frac{\pi}{4}(x_3^2 - x_4^2); \quad A_2 = \frac{\pi}{4}(x_5^2 - x_6^2) \tag{c}$$

Once the problem is formulated in terms of the six selected design variables, it is always possible to modify it to meet more specialized needs. For example, the height x_1 may be assigned a fixed numerical value, thus eliminating it from the problem formulation. In addition, complete symmetry of the structure may be required to make its fabrication easier; that is, it may be necessary for the two bars to have the same cross-section, size, and material. In such a case, we set $x_3 = x_5$ and $x_4 = x_6$ in all expressions of the problem formulation. Such modifications are left as exercises.

STEP 4: OPTIMIZATION CRITERION The structure's mass is identified as the objective function in the problem statement. Since it is to be minimized, it is called the *cost function* for the problem. An expression for the mass is determined by the cross-sectional shape of the bars and associated design variables. For the hollow circular tubes and selected design variables, the total mass of the structure is calculated as (density × material volume):

$$Mass = \rho[l(A_1 + A_2)] = \left[\rho\sqrt{x_1^2 + (0.5x_2)^2}\right]\frac{\pi}{4}(x_3^2 - x_4^2 + x_5^2 - x_6^2) \tag{d}$$

Note that if the outer diameter and the ratio between the inner and outer diameters are selected as design variables, the form of the mass function changes. Thus, the *final form* depends on the design variables selected for the problem.

STEP 5: FORMULATION OF CONSTRAINTS It is important to include all constraints in the problem formulation because the final solution depends on them. For the two-bar structure, the constraints are on the stress in the bars and on the design variables themselves. These constraints will be formulated for hollow circular tubes using the previously defined design variables. They can be similarly formulated for other sets of design variables and cross-sectional shapes.

To avoid overstressing a bar, the calculated stress σ (tensile or compressive) must not exceed the material allowable stress $\sigma_a > 0$. The stresses σ_1 and σ_2 in the two bars are calculated as force/area:

$$\begin{aligned} \sigma_1 &= \frac{F_1}{A_1} \text{ (stress in bar 1)} \\ \sigma_2 &= \frac{F_2}{A_2} \text{ (stress in bar 2)} \end{aligned} \tag{e}$$

Note that to treat positive and negative stresses (tension and compression), we must use the absolute value of the calculated stress in writing the constraints (e.g., $|\sigma| \le \sigma_a$). The absolute-value constraints can be treated by different approaches in optimization methods. Here we split each absolute-value constraint into two constraints. For example, the stress constraint for bar 1 is written as the following two constraints:

$$\begin{aligned} \sigma_1 &\le \sigma_a \text{ (tensile stress in bar 1)} \\ -\sigma_1 &\le \sigma_a \text{ (compressive stress in bar 1)} \end{aligned} \tag{f}$$

With this approach, the second constraint is satisfied automatically if bar 1 is in tension, and the first constraint is automatically satisfied if bar 1 is in compression. Similarly, the stress constraint for bar 2 is written as

$$\begin{aligned} \sigma_2 &\le \sigma_a \text{ (tensile stress in bar 2)} \\ -\sigma_2 &\le \sigma_a \text{ (compressive stress in bar 2)} \end{aligned} \tag{g}$$

Finally, to impose fabrication and space limitations, constraints on the design variables are imposed as

$$x_{iL} \le x_i \le x_{iU}; \quad i = 1 \text{ to } 6 \tag{h}$$

where x_{iL} and x_{iU} are the minimum and maximum allowed values for the ith design variable. Their numerical values must be specified before the problem can be solved.

Note that the expression for bar stress changes if different design variables are chosen for circular tubes, or if a different cross-sectional shape is chosen for the bars. For example, inner and outer radii, mean radius and wall thickness, or outside diameter and the ratio of inside to outside diameter as design variables will all produce different expressions for the cross-sectional areas and stresses. *These results show that the choice of design variables greatly influences the problem formulation.*

Note also that we had to first *analyze* the structure (calculate its response to given inputs) to write the constraints properly. It was only after we had calculated the forces in the bars that we were able to write the constraints. This is an important step in any engineering design problem formulation: *We must be able to analyze the system before we can formulate the design optimization problem.*

In the following examples, we summarize two formulations of the problem. The first uses several intermediate variables, which is useful when the problem is transcribed into a computer program. Because this formulation involves simpler expressions of various quantities, it is easier to write and debug a computer program. In the second formulation, all intermediate variables are eliminated to obtain the formulation exclusively in terms of design variables. This formulation has slightly more complex expressions. It is important to note that the second formulation may not be possible for all applications because some problem functions may only be implicit functions of the design variables. One such formulation is presented in Chapter 14.

EXAMPLE 2.8 FORMULATION OF THE TWO-BAR BRACKET PROBLEM WITH INTERMEDIATE VARIABLES

A summary of the problem formulation for optimum design of the two-bar bracket using intermediate variables is as follows:

Specified data: W, θ, $\sigma_a > 0$, x_{iL} and x_{iU}, $i = 1$ to 6

Design variables: x_1, x_2, x_3, x_4, x_5, x_6

Intermediate variables:

Bar cross-sectional areas:

$$A_1 = \frac{\pi}{4}(x_3^2 - x_4^2); \quad A_2 = \frac{\pi}{4}(x_5^2 - x_6^2) \tag{a}$$

Length of bars:

$$l = \sqrt{x_1^2 + (0.5x_2)^2} \tag{b}$$

Forces in bars:

$$F_1 = -0.5Wl\left[\frac{\sin\theta}{x_1} + \frac{2\cos\theta}{x_2}\right]$$

$$F_2 = -0.5Wl\left[\frac{\sin\theta}{x_1} - \frac{2\cos\theta}{x_2}\right] \tag{c}$$

Bar stresses:

$$\sigma_1 = \frac{F_1}{A_1}; \quad \sigma_2 = \frac{F_2}{A_2} \tag{d}$$

Cost function: Minimize the total mass of the bars,

$$Mass = \rho l(A_1 + A_2) \tag{e}$$

Constraints:

Bar stress:

$$-\sigma_1 \le \sigma_a; \quad \sigma_1 \le \sigma_a; \quad -\sigma_2 \le \sigma_a; \quad \sigma_2 \le \sigma_a \tag{f}$$

Design variable limits:

$$x_{iL} \le x_i \le x_{iU}; \quad i = 1 \text{ to } 6 \tag{g}$$

Note that the intermediate variables, such as A_1, A_2, F_1, F_2, σ_1, and σ_2, may also be treated as optimization variables. However, in that case, we have six equality constraints between the variables, in addition to the other constraints.

EXAMPLE 2.9 FORMULATION OF THE TWO-BAR BRACKET WITH DESIGN VARIABLES ONLY

A summary of the problem formulation for optimum design of the two-bar bracket in terms of design variables only is obtained by eliminating the intermediate variables from all the expressions as follows:

Specified data: W, θ, $\sigma_a > 0$, x_{iL} and x_{iU}, $i = 1$ to 6
Design variables: x_1, x_2, x_3, x_4, x_5, x_6
Cost function: Minimize total mass of the bars,

$$Mass = \frac{\pi\rho}{4}\sqrt{x_1^2 + (0.5x_2)^2}\,(x_3^2 - x_4^2 + x_5^2 - x_6^2) \tag{a}$$

Constraints:

Bar stress:

$$\frac{2W\sqrt{x_1^2 + (0.5x_2)^2}}{\pi(x_3^2 - x_4^2)}\left[\frac{\sin\theta}{x_1} + \frac{2\cos\theta}{x_2}\right] \le \sigma_a \tag{b}$$

$$\frac{-2W\sqrt{x_1^2 + (0.5x_2)^2}}{\pi(x_3^2 - x_4^2)}\left[\frac{\sin\theta}{x_1} + \frac{2\cos\theta}{x_2}\right] \le \sigma_a \tag{c}$$

$$\frac{2W\sqrt{x_1^2 + (0.5x_2)^2}}{\pi(x_5^2 - x_6^2)}\left[\frac{\sin\theta}{x_1} - \frac{2\cos\theta}{x_2}\right] \le \sigma_a \tag{d}$$

$$\frac{-2W\sqrt{x_1^2 + (0.5x_2)^2}}{\pi(x_5^2 - x_6^2)}\left[\frac{\sin\theta}{x_1} - \frac{2\cos\theta}{x_2}\right] \le \sigma_a \tag{e}$$

Design variable limits:

$$x_{iL} \le x_i \le x_{iU}; \quad i = 1 \text{ to } 6 \tag{f}$$

2.6 DESIGN OF A CABINET

STEP 1: PROJECT/PROBLEM DESCRIPTION A cabinet is assembled from components C_1, C_2, and C_3. Each cabinet requires 8 C_1, 5 C_2, and 15 C_3 components. The assembly of C_1 requires either 5 bolts or 5 rivets, whereas C_2 requires 6 bolts or 6 rivets, and C_3 requires 3 bolts or 3 rivets. The cost of installing a bolt, including the cost of the bolt itself, is $0.70 for C_1, $1.00 for C_2, and $0.60 for C_3. Similarly, riveting costs are $0.60 for C_1, $0.80 for C_2, and $1.00 for C_3. Bolting and riveting capacities per day are 6000 and 8000, respectively. To minimize the cost for the 100 cabinets that must be assembled each day, we wish to determine the number of components to be bolted and riveted (after Siddall, 1972).

STEP 2: DATA AND INFORMATION COLLECTION All data for the problem are given in the project statement. This problem can be formulated in several different ways depending on the assumptions made and the definition of the design variables. Three formulations are presented, and for each one, the design variables are identified and expressions for the cost and constraint functions are derived; that is, steps 3 through 5 are presented.

2.6.1 Formulation 1 for Cabinet Design

STEP 3: DEFINITION OF DESIGN VARIABLES In the first formulation, the following design variables are identified for 100 cabinets:

x_1 = number of C_1 to be bolted for 100 cabinets
x_2 = number of C_1 to be riveted for 100 cabinets
x_3 = number of C_2 to be bolted for 100 cabinets
x_4 = number of C_2 to be riveted for 100 cabinets
x_5 = number of C_3 to be bolted for 100 cabinets
x_6 = number of C_3 to be riveted for 100 cabinets

STEP 4: OPTIMIZATION CRITERION The design objective is to minimize the total cost of cabinet fabrication, which is obtained from the specified costs for bolting and riveting each component:

$$\begin{aligned} Cost &= 0.70(5)x_1 + 0.60(5)x_2 + 1.00(6)x_3 + 0.80(6)x_4 + 0.60(3)x_5 + 1.00(3)x_6 \\ &= 3.5x_1 + 3.0x_2 + 6.0x_3 + 4.8x_4 + 1.8x_5 + 3.0x_6 \end{aligned} \tag{a}$$

STEP 5: FORMULATION OF CONSTRAINTS The constraints for the problem consist of riveting and bolting capacities and the number of cabinets fabricated each day. Since 100 cabinets must be fabricated, the required numbers of C_1, C_2, and C_3 are given in the following constraints:

$$\begin{aligned} x_1 + x_2 &= 8 \times 100 \quad \text{(number of } C_1 \text{ needed)} \\ x_3 + x_4 &= 5 \times 100 \quad \text{(number of } C_2 \text{ needed)} \\ x_5 + x_6 &= 15 \times 100 \quad \text{(number of } C_3 \text{ needed)} \end{aligned} \tag{b}$$

Bolting and riveting capacities must not be exceeded. Thus,

$$\begin{aligned} 5x_1 + 6x_3 + 3x_5 &\le 6000 \quad \text{(bolting capacity)} \\ 5x_2 + 6x_4 + 3x_6 &\le 8000 \quad \text{(riveting capacity)} \end{aligned} \tag{c}$$

Finally, all design variables must be non-negative to find a meaningful solution:

$$x_i \ge 0; \quad i = 1 \text{ to } 6 \tag{d}$$

2.6.2 Formulation 2 for Cabinet Design

STEP 3: DEFINITION OF DESIGN VARIABLES If we relax the constraint that each component must be bolted or riveted, then the following design variables can be defined:

x_1 = total number of bolts required for all C_1
x_2 = total number of bolts required for all C_2
x_3 = total number of bolts required for all C_3
x_4 = total number of rivets required for all C_1
x_5 = total number of rivets required for all C_2
x_6 = total number of rivets required for all C_3

STEP 4: OPTIMIZATION CRITERION The objective is still to minimize the total cost of fabricating 100 cabinets, given as

$$Cost = 0.70x_1 + 1.00x_2 + 0.60x_3 + 0.60x_4 + 0.80x_5 + 1.00x_6 \tag{e}$$

STEP 5: FORMULATION OF CONSTRAINTS Since 100 cabinets must be built every day, it will be necessary to have 800 C_1, 500 C_2, and 1500 C_3 components. The total number of bolts and rivets needed for all C_1, C_2, and C_3 components is indicated by the following equality constraints:

$$\begin{aligned} x_1 + x_4 &= 5 \times 800 \quad \text{(bolts and rivets needed for } C_1) \\ x_2 + x_5 &= 6 \times 500 \quad \text{(bolts and rivets needed for } C_2) \\ x_3 + x_6 &= 3 \times 1500 \quad \text{(bolts and rivets needed for } C_3) \end{aligned} \tag{f}$$

Constraints on capacity for bolting and riveting are

$$\begin{aligned} x_1 + x_2 + x_3 &\le 6000 \quad \text{(bolting capacity)} \\ x_4 + x_5 + x_6 &\le 8000 \quad \text{(riveting capacity)} \end{aligned} \tag{g}$$

Finally, all design variables must be non-negative:

$$x_i \ge 0; \quad i = 1 \text{ to } 6 \tag{h}$$

Thus, this formulation also has six design variables, three equality constraints, and two inequality constraints. After an optimum solution has been obtained, we can decide how many components to bolt and how many to rivet.

2.6.3 Formulation 3 for Cabinet Design

STEP 3: DEFINITION OF DESIGN VARIABLES Another formulation of the problem is possible if we require that all cabinets be identical. The following design variables can be identified:

x_1 = number of C_1 to be bolted on one cabinet
x_2 = number of C_1 to be riveted on one cabinet
x_3 = number of C_2 to be bolted on one cabinet
x_4 = number of C_2 to be riveted on one cabinet
x_5 = number of C_3 to be bolted on one cabinet
x_6 = number of C_3 to be riveted on one cabinet

STEP 4: OPTIMIZATION CRITERION With these design variables, the cost of fabricating 100 cabinets each day is given as

$$\begin{aligned} Cost &= 100[0.70(5)x_1 + 0.60(5)x_2 + 1.00(6)x_3 + 0.80(6)x_4 + 0.60(3)x_5 + 1.00(3)x_6] \\ &= 350x_1 + 300x_2 + 600x_3 + 480x_4 + 180x_5 + 300x_6 \end{aligned} \tag{i}$$

STEP 5: FORMULATION OF CONSTRAINTS Since each cabinet needs 8 C_1, 5 C_2, and 15 C_3 components, the following equality constraints can be identified:

$$\begin{aligned} x_1 + x_2 &= 8 \quad \text{(number of } C_1 \text{ needed)} \\ x_3 + x_4 &= 5 \quad \text{(number of } C_2 \text{ needed)} \\ x_5 + x_6 &= 15 \quad \text{(number of } C_3 \text{ needed)} \end{aligned} \tag{j}$$

Constraints on the capacity to rivet and bolt are expressed as the following inequalities:

$$\begin{aligned} (5x_1 + 6x_3 + 3x_5)100 &\le 6000 \quad \text{(bolting capacity)} \\ (5x_2 + 6x_4 + 3x_6)100 &\le 8000 \quad \text{(riveting capacity)} \end{aligned} \tag{k}$$

Finally, all design variables must be non-negative:

$$x_i \ge 0; \quad i = 1 \text{ to } 6 \tag{l}$$

The following points are noted for the three formulations:

1. Because cost and constraint functions are *linear* in all three formulations, they are linear programming problems. It is conceivable that each formulation will yield a different optimum solution. After solving the problems, the designer can select the best strategy for fabricating cabinets.
2. All formulations have *three equality constraints*, each involving two design variables. Using these constraints, we can eliminate three variables from the problem and thus reduce its dimension. This is desirable from a computational standpoint because the number of variables and constraints is reduced. However, because the elimination of variables is not possible for many complex problems, we must develop and use methods to treat both equality and inequality constraints.

3. For a meaningful solution for these formulations, all design variables must have integer values. These are called *integer programming problems*. Some numerical methods to treat this class of problem are discussed in Chapter 15.

2.7 MINIMUM-WEIGHT TUBULAR COLUMN DESIGN

STEP 1: PROJECT/PROBLEM DESCRIPTION Straight columns are used as structural elements in civil, mechanical, aerospace, agricultural, and automotive structures. Many such applications can be observed in daily life—for example, a street light pole, a traffic light post, a flag pole, a water tower support, a highway sign post, a power transmission pole. It is important to optimize the design of a straight column since it may be mass-produced. The objective of this project is to design a minimum-mass *tubular* column of length l supporting a load P without buckling or overstressing. The column is fixed at the base and free at the top, as shown in Figure 2.7. This type of structure is called a cantilever column.

STEP 2: DATA AND INFORMATION COLLECTION The *buckling load* (also called the *critical load*) for a cantilever column is given as

$$P_{\text{cr}} = \frac{\pi^2 EI}{4l^2} \tag{a}$$

The buckling load formula for a column with other support conditions is different from this formula (Crandall, Dahl, and Lardner, 1999). Here, I is the moment of inertia for the cross-section of the column and E is the material property, called the modulus of elasticity (Young's modulus). Note that the buckling load depends on the design of the column (i.e., the moment of inertia I). It imposes a limit on the applied load; that is, the column fails if the applied load exceeds the buckling load. The material stress σ for the column is defined as P/A, where A is the cross-sectional area of the column. The material allowable stress under the axial load is σ_a, and the material mass density is ρ (mass per unit volume).

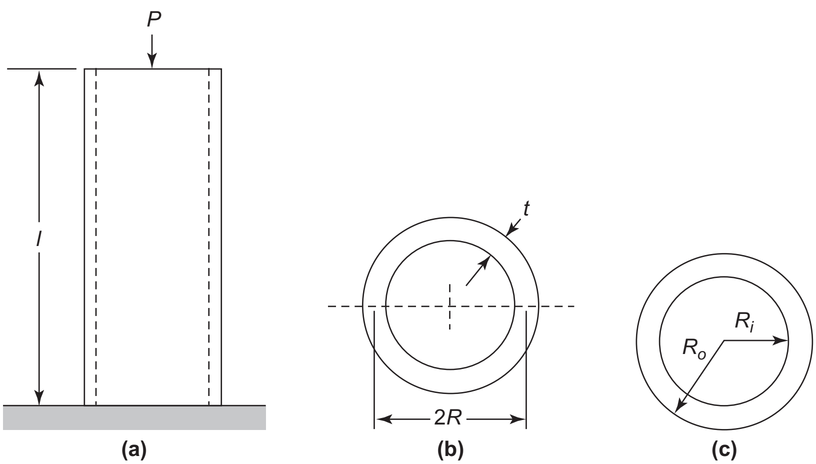

FIGURE 2.7 (a) Tubular column; (b) formulation 1; (c) formulation 2.

A cross-section of the tubular column is shown in Figure 2.7. Many formulations for the design problem are possible depending on how the design variables are defined. Two such formulations are described here.

2.7.1 Formulation 1 for Column Design

STEP 3: DEFINITION OF DESIGN VARIABLES For the first formulation, the following design variables are defined:

R = mean radius of the column
t = wall thickness

Assuming that the column wall is thin ($R \gg t$), the material cross-sectional area and moment of inertia are

$$A = 2\pi Rt; \quad I = \pi R^3 t \tag{b}$$

STEP 4: OPTIMIZATION CRITERION The total mass of the column to be minimized is given as

$$Mass = \rho(lA) = 2\rho l\pi Rt \tag{c}$$

STEP 5: FORMULATION OF CONSTRAINTS The first constraint is that the stress (P/A) should not exceed the material allowable stress σ_a to avoid material failure. This is expressed as the inequality $\sigma \le \sigma_a$. Replacing σ with P/A and then substituting for A, we obtain

$$\frac{P}{2\pi Rt} \le \sigma_a \tag{d}$$

The column should not buckle under the applied load P, which implies that the applied load should not exceed the buckling load (i.e., $P \le P_{cr}$). Using the given expression for the buckling load in Eq. (a) and substituting for I, we obtain

$$P \le \frac{\pi^3 ER^3 t}{4l^2} \tag{e}$$

Finally, the design variables R and t must be within the specified minimum and maximum values:

$$R_{min} \le R \le R_{max}; \quad t_{min} \le t \le t_{max} \tag{f}$$

2.7.2 Formulation 2 for Column Design

STEP 3: DEFINITION OF DESIGN VARIABLES Another formulation of the design problem is possible if the following design variables are defined:

R_o = outer radius of the column
R_i = inner radius of the column

In terms of these design variables, the cross-sectional area A and the moment of inertia I are

$$A = \pi(R_o^2 - R_i^2); \quad I = \frac{\pi}{4}(R_o^4 - R_i^4). \tag{g}$$

STEP 4: OPTIMIZATION CRITERION Minimize the total mass of the column:

$$Mass = \rho(lA) = \pi\rho l(R_o^2 - R_i^2) \tag{h}$$

STEP 5: FORMULATION OF THE CONSTRAINTS The material crushing constraint is ($P/A \leq \sigma_a$):

$$\frac{P}{\pi(R_o^2 - R_i^2)} \leq \sigma_a \tag{i}$$

Using the foregoing expression for I, the buckling load constraint is ($P \leq P_{\mathrm{cr}}$):

$$P \leq \frac{\pi^3 E}{16 l^3}(R_o^4 - R_i^4) \tag{j}$$

Finally, the design variables R_o and R_i must be within specified limits:

$$R_{o\ min} \leq R_o \leq R_{o\ max}; \quad R_{i\ min} \leq R_i \leq R_{o\ max} \tag{k}$$

When this problem is solved using a numerical method, a constraint $R_o > R_i$ must also be imposed. Otherwise, some methods may take the design to the point where $R_o < R_i$. This situation is not physically possible and must be explicitly excluded to numerically solve the design problem.

In addition to the foregoing constraints, local buckling of the column wall needs to be considered for both formulations. Local buckling can occur if the wall thickness becomes too small. This can be avoided if the ratio of mean radius to wall thickness is required to be smaller than a limiting value, that is,

$$\frac{(R_o + R_i)}{2(R_o - R_i)} \leq k \quad \text{or} \quad \frac{R}{t} \leq k \tag{l}$$

where k is a specified value that depends on Young's modulus and the yield stress of the material. For steel with E = 29,000 ksi and a yield stress of 50 ksi, k is given as 32 (AISC, 2005).

2.8 MINIMUM-COST CYLINDRICAL TANK DESIGN

STEP 1: PROJECT/PROBLEM DESCRIPTION Design a minimum-cost cylindrical tank closed at both ends to contain a fixed volume of fluid V. The cost is found to depend directly on the area of sheet metal used.

STEP 2: DATA AND INFORMATION COLLECTION Let c be the dollar cost per unit area of the sheet metal. Other data are given in the project statement.

STEP 3: DEFINITION OF DESIGN VARIABLES The design variables for the problem are identified as

R = radius of the tank
H = height of the tank

STEP 4: OPTIMIZATION CRITERION The cost function for the problem is the dollar cost of the sheet metal for the tank. Total surface area of the sheet metal consisting of the end plates and cylinder is given as

$$A = 2\pi R^2 + 2\pi RH \tag{a}$$

Therefore, the cost function for the problem is given as

$$f = c(2\pi R^2 + 2\pi RH) \tag{b}$$

STEP 5: FORMULATION OF CONSTRAINTS The volume of the tank ($\pi R^2 H$) is required to be V. Therefore,

$$\pi R^2 H = V \tag{c}$$

Also, both of the design variables R and H must be within some minimum and maximum values:

$$R_{min} \leq R \leq R_{max}; \quad H_{min} \leq H \leq H_{max} \tag{d}$$

This problem is quite similar to the can problem discussed in Section 2.2. The only difference is in the volume constraint. There the constraint is an inequality and here it is an equality.

2.9 DESIGN OF COIL SPRINGS

STEP 1: PROJECT/PROBLEM DESCRIPTION Coil springs are used in numerous practical applications. Detailed methods for analyzing and designing such mechanical components have been developed over the years (e.g., Spotts, 1953; Wahl, 1963; Shigley, Mischke, and Budynas, 2004; Haug and Arora, 1979). The purpose of this project is to design a minimum-mass spring (shown in Figure 2.8) to carry a given axial load (called a tension-compression spring) without material failure and while satisfying two performance requirements: The spring must deflect by at least Δ (in) and the frequency of surge waves must not be less than ω_0 (Hertz, Hz).

STEP 2: DATA AND INFORMATION COLLECTION To formulate the problem of designing a coil spring, see the notation and data defined in Table 2.2.

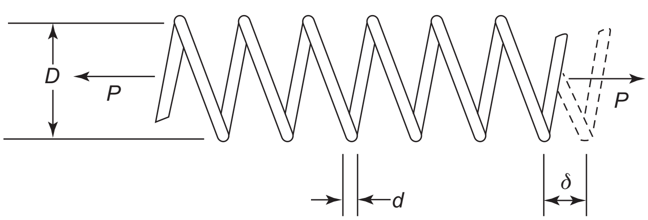

FIGURE 2.8 Coil spring.

TABLE 2.2 Information to design a coil spring

Notation	Data
Deflection along the axis of spring	δ, in
Mean coil diameter	D, in
Wire diameter	d, in
Number of active coils	N
Gravitational constant	$g = 386$ in/s^2
Frequency of surge waves	ω, Hz
Weight density of spring material	$\gamma = 0.285$ lb/in^3
Shear modulus	$G = (1.15 \times 10^7)$ lb/in^2
Mass density of material ($\rho = \gamma/g$)	$\rho = (7.38342 \times 10^{-4})$ lb-s^2/in^4
Allowable shear stress	$\tau_a = 80{,}000$ lb/in^2
Number of inactive coils	$Q = 2$
Applied load	$P = 10$ lb
Minimum spring deflection	$\Delta = 0.5$ in
Lower limit on surge wave frequency	$\omega_0 = 100$ Hz
Limit on outer diameter of coil	$D_o = 1.5$ in

The wire twists when the spring is subjected to a tensile or a compressive load. Therefore, shear stress needs to be calculated so that a constraint on it can be included in the formulation. In addition, surge wave frequency needs to be calculated. These and other design equations for the spring are given as

Load deflection equation:
$$P = K\delta \tag{a}$$

Spring constant:
$$K = \frac{d^4 G}{8D^3 N} \tag{b}$$

Shear stress:
$$\tau = \frac{8kPD}{\pi d^3} \tag{c}$$

Wahl stress concentration factor:
$$k = \frac{(4D - d)}{4(D - d)} + \frac{0.615d}{D} \tag{d}$$

Frequency of surge waves:
$$\omega = \frac{d}{2\pi N D^2}\sqrt{\frac{G}{2\rho}} \tag{e}$$

The expression for the Wahl stress concentration factor k in Eq. (d) has been determined experimentally to account for unusually high stresses at certain points on the spring. These analysis equations are used to define the constraints.

STEP 3: DEFINITION OF DESIGN VARIABLES The three design variables for the problem are defined as

d = wire diameter, in
D = mean coil diameter, in
N = number of active coils, integer

STEP 4: OPTIMIZATION CRITERION The problem is to *minimize the mass* of the spring, given as volume × mass density:

$$Mass = \left(\frac{\pi}{4}d^2\right)[(N+Q)\pi D]\rho = \frac{1}{4}(N+Q)\pi^2 D d^2 \rho \tag{f}$$

STEP 5: FORMULATION OF CONSTRAINTS

Deflection constraint. It is often a requirement that *deflection* under a load P be at least Δ. Therefore, the constraint is that the calculated deflection δ must be greater than or equal to Δ. Such a constraint is common to spring design. The function of the spring in many applications is to provide a modest restoring force as parts undergo large displacement in carrying out kinematic functions. Mathematically, this performance requirement ($\delta \geq \Delta$) is stated in an inequality form, using Eq. (a), as

$$\frac{P}{K} \geq \Delta \tag{g}$$

Shear stress constraint. To prevent material overstressing, *shear stress* in the wire must be no greater than τ_a, which is expressed in mathematical form as

$$\tau \leq \tau_a \tag{h}$$

Constraint on the frequency of surge waves. We also wish to avoid resonance in dynamic applications by making the *frequency of surge waves* (along the spring) as great as possible. For the present problem, we require the frequency of surge waves for the spring to be at least ω_0 (Hz). The constraint is expressed in mathematical form as

$$\omega \geq \omega_0 \tag{i}$$

Diameter constraint. The *outer diameter* of the spring should not be greater than D_o, so

$$D + d \leq D_0 \tag{j}$$

Explicit bounds on design variables. To avoid fabrication and other practical difficulties, we put *minimum and maximum size limits* on the wire diameter, coil diameter, and number of turns:

$$\begin{aligned} d_{\min} &\leq d \leq d_{\max} \\ D_{\min} &\leq D \leq D_{\max} \\ N_{\min} &\leq N \leq N_{\max} \end{aligned} \tag{k}$$

Thus, the purpose of the minimum-mass spring design problem is to select the design variables d, D, and N to minimize the mass of Eq. (f), while satisfying the ten inequality constraints of Eqs. (g) through (k). If the intermediate variables are eliminated, the problem formulation can be summarized in terms of the design variables only.

EXAMPLE 2.10 FORMULATION OF THE SPRING DESIGN PROBLEM WITH DESIGN VARIABLES ONLY

A summary of the problem formulation for the optimum design of coil springs is as follows:

Specified data: $Q, P, \rho, \gamma, \tau_a, G, \Delta, \omega_0, D_0, d_{min}, d_{max}, D_{min}, D_{max}, N_{min}, N_{max}$
Design variables: d, D, N
Cost function: Minimize the mass of the spring given in Eq. (f).
Constraints:

Deflection limit:

$$\frac{8PD^3N}{d^4G} \geq \Delta \tag{l}$$

Shear stress:

$$\frac{8PD}{\pi d^3}\left[\frac{(4D-d)}{4(D-d)} + \frac{0.615d}{D}\right] \leq \tau_a \tag{m}$$

Frequency of surge waves:

$$\frac{d}{2\pi ND^2}\sqrt{\frac{G}{2\rho}} \geq \omega_0 \tag{n}$$

Diameter constraint: Given in Eq. (j).
Design variable bounds: Given in Eqs. (k).

2.10 MINIMUM-WEIGHT DESIGN OF A SYMMETRIC THREE-BAR TRUSS

STEP 1: PROJECT/PROBLEM DESCRIPTION As an example of a slightly more complex design problem, consider the three-bar structure shown in Figure 2.9 (Schmit, 1960; Haug and Arora, 1979). This is a statically indeterminate structure for which the member forces cannot be calculated solely from equilibrium equations. The structure is to be designed for minimum volume (or, equivalently, minimum mass) to support a force P. It must satisfy various performance and technological constraints, such as member crushing, member buckling, failure by excessive deflection of node 4, and failure by resonance when the natural frequency of the structure is below a given threshold.

STEP 2: DATA AND INFORMATION COLLECTION Needed to solve the problem are geometry data, properties of the material used, and loading data. In addition, since the structure is statically indeterminate, the static equilibrium equations alone are not enough to analyze it. We need to use advanced analysis procedures to obtain expressions for member forces,

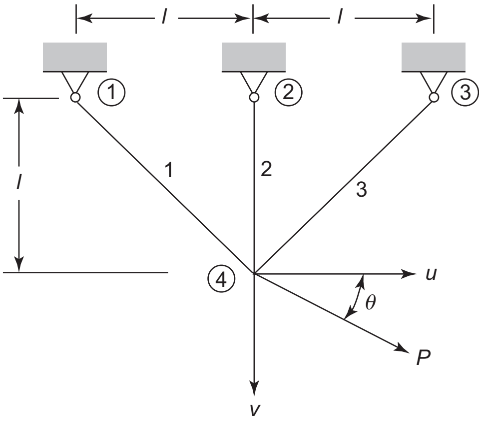

FIGURE 2.9 Three-bar truss.

nodal displacements, and the natural frequency to formulate constraints for the problem. Here we will give such expressions.

Since the structure must be symmetric, members 1 and 3 will have the same cross-sectional area, say A_1. Let A_2 be the cross-sectional area of member 2. Using analysis procedures for statically indeterminate structures, horizontal and vertical displacements u and v of node 4 are calculated as

$$u = \frac{\sqrt{2}lP_u}{A_1E}; \quad v = \frac{\sqrt{2}lP_v}{(A_1 + \sqrt{2}A_2)E} \tag{a}$$

where E is the modulus of elasticity for the material, P_u and P_v are the horizontal and vertical components of the applied load P given as $P_u = P\cos\theta$ and $P_v = P\sin\theta$, and l is the height of the truss as shown in Figure 2.9. Using the displacements, forces carried by the members of the truss can be calculated. Then the stresses σ_1, σ_2, and σ_3 in members 1, 2, and 3 under the applied load P can be computed from member forces as (stress = force/area; $\sigma_i = F_i/A_i$):

$$\sigma_1 = \frac{1}{\sqrt{2}}\left[\frac{P_u}{A_1} + \frac{P_v}{(A_1 + \sqrt{2}A_2)}\right] \tag{b}$$

$$\sigma_2 = \frac{\sqrt{2}P_v}{(A_1 + \sqrt{2}A_2)} \tag{c}$$

$$\sigma_3 = \frac{1}{\sqrt{2}}\left[-\frac{P_u}{A_1} + \frac{P_v}{(A_1 + \sqrt{2}A_2)}\right] \tag{d}$$

Note that the member forces, and hence stresses, are dependent on the design of the structure, that is, the member areas.

Many structures support moving machinery and other dynamic loads. These structures vibrate with a certain frequency known as *natural frequency*. This is an intrinsic dynamic property of a structural system. There can be several modes of vibration, each having its own frequency. *Resonance* causes catastrophic failure of the structure, which occurs when any one of its vibration frequencies coincides with the frequency of the operating machinery it supports.

Therefore, it is reasonable to demand that no structural frequency be close to the frequency of the operating machinery. The mode of vibration corresponding to the lowest natural frequency is important because that mode is excited first. It is important to make the lowest (fundamental) natural frequency of the structure as high as possible to avoid any possibility of resonance. This also makes the structure stiffer. Frequencies of a structure are obtained by solving an eigenvalue problem involving the structure's stiffness and mass properties. The lowest eigenvalue ζ related to the lowest natural frequency of the symmetric three-bar truss is computed using a consistent-mass model:

$$\zeta = \frac{3EA_1}{\rho l^2\left(4A_1 + \sqrt{2}A_2\right)} \tag{e}$$

where ρ is the material mass per unit volume (mass density). This completes the analysis of the structure.

STEP 3: DEFINITION OF DESIGN VARIABLES The following design variables are defined for the symmetric structure:

A_1 = cross-sectional area of material for members 1 and 3
A_2 = cross-sectional area of material for member 2

Other design variables for the problem are possible depending on the cross-sectional shape of members, as shown in Figure 2.6.

STEP 4: OPTIMIZATION CRITERION The relative merit of any design for the problem is measured in its material weight. Therefore, the total weight of the structure serves as a cost function (weight of member = cross-sectional area × length × weight density):

$$Volume = l\gamma\left(2\sqrt{2}A_1 + A_2\right) \tag{f}$$

where γ is the weight density.

STEP 5: FORMULATION OF CONSTRAINTS The structure is designed for use in two applications. In each application, it supports different loads. These are called loading conditions for the structure. In the present application, a symmetric structure is obtained if the following two loading conditions are considered. The first load is applied at an angle θ and the second one, of same magnitude, at an angle $(\pi - \theta)$, where the angle θ ($0° \le \theta \le 90°$) is shown earlier in Figure 2.9. If we let member 1 be the same as member 3, then the second loading condition can be ignored. Therefore, we consider only one load applied at an angle θ ($0° \le \theta \le 90°$).

Note from Eqs. (b) and (c) that the stresses σ_1 and σ_2 are always positive (tensile). If $\sigma_a > 0$ is an allowable stress for the material, then the *stress constraints* for members 1 and 2 are

$$\sigma_1 \le \sigma_a; \quad \sigma_2 \le \sigma_a \tag{g}$$

However, from Eq. (c), stress in member 3 can be positive (tensile) or negative (compressive) depending on the load angle. Therefore, both possibilities need to be considered in formulating the stress constraint for member 3. One way to formulate such a constraint was explained in Section 2.5. Another way is as follows:

$$\text{IF } (\sigma_3 < 0) \text{ THEN } -\sigma_3 \le \sigma_a \text{ ELSE } \sigma_3 \le \sigma_a \tag{h}$$

Since the sign of the stress does not change with design, if the member is in compression, it remains in compression throughout the optimization process. Therefore, the constraint function remains continuous and differentiable.

A similar procedure can be used for stresses in bars 1 and 2 if the stresses can reverse their sign (e.g., when the load direction is reversed). Horizontal and vertical deflections of node 4 must be within the specified limits Δ_u and Δ_v, respectively. Using Eq. (a), the *deflection constraints* are

$$u \le \Delta_u; \quad v \le \Delta_v \tag{i}$$

As discussed previously, the *fundamental natural frequency* of the structure should be higher than a specified frequency ω_0 (Hz). This constraint can be written in terms of the lowest eigenvalue for the structure. The eigenvalue corresponding to a frequency of ω_0 (Hz) is given as $(2\pi\omega_0)^2$. The lowest eigenvalue ζ for the structure given in Eq. (e) should be higher than $(2\pi\omega_0)^2$, that is,

$$\zeta \ge (2\pi\omega_0)^2 \tag{j}$$

To impose *buckling constraints* for members under compression, an expression for the moment of inertia of the cross-section is needed. This expression cannot be obtained because the cross-sectional shape and dimensions are not specified. However, the moment of inertia I can be related to the cross-sectional area of the members as $I = \beta A^2$, where A is the cross-sectional area and β is a nondimensional constant. This relation follows if the shape of the cross-section is fixed and all of its dimensions are varied in the same proportion.

The axial force for the ith member is given as $F_i = A_i\sigma_i$, where $i = 1, 2, 3$ with tensile force taken as positive. Members of the truss are considered columns with pin ends. Therefore, the buckling load for the ith member is given as π^2EI/l_i^2, where l_i is the length of the ith member (Crandall, Dahl, and Lardner, 1999). Buckling constraints are expressed as $-F_i \le \pi^2EI/l_i^2$, where $i = 1, 2, 3$. The negative sign for F_i is used to make the left side of the constraints positive when the member is in compression. Also, there is no need to impose buckling constraints for members in tension. With the foregoing formulation, the buckling constraint for tensile members is automatically satisfied. Substituting various quantities, member buckling constraints are

$$-\sigma_1 \le \frac{\pi^2 E\beta A_1}{2l^2} \le \sigma_a; \quad -\sigma_2 \le \frac{\pi^2 E\beta A_2}{l^2} \le \sigma_a; \quad -\sigma_3 \le \frac{\pi^2 E\beta A_1}{2l^2} \le \sigma_a \tag{k}$$

Note that the buckling load has been divided by the member area to obtain the buckling stress in Eqs. (k). The buckling stress is required not to exceed the material allowable stress σ_a. It is additionally noted that with the foregoing formulation, the load P in Figure 2.9 can be applied in the positive or negative direction. When the load is applied in the opposite direction, the member forces are also reversed. The foregoing formulation for the buckling constraints can treat both positive and negative load in the solution process.

Finally, A_1 and A_2 must both be non-negative, that is, A_1, $A_2 \geq 0$. Most practical design problems require each member to have a certain minimum area, A_{min}. The minimum area constraints can be written as

$$A_1, \quad A_2 \geq A_{min} \tag{l}$$

The optimum design problem, then, is to find cross-sectional areas A_1, $A_2 \geq A_{min}$ to minimize the volume of Eq. (f) subject to the constraints of Eqs. (g) through (l). This small-scale problem has 11 inequality constraints and 2 design variables.

2.11 A GENERAL MATHEMATICAL MODEL FOR OPTIMUM DESIGN

To describe optimization concepts and methods, we need a general mathematical statement for the optimum design problem. Such a mathematical model is defined as the minimization of a cost function while satisfying all equality and inequality constraints. The inequality constraints in the model are always transformed as "$\leq$ types." This will be called the *standard design optimization model* that is treated throughout this text. It will be shown that all design problems can easily be transcribed into the standard form.

2.11.1 Standard Design Optimization Model

In previous sections, several design problems were formulated. All problems have an optimization criterion that can be used to compare various designs and to determine an optimum or the best one. Most design problems must also satisfy certain constraints. Some design problems have only inequality constraints, others have only equality constraints, and some have both. We can define a general mathematical model for optimum design to encompass all of the possibilities. A standard form of the model is first stated, and then transformation of various problems into the standard form is explained.

Standard Design Optimization Model

Find an n-vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of design variables to

Minimize a cost function:

$$f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n) \tag{2.1}$$

subject to the p equality constraints:

$$h_j(\mathbf{x}) = h_j(x_1, x_2, \ldots, x_n) = 0; \quad j = 1 \text{ to } p \tag{2.2}$$

and the m inequality constraints:

$$g_i(\mathbf{x}) = g_i(x_1, x_2, \ldots, x_n) \leq 0; \quad i = 1 \text{ to } m \tag{2.3}$$

Note that the simple bounds on design variables, such as $x_i \geq 0$, or $x_{iL} \leq x_i \leq x_{iU}$, where x_{iL} and x_{iU} are the smallest and largest allowed values for x_i, are assumed to be included in the inequalities of Eq. (2.3).

In numerical methods, these constraints are treated explicitly to take advantage of their simple form to achieve efficiency. However, in discussing the basic optimization concepts, we assume that the inequalities in Eq. (2.3) include these constraints as well.

2.11.2 Maximization Problem Treatment

The general design model treats only minimization problems. This is no restriction, as maximization of a function $F(\mathbf{x})$ is the same as minimization of a transformed function $f(\mathbf{x}) = -F(\mathbf{x})$. To see this graphically, consider a plot of the function of one variable $F(x)$, shown in Figure 2.10(a). The function $F(x)$ takes its maximum value at the point x^*. Next consider a graph of the function $f(x) = -F(x)$, shown in Figure 2.10(b). It is clear that $f(x)$ is a reflection of $F(x)$ about the x-axis. It is also clear from the graph that $f(x)$ takes on a minimum value at the same point x^* where the maximum of $F(x)$ occurs. Therefore, minimization of $f(x)$ is equivalent to maximization of $F(x)$.

2.11.3 Treatment of "Greater Than Type" Constraints

The standard design optimization model treats only "≤ type" inequality constraints. Many design problems may also have "≥ type" inequalities. Such constraints can be

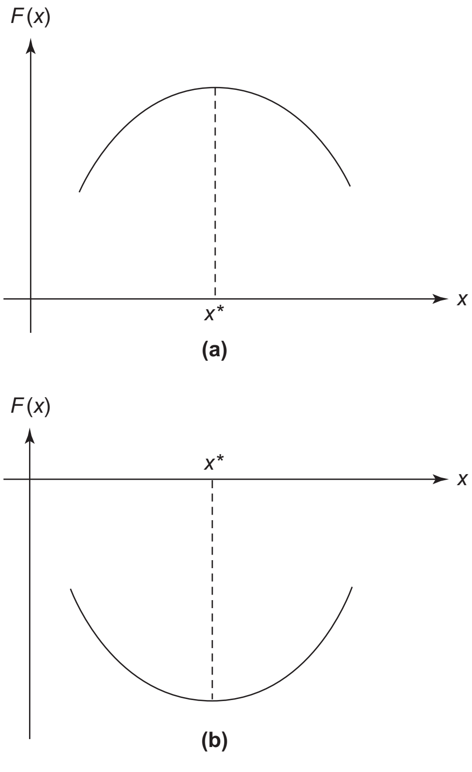

FIGURE 2.10 Point maximizing $F(x)$ equals point minimizing $-F(x)$: (a) plot of $F(x)$; (b) plot of $f(x) = -F(x)$.

converted to the standard form without much difficulty. The "$\geq$ type" constraint $G_j(\mathbf{x}) \geq 0$ is equivalent to the "$\leq$ type" inequality $g_j(\mathbf{x}) = -G_j(\mathbf{x}) \leq 0$. Therefore, we can multiply any "$\geq$ type" constraint by -1 to convert it to a "$\leq$ type."

2.11.4 Application to Different Engineering Fields

Design optimization problems from different fields of engineering can be transcribed into the standard model. It must be realized that *the overall process of designing different engineering systems is the same*. Analytical and numerical methods for analyzing systems can differ. Formulation of the design problem can contain terminology that is specific to the particular domain of application. For example, in the fields of structural, mechanical, and aerospace engineering, we are concerned with the integrity of the structure and its components. The performance requirements involve constraints on member stresses, strains, deflections at key points, frequencies of vibration, buckling failure, and so on. These concepts are specific to each field, and designers working in the particular field understand their meaning and the constraints.

Other fields of engineering also have their own terminology to describe design optimization problems. However, once the problems from different fields have been transcribed into mathematical statements using a standard notation, they have the same mathematical form. They are contained in the standard design optimization model defined in Eqs. (2.1) through (2.3). For example, all of the problems formulated earlier in this chapter can be transformed into the form of Eqs. (2.1) through (2.3). Therefore, the optimization concepts and methods described in the text are quite general and can be used to solve problems from diverse fields. *The methods can be developed without reference to any design application.* This is a key point and must be kept in mind while studying the optimization concepts and methods.

2.11.5 Important Observations about the Standard Model

Several points must be clearly understood about the standard model:

1. *Dependence of functions on design variables:* First of all, the functions $f(\mathbf{x})$, $h_j(\mathbf{x})$, and $g_i(\mathbf{x})$ must *depend*, explicitly or implicitly, on some of the *design variables*. Only then are they valid for the design problem. Functions that do not depend on any variable have no relation to the problem and can be safely ignored.
2. *Number of equality constraints:* The number of *independent equality constraints* must be less than, or at the most equal to, the number of design variables (i.e., $p \leq n$). When $p > n$, we have an *overdetermined system* of equations. In that case, either some *equality constraints* are *redundant* (linearly dependent on other constraints) or they are *inconsistent*. In the former case, redundant constraints can be deleted and, if $p < n$, the optimum solution for the problem is possible. In the latter case, no solution for the design problem is possible and the problem formulation needs to be closely reexamined. When $p = n$, no optimization of the system is necessary because the roots of the equality constraints are the only candidate points for optimum design.
3. *Number of inequality constraints:* While there is a restriction on the number of independent equality constraints, *there is no restriction on the number of inequality*

constraints. However, the total number of active constraints (satisfied at equality) must, at the optimum, be less than or at the most equal to the number of design variables.

4. *Unconstrained problems:* Some design problems may not have any constraints. These are called *unconstrained*; those with constraints are called *constrained*.

5. *Linear programming problems:* If all of the functions $f(\mathbf{x})$, $h_j(\mathbf{x})$, and $g_i(\mathbf{x})$ are linear in design variables $\mathbf{x}$, then the problem is called a *linear programming problem*. If any of these functions is nonlinear, the problem is called a *nonlinear programming problem*.

6. *Scaling of problem functions:* It is important to note that if the *cost function is scaled* by multiplying it with a positive constant, the optimum design does not change. However, the optimum cost function value does change. Also, any constant can be added to the cost function without affecting the optimum design. Similarly, the inequality *constraints can be scaled* by any positive constant and the equalities by any constant. This will not affect the feasible region and hence the optimum solution. All the foregoing transformations, however, affect the values of the *Lagrange multipliers* (defined in Chapter 4). Also, performance of the numerical algorithms for a solution to the optimization problem may be affected by these transformations.

2.11.6 Feasible Set

The term *feasible set* will be used throughout the text. *A feasible set for the design problem is a collection of all feasible designs.* The terms *constraint set* and *feasible design space* are also used to represent the feasible set of designs. The letter S is used to represent the feasible set. Mathematically, the set S is a collection of design points satisfying all constraints:

$$S = \{\mathbf{x} \mid h_j(\mathbf{x}) = 0, \;\; j = 1 \text{ to } p; \;\; g_i(\mathbf{x}) \le 0, \;\; i = 1 \text{ to } m\} \tag{2.4}$$

The *set of feasible designs* is sometimes referred to as the *feasible region*, especially for optimization problems with two design variables. It is important to note that the *feasible region usually shrinks when more constraints are added to the design model and expands when some constraints are deleted*. When the feasible region shrinks, the number of possible designs that can optimize the cost function is reduced; that is, there are fewer feasible designs. In this event, the minimum value of the cost function is likely to increase. The effect is completely the opposite when some constraints are dropped. This observation is significant for practical design problems and should be clearly understood.

2.11.7 Active/Inactive/Violated Constraints

We will quite frequently refer to a constraint as *active*, *tight*, *inactive*, or *violated*. We define these terms precisely. An inequality constraint $g_j(\mathbf{x}) \le 0$ is said to be *active* at a design point $\mathbf{x}^*$ if it is satisfied at equality (i.e., $g_j(\mathbf{x}^*) = 0$). This is also called a *tight* or *binding* constraint. For a feasible design, an inequality constraint may or may not be active. However, all equality constraints are active for all feasible designs.

An inequality constraint $g_j(\mathbf{x}) \le 0$ is said to be *inactive* at a design point $\mathbf{x}^*$ if it is strictly satisfied (i.e., $g_j(\mathbf{x}^*) < 0$). It is said to be *violated* at a design point $\mathbf{x}^*$ if its value is positive (i.e., $g_j(\mathbf{x}^*) > 0$). An *equality constraint* $h_i(\mathbf{x}) = 0$ is violated at a design point $\mathbf{x}^*$ if $h_i(\mathbf{x}^*)$ is not identically zero. Note that by these definitions, an equality constraint is either active or violated at a given design point.

2.11.8 Discrete and Integer Design Variables

So far, we have assumed in the standard model that variables x_i can have any numerical value within the feasible region. Many times, however, some variables are required to have discrete or integer values. Such variables appear quite often in engineering design problems. We encountered problems in Sections 2.4, 2.6, and 2.9 that have integer design variables. Before describing how to treat them, let us define what we mean by discrete and integer variables.

A design variable is called *discrete* if its value must be selected from a given finite set of values. For example, a plate thickness must be one that is available commercially: 1/8, 1/4, 3/8, 1/2, 5/8, 3/4, 1 in, and so on. Similarly, structural members must be selected from a catalog to reduce fabrication cost. Such variables must be treated as discrete in the standard formulation.

An *integer variable*, as the name implies, must have an integer value, for example, the number of logs to be shipped, the number of bolts used, the number of coils in a spring, the number of items to be shipped, and so on. Problems with such variables are called *discrete* and *integer programming problems*. Depending on the type of problem functions, the problems can be classified into five different categories. These classifications and the methods to solve them are discussed in Chapter 15.

In some sense, discrete and integer variables impose additional constraints on the design problem. Therefore, as noted before, the optimum value of the cost function is likely to increase with these variables compared with the same problem that is solved with continuous variables. If we treat all design variables as continuous, the minimum value of the cost function represents a lower bound on the true minimum value when discrete or integer variables are used. This gives some idea of the "best" optimum solution if all design variables are continuous. The optimum cost function value is likely to increase when discrete values are assigned to variables. Thus, the first suggested procedure is to solve the problem assuming continuous design variables if possible. Then the nearest discrete/integer values are assigned to the variables and the design is checked for feasibility. With a few trials, the best feasible design close to the continuous optimum can be obtained.

As a second approach for solving such problems, an *adaptive numerical optimization procedure* may be used. An optimum solution with continuous variables is first obtained if possible. Then only the variables that are close to their discrete or integer value are assigned that value. They are held fixed and the problem is optimized again. The procedure is continued until all variables have been assigned discrete or integer values. A few further trials may be carried out to improve the optimum cost function value. This procedure has been demonstrated by Arora and Tseng (1988).

The foregoing procedures require additional computational effort and do not guarantee a true minimum solution. However, they are quite straightforward and do not require any additional methods or software for solution of discrete/integer variable problems.

2.11.9 Types of Optimization Problems

The standard design optimization model can represent many different problem types. We saw that it can be used to represent unconstrained, constrained, linear programming, and nonlinear programming optimization problems. It is important to understand other optimization problems that are encountered in practical applications. Many times these problems can be transformed into the standard model and solved by the optimization methods presented and discussed in this text. Here we present an overview of the types of optimization problems.

Continuous/Discrete-Variable Optimization Problems

When the design variables can have any numerical value within their allowable range, the problem is called a *continuous-variable* optimization problem. When the problem has only discrete/integer variables, it is called a *discrete/integer-variable* optimization problem. When the problem has both continuous and discrete variables, it is called a mixed-variable optimization problem. Numerical methods for these types of problems have been developed, as discussed in later chapters.

Smooth/Nonsmooth Optimization Problems

When its functions are continuous and differentiable, the problem is referred to as smooth (*differentiable*). There are numerous practical optimization problems in which the functions can be formulated as continuous and differentiable. There are also many practical applications where the problem functions are not differentiable or even discontinuous. Such problems are called nonsmooth (*nondifferentiable*).

Numerical methods to solve these two classes of problems can be different. Theory and numerical methods for smooth problems are well developed. Therefore, it is most desirable to formulate the problem with continuous and differentiable functions as far as possible. Sometimes, a problem with discontinuous or nondifferentiable functions can be transformed into one that has continuous and differentiable functions so that optimization methods for smooth problems can be used. Such applications are discussed in Chapter 6.

Network Optimization Problems

A network or a graph consists of points and lines connecting pairs of points. Network models are used to represent many practical problems and processes from different branches of engineering, computer science, operations research, transportation, telecommunication, decision support, manufacturing, airline scheduling, and many other disciplines. Depending on the application type, network optimization problems have been classified as transportation problems, assignment problems, shortest-path problems, maximum-flow problems, minimum-cost-flow problems, and critical path problems.

To understand the concept of network problems, let us describe the transportation problem in more detail. Transportation models play an important role in logistics and supply chain management for reducing cost and improving service. Therefore the goal is to find the most effective way to transport goods. A shipper having m warehouses with supply s_i of goods at the ith warehouse must ship goods to n geographically dispersed retail centers, each with a customer demand d_j that must be met. The objective is to determine the minimum cost distribution system, given that the unit cost of transportation between the ith warehouse and the jth retail center is c_{ij}.

This problem can be formulated as one of linear programming. Since such network optimization problems are encountered in diverse fields, special methods have been developed to solve them more efficiently and perhaps in real time. Many textbooks are available on this subject. We do not address these problems in any detail, although some of the methods presented in Chapters 15 through 20 can be used to solve them.

Dynamic-Response Optimization Problems

Many practical systems are subjected to transient dynamic inputs. In such cases, some of the problem constraints are time-dependent. Each of these constraints must be imposed for the entire time interval of interest. Therefore each represents an infinite set of constraints because the constraint must be imposed at each time point in the given interval. The usual approach to treating such a constraint is to impose it at a finite number of time points in the given interval. This way the problem is transformed into the standard form and treated with the methods presented in this textbook.

Design Variables as Functions

In some applications, the design variables are not parameters but functions of one, two, or even three variables. Such design variables arise in optimal control problems where the input needs to be determined over the desired range of time to control the behavior of the system. The usual treatment of design functions is to parameterize them. In other words, each function is represented in terms of some known functions, called the *basis functions*, and the parameters multiplying them. The parameters are then treated as design variables. In this way the problem is transformed into the standard form and the methods presented in this textbook can be used to solve it.

EXERCISES FOR CHAPTER 2

2.1 A 100×100-m lot is available to construct a multistory office building. At least 20,000 m^2 of total floor space is needed. According to a zoning ordinance, the maximum height of the building can be only 21 m, and the parking area outside the building must be at least 25 percent of the total floor area. It has been decided to fix the height of each story at 3.5 m. The cost of the building in millions of dollars is estimated at $0.6h + 0.001A$, where A is the cross-sectional area of the building per floor and h is the height of the building. Formulate the minimum-cost design problem.

2.2 A refinery has two crude oils:

1. Crude A costs \$120/barrel (bbl) and 20,000 bbl are available.
2. Crude B costs \$150/bbl and 30,000 bbl are available.

The company manufactures gasoline and lube oil from its crudes. Yield and sale price per barrel and markets are shown in Table E2.2. How much crude oil should the company use to maximize its profit? Formulate the optimum design problem.

TABLE E2.2 Data for refinery operations

	Yield/bbl			
Product	**Crude A**	**Crude B**	**Sale price per bbl ($)**	**Market (bbl)**
Gasoline	0.6	0.8	200	20,000
Lube oil	0.4	0.2	450	10,000

2.3 Design a beer mug, shown in Figure E2.3, to hold as much beer as possible. The height and radius of the mug should be no more than 20 cm. The mug must be at least 5 cm in radius. The surface area of the sides must be no greater than 900 cm^2 (ignore the bottom area of the mug and mug handle). Formulate the optimum design problem.

2.4 A company is redesigning its parallel-flow heat exchanger of length l to increase its heat transfer. An end view of the unit is shown in Figure E2.4. There are certain limitations on the design problem. The smallest available conducting tube has a radius of 0.5 cm, and all tubes must be of the same size. Further, the total cross-sectional area of all of the tubes cannot exceed 2000 cm^2 to ensure adequate space inside the outer shell. Formulate the problem to determine the number of tubes and the radius of each one to maximize the surface area of the tubes in the exchanger.

2.5 Proposals for a parking ramp have been defeated, so we plan to build a parking lot in the downtown urban renewal section. The cost of land is $200W + 100D$, where W is the width along the street and D is the depth of the lot in meters. The available width along the street is 100 m, while the maximum depth available is 200 m. We want the size of the lot to be at least 10,000 m^2. To avoid unsightliness, the city requires that the longer dimension of any

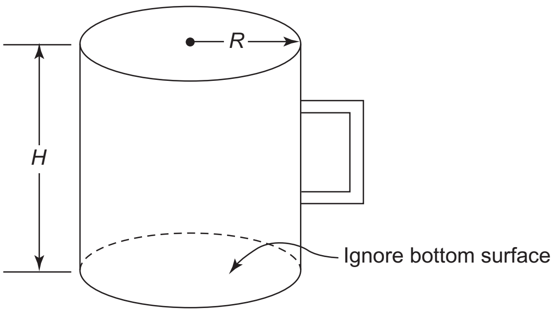

FIGURE E2.3 Beer mug.

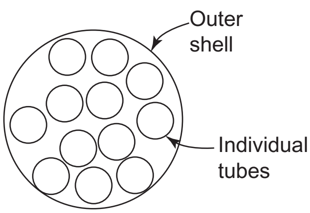

FIGURE E2.4 Cross-section of a heat exchanger.

lot be no more than twice the shorter dimension. Formulate the minimum-cost design problem.

2.6 A manufacturer sells products A and B. Profit from A is \$10/kg and is \$8/kg from B. Available raw materials for the products are 100 kg of C and 80 kg of D. To produce 1 kg of A, we need 0.4 kg of C and 0.6 kg of D. To produce 1 kg of B, we need 0.5 kg of C and 0.5 kg of D. The markets for the products are 70 kg for A and 110 kg for B. How much of A and B should be produced to maximize profit? Formulate the design optimization problem.

2.7 Design a diet of bread and milk to get at least 5 units of vitamin A and 4 units of vitamin B daily. The amount of vitamins A and B in 1 kg of each food and the cost per kilogram of the food are given in Table E2.7. Formulate the design optimization problem so that we get at least the basic requirements of vitamins at the minimum cost.

TABLE E2.7 Data for the diet problem

Vitamin	Bread	Milk
A	1	2
B	3	2
Cost/kg, \$	2	1

2.8 Enterprising chemical engineering students have set up a still in a bathtub. They can produce 225 bottles of pure alcohol each week. They bottle two products from alcohol: (1) wine, at 20 proof, and (2) whiskey, at 80 proof. Recall that pure alcohol is 200 proof. They have an unlimited supply of water, but can only obtain 800 empty bottles per week because of stiff competition. The weekly supply of sugar is enough for either 600 bottles of wine or 1200 bottles of whiskey. They make a \$1.00 profit on each bottle of wine and a \$2.00 profit on each bottle of whiskey. They can sell whatever they produce. How many bottles of wine and whiskey should they produce each week to maximize profit? Formulate the design optimization problem (created by D. Levy).

2.9 Design a can closed at one end using the smallest area of sheet metal for a specified interior volume of 600 cm^3. The can is a right-circular cylinder with interior height h and radius r. The ratio of height to diameter must not be less than 1.0 nor greater than 1.5. The height cannot be more than 20 cm. Formulate the design optimization problem.

2.10 Design a shipping container closed at both ends with dimensions $b \times b \times h$ to minimize the ratio: (round-trip cost of shipping container only)/(one-way cost of shipping contents only). Use the data in the following table. Formulate the design optimization problem.

Mass of container/surface area	80 kg/m^2
Maximum b	10 m
Maximum h	18 m
One-way shipping cost, full or empty	\$18/kg gross mass
Mass of contents	150 kg/m^3

2.11 Certain mining operations require an open-top rectangular container to transport materials. The data for the problem are as follows:

Construction costs:
– Sides: \$50/m^2
– Ends: \$60/m^2
– Bottom: \$90/m^2
Minimum volume needed: 150 m^3

Formulate the problem of determining the container dimensions at a minimum cost.

2.12 Design a circular tank closed at both ends to have a volume of 250 m^3. The fabrication cost is proportional to the surface area of the sheet metal and is \$400/m^2. The tank is to be housed in a shed with a sloping roof. Therefore, height H of the tank is limited by the relation $H \leq (10 - D/2)$, where D is the tank's diameter. Formulate the minimum-cost design problem.

2.13 Design the steel framework shown in Figure E2.13 at a minimum cost. The cost of a horizontal member in one direction is \$20 w and in the other direction it is \$30 d. The cost of a vertical column is \$50 h. The frame must enclose a total volume of at least 600 m^3. Formulate the design optimization problem.

2.14 Two electric generators are interconnected to provide total power to meet the load. Each generator's cost is a function of the power output, as shown in Figure E2.14. All costs and power are expressed on a per-unit basis. The total power needed is at least 60 units. Formulate a minimum-cost design problem to determine the power outputs P_1 and P_2.

2.15 **Transportation problem.** A company has m manufacturing facilities. The facility at the ith location has capacity to produce b_i units of an item. The product should be shipped to n distribution centers. The distribution center at the jth location requires at least a_j units of the item to satisfy demand. The cost of shipping an item from the ith plant to the jth distribution center is c_{ij}. Formulate a minimum-cost transportation system to meet each of the distribution center's demands without exceeding the capacity of any manufacturing facility.

2.16 **Design of a two-bar truss.** Design a symmetric two-bar truss (both members have the same cross-section), as shown in Figure E2.16, to support a load W. The truss consists of two steel tubes pinned together at one end and supported on the ground at the other. The span of

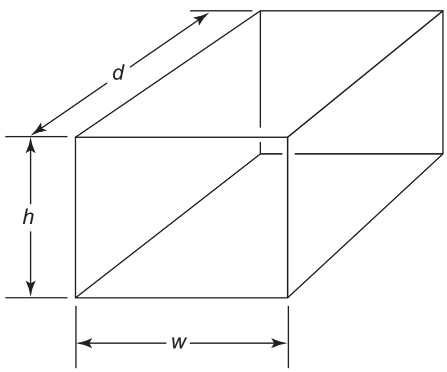

FIGURE E2.13 Steel frame.

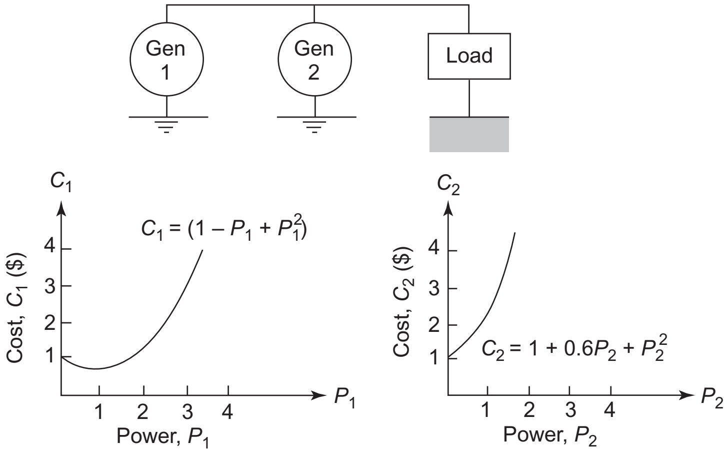

FIGURE E2.14 Graphic of a power generator.

the truss is fixed at s. Formulate the minimum-mass truss design problem using height and cross-sectional dimensions as design variables. The design should satisfy the following constraints:

1. Because of space limitations, the height of the truss must not exceed b_1 and must not be less than b_2.
2. The ratio of mean diameter to thickness of the tube must not exceed b_3.
3. The compressive stress in the tubes must not exceed the allowable stress σ_a for steel.
4. The height, diameter, and thickness must be chosen to safeguard against member buckling.

Use the following data: $W = 10$ kN; span $s = 2$ m; $b_1 = 5$ m; $b_2 = 2$ m; $b_3 = 90$; allowable stress $\sigma_a = 250$ MPa; modulus of elasticity $E = 210$ GPa; mass density $\rho = 7850$ kg/m^3; factor of safety against buckling $FS = 2$; $0.1 \le D \le 2$ (m); and $0.01 \le t \le 0.1$ (m).

2.17 A beam of rectangular cross-section (Figure E2.17) is subjected to a maximum bending moment of M and a maximum shear of V. The allowable bending and shearing stresses are σ_a and τ_a, respectively. The bending stress in the beam is calculated as

$$\sigma = \frac{6M}{bd^2}$$

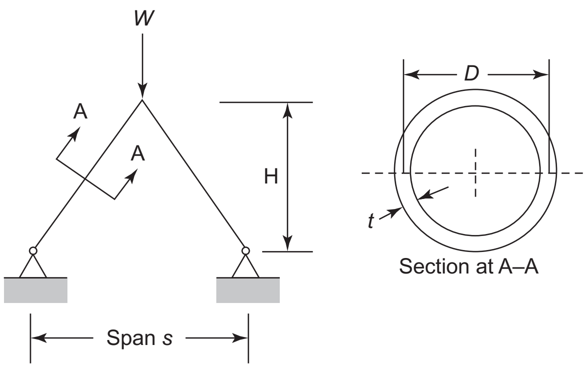

FIGURE E2.16 Two-bar structure.

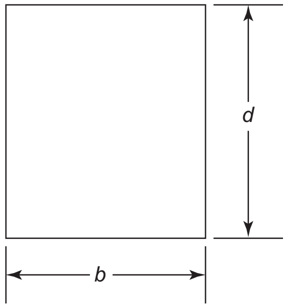

FIGURE E2.17 Cross-section of a rectangular beam.

and the average shear stress in the beam is calculated as

$$\tau = \frac{3V}{2bd}$$

where d is the depth and b is the width of the beam. It is also desirable to have the depth of the beam not exceed twice its width. Formulate the design problem for minimum cross-sectional area using this data: $M = 140$ kN • m, $V = 24$ kN, $\sigma_a = 165$ MPa, $\tau_a = 50$ MPa.

2.18 A vegetable oil processor wishes to determine how much shortening, salad oil, and margarine to produce to optimize the use its current oil stock supply. At the present time, he has 250,000 kg of soybean oil, 110,000 kg of cottonseed oil, and 2000 kg of milk-base substances. The milk-base substances are required only in the production of margarine. There are certain processing losses associated with each product: 10 percent for shortening, 5 percent for salad oil, and no loss for margarine. The producer's back orders require him to produce at least 100,000 kg of shortening, 50,000 kg of salad oil, and 10,000 kg of margarine. In addition, sales forecasts indicate a strong demand for all products in the near future. The profit per kilogram and the base stock required per kilogram of each product are given in Table E2.18. Formulate the problem to maximize profit over the next production scheduling period. (created by J. Liittschwager)

TABLE E2.18 Data for the vegetable oil processing problem

		Parts per kg of base stock requirements		
Product	***Profit per kg***	***Soybean***	***Cottonseed***	***Milk base***
Shortening	1.00	2	1	0
Salad oil	0.80	0	1	0
Margarine	0.50	3	1	1

Section 2.11: A General Mathematical Model for Optimum Design

2.19 *Answer True or False:*

1. Design of a system implies specification of the design variable values.
2. All design problems have only linear inequality constraints.
3. All design variables should be independent of each other as far as possible.

4. If there is an equality constraint in the design problem, the optimum solution must satisfy it.
5. Each optimization problem must have certain parameters called the design variables.
6. A feasible design may violate equality constraints.
7. A feasible design may violate "$\geq$ type" constraints.
8. A "$\leq$ type" constraint expressed in the standard form is active at a design point if it has zero value there.
9. The constraint set for a design problem consists of all feasible points.
10. The number of independent equality constraints can be larger than the number of design variables for the problem.
11. The number of "$\leq$ type" constraints must be less than the number of design variables for a valid problem formulation.
12. The feasible region for an equality constraint is a subset of that for the same constraint expressed as an inequality.
13. Maximization of $f(x)$ is equivalent to minimization of $1/f(x)$.
14. A lower minimum value for the cost function is obtained if more constraints are added to the problem formulation.
15. Let f_n be the minimum value for the cost function with n design variables for a problem. If the number of design variables for the same problem is increased to, say, $m = 2n$, then $f_m > f_n$, where f_m is the minimum value for the cost function with m design variables.

***2.20** A trucking company wants to purchase several new trucks. It has $2 million to spend. The investment should yield a maximum of trucking capacity for each day in tonnes $\times$ kilometers. Data for the three available truck models are given in Table E2.20: truck load capacity, average speed, crew required per shift, hours of operation for three shifts, and cost of each truck. There are some limitations on the operations that need to be considered. The labor market is such that the company can hire at most 150 truck drivers. Garage and maintenance facilities can handle at the most 25 trucks. How many trucks of each type should the company purchase? Formulate the design optimization problem.

TABLE E2.20 Data for available trucks

Truck model	Truck load capacity (tonnes)	Average truck speed (km/h)	Crew required per shift	No. of hours of operations per day (3 shifts)	Cost of each truck ($)
A	10	55	1	18	40,000
B	20	50	2	18	60,000
C	18	50	2	21	70,000

***2.21** A large steel corporation has two iron ore reduction plants. Each plant processes iron ore into two different ingot stocks, which are shipped to any of three fabricating plants where they are made into either of two finished products. In total, there are two reduction plants, two ingot stocks, three fabricating plants, and two finished products. For the upcoming season, the company wants to minimize total tonnage of iron ore processed in its reduction plants, subject to production and demand constraints. Formulate the design optimization problem and transcribe it into the standard model.

Nomenclature

$a(r, s)$ = tonnage yield of ingot stock s from 1 ton of iron ore processed at reduction plant r
$b(s, f, p)$ = total yield from 1 ton of ingot stock s shipped to fabricating plant f and manufactured into product p
$c(r)$ = ore-processing capacity in tonnage at reduction plant r
$k(f)$ = capacity of fabricating plant f in tonnage for all stocks
$D(p)$ = tonnage demand requirement for product p

Production and Demand Constraints

1. The total tonnage of iron ore processed by both reduction plants must equal the total tonnage processed into ingot stocks for shipment to the fabricating plants.
2. The total tonnage of iron ore processed by each reduction plant cannot exceed its capacity.
3. The total tonnage of ingot stock manufactured into products at each fabricating plant must equal the tonnage of ingot stock shipped to it by the reduction plants.
4. The total tonnage of ingot stock manufactured into products at each fabricating plant cannot exceed the plant's available capacity.
5. The total tonnage of each product must equal its demand.

Constants for the Problem

$a(1,1) = 0.39$	$c(1) = 1{,}200{,}000$	$k(1) = 190{,}000$	$D(1) = 330{,}000$
$a(1,2) = 0.46$	$c(2) = 1{,}000{,}0\ 00$	$k(2) = 240{,}000$	$D(2) = 125{,}000$
$a(2,1) = 0.44$		$k(3) = 290{,}000$	
$a(2,2) = 0.48$			
		$b(1,1,1) = 0.79$	$b(1,1,2) = 0.84$
		$b(2,1,1) = 0.68$	$b(2,1,2) = 0.81$
		$b(1,2,1) = 0.73$	$b(1,2,2) = 0.85$
		$b(2,2,1) = 0.67$	$b(2,2,2) = 0.77$
		$b(1,3,1) = 0.74$	$b(1,3,2) = 0.72$
		$b(2,3,1) = 0.62$	$b(2,3,2) = 0.78$

2.22 Optimization of a water canal. Design a water canal having a cross-sectional area of 150 m^2. The lowest construction costs occur when the volume of the excavated material equals the amount of material required for the dykes, as shown in Figure E2.22. Formulate the problem to minimize the dug-out material A_1. Transcribe the problem into the standard design optimization model.

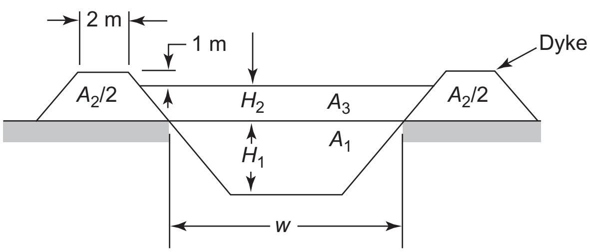

FIGURE E2.22 Graphic of a cross-section of a canal. *(Created by V. K. Goel.)*

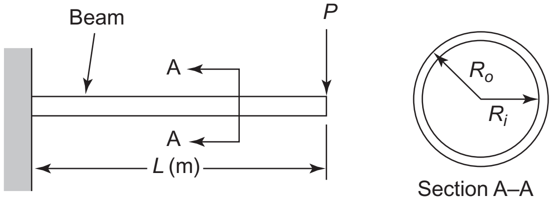

FIGURE E2.23 Cantilever beam.

2.23 A cantilever beam is subjected to the point load P (kN), as shown in Figure E2.23. The maximum bending moment in the beam is PL (kN • m) and the maximum shear is P (kN). Formulate the minimum-mass design problem using a hollow circular cross-section. The material should not fail under bending or shear stress. The maximum bending stress is calculated as

$$\sigma = \frac{PL}{I} R_o$$

where I = moment of inertia of the cross-section. The maximum shearing stress is calculated as

$$\tau = \frac{P}{3I}(R_o^2 + R_o R_i + R_i^2)$$

Transcribe the problem into the standard design optimization model (also use $R_o \leq$ 40.0 cm, $R_i \leq$ 40.0 cm). Use this data: $P = 14$ kN; $L = 10$ m; mass density $\rho = 7850$ kg/m^3; allowable bending stress $\sigma_b = 165$ MPa; allowable shear stress $\tau_a = 50$ MPa.

2.24 Design a hollow circular beam-column, shown in Figure E2.24, for two conditions: When $P = 50$ (kN), the axial stress σ must not exceed an allowable value σ_a, and when $P = 0$, deflection δ due to self-weight should satisfy $\delta \leq 0.001L$. The limits for dimensions are $t = 0.10$ to 1.0 cm, $R = 2.0$ to 20.0 cm, and $R/t \leq 20$ (AISC, 2005). Formulate the minimum-weight design problem and transcribe it into the standard form. Use the following data: $\delta = 5wL^4/384EI$; w = self-weight force/length (N/m); $\sigma_a = 250$ MPa; modulus of elasticity $E = 210$ GPa; mass density $\rho = 7800$ kg/m^3; $\sigma = P/A$; gravitational constant $g = 9.80$ m/s^2; moment of inertia $I = \pi R^3 t$ (m^4).

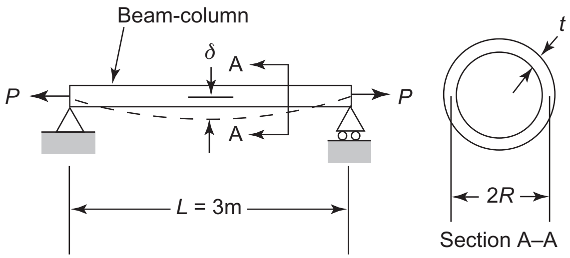

FIGURE E2.24 Graphic of a hollow circular beam-column.

CHAPTER

3

Graphical Optimization and Basic Concepts

Upon completion of this chapter, you will be able to

- Graphically solve any optimization problem having two design variables
- Plot constraints and identify their feasible/infeasible side
- Identify the feasible region (feasible set) for a problem
- Plot objective function contours through the feasible region
- Graphically locate the optimum solution for a problem and identify active/inactive constraints
- Identify problems that may have multiple, unbounded, or infeasible solutions
- Explain basic concepts and terms associated with optimum design

Optimization problems having only two design variables can be solved by observing how they are graphically represented. All constraint functions are plotted, and a set of feasible designs (the feasible set) for the problem is identified. Objective function contours are then drawn, and the optimum design is determined by visual inspection. In this chapter, we illustrate the graphical solution process and introduce several concepts related to optimum design problems. In Section 3.1, a design optimization problem is formulated and used to describe the solution process. Several more example problems are solved in later sections to illustrate concepts and the procedure.

3.1 GRAPHICAL SOLUTION PROCESS

3.1.1 Profit Maximization Problem

STEP 1: PROJECT/PROBLEM DESCRIPTION A company manufactures two machines, A and B. Using available resources, either 28 A or 14 B can be manufactured daily. The sales department can sell up to 14 A machines or 24 B machines. The shipping facility can handle no more than 16 machines per day. The company makes a profit of \$400 on each A machine and \$600 on each B machine. How many A and B machines should the company manufacture every day to maximize its profit?

STEP 2: DATA AND INFORMATION COLLECTION Data and information are defined in the project statement.

STEP 3: DEFINITION OF DESIGN VARIABLES The following two design variables are identified in the problem statement:

x_1 = number of A machines manufactured each day
x_2 = number of B machines manufactured each day

STEP 4: OPTIMIZATION CRITERION The objective is to maximize daily profit, which can be expressed in terms of design variables as

$$P = 400x_1 + 600x_2 \tag{a}$$

STEP 5: FORMULATION OF CONSTRAINTS Design constraints are placed on manufacturing capacity, on sales personnel, and on the shipping and handling facility. The constraint on the shipping and handling facility is quite straightforward:

$$x_1 + x_2 \le 16 \text{ (shipping and handling constraint)} \tag{b}$$

Constraints on manufacturing and sales facilities are a bit tricky. First, consider the manufacturing limitation. It is assumed that if the company is manufacturing x_1 A machines per day, then the remaining resources and equipment can be proportionately used to manufacture x_2 B machines, and vice versa. Therefore, noting that $x_1/28$ is the fraction of resources used to produce A and $x_2/14$ is the fraction used to produce B, the constraint is expressed as

$$\frac{x_1}{28} + \frac{x_2}{14} \le 1 \text{ (manufacturing constraint)} \tag{c}$$

Similarly, the constraint on sales department resources is given as

$$\frac{x_1}{14} + \frac{x_2}{24} \le 1 \text{ (limitation on sales department)} \tag{d}$$

Finally, the design variables must be non-negative as

$$x_1, \ x_2 \ge 0 \tag{e}$$

Note that for this problem, the formulation remains valid even when a design variable has zero value. The problem has two design variables and five inequality constraints. All functions of the problem are linear in variables x_1 and x_2. Therefore, it is a *linear programming problem*. Note also that for a meaningful solution, both design variables must have integer values at the optimum point.

3.1.2 Step-by-Step Graphical Solution Procedure

STEP 1: COORDINATE SYSTEM SET-UP The first step in the solution process is to set up an origin for the x-y coordinate system and scales along the x- and y-axes. By looking at the constraint functions, a coordinate system for the profit maximization problem can be set up using a range of 0 to 25 along both the x and y axes. In some cases, the scale may need to be adjusted after the problem has been graphed because the original scale may provide too small or too large a graph for the problem.

STEP 2: INEQUALITY CONSTRAINT BOUNDARY PLOT To illustrate the graphing of a constraint, let us consider the inequality $x_1 + x_2 \le 16$ given in Eq. (b). To represent the constraint graphically, we first need to plot the constraint boundary; that is, the points that satisfy the constraint as an equality $x_1 + x_2 = 16$. This is a linear function of the variables x_1 and x_2. To plot such a function, we need two points that satisfy the equation $x_1 + x_2 = 16$. Let these points be calculated as (16,0) and (0,16). Locating them on the graph and joining them by a straight line produces the line F—J, as shown in Figure 3.1. Line F—J then represents the boundary of the feasible region for the inequality constraint $x_1 + x_2 \le 16$. Points on one side of this line violate the constraint, while those on the other side satisfy it.

STEP 3: IDENTIFICATION OF THE FEASIBLE REGION FOR AN INEQUALITY The next task is to determine which side of constraint boundary F—J is feasible for the constraint $x_1 + x_2 \le 16$. To accomplish this, we select a point on either side of F—J and evaluate the constraint function there. For example, at point (0,0), the left side of the constraint has a value of 0. Because the value is less than 16, the constraint is satisfied and the region

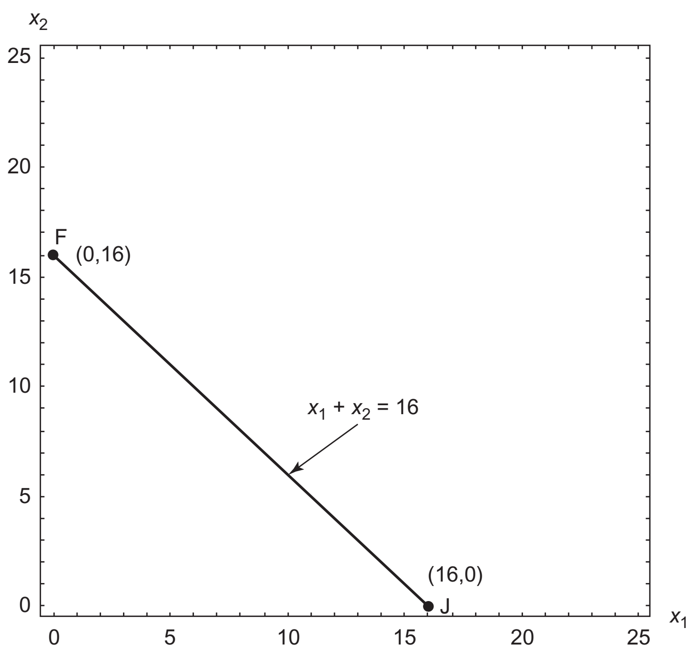

FIGURE 3.1 Constraint boundary for the inequality $x_1 + x_2 \le 16$ in the profit maximization problem.

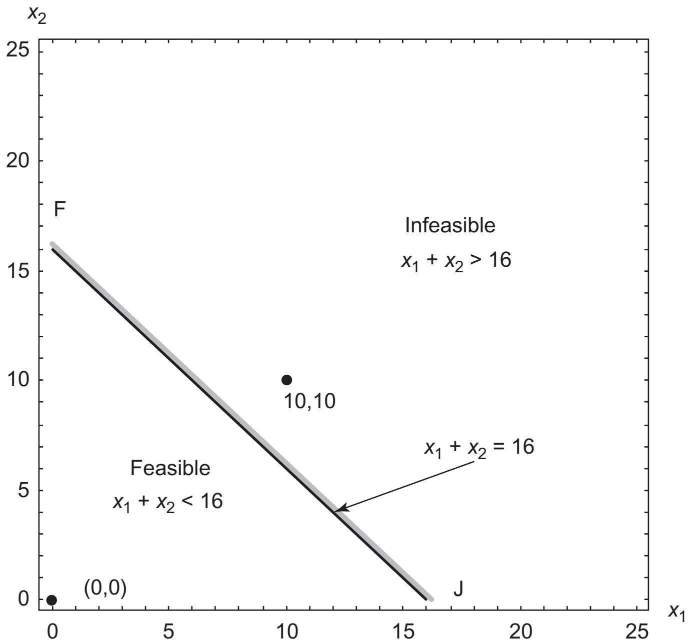

FIGURE 3.2 Feasible/infeasible side for the inequality $x_1 + x_2 \leq 16$ in the profit maximization problem.

below F–J is feasible. We can test the constraint at another point on the opposite side of F–J, say at point (10,10). At this point the constraint is violated because the left side of the constraint function is 20, which is larger than 16. Therefore, the region above F–J is infeasible with respect to the constraint, as shown in Figure 3.2. The infeasible region is "shaded-out," *a convention that is used throughout this text*.

Note that if this were an *equality* constraint $x_1 + x_2 = 16$, the feasible region for it would only be the points on line F–J. Although there are infinite points on F–J, the feasible region for the equality constraint is much smaller than that for the same constraint written as an inequality. This shows the importance of properly formulating all the constraints of the problem.

STEP 4: IDENTIFICATION OF THE FEASIBLE REGION By following the procedure that is described in step 3, all inequalities are plotted on the graph and the feasible side of each one is identified (if equality constraints were present, they would also be plotted at this stage). Note that the constraints $x_1, x_2 \geq 0$ restrict the feasible region to the first quadrant of the coordinate system. The intersection of feasible regions for all constraints provides the feasible region for the profit maximization problem, indicated as ABCDE in Figure 3.3. Any point in this region or on its boundary provides a feasible solution to the problem.

STEP 5: PLOTTING OF OBJECTIVE FUNCTION CONTOURS The next task is to plot the objective function on the graph and locate its optimum points. For the present problem, the objective is to maximize the profit $P = 400x_1 + 600x_2$, which involves three variables: P, x_1, and x_2. The function needs to be represented on the graph so that the value of P can be

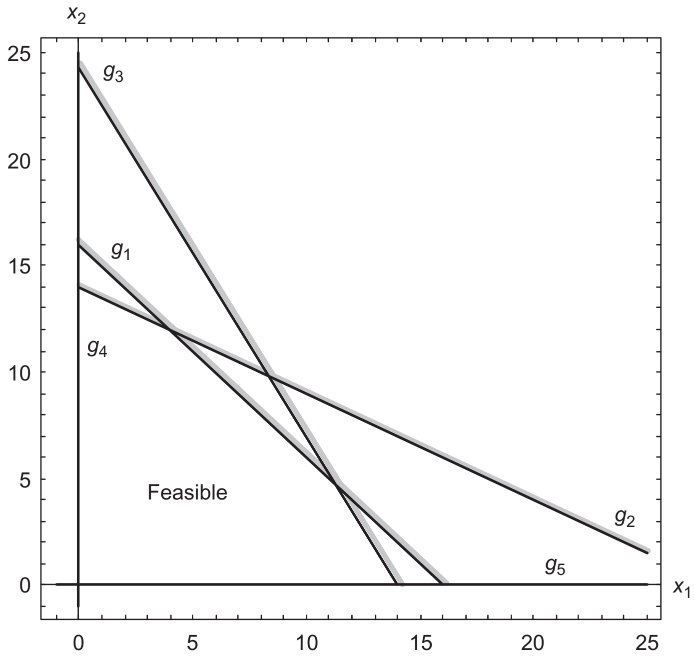

FIGURE 3.3 Feasible region for the profit maximization problem.

compared for different feasible designs to locate the best design. However, because there are infinite feasible points, it is not possible to evaluate the objective function at every point. One way of overcoming this impasse is to plot the contours of the objective function.

A *contour* is a curve on the graph that connects all points having the same objective function value. A collection of points on a contour is also called the *level set*. If the objective function is to be minimized, the contours are also called *isocost curves*. To plot a contour through the feasible region, we need to assign it a value. To obtain this value, consider a point in the feasible region and evaluate the profit function there. For example, at point (6,4), P is $P = 6 \times 400 + 4 \times 600 = 4800$. To plot the $P = 4800$ contour, we plot the function $400x_1 + 600x_2 = 4800$. This contour is a straight line, as shown in Figure 3.4.

STEP 6: IDENTIFICATION OF THE OPTIMUM SOLUTION To locate an optimum point for the objective function, we need at least two contours that pass through the feasible region. We can then observe trends for the values of the objective function at different feasible points to locate the best solution point. Contours for $P = 2400$, 4800, and 7200 are plotted in Figure 3.5. We now observe the following trend: As the contours move up toward point D, feasible designs can be found with larger values for P. It is clear from observation that point D has the largest value for P in the feasible region. We now simply read the coordinates of point D (4, 12) to obtain the optimum design, having a maximum value for the profit function as $P = 8800$.

Thus, the best strategy for the company is to manufacture 4 A and 12 B machines to maximize its daily profit. The inequality constraints in Eqs. (b) and (c) are *active* at the optimum; that is, they are satisfied at equality. These represent limitations on shipping and handling

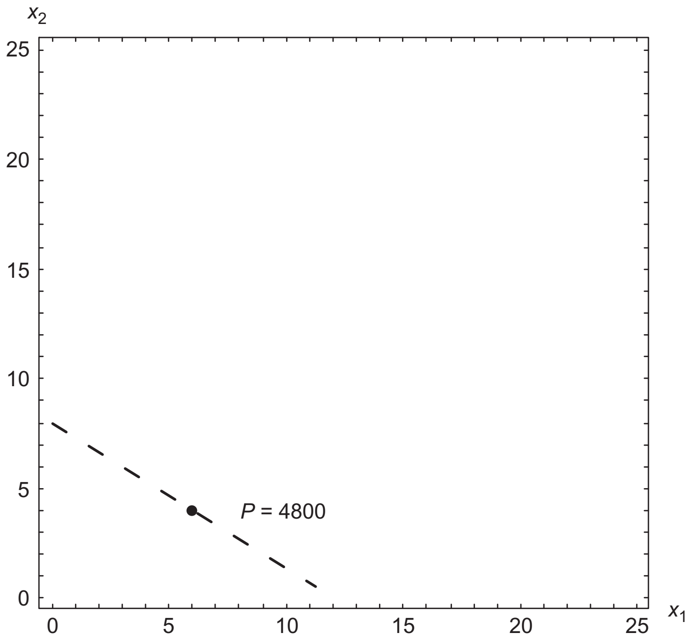

FIGURE 3.4 Plot of $P = 4800$ objective function contour for the profit maximization problem.

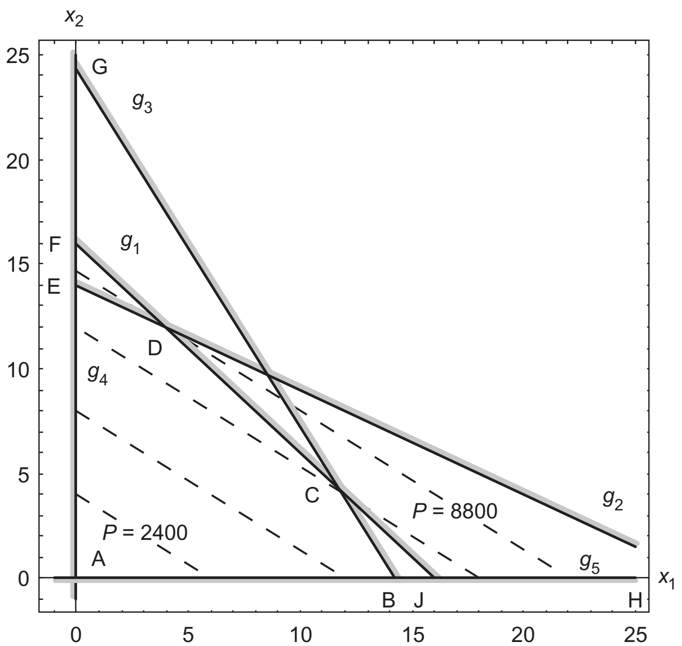

FIGURE 3.5 Graphical solution to the profit maximization problem: optimum point D = (4, 12); maximum profit, $P = 8800$.

facilities, and on manufacturing. The company can think about relaxing these constraints to improve its profit. All other inequalities are strictly satisfied and therefore *inactive*.

Note that in this example the design variables must have integer values. Fortunately, the optimum solution has integer values for the variables. If this had not been the case, we would have used the procedure suggested in Section 2.11.8 or in Chapter 15 to solve this problem. Note also that for this example all functions are linear in design variables. Therefore, all curves in Figures 3.1 through 3.5 are *straight lines*. In general, the functions of a design problem may not be linear, in which case curves must be plotted to identify the feasible region, and contours or *isocost curves* must be drawn to identify the optimum design. To *plot a nonlinear function*, a table of numerical values for x_1 and x_2 must be generated. These points must be then plotted on a graph and connected by a smooth curve.

3.2 USE OF MATHEMATICA FOR GRAPHICAL OPTIMIZATION

It turns out that good programs, such as Mathematica and MATLAB®, are available to implement the step-by-step procedure of the previous section and obtain a graphical solution for the problem on the computer screen. Mathematica is an interactive software package with many capabilities; however, we will explain its use to solve a two-variable optimization problem by plotting all functions on the computer screen. Although other commands for plotting functions are available, the most convenient for working with inequality constraints and objective function contours is the `ContourPlot` command. As with most Mathematica commands, this one is followed by what we call subcommands as "arguments" that define the nature of the plot. All Mathematica commands are case-sensitive, so it is important to pay attention to which letters are capitalized.

Mathematica input is organized into what is called a *notebook*. A notebook is divided into *cells*, with each cell containing input that can be executed independently. To explain the graphical optimization capability of Mathematica, we will again use the profit maximization problem. (*Note that the commands used here may change in future releases of the program*.) We start by entering in the notebook the problem functions as follows (the first two commands are for initialization of the program):

```
<<Graphics`Arrow`
Clear[x1,x2];

P=400*x1+600*x2;
g1=x1+x2-16;          (*shipping and handling constraint*)
g2=x1/28+x2/14−1;     (*manufacturing constraint*)
g3=x1/14+x2/24−1;     (*limitation on sales department*)
g4=−x1;               (*non-negativity*)
g5=−x2;               (*non-negativity*)
```

This input illustrates some basic features concerning Mathematica format. Note that the ENTER key acts simply as a carriage return, taking the blinking cursor to the next line. Pressing SHIFT and ENTER actually inputs the typed information into Mathematica. When no immediate output from Mathematica is desired, the input line must end with a semicolon (;). If the semicolon is omitted, Mathematica will simplify the input and display it on the screen or execute an arithmetic expression and display the result. Comments are bracketed as `(*Comment*)`. Note also that all constraints are assumed to be in the standard "≤" form. This helps in identifying the infeasible region for constraints on the screen using the *ContourPlot* command.

3.2.1 Plotting Functions

The Mathematica command used to plot the contour of a function, say `g1=0`, is entered as follows:

```
Plotg1=ContourPlot[g1,{x1,0,25},{x2,0,25}, ContourShading→False, Contours→{0},
ContourStyle→{{Thickness[.01]}}, Axes→True, AxesLabel→{"x1","x2"},
PlotLabel→"Profit Maximization Problem", Epilog→{Disk[{0,16},{.4,.4}],
Text["(0,16)",{2,16}], Disk[{16,0},{.4,.4}], Text["(16,0)",{17,1.5}],
Text["F",{0,17}], Text["J",{17,0}], Text["x1+x2=16",{13,9}], Arrow[{13,8.3},{10,6}]},
DefaultFont→{"Times",12}, ImageSize→72.5];
```

Plotg1 is simply an arbitrary name referring to the data points for the function `g1` determined by the *ContourPlot* command; it is used in future commands to refer to this particular plot. This *ContourPlot* command plots a contour defined by the equation `g1=0` as shown earlier in Figure 3.1. Arguments of the *ContourPlot* command containing various subcommands are explained as follows (note that the arguments are separated by commas and are enclosed in square brackets (`[ ]`):

`g1`: function to be plotted.
`{x1,0,25}`, `{x2,0,25}`: ranges for the variables `x1` and `x2`; `0` to `25`.
ContourShading→`False`: indicates that shading will not be used to plot contours, whereas *ContourShading*→`True` would indicate that shading will be used. Note that most subcommands are followed by an arrow (→) or (->) and a set of parameters enclosed in braces (`{}`).
Contours→`{0}`: contour values for `g1`; one contour is requested having `0` value.
ContourStyle→`{{Thickness[.01]}}`: defines characteristics of the contour such as thickness and color. Here, the thickness of the contour is specified as "`.01`". It is given as a fraction of the total width of the graph and needs to be determined by trial and error.
Axes→`True`: indicates whether axes should be drawn at the origin; in the present case, where the origin `(0,0)` is located at the bottom left corner of the graph, the *Axes* subcommand is irrelevant except that it allows for the use of the *AxesLabel* command.
AxesLabel→`{"x1","x2"}`: allows one to indicate labels for each axis.
PlotLabel→`"Profit Maximization Problem"`: places a label at the top of the graph.
Epilog→`{...}`: allows insertion of additional graphics primitives and text in the figure on the screen figure on the screen; `Disk [{0,16}, {.4,.4}]` allows insertion of a dot at the location `(0,16)` of radius .4 in both directions; `Text ["(0,16)", (2,16)]` allows `"(0,16)"` to be placed at the location `(2,16)`.

`ImageSize→72 5`: indicates that the width of the plot should be 5 inches; the size of the plot can be adjusted by selecting the image and dragging one of the black square control points; the images in Mathematica can be copied and pasted to a word processor file.
`DefaultFont→{"Times",12}`: specifies the preferred font and size for the text.

3.2.2 Identification and Shading of Infeasible Region for an Inequality

Figure 3.2 is created using a slightly modified *ContourPlot* command used earlier for Figure 3.1:

```
Plotg1=ContourPlot[g1,{x1,0,25},{x2,0,25}, ContourShading→False, Contours→{0,.65},
ContourStyle→{{Thickness[.01]}, {GrayLevel[.8],Thickness[.025]}}, Axes→True,
AxesLabel→{"x1","x2"}, PlotLabel→"Profit Maximization Problem",
Epilog→{Disk[{10,10},{.4,.4}], Text["(10,10)",{11,9}], Disk[{0,0},{.4,.4}],
Text["(0,0)",{2,.5}], Text["x1+x2=16",{18,7}], Arrow[{18,6.3},{12,4}],
Text["Infeasible",{17,17}], Text["x1+x2>16",{17,15.5}], Text["Feasible",{5,6}],
Text["x1+x2<16",{5,4.5}]}, DefaultFont→{"Times",12}, ImageSize→72.5];
```

Here, two contour lines are specified, the second one having a small positive value. This is indicated by the command: `Contours→{0,.65}`. The constraint boundary is represented by the contour `g1=0`. The contour `g1=0.65` will pass through the infeasible region, where the positive number `0.65` is determined by trial and error.

To shade the infeasible region, the characteristics of the contour are changed. Each set of brackets `{}` with the *ContourStyle* subcommand corresponds to a specific contour. In this case, `{Thickness[.01]}` provides characteristics for the first contour `g1=0`, and `{GrayLevel[.8],Thickness[0.025]}` provides characteristics for the second contour `g1=0.65`. *GrayLevel* specifies a color for the contour line. A gray level of `0` yields a black line, whereas a gray level of 1 yields a white line. Thus, this *ContourPlot* command essentially draws one thin, black line and one thick, gray line. This way the infeasible side of an inequality is shaded out.

3.2.3 Identification of Feasible Region

By using the foregoing procedure, all constraint functions for the problem are plotted and their feasible sides are identified. The plot functions for the five constraints `g1` through `g5` are named `Plotg1`, `Plotg2`, `Plotg3`, `Plotg4`, and `Plotg5`. All of these functions are quite similar to the one that was created using the *ContourPlot* command explained earlier. As an example, the `Plotg4` function is given as

```
Plotg4=ContourPlot[g4,{x1,-1,25},{x2,-1,25}, ContourShading→False, Contours→{0,.35},
ContourStyle→{{Thickness[.01]}, {GrayLevel[.8],Thickness[.02]}},
DisplayFunction→Identity];
```

The *DisplayFunction→Identity* subcommand is added to the *ContourPlot* command to suppress display of output from each `Plotgi` function; without that, Mathematica executes each `Plotgi` function and displays the results. Next, with the following *Show* command, the five plots are combined to display the complete feasible set in Figure 3.3:

```
Show[{Plotg1,Plotg2,Plotg3,Plotg4,Plotg5}, Axes→True,AxesLabel→{"x1","x2"},
PlotLabel→"Profit Maximization Problem", DefaultFont→{"Times",12}, Epilog→
{Text["g1",{2.5,16.2}], Text["g2",{24,4}], Text["g3",{2,24}], Text["g5",{21,1}],
Text["g4",{1,10}], Text["Feasible",{5,6}]}, DefaultFont→{"Times",12},
ImageSize→72.5,DisplayFunction→ $DisplayFunction];
```

The *Text* subcommands are included to add text to the graph at various locations. The *DisplayFunction→$DisplayFunction* subcommand is added to display the final graph; without that it is not displayed.

3.2.4 Plotting of Objective Function Contours

The next task is to plot the objective function contours and locate its optimum point. The objective function contours of values 2400, 4800, 7200, and 8800, shown in Figure 3.4, are drawn by using the *ContourPlot* command as follows:

```
PlotP=ContourPlot[P,{x1,0,25},{x2,0,25}, ContourShading→False, Contours→{4800},
ContourStyle→{{Dashing[{.03,.04}], Thickness[.007]}}, Axes→True,
AxesLabel→{"x1","x2"}, PlotLabel→"Profit Maximization Problem",
DefaultFont→{"Times",12}, Epilog→{Disk[{6,4},{.4,.4}], Text["P= 4800",{9.75,4}]},
ImageSize→72.5];
```

The *ContourStyle* subcommand provides four sets of characteristics, one for each contour. *Dashing*`[{a,b}]` yields a dashed line with `"a"` as the length of each dash and `"b"` as the space between dashes. These parameters represent a fraction of the total width of the graph.

3.2.5 Identification of Optimum Solution

The *Show* command used to plot the feasible region for the problem in Figure 3.3 can be extended to plot the profit function contours as well. Figure 3.5 contains the graphical representation of the problem, obtained using the following *Show* command:

```
Show[{Plotg1,Plotg2,Plotg3,Plotg4,Plotg5, PlotP}, Axes→True, AxesLabel→{"x1","x2"},
PlotLabel→"Profit Maximization Problem", DefaultFont→{"Times",12},
Epilog→{Text["g1",{2.5,16.2}], Text["g2",{24,4}], Text["g3",{3,23}], Text["g5",{23,1}],
Text["g4",{1,10}], Text["P= 2400",{3.5,2}], Text["P= 8800",{17,3.5}], Text["G",{1,24.5}],
Text["C",{10.5,4}], Text["D",{3.5,11}], Text["A",{1,1}], Text["B",{14,−1}],
Text["J",{16,−1}], Text["H",{25,−1}], Text["E",{−1,14}], Text["F",{−1,16}]},
DefaultFont→{"Times",12}, ImageSize→72.5, DisplayFunction→ $DisplayFunction];
```

Additional *Text* subcommands have been added to label different objective function contours and different points. The final graph is used to obtain the graphical solution. The *Disk* subcommand can be added to the *Epilog* command to put a dot at the optimum point.

3.3 USE OF MATLAB FOR GRAPHICAL OPTIMIZATION

MATLAB has many capabilities for solving engineering problems. For example, it can plot problem functions and graphically solve a two-variable optimization problem. In this section, we explain how to use the program for this purpose; other uses of the program for solving optimization problems are explained in Chapter 7.

There are two modes of input with MATLAB. We can enter commands interactively, one at a time, with results displayed immediately after each one. Alternatively, we can create an input file, called an *m-file* that is executed in batch mode. The m-file can be created using the text editor in MATLAB. To access this editor, select "File," "New," and "m-file." When saved, this file will have the suffix ".m" (dot m). To submit or run the file, after starting MATLAB, we simply type the name of the file we wish to run in the command window, without the suffix (the *current directory* in the MATLAB program must be one where the file is located). In this section, we will solve the profit maximization problem of the previous section using MATLAB7.6. It is important to note that with future releases, the commands we will discuss may change.

3.3.1 Plotting of Function Contours

For plotting all of the constraints with MATLAB and identifying the feasible region, it is assumed that all inequality constraints are written in the standard "≤" form. The M-file for the profit maximization problem with explanatory comments is displayed in Table 3.1. Note that the file comments are preceded by the percent sign, %. The comments are ignored during MATLAB execution. For contour plots, the first command in the input file is entered as follows:

```
[x1,x2]=meshgrid(-1.0:0.5:25.0, -1.0:0.5:25.0);
```

This command creates a grid or array of points where all functions to be plotted are evaluated. The command indicates that x1 and x2 will start at −1.0 and increase in increments of 0.5 up to 25.0. These variables now represent two-dimensional arrays and require special attention in operations using them. "*" (star) and "/" (slash) indicate scalar multiplication and division, respectively, whereas ".*" (dot star) and "./" (dot slash) indicate element-by-element multiplication and division. The ".^" (dot hat) is used to apply an exponent to each element of a vector or a matrix. The semicolon ";" after a command prevents MATLAB from displaying the numerical results immediately (i.e., all of the values for x1 and x2).

This use of a semicolon is a convention in MATLAB for most commands. Note that matrix division and multiplication capabilities are not used in the present example, as the variables in the problem functions are only multiplied or divided by a scalar rather than another variable. If, for instance, a term such as x_1x_2 is present, then the element-by-element operation x1.*x2 is necessary. The "contour" command is used for plotting all problem functions on the screen.

The procedure for identifying the infeasible side of an inequality is to plot two contours for the inequality: one of value 0 and the other of a small positive value. The second

TABLE 3.1 MATLAB file for the profit maximization problem

m-file with explanatory comments

```
%Create a grid from −1 to 25 with an increment of 0.5 for the variables x1 and x2
[x1,x2]=meshgrid(−1:0.5:25.0,−1:0.5:25.0);
%Enter functions for the profit maximization problem
f=400*x1+600*x2;
g1=x1+x2−16;
g2=x1/28+x2/14−1;
g3=x1/14+x2/24−1;
g4=−x1;
g5=−x2;
%Initialization statements; these need not end with a semicolon
   cla reset
   axis auto   %Minimum and maximum values for axes are determined automatically
               %Limits for x- and y-axes may also be specified with the command
               %axis ([xmin xmax ymin ymax])
   xlabel('x1'),ylabel('x2') %Specifies labels for x- and y-axes
   title ('Profit Maximization Problem') %Displays a title for the problem
   hold on    %retains the current plot and axes properties for all subsequent plots
%Use the "contour" command to plot constraint and cost functions
   cv1=[0 .5];   %Specifies two contour values, 0 and .5
   const1=contour(x1,x2,g1,cv1,'k');   %Plots two specified contours of g1; k=black
                                         color
   clabel(const1)   %Automatically puts the contour value on the graph
   text(1,16,'g1')   %Writes g1 at the location (1, 16)
   cv2=[0 .03];
   const2=contour(x1,x2,g2,cv2,'k');
   clabel(const2)
   text(23,3,'g2')
   const3=contour(x1,x2,g3,cv2,'k');
   clabel(const3)
   text(1,23,'g3')
   cv3=[0 .5];
   const4=contour(x1,x2,g4,cv3,'k');
   clabel(const4)
   text(.25,20,'g4')
   const5=contour(x1,x2,g5,cv3,'k');
   clabel(const5)
   text(19,.5,'g5')
   text(1.5,7,'Feasible Region')
   fv=[2400, 4800, 7200, 8800];   %Defines 4 contours for the profit function
   fs=contour(x1,x2,f,fv,'k−');   %'k−' specifies black dashed lines for profit
                                    function contours
   clabel(fs)
   hold off   %Indicates end of this plotting sequence
              %Subsequent plots will appear in separate windows
```

contour will pass through the problem's infeasible region. The thickness of the infeasible contour is changed to indicate the infeasible side of the inequality using the graph-editing capability, which is explained in the following subsection.

In this way all constraint functions are plotted and the problem's feasible region is identified. By observing the trend of the objective function contours, the optimum point for the problem is identified.

3.3.2 Editing of Graph

Once the graph has been created using the commands just described, we can edit it before printing it or copying it to a text editor. In particular, we may need to modify the appearance of the constraints' infeasible contours and edit any text. To do this, first select "Current Object Properties..." under the "Edit" tab on the graph window. Then double-click on any item in the graph to edit its properties. For instance, we can increase the thickness of the infeasible contours to shade out the infeasible region. In addition, text may be added, deleted, or moved as desired. Note that if MATLAB is re-run, any changes made directly to the graph are lost. For this reason, it is a good idea to save the graph as a `".fig"` file, which may be recalled with MATLAB.

Another way to shade out the infeasible region is to plot several closely spaced contours in it using the following commands:

```
cv1=[0:0.01:0.5];     %[Starting contour: Increment: Final contour]
const1=contour(x1,x2,g1,cv1,'g');     % g = green color
```

There are two ways to transfer the graph to a text document. First, select "Copy Figure" under the "Edit" tab so that the figure can be pasted as a bitmap into a document. Alternatively, select "Export..." under the "File" tab. The figure is exported as the specified file type and can be inserted into another document through the "Insert" command. The final MATLAB graph for the profit maximization problem is shown in Figure 3.6.

3.4 DESIGN PROBLEM WITH MULTIPLE SOLUTIONS

A situation can arise in which a constraint is parallel to the cost function. If the constraint is active at the optimum, there are multiple solutions to the problem. To illustrate this situation, consider the following design problem:

Minimize

$$f(\mathbf{x}) = -x_1 - 0.5x_2 \tag{a}$$

subject to

$$2x_1 + 3x_2 \le 12, \;\; 2x_1 + x_2 \le 8, \;\; -x_1 \le 0, \;\; -x_2 \le 0 \tag{b}$$

In this problem, the second constraint is parallel to the cost function. Therefore, there is a possibility of *multiple optimum designs*. Figure 3.7 provides a graphical solution to the problem. It is seen that any point on the line B–C gives an optimum design, giving the problem infinite optimum solutions.

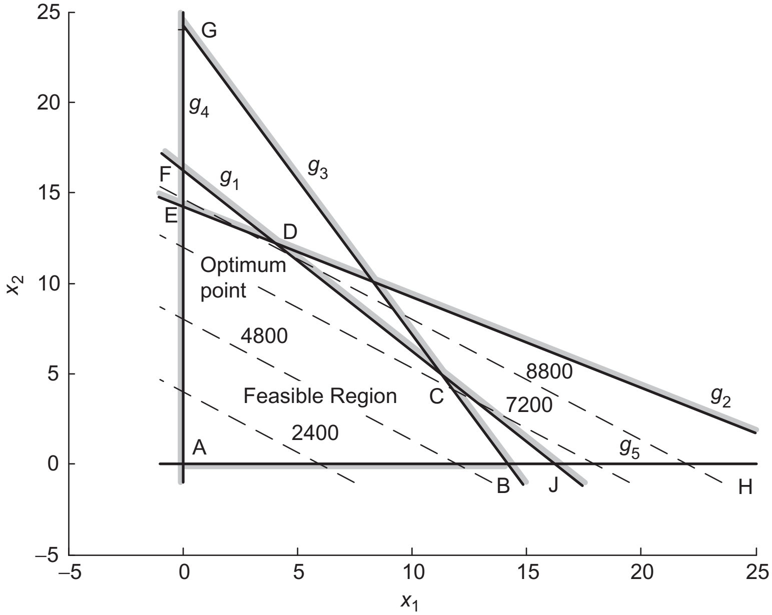

FIGURE 3.6 This shows a graphical representation of the profit maximization problem with MATLAB.

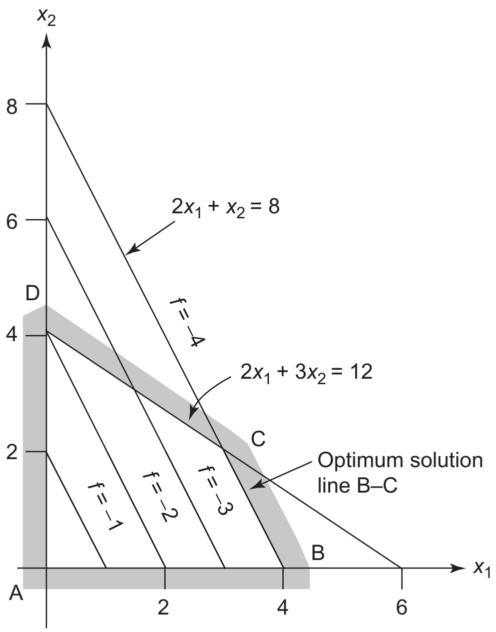

FIGURE 3.7 Example problem with multiple solutions.

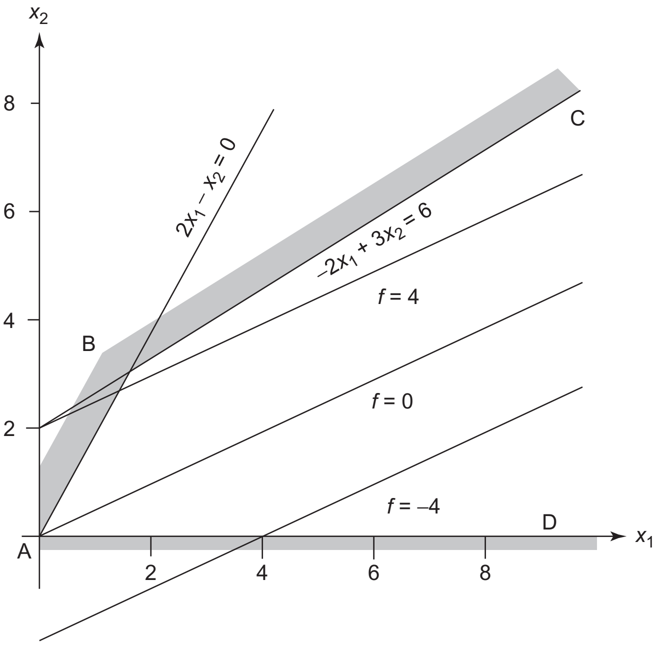

FIGURE 3.8 Example problem with an unbounded solution.

3.5 PROBLEM WITH UNBOUNDED SOLUTIONS

Some design problems may not have a bounded solution. This situation can arise if we forget a constraint or incorrectly formulate the problem. To illustrate such a situation, consider the following design problem:

Minimize

$$f(\mathbf{x}) = -x_1 + 2x_2 \tag{c}$$

subject to

$$-2x_1 + x_2 \le 0, \;\; -2x_1 + 3x_2 \le 6, \;\; -x_1 \le 0, \;\; -x_2 \le 0 \tag{d}$$

The feasible set for the problem is shown in Figure 3.8 with several cost function contours. It is seen that the feasible set is unbounded. Therefore, there is no finite optimum solution, and we must re-examine the way the problem was formulated to correct the situation. Figure 3.8 shows that the problem is underconstrained.

3.6 INFEASIBLE PROBLEM

If we are not careful in formulating it, a design problem may not have a solution, which happens when there are conflicting requirements or inconsistent constraint equations. There may also be no solution when we put *too many constraints* on the system; that is, the

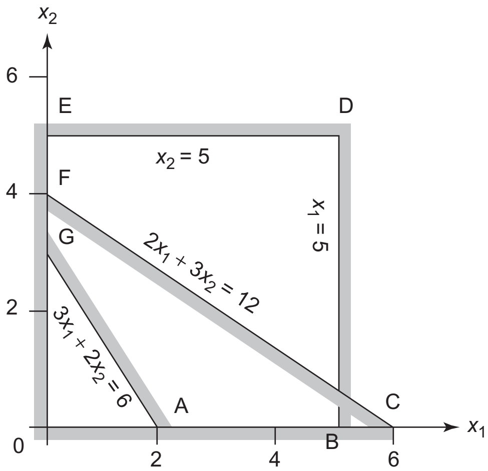

FIGURE 3.9 Infeasible design optimization problem.

constraints are so restrictive that no feasible solution is possible. These are called *infeasible problems*. To illustrate them, consider the following:

Minimize

$$f(\mathbf{x}) = x_1 + 2x_2 \tag{e}$$

subject to

$$3x_1 + 2x_2 \le 6, \quad 2x_1 + 3x_2 \ge 12, \quad x_1,\ x_2 \le 5,\ \ x_1,\ x_2 \ge 0 \tag{f}$$

Constraints for the problem are plotted in Figure 3.9 and their infeasible side is shaded-out. It is evident that there is no region within the design space that satisfies all constraints; that is, there is no feasible region for the problem. Thus, the problem is infeasible. Basically, the first two constraints impose conflicting requirements. The first requires the feasible design to be below the line A–G, whereas the second requires it to be above the line C–F. Since the two lines do not intersect in the first quadrant, the problem has no feasible region.

3.7 GRAPHICAL SOLUTION FOR THE MINIMUM-WEIGHT TUBULAR COLUMN

The design problem formulated in Section 2.7 will now be solved by the graphical method using the following data: $P = 10$ MN, $E = 207$ GPa, $\rho = 7833\ \text{kg/m}^3$, $l = 5.0$ m, and $\sigma_a = 248$ MPa. Using these data, formulation 1 for the problem is defined as "Find mean radius R (m) and thickness t (m) to minimize the mass function":

$$f(R,t) = 2\rho l \pi R t = 2(7833)(5)\pi R t = 2.4608 \times 10^5 Rt,\ \text{kg} \tag{a}$$

subject to the four inequality constraints

$$g_1(R,t) = \frac{P}{2\pi Rt} - \sigma_a = \frac{10 \times 10^6}{2\pi Rt} - 248 \times 10^6 \le 0 \quad \text{(stress constraint)} \tag{b}$$

$$g_2(R,t) = P - \frac{\pi^3 ER^3 t}{4l^2} = 10 \times 10^6 - \frac{\pi^3(207 \times 10^9)R^3 t}{4(5)(5)} \le 0 \quad \text{(buckling load constraint)} \tag{c}$$

$$g_3(R,t) = -R \le 0 \tag{d}$$

$$g_4(R,t) = -t \le 0 \tag{e}$$

Note that the explicit bound constraints discussed in Section 2.7 are simply replaced by the non-negativity constraints g_3 and g_4. The constraints for the problem are plotted in Figure 3.10, and the feasible region is indicated. Cost function contours for $f = 1000$ kg, 1500 kg, and 1579 kg are also shown. In this example the cost function contours run parallel to the stress constraint g_1. Since g_1 is active at the optimum, the problem has infinite optimum designs, that is, the entire curve A–B in Figure 3.10. We can read the coordinates of any point on the curve A–B as an optimum solution. In particular, point A, where constraints g_1 and g_2 intersect, is also an optimum point where $R^* = 0.1575$ m and $t^* = 0.0405$ m.

The superscript "*" on a variable indicates its optimum value, a notation that will be used throughout this text.

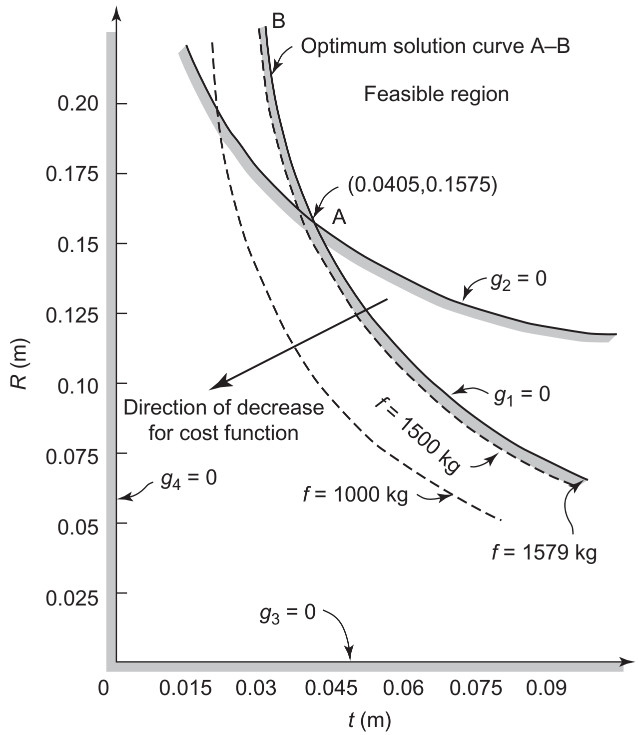

FIGURE 3.10 A graphical solution to the problem of designing a minimum-weight tubular column.

3.8 GRAPHICAL SOLUTION FOR A BEAM DESIGN PROBLEM

STEP 1: PROJECT/PROBLEM DESCRIPTION A beam of rectangular cross-section is subjected to a bending moment M ($N \cdot m$) and a maximum shear force V (N). The bending stress in the beam is calculated as $\sigma = 6M/bd^2$ (Pa), and average shear stress is calculated as $\tau = 3V/2bd$ (Pa), where b is the width and d is the depth of the beam. The allowable stresses in bending and shear are 10 MPa and 2 MPa, respectively. It is also desirable that the depth of the beam not exceed twice its width and that the cross-sectional area of the beam be minimized. In this section, we formulate and solve the problem using the graphical method.

STEP 2: DATA AND INFORMATION COLLECTION Let bending moment $M = 40\ kN \cdot m$ and the shear force $V = 150$ kN. All other data and necessary equations are given in the project statement. We shall formulate the problem using a consistent set of units, N and mm.

STEP 3: DEFINITION OF DESIGN VARIABLES The two design variables are

d = depth of beam, mm
b = width of beam, mm

STEP 4: OPTIMIZATION CRITERION The cost function for the problem is the cross-sectional area, which is expressed as

$$f(b, d) = bd \tag{a}$$

STEP 5: FORMULATION OF CONSTRAINTS Constraints for the problem consist of bending stress, shear stress, and depth-to-width ratio. Bending and shear stresses are calculated as

$$\sigma = \frac{6M}{bd^2} = \frac{6(40)(1000)(1000)}{bd^2}, \ \ N/mm^2 \tag{b}$$

$$\tau = \frac{3V}{2bd} = \frac{3(150)(1000)}{2bd}, \ \ N/mm^2 \tag{c}$$

Allowable bending stress σ_a and allowable shear stress τ_a are given as

$$\sigma_a = 10 \ \ MPa = 10 \times 10^6 \ \ N/m^2 = 10 \ N/mm^2 \tag{d}$$

$$\tau_a = 2 \ \ MPa = 2 \times 10^6 \ \ N/m^2 = 2 \ N/mm^2 \tag{e}$$

Using Eqs. (b) through (e), we obtain the bending and shear stress constraints as

$$g_1 = \frac{6(40)(1000)(1000)}{bd^2} - 10 \le 0 \ \ \text{(bending stress)} \tag{f}$$

$$g_2 = \frac{3(150)(1000)}{2bd} - 2 \le 0 \ \ \text{(shear stress)} \tag{g}$$

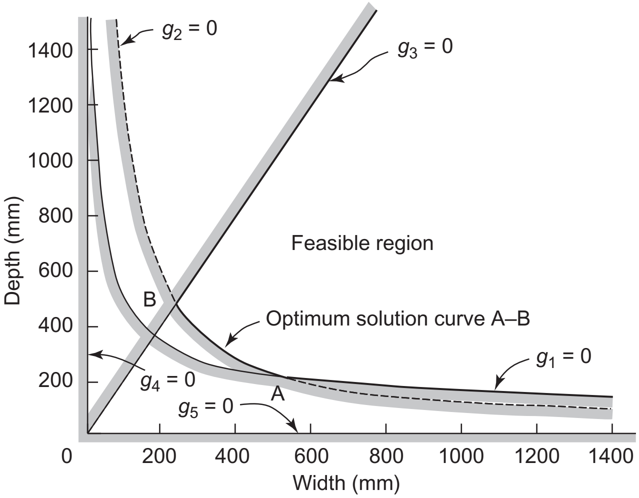

FIGURE 3.11 Graphical solution to the minimum-area beam design problem.

The constraint that requires that the depth be no more than twice the width can be expressed as

$$g_3 = d - 2b \leq 0 \tag{h}$$

Finally, both design variables should be non-negative:

$$g_4 = -b \leq 0; \quad g_5 = -d \leq 0 \tag{i}$$

In reality, b and d cannot both have zero value, so we should use some minimum value as a lower bound on them (i.e., $b \geq b_{min}$ and $d \geq d_{min}$)

Graphical Solution

Using MATLAB, the constraints for the problem are plotted in Figure 3.11, and the feasible region is identified. Note that the cost function is parallel to the constraint g_2 (both functions have the same form: bd = constant). Therefore, any point along the curve A–B represents an optimum solution, so there are infinite optimum designs. This is a desirable situation since a wide choice of optimum solutions is available to meet a designer's needs.

The optimum cross-sectional area is 112,500 mm^2. Point B corresponds to an optimum design of $b = 237$ mm and $d = 474$ mm. Point A corresponds to $b = 527.3$ mm and $d = 213.3$ mm. These points represent the two extreme optimum solutions; all other solutions lie between these two points on the curve A–B.

EXERCISES FOR CHAPTER 3

Solve the following problems using the graphical method.

3.1 Minimize $f(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 3)^2$
subject to $x_1 + x_2 \leq 4$
$x_1, x_2 \geq 0$

3.2 Maximize $F(x_1, x_2) = x_1 + 2x_2$
subject to $2x_1 + x_2 \leq 4$
$x_1, x_2 \geq 0$

3.3 Minimize $f(x_1, x_2) = x_1 + 3x_2$
subject to $x_1 + 4x_2 \geq 48$
$5x_1 + x_2 \geq 50$
$x_1, x_2 \geq 0$

3.4 Maximize $F(x_1, x_2) = x_1 + x_2 + 2x_3$
subject to $1 \leq x_1 \leq 4$
$3x_2 - 2x_3 = 6$
$-1 \leq x_3 \leq 2$
$x_2 \geq 0$

3.5 Maximize $F(x_1, x_2) = 4x_1x_2$
subject to $x_1 + x_2 \leq 20$
$x_2 - x_1 \leq 10$
$x_1, x_2 \geq 0$

3.6 Minimize $f(x_1, x_2) = 5x_1 + 10x_2$
subject to $10x_1 + 5x_2 \leq 50$
$5x_1 - 5x_2 \geq -20$
$x_1, x_2 \geq 0$

3.7 Minimize $f(x_1, x_2) = 3x_1 + x_2$
subject to $2x_1 + 4x_2 \leq 21$
$5x_1 + 3x_2 \leq 18$
$x_1, x_2 \geq 0$

3.8 Minimize $f(x_1, x_2) = x_1^2 - 2x_2^2 - 4x_1$
subject to $x_1 + x_2 \leq 6$
$x_2 \leq 3$
$x_1, x_2 \geq 0$

3.9 Minimize $f(x_1, x_2) = x_1x_2$
subject to $x_1 + x_2^2 \leq 0$
$x_1^2 + x_2^2 \leq 9$

3.10 Minimize $f(x_1, x_2) = 3x_1 + 6x_2$
subject to $-3x_1 + 3x_2 \leq 2$
$4x_1 + 2x_2 \leq 4$
$-x_1 + 3x_2 \geq 1$

Develop an appropriate graphical representation for the following problems and determine the minimum and the maximum points for the objective function.

3.11 $f(x, y) = 2x^2 + y^2 - 2xy - 3x - 2y$
subject to $y - x \leq 0$
$x^2 + y^2 - 1 = 0$

3.12 $f(x, y) = 4x^2 + 3y^2 - 5xy - 8x$
subject to $x + y = 4$

3.13 $f(x, y) = 9x^2 + 13y^2 + 18xy - 4$
subject to $x^2 + y^2 + 2x = 16$

3.14 $f(x, y) = 2x + 3y - x^3 - 2y^2$
subject to $x + 3y \leq 6$
$5x + 2y \leq 10$
$x, y \geq 0$

3.15 $f(r, t) = (r - 8)^2 + (t - 8)^2$
subject to $12 \geq r + t$
$t \leq 5$
$r, t \geq 0$

3.16 $f(x_1, x_2) = x_1^3 - 16x_1 + 2x_2 - 3x_2^2$
subject to $x_1 + x_2 \leq 3$

3.17 $f(x, y) = 9x^2 + 13y^2 + 18xy - 4$
subject to $x^2 + y^2 + 2x \geq 16$

3.18 $f(r, t) = (r - 4)^2 + (t - 4)^2$
subject to $10 - r - t \geq 0$
$5 \geq r$
$r, t \geq 0$

3.19 $f(x, y) = -x + 2y$
subject to $-x^2 + 6x + 3y \leq 27$
$18x - y^2 \geq 180$
$x, y \geq 0$

3.20 $f(x_1, x_2) = (x_1 - 4)^2 + (x_2 - 2)^2$
subject to $10 \geq x_1 + 2x_2$
$0 \leq x_1 \leq 3$
$x_2 \geq 0$

3.21 Solve the rectangular beam problem of Exercise 2.17 graphically for the following data: $M = 80$ kN • m, $V = 150$ kN, $\sigma_a = 8$ MPa, and $\tau_a = 3$ MPa.

3.22 Solve the cantilever beam problem of Exercise 2.23 graphically for the following data: $P = 10$ kN; $l = 5.0$ m; modulus of elasticity, $E = 210$ Gpa; allowable bending stress, $\sigma_a = 250$ MPa; allowable shear stress, $\tau_a = 90$ MPa; mass density, $\rho = 7850$ kg/m^3; $R_o \leq$ 20.0 cm; $R_i \leq$ 20.0 cm.

3.23 For the minimum-mass tubular column design problem formulated in Section 2.7, consider the following data: $P = 50$ kN; $l = 5.0$ m; modulus of elasticity, $E = 210$ Gpa; allowable stress, $\sigma_a = 250$ MPa; mass density $\rho = 7850$ kg/m^3.

Treating mean radius R and wall thickness t as design variables, solve the design problem graphically, imposing an additional constraint $R/t \leq 50$. This constraint is needed to avoid local crippling of the column. Also impose the member size constraints as

$$0.01 \leq R \leq 1.0 \text{ m}; \quad 5 \leq t \leq 200 \text{ mm}$$

3.24 For Exercise 3.23, treat outer radius R_o and inner radius R_i as design variables, and solve the design problem graphically. Impose the same constraints as in Exercise 3.23.

3.25 Formulate the minimum-mass column design problem of Section 2.7 using a hollow square cross-section with outside dimension w and thickness t as design variables. Solve the problem graphically using the constraints and the data given in Exercise 3.23.

3.26 Consider the symmetric (members are identical) case of the two-bar truss problem discussed in Section 2.5 with the following data: $W = 10$ kN; $\theta = 30°$; height $h = 1.0$ m; span $s = 1.5$ m; allowable stress, $\sigma_a = 250$ MPa; modulus of elasticity, $E = 210$ GPa.

Formulate the minimum-mass design problem with constraints on member stresses and bounds on design variables. Solve the problem graphically using circular tubes as members.

3.27 Formulate and solve the problem of Exercise 2.1 graphically.

3.28 In the design of the closed-end, thin-walled cylindrical pressure vessel shown in Figure E3.28, the design objective is to select the mean radius R and wall thickness t to minimize the total mass. The vessel should contain at least 25.0 m^3 of gas at an internal pressure of 3.5 MPa. It is required that the circumferential stress in the pressure vessel not exceed 210 MPa and the circumferential strain not exceed (1.0E−03). The circumferential stress and strain are calculated from the equations

$$\sigma_c = \frac{PR}{t}, \quad \varepsilon_c = \frac{PR(2-\nu)}{2Et}$$

where ρ = mass density (7850 kg/m^3), σ_c = circumferential stress (Pa), ε_c = circumferential strain, P = internal pressure (Pa), E = Young's modulus (210 GPa), and ν = Poisson's ratio (0.3). Formulate the optimum design problem, and solve it graphically.

3.29 Consider the symmetric three-bar truss design problem formulated in Section 2.10. Formulate and solve the problem graphically for the following data: $l = 1.0$ m; $P = 100$ kN; $\theta = 30°$; mass density, $\rho = 2800$ kg/m^3; modulus of elasticity, $E = 70$ GPa; allowable stress, $\sigma_a = 140$ MPa; $\Delta_u = 0.5$ cm; $\Delta_v = 0.5$ cm; $\omega_o = 50$ Hz; $\beta = 1.0$; $A_1, A_2 \geq 2$ cm^2.

3.30 Consider the cabinet design problem in Section 2.6. Use the equality constraints to eliminate three design variables from the problem. Restate the problem in terms of the remaining three variables, transcribing it into the standard form.

3.31 Graphically solve the insulated spherical tank design problem formulated in Section 2.3 for the following data: $r = 3.0$ m, $c_1 = \$10{,}000$, $c_2 = \$1000$, $c_3 = \$1$, $c_4 = \$0.1$, $\Delta T = 5$.

3.32 Solve the cylindrical tank design problem given in Section 2.8 graphically for the following data: $c = \$1500/m^2$, $V = 3000$ m^3.

3.33 Consider the minimum-mass tubular column problem formulated in Section 2.7. Find the optimum solution for it using the graphical method for the data: load, $P = 100$ kN; length, $l = 5.0$ m; Young's modulus, $E = 210$ GPa; allowable stress, $\sigma_a = 250$ MPa; mass density, $\rho = 7850$ kg/m^3; $R \leq 0.4$ m; $t \leq 0.1$ m; $R, t \geq 0$.

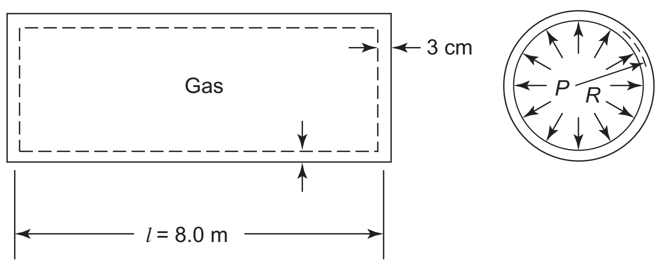

FIGURE E3.28 Graphic of a cylindrical pressure vessel.

***3.34** Design a hollow torsion rod, shown in Figure E3.34, to satisfy the following requirements (created by J. M. Trummel):

1. The calculated shear stress τ shall not exceed the allowable shear stress τ_a under the normal operating torque T_o (N • m).
2. The calculated angle of twist, θ, shall not exceed the allowable twist, θ_a (radians).
3. The member shall not buckle under a short duration torque of T_{max} (N • m).

Requirements for the rod and material properties are given in Tables E3.34 (select a material for one rod). Use the following design variables: x_1 = outside diameter of the rod; x_2 = ratio of inside/outside diameter, d_i/d_o.

Using graphical optimization, determine the inside and outside diameters for a minimum-mass rod to meet the preceding design requirements. Compare the hollow rod

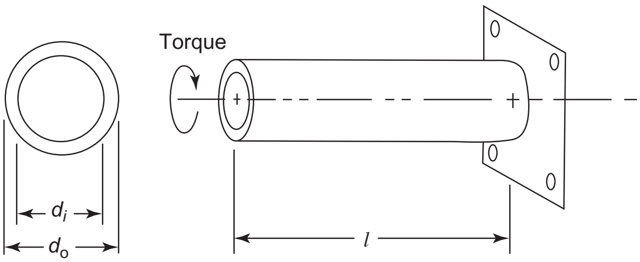

FIGURE E3.34 Graphic of a hollow torsion rod.

TABLE E3.34(a) Rod requirements

Torsion rod no.	**Length l (m)**	**Normal torque T_o (kN · m)**	**Maximum T_{max} (kN · m)**	**Allowable twist θ_a (degrees)**
1	0.50	10.0	20.0	2
2	0.75	15.0	25.0	2
3	1.00	20.0	30.0	2

TABLE E3.34(b) Materials and properties for the torsion rod

Material	**Density, ρ (kg/m^3)**	**Allowable shear stress, τ_a (MPa)**	**Elastic modulus, E (GPa)**	**Shear modulus, G (GPa)**	**Poisson ratio (ν)**
1. 4140 alloy steel	7850	275	210	80	0.30
2. Aluminum alloy 24 ST4	2750	165	75	28	0.32
3. Magnesium alloy A261	1800	90	45	16	0.35
4. Berylium	1850	110	300	147	0.02
5. Titanium	4500	165	110	42	0.30

with an equivalent solid rod ($d_i/d_o = 0$). Use a consistent set of units (e.g., Newtons and millimeters) and let the minimum and maximum values for design variables be given as

$$0.02 \le d_o \le 0.5\text{ m}, \quad 0.60 \le \frac{d_i}{d_o} \le 0.999$$

Useful expressions	
Mass	$M = \frac{\pi}{4}\rho l(d_o^2 - d_i^2)$, kg
Calculated shear stress	$\tau = \frac{c}{J}T_o$, Pa
Calculated angle of twist	$\theta = \frac{l}{GJ}T_o$, radians
Critical buckling torque	$T_{cr} = \frac{\pi d_o^3 E}{12\sqrt{2}(1-\nu^2)^{0.75}}\left(1 - \frac{d_i}{d_o}\right)^{2.5}$, N $\cdot$ m

Notation	
M	mass (kg)
d_o	outside diameter (m)
d_i	inside diameter (m)
ρ	mass density of material (kg/m^3)
l	length (m)
T_o	normal operating torque (N $\cdot$ m)
c	distance from rod axis to extreme fiber (m)
J	polar moment of inertia (m^4)
θ	angle of twist (radians)
G	modulus of rigidity (Pa)
T_{cr}	critical buckling torque (N $\cdot$ m)
E	modulus of elasticity (Pa)
ν	Poisson's ratio

***3.35** Formulate and solve Exercise 3.34 using the outside diameter d_o and the inside diameter d_i as design variables.

***3.36** Formulate and solve Exercise 3.34 using the mean radius R and wall thickness t as design variables. Let the bounds on design variables be given as $5 \le R \le 20$ cm and $0.2 \le t \le 4$ cm.

3.37 Formulate the problem in Exercise 2.3 and solve it using the graphical method.

3.38 Formulate the problem in Exercise 2.4 and solve it using the graphical method.

3.39 Solve Exercise 3.23 for a column pinned at both ends. The buckling load for such a column is given as $\pi^2 EI/l^2$. Use the graphical method.

3.40 Solve Exercise 3.23 for a column fixed at both ends. The buckling load for such a column is given as $4\pi^2 EI/l^2$. Use the graphical method.

3.41 Solve Exercise 3.23 for a column fixed at one end and pinned at the other. The buckling load for such a column is given as $2\pi^2 EI/l^2$. Use the graphical method.

3.42 Solve Exercise 3.24 for a column pinned at both ends. The buckling load for such a column is given as $\pi^2 EI/l^2$. Use the graphical method.

3.43 Solve Exercise 3.24 for a column fixed at both ends. The buckling load for such a column is given as $4\pi^2 EI/l^2$. Use the graphical method.

3.44 Solve Exercise 3.24 for a column fixed at one end and pinned at the other. The buckling load for such a column is given as $2\pi^2 EI/l^2$. Use the graphical method.

3.45 Solve the can design problem formulated in Section 2.2 using the graphical method.

3.46 Consider the two-bar truss shown in Figure 2.5. Using the given data, design a minimum-mass structure where $W = 100$ kN; $\theta = 30°$; $h = 1$ m; $s = 1.5$ m; modulus of elasticity $E = 210$ GPa; allowable stress $\sigma_a = 250$ MPa; mass density $\rho = 7850$ kg/m^3. Use Newtons and millimeters as units. The members should not fail in stress and their buckling should be avoided. Deflection at the top in either direction should not be more than 5 cm.

Use cross-sectional areas A_1 and A_2 of the two members as design variables and let the moment of inertia of the members be given as $I = A^2$. Areas must also satisfy the constraint $1 \le A_i \le 50$ cm^2.

3.47 For Exercise 3.46, use hollow circular tubes as members with mean radius R and wall thickness t as design variables. Make sure that $R/t \le 50$. Design the structure so that member 1 is symmetric with member 2. The radius and thickness must also satisfy the constraints $2 \le t \le 40$ mm and $2 \le R \le 40$ cm.

3.48 Design a symmetric structure defined in Exercise 3.46, treating cross-sectional area A and height h as design variables. The design variables must also satisfy the constraints $1 \le A \le 50$ cm^2 and $0.5 \le h \le 3$ m.

3.49 Design a symmetric structure defined in Exercise 3.46, treating cross-sectional area A and span s as design variables. The design variables must also satisfy the constraints $1 \le A \le 50$ cm^2 and $0.5 \le s \le 4$ m.

3.50 Design a minimum-mass symmetric three-bar truss (the area of member 1 and that of member 3 are the same) to support a load P, as was shown in Figure 2.9. The following notation may be used: $P_u = P \cos\theta$, $P_v = P \sin\theta$, A_1 = cross-sectional area of members 1 and 3, A_2 = cross-sectional area of member 2.

The members must not fail under the stress, and the deflection at node 4 must not exceed 2 cm in either direction. Use Newtons and millimeters as units. The data is given as $P = 50$ kN; $\theta = 30°$; mass density, $\rho = 7850$ kg/m^3; $l = 1$ m; modulus of elasticity, $E = 210$ GPa; allowable stress, $\sigma_a = 150$ MPa. The design variables must also satisfy the constraints $50 \le A_i \le 5000$ mm^2.

***3.51** **Design of a water tower support column.** As an employee of ABC Consulting Engineers, you have been asked to design a cantilever cylindrical support column of minimum mass for a new water tank. The tank itself has already been designed in the teardrop shape, shown in Figure E3.51. The height of the base of the tank (H), the diameter of the tank (D), and the wind pressure on the tank (w) are given as $H = 30$ m, $D = 10$ m, and $w = 700$ N/m^2. Formulate the design optimization problem and then solve it graphically (created by G. Baenziger).

In addition to designing for combined axial and bending stresses and buckling, several limitations have been placed on the design. The support column must have an inside diameter of at least 0.70 m (d_i) to allow for piping and ladder access to the interior

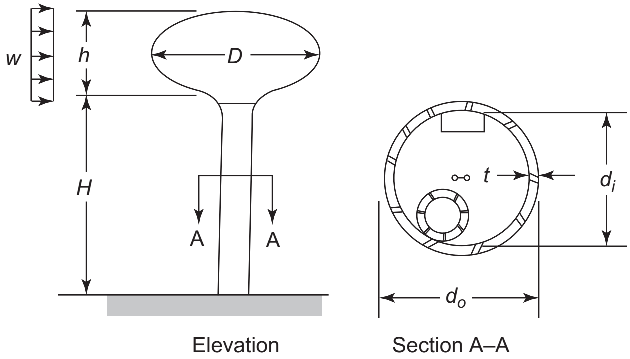

FIGURE E3.51 Graphic of a water tower support column.

of the tank. To prevent local buckling of the column walls, the diameter/thickness ratio (d_o/t) cannot be greater than 92. The large mass of water and steel makes deflections critical, as they add to the bending moment. The deflection effects, as well as an assumed construction eccentricity (e) of 10 cm, must be accounted for in the design process. Deflection at the center of gravity (C.G.) of the tank should not be greater than Δ.

Limits on the inner radius and wall thickness are $0.35 \leq R \leq 2.0$ m and $1.0 \leq t \leq 20$ cm.

Pertinent constants and formulas	
Height of water tank	$h = 10$ m
Allowable deflection	$\Delta = 20$ cm
Unit weight of water	$\gamma_w = 10\ \text{kN/m}^3$
Unit weight of steel	$\gamma_s = 80\ \text{kN/m}^3$
Modulus of elasticity	$E = 210$ GPa
Moment of inertia of the column	$I = \frac{\pi}{64}\left[d_o^4 - (d_o - 2t)^4\right]$
Cross-sectional area of column material	$A = \pi t(d_o - t)$
Allowable bending stress	$\sigma_b = 165$ MPa
Allowable axial stress	$\sigma_a = \dfrac{12\pi^2 E}{92(H/r)^2}$ (calculated using the critical buckling load with a factor of safety of 23/12)
Radius of gyration	$r = \sqrt{I/A}$
Average thickness of tank wall	$t_t = 1.5$ cm
Volume of tank	$V = 1.2\pi D^2 h$
Surface area of tank	$A_s = 1.25\pi D^2$
Projected area of tank, for wind loading	$A_p = \dfrac{2Dh}{3}$
Load on the column due to weight of water and steel tank	$P = V\gamma_w + A_s t_t \gamma_s$
Lateral load at the tank C.G. due to wind pressure	$W = wA_p$

Deflection at C.G. of tank	$\delta = \delta_1 + \delta_2$, where
	$\delta_1 = \frac{WH^2}{12EI}(4H + 3h)$
	$\delta_2 = \frac{H}{2EI}(0.5Wh + Pe)(H + h)$
Moment at base	$M = W(H + 0.5h) + (\delta + e)P$
Bending stress	$f_b = \frac{M}{2I} d_o$
Axial stress	$f_a(= P/A) = \frac{V\gamma_w + A_s\gamma_s t_t}{\pi t(d_o - t)}$
Combined stress constraint	$\frac{f_a}{\sigma_a} + \frac{f_b}{\sigma_b} \leq 1$
Gravitational acceleration	$g = 9.81\ \text{m/s}^2$

***3.52 Design of a flag pole.** Your consulting firm has been asked to design a minimum-mass flag pole of height H. The pole will be made of uniform hollow circular tubing with d_o and d_i as outer and inner diameters, respectively. The pole must not fail under the action of high winds.

For design purposes, the pole will be treated as a cantilever that is subjected to a uniform lateral wind load of w (kN/m). In addition to the uniform load, the wind induces a concentrated load of P (kN) at the top of the pole, as shown in Figure E3.52. The flag pole must not fail in bending or shear. The deflection at the top should not exceed 10 cm. The ratio of mean diameter to thickness must not exceed 60. The pertinent data are given in the table that follows. Assume any other data if needed. The minimum and maximum values of design variables are $5 \leq d_o \leq 50$ cm and $4 \leq d_i \leq 45$ cm.

Formulate the design problem and solve it using the graphical optimization technique.

Pertinent constants and equations	
Cross-sectional area	$A = \frac{\pi}{4}(d_o^2 - d_i^2)$
Moment of inertia	$I = \frac{\pi}{64}(d_o^4 - d_i^4)$
Modulus of elasticity	$E = 210$ GPa
Allowable bending stress	$\sigma_b = 165$ MPa
Allowable shear stress	$\tau_s = 50$ MPa
Mass density of pole material	$\rho = 7800\ \text{kg/m}^3$
Wind load	$w = 2.0$ kN/m
Height of flag pole	$H = 10$ m
Concentrated load at top	$P = 4.0$ kN
Moment at base	$M = (PH + 0.5wH^2)$, kN • m
Bending stress	$\sigma = \frac{M}{2I} d_o$, kPa
Shear at base	$S = (P + wH)$, kN
Shear stress	$\tau = \frac{S}{12I}(d_o^2 + d_o d_i + d_i^2)$, kPa
Deflection at top	$\delta = \frac{PH^3}{3EI} + \frac{wH^4}{8EI}$
Minimum and maximum thickness	0.5 and 2 cm

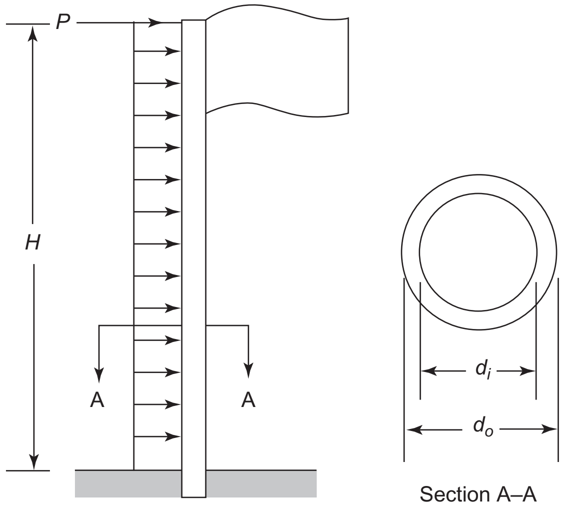

FIGURE E3.52 Flag pole.

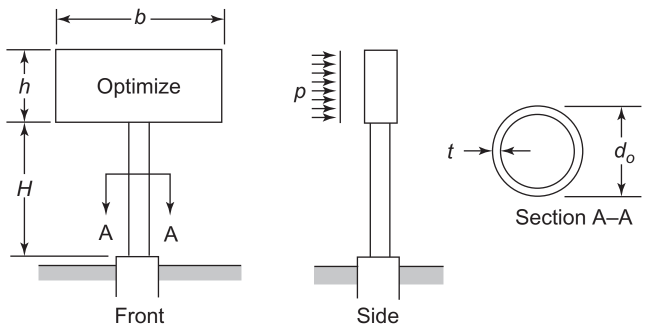

FIGURE E3.53 A sign support column.

***3.53 Design of a sign support column.** A company's design department has been asked to design a support column of minimum weight for the sign shown in Figure E3.53. The height to the bottom of the sign H, the width b, and the wind pressure p on the sign are as follows: $H = 20$ m, $b = 8$ m, $p = 800\ \text{N/m}^2$.

The sign itself weighs $2.5\ \text{kN/m}^2(w)$. The column must be safe with respect to combined axial and bending stresses. The allowable axial stress includes a factor of safety with respect to buckling. To prevent local buckling of the plate, the diameter/thickness ratio d_o/t must not exceed 92. Note that the bending stress in the column will increase as a result of the deflection of the sign under the wind load. The maximum deflection at the sign's center of gravity should not exceed 0.1 m. The minimum and maximum values of design variables are $25 \leq d_o \leq 150$ cm and $0.5 \leq t \leq 10$ cm (created by H. Kane).

Pertinent constants and equations	
Height of sign	$h = 4.0$ m
Cross-sectional area	$A = \frac{\pi}{4}\left[d_o^2 - (d_o - 2t)^2\right]$
Moment of inertia	$I = \frac{\pi}{64}(d_o^4 - (d_o - 2t)^4)$
Radius of gyration	$r = \sqrt{I/A}$
Young's modulus (aluminum alloy)	$E = 75$ GPa
Unit weight of aluminum	$\gamma = 27$ kN/m^3
Allowable bending stress	$\sigma_b = 140$ MPa
Allowable axial stress	$\sigma_a = \frac{12\pi^2 E}{92(H/r)^2}$
Wind force	$F = pbh$
Weight of sign	$W = wbh$
Deflection at center of gravity of sign	$\delta = \frac{F}{EI}\left(\frac{H^3}{3} + \frac{H^2 h}{2} + \frac{Hh^2}{4}\right)$
Bending stress in column	$f_b = \frac{M}{2I}d_o$
Axial stress	$f_a = \frac{W}{A}$
Moment at base	$M = F\left(H + \frac{h}{2}\right) + W\delta$
Combined stress requirement	$\frac{f_a}{\sigma_a} + \frac{f_b}{\sigma_b} \le 1$

***3.54 Design of a tripod.** Design a minimum mass tripod of height H to support a vertical load $W = 60$ kN. The tripod base is an equilateral triangle with sides $B = 1200$ mm. The struts have a solid circular cross-section of diameter D (Figure E3.54).

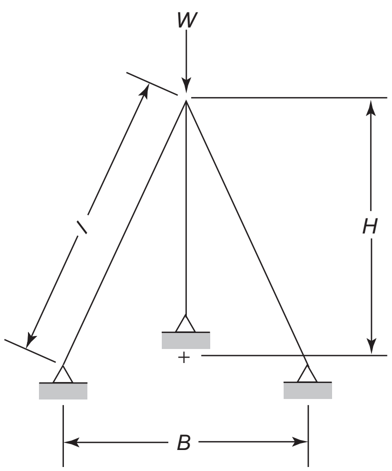

FIGURE E3.54 Tripod.

The axial stress in the struts must not exceed the allowable stress in compression, and the axial load in the strut P must not exceed the critical buckling load P_{cr} divided by a safety factor FS = 2. Use consistent units of Newtons and centimeters. The minimum and maximum values for the design variables are $0.5 \le H \le 5$ m and $0.5 \le D \le 50$ cm. Material properties and other relationships are given next:

Material	aluminum alloy 2014-T6
Allowable compressive stress	$\sigma_a = 150$ MPa
Young's modulus	$E = 75$ GPa
Mass density	$\rho = 2800\ \mathrm{kg/m^3}$
Strut length	$l = \left(H^2 + \frac{1}{3}B^2\right)^{0.5}$
Critical buckling load	$P_{cr} = \frac{\pi^2 EI}{l^2}$
Moment of inertia	$I = \frac{\pi}{64}D^4$
Strut load	$P = \frac{Wl}{3H}$