MA 110 - Ordinary Differential Equations

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Outline of the lecture

- Annihilator Method

Annihilator Operator

If A is a linear differential operator with constant coefficients and $f(x)$ is a sufficiently smooth differentiable function such that

$$A(f(x)) = 0,$$

then A is said to be the annihilator of the function $f(x)$.

Examples :

1. D^4 annihilates $1 - 5x^2 + 8x^3$. (i.e., it is a solution of DE $D^4 y = 0$.)
2. $D + 3$ annihilates e^{-3x}.
3. $(D-2)^2$ annihilates $4e^{2x} - 10xe^{2x}$.
4. $D^2 + 16$ annihilates $\cos 4x,\ \sin 4x$ or any of their linear combinations.
5. $D^2 + 2D + 5$ annihilates $5e^{-x}\cos 2x - 9e^{-x}\sin 2x$.
6. D^n annihilates $1, x, x^2, \cdots, x^{n-1}$.
7. $(D-\alpha)^n$ annihilates $e^{\alpha x},\ xe^{\alpha x}, \cdots, x^{n-1}e^{\alpha x}$.
8. $(D^2 - 2\alpha D + (\alpha^2 + \beta^2))^n$ annihilates $e^{\alpha x}\cos\beta x,\ xe^{\alpha x}\cos\beta x, \cdots$ $x^{n-1}e^{\alpha x}\cos\beta x,\ e^{\alpha x}\sin\beta x,\ xe^{\alpha x}\sin\beta x, \cdots\ x^{n-1}e^{\alpha x}\sin\beta x$.

Example

Find a particular solution of

$$y^{(4)} - 16y = x^4 + x + 1.$$

Here,

$$L = D^4 - 16,$$

and let us take

$$A = D^5.$$

Hence a solution y of $L(y) = r(x)$ is also a solution of

$$D^5(D^4 - 16)y = 0.$$

This has characteristic equation

$$m^5(m^4 - 16) = m^5(m-2)(m+2)(m^2+4) = 0.$$

Thus, a general solution of $(AL)(y) = 0$ is of the form

$$c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4 + c_6e^{2x} + c_7e^{-2x} + c_8\cos 2x + c_9\sin 2x.$$

Annihilator Method

So in order to solve

$$L(y) = x^4 + x + 1,$$

we should look for a solution of the form

$$c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4.$$

(Why?) Now,

$$(D^4 - 16)(c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4) = x^4 + x + 1$$

gives

$$24c_5 - 16(c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4) = x^4 + x + 1.$$

Comparing coefficients, get:

$$c_5 = -\frac{1}{16}, c_4 = 0, c_3 = 0, c_2 = -\frac{1}{16}, c_1 = -\frac{5}{32}.$$

Example

Solve the DE:

$$L(y) = (D^2 - 5D + 6)(y) = xe^x.$$

Characteristic polynomial for the associated homogeneous equation is

$$m^2 - 5m + 6 = (m-2)(m-3).$$

Thus, $\mathrm{Ker}\ L = \langle e^{2x}, e^{3x} \rangle$. Take,

$$A = (D-1)^2.$$

So $y_p = ae^x + bxe^x$, i.e., we need to find a, b such that

$$L(ae^x + bxe^x) = xe^x.$$

(Actually, $L(ae^x + bxe^x + ce^{2x} + de^{3x}) = xe^x$, but then $L(ce^{2x} + de^{3x}) = 0$.) This gives $(2a - 3b)e^x + 2bxe^x = xe^x$. Thus,

$$y = \frac{3}{4}e^x + \frac{1}{2}xe^x$$

is a particular solution.

Example

Find the candidate solution of

$$y^{(4)} + 2y'' + y = 3\sin x - 5\cos x.$$

The auxiliary equation is $m^4 + 2m^2 + 1 = (m^2+1)^2 = 0$, and therefore a basis of Ker L is

$$\{\cos x, \sin x, x\cos x, x\sin x\}.$$

Let $A = (D^2+1)$. The general solution of $(AL)(y) = (D^2+1)(D^2+1)(D^2+1)(y) = 0$ is of the form

$$a\cos x + b\sin x + x(c\cos x + d\sin x) + x^2(e\cos x + f\sin x).$$

So a candidate of particular solution of the non-homogeneous D.E. is $y = x^2(e\cos x + f\sin x)$.

Exercise: Get candidate solutions by the annihilator method:

1. $y^{(4)} - y^{(3)} - y'' + y' = x^2 + 4 + x \sin x$.
2. $y^{(4)} - 2y'' + y = x^2 e^x + e^{2x}$.

Annihilator Method - Formalising the method of undetermined coefficients

- Consider the set

$$\mathcal{U} = \{x^p e^{kx} \cos ax,\ x^q e^{lx} \sin bx | a, b, k, l \in \mathbb{R},\ p, q = 0, 1, 2, \ldots\}.$$

Given a polynomial P in $D = d/dx$, let

$$\mathcal{B}(P) = \{\varphi(x) \in \mathcal{U} | P\varphi(x) = 0\}.$$

- We are interested in solving the ODE $L(y) = r(x)$.
- We find a polynomial in D, say A, of the smallest order so that $Ar(x) = 0$. So we get $(AL)(y) = A(r(x)) = 0$.
- A solution y_p of $(AL)y = 0$ should be chosen to be of the form

$$y_p(x) = \sum_{\varphi \in \mathcal{B}(AL) \setminus \mathcal{B}(L)} a_\varphi \, \varphi(x)$$

and the coefficients a_φ to be found by substituting $y = y_p$ in $L(y) = r(x)$.

Let a be a complex number and $L(y) = r(x)$ be a linear differential equation. If $r(x) = x^d e^{ax}$ or even $p(x)e^{ax}$ with $p(x)$ being polynomial of degree d, still we can apply Annihilator method.
Q9(vii) (Tut-sheet 3): Solve

$$y'' + y = x\cos x + \sin x \tag{1}$$

(Here $y_p = x(Cx + D)\sin x + x(Ex + F)\cos x$. Instead of this y_p we can solve it in a simpler way.)
Sol: Note that the right hand side of (1) can be written as

$$x\cos x + \sin x = f(x) + \overline{f(x)} \tag{2}$$

where $f(x) = \dfrac{1}{2}(xe^{ix} - ie^{ix})$ and $\overline{f(x)}$ denotes the complex conjugate of $f(x)$.

So it is enough to solve individually the equations

$$y'' + y = f(x), \quad y'' + y = \overline{f(x)}$$

and add up the results.
Now assume that $Y(x)$ is a solution of

$$y'' + y = f(x), \tag{3}$$

namely $Y'' + Y = f(x)$, then $\overline{Y(x)}$ is a solution of

$$y'' + y = \overline{f(x)} \tag{4}$$

We now proceed to solve (3) in the usual manner indicating a further useful simplification as we go along.

The annihilator of $f(x)$ is $(D-i)^2$ and so

$$(D-i)^2(Y''+Y)=0$$

which is a fourth order linear constant coeff. ODE with zero right hand side. So the form of $Y(x)$ is:

$$Y(x)=Axe^{ix}+Bx^2e^{ix} \tag{5}$$

Substituting (5) into (3),

$$A(D+i)(D-i)xe^{ix}+B(D+i)(D-i)x^2e^{ix}=\frac{1}{2}(xe^{ix}-ie^{ix}). \tag{6}$$

Tutorial

PAY ATTENTION: We are first going to apply $(D - i)$ before applying $(D + i)$ as you must pause and think why do we do this. Now observe the useful facts:

$$\begin{aligned}
(D-a)(x^k e^{ax}) &= kx^{k-1}e^{ax}, \quad \text{more generally} \\
(D-a)(P(x)e^{ax}) &= P'(x)e^{ax}, \quad \text{for any polynomial } P(x). \\
(D-b)(P(x)e^{ax}) &= (D-a+a-b)(P(x)e^{ax}) \\
&= (D-a)(P(x)e^{ax}) + (a-b)P(x)e^{ax} \text{ etc.,}
\end{aligned}$$

So with these we continue to simplify left hand side of (6):

$$\begin{aligned} A(D+i)e^{ix} + B(D+i)(2xe^{ix}) &= \\ A(D-i)e^{ix} + 2iAe^{ix} + 2B(D-i)xe^{ix} + 4iBxe^{ix} &= \\ 2iAe^{ix} + 2Be^{ix} + 4iBxe^{ix}. \end{aligned}$$

Comparing with right hand side of (6) gives us immediately $A = -1/8$ and $B = -i/8$. Thus

$$Y(x) = -\frac{1}{8}(xe^{ix} + ix^2e^{ix})$$

and finally the particular solution of the original problem is

$$Y(x) + \overline{Y(x)} = -\frac{x}{4}\cos x + \frac{x^2}{4}\sin x.$$