Hierarchical Forecasting

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1 Introduction

- Importance of coherency
- Point forecasting
- Probabilistic forecasting

The key macroeconomic indicators such as Gross Domestic Product (GDP), inflation and monetary policies which are used to study the behavior and performance of an economy as a whole are it self aggregates of various other components. For example, if we take the GDP growth, it is the aggregate of consumption, government expenditure, investments and net exports. These four components are again aggregates of some sub components. When we collect data for each of these individual variab les over some time period, we will observe a collection of multiple time series that are bounded with some aggregation constraints. Thus the macroeconomic data are naturally forming cross sectional hierarchical time series.

If the interest is on a single macroeconomic variable along different time granularities, then it can be considered as a temporal hierarchy. For example, suppose we have monthly consumer product index (CPI) of a particular country. The quarterly CPI is then the aggregate of corresponding monthly CPI of each quarter. Similarly the yearly CPI is the aggregate of quarterly CPI of each year. Hence it will form a temporal hierarchy.

Macroeconomic forecasts are crucial for economic and business activities of any economy. Therefore this area of study has a long history in literature. Econometricians have developed various approaches for getting reliable economic forecasts using macroeconomic data. However, the information of aggregation structure in

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real data is limitedly used in literature. Moreover, having coherent forecasts will help the economists and policy makers for align decision making that impact for the whole economy. Therefore, our focus in this chapter is to introduce hierarchical forecasting methods for macroeconomic forecasting particularly for cross-sectional hierarchical data structures.

Obtaining coherent forecasts are independent from the forecasting models. That means forecasters were given the freedom to use any reliable forecasting method to obtain the forecasts for individual series in the hierarchy. Getting coherent forecasts is a post-processing technique which ensures the aggregation properties are preserved in the forecasts.

briefly discuss the point forecasts as well as probabilistic forecasts in the sense of macroeconomic data

- Importance of coherency
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2 Hierarchical time series

Fix this depending on Section 2 To simplify the introduction of some notation we use the simple two-level hierarchical structure shown in Figure 1. Denote as $y_{Tot,t}$ the value observed at time t for the most aggregate (Total) series corresponding to level 0 of the hierarchy. Below level 0, denote as $y_{i,t}$ the value of the series corresponding to node i, observed at time t. For example, $y_{A,t}$ denotes the tth observation of the series corresponding to node A at level 1, $y_{AB,t}$ denotes the tth observation of the series corresponding to node AB at level 2, and so on.

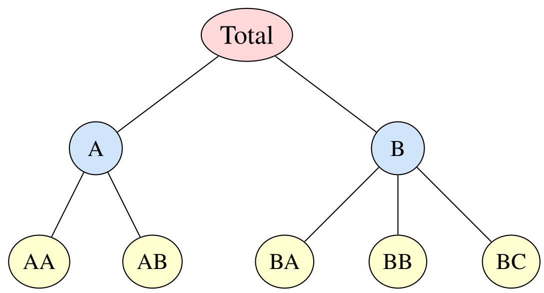

Fig. 1 A simple two-level hierarchical structure.

Let $\mathbf{y}_t = (y_{Tot,t}, y_{A,t}, y_{B,t}, y_{AA,t}, y_{AB,t}, y_{BA,t}, y_{BB,t}, y_{BC,t})'$, a vector containing observations across all series of the hierarchy at t. Similarly denote as $\mathbf{b}_t = (y_{AA,t}, y_{AB,t}, y_{BA,t}, y_{BB,t}, y_{BC,t})'$ a vector containing observations only for the bottom-level series. In general, $\mathbf{y}_t \in \mathbb{R}^n$ and $\mathbf{b}_t \in \mathbb{R}^m$ where n denotes the number of total series in the structure, m the number of series at the bottom level, and $n > m$ always. In the simple example of Figure 1, $n = 8$ and $m = 5$.

Aggregation constraints dictate that $y_{Tot} = y_{A,t} + y_{B,t} = y_{AA,t} + y_{AB,t} + y_{BA,t} + y_{BB,t} + y_{BC,t}$, $y_{A,t} = y_{AA,t} + y_{AB,t}$ and $y_B = y_{BA,t} + y_{BB,t} + y_{BC,t}$. Hence we can write

$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t, \tag{1}$$

where

$$\mathbf{S} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ & & \mathbf{I}_5 & & \end{pmatrix}$$

an $n \times m$ matrix referred to as the *summing matrix* and $\mathbf{I}_m$ is an m-dimensional identity matrix. $\mathbf{S}$ reflects the linear aggregation constraints and in particular how the bottom-level series aggregate to levels above. Thus, columns of $\mathbf{S}$ span the linear subspace of $\mathbb{R}^n$ for which the aggregation constraints hold. We refer to this as the *coherent subspace* and denote it by $\mathfrak{s}$. Notice that pre-multiplying a vector in $\mathbb{R}^m$ by $\mathbf{S}$ will result in an n-dimensional vector that lies in $\mathfrak{s}$.

Property 1. A hierarchical time series has observations that are *coherent*, i.e., $\mathbf{y}_t \in \mathfrak{s}$ for all t. We use the term coherent to describe not just $\mathbf{y}_t$ but any vector in $\mathfrak{s}$.

Structures similar to the one portrayed in Figure 1 can be found in macroeconomics. For instance in Section **??** we consider the case of GDP and its components. However, while this motivating example involves aggregation constraints, the mathematical framework we use can be applied for any general linear constraints, examples of which are ubiquitous in macroeconomics. For instance, the trade balance is computed as exports minus imports, while the consumer price index is computed as a weighted average of sub-indices, which are in turn weighted averages of sub-sub-indices and so on. These structures can also be captured by an appropriately designed $\mathbf{S}$ matrix.

An important alternative aggregation structure also commonly found in macroeconomics, is one for which the most aggregate series is disaggregated by attributes of interest that are crossed, as distinct to nested which is the case for hierarchical time series. For example, industrial production may be disaggregated along the lines of geography or sector or both. We refer to this as a *grouped* structure. Figure 2 shows a simple example of such a structure. The Total series disaggregates into $y_{A,t}$ and $y_{B,t}$, but also into $y_{X,t}$ and $y_{Y,t}$, at level 1, and then into the bottom-level series, $\mathbf{b}_t = (y_{AX}, y_{AY}, y_{BX}, y_{BY})'$. Hence, in contrast to hierarchical, grouped time series do not naturally disaggregate in a unique manner.

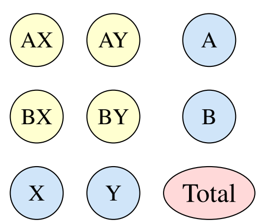

Fig. 2 A simple two-level grouped structure.

An important implementation of aggregation structures are *temporal hierarchies* introduced by **?**. In this case the aggregation structure spans the time dimension and dictates how higher frequency data (e.g., monthly) are aggregated to lower frequencies. There is a vast literature that studies the effects of temporal aggregation, going back to the seminal work of **????** and others such as, **????**. The main aim of this work is to find the single most optimum level of aggregation for modelling and forecasting time series. In this literature, the analyses, results (whether theoretical or empirical) and inferences, are extremely heterogeneous, making it very challenging to reach a consensus or some concrete conclusions. For example, **?** who study the effect of aggregation on several key macroeconomic variables state, “Quarterly data do not seem to suffer badly from temporal aggregation distortion, nor are they subject to the construction problems affecting monthly data. They therefore may be the optimal data for econometric analysis.” A similar conclusion is reached by

?. **?** consider forecasting French cash state deficit and provide empirical evidence of forecast accuracy gains from forecasting with the aggregate model rather than aggregating forecasts from the disaggregate model.

The overwhelming majority of the literature concentrates on a single level of temporal aggregation (there are some notable exceptions such as, **??**). **?** show that considering multiple levels of aggregation via temporal hierarchies and implementing forecast reconciliation approaches rather than single level approaches results in substantial gains in forecast accuracy across all levels of temporal aggregation. This is an example of the benefits of forecast reconciliation to which we now turn out attention to.

3 Point forecasting

A requirement when forecasting hierarchial time series is that the forecasts adhere to the same aggregation constraints as the observed data, i.e., they are coherent.

Definition 1. A set of h-step ahead forecasts $\tilde{\mathbf{y}}_{T+h|T}$, stacked in the same order as $\mathbf{y}_t$ and generated using information up to and including time T, are said to be *coherent* if $\tilde{\mathbf{y}}_{T+h|T} \in \mathfrak{s}$.

Hence, coherent forecasts of lower level series aggregate up to their corresponding upper level series and vice versa.

Add the picture here. Let us consider the smallest possible hierarchy with two bottom level series, A and B that add up to the top level Tot. Suppose $\breve{\mathbf{y}}_{T+h|T}$ of this hierarchy is given by $\breve{\mathbf{y}}_{T+h|T} = [\breve{y}_{Tot,T+h|T}, \breve{y}_{A,T+h|T}, \breve{y}_{B,T+h|T}]$. Due to the aggregation structure we have $\breve{y}_{Tot,T+h|T} = \breve{y}_{A,T+h|T} + \breve{y}_{B,T+h|T}$. This implies that, even though $\breve{\mathbf{y}}_{Tot,T+h|T} \in \mathbb{R}^3$, the points actually lie in $\mathfrak{s} \subset \mathbb{R}^3$, which is a two dimensional subspace within $\mathbb{R}^3$ space.

3.1 Single-level approaches

A common theme across all traditional approaches for forecasting hierarchical time series is that a single-level of aggregation is first selected and forecasts for that level are generated. These are then linearly combined to generate a set of coherent forecasts the rest of the structure.

3.1.1 Bottom-up

In the *bottom-up* approach, forecasts for the most disaggregate are first generated. These are then aggregated to obtain forecasts for all other series of the hierarchy (Dunn et al. 1976). In general, this consists of first generating $\hat{\mathbf{b}}_{T+h|T} \in \mathbb{R}^m$, a set of

h-step ahead forecasts for the bottom-level series. For the simple hierarchical structure of Figure 1, $\hat{\mathbf{b}}_{T+h|T} = (\hat{y}_{AA,T+h|T}, \hat{y}_{AB,T+h|T}, \hat{y}_{BA,T+h|T}, \hat{y}_{BB,T+h|T}, \hat{y}_{BC,T+h|T})$, where, $\hat{y}_{i,T+h|T}$ is the h-step ahead forecast of the series corresponding to node i. A set of coherent forecasts for the whole hierarchy is then given by,

$$\tilde{\mathbf{y}}^{BU}_{T+h|T} = \mathbf{S}\hat{\mathbf{b}}_{T+h|T}.$$

Generating bottom-up forecasts has the advantage of no information being lost due to aggregation. However, bottom-level data can potentially be highly volatile or very noisy and therefore challenging to forecast.

3.1.2 Top-down

In contrast *top-down* approaches involve first generating forecasts for the most aggregate level and then disaggregating these down the hierarchy. In general, coherent forecasts generated from top-down approaches are given by,

$$\tilde{\mathbf{y}}^{TD}_{T+h|T} = \mathbf{S}\mathbf{p}\hat{y}_{Tot,T+h|T},$$

where $\mathbf{p} = (p_1, ..., p_m)'$ is an m-dimensional vector consisting of a set of proportions which disaggregate the top-level forecast $\hat{y}_{Tot,T+h|T}$ to forecasts for the bottom-level series, hence $\mathbf{p}\hat{y}_{Tot,T+h|T} = \hat{\mathbf{b}}_{T+h|T}$. These are then aggregated up by the summing matrix $\mathbf{S}$.

Traditionally proportions have been calculated based on the observed historical data. Gross & Sohl (1990) present and evaluate twenty-one alternative approaches. The most convenient attribute of these approaches is their simplicity. Generating a set of coherent forecasts involves only modelling and generating forecasts for the most aggregate top-level series. In general, such top-down approaches seem to produce quite reliable forecasts for the aggregate levels and they are useful with low count data. However, a significant disadvantage is the loss of information due to aggregation. Using such top-down approaches, is limited as it does not allow to capture and model individual series characteristics. To overcome this limitation, **?** introduced a new top-down approach which disaggregates the top-level forecasts based on proportions of forecasts rather than the historical data and show evidence that this method outperforms the conventional top-down approaches. However, a limitation of all top-down and by implication middle-out approaches that follow next, is that they introduce bias to the forecasts. We discuss this in detail in Section 3.2 that follows.

3.1.3 Middle-out

A compromise between bottom-up and top-down approaches is the middle-out approach. It entails first forecasting the series of a selected middle-level. For series above the middle-level, coherent forecasts are generated using the bottom-up approach by aggregating the middle-level forecasts upwards. For series below the middle level, coherent forecasts are generated using a top-down approach by disaggregating the middle-level forecasts downwards. As mentioned above As the since the middle-out approach involves generating top-down forecasts, it also introduces bias to the forecasts. We discuss this in detail in Section 3.2 that follows.

3.2 Point forecast reconciliation

All the traditional approaches discussed so far are limited to only using information from a single-level of aggregation and furthermore ignoring any cross-correlations across levels of a hierarchy. An alternative framework that overcomes these limitations, first proposed by **?** and implemented by **?**, is one that involves forecast reconciliation. In a first step ignoring any aggregation constraints, forecasts for all the series across all levels of the hierarchy are generated. We refer to these as *base* forecasts and denote them by $\hat{\mathbf{y}}_{T+h|T}$. In general, base forecasts will not be coherent. An example of an exception is when a simple method such as a random walk is used to generate all base forecasts so that the coherent nature of the data is extended to the forecasts.

In a second step, base forecasts are reconciled so that they become coherent.

ex-post adjustment

This is achieved by projecting the base forecasts $\hat{\mathbf{y}}_{T+h|T}$ onto the coherent subspace $\mathfrak{s}$, via a projection matrix $\mathbf{SG}$, resulting in a set of coherent forecasts $\tilde{\mathbf{y}}_{T+h|T}$. More specifically,

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{SG}\hat{\mathbf{y}}_{T+h|T}, \tag{2}$$

where $\mathbf{G}$ is an $m \times n$ matrix that maps $\hat{\mathbf{y}}_{T+h|T}$ to the $\mathbb{R}^m$ space, producing a set of coherent forecasts for the bottom-level which are in turn mapped to the coherent subspace by the summing matrix $\mathbf{S}$ as defined in (1). We restrict our attention to projections on $\mathfrak{s}$ in which case $\mathbf{SGS} = \mathbf{S}$. This ensures that unbiasedness is preserved, i.e., for a set of unbiased base forecasts reconciled forecasts will also be unbiased.

Note that all single-level approaches discussed so far can also be represented by (2) using appropriately designed $\mathbf{G}$ matrices, however not all of these will be projections. For example, for the bottom-up approach, $\mathbf{G} = \big(\mathbf{0}_{(m\times n-m)}\ \mathbf{I}_m\big)$ in which case $\mathbf{SGS} = \mathbf{S}$. For the top-down approach $\mathbf{G} = (\mathbf{p}\ \mathbf{0}_{(m\times n-1)})$, for which case $\mathbf{SGS} \neq \mathbf{S}$.

3.2.1 OLS reconciliation

Assume that $\hat{\mathbf{y}}_{T+h|T}$ is a set of unbiased base forecasts, i.e., $E_{1:t}(\hat{\mathbf{y}}_{T+h|T}) = E_{1:t}[\mathbf{y}_{T+h}|\mathbf{y}_1,\ldots,\mathbf{y}_T]$, the true mean with the expectation taken over the observed sample up to time T. For any $\mathbf{G}$ such that $\mathbf{SGS} = \mathbf{S}$ or equivalently $\mathbf{SG} = \mathbf{I}_m$ the resulting coherent forecasts are also unbiased. Can we tie in here this? Can we say: More generally (Gamakumara et al. 2018) show that any **SG** that is a projection matrix will result to unbiased coherent forecasts.

? proposed to reconcile the unbiased base forecasts through the following regression model. From (1),

$$\hat{\mathbf{y}}_{T+h|T} = \mathbf{S}\beta_{T+h|T} + \varepsilon_{T+h|T}, \tag{3}$$

where $\beta_{T+h|T} = E[\mathbf{b}_{t+h}|\mathbf{b}_1, \ldots.., \mathbf{b}_t]$ is the unknown conditional mean of the bottom-level series and $\varepsilon_{T+h|T}$ is the coherence or reconciliation error with mean zero and variance $\mathbf{V}$. The ordinary least squares (OLS) solution leads to the usual projection matrix $\mathbf{S}(\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'$, so that a set of coherent forecasts are obtained by,

$$\tilde{\mathbf{y}}^{\text{OLS}}_{T+h|T} = \mathbf{S}\mathbf{G}\hat{\mathbf{y}}_{T+h|T}$$

where $\mathbf{G} = (\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'$. In this reconciliation, the base forecasts are orthogonally projected to the coherent subspace $\mathfrak{s}$. Hence the OLS projection minimises the Euclidean distance between $\hat{\mathbf{y}}_{T+h|T}$ and $\tilde{\mathbf{y}}_{T+h|T}$. The OLS reconciled forecasts are also unbiased since $\mathbf{SGS} = \mathbf{S}$. We should note that using a GLS estimator in this context is not possible since $\mathbf{V}$ is not identifiable as shown by **?**.

Can we add Tas's picture and talk about optimality in this sense.

3.2.2 Optimal MinT reconciliation

? build a unifying framework for much of the previous literature on forecast reconciliation.

and introduce the MinT approach. Assume again that $\hat{\mathbf{y}}_{T+h|T}$ is a set of unbiased base forecasts, i.e., $E_{1:t}(\hat{\mathbf{y}}_{T+h|T}) = E_{1:t}[\mathbf{y}_{T+h}|\mathbf{y}_1, \ldots, \mathbf{y}_T]$, the true mean with the expectation taken over the observed sample up to time T. Let

$$\hat{\mathbf{e}}_{T+h|T} = \mathbf{y}_{T+h|T} - \hat{\mathbf{y}}_{T+h|T} \tag{4}$$

denote a set of base forecast errors, with $\text{Var}(\hat{\mathbf{e}}_{T+h|T}) = \mathbf{W}_h$, and

$$\tilde{\mathbf{e}}_{T+h|T} = \mathbf{y}_{T+h|T} - \tilde{\mathbf{y}}_{T+h|T}$$

denote a set of coherent forecast errors. Lemma 1 in **?** shows that for any matrix $\mathbf{G}$ such that $\mathbf{SGS} = \mathbf{S}$, $\text{Var}(\tilde{\mathbf{e}}_{T+h|T}) = \mathbf{SGW}_h\mathbf{S}'\mathbf{G}'$. Furthermore Theorem 1 shows that

$$\mathbf{G} = (\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1} \tag{5}$$

is the unique solution that minimises the $\text{tr}[\mathbf{SGW}_h\mathbf{S}'\mathbf{G}']$ subject to $\mathbf{SGS} = \mathbf{S}$. Note that unlike the GLS solution to (3), the MinT solution is a function of $\mathbf{W}_h$, the variance of the base forecast errors. MinT is optimal in the sense that given a set of unbiased base forecasts, it returns a set of best linear unbiased reconciled forecasts using as **G** the unique solution that minimises the trace (hence MinT) of the variance of the forecast error of the reconciled forecasts.

A significant advantage of the MinT reconciliation solution is that it is the first to incorporate the full correlation structure of the hierarchy via $\mathbf{W}_h$. However, estimating $\mathbf{W}_h$ is challenging, especially for $h > 1$. Of course setting $\mathbf{W}_h = k_h\mathbf{I}_n$ for all h where $k_h > 0$ is a proportionality constant, leads to the OLS solution of **?**. A disadvantage of this simplifying solution, further to not accounting for the correlations across series, is that the homoscedastic diagonal entries do not account for the scale differences between the levels of the hierarchy due to aggregation. However OLS does well in practice because as discussed it minimises the Euclidean distance and blah blah. Not sure how much we want to say here.

? present possible alternative estimators for $\mathbf{W}_h$ and show that these lead to different **G** matrices. We summarise these below.

- Set $\mathbf{W}_h = \text{diag}(\hat{\mathbf{W}}_1)$ for all h, where $k_h > 0$ and

$$\hat{\mathbf{W}}_1 = \frac{1}{t}\sum_{k=1}^{t}\hat{\mathbf{e}}_k\hat{\mathbf{e}}_k'$$

  is the unbiased sample estimator of the in-sample one-step-ahead base forecast errors as defined in (4). This estimator scales the base forecasts using the variance of the in-sample residuals and is therefore describes and referred to as a WLS estimator.
- Set $\mathbf{W}_h = k_h\hat{\mathbf{W}}_1$, for all h, where $k_h > 0$, the unrestricted sample covariance estimator for $h = 1$. Although this is relatively simple to obtain and provides a good solution for small hierarchies, it does not provide reliable results a m grows compared to t. We refer to this a the MinT(Sample) estimator.
- Set $\mathbf{W}_h = k_h\hat{\mathbf{W}}_1^D$, for all h, where $k_h > 0$, $\hat{\mathbf{W}}_1^D = \lambda_D\text{diag}(\hat{\mathbf{W}}_1) + (1-\lambda_D)\hat{\mathbf{W}}_1$ is a shrinkage estimator with diagonal target, and shrinkage intensity parameter

$$\hat{\lambda}_D = \frac{\sum_{i\neq j}\hat{Var}(\hat{r}_{ij})}{\sum_{i\neq j}\hat{r}_{ij}^2},$$

  where $\hat{r}_{ij}$ is the ijth element of $\hat{\boldsymbol{R}}_1$, the 1-step-ahead sample correlation matrix as proposed by Schäfer & Strimmer (2005). Hence, off-diagonal elements of $\hat{\mathbf{W}}_1$ are shrunk towards zero while diagonal elements (variances) remain unchanged. We refer to this as the MinT(Shrink) estimator.

4 Hierarchical probabilistic forecasting

Point forecasts are limited since they provide no indication of uncertainty around the forecast. A richer description of forecast uncertainty can be obtained by providing a "probabilistic forecasts", that is a full density for the target of interest. For a review of probabilistic forecasts, and methods for evaluating such forecasts known as *scoring rules* see (Gneiting & Katzfuss 2014). In recent years, the use of probabilistic forecasts and their evaluation via scoring rules has become pervasive in macroeconomic forecasting, for example need to find some references that use scoring rules for macro forecasting. Check Bayesian macro guys like Koop Korobilis, Josh Chan also Mike Smith's work with Shaun Vahey.

The literature on hierarchical probabilistic forecasting is still an emerging area of interest. To the best of our knowledge the first attempt to even define coherence in the setting of probabilistic forecasting is provided by Taieb et al. (2017) who define a coherent forecast in terms of a convolution. An equivalent definition, provided by Gamakumara et al. (2018) defines a coherent probabilistic forecast as a probability measure on the coherent subspace $\mathfrak{s}$. Gamakumara et al. (2018) also generalise the concept of forecast reconciliation to the probabilistic setting.

Definition 2. Let $\mathscr{A}$ be a subset[1] of $\mathfrak{s}$ and let $\mathscr{B}$ be all points in $\mathbb{R}^n$ that are mapped onto $\mathscr{A}$ after premultiplication by $\boldsymbol{SG}$. Letting $\hat{\nu}$ be a 'base' probabilistic forecast for the full hierarchy, the coherent measure $\tilde{\nu}$ 'reconciles' $\hat{\nu}$ if $\tilde{\nu}(\mathscr{A}) = \hat{\nu}(\mathscr{B})$ for all $\mathscr{A}$.

In practice this definition suggests two approaches. For some parametric distributions, for instance the multivariate normal, it may be possible to derive a reconciled probabilistic forecast analytically. However, in macroeconomic forecasting, non-standard distributions such as bimodal distribution are often required to take different policy regimes into account worth checking if any (marginal) predictives are bimodal before we include this statement. In such cases a non-parametric approach based on bootstrapping in-sample errors proposed Gamakumara et al. (2018) can be used as long as a sample from the predictive distribution is available. Each of these scenarios is now covered in detail.

4.1 Probabilistic forecast reconciliation in the Gaussian framework

In the case where the base forecasts are probabilistic forecasts characterised by elliptical distributions Gamakumara et al. (2018) show that reconciled probabilistic forecasts will also be elliptical. This is particularly straightforward for the Gaussian distribution which is completely characterised by two moments. Letting the base probabilistic forecast be $\mathscr{N}(\hat{\mathbf{y}}_{T+h|T}, \hat{\boldsymbol{\Sigma}}_{T+h|T})$, then the reconciled probabilistic forecast will be $\mathscr{N}(\tilde{\mathbf{y}}_{T+h|T}, \tilde{\boldsymbol{\Sigma}}_{T+h|T})$, where,

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{S}\mathbf{G}\hat{\mathbf{y}}_{T+h|T}, \tag{6}$$

and

$$\tilde{\Sigma}_{T+h|T} = \mathbf{S}\mathbf{G}\hat{\boldsymbol{\Sigma}}_{T+h|T}\mathbf{G}'\mathbf{S}'. \tag{7}$$

[1] Strictly speaking $\mathscr{A}$ is a Borel set

There are several options for obtaining the base probabilistic forecast and in particular the variance covariance matrix $\hat{\boldsymbol{\Sigma}}$. One option is to fit multivariate models level by level or for the hierarchy as a whole leading respectively to a $\hat{\boldsymbol{\Sigma}}$ that is block diagonal or dense. Another alternative is to fit univariate models for each individual series in which case $\hat{\boldsymbol{\Sigma}}$ is a diagonal matrix. Due to the large number of series under investigation here we consider the latter option. However we emphasise that correlation will enter the probabilistic forecast after reconciliation. The reconciled probabilistic forecast will ultimately depending on the choice of $\mathbf{G}$; the same choices of $\mathbf{G}$ matrices used in section 3 are be used here.

Need to check with Puwasala that base forecasts have diagonal sigma hat

4.2 Probabilistic forecast reconciliation in the non-parametric framework

In many applications, including macroeconomic forecasting, it may not reasonable to assume Gaussian predictive distributions. Therefore, non-parametric approaches has been widely used for probabilistic forecasts in different disciplines. For example, ensemble forecasting in weather applications (Gneiting & Raftery (2005), Gneiting & Katzfuss (2014), Gneiting et al. (2008)), bootstrap based approaches (Manzan & Zerom (2008), Vilar & Vilar (2013)). Check/replace these references with references that show heavy tails/skewness in macro applications.

Due to these concerns, we employ a reconciliation method proposed by Gamakumara et al. (2018) that does not make parametric assumptions about the predictive distribution. An important result that this method exploits is that applying methods for point forecast reconciliation to the draws from incoherent base predictive distribution results in a sample from the reconciled predictive distribution. This process, is summarised

1. Fit univariate models to each series in the hierarchy over a training set from $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_T$.
2. For each series compute h-step ahead point forecasts, for all h up to H. Collect these into a $n \times H$ matrix $\hat{\boldsymbol{Y}} := (\hat{\boldsymbol{y}}_{T+1|T}, \ldots, \hat{\boldsymbol{y}}_{T+H|T})$, where $\hat{\boldsymbol{y}}_{T+h|T}$ is a $n \times 1$ vector of h-step point forecasts for all series in the hierarchy.
3. Compute one-step ahead in-sample forecasting errors. Collect these into an $n \times T$ matrix $\hat{\boldsymbol{E}} = (\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots.., \hat{\mathbf{e}}_T)$, where the $n \times 1$ vector $\hat{\mathbf{e}}_t = \boldsymbol{y}_t - \hat{\boldsymbol{y}}_{t|t-1}$. Here, $\hat{\boldsymbol{y}}_{t|t-1}$ is a vector of forecasts made for time t using information up to and including $t-1$. Information from $t = 1, \ldots, T$ will be used to train the model used to form these forecasts.
4. Block bootstrap from $\hat{\boldsymbol{E}}$, that is choose H consecutive columns of $\hat{\boldsymbol{E}}$ at random, repeating this process B times. Denote the $n \times H$ matrix obtained at iteration b as $\hat{\boldsymbol{E}}^b$ for b=1,...,B.

5. For all b, compute $\hat{\boldsymbol{\Upsilon}}^b := \hat{\boldsymbol{Y}} + \hat{E}^b$. Each row of $\hat{\boldsymbol{\Upsilon}}^b$ is a sample path of h forecasts for a single series. Each column of $\hat{\boldsymbol{\Upsilon}}^b$ is a realisation from the joint predictive distribution at a particular horizon.
6. For each $b = 1,\dots,B$ select the h^{th} column of $\hat{\boldsymbol{\Upsilon}}^b$ and stack these to form a $n \times B$ matrix $\hat{\boldsymbol{\Upsilon}}_{T+h|T}$
7. For a given $\boldsymbol{G}$ matrix and for each $h = 1,\dots,H$ compute $\tilde{\boldsymbol{\Upsilon}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{\Upsilon}}_{T+h|T}$. Each column of $\tilde{\boldsymbol{\Upsilon}}_{T+h|T}$ is a realisation from the joint h-step ahead reconciled predictive distribution.

Check with Puwasala that this is exactly what she has done. Notation may need work to bring in line with previous sections.

5 Empirical Study: Australian GDP

In our empirical data we consider Gross Domestic Product (GDP) of Australia with quarterly data available from the December quarter of 1984 until the March quarter of 2018. The Australian Bureau of Statistics (ABS) measures GDP using three main approaches namely, Production, Income and Expenditure. The final GDP figure is obtained as an average of these three figures. Each of these measures can be disaggregated into additional series, which themselves could be targets of interests to forecasters. This suggests a hierarchical approach to forecasting could be used to improve forecasts of all series in the hierarchy including headline GDP.

For two of the three approaches, namely the Income approach and Expenditure approach, nominal data are available. Nominal data are the focus of our study rather than real data. This is due to the fact that real data are constructed via a chain price index approach with different price deflators used for each series. As a result, real GDP data are not coherent - the aggregate series is not a linear combination of the disaggregate series. For similar reasons we do not use seasonally adjusted data; the process of seasonal adjustment results in data that are not coherent. Finally, although there is a small statistical discrepancy between each series and the headline GDP figure, we can simply treat this statistical discrepancy, which is also published by the ABS, as a time series in its own right. For further details on the data we refer the reader to (Australian Bureau of Statistics 2018).

Following paragraph to be separated over next two sections

Figures 15, 17 and 18 depict these hierarchies. In each hierarchy, the most aggregate level is denoted in grey whereas, the most disaggregate level is denoted in red. Intermediate levels are denoted in orange and blue. Levels denoted in orange continues to disaggregate further and these are separately depicted in different tree diagrams. Further, a description of each series in these hierarchies along with the series ID assigned by the ABS is given in the tables 1, 2, 3 and 4 in appendix 1.

Following subsections will give a brief description of income and expenditure hierarchies.

5.1 Income approach

In the income approach, the GDP is measured by the aggregation of all income flows. That is the aggregation of all factor incomes and the taxes less subsidies on production and imports at purchaser's price (Australian Bureau of Statistics 2015). Underline equation is given as,

$$\begin{aligned} GDP(I) = &\, \textit{Compensation of employees} + \textit{Gross operating surplus} + \textit{Gross mixed income} \\ &+ \textit{Taxes on production and imports} - \textit{Subsidies on production and imports} \\ &+ \textit{Statistical decrepency (I)} \end{aligned}$$

Hierarchy shown in figure 17 in appendix 1 reflects how these are further disaggregated.

5.2 Expenditure approach

In the expenditure approach, the GDP is calculated as the aggregation of final consumption expenditure, gross fixed capital formation (GFCF), changes in inventories of finished goods, work-in-progress and raw materials and the value of exports less imports of the goods and services (Australian Bureau of Statistics 2015). Underline equation is,

$$\begin{aligned} GDP(E) = &\, \textit{Final consumption expenditure} + \textit{Gross fixed capital formation} + \textit{Changes in inventories} \\ &+ \textit{Exports of goods and services} - \textit{Imports of goods and services} + \textit{Statistical decrepency (E)} \end{aligned}$$

Associated hierarchical structure is given in figure 18, 19 and 20 in appendix 1.

Income and expenditure hierarchies consist 16 and 81 series respectively. All quarterly data for these series were obtained from the ABS and used to estimate coherent forecasts for Australian GDP along with its disaggregate components. In the following section we describe the hierarchical forecasting methods that we are using to get these forecasts.

- GDP of Australia
- How GDP is measured
  - Production, Income and Expenditure approach
  - Explain the hierarchy
- Issues with data
  - Why use current price rather than constant price (This is why we ignores production approach)
  - What is statistical discrepancy

  - Frequency of data
  - Does the data satisfies coherency

- Forecasting methods

  - Add a figure of all time series in income approach. Explain why traditional methods might not work using these time series as forecasting one layer of the hierarchy will ignore the structural information of individual time series to be used in the forecasts.
  - Explain ETS and ARIMA forecasting methods briefly
  - Hierarchical forecasting

In this empirical study we apply above discussed hierarchical methods to obtain coherent point and probabilistic forecasts for Australian GDP from income and expenditure approach along with the forecasts for its disaggregate components.

Let us first observe the time series plots for each approach. Figure 3 and 4 depicts these plots for income and expenditure hierarchies respectively. The upper panel of each figure shows the time series of all aggregate level series whereas lower panel shows the bottom level series. We can see that different series reflect different characteristics. For example, in the expenditure hierarchy, some series reflects an upward trend while some others reflect a downward trend or no trend at all. Further some series are having seasonal pattern whereas some does not have any seasonality. Moreover the bottom level series reflects some noise level compared to aggregate series in both hierarchies. Therefore, coherent forecasts through traditional methods such as top-down or bottom-up methods would not be accurate as they will ignore part of the information in generating forecasts. Thus forecast reconciliation is important in this empirical study.

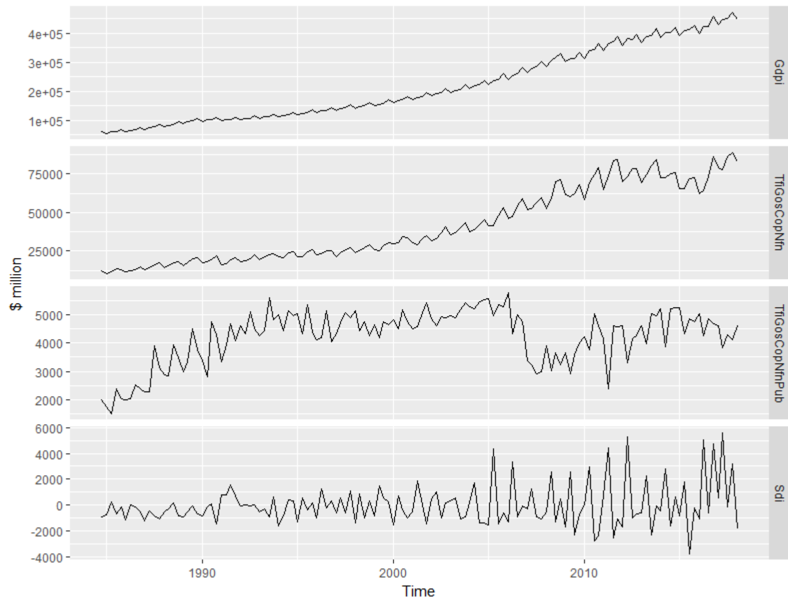

Fig. 3 Time plots for series from different levels of income hierarchy.

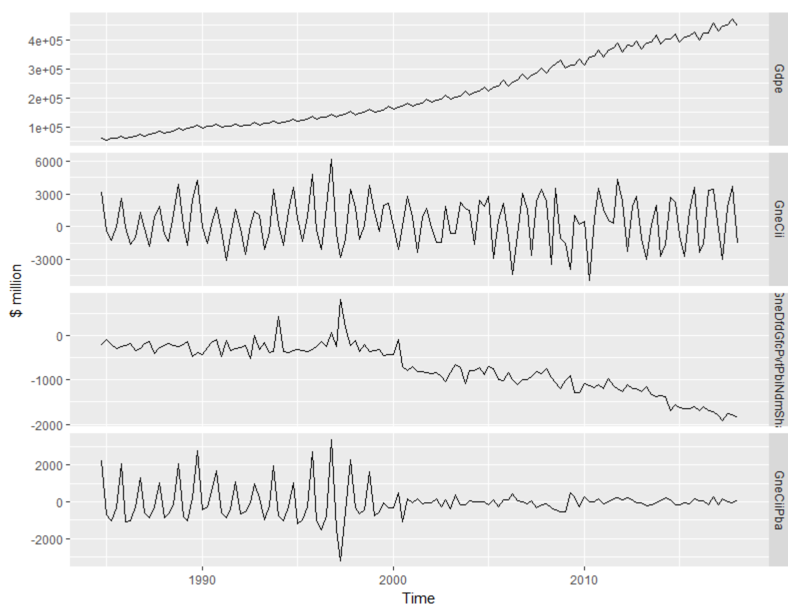

Fig. 4 Time plots for series from different levels of expenditure hierarchy.

6. Experimental Design and - setup: Time frame of Rolling window, - alternative methods: ARIMA/base vs naïve , BU, OLS, WLS, MinT(Shrink). - evaluation

- Point: MASE, MASE - Probabilistic: CRPS, Energy Score Skill score.

7. Results

6 Methodology

We now demonstrate the potential for reconciliation methods to improve forecast accuracy for Australian GDP data. We consider forecasts from $H = 1$ quarter ahead forecasts up to $H = 4$ quarter ahead using an *expanding* window. First, the training sample is set to QX of XXXX to QX of XXXX and forecasts are produced for QX of XXXX to QX of XXXX. Then the training window is expanded one period ahead, i.e. from QX of XXXX to QX of XXXX with forecasts produces for QX of XXXX to QX of XXXX. All up this leads to XX forecasts.

Need to get these dates off Puwasala. Also last section may need to be changed

6.1 Models

The first task in forecasting time series is to obtain base forecasts for all series in the hierarchy. In the case of the income approach this necessitates forecasting XX time series while in the case of the expenditure approach forecasts for XX time series must be obtained. As such our focus was on a methodology that was fast but flexible. We consider simple univariate ARIMA models, where model order is selected via a combination of unit root testing and AIC using an algorithm developed by XXX and implement in the auto.arima function in XXX. Cite this to Rob's satisfaction. A similar approach was also undertaken using the ETS framework to produce base forecasts. This had minimal impact on our conclusions with respect to forecast reconciliation methods, and in most cases ARIMA forecats outperformed ETS forecasts. Consequently, results for ETS models are excluded but are available from the authors upon request again do we put this in an appendix?. We note that a number of more complicated approaches could have been used to obtain base forecasts including multivariate models such as vector autoregressions and models and methods that handle a large number of predictors such as factor models or least angle regresssion. However, **?** show that univariate ARIMA models are highly competitive for forecasting Australian GDP even compared to these methods, and in any case our primary motivation is to demonstrate the potential of forecast reconciliation.

The forecast reconciliation approaches that we consider are bottom up, OLS, WLS with What scaling did we use and the MinT (shrink) approach. The MinT (sample) approach was also used but due to the size of the hierarchy forecasts reconciled via this approach were less stable. Finally, all forecasts both base and reconciled are compared to a naïve benchmark. Since the data are not deseasonalised, the naïve benchmark is a seasonal random walk, i.e. the forecast for GDP (or one of its components) is the realised GDP in the same quarter of the previous year. The naïve forecast is by construction coherent and therefore does not need to be reconciled.

6.2 Evaluation

For evaluating point forecasts we consider two metrics, the Root Mean Squared Error (RMSE) and the Mean Absolute Scaled Error (MASE). The absolute scaled error is defined as

$$q_{t+h} = \sum \frac{|\breve{e}_{t+h|t}|}{(T-m)^{-1}\sum_{j=m+1}^{T}|y_j - y_{j-m}|}\,,$$

where $\breve{e}_{t+h}$[2] is the difference between any forecast and the realisation check m is and change since m is bottom level dimension. An advantage of using MASE is that it is a scale independent measure. This is particularly relevant for hierarchical time series, since aggregate series by their very nature are on a larger scale than disaggregate series. As such scale dependent metrics may unfairly favour methods that perform well for the aggregate series but poorly for disaggregate series. For more details on different point forecast accuracy measures refer to **?**.

6.2.1 Probabilistic forecast evaluation

Forecast accuracy of probabilistic forecasts can be evaluated using scoring rules Gneiting & Katzfuss (2014). Let $\breve{F}$ be a probabilistic forecast and let $\breve{\mathbf{y}} \sim \breve{F}$ where breve is used to denote that either base forecast or reconciled forecast can be evaluated. The accuracy of multivariate probabilistic forecasts will be measured by the energy score given by

$$eS(\breve{F}_{T+h|T}, \mathbf{y}_{T+h}) = E_{\breve{F}}\|\breve{\mathbf{y}}_{T+h} - \mathbf{y}_{T+h}\|^{\alpha} - \frac{1}{2}E_{\breve{F}}\|\breve{\mathbf{y}}_{T+h} - \breve{\mathbf{y}}^{*}_{T+h}\|^{\alpha}\,,$$

where $\mathbf{y}_{T+h}$ is the realisation at time $T+h$, $\alpha \in (0,2]$. What did we use for alpha?. The expectations can be evaluated numerically as long as a sample from $\breve{F}$ is available which is the case for all methods we employ. An advantage of using energy score is that in the univariate case it simplifies to the commonly used cumulative rank probability score (CRPS) given by

$$\text{CRPS}(\breve{F}_i, y_{i,T+h}) = E_{\breve{F}_i}|\breve{y}_{i,T+h} - y_{i,T+h}| - \frac{1}{2}E_{\breve{F}_i}|\breve{y}_{i,T+h} - \breve{y}^{*}_{i,T+h}|, \tag{8}$$

[2] breve is used intead of a hat or tilde to denote that this can be the error either a base or reconciled forecast

where the subscript i is used to denote that CRPS measures forecast accuracy for a single variable in the hierarchy.

As an alternative to the energy score, log score and variogram scores were also considered. The log score was disregarded since Gamakumara et al. (2018) prove that the log score is improper with respect to the class of incoherent probabilistic forecasts when the true DGP is coherent. The variogram score gave similar results to the energy score; variogram score results are omitted for brevity but are available from the authors upon request. or we put them in an appendix

Discuss results

Income approach

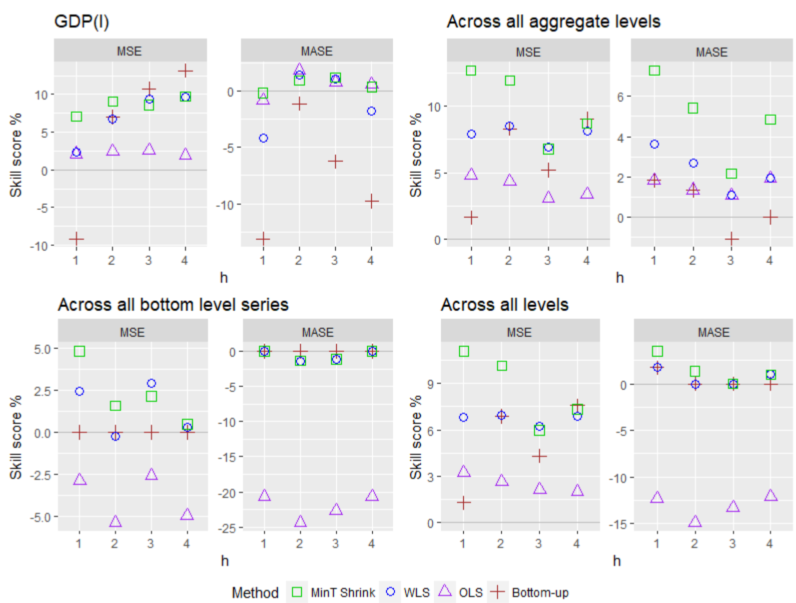

Fig. 5 Summary of point forecasts in income approach

Expenditure approach

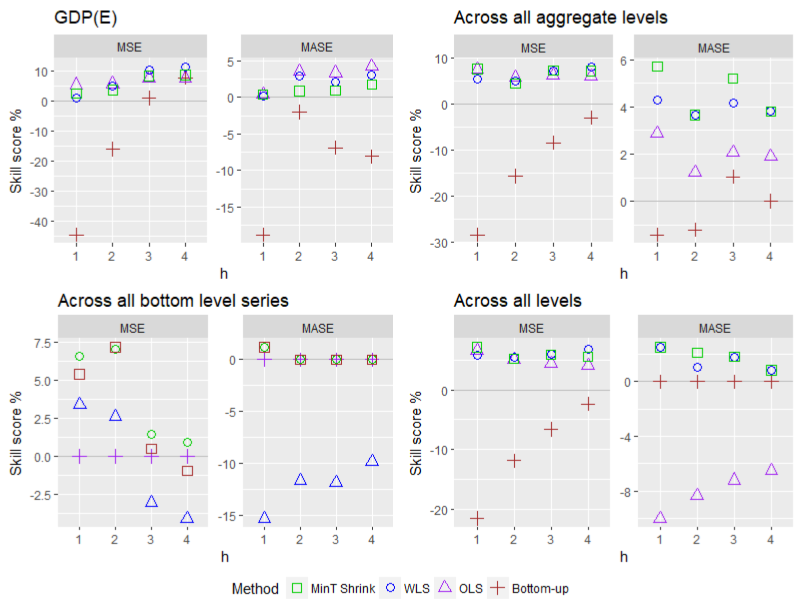

Fig. 6 Summary of point forecasts in expenditure approach

7 Results

Results are presented in **??** and **??** for income and expenditure approaches respectively.

Results are presented in tables **??**, **??**, **??** for income approach and **??**, **??**, **??** for expenditure approach.

Discuss results
Possible discuss skill scores
Discuss breakdown into looking at different levels.

Income approach

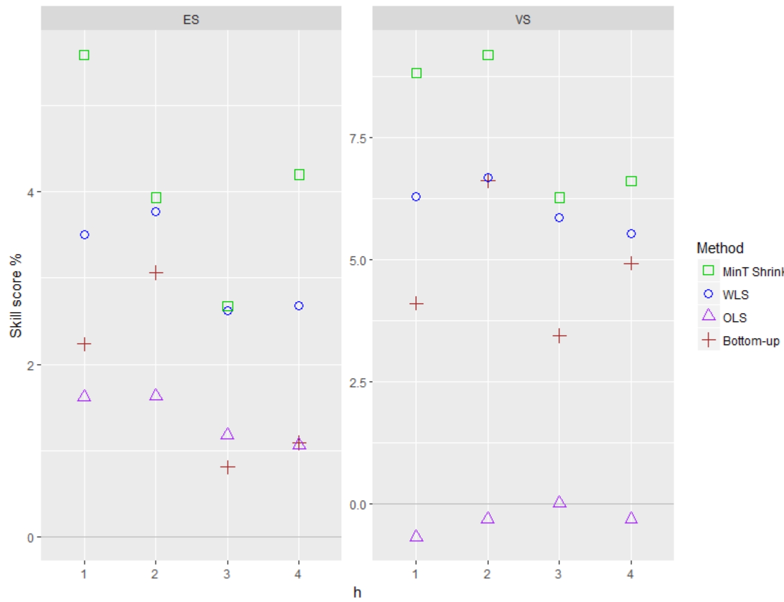

Fig. 7 Skill scores with respect to energy score and variogram score for multivariate Gaussian forecast distribution of income hierarchy

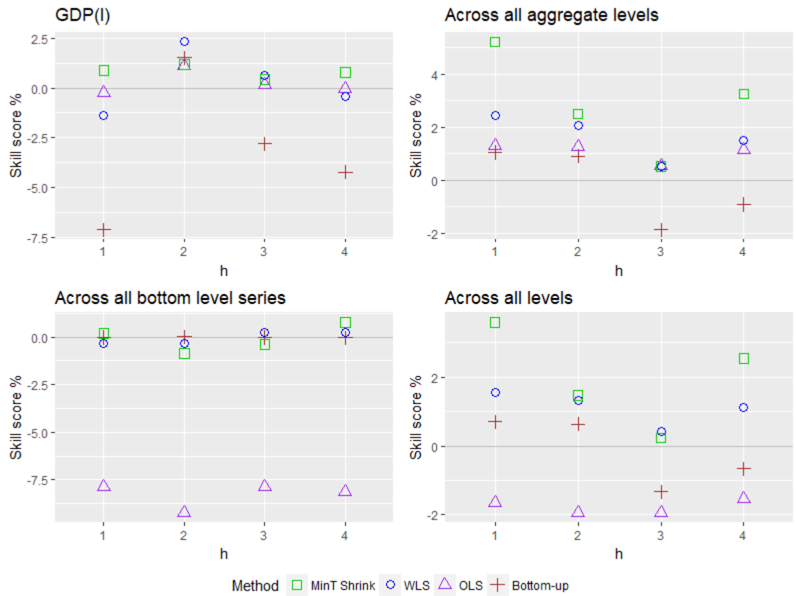

Fig. 8 Skill scores for univariate Gaussian forecast distributions of individual series of income hierarchy

Expenditure approach

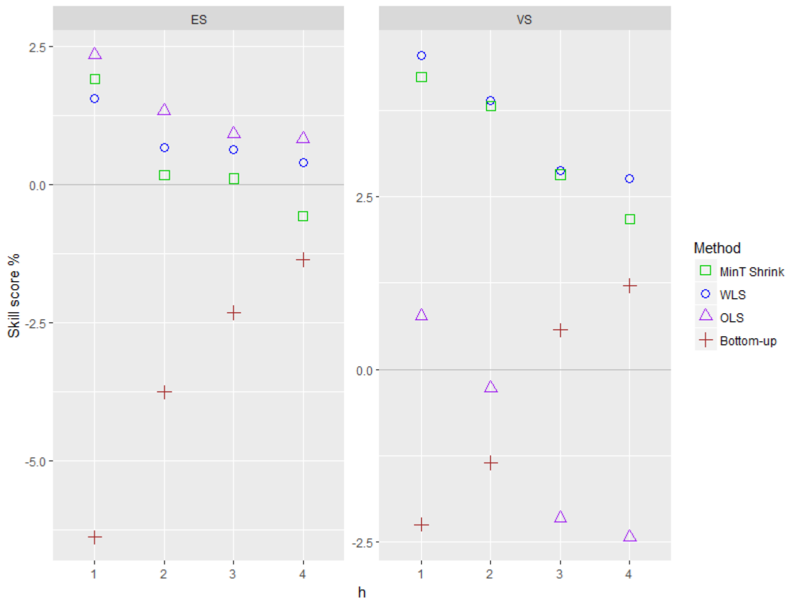

Fig. 9 Skill scores with respect to energy score and variogram score for multivariate Gaussian forecast distribution of expenditure hierarchy

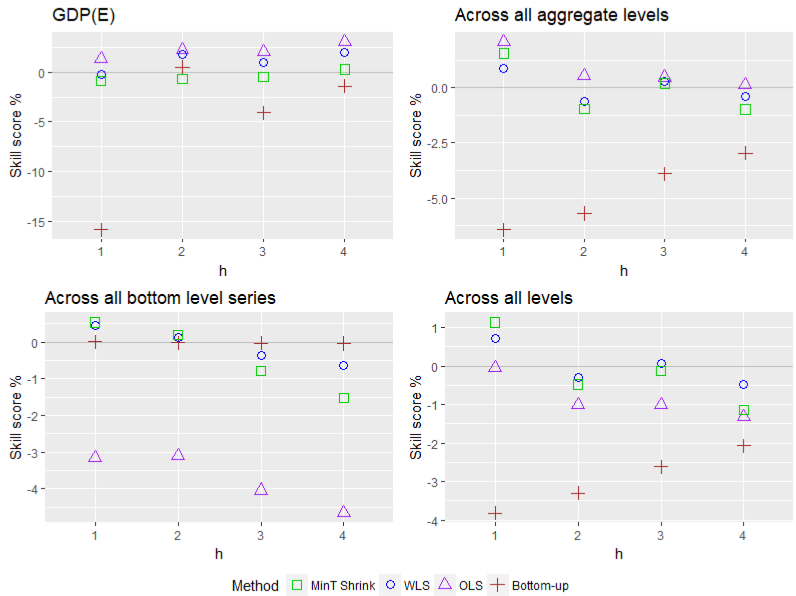

Fig. 10 Skill scores for univariate Gaussian forecast distributions of individual series of expenditure hierarchy

7.0.1 Non-parametric probabilistic forecasts for Australian GDP

We also estimate the coherent probabilistic forecasts for GDP and its disaggregate components by using the non-parametric bootstrap approach explained in the section (?).

Income approach

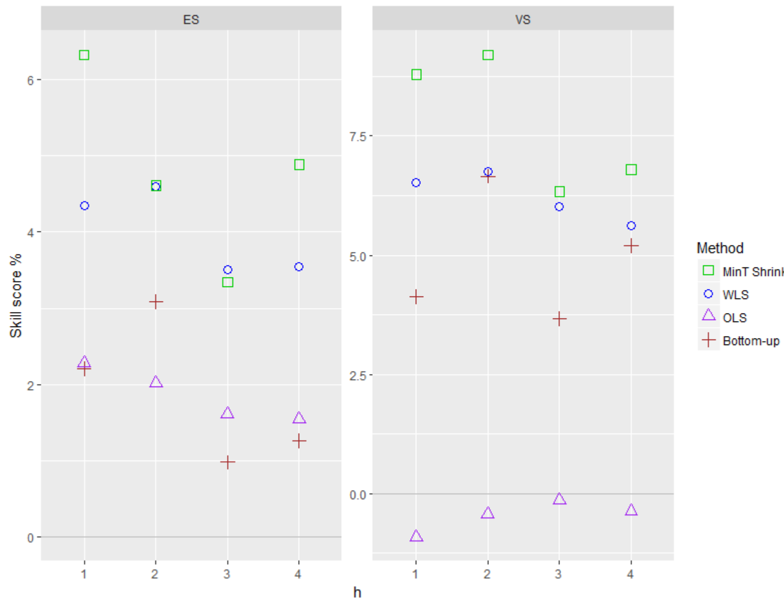

Fig. 11 Skill scores with respect to energy score and variogram score for multivariate Gaussian forecast distribution of income hierarchy

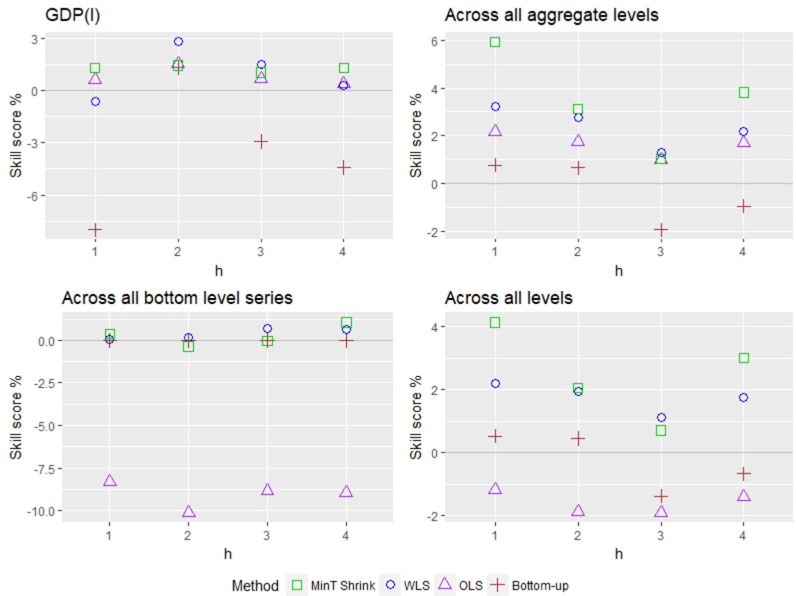

Fig. 12 Skill scores for univariate Gaussian forecast distributions of individual series of income hierarchy

Expenditure approach

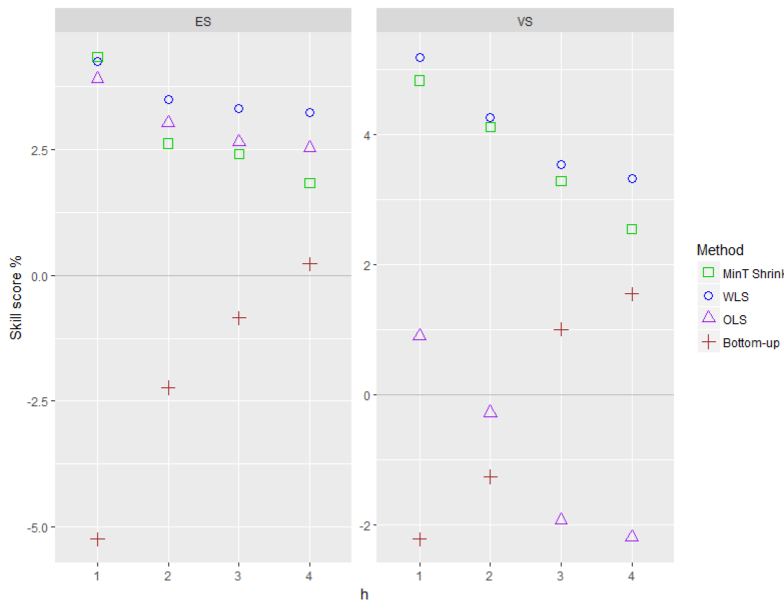

Fig. 13 Skill scores with respect to energy score and variogram score for multivariate Gaussian forecast distribution of expenditure hierarchy

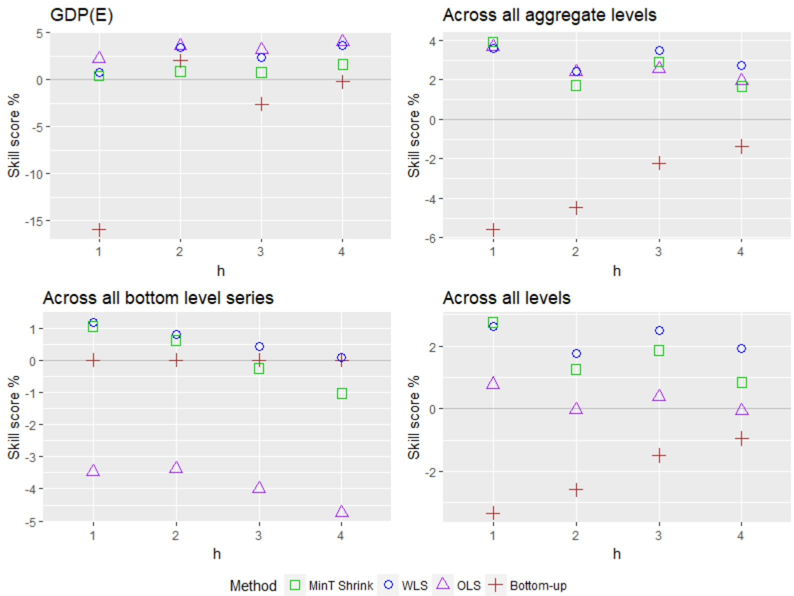

Fig. 14 Skill scores for univariate Gaussian forecast distributions of individual series of expenditure hierarchy

Appendix

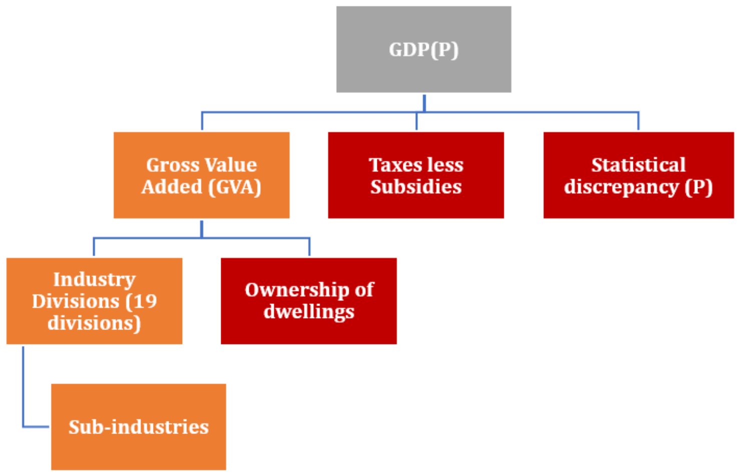

Fig. 15 Hierarchy of production approach.

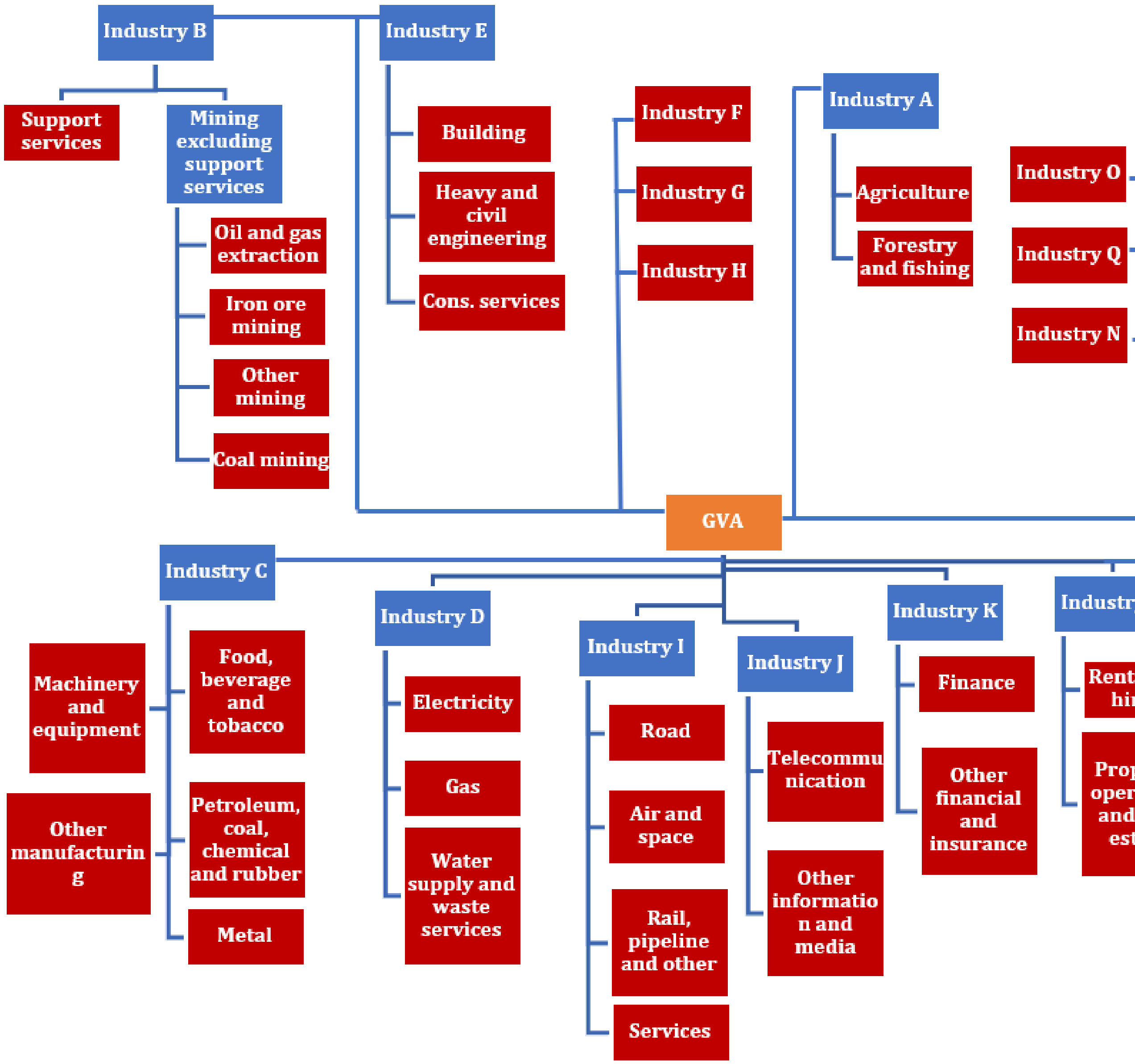

Fig. 16 Hierarchy of GVA under production approach.

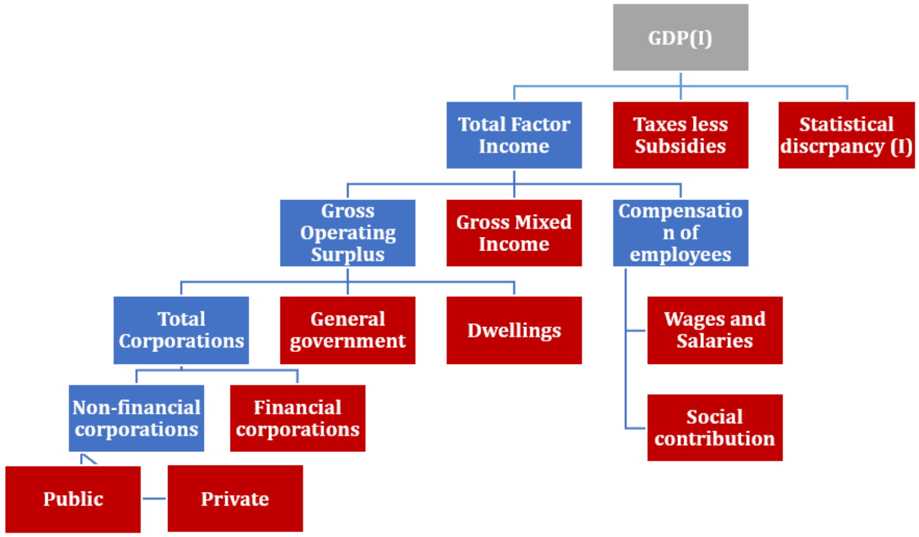

Fig. 17 Hierarchy of income approach.

$$GDP(E) = Final\ consumption\ expenditure + Gross\ fixed\ capital\ formation + Changes\ in\ inventories + Exports\ of\ goods\ and\ services - Imports\ of\ goods\ and\ services + Statistical\ decrepency\ (E)$$

Associated hierarchical structure is given in figure 18, 19 and 20.

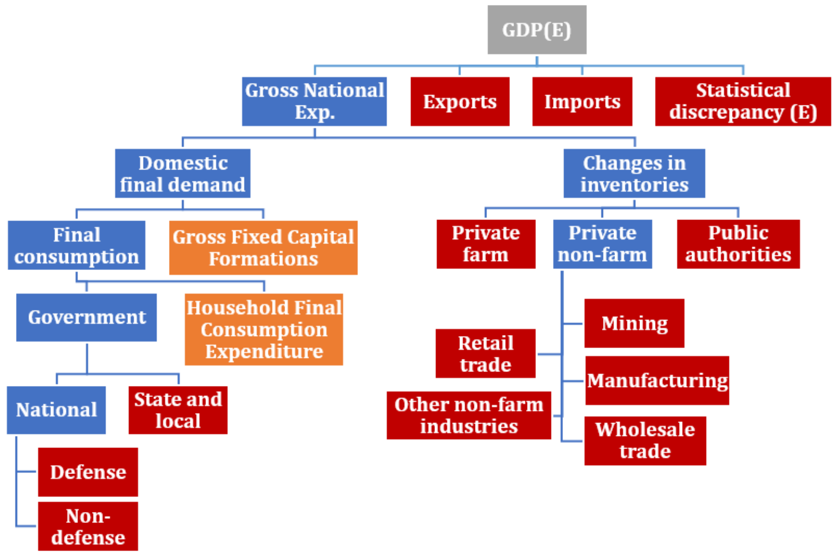

Fig. 18 Hierarchy of expenditure approach.

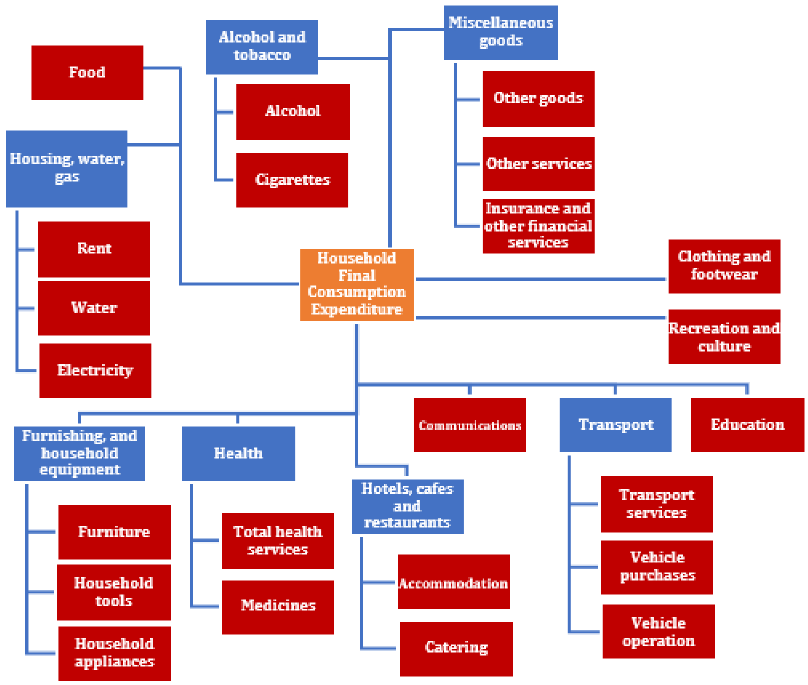

Fig. 19 Household final consumption expenditure under expenditure approach.

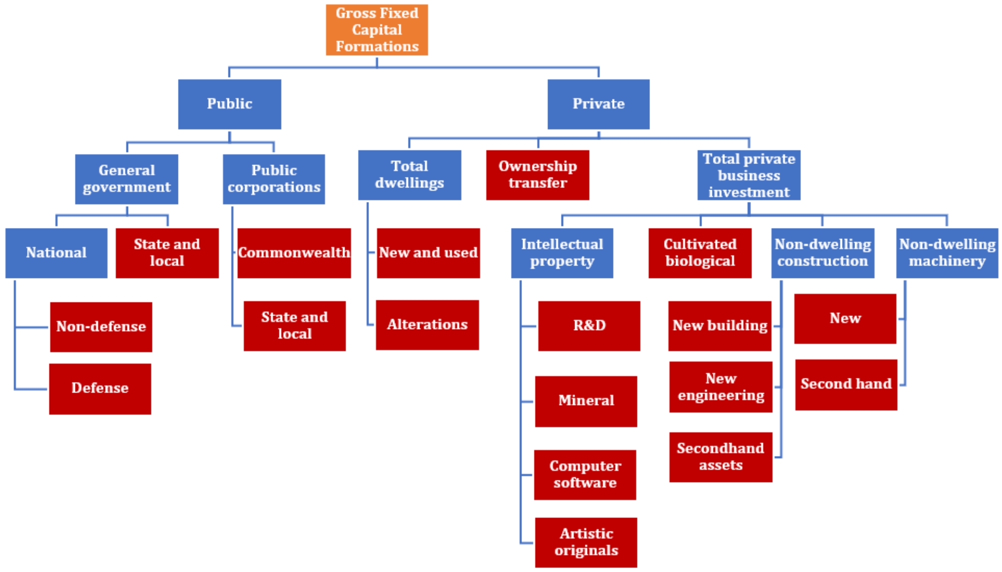

Fig. 20 Gross fixed capital formation (GFCF) under expenditure approach.

Table 1 Variables, Series IDs and their descriptions for Income Approach

Variable	Series ID	Description
Gdpi	A2302467A	GDP(I)
Sdi	A2302413V	Statistical discrepancy (I)
Tsi	A2302412T	Taxes less subsidies (I)
TfiCoeWns	A2302399K	Compensation of employees; Wages and salaries
TfiCoeEsc	A2302400J	Compensation of employees; Employers' social contributions
TfiCoe	A2302401K	Compensation of employees
TfiGosCopNfnPvt	A2323369L	Private non-financial corporations; Gross operating surplus
TfiGosCopNfnPub	A2302403R	Public non-financial corporations; Gross operating surplus
TfiGosCopNfn	A2302404T	Non-financial corporations; Gross operating surplus
TfiGosCopFin	A2302405V	Financial corporations; Gross operating surplus
TfiGosCop	A2302406W	Total corporations; Gross operating surplus
TfiGosGvt	A2298711F	General government; Gross operating surplus
TfiGosDwl	A2302408A	Dwellings owned by persons; Gross operating surplus
TfiGos	A2302409C	All sectors; Gross operating surplus
TfiGmi	A2302410L	Gross mixed income
Tfi	A2302411R	Total factor income

Table 2 Variables, Series IDs and their descriptions for Expenditure Approach

Variable	Series ID	Description
Gdpe	A2302467A	GDP(E)
Sde	A2302566J	Statistical Discrepancy(E)
Exp	A2302564C	Exports of goods and services
Imp	A2302565F	Imports of goods and services
Gne	A2302563A	Gross national exp.
GneDfdFceGvtNatDef	A2302523J	Gen. gov. - National; Final consumption exp. - Defence
GneDfdFceGvtNatNdf	A2302524K	Gen. gov. - National; Final consumption exp. - Non-defence
GneDfdFceGvtNat	A2302525L	Gen. gov. - National; Final consumption exp.
GneDfdFceGvtSnl	A2302526R	Gen. gov. - State and local; Final consumption exp,
GneDfdFceGvt	A2302527T	Gen. gov.; Final consumption exp.
GneDfdFce	A2302529W	All sectors; Final consumption exp.
GneDfdGfcPvtTdwNnu	A2302543T	Pvt.; Gross fixed capital formation (GFCF)
GneDfdGfcPvtTdwAna	A2302544V	Pvt.; GFCF - Dwellings - Alterations and additions
GneDfdGfcPvtTdw	A2302545W	Pvt.; GFCF - Dwellings - Total
GneDfdGfcPvtOtc	A2302546X	Pvt.; GFCF - Ownership transfer costs
GneDfdGfcPvtPbiNdcNbd	A2302533L	Pvt. GFCF - Non-dwelling construction - New building
GneDfdGfcPvtPbiNdcNec	A2302534R	Pvt.; GFCF - Non-dwelling construction - New engineering construction
GneDfdGfcPvtPbiNdcSha	A2302535T	Pvt.; GFCF - Non-dwelling construction - Net purchase of second hand assets
GneDfdGfcPvtPbiNdc	A2302536V	Pvt.; GFCF - Non-dwelling construction - Total
GneDfdGfcPvtPbiNdmNew	A2302530F	Pvt.; GFCF - Machinery and equipment - New
GneDfdGfcPvtPbiNdmSha	A2302531J	Pvt.; GFCF - Machinery and equipment - Net purchase of second hand assets
GneDfdGfcPvtPbiNdm	A2302532K	Pvt.; GFCF - Machinery and equipment - Total
GneDfdGfcPvtPbiCbr	A2716219R	Pvt.; GFCF - Cultivated biological resources
GneDfdGfcPvtPbiIprRnd	A2716221A	Pvt.; GFCF - Intellectual property products - Research and development
GneDfdGfcPvtPbiIprMnp	A2302539A	Pvt.; GFCF - Intellectual property products - Mineral and petroleum exploration
GneDfdGfcPvtPbiIprCom	A2302538X	Pvt.; GFCF - Intellectual property products - Computer software
GneDfdGfcPvtPbiIprArt	A2302540K	Pvt.; GFCF - Intellectual property products - Artistic originals
GneDfdGfcPvtPbiIpr	A2716220X	Pvt.; GFCF - Intellectual property products Total
GneDfdGfcPvtPbi	A2302542R	Pvt.; GFCF - Total private business investment
GneDfdGfcPvt	A2302547A	Pvt.; GFCF
GneDfdGfcPubPcpCmw	A2302548C	Plc. corporations - Commonwealth; GFCF
GneDfdGfcPubPcpSnl	A2302549F	Plc. corporations - State and local; GFCF
GneDfdGfcPubPcp	A2302550R	Plc. corporations; GFCF Total
GneDfdGfcPubGvtNatDef	A2302551T	Gen. gov. - National; GFCF - Defence
GneDfdGfcPubGvtNatNdf	A2302552V	Gen. gov. - National ; GFCF - Non-defence
GneDfdGfcPubGvtNat	A2302553W	Gen. gov. - National ; GFCF Total
GneDfdGfcPubGvtSnl	A2302554X	Gen. gov. - State and local; GFCF
GneDfdGfcPubGvt	A2302555A	Gen. gov.; GFCF
GneDfdGfcPub	A2302556C	Plc.; GFCF
GneDfdGfc	A2302557F	All sectors; GFCF

Table 3 Variables, Series IDs and their descriptions for Changes in Inventories - Expenditure Approach

Variable	Series ID	Description
GneCii	A2302562X	Changes in Inventories
GneCiiPfm	A2302560V	Farm
GneCiiPba	A2302561W	Public authorities
GneCiiPnf	A2302559K	Private; Non-farm Total
GneCiiPnfMin	A83722619L	Private; Mining (B)
GneCiiPnfMan	A3348511X	Private; Manufacturing (C)
GneCiiPnfWht	A3348512A	Private; Wholesale trade (F)
GneCiiPnfRet	A3348513C	Private; Retail trade (G)
GneCiiPnfOnf	A2302273C	Private; Non-farm; Other non-farm industries

Table 4 Variables, Series IDs and their descriptions for Household Final Consumption - Expenditure Approach

Variable	Series ID	Description
GneDfdHfc	A2302254W	Household Final Consumption Expenditure
GneDfdFceHfcFud	A2302237V	Food
GneDfdFceHfcAbt	A3605816F	Alcoholic beverages and tobacco
GneDfdFceHfcAbtCig	A2302238W	Cigarettes and tobacco
GneDfdFceHfcAbtAlc	A2302239X	Alcoholic beverages
GneDfdFceHfcCnf	A2302240J	Clothing and footwear
GneDfdFceHfcHwe	A3605680F	Housing, water, electricity, gas and other fuels
GneDfdFceHfcHweRnt	A3605681J	Actual and imputed rent for housing
GneDfdFceHfcHweWsc	A3605682K	Water and sewerage charges
GneDfdFceHfcHweEgf	A2302242L	Electricity, gas and other fuel
GneDfdFceHfcFhe	A2302243R	Furnishings and household equipment
GneDfdFceHfcFheFnt	A3605683L	Furniture, floor coverings and household goods
GneDfdFceHfcFheApp	A3605684R	Household appliances
GneDfdFceHfcFheTls	A3605685T	Household tools
GneDfdFceHfcHlt	A2302244T	Health
GneDfdFceHfcHltMed	A3605686V	Medicines, medical aids and therapeutic appliances
GneDfdFceHfcHltHsv	A3605687W	Total health services
GneDfdFceHfcTpt	A3605688X	Transport
GneDfdFceHfcTptPvh	A2302245V	Purchase of vehicles
GneDfdFceHfcTptOvh	A2302246W	Operation of vehicles
GneDfdFceHfcTptTsv	A2302247X	Transport services
GneDfdFceHfcCom	A2302248A	Communications
GneDfdFceHfcRnc	A2302249C	Recreation and culture
GneDfdFceHfcEdc	A2302250L	Education services
GneDfdFceHfcHcr	A2302251R	Hotels, cafes and restaurants
GneDfdFceHfcHcrCsv	A3605694V	Catering services
GneDfdFceHfcHcrAsv	A3605695W	Accommodation services
GneDfdFceHfcMis	A3605696X	Miscellaneous goods and services
GneDfdFceHfcMisOgd	A3605697A	Other goods
GneDfdFceHfcMisIfs	A2302252T	Insurance and other financial services
GneDfdFceHfcMisOsv	A3606485T	Other services

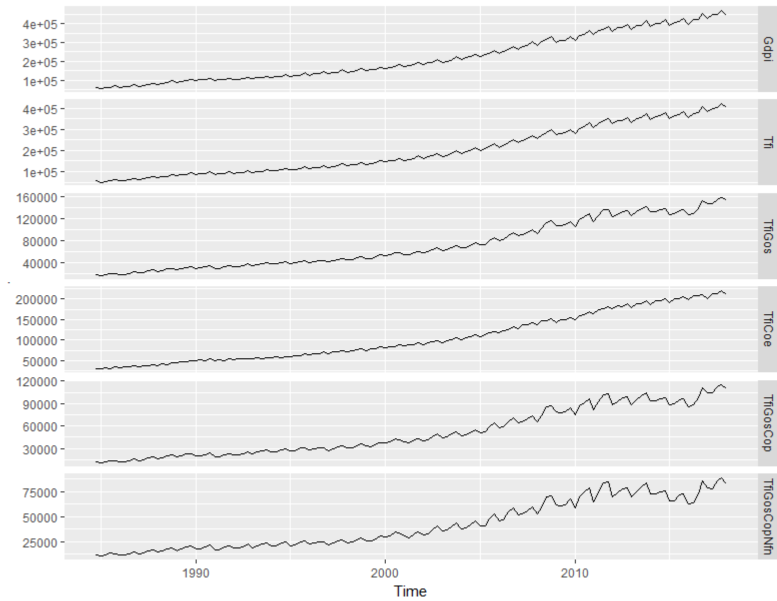

Fig. 21 All aggregate level series of income hierarchy.

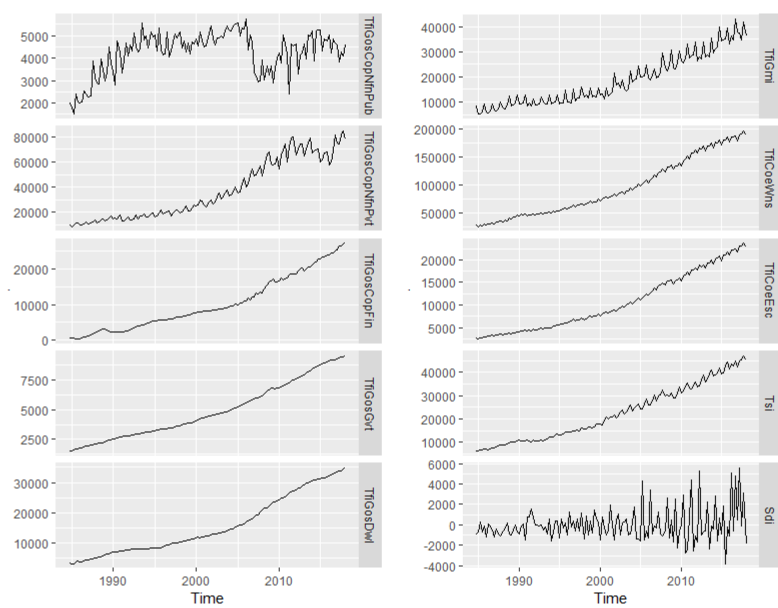

Fig. 22 All bottom level series of income hierarchy.

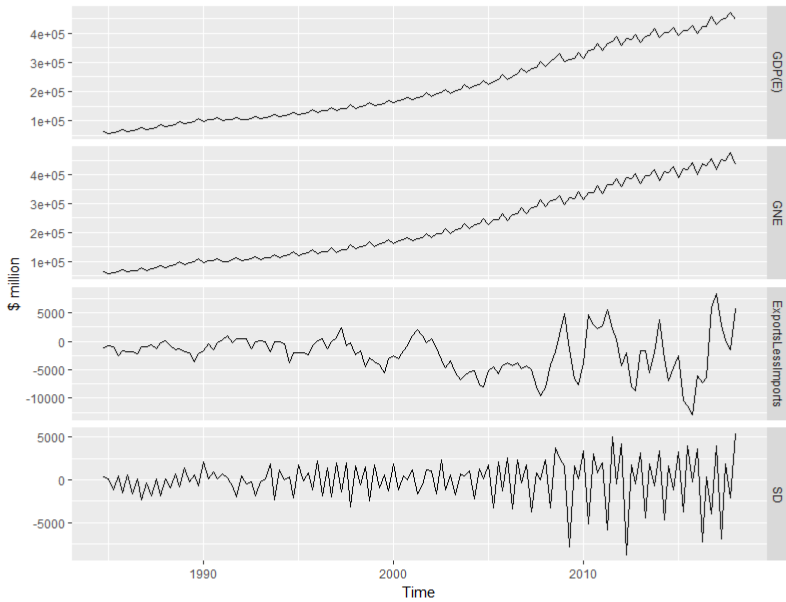

Fig. 23 GDP(E), GNE, Experts less Imports and Statistical discrepancy in expenditure hierarchy.

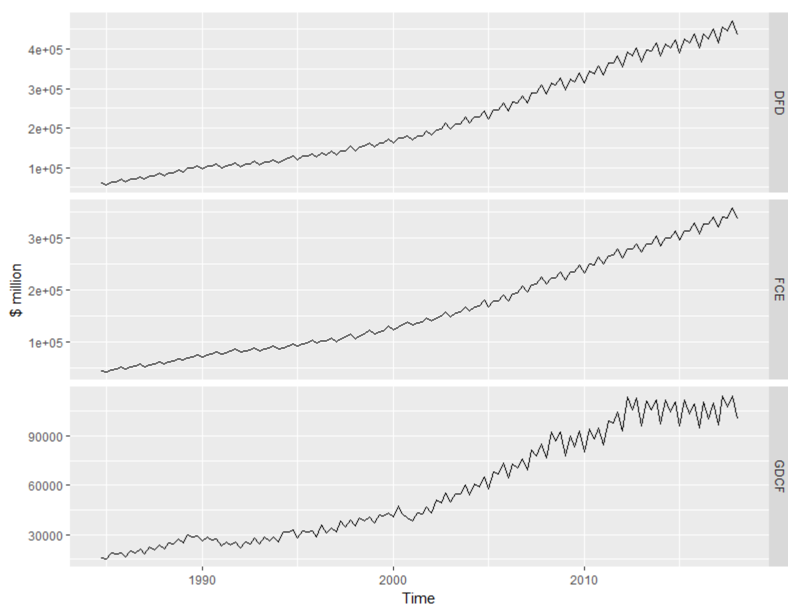

Fig. 24 Domestic Final Demand, Final Consumption Expenditure and Gross Fixed Capital Formations in expenditure hierarchy.

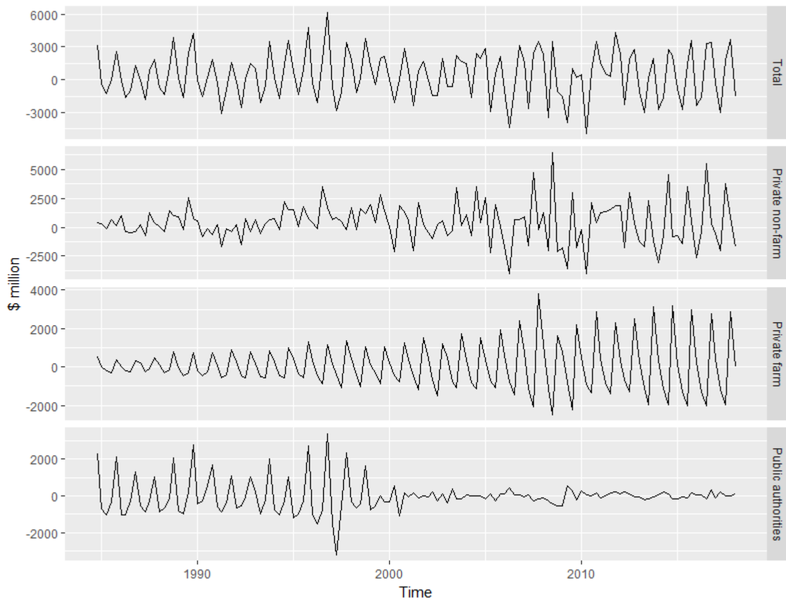

Fig. 25 Total changes in inventory, Private non-farm, Private farm and Public authorities in expenditure hierarchy.

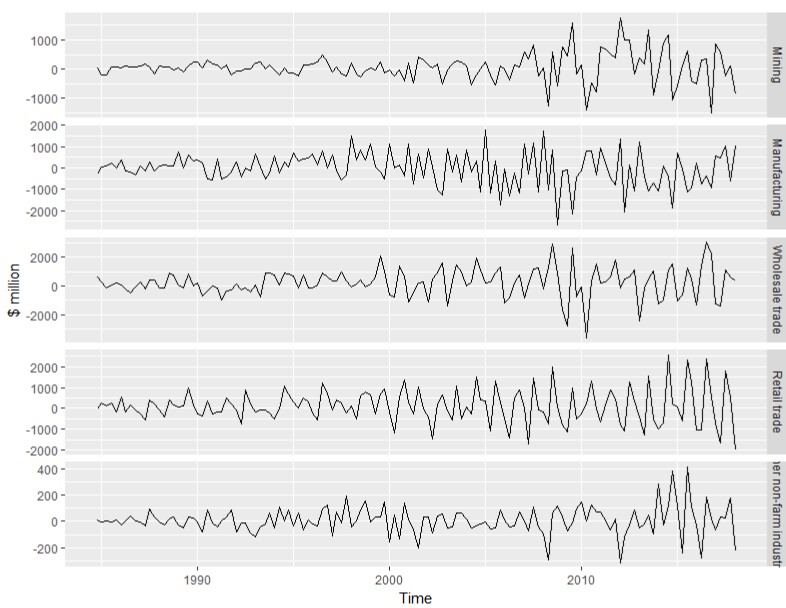

Fig. 26 Disaggregation of private non-farm in expenditure hierarchy.

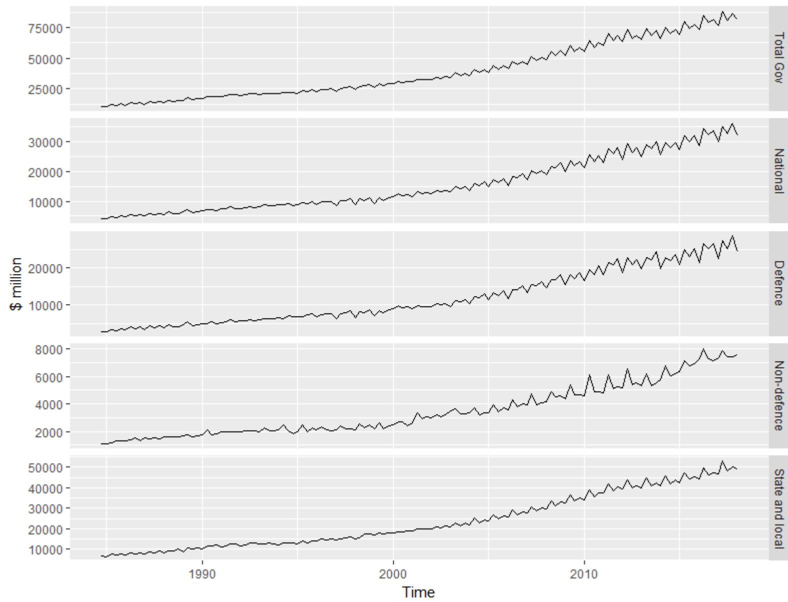

Fig. 27 Disaggregation of government final consumption expenditure.

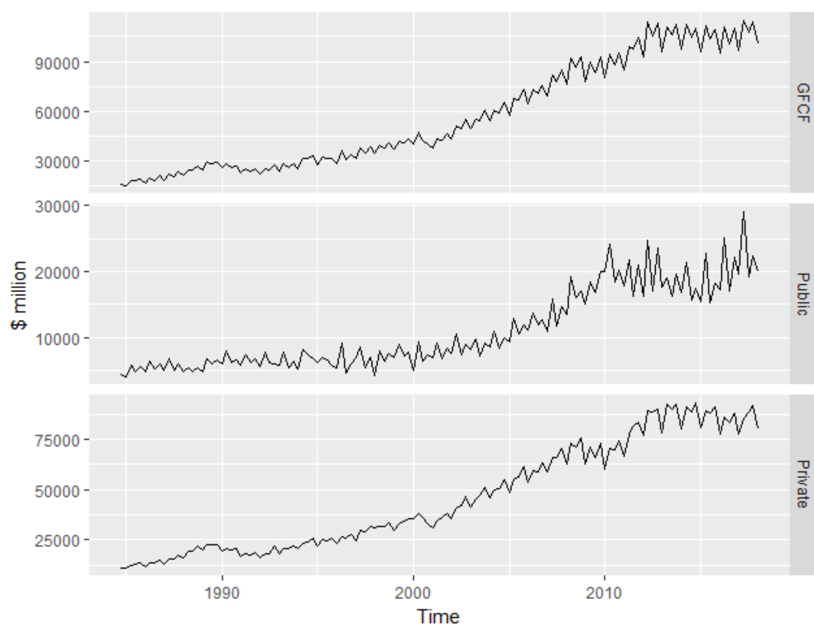

Fig. 28 Public, private and total fixed capital formations.

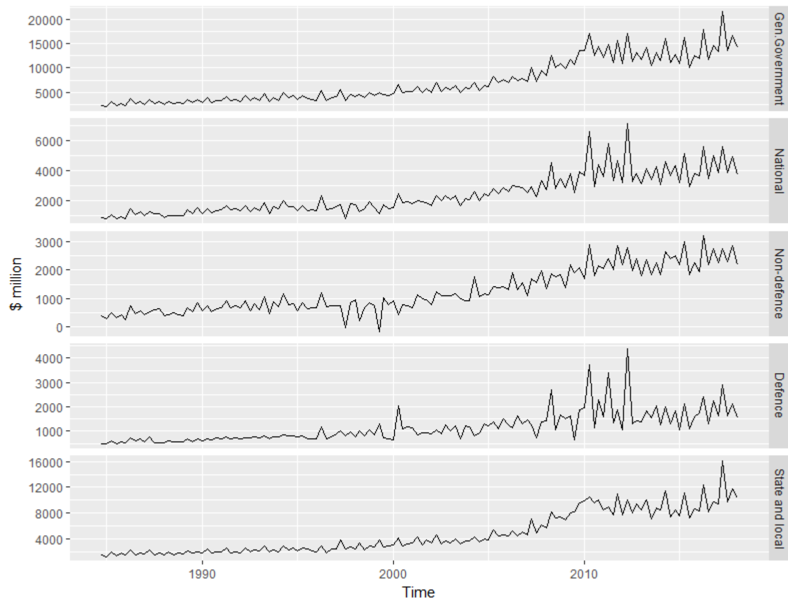

Fig. 29 Disaggregation of general government of Gross fixed capital formations.

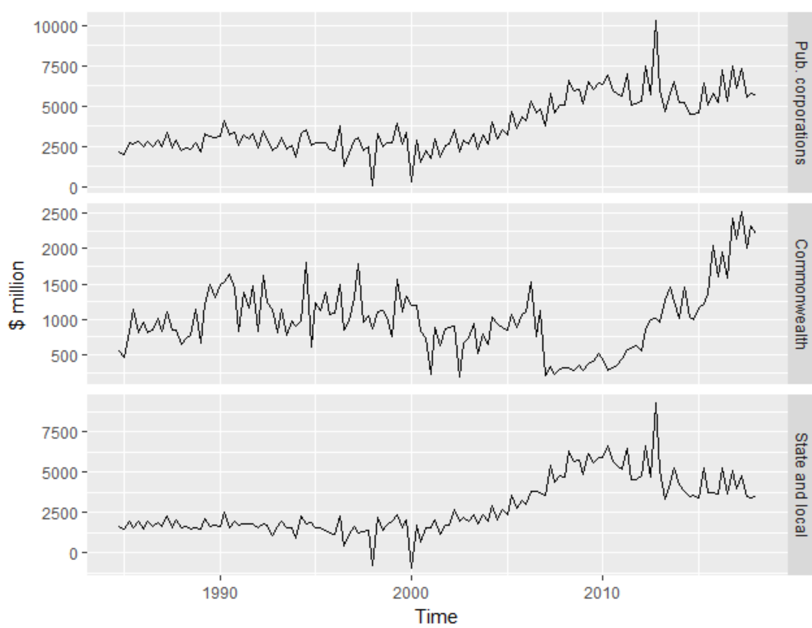

Fig. 30 Disaggregation of public corporations of Gross fixed capital formations.

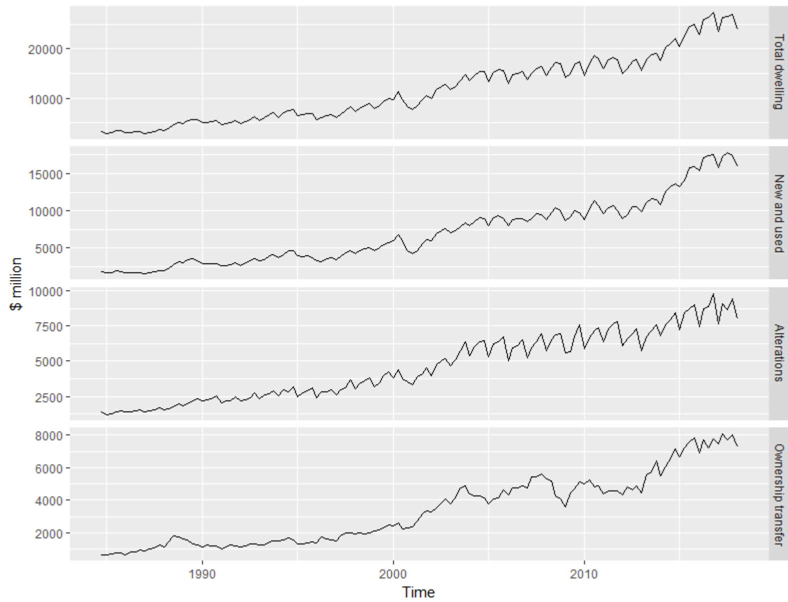

Fig. 31 Disaggregation of total dwelling and ownership transfer.

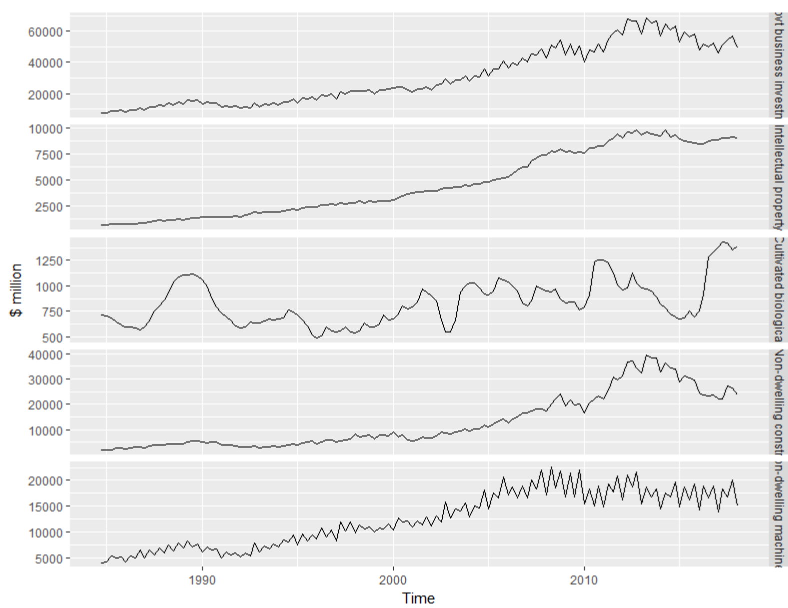

Fig. 32 Main disaggregation of total private business investments.

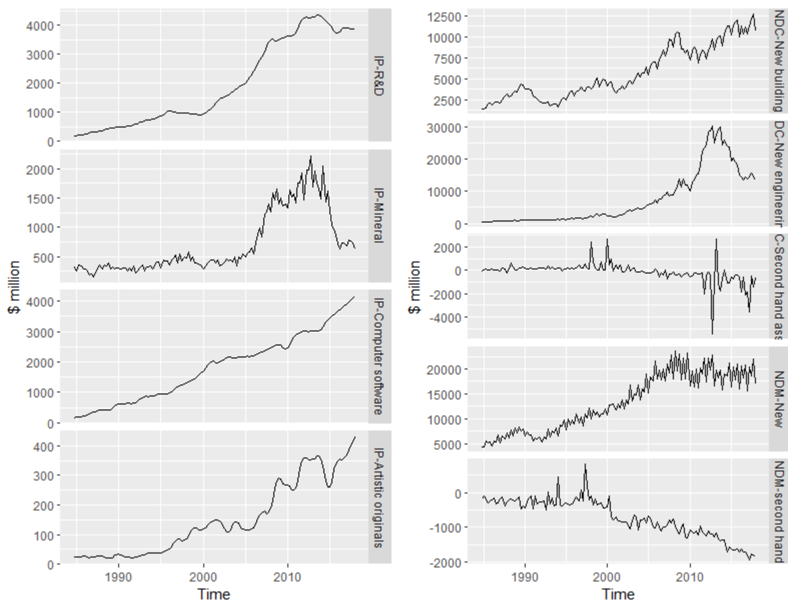

Fig. 33 Remaining disaggregation of total private business investments.

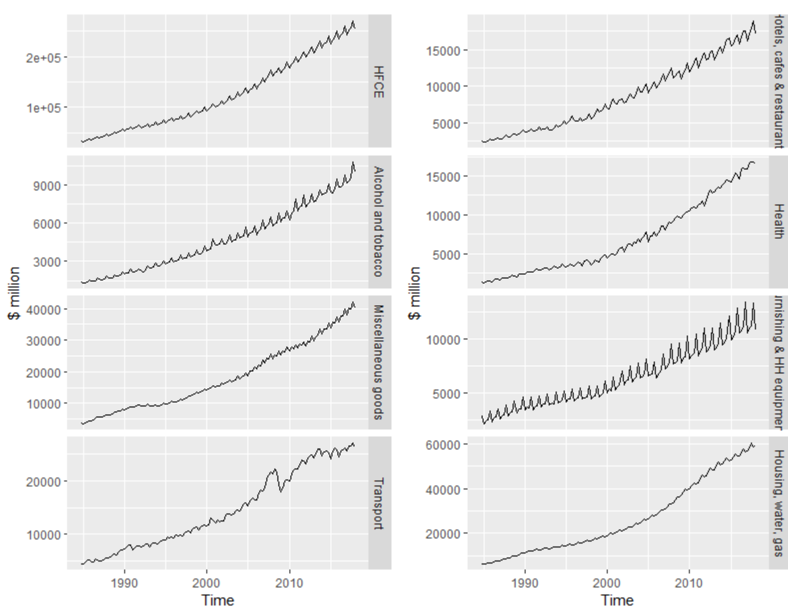

Fig. 34 Main disaggregation of household final consumption expenditure.

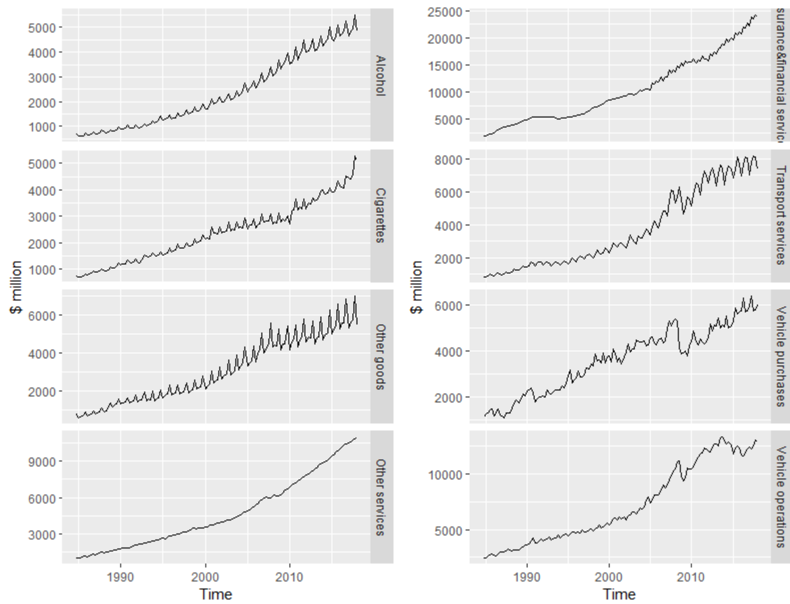

Fig. 35 Disaggregation of household final consumption expenditure.

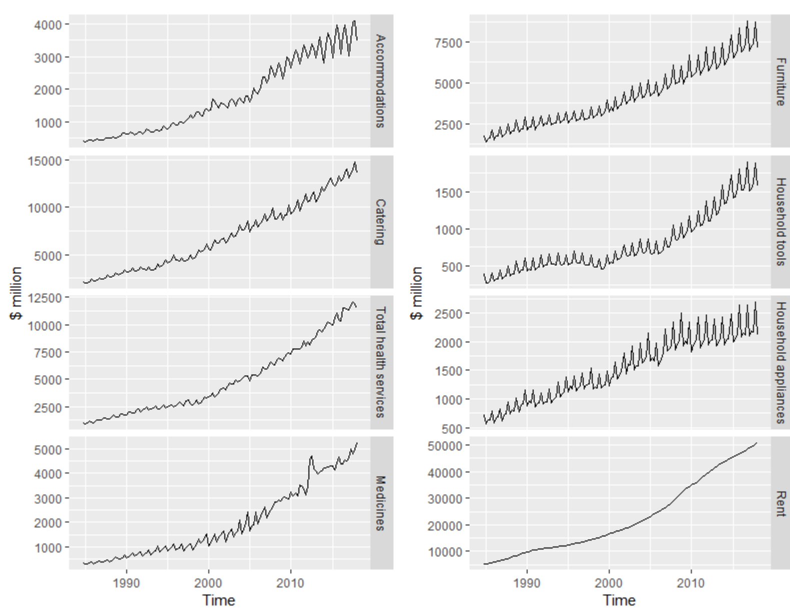

Fig. 36 Disaggregation of household final consumption expenditure.

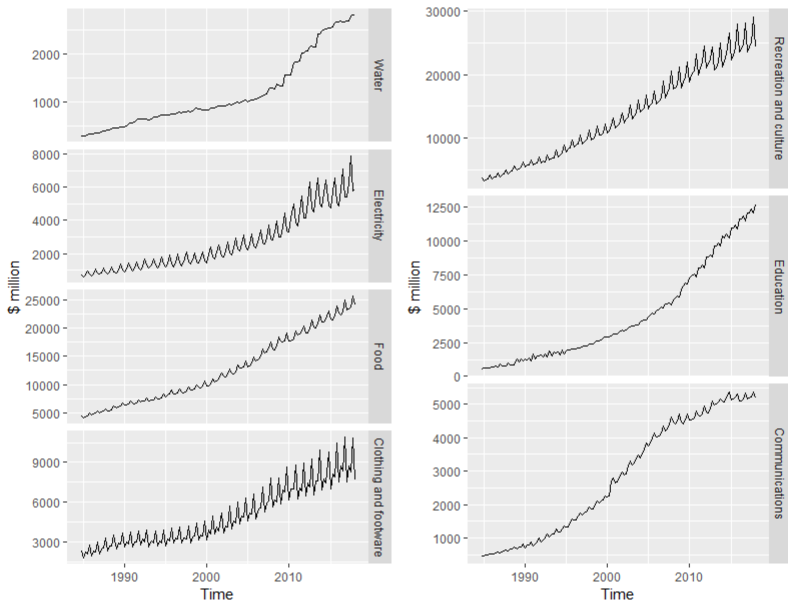

Fig. 37 Disaggregation of household final consumption expenditure.

References

Australian Bureau of Statistics (2015), Australian System of National Accounts: Concepts, Sources and Methods, Technical report.
URL: *http://www.abs.gov.au/AUSSTATS/abs@.nsf/DetailsPage/5216.02015?OpenDocument*

Australian Bureau of Statistics (2018), Australian National Accounts : National Income, Expenditure and Product, Technical report.
URL: *http://www.abs.gov.au/AUSSTATS/abs@.nsf/DetailsPage/5206.0Sep 2018?OpenDocument*

Dunn, D. M., Williams, W. H. & Dechaine, T. L. (1976), 'Aggregate Versus Subaggregate Models in Local Area Forecasting', *Journal of American Statistical Association* **71**(353), 68–71.

Gamakumara, P., Panagiotelis, A., Athanasopoulos, G. & Hyndman, R. J. (2018), Probabilisitic Forecasts in Hierarchical Time Series.

Gneiting, T. & Katzfuss, M. (2014), 'Probabilistic Forecasting', *Annual Review of Statistics and Its Application* **1**, 125–151.

Gneiting, T. & Raftery, A. E. (2005), 'Weather forecasting with ensemble methods', *Science* **310.5746**, 248–249.

Gneiting, T. & Raftery, A. E. (2007), 'Strictly Proper Scoring Rules, Prediction, and Estimation', *Journal of the American Statistical Association* **102**(477), 359–378.

Gneiting, T., Stanberry, L. I., Grimit, E. P., Held, L. & Johnson, N. A. (2008), 'Assessing probabilistic forecasts of multivariate quantities, with an application to ensemble predictions of surface winds', *Test* **17**(2), 211–235.

Gross, C. W. & Sohl, J. E. (1990), 'Disaggregation methods to expedite product line forecasting', *Journal of Forecasting* **9**(3), 233–254.

Hyndman, R., Yasmeen, F., Ihaka, R., Reid, D. & Shaub, D. (2018), 'forecast: Forecasting Function for Time Series and Linear Models'.
URL: *https://pkg.robjhyndman.com/forecast*

Manzan, S. & Zerom, D. (2008), 'A bootstrap-based non-parametric forecast density', *International Journal of Forecasting* **24**(3), 535–550.

Schäfer, J. & Strimmer, K. (2005), 'A Shrinkage Approach to Large-Scale Covariance Matrix Estimation and Implications for Functional Genomics', *Statistical Applications in Genetics and Molecular Biology* **4**(1).
URL: *https://www.degruyter.com/view/j/sagmb.2005.4.issue-1/sagmb.2005.4.1.1175/sagmb.2005.4.1.1175.xml*

Scheuerer, M. & Hamill, T. M. (2015), 'Variogram-Based Proper Scoring Rules for Probabilistic Forecasts of Multivariate Quantities *', *Monthly Weather Review* **143**(4), 1321–1334.

Taieb, S. B., Taylor, J. W. & Hyndman, R. J. (2017), 'Hierarchical Probabilistic Forecasting of Electricity Demand with Smart Meter Data', pp. 1–30.

Vilar, J. A. & Vilar, J. A. (2013), 'Time series clustering based on nonparametric multidimensional forecast densities', *Electronic Journal of Statistics* **7**(1), 1019–1046.