MA 110 - Ordinary Differential Equations

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Outline of the lecture

- n^{th} order
- Method of variation of parameters
- Method of undetermined coefficients

Non-homogeneous n$^{\text{th}}$ order Linear ODE's

Consider the non-homogeneous DE

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x)$$

where $p_1(x), \cdots, p_n(x), r(x)$ are continuous functions on an interval I.
Let $y_p(x)$ be any solution of

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x)$$

and $y_1(x), \cdots, y_n(x)$ be a basis of the solution space of the corresponding homogeneous DE.
Then the set of solutions of the non-homogeneous DE is

$$\{c_1y_1(x) + \cdots + c_ny_n(x) + y_p(x) \mid c_1, \cdots, c_n \in \mathbb{R}\}.$$

Summary:

In order to find the general solution of a non-homogeneous DE, we need to

- get the general solution of the corresponding homogeneous DE.
- get one particular solution of the non-homogeneous DE

General Solution of Homogeneous Equations with Constant Coefficients

Consider

$$y^{(n)} + p_1 y^{(n-1)} + \ldots + p_n y = 0$$

where $p_1, \cdots, p_n$ are in $\mathbb{R}$; that is, an n$^{\text{th}}$ order homogeneous linear ODE with constant coefficients.

Suppose e^{mx} is a solution of this equation. Then,

$$m^n e^{mx} + p_1 m^{n-1} e^{mx} + \cdots + p_n e^{mx} = 0,$$

and this implies

$$m^n + p_1 m^{n-1} + \cdots + p_n = 0.$$

This is called the characteristic equation or auxiliary equation of the linear homogeneous ODE with constant coefficients.

This polynomial of degree n has n zeros say $m_1, \cdots, m_n$, some of which may be equal and hence the characteristic polynomial can be written in the form

$$(m - m_1)(m - m_2) \cdots (m - m_n).$$

Depending on the the nature of these zeros (real & unequal, real & equal, complex), we write down the general solution of the homogeneous DE in a similar way as we did for 2^{nd} order equations.

Example 1

Find a basis of solutions and general solution of the DE:

$$y^{(3)} - 7y' + 6y = 0.$$

1. The auxiliary equation is

$$m^3 - 7m + 6 = 0.$$

2. The roots are $1, 2, -3$.
3. Hence a basis for solutions is $\{e^x, e^{2x}, e^{-3x}\}$ Why?.
4. Thus, the general solution is of the form

$$c_1 e^x + c_2 e^{2x} + c_3 e^{-3x},$$

where $c_1, c_2, c_3 \in \mathbb{R}$.

Example 2

Find the general solution of the DE:

$$L(y) = (D^3 - D^2 - 8D + 12)(y) = 0.$$

1. The auxiliary equation is

$$m^3 - m^2 - 8m + 12 = (m-2)^2(m+3) = 0.$$

2. The roots are $2, 2, -3$.
3. Hence a basis for solutions is $\{e^{2x}, xe^{2x}, e^{-3x}\}$. Why?
4. Thus, the general solution is of the form

$$c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-3x},$$

where $c_1, c_2, c_3 \in \mathbb{R}$.

Examples 3 & 4

Find the general solution of the DE:

$$L(y) = (D^6 + 2D^5 - 2D^3 - D^2)(y) = 0.$$

The general solution is

$$c_1 + c_2 x + c_3 e^x + c_4 e^{-x} + c_5 x e^{-x} + c_6 x^2 e^{-x},$$

with $c_i \in \mathbb{R}$.

Find the general solution of

$$y^{(5)} - 9y^{(4)} + 34y^{(3)} - 66y^{(2)} + 65y' - 25y = 0.$$

The characteristic polynomial is

$$(m-1)(m^2 - 4m + 5)^2.$$

The roots are

$$1, 2 \pm \imath, 2 \pm \imath.$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{2x} \cos x + c_3 x e^{2x} \cos x + c_4 e^{2x} \sin x + c_5 x e^{2x} \sin x,$$

where $c_i \in \mathbb{R}$.

For $y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x)$, we proceed exactly the same way as $n = 2$. Let the solution of the associated homogeneous DE be $y = c_1y_1 + c_2y_2 + \ldots + c_ny_n$. Let the candidate for particular solution be

$$y = v_1y_1 + v_2y_2 + \ldots + v_ny_n$$

$$y' = v_1y_1' + \ldots + v_ny_n' + v_1'y_1 + \ldots + v_n'y_n$$

Demand

$$v_1'y_1 + \ldots + v_n'y_n = 0.$$

Take

$$y'' = v_1y_1'' + \ldots + v_ny_n'' + v_1'y_1' + \ldots + v_n'y_n'$$

and demand

$$v_1'y_1' + \ldots + v_n'y_n' = 0.$$

$$\ldots\ldots$$

Variation of Parameters

Take

$$y^{(n-1)} = v_1 y_1^{(n-1)} + \ldots + v_n y_n^{(n-1)} + v_1' y_1^{(n-2)} + \ldots + v_n' y_n^{(n-2)}$$

and demand

$$v_1' y_1^{(n-2)} + \ldots + v_n' y_n^{(n-2)} = 0.$$

$$y^{(n)} = v_1 y_1^{(n)} + \ldots + v_n y_n^{(n)} + v_1' y_1^{(n-1)} + \ldots + v_n' y_n^{(n-1)}$$

$$\begin{aligned}\text{Substituting } y, y', \ldots, y^{(n)}\\ \text{in the non-homogeneous DE, rearranging}\\ \text{and using } L(y_1) = \ldots = L(y_n) = 0\\ \text{we get } v_1' y_1^{(n-1)} + \ldots + v_n' y_n^{(n-1)} = r(x).\end{aligned}$$

Variation of Parameters

Thus,

$$\begin{bmatrix} y_1 & y_2 & \cdot & y_n \\ y_1' & y_2' & \cdot & y_n' \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdot & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ \cdot \\ v_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ r(x) \end{bmatrix}.$$

Use Cramer's rule to solve for

$$v_1', v_2', \ldots, v_n',$$

and thus get

$$v_1, v_2, \ldots, v_n,$$

and form

$$y = v_1 y_1 + v_2 y_2 + \ldots + v_n y_n.$$

Example 5

Solve

$$y^{(3)} - y^{(2)} - y^{(1)} + y = r(x).$$

Characteristic polynomial for the homogeneous equation is

$$m^3 - m^2 - m + 1 = (m-1)^2(m+1).$$

Hence, a basis of solutions is

$$\{e^x, xe^x, e^{-x}\}.$$

We need to calculate $W(x)$. Use Abel's formula:

$$W(x) = W(0)e^{-\int_0^x p_1(t)dt} = W(0) \cdot e^x.$$

Now,

$$W(x) = \begin{vmatrix} e^x & xe^x & e^{-x} \\ e^x & e^x + xe^x & -e^{-x} \\ e^x & 2e^x + xe^x & e^{-x} \end{vmatrix}.$$

Thus,

$$W(0) = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 4.$$

Hence,

$$W(x) = 4e^x.$$

Variation of Parameters

Let

$$W_1(x) = \begin{vmatrix} 0 & xe^x & e^{-x} \\ 0 & e^x + xe^x & -e^{-x} \\ r(x) & 2e^x + xe^x & e^{-x} \end{vmatrix} = -r(x)(2x+1).$$

Similarly,

$$W_2(x) = 2r(x), \;\; W_3(x) = r(x)e^{2x}.$$

Therefore,

$$y(x) = e^x \int_0^x \frac{-r(t)(2t+1)}{4e^t} dt + xe^x \int_0^x \frac{2r(t)}{4e^t} dt + e^{-x} \int_0^x \frac{r(t)e^{2t}}{4e^t} dt.$$

Undetermined Coefficients - Example 6

Find a particular solution of

$$y^{(4)} + 2y'' + y = 3\sin x - 5\cos x.$$

What's the general solution of the homogeneous equation?
The auxiliary equation is

$$m^4 + 2m^2 + 1 = (m^2+1)^2 = 0,$$

and therefore a basis of Ker L is

$$\{\cos x, \sin x, x\cos x, x\sin x\}.$$

So the candidate is

$$y = x^2(a\cos x + b\sin x).$$

Substituting, we get:

$$-8a\cos x - 8b\sin x = 3\sin x - 5\cos x.$$

Thus,

$$y(x) = x^2(\frac{5}{8}\cos x - \frac{3}{8}\sin x).$$

Example 7

Find the candidate solution for

$$y^{(4)} - y^{(3)} - y'' + y' = x^2 + 4 + x \sin x.$$

Note that the auxiliary/ characteristic equation is

$$m^4 - m^3 - m^2 + m = m(m-1)^2(m+1) = 0.$$

Thus a basis of Ker L is $\{1, e^x, xe^x, e^{-x}\}$. What's the candidate for solution?

$x^2 + 4$ on the right suggests $ax^2 + bx + c$, but we should modify this to

$$x(ax^2 + bx + c),$$

since a constant is a solution of the homogeneous equation.

$x \sin x$ suggests

$$(\alpha x + \beta) \cos x + (\gamma x + \delta) \sin x.$$

Do we need to modify this? No. So our final candidate is

$$x(ax^2 + bx + c) + (\alpha x + \beta) \cos x + (\gamma x + \delta) \sin x.$$

Example 8

Find the candidate solution for

$$y^{(4)} - 2y'' + y = x^2 e^x + e^{2x}.$$

Note that the AE is

$$m^4 - 2m^2 + 1 = (m-1)^2(m+1)^2 = 0.$$

So a basis of Ker L is $\{e^x, xe^x, e^{-x}, xe^{-x}\}$.
e^{2x} on the right suggests

$$ae^{2x},$$

and no modification required.
$x^2 e^x$ initially suggests $(bx^2 + cx + d)e^x$, but this should be modified to

$$x^2(bx^2 + cx + d)e^x.$$

So final candidate would be

$$ae^{2x} + x^2(bx^2 + cx + d)e^x.$$

Review Question 1

For the non-homogeneous equation $y'' - 5y' + 6y = x\cos x$, the form of y_p using the method of undetermined coefficients is __________.

(a) $A_1 x \ \cos x + B_1 x \ \sin x$
(b) $(A_0 + A_1 x)\cos x + (B_0 + B_1 x)\sin x$
(c) $x((A_0 + A_1 x)\cos x + (B_0 + B_1 x)\sin x)$
(d) None of the above.

Solution : ✔ (b)

Review Question 2

Solve the IVP : $y'' + (1 + \frac{1}{y})(y')^2 = 0, \; y(0) = 1, \; y'(0) = 1/e.$

- Put $y' = v$. So $y'' = \frac{dv}{dx} = \frac{dy}{dx}\frac{dv}{dy} = v\frac{dv}{dy}$
- The DE becomes $v\frac{dv}{dy} + (1 + \frac{1}{y})v^2 = 0.$
- Solve for v : $v(y) = Ce^{-y}y^{-1}.$
- Use initial conditions to obtain $C = 1.$
- Thus $ye^y v = 1$. Solve to obtain $ye^y - e^y = x + C_1.$
- Use $y(0) = 1$ to obtain $C_1 = 0.$

Hence, $\boxed{(y-1)e^y = x}$.

If the Wronskian $W(y_1, y_2)(x) = x^2 e^x,\ x > 0$ and $y_1(x) = x$, then y_2 is ——.

Solution : ✔ xe^x.

Solve using the method of variation of parameters :

$$(x^2 + 1)(y'' - 2y + 1) = e^x, \qquad y(0) = y'(0) = 1.$$