MA 110 - Ordinary Differential Equations

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April 18, 2024

Outline of the lecture

- Recall : Definition & Properties of Laplace transforms
- Gamma function
- Partial Fractions

Variable coefficients - an example

Compute the Laplace transform of a solution of

$$ty'' + y' + ty = 0,\ t > 0,\ \ y(0) = k,\ Y(1) = 1/\sqrt{2}.$$

$$L(ty'' + y' + ty) = 0$$

$$-\frac{d}{ds}L(y'') + (sL(y) - y(0)) - \frac{d}{ds}(L(y)) = 0$$

$$-\frac{d}{ds}(s^2L(y) - sy(0) - y'(0)) + (sL(y) - y(0)) - \frac{d}{ds}(L(y)) = 0$$

.... Derive

$$(s^2+1)Y'(s) + sY(s) = 0 \Longrightarrow Y(s) = \frac{C}{\sqrt{s^2+1}}$$

$$Y(1) = 1/\sqrt{2} \Longrightarrow Y(s) = \frac{1}{\sqrt{s^2+1}}.$$

Properties

1.	Linearity	$L(af(t)+bg(t)) = aL(f(t)) + bL(g(t))$
2.	I Shifting theorem	$L(e^{at}f(t)) = F(s-a)$
3.	Scaling	$L(f(ct)) = \frac{1}{c}F\left(\frac{s}{c}\right), \; c > 0$
4.	Laplace transform of derivative	$L(f') = sL(f) - f(0)$ $L(f'') = s^2L(f) - sf(0) - f'(0)$
5.	L.T. of integral	$L\left(\int_0^t f(\tau)\,d\tau\right) = \frac{F(s)}{s}, \qquad \text{for } s > \alpha.$
6.	Dervative of L.T.	$F'(s) = -L(tf(t))$ $L(t^n f(t)) = (-1)^n F^{(n)}(s)$
7.	Integral of L.T.	$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\tilde{s})\,d\tilde{s}, \;\; s > \alpha.$
8.	II shifting theorem	$L(u_c(t)f(t-c)) = e^{-cs}F(s)$
9.	Convolution & L.T.	$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$ $L(f*g) = L(f)\cdot L(g)$
10.	L.T. of Periodic function	$L(f) = \frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$

Gamma Function

Let us now introduce the gamma function.
$\Gamma : (0, \infty) \to \mathbb{R}$ is defined by

$$\Gamma(y) = \int_0^\infty e^{-x} x^{y-1} dx.$$

Now we show that the right hand side integral converges.
Write it as

$$\int_0^1 e^{-x} x^{y-1} dx + \int_1^\infty e^{-x} x^{y-1} dx,$$

and we need to check that both these integrals do converge.
These integrals can be shown to converge by using comparison tests.

The gamma function satisfies a nice functional equation:

$$\Gamma(y+1) = y\Gamma(y).$$

Proof: Let $0 < a < b$. Use integration by parts to see:

$$\begin{aligned} \int_a^b e^{-x} x^y \, dx &= [-x^y e^{-x}]_a^b + y \int_a^b e^{-x} x^{y-1} \, dx \\ &= a^y e^{-a} - b^y e^{-b} + y \int_a^b e^{-x} x^{y-1} \, dx. \end{aligned}$$

Take limit as $b \to \infty$ and $a \to 0^+$ to get the functional equation. In particular,

$$\Gamma(n+1) = n!.$$

Thus, the gamma function interpolates the factorial function.

Exercise : Gamma Function

1. Prove that

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Hint: Let $I =$ lhs. Compute I^2 as a double integral by changing to polar coordinates.

2. Find $\Gamma(\frac{1}{2}), \Gamma(\frac{3}{2}), \ldots$.

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-x} x^{-\frac{1}{2}} dx.$$

Put $x = t^2$. Thus,

$$\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-t^2} dt = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

Now,

$$\Gamma(\frac{3}{2}) = \frac{1}{2} \cdot \Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}.$$

Determine $L(t^p), p > -1$.

$$L(t^p) = \int_0^\infty e^{-st} t^p dt.$$

Put $x = st$. Thus, $dt = \dfrac{dx}{s}$. Thus,

$$\begin{aligned} L(t^p)(s) &= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^p \cdot \frac{dx}{s} \\ &= \frac{1}{s^{p+1}} \int_0^\infty e^{-x} x^p dx \\ &= \frac{\Gamma(p+1)}{s^{p+1}}, \end{aligned}$$

where $s > 0$. Hence $L(t^n) = \dfrac{n!}{s^{n+1}}$, $n = 0, 1, \ldots$. For $p = \pm\dfrac{1}{2}$, we get, for $s > 0$,

$$L(t^{-1/2}) = \frac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\frac{\pi}{s}}, \; L(t^{1/2}) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}.$$

Suppose f is piecewise continuous on all $[0, \alpha]$ and is of exponential order. Then we claim that there exists $K > 0$ such that

$$|f(t)| \leq Ke^{at},$$

for all $t > 0$ and for some $a \in \mathbb{R}$.

Proof: Exponential order implies there is $K_1 > 0$ such that

$$|f(t)| \leq K_1 e^{at},$$

for all $t \geq M > 0$ and for some $a \in \mathbb{R}$. Piecewise continuity implies that

$$|f(t)| \leq K_2,$$

on $[0, M]$. Thus, on $[0, M]$,

$$|f(t)| \leq K_3 e^{at},$$

for some $K_3 > 0$. (Why?) Choose $K = \max(K_1, K_3)$ to obtain the result.

Thus,

$$|L(f)(s)| \leq \int_0^\infty |e^{-st} f(t)| dt \leq K \int_0^\infty e^{-(s-a)t} dt = \frac{K}{s-a},$$

for $s > a$.
In particular, it follows that

$$L(f)(s) \to 0,$$

as $s \to \infty$.
Remark: This limiting behaviour is true for any f such that $L(f)$ exists; i.e., even without assuming exponential order etc. Proof is tough!
Remark: Thus, $\frac{s-1}{s+1}, \frac{e^s}{s}, s^2, \frac{s}{\ln s}$ etc are not the Laplace transform of any function!

Example

Solve $y'' + ty' - 2y = 4,\ y(0) = -1,\ y'(0) = 0.$

$$L(y'') + L(ty') - 2L(y) = L(4)$$
$$(s^2L(y) - sy(0) - y'(0)) - (sL(y) - y(0))' - 2L(y) = \frac{4}{s}$$

Denoting $Y(s) = L(y)$, by simplifying the above expression and using the initial conditions, we obtain

$$Y'(s) + (\frac{3}{s} - s)Y(s) = 1 - \frac{4}{s^2}.$$

Solving this DE, we obtain :

$$Y(s) = \frac{2}{s^3} - \frac{1}{s} + \frac{c}{s^3}e^{s^2/2},$$

where c is a constant.

By using the remark in the previous slide, we have $c = 0$!

Partial Fractions

Suppose $f(x) = \dfrac{P(x)}{(x-x_1)^{k_1}(x-x_2)^{k_2}\cdots(x-x_d)^{k_n}}$ takes the form

$$f(x) = \sum_{i=1}^{n} \left(\frac{a_{i1}}{x-x_i} + \frac{a_{i2}}{(x-x_i)^2} + \cdots + \frac{a_{ik_i}}{(x-x_i)^{k_i}} \right).$$

Then,

$$a_{ij} = \frac{1}{(k_i-j)!} \lim_{x\to x_i} \frac{d^{k_i-j}}{dx^{k_i-j}} \left((x-x_i)^{k_i} f(x) \right)$$

for $j = 1, 2, \cdots, k_i$. In the special case when x_i is a simple root,

$$a_{i1} = \frac{P(x_i)}{Q'(x_i)},$$

when $f(x) = \dfrac{P(x)}{Q(x)}$.

Remark: Note that any $f(x)$ can be put into this form over $\mathbb{C}$. So we can do this over $\mathbb{C}$, and then club conjugate terms to get partial fractions over $\mathbb{R}$.

Example:

$$f(x) = \frac{x^2 - 5}{(x^2 - 1)(x^2 + 1)} = \frac{x^2 - 5}{(x+1)(x-1)(x+\imath)(x-\imath)}.$$

This can be decomposed into rational functions whose denominators are $x+1, x-1, x+\imath, x-\imath$. Note that each term is of power one. Let $x_i = -1, 1, -\imath, \imath$. Note that

$$\frac{P(x_i)}{Q'(x_i)} = \frac{x_i^2 - 5}{4x_i^3},$$

and we get $1, -1, \frac{3\imath}{2}, -\frac{3\imath}{2}$ respectively. Thus,

$$\begin{aligned} f(x) &= \frac{1}{x+1} - \frac{1}{x-1} + \frac{3\imath}{2}\frac{1}{x+\imath} - \frac{3\imath}{2}\frac{1}{x-\imath} \\ &= \frac{1}{x+1} - \frac{1}{x-1} + \frac{3}{x^2+1}. \end{aligned}$$

Example: Solve the IVP:

$$y'' - 3y' + 2y = 4t, \;\; y(0) = 1, y'(0) = -1.$$

Apply Laplace transform:

$$s^2 L(y) - sy(0) - y'(0) - 3sL(y) + 3y(0) + 2L(y) = \frac{4}{s^2}.$$

Thus,

$$L(y) = \frac{s^3 - 4s^2 + 4}{s^2(s-1)(s-2)}.$$

Need to write
$$\frac{s^3-4s^2+4}{s^2(s-1)(s-2)}$$
in partial fractions:

Coefficient of $\dfrac{1}{s-1}$ is $\dfrac{s^3-4s^2+4}{4s^3-9s^2+4s}(1) = -1.$

Coefficient of $\dfrac{1}{s-2}$ is $\dfrac{s^3-4s^2+4}{4s^3-9s^2+4s}(2) = -1.$

Coefficient of $\dfrac{1}{s}$ is $\dfrac{1}{1!}\lim_{s\to 0}\dfrac{d}{ds}\dfrac{s^3-4s^2+4}{(s-1)(s-2)} = 3.$

Coefficient of $\dfrac{1}{s^2}$ is $\dfrac{1}{0!}\lim_{s\to 0}\dfrac{s^3-4s^2+4}{(s-1)(s-2)} = 2.$

Thus,
$$L(y) = \frac{3}{s} + \frac{2}{s^2} - \frac{1}{s-1} - \frac{1}{s-2}.$$

So,
$$y = 3 + 2t - e^t - e^{2t}.$$

Additional Examples

Example:

$$\begin{aligned} L(e^{-t}\sin^2 t) &= L(\frac{1}{2}(e^{-t}(1-\cos 2t)) \\ &= \frac{1}{2}\left[\frac{1}{s+1} - \frac{s+1}{(s+1)^2+4}\right]. \end{aligned}$$

Example:

$$\begin{aligned} L(t^2 e^{-at}) &= \frac{d^2}{ds^2}\left(\frac{1}{s+a}\right) \\ &= \frac{2}{(s+a)^3}. \end{aligned}$$

Example:

$$L(t^a e^{-bt}) = \frac{\Gamma(a+1)}{(s+b)^{a+1}}.$$

WISH YOU ALL THE VERY BEST