Annihilator Method

Exercise: Get candidate solutions by the annihilator method:

1. $y^{(4)} - y^{(3)} - y'' + y' = x^2 + 4 + x \sin x$.
2. $y^{(4)} - 2y'' + y = x^2 e^x + e^{2x}$.

Find the candidate solution for

$$y^{(4)} - y^{(3)} - y'' + y' = x^2 + 4 + x\sin x.$$

Note that the auxiliary/ characteristic equation is

$$m^4 - m^3 - m^2 + m = m(m-1)^2(m+1).$$

Thus a basis of Ker L is $\{1, e^x, xe^x, e^{-x}\}$.
$A_1 = D^3$ is the annihilator of $r_1(x) = x^2 + 4$. Since
$A_1 L = D^4(D-1)^2(D+1)$, the candidate solution w.r.t. $r_1(x)$ is

$$x(ax^2 + bx + c),$$

since a constant is a solution of the homogeneous DE $Ly = 0$.
$A_2 = (D^2+1)^2$ is the annihilator of $r_2(x) = x\sin x$. Since
$A_2 L = (D^2+1)^2 D(D-1)^2(D+1)$, the candidate solution w.r.t. $r_2(x)$ is

$$(\alpha x + \beta)\cos x + (\gamma x + \delta)\sin x.$$

So our final candidate is

$$x(ax^2 + bx + c) + (\alpha x + \beta)\cos x + (\gamma x + \delta)\sin x.$$

Review Question - recap

Find the candidate solution for

$$y^{(4)} - 2y'' + y = x^2 e^x + e^{2x}.$$

Note that the AE is

$$m^4 - 2m^2 + 1 = (m-1)^2(m+1)^2.$$

So a basis of Ker L is $\{e^x, xe^x, e^{-x}, xe^{-x}\}$.
$A_1 = D - 2$ is the annihilator of $r_1(x) = e^{2x}$. Since
$A_1 L = (D-2)(D-1)^2(D+1)^2$, the candidate solution w.r.t. $r_1(x)$ is

$$ae^{2x}.$$

$A_2 = (D-1)^3$ is the annihilator of $r_2(x) = x^2 e^x$. Since
$A_2 L = (D-1)^5(D+1)^2$, the candidate solution w.r.t. $r_2(x)$ is

$$x^2(bx^2 + cx + d)e^x.$$

So final candidate would be

$$ae^{2x} + x^2(bx^2 + cx + d)e^x.$$

Example 6

Find a particular solution of

$$Ly = y'' - 3y' - 4y = -8e^t \cos 2t.$$

Roots of the characteristic equation are $-1, 4$. So the form of y_p is $y_p = Ae^t \cos 2t + Be^t \sin 2t$. We will use complex functions. Let $h(t) = -4e^{(1+i2)t}$. Then $-8e^t \cos 2t = h(t) + \overline{h(t)}$. $D - (1+i2)$ annihilates $h(t)$. The basis for Ker L is $\{e^{-t}, e^{4t}\}$ and the basis for Ker $(D-(1+i2))L$ is $\{e^{-t}, e^{4t}, e^{(1+2i)t}\}$. Hence the form of the particular solution to $Ly = h$ is $Y(t) = Ae^{(1+2i)t}$.

$$Y'' - 3Y' - 4Y \ = (-3+4i)Y - 3(1+2i)Y - 4Y$$

$$\Rightarrow -2A(5+i)e^{(1+2i)t} = -4e^{(1+2i)t} \Rightarrow A = \frac{5}{13} - i\frac{1}{13}.$$

Our particular solution is $y_p = Y + \bar{Y} = 2Re(Y)$. Hence

$$y_p(t) = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t.$$

- $(xD)^k := (xD)^{k-1}(xD), k \geq 2.$
- $(xD)^2 y = (xD)(xDy) = xDy + x^2D^2y \Rightarrow \; x^2D^2 = (xD)^2 - xD = xD(xD-1).$
- Observe that $(xD)^k(x^m) = m^k x^m, k = 1, 2, \cdots.$ Hence for any polynomial $P(x)$, $P(xD)(x^m) = P(m)x^m$.
- If α, β are the roots of the auxilliary equation $m(m-1) + am + b = 0$, then

$$(xD-\alpha)(xD-\beta)y = x^2D^2y + axDy + by \; = \; (xD-\beta)(xD-\alpha)y.$$

- $(xD-a)(x^a) = 0, (xD-a)(x^a \ln x) = x^a,$
  $(xD-a)^2(x^a \ln x) = 0$ for $a \in \mathbb{R}$.

Cauchy Euler equation-Example 2

Solve $Ly = x^2y'' + 4xy' + 2y = 2\ln x, x > 0$

$$
\begin{aligned}
Ly &= (x^2D^2 + 4xD + 2)y \\
&= (xD(xD-1) + 4xD + 2)y \;=\; ((xD)^2 + 3(xD) + 2)y \\
&= (xD+1)(xD+2)y.
\end{aligned}
$$

A basis of Ker L is $\{\frac{1}{x}, \frac{1}{x^2}\}$. Also, $(xD)^2$ annihilates $\ln x$ and a basis for Ker $(xD)^2L$ is $\{\frac{1}{x}, \frac{1}{x^2}, 1, \ln x\}$. The form of the particular solution is $y_p = A + B\ln x$. Substituting y_p is the DE, we get

$$(x^2D^2 + 4xD + 2)(A + B\ln x) = 2\ln x,$$

i.e., $((xD)^2 + 3(xD) + 2)(A + B\ln x) = 2\ln x$
i.e., $3B + 2A + 2B\ln x = 2\ln x$.
So, $A = -\frac{3}{2}, B = 1$. Therefore, $y_p(x) = -\frac{3}{2} + \ln x$.

Solve $Ly = x^3y''' + 4x^2y'' + xy' - y = x, x > 0$
Auxilliary equation is $m^3 + m^2 - m - 1 = (m-1)(m+1)^2 = 0$.
A basis for Ker L is $\{x, \frac{1}{x}, \frac{\ln x}{x}\}$. Also, $(xD-1)$ annihilates x. So a basis for Ker $(xD-1)L$ is $\{\frac{1}{x}, \frac{\ln x}{x}, x, x\ln x\}$. The form of the particular solution is $y_p(x) = Ax\ln x$. Substituting y_p is the DE, we get

$$(xD-1)(xD+1)^2(Ax\ln x) = x$$

i.e., $(xD+1)^2(Ax) = x$,
i.e., $(xD+1)(2Ax) = x$, i.e.,$4Ax = x$.
So, $A = \frac{1}{4}$. Therefore, $y_p(x) = \frac{1}{4}\ln x, x > 0$.