# MA 110 - Ordinary Differential Equations


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# Outline of the lecture

1. Basic Concepts
2. Solutions of DEs

# Differential equations

## Definition

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.

## Definition

Let $y(x)$ denote a function in the variable $x$. An ordinary differential equation (ODE) is an equation containing one or more derivatives of an unknown function $y$.

In general, a differential equation involving derivative of one or more dependent variables with respect to a single independent variable is called an ODE.

## Definition

A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a partial differential equation (PDE).

Note that, the ODE may contain $y$ itself (the $0^{\text{th}}$ derivative), and known functions of $x$ (including constants). In other words, an ODE is a relation between the derivatives $y, y'$ or $\dfrac{dy}{dx}, \ldots, y^{(n)}$ or $\dfrac{d^n y}{dx^n}$ and functions of $x$:

$$\boxed{F(x, y, y', \ldots, y^{(n)}) = 0.}$$

DE's occur naturally in physics, engineering and so on.

# Examples

Further classification according to the appearance of the highest derivative in the equation is done now.

**Definition**

The order of a differential equation is the order of the highest derivative in the equation.

Examples :

1. $\dfrac{d^2y}{dx^2} + xy\left(\dfrac{dy}{dx}\right)^2 = 0$ (ODE, 2nd order)
2. $\dfrac{d^4x}{dt^4} + 5\dfrac{d^2x}{dt^2} + 3x = \sin t$ (ODE, 4th order)
3. $\dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial s} = v$ (PDE, 1st order)
4. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ (PDE, 2nd order)
5. $\dfrac{dx}{dt} = f(x,y), \ \dfrac{dy}{dt} = g(x,y), \ \ x = x(t), \ y = y(t).$ (System of ODEs, 1st order)

Linear equations - A linear ODE of order $n$ is of the form

$$\boxed{a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = b(x)}$$

where $a_0, a_1, \ldots, a_n, b$ are functions of $x$ and $a_0(x) \neq 0$.

Check list : If the dependent variable $y$ and it's derivatives occur with maximum power 1, no products of $y$ and/or its derivatives are there.

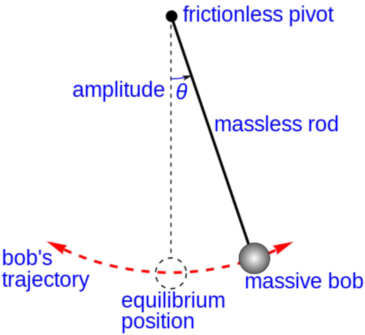
# Example : Radioactive decay

A radioactive substance decomposes at a rate proportional to the amount present. Let $y(t)$ be the amount present at time $t$. Then

$$\frac{dy}{dt} = -k \cdot y$$

where $k$ is a physical constant whose value is found by experiments ($-k$ is called the decay constant).
Linear ODE of first order.

Consider an oscillating pendulum of length $L$. Let $\theta$ be the angle it makes with the vertical direction.




$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.$$

ODE of second order. not linear - Non-linear DE.

# Example : A falling object

A body of mass $m$ falls under the force of gravity. The drag force due to air resistance is $c \cdot v^2$ where $v$ is the velocity and $c$ is a constant Then,

$$m\frac{dv}{dt} = mg - c \cdot v^2.$$

An ODE of first order. Linear or non-linear? (NL)

Examples :

1. $y'' + 5y' + 6y = 0$ - 2nd order, linear
2. $y^{(4)} + x^2 y^{(3)} + x^3 y' = xe^x$ - 4th order, linear
3. $y'' + 5(y')^3 + 6y = 0$ - 2nd order, non-linear.

# Can we solve it?

Given an equation, you would like to solve it. At least, try to solve it.

Questions:

1. What is a solution?
2. Does an equation always have a solution? If so, how many?
3. Can the solutions be expressed in a nice form? If not, how to get a feel for it?
4. How much can we proceed in a systematic manner?

order - first, second, ..., $n^{\text{th}}$, ...
linear or non-linear?

# What is a solution?

Consider $F(x, y, y', \ldots, y^{(n)}) = 0$. We assume that it is always possible to solve a differential equation for the highest derivative, obtaining

$$y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$$

and study equations of this form. This is to avoid the ambiguity which may arise because a single equation $F(x, y, y', \ldots, y^{(n)}) = 0$ may correspond to several equations of the form $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$. For example, the equation $y'^2 + xy' + 4y = 0$ leads to the two equations

$$y' = \frac{-x + \sqrt{x^2 - 16y}}{2} \text{ or } y' = \frac{-x - \sqrt{x^2 - 16y}}{2}.$$

Explicit solution is some Y = F(x), satisfying the DE.

### Definition

A explicit solution of the ODE $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$ on the interval $\alpha < x < \beta$ is a function $\phi(x)$ such that $\phi', \phi'', \cdots, \phi^{(n)}$ exist and satisfy

$$\phi^{(n)}(x) = f(x, \phi, \phi', \cdots, \phi^{(n-1)}),$$

for every $x$ in $\alpha < x < \beta$.

# Implicit solution & Formal solution

Implicit solution is the relation g(x,y) = 0, satisfying the DE.

**Definition**

A relation $g(x, y) = 0$ is called an implicit solution of $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$ if this relation defines at least one function $\phi(x)$ on an interval $\alpha < x < \beta$, such that, this function is an explicit solution of $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$ in this interval.

Examples :

1. $x^2 + y^2 - 25 = 0$ is an implicit solution of $x + yy' = 0$ in $-5 < x < 5$, because it defines two functions

$$\phi_1(x) = \sqrt{25 - x^2}, \ \phi_2(x) = -\sqrt{25 - x^2}$$

   which are solutions of the DE in the given interval. Verify!

2. Consider $x^2 + y^2 + 25 = 0 \implies x + yy' = 0 \implies y' = -\dfrac{x}{y}$. We say $x^2 + y^2 + 25 = 0$ formally satisfies $x + yy' = 0$. But it is NOT an implicit solution of DE as this relation doesn't yield $\phi$ which is an explicit solution of the DE on any real interval $I$.

# First order ODE & Initial Value Problem for first order ODE

Consider a linear first order ODE of the form $\boxed{y' + a(x)y = b(x)}$.
If $b(x) = 0$, then we say that the equation is homogeneous.

Note that the solutions of a homogeneous differential equation form a vector space under usual addition and scalar multiplication

We now consider first order ODE of the form $\boxed{F(x, y, y') = 0}$ or $\boxed{y' = f(x, y)}$.

**Definition**

Initial value problem (IVP) : A DE along with an initial condition is an IVP.

$$y' = f(x, y), \; y(x_0) = y_0.$$

# Examples

Given an amount of a radioactive substance, say 1 gm, find the amount present at any later time.

The relevant ODE is

$$\frac{dy}{dt} = -k \cdot y.$$

Initial amount given is 1 gm at time $t = 0$. i.e.,

$$y(0) = 1.$$

By inspection, $y = ce^{-kt}$, for an arbitrary constant $c$, is a solution of the above ODE. The initial condition determines $c = 1$. Hence

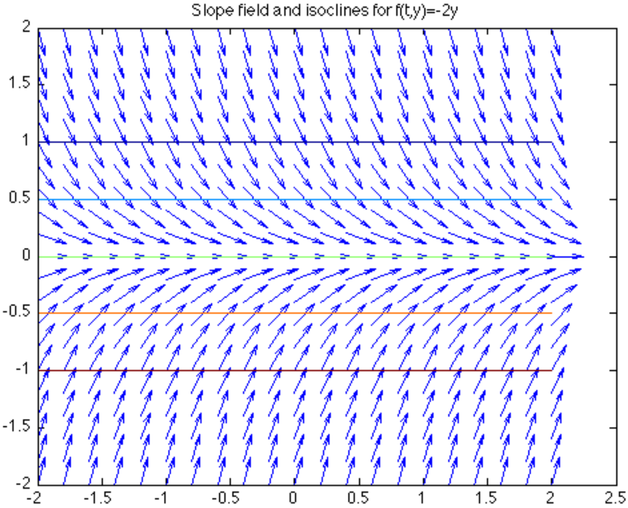
$$y = e^{-kt}$$

is a particular solution to the above ODE with the given initial condition.

# Geometrical meaning : $\dfrac{dy}{dt} = -2 \cdot y$

1. Suppose that $y$ has certain value. From the RHS of the DE, we obtain $\dfrac{dy}{dt}$. For instance, if $y = 1.5$, $\dfrac{dy}{dt} = -3$. This means that the slope of a solution $y = y(t)$ has the value $-3$ at any point where $y = 1.5$.
2. Display this information graphically in $ty$-plane by drawing short line segments or arrows of slope $-3$ at several points on $y = 1.5$.
3. Similarly proceed for other values of $y$.
4. The figures given in the next slide and the slide after two slides are examples of direction fields or slope fields.
5. An isocline (lines/curves along which solutions have the same slope) is often used to supplement the slope field. In an equation of the form $\dfrac{dy}{dt} = f(t, y)$, the isocline is a line in the $ty$-plane obtained by setting $f(t, y)$ equal to a constant.

# Slope field


Slope field and isoclines for f(t,y)=-2y
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5

## Examples

Find the curve through the point $(1, 1)$ in the $xy$-plane having at each of its points, the slope $-\frac{y}{x}$.
The relevant ODE is

$$y' = -\frac{y}{x}.$$

By inspection,

$$y = \frac{c}{x}$$

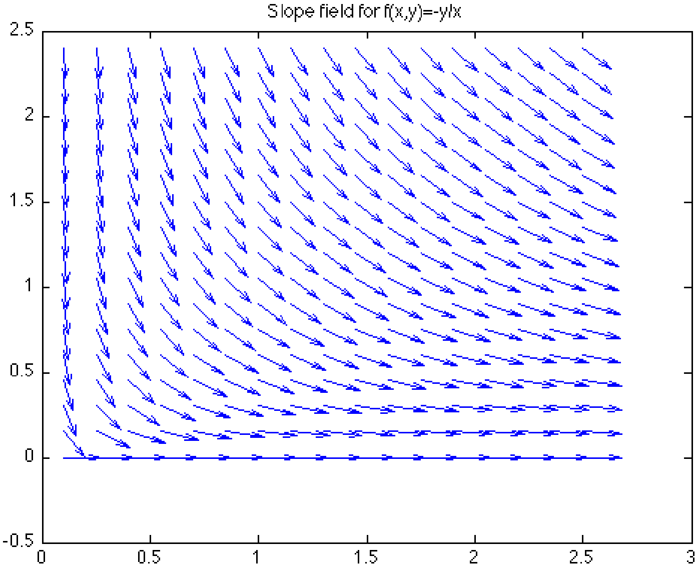
is its general solution for an arbitrary constant $c$; that is, a family of hyperbolas.
The initial condition given is

$$y(1) = 1,$$

which implies $c = 1$. Hence the particular solution for the above problem is

$$y = \frac{1}{x}.$$

# Slope field



# Examples

A first order IVP can have

1. NO solution : $|y'| + |y| = 0,\ y(0) = 3.$
2. Precisely one solution : $y' = x,\ y(0) = 1.$ What is the solution?
3. Infinitely many solutions: $xy' = y - 1,\ y(0) = 1$ The solutions are $y = 1 + cx$.

Motivation to study conditions under which the solution would exist and the conditions under which it will be unique!

We first start with a few methods for finding out the solution of first order ODEs, discuss the geometric meaning of solutions and then proceed to study existence-uniqueness results.

# Separable ODE's

An ODE of the form

$$M(x) + N(y)y' = 0$$

is called a separable ODE.
Let $H_1(x)$ and $H_2(y)$ be any functions such that $H_1'(x) = M(x)$ and $H_2'(y) = N(y)$.
Substituting in the DE, we obtain

$$H_1'(x) + H_2'(y)y' = 0.$$

Using chain rule, $\boxed{\dfrac{d}{dx}H_2(y) = H_2'(y)\dfrac{dy}{dx}}$.
Hence,

$$\frac{d}{dx}(H_1(x) + H_2(y)) = 0.$$

Integrating, $\boxed{H_1(x) + H_2(y) = c,}$ where $c$ is an arbitrary constant.
Note: This method many times gives us an implicit solution and not necessarily an explicit one!

# Separable ODE - Example 1

Solve the DE :

$$y' = -2xy.$$

Separating the variables, we get :

$$\frac{dy}{y} = -2xdx.$$

Integrating both sides, we obtain :

$$\ln|y| = -x^2 + c_1.$$

Thus, the solutions are

$$y = ce^{-x^2}.$$

How do they look?

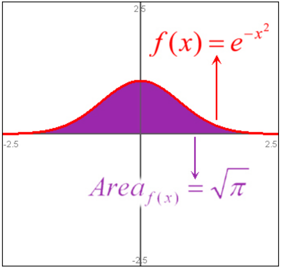
If we are given an initial condition

$$y(x_0) = y_0,$$

then we get:

$$c = y_0 e^{x_0^2}$$

and $\boxed{y = y_0 e^{x_0^2 - x^2}.}$




$$(y_0 = e^{-x_0^2})$$