MA-110 Linear Algebra and Differential Equations

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Linear Independence: Definition

The vectors $v_1, v_2, \ldots, v_n$ in a vector space V, are *linearly independent*

if $\boxed{a_1 v_1 + \cdots + a_n v_n = 0 \Rightarrow a_1 = 0, \ \ldots, \ a_n = 0}$.

Equivalently, for every nonezero $(a_1, \ldots, a_n)^T$ in $\mathbb{R}^n$,

we have $a_1 v_1 + \cdots + a_n v_n \neq 0$ in V.

The vectors $v_1, \ldots, v_n$ are *linearly dependent* if they are not linearly independent. i.e., we can find $(a_1, \ldots, a_n)^T \neq 0$ in $\mathbb{R}^n$, such that $a_1 v_1 + \cdots + a_n v_n = 0$ in V.

Observe: When $V = \mathbb{R}^m$, if $A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$, then

$\boxed{Ax = x_1 v_1 + \cdots + x_n v_n = 0 \text{ has a } \textbf{non-trivial} \text{ solution,}}$

$\Leftrightarrow N(A) \neq 0 \Leftrightarrow v_1, \ldots, v_n$ are linearly **dependent** and

$\boxed{Ax = x_1 v_1 + \cdots + x_n v_n = 0 \text{ has only the } \textbf{trivial} \text{ solution}}$

$\Leftrightarrow N(A) = 0 \Leftrightarrow v_1, \ldots, v_n$ are linearly **independent**.

Linear Independence: Remarks

Remarks/Examples:

1. The zero vector 0 is not linearly independent. Why?
2. If $v \neq 0$, then it is linearly independent. Why?
3. v, w are not linearly independent $\iff$ one is a multiple of the other $\iff$ (for $V = \mathbb{R}^m$) they lie on the same line through the origin.
4. More generally, $v_1, \ldots, v_n$ are not linearly independent $\iff$ one of the v_i's can be written as a linear combination of the others, i.e., v_i is in $\mathrm{Span}\{v_j : j = 1, \ldots n, j \neq i\}$.
5. Let A be $m \times n$. Then $\mathrm{rank}(A) = n \iff N(A) = 0$ $\iff A_{*1}, \cdots, A_{*n}$ are linearly independent.

   In particular, if A is $n \times n$, A is invertible $\iff A_{*1}, \cdots, A_{*n}$ are linearly independent.

Example: $e_1, \ldots, e_n$ are linearly independent vectors in $\mathbb{R}^n$.

Linear Independence: Example

Example: Are the vectors $v_1 = \begin{pmatrix} 2 & 2 & 2 \end{pmatrix}^T$, $v_2 = \begin{pmatrix} 4 & 5 & 3 \end{pmatrix}^T$, $v_3 = \begin{pmatrix} 6 & 7 & 5 \end{pmatrix}^T$ and $v_4 = \begin{pmatrix} 4 & 6 & 2 \end{pmatrix}^T$ linearly independent?

For $A = \begin{pmatrix} v_1 & \cdots & v_4 \end{pmatrix}$, reduced form $R = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

A has only 2 pivots $\Rightarrow N(A) \neq 0$, so v_1, v_2, v_3, v_4 are not independent. A non-trivial linear combination which is zero is $(1)v_1 + (1)v_2 + (-1)v_3 + (0)v_4$, or $(2)v_1 + (-2)v_2 + (0)v_3 + (1)v_4$.

- More generally, if $v_1, \ldots, v_n$ are vectors in $\mathbb{R}^m$, then

$$A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \text{ is } m \times n.$$

If $m < n$, then $\mathrm{rank}(A) < n \Rightarrow N(A) \neq 0$. Thus

In $\mathbb{R}^m$, any set with more than m vectors is linearly dependent.

Important point

Summary: Vector Spaces, Span and Independence

- **Vector space**: A triple $(V, +, *)$ which is closed under $+$ and $*$ with some additional properties satisfied by $+$ and $*$.
- **Subspace:** A non-empty subset W of V closed under linear combinations.

Let $V = \mathbb{R}^m$, $v_1, \dots, v_n$ be in V, and $A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$.

- For v in V, v is in $\text{Span}\{v_1, \dots, v_n\}$
$$\Leftrightarrow Ax = v \text{ is consistent}$$
- $v_1, \dots, v_n$ are linearly independent $\Leftrightarrow N(A) = 0$
$$\Leftrightarrow \text{rank}(A) = n.$$
- In particular, with $n = m$, A is invertible
$$\Leftrightarrow Ax = v \text{ is consistent for every } v$$
$$\Leftrightarrow \text{Span}\{v_1, \dots, v_n\} = \mathbb{R}^n \Leftrightarrow \text{rank}(A) = n$$
$$\Leftrightarrow N(A) = 0 \Leftrightarrow v_1, \dots, v_n \text{ are linearly independent.}$$
- If $\text{Span}\{v_1, \dots, v_n\} = \mathbb{R}^m$, then $m \leq n$, and any subset of $\mathbb{R}^m$ with more than m vectors is dependent.

Basis: Introduction

Let $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$, $v_3 = \begin{pmatrix} 3 \\ 8 \\ 7 \end{pmatrix}$, $v_4 = \begin{pmatrix} 5 \\ 12 \\ 13 \end{pmatrix}$, and $A = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix}$. Can $C(A) =$Span$\{v_1, v_2, v_3, v_4\}$ be spanned by less than 4 vectors?

Observe:

- $v_2 = 2v_1$ & $v_4 = 2v_1 + v_3 \Rightarrow C(A) = \text{Span}\{v_1, v_3\}$.

Moreover, the span of only v_1 or only v_3 is a line.

Thus, $\{v_1, v_3\}$ is a *minimal spanning set* for $C(A)$.

- Clearly v_1 is not on the line spanned by v_3 or vice versa.

Hence, v_1 and v_3 are linearly independent vectors in $C(A)$. Moreover, if v is in $C(A) = \text{Span}\{v_1, v_3\}$, then v_1, v_3, v are linearly dependent. Why?

Thus, $\{v_1, v_3\}$ is a *maximal linearly independent set* in $C(A)$.

Any such set of vectors gives a *basis* of $C(A)$.

Defn. A subset $\mathscr{B}$ of a vector space V, is said to be a *basis* of V, if it is linearly independent and $\text{Span}(\mathscr{B}) = V$.

Theorem: For any subset S of a vector space V, the following are equivalent:

(i) S is a maximal linearly independent set in V
(ii) S is linearly independent and $\text{Span}(S) = V$.
(iii) S is a minimal spanning set of V.

Remark/Examples:

- Every vector space V has a basis.
- By convention, the empty set is a basis for $V = \{0\}$.
- $\{e_1, \ldots, e_n\}$ is a basis for $\mathbb{R}^n$, called the *standard basis*.
- A basis of $\mathbb{R}$ is just $\{1\}$. Is this unique?
- $\left\{\begin{pmatrix}-1 & 1\end{pmatrix}^T, \begin{pmatrix}0 & 1\end{pmatrix}^T\right\}$ is a basis for $\mathbb{R}^2$. So is $\{e_1, e_2\}$, as is the set consisting of columns of a 2×2 invertible matrix.
- Find a basis in all the examples seen so far.

Coordinate Vector: Definition

• Let $\mathscr{B} = \{v_1, \ldots, v_n\}$ be a basis for V and v a vector in V. $\text{Span}(\mathscr{B}) = V \Rightarrow v = a_1 v_1 + \cdots + a_n v_n$ for scalars $a_1, \ldots, a_n$. Linear independence $\Rightarrow$ this expression for v is unique. Thus

Every $v \in V$ can be *uniquely* written

as a linear combination of $\{v_1, \ldots, v_n\}$.

Exercise: Prove this!

Definition: If $v = a_1 v_1 + \cdots + a_n v_n$, then $(a_1, \ldots, a_n)^T \in \mathbb{R}^n$ is called the *coordinate vector* of v w.r.t. $\mathscr{B}$, denoted $[v]_{\mathscr{B}}$.

Note: $[v]_{\mathscr{B}}$ depends not only on the basis $\mathscr{B}$, but also the order of the elements in $\mathscr{B}$.

Question:

How does $[v]_{\mathscr{B}}$ change, if $\mathscr{B}$ is rewritten as $\{v_2, v_1, v_3, \ldots, v_n\}$?