Hierarchical Forecasting

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1 Introduction

Structure of the data with an macroeconomic example

Temporal hierarchies

Forecasting hierarchical time series

- Importance of coherency
- Point forecasting
- Probabilistic forecasting

2 Hierarchical forecasting in macroeconomic data

The key macroeconomic indicators such as Gross Domestic Product (GDP), inflation and monetary policies which are used to study the behavior and performance of an economy as a whole are it self aggregates of various other components. For example, if we take the GDP growth, it is the aggregate of consumption, government expenditure, investments and net exports. These four components are again aggregates of some sub components. When we collect data for each of these individual variables over some time period, we will observe a collection of multiple time series that are bounded with some aggregation constraints. Thus the macroeconomic data are naturally forming cross sectional hierarchical time series.

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If the interest is on a single macroeconomic variable along different time granularities, then it can be considered as a temporal hierarchy. For example, suppose we have monthly consumer product index (CPI) of a particular country. The quarterly CPI is then the aggregate of corresponding monthly CPI of each quarter. Similarly the yearly CPI is the aggregate of quarterly CPI of each year. Hence it will form a temporal hierarchy.

Macroeconomic forecasts are crucial for economic and business activities of any economy. Therefore this area of study has a long history in literature. Econometricians have developed various approaches for getting reliable economic forecasts using macroeconomic data. However, the information of aggregation structure in real data is limitedly used in literature. Moreover, having coherent forecasts will help the economists and policy makers for align decision making that impact for the whole economy. Therefore, our focus in this chapter is to introduce hierarchical forecasting methods for macroeconomic forecasting particularly for cross-sectional hierarchical data structures.

Obtaining coherent forecasts are independent from the forecasting models. That means forecasters were given the freedom to use any reliable forecasting method to obtain the forecasts for individual series in the hierarchy. Getting coherent forecasts is a post-processing technique which ensures the aggregation properties are preserved in the forecasts.

briefly discuss the point forecasts as well as probabilistic forecasts in the sense of macroeconomic data

3 Point forecasting

Fix this depending on Section 2 We first introduce some general notation which can be used to capture general macroeconomic variables that are formed by hierarchical structures as discussed in 2.

To simplify the notation we use the simple two-level hierarchical structure shown in Figure 1. Denote as $y_{Tot,t}$ the value observed at time t for the most aggregate (Total) series corresponding to level 0 of the hierarchy. Below level 0, denote as $y_{i,t}$ the value of the series corresponding to node i, observed at time t. For example, $y_{A,t}$ denotes the tth observation of the series corresponding to node A at level 1, $y_{AB,t}$ denotes the tth observation of the series corresponding to node AB at level 2, and so on.
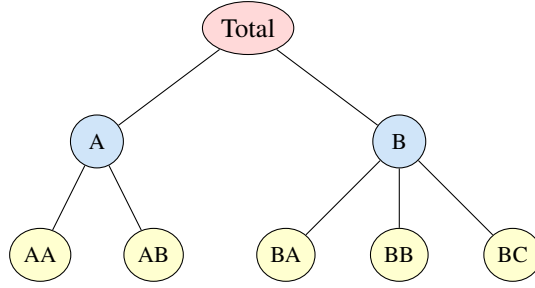

Fig. 1 A simple two-level hierarchical structure.

Let $\mathbf{y}_t = (y_{Tot,t}, y_{A,t}, y_{B,t}, y_{AA,t}, y_{AB,t}, y_{BA,t}, y_{BB,t}, y_{BC,t})'$, a vector containing observations across all series of the hierarchy at t. Similarly denote as $\mathbf{b}_t = (y_{AA,t}, y_{AB,t}, y_{BA,t}, y_{BB,t}, y_{BC,t})'$ a vector containing observations only for the bottom-level series. In general, $\mathbf{y}_t \in \mathbb{R}^n$ and $\mathbf{b}_t \in \mathbb{R}^m$ where n denotes the number of total series in the structure, m the number of series at the bottom level, and $n > m$ always. In the simple example of Figure 1, $n = 8$ and $m = 5$.

Aggregation constraints dictate that $y_{Tot} = y_{A,t} + y_{B,t} = y_{AA,t} + y_{AB,t} + y_{BA,t} + y_{BB,t} + y_{BC,t}$, $y_{A,t} = y_{AA,t} + y_{AB,t}$ and $y_B = y_{BA,t} + y_{BB,t} + y_{BC,t}$. Hence we can write

$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t, \tag{1}$$

where

$$\mathbf{S} = \begin{pmatrix} 1\ 1\ 1\ 1\ 1 \\ 1\ 1\ 0\ 0\ 0 \\ 0\ 0\ 1\ 1\ 1 \\ \mathbf{I}_5 \end{pmatrix}$$

an $n \times m$ matrix referred to as the *summing matrix* and $\mathbf{I}_m$ is an m-dimensional identity matrix. $\mathbf{S}$ reflects the linear aggregation constraints and in particular how the bottom-level series aggregate to levels above. Thus, columns of $\mathbf{S}$ span the linear

subspace of $\mathbb{R}^n$ for which the aggregation constraints hold. We refer to this as the *coherent subspace* and denote it by $\mathfrak{s}$. Notice that pre-multiplying a vector in $\mathbb{R}^m$ by $\mathbf{S}$ will result to an n-dimensional vector that lies in $\mathfrak{s}$.

The notation introduced here is very general and can capture any form of linear constraints useful for forecasting economic variables. GDP which we discussed in Section 2 is an example a hierarchy where variables at lower levels of aggregation are simply summed to form variables at higher levels.

Examples of diverse types of constraints include trade balance, which equals exports minus imports, or inflation measures such are the consumer price index which is computed as a weighted average of sub-indices, which are in turn weighted averages of sub-sub-indices and so on.

A commonly found aggregation structure is the one shown in Figure 2 referred to as a grouped structure. In this simple example we have two groupings or attributes of the Total series which we are interested in. The Total series can be alternatively disaggregated into, A and B, or X and Y. Hence with grouped time series, the structure does not naturally disaggregate in a unique hierarchical manner, as the disaggregating factors are both nested and crossed.

TALK TO TAS ABOUT THIS ONE. An example is state domestic product of a country versus gross domestic product where the economic variables interact with geographical location.
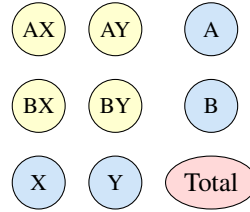
Fig. 2 A simple two-level grouped structure.

An interesting case of grouped structures are temporal hierarchies introduced by **?**. In this case the structure spans the time dimension and we interested in temporal aggregation.

? study the effect of aggregation on several key macroeconomic variables and conclude that temporal aggregation to the annual level greatly simplifies the complex low-frequency cyclical ARIMA dynamics found in monthly and quarterly data.

They state that "quarterly data may be the best compromise among frequency of observation, measurement error, and temporal aggregation distortion". A similar conclusion is reached by **?**.

To demonstrate the grouping nature of the data consider a monthly series, such as industrial production. An example is monthly data which

There is a huge related
??????

and hence we are interested on the effects of temporal aggregation on forecasting economic variables.

Add something to jump to coherent forecasts depending on the intro.

Definition 1. A set of h-step ahead forecasts $\tilde{\mathbf{y}}_{t+h|t}$, stacked in the same order as $\mathbf{y}_t$ and generated using information up to and including time t, are said to be coherent if $\tilde{\mathbf{y}}_{t+h|t} \in \mathfrak{s}$.

Hence a set of coherent forecasts respect the aggregation constraints across the hierarchy. That is the forecasts of lower level series aggregate up to their corresponding upper level series of the hierarchy and vice versa.

Do we need the following? Let us consider the smallest possible hierarchy with two bottom level series, A and B that add up to the top level Tot. Suppose $\breve{\mathbf{y}}_{t+h|t}$ of this hierarchy is given by $\breve{\mathbf{y}}_{t+h|t} = [\breve{y}_{Tot,t+h|t}, \breve{y}_{A,t+h|t}, \breve{y}_{B,t+h|t}]$. Due to the aggregation structure we have $\breve{y}_{Tot,t+h|t} = \breve{y}_{A,t+h|t} + \breve{y}_{B,t+h|t}$. This implies that, even though $\breve{y}_{Tot,t+h|t} \in \mathbb{R}^3$, the points actually lie in $\mathfrak{s} \subset \mathbb{R}^3$, which is a two dimensional subspace within $\mathbb{R}^3$ space.

I think the 3-series example talking about *basis series* in the Prob Fore paper is useful here. Should we add?

3.1 Traditional forecasting approaches

Traditionally hierarchical structures were forecasted using these. We briefly present these here. For more details - give references.

Traditional approaches involve selecting a level of aggregation, generating forecasts at the level and then linearly combining these to generate a set of coherent forecasts the rest of the structure.

3.1.1 Bottom-up

Maybe we add these section headings here.

In the *bottom-up* approach, forecasts for the lowest level series are first generated. These are then aggregated to obtain forecasts for all the other levels of the hierarchy (Dunn et al. 1976). In general this consists of first generating $\hat{\mathbf{b}}_{t+h|t} \in \mathbb{R}^m$, a set of h-step ahead forecasts for the bottom-level series. For the simple hierarchical structure of Figure 1, $\hat{\mathbf{b}}_{t+h|t} = (\hat{y}_{AA,t+h|t}, \hat{y}_{AB,t+h|t}, \hat{y}_{BA,t+h|t}, \hat{y}_{BB,t+h|t}, \hat{y}_{BC,t+h|t})$, where, $\hat{y}_{i,t+h|t}$ is the h-step ahead forecast of the series corresponding to node i. A set of coherent forecasts for the whole hierarchy is given by,

$$\tilde{\mathbf{y}}_{t+h|t}^{BU} = \mathbf{S}\hat{\mathbf{b}}_{t+h|t}.$$

Generating bottom-up forecasts has the advantage of no information being lost due to aggregation. However, bottom-level data can potentially be highly volatile or very noisy and therefore challenging to forecast.

3.1.2 Top-down

In contrast to bottom-up, *top-down* approaches involve first generating forecasts for the most aggregated series and then disaggregating these down the hierarchy. In general, coherent forecasts generated from top-down approaches can be written as,

$$\tilde{\mathbf{y}}_{t+h|t}^{TD} = \mathbf{Sp}\hat{y}_{Tot,t+h|t},$$

where $\mathbf{p} = (p_1, ..., p_m)'$ is an m-dimensional vector consisting of a set of proportions which disaggregate the top-level forecast $\hat{y}_{Tot,t+h|t}$ to forecasts for the bottom-level series. These are then aggregated up by the summing matrix.

Two commonly used sets of proportions are:

$$p_j = \frac{1}{T} \sum_{t=1}^{T} \frac{y_{j,t}}{y_{tot,t}},$$

for $j = 1, \ldots, m$, where each p_j reflects the average of the historical proportions of the bottom-level series $y_{j,t}$ over the period $t = 1, \ldots, T$ relative to the total aggregate $y_{tot,t}$ and hence referred to as *average historical proportions*, and

$$p_j = \frac{\frac{1}{T} \sum_{t=1}^{T} y_{j,t}}{\frac{1}{T} \sum_{t=1}^{T} y_{Tot,t}}, \tag{2}$$

where each proportion p_j captures the average historical value of the bottom-level series $y_{j,t}$ relative to the average value of the total aggregate $y_{tot,t}$ and hence referred to as *proportions of the historical averages*.

Both these approaches performed well in Gross & Sohl (1990). Their most convenient attribute is their simplicity. Generating a set of coherent forecasts involves only modelling and generating forecasts for the most aggregated top-level series. In general, such top-down approaches seem to produce quite reliable forecasts for the aggregate levels and they are useful with low count data. However, a significant disadvantage is the loss of information due to aggregation. Using such top-down approaches, is limited as does not allow to capture and model individual series characteristics such as time dynamics, special events, etc.
Complete this if we decide to leave these in.

To overcome this limitation, **?** introduced a new top-down approach which disaggregate the top level forecasts according to the proportions of forecasts rather than historical proportions. They found that their method outperforms the conventional top-down approach through an empirical application.

This seems to be simply of combination of top and bottom level forecasts. We may drop it.

Recently, Mirčetić et al. (2017) proposed a modified approach to the conventional top-down methods, which involves forecasting h-step ahead proportions of bottom level series relative to the top level series. That is, let the ratio between the jth bottom level series and the top level series is given by, $p_{j,t} = y_{j,t}/y_{Tot,t}$, for $t = 1, ..., T$. Then this series of proportions will be forecast for the h-step ahead period, which is denoted by $\hat{p}_{j,T+h|T}$. These proportions will be then used as the disaggregate proportions to construct coherent top-down forecasts. Thus giving the elements of vector **p** as,

$$p_j = \hat{p}_{j,T+h|T}. \tag{3}$$

This approach is much reliable, since it calculates the proportions as a function of time, rather than using the simple averages in traditional top-down methods.

3.1.3 Middle-out

Add middle-out

As shown by **?**, top-down methods are adding a bias component to each disaggregate level. Therefore, all these top-down approaches are producing biased forecasts even if the top level base forecasts are unbiased.

A compromise between these two approaches is the middle-out method which entails forecasting each series of a selected middle level in the hierarchy and then forecasting upper levels by the bottom-up method and lower levels by the top-down method.

Since this is a hybrid approach of bottom-up and top-down approaches it still remains the limitations of those two.

3.2 Forecast reconciliation approaches

A common theme of the traditional approaches discussed so far is that a single level of aggregation is first selected and then forecasts for that level are either aggregated-up, disaggregated-down or a combination of the two is used, in order to generate coherent forecasts for the whole hierarchy. Hence, these approaches are limited in only using information from a single level of aggregation and furthermore ignoring any correlations across levels.

An alternative framework that overcomes these limitations, first proposed by **?** and implemented by **?**, is one that involves forecast reconciliation. In a first step forecasts for all the series across all levels of the hierarchy are generated, ignoring the aggregation. We refer to these as *base* forecasts and denote them as $\hat{\mathbf{y}}_{t+h|t}$. In general, base forecasts will not be coherent. An example of an exception is when a

simple method such as a random walk is used to generate all base forecasts so that the coherent nature of the data is extended to the forecasts.

In a second step, base forecasts are revised and reconciled so that they become coherent. This is achieved by projecting the base forecasts $\hat{\mathbf{y}}_{t+h|t}$ onto the coherent subspace $\mathfrak{s}$, via a projection matrix $\mathbf{SG}$, resulting in a set of coherent forecasts $\tilde{\mathbf{y}}_{t+h|t}$. More specifically,

$$\tilde{\mathbf{y}}_{t+h|t} = \mathbf{SG}\hat{\mathbf{y}}_{t+h|t}, \tag{4}$$

where $\mathbf{G}$ is an $m \times n$ matrix that maps $\hat{\mathbf{y}}_{t+h|t}$ to the $\mathbb{R}^m$ space, producing a set of coherent forecasts for the bottom-level series, which are in turn mapped to the coherent subspace by the summing matrix $\mathbf{S}$ as defined in (1).

Note that all traditional approaches discussed so far can also be represented by (4) using an appropriately designed $\mathbf{G}$. For example, for the bottom-up approach, $\mathbf{G} = \left(\mathbf{0}_{(m \times n-m)} \ \mathbf{I}_m \right)$. We do not define these as reconciliation approaches as only forecasts from a single level of the hierarchy are used and subsequently these are either aggregated or disaggregated with no revision or reconciliation taking place. Revisit the following: It is worth discussing the unbiased property of these reconciled point forecasts.

Assume that the base forecasts are unbiased. That is, $E_{1:t}(\hat{\mathbf{y}}_{t+h|t}) = \mu_{t+h|t}$ where $\mu_{t+h|t} = E_{1:t}[\mathbf{y}_{t+h}|\mathbf{y}_1,...,\mathbf{y}_t]$ is the true mean with the expectation taken over the observed sample up to time t. Then for any $\mathbf{G}$ such that $\mathbf{SGS} = \mathbf{S}$ or equivalently $\mathbf{SG} = \mathbf{I}_m$ the resulting coherent forecasts are also unbiased. Can we tie in here this? Can we say: More generally (Gamakumara et al. 2018) show that any $\mathbf{SG}$ that is a projection matrix will result to unbiased coherent forecasts.

3.2.1 OLS reconciliation

? proposed to "optimally" reconcile the base forecasts through the following regression model:

$$\hat{\mathbf{y}}_{t+h|t} = \mathbf{S}\beta_{t+h|t} + \varepsilon_{t+h|t}, \tag{5}$$

where $\beta_{t+h|t} = E[\mathbf{b}_{t+h}|\mathbf{b}_1,.....,\mathbf{b}_t]$ is the unknown conditional mean of the bottom-level series and $\varepsilon_{t+h|t}$ is the coherence or reconciliation error with mean zero and variance $\mathbf{V}$. The ordinary least squares (OLS) solution leads to the usual projection matrix $\mathbf{S}(\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'$, so that a set of coherent forecasts are obtained by,

$$\tilde{\mathbf{y}}_{t+h|t}^{\text{OLS}} = \mathbf{S}(\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'\hat{\mathbf{y}}_{t+h|t}.$$

In this reconciliation, the incoherent forecasts are orthogonally projected to the coherent subspace $\mathfrak{s}$. Hence the OLS projection minimises the Euclidean distance between $\hat{\mathbf{y}}_{t+h|t}$ and $\tilde{\mathbf{y}}_{t+h|t}$.

Can we add Tas's picture and talk about optimality in this sense.

We should note that using the GLS solution in this context is not possible as **?** show that $\mathbf{V}$ is not identifiable.

3.2.2 Optimal MinT reconciliation

? build on the work of **?** on forecast reconciliation and introduce the MinT approach. MinT is optimal in the sense that given a set of unbiased base forecasts, MinT returns a set of best linear unbiased coherent forecasts using as **G** the unique solution that minimises the trace (hence MinT) of the variance of the forecast error of the coherent forecasts.

Let

$$\hat{\mathbf{e}}_{t+h|t} = \mathbf{y}_{t+h|t} - \hat{\mathbf{y}}_{t+h|t} \tag{6}$$

denote the forecast errors for a set of unbiased base forecasts $\hat{\mathbf{y}}_{t+h|t}$, with $\text{Var}(\hat{\mathbf{e}}_{t+h|t}) = \mathbf{W}_h$. Also let

$$\tilde{\mathbf{e}}_{t+h|t} = \mathbf{y}_{t+h|t} - \tilde{\mathbf{y}}_{t+h|t}$$

denote a set of coherent forecast errors. Lemma 1 in **?** shows that for any matrix **G** such that $\mathbf{SGS} = \mathbf{S}$, $\text{Var}(\tilde{\mathbf{e}}_{t+h|t}) = \mathbf{SGW}_h\mathbf{S}'\mathbf{G}'$. Furthermore Theorem 1 shows that

$$\mathbf{G} = (\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1} \tag{7}$$

is the unique solution that minimises the $\text{tr}[\mathbf{SGW}_h\mathbf{S}'\mathbf{G}']$ subject to $\mathbf{SGS} = \mathbf{S}$. The unbiasedness constraint ensures that for a set of unbiased base forecasts the resulting reconciled forecasts are also unbiased. Note that unlike the GLS solution to (5), the MinT solution is a function of $\mathbf{W}_h$, the variance of the base forecast errors.

The MinT solution is an oblique projection of $\hat{\mathbf{y}}_{t+h|t}$ onto the coherent subspace $\mathfrak{s}$ and as shown by **?**, it minimises the Mahalonobis distance between $\hat{\mathbf{y}}_{t+h|t}$ and $\tilde{\mathbf{y}}_{t+h|t}$.

A significant advantage of the MinT reconciliation solution is that it is the first to incorporate the full correlation structure of the hierarchy via $\mathbf{W}_h$. However, estimating $\mathbf{W}_h$ is challenging, especially for $h > 1$. Of course setting $\mathbf{W}_h = k_h\mathbf{I}_n$ for all h where $k_h > 0$ is a proportionality constant, leads to the OLS solution of **?**. A disadvantage of this simplifying solution, further to not accounting for the correlations across series, is that the homoscedastic diagonal does not account for the scale differences between the levels of the hierarchy due to aggregation. However OLS does well in practice because as discussed it minimises the Euclidean distance and blah blah. Not sure how much we want to say here.

? present possible alternative estimators for $\mathbf{W}_h$ and show that these lead to different **G** matrices. We summarise these below.

- Set $\mathbf{W}_h = \text{diag}(\hat{\mathbf{W}}_1)$ for all h, where $k_h > 0$ and

$$\hat{\mathbf{W}}_1 = \frac{1}{t}\sum_{k=1}^{t}\hat{\mathbf{e}}_k\hat{\mathbf{e}}_k'$$

is the unbiased sample estimator of the in-sample one-step-ahead base forecast errors as defined in (6). This estimator scales the base forecasts using the variance of the in-sample residuals and is therefore describes and referred to as a WLS estimator.

- Set $\mathbf{W}_h = k_h\hat{\mathbf{W}}_1$, for all h, where $k_h > 0$, the unrestricted sample covariance estimator for $h = 1$. Although this is relatively simple to obtain and provides a good solution for small hierarchies, it does not provide reliable results a m grows compared to t. We refer to this a the MinT(Sample) estimator.
- Set $\mathbf{W}_h = k_h\hat{\mathbf{W}}_1^D$, for all h, where $k_h > 0$, $\hat{\mathbf{W}}_1^D = \lambda_D\text{diag}(\hat{\mathbf{W}}_1) + (1 - \lambda_D)\hat{\mathbf{W}}_1$ is a shrinkage estimator with diagonal target, and shrinkage intensity parameter

$$\hat{\lambda}_D = \frac{\sum_{i\neq j}\hat{Var}(\hat{r}_{ij})}{\sum_{i\neq j}\hat{r}_{ij}^2},$$

where $\hat{r}_{ij}$ is the ijth element of $\hat{\boldsymbol{R}}_1$, the 1-step-ahead sample correlation matrix as proposed by Schäfer & Strimmer (2005). Hence, off-diagonal elements of $\hat{\mathbf{W}}_1$ are shrunk towards zero while diagonal elements (variances) remain unchanged. We refer to this as the MinT(Shrink) estimator.

4 Hierarchical probabilistic forecasting

Point forecasts are limited since they provide no indication of uncertainty around the forecast. A richer description of forecast uncertainty can be obtained by providing a "probabilistic forecasts", that is a full density for the target of interest. For a review of probabilistic forecasts, and methods for evaluating such forecasts known as *scoring rules* see (Gneiting & Katzfuss 2014). In recent years, the use of probabilistic forecasts and their evaluation via scoring rules has become pervasive in macroeconomic forecasting, for example need to find some references that use scoring rules for macro forecasting. Check Bayesian macro guys like Koop Korobilis, Josh Chan also Mike Smith's work with Shaun Vahey.

The literature on hierarchical probabilistic forecasting is still an emerging area of interest. To the best of our knowledge the first attempt to even define coherence in the setting of probabilistic forecasting is provided by Taieb et al. (2017) who define a coherent forecast in terms of a convolution. An equivalent definition, provided by Gamakumara et al. (2018) defines a coherent probabilistic forecast as a probability measure on the coherent subspace $\mathfrak{s}$. Gamakumara et al. (2018) also generalise the concept of forecast reconciliation to the probabilistic setting. In brief, let $\mathscr{A}$ be a subset[1] of $\mathfrak{s}$ and let $\mathscr{B}$ be all points in $\mathbb{R}^n$ that are mapped onto $\mathscr{A}$ after premultiplication by $\boldsymbol{SG}$. Letting $\hat{v}$ be a 'base' probabilistic forecast for the full hierarchy, the coherent measure $\tilde{v}$ 'reconciles' $\hat{v}$ if $\tilde{v}(\mathscr{A}) = \hat{v}(\mathscr{B})$ for all $\mathscr{A}$.

In practice this definition suggests two approaches. For some parametric distributions, for instance the multivariate normal, it may be possible to derive a reconciled probabilistic forecast analytically. However, in macroeconomic forecasting, non-standard distributions such as bimodal distribution are often required to take different policy regimes into account worth checking if any (marginal) predictives are

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[1] Strictly speaking $\mathscr{A}$ is a Borel set

bimodal before we include this statement. In such cases a non-parametric approach based on bootstrapping in-sample errors proposed Gamakumara et al. (2018) can be used as long as a sample from the predictive distribution is available. Each of these scenarios is now covered in detail.

4.1 Probabilistic forecast reconciliation in the Gaussian framework

In the case where the base forecasts are probabilistic forecasts characterised by elliptical distributions Gamakumara et al. (2018) show that reconciled probabilistic forecasts will also be elliptical. This is particularly straightforward for the Gaussian distribution which is completely characterised by two moments. Letting the base probabilistic forecast be $\mathcal{N}(\hat{\mu}_{T+h}, \hat{\Sigma}_{T+h})$, then the reconciled probabilistic forecast will be $\mathcal{N}(\tilde{\mu}_{T+h}, \tilde{\Sigma}_{T+h})$, where,

$$\tilde{\mu}_{T+h} = \mathbf{SG}\hat{\mu}_{T+h}, \tag{8}$$

and

$$\tilde{\Sigma}_{T+h} = \mathbf{SG}\hat{\Sigma}_{T+h}\mathbf{G}'\mathbf{S}'. \tag{9}$$

There are several options for obtaining the base probabilistic forecast and in particular the variance covariance matrix $\hat{\Sigma}$. One option is to fit multivariate models level by level or for the hierarchy as a whole leading respectively to a $\hat{\Sigma}$ that is block diagonal or dense. Another alternative is to fit univariate models for each individual series in which case $\hat{\Sigma}$ is a diagonal matrix. Due to the large number of series under investigation here we consider the latter option. However we emphasise that correlation will enter the probabilistic forecast after reconciliation. The reconciled probabilistic forecast will ultimately depending on the choice of $\mathbf{G}$; the same choices of $\mathbf{G}$ matrices used in section 3 are be used here.

Need to check with Puwasala that base forecasts have diagonal sigma hat

4.2 Probabilistic forecast reconciliation in the non-parametric framework

In many applications, including macroeconomic forecasting, it may not reasonable to assume Gaussian predictive distributions. Therefore, non-parametric approaches has been widely used for probabilistic forecasts in different disciplines. For example, ensemble forecasting in weather applications (Gneiting & Raftery (2005), Gneiting & Katzfuss (2014), Gneiting et al. (2008)), bootstrap based approaches (Manzan & Zerom (2008), Vilar & Vilar (2013)). Check/replace these references with references that show heavy tails/skewness in macro applications.

Due to these concerns, we employ a reconciliation method proposed by Gamakumara et al. (2018) that does not make parametric assumptions about the predictive

distribution. An important result that this method exploits is that applying methods for point forecast reconciliation to the draws from incoherent base predictive distribution results in a sample from the reconciled predictive distribution. This involves a two step process, which is summarised below

The first step is to fit univariate models to each series in the hierarchy over a training set from $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_T$. Any model can be used as long as it is straightforward to produce h-step ahead point forecasts $\hat{\boldsymbol{y}}_{T+1}, \ldots, \hat{\boldsymbol{y}}_{T+h}$ and a matrix of one-step ahead in-sample forecasting errors $\Gamma_{(T \times n)} = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_T)'$. We then block bootstrap a sample of size h from Γ, and add these to the point forecasts $\hat{\boldsymbol{y}}_{T+1}, \ldots, \hat{\boldsymbol{y}}_{T+h}$. Repeating this process for N bootstrap samples and stacking in an $(N \times n)$ gives,

$$
\hat{\mathbf{Y}}_{T+h|T}^{b} = \begin{pmatrix} \hat{\boldsymbol{y}}_{1,T+h|T}^{b} \\ \hat{\boldsymbol{y}}_{2,T+h|T}^{b} \\ \vdots \\ \hat{\boldsymbol{y}}_{N,T+h|T}^{b} \end{pmatrix}.
\tag{10}
$$

An advantage to this approach is that the dependence structure of the hierarchy as well as serial correlation, are captured through the bootstrapping. However future paths constructed in this way will not be coherent and thus require reconciliation.

Step 2: Reconciling future sample paths

The second step is to project these sample paths to the coherent subspace. Then we get a set of reconciled future paths which form a possible sample from the reconciled probabilistic forecast distribution. Similar to the projection used in the point forecast reconciliation, we project each sample path through the projection $\mathbf{SG}$. Then we get,

$$
\tilde{\boldsymbol{y}}_{i,T+h}^{b} = \mathbf{SG}\hat{\boldsymbol{y}}_{i,T+h}^{b}, \quad i = 1, \ldots, N
\tag{11}
$$

and let

$$
\tilde{\mathbf{Y}}_{T+h|T}^{b} = \begin{pmatrix} \tilde{\boldsymbol{y}}_{1,T+h|T}^{b} \\ \tilde{\boldsymbol{y}}_{2,T+h|T}^{b} \\ \vdots \\ \tilde{\boldsymbol{y}}_{N,T+h|T}^{b} \end{pmatrix}
\tag{12}
$$

where, $\tilde{\boldsymbol{y}}_{i,T+h}^{b} \in \mathfrak{s}$ denote a h-step-ahead reconciled future paths for $i = 1, \ldots, N$. $\tilde{\mathbf{Y}}_{T+h|T}^{b}$ form an empirical coherent forecast distribution of the hierarchy that lies in the coherent subspace $\mathfrak{s}$. As we discussed in point forecast reconciliation methods, different estimates of $\mathbf{G}$ provide alternative estimates of reconciled future paths. We can also obtain bottom-up based future paths by simply aggregating the future paths of bottom level series for their respective upper levels. This is referred to as bottom-up future paths. Even though the bottom-up approach generates coherent

probabilistic forecasts, it cannot be considered as a reconciliation method since it use only half of the information.

5 Evaluation

In this section we briefly discuss the methods that we use to measure the forecast accuracy of hierarchical forecasts.

5.1 Point forecast evaluation

Any existing point forecast evaluation method can be used to evaluate the accuracy of hierarchical point forecast. We are mainly using Mean Squared Error (MSE) and Mean Absolute Scaled Error (MASE) in our empirical application. Mathematical expression for these measures are given below.

$$MSE = mean(e_{t+h}^2), \tag{13}$$

and

$$MASE = mean(q_{t+h}), \tag{14}$$

where $q_{t+h} = \frac{e_{t+h}}{\frac{1}{T-m}\Sigma_{j=m+1}^{T}|y_j-y_{j-m}|}$. Further e_{t+h} is the forecast error, which is the difference between forecast and realisation.

MSE is a scaled dependent measure. Therefore to compare series in different scales or with different units, MASE is preferred. For more details on different point forecast accuracy measures refer to Hyndman & Athanasopoulos (2018).

5.2 Probabilistic forecast evaluation

Forecast accuracy of probabilistic forecasts can be evaluated using scoring rules Gneiting & Katzfuss (2014). Scoring rules provides a summary measurement based on the relationship between realisation and the forecast distribution. However, not all scoring rules are applicable for hierarchical probabilistic forecast evaluation. An comprehensive discussion on this can be found in Gamakumara et al. (2018).

Since the whole forecast distribution of the hierarchy is a multivariate distribution, the forecast accuracy should also be assessed in the multivariate framework. Therefore using multivariate scoring rules is helpful. Energy score (Gneiting et al. 2008) and variogram scores (Scheuerer & Hamill 2015) are the mainly using multivariate scoring rules in hierarchical framework. Further, multivariate log score (Gneiting & Raftery 2007) can also be used for evaluating parametric forecast distributions. The mathematical expressions for these are given table 1.

Log score is easy to use if the parametric forecast distributions are available. However, Gamakumara et al. (2018) showed that the multivariate log scores are inappropriate for the comparisons between incoherent and coherent forecast distributions. This is due to the degeneracy of multivariate coherent forecast distributions.

Energy score and variogram score can be used in any comparisons between multivariate hierarchical forecast distributions. The expectations in these scoring rules can be approximated by the sample means of simulated samples.

It would be also interested to see the performance of forecast distributions of individual series in the hierarchy. To measure these we use univariate scoring rules such as Continuous Rank Probability Score (CRPS) and univariate log score (Gneiting et al. 2008). The mathematical expression for CRPS is given by,

$$\text{CRPS}(\breve{F}_i, y_{T+h,i}) = E_{\breve{F}_i} |\breve{Y}_{T+h,i} - y_{T+h,i}| - \frac{1}{2} E_{\breve{F}_i} |\breve{Y}_{T+h,i} - \breve{Y}^*_{T+h,i}|, \qquad (15)$$

where $\breve{Y}_{T+h,i}$ and $\breve{Y}^*_{T+h,i}$ are two independent copies from the ith reconciled marginal forecast distribution $\tilde{F}_i$ of the hierarchy and $y_{T+h,i}$ is the ith realization from the true marginal distribution G_i. As in multivariate scoring rules, the expectations can be approximated by the sample average.

5.3 Comparison between different forecasting methods

We are mainly interested to evaluate the hierarchical forecasts in two aspects. One is to examine whether the having coherent forecasts is improving the forecast accuracy. This can be evaluated by comparing the incoherent forecasts with any coherent forecasts in both point as well as probabilistic framework. Secondly we are inter-

Table 1 Scoring rules to evaluate multivariate forecast densities. $\breve{\mathbf{y}}_{T+h}$ and $\breve{\mathbf{y}}^*_{T+h}$ be two independent random vectors from the coherent forecast distribution $\breve{\mathbf{F}}$ with the density function $\breve{\mathbf{f}}(\cdot)$ at time $T+h$ and $\mathbf{y}_{T+h}$ is the vector of realizations. Further $\breve{Y}_{T+h,i}$ and $\breve{Y}_{T+h,j}$ are ith and jth components of the vector $\breve{\mathbf{Y}}_{T+h}$. Moreover, the variogram score is given for order p where, w_{ij} are non-negative weights.

Scoring rule	Expression
Log score	$\text{LS}(\breve{\mathbf{F}}, \mathbf{y}_{T+h}) = -\log \breve{\mathbf{f}}(\mathbf{y}_{T+h})$
Energy score	$\text{eS}(\breve{\mathbf{Y}}_{T+h}, \mathbf{y}_{T+h}) = E_{\breve{\mathbf{F}}} \|\breve{\mathbf{Y}}_{T+h} - \mathbf{y}_{T+h}\|^\alpha - \frac{1}{2} E_{\breve{\mathbf{F}}} \|\breve{\mathbf{Y}}_{T+h} - \breve{\mathbf{Y}}^*_{T+h}\|^\alpha, \quad \alpha \in (0,2]$
Variogram score	$\text{VS}(\breve{\mathbf{F}}, \mathbf{y}_{T+h}) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \left(|y_{T+h,i} - y_{T+h,j}|^p - E_{\breve{\mathbf{F}}} |\breve{Y}_{T+h,i} - \breve{Y}_{T+h,j}|^p \right)^2$

ested in finding the best reconciliation method by comparing reconciled forecast from different reconciliation methods.

For any comparison we use Skill score as defined in (Gneiting & Raftery 2007). For a given forecasting method, evaluated by a particular scoring rule $S(\cdot)$, the skill score can be calculated as follows,

$$Ss[S_B(\cdot)] = \frac{S_B(\boldsymbol{Y},\boldsymbol{y})^{\text{ref}} - S_B(\check{\boldsymbol{Y}},\boldsymbol{y})}{S_B(\boldsymbol{Y},\boldsymbol{y})^{\text{ref}}} \times 100\%, \tag{16}$$

where $S_B(\cdot)$ is average score over B replicates and $S_B(\boldsymbol{Y},\boldsymbol{y})^{\text{ref}}$ is the average score of the reference forecasting methods. Thus $Ss[S_B(\cdot)]$ gives the percentage improvement of the preferred forecasting method relative to the reference method. Any positive value indicates that method is superior to the reference method, whereas any negative value of $Ss[S_B(\cdot)]$ indicate that the method we compared is poor than the reference method.

6 Empirical study

- GDP of Australia

- How GDP is measured

  - Production, Income and Expenditure approach
  - Explain the hierarchy

- Issues with data

  - Why use current price rather than constant price (This is why we ignores production approach)
  - What is statistical discrepancy
  - Frequency of data
  - Does the data satisfies coherency

- Forecasting methods

  - Add a figure of all time series in income approach. Explain why traditional methods might not work using these time series as forecasting one layer of the hierarchy will ignore the structural information of individual time series to be used in the forecasts.
  - Explain ETS and ARIMA forecasting methods briefly
  - Hierarchical forecasting

6.1 Point forecasting

6.1.1 Income approach

Table 2 Summary of point forecast in income approach

| | ETS | | | | | | | | ARIMA | | | | | | | |
| | MSE | | | | MASE | | | | MSE | | | | MASE | | | |
Method	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4
								Most aggregate level								
Benchmark	−418.74	−135.45	−36.30	−3.10	−127.28	−69.39	−28.70	−21.03	−420.06	−128.25	−39.11	−15.49	−140.89	−67.24	−38.84	−28.69
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Bottom-up	−17.93	−3.68	7.80	9.35	−5.62	−3.49	−4.36	−15.49	−9.15	7.01	10.74	13.18	−13.13	−1.12	−6.25	−9.78
OLS	−0.18	3.45	6.05	6.70	0.05	0.96	3.47	1.50	2.12	2.46	2.54	1.92	−0.83	1.84	0.75	0.62
WLS	−4.05	5.59	11.57	13.09	−0.67	2.96	3.32	−2.27	2.41	6.80	9.35	9.71	−4.14	1.41	1.05	−1.78
MinT Shrink	−0.72	5.86	9.33	9.70	3.25	5.14	4.63	−1.15	7.06	9.05	8.63	9.73	−0.14	0.93	1.23	0.39
								Average across all aggregate levels								
Benchmark	−320.17	−101.63	−29.68	−2.32	−96	−40.28	−11.83	−1.92	−298.21	−86.44	−19.78	0.71	−78.18	−36.49	−11.83	−2.91
Base	0.00	0.00	0.00	0.00	0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Bottom-up	−7.64	−2.49	0.73	1.55	−2	0.00	−2.15	−1.92	1.61	8.32	5.18	9.05	1.82	1.35	−1.08	0.00
OLS	3.16	3.29	2.91	2.92	2	2.78	2.15	1.92	4.79	4.33	3.05	3.37	1.82	1.35	1.08	1.94
WLS	3.61	4.84	4.39	4.45	4	4.17	2.15	1.92	7.89	8.49	6.94	8.15	3.64	2.70	1.08	1.94
MinT Shrink	6.46	5.18	2.39	1.54	6	4.17	3.23	2.88	12.69	11.96	6.76	8.69	7.27	5.41	2.15	4.85
								Average across bottom level								
Benchmark	−200.06	−71.72	−21.93	−0.40	−61.02	−30.67	−11.11	−2.02	−191.76	−64.02	−7.71	6.65	−63.79	−32.43	−13.64	−4.12
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Bottom-up	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
OLS	0.66	−2.57	−3.50	−3.73	−23.73	−26.67	−18.89	−12.12	−2.88	−5.33	−2.55	−4.91	−20.69	−24.32	−22.73	−20.62
WLS	6.18	3.62	1.38	0.28	1.69	0.00	−1.11	0.00	2.43	−0.25	2.93	0.29	0.00	−1.35	−1.14	0.00
MinT Shrink	8.78	4.34	0.58	−1.43	3.39	1.33	1.11	1.01	4.81	1.57	2.15	0.47	0.00	−1.35	−1.14	0.00
								Average across all levels								
Benchmark	−295.40	−96.29	−28.41	−2.01	−71.43	−33.78	−10.99	−1.98	−276.70	−82.54	−17.73	1.68	−68.42	−33.78	−12.22	−4.04
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Bottom-up	−6.06	−2.05	0.61	1.31	0.00	0.00	−1.10	0.00	1.28	6.88	4.30	7.57	1.75	0.00	0.00	0.00
OLS	2.64	2.24	1.86	1.87	−14.29	−14.86	−10.99	−6.93	3.24	2.65	2.09	2.01	−12.28	−14.86	−13.33	−12.12
WLS	4.14	4.62	3.89	3.79	3.57	1.35	0.00	0.99	6.79	6.97	6.25	6.86	1.75	0.00	0.00	1.01
MinT Shrink	6.94	5.03	2.09	1.07	5.36	2.70	2.20	1.98	11.10	10.15	5.98	7.34	3.51	1.35	0.00	1.01

6.1.2 Expenditure approach

Table 3 Summary of point forecast in expenditure approach

| | ETS | | | | | | | | ARIMA | | | | | | | |
| | MSE | | | | MASE | | | | MSE | | | | MASE | | | |
Method	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4
								Most aggregate level								
Benchmark	−418.74	−135.45	−36.30	−3.10	−127.28	−69.39	−28.70	−21.03	−420.06	−128.25	−39.11	−15.49	−140.89	−67.24	−38.84	−28.69
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Bottom-up	−24.62	−6.94	13.89	19.17	−6.68	−2.54	3.99	−2.73	−44.70	−16.08	0.96	7.78	−18.84	−1.99	−6.89	−8.10
OLS	8.32	8.19	11.30	11.18	6.06	5.67	6.74	4.30	5.37	5.72	7.59	7.70	0.43	3.56	3.38	4.25
WLS	8.75	10.69	17.96	18.90	8.48	7.91	9.37	4.83	0.97	5.08	10.24	11.50	0.15	2.91	2.11	3.12
MinT Shrink	9.80	10.20	16.71	18.44	9.02	6.77	9.30	5.88	2.45	3.64	8.33	8.62	0.42	0.89	0.94	1.78
								Average across all aggregate levels								
Benchmark	−342.16	−149.39	−58.77	−21.04	−56.72	−32.10	−14.58	−2.78	−311.20	−147.72	−57.31	−28.38	−50.00	−30.49	−14.58	−5.71
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Bottom-up	−9.67	1.15	6.67	10.92	2.99	4.94	4.17	5.56	−28.47	−15.65	−8.36	−2.91	−1.43	−1.22	1.04	0.00
OLS	9.39	9.12	9.29	8.96	1.49	1.23	1.04	0.93	7.67	5.77	6.24	6.01	2.86	1.22	2.08	1.90
WLS	12.33	14.06	13.73	14.24	5.97	6.17	5.21	5.56	5.56	5.02	7.25	7.20	4.29	3.66	4.17	3.81
MinT Shrink	13.99	16.04	14.26	14.47	7.46	7.41	6.25	5.56	7.71	4.56	7.32	7.20	5.71	3.66	5.21	3.81
								Average across bottom level								
Benchmark	−167.10	−69.04	−24.00	−3.48	−53.01	−27.45	−11.76	−2.27	−136.49	−43.61	−10.04	6.21	−49.41	−26.21	−12.71	−2.27
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Bottom-up	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
OLS	−1.06	−5.53	−11.11	−15.14	−14.46	−13.73	−16.81	−17.42	3.40	2.61	−3.02	−4.06	−15.29	−11.65	−11.86	−9.85
WLS	1.91	−0.99	−4.84	−7.77	0.00	0.00	0.00	−1.52	6.53	7.03	1.43	0.90	1.18	0.00	0.00	0.00
MinT Shrink	2.36	−2.43	−6.15	−8.77	1.20	0.00	0.00	−1.52	5.37	7.14	0.48	−0.96	1.18	0.00	0.00	0.00
								Average across all levels								
Benchmark	−301.11	−132.24	−52.19	−17.98	−55.84	−28.42	−12.61	−2.42	−268.67	−122.65	−47.54	−21.54	−50.0	−27.08	−12.61	−3.25
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Bottom-up	−7.41	0.91	5.41	9.02	0.00	1.05	0.90	1.61	−21.54	−11.88	−6.64	−2.34	0.0	0.00	0.00	0.00
OLS	6.94	5.99	5.43	4.77	−10.39	−9.47	−11.71	−12.10	6.63	5.01	4.33	4.02	−10.0	−8.33	−7.21	−6.50
WLS	9.89	10.84	10.22	10.41	1.30	2.11	0.90	0.81	5.79	5.50	6.05	6.74	2.5	1.04	1.80	0.81
MinT Shrink	11.26	12.10	10.40	10.42	2.60	2.11	1.80	0.81	7.14	5.18	5.91	5.58	2.5	2.08	1.80	0.81

6.2 Probabilistic forecasting: Gaussian approach

6.2.1 Income approach

Table 4 Skill scores with respect to energy score and variogram score for multivariate Gaussian forecast distribution

| | ETS | | | | | | | | ARIMA | | | | | | | |
| | ES | | | | VS | | | | ES | | | | VS | | | |
Method	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4
Benchmark	−96.14	−36.28	−6.59	5.26	−156.30	−50.32	−9.15	6.00	−88.32	−32.30	−3.07	6.61	−145.36	−40.81	0.21	13.59
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Bottom-up	−0.09	0.84	−1.11	−2.53	1.08	2.99	2.51	2.44	2.24	3.06	0.81	1.09	4.11	6.63	3.45	4.93
OLS	1.87	1.93	1.33	1.41	1.01	−0.62	−0.60	0.00	1.62	1.63	1.18	1.06	−0.68	−0.31	0.02	−0.31
WLS	3.85	4.00	2.05	1.16	7.41	5.61	3.62	2.64	3.51	3.77	2.62	2.68	6.31	6.69	5.86	5.54
MinT Sample	4.58	−0.14	−2.41	−2.78	8.48	−1.36	−8.09	−13.39	−0.44	−4.27	−5.78	−2.11	−0.65	−0.55	−6.82	−4.65
MinT Shrink	5.43	4.54	2.40	1.61	9.84	6.16	3.39	1.13	5.59	3.93	2.67	4.20	8.83	9.20	6.28	6.62

Table 5 Skill scores with respect to log score for multivariate Gaussian forecast distribution

| | ETS | | | | ARIMA | | | |
Method	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4
Bottom-up	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
OLS	−2.62	5.65	20.17	34.21	−1.71	7.24	21.16	34.33
WLS	−0.26	−1.22	−1.60	−1.81	−0.40	−1.21	−1.70	−2.04
MinT Sample	−3.43	−10.18	−15.72	−20.50	−4.98	−11.25	−15.98	−18.74
MinT Shrink	−0.16	−1.60	−2.40	−2.91	−0.82	−2.35	−2.91	−3.31

Table 6 Skill scores for univariate Gaussian forecast distributions of individual series of the hierarchy

| | ETS | | | | | | | | ARIMA | | | | | | | |
| | CRPS | | | | LS | | | | CRPS | | | | LS | | | |
Method	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4
							Most aggregate level									
Benchmark	−129.45	−55.66	−14.67	−4.17	−5.53	11.47	28.94	36.32	−133.77	−50.95	−19.67	−7.94	−4.15	15.67	31.67	37.55
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
OLS	−0.82	1.61	2.64	1.09	−1.27	−1.57	−1.10	−1.18	−0.25	1.11	0.15	−0.06	−0.73	−1.07	−1.37	−1.82
WLS	−1.80	2.88	2.92	−2.21	−2.55	−2.78	−1.76	−2.72	−1.39	2.32	0.65	−0.41	−0.27	−1.18	−1.70	−2.04
MinT Sample	2.88	0.92	−0.49	−4.32	−2.22	−5.82	−12.33	−17.64	−4.91	−9.54	−9.82	−6.08	−3.92	−10.06	−13.91	−13.34
MinT Shrink	1.01	4.31	3.22	−1.83	−1.95	−2.81	−3.96	−6.27	0.89	1.17	0.43	0.78	−1.11	−1.45	−1.69	−1.87
						Average across all aggregate levels										
Benchmark	−108.81	−41.06	−7.27	4.76	−6.09	10.39	23.15	30.46	−97.82	−37.19	−5.84	4.74	−5.03	11.88	24.56	30.37
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
OLS	1.43	1.86	1.48	1.33	−1.12	−2.70	−4.01	−4.50	1.29	1.27	0.53	1.15	−1.21	−2.91	−4.50	−5.15
WLS	2.47	2.69	0.97	−0.11	−1.73	−3.80	−6.31	−7.52	2.45	2.07	0.50	1.49	−1.61	−3.65	−5.77	−6.11
MinT Sample	3.19	−2.24	−3.30	−3.63	−1.73	−7.01	−14.84	−18.70	−1.34	−6.29	−8.33	−3.23	−4.72	−10.96	−18.30	−18.60
MinT Shrink	4.02	3.13	1.40	0.59	−1.52	−4.22	−8.03	−9.58	5.22	2.49	0.51	3.26	−1.61	−4.40	−8.44	−8.88
						Average across all bottom level										
Benchmark	−74.79	−29.85	−4.80	5.13	−8.12	9.06	28.41	43.29	−72.44	−28.79	−4.24	4.85	−7.85	10.01	29.56	44.08
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
OLS	−6.37	−7.66	−5.73	−3.82	−2.79	6.22	21.18	34.84	−7.87	−9.25	−7.89	−8.12	−2.28	6.57	19.63	32.19
WLS	1.64	0.61	0.28	0.20	−0.13	−0.84	−1.07	−0.90	−0.31	−0.35	0.24	0.26	−0.25	−0.94	−1.05	−1.15
MinT Sample	0.72	−3.14	−2.34	−4.16	−1.65	−5.48	−8.13	−10.65	−6.11	−8.48	−9.33	−6.84	−3.54	−7.92	−10.66	−10.88
MinT Shrink	3.45	1.61	1.58	1.00	0.13	−1.05	−1.72	−2.13	0.20	−0.86	−0.39	0.78	−0.76	−2.09	−2.75	−2.80
						Average across all levels										
Benchmark	−97.58	−37.60	−6.54	4.87	−7.31	9.64	26.30	38.47	−89.67	−34.63	−5.37	4.77	−6.69	10.85	27.54	38.98
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
OLS	−1.15	−1.08	−0.65	−0.17	−2.09	2.41	10.97	20.04	−1.65	−1.94	−1.95	−1.54	−1.73	2.47	9.79	18.27
WLS	2.20	2.04	0.77	−0.02	−0.81	−2.12	−3.11	−3.41	1.56	1.33	0.42	1.13	−0.81	−2.09	−2.91	−3.00
MinT Sample	2.38	−2.52	−3.02	−3.78	−1.74	−6.17	−10.82	−13.66	−2.87	−6.96	−8.63	−4.28	−4.04	−9.23	−13.70	−13.74
MinT Shrink	3.83	2.66	1.46	0.71	−0.58	−2.41	−4.20	−4.96	3.60	1.47	0.24	2.54	−1.04	−3.04	−5.05	−5.05

6.2.2 Expenditure approach

Table 7 Skill scores with respect to energy score and variogram score for multivariate Gaussian forecast distribution

| | ETS | | | | | | | | ARIMA | | | | | | | |
| | ES | | | | VS | | | | ES | | | | VS | | | |
Method	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4
Benchmark	−93.6361	−44.9789	−13.7444	1.0137	−91.7685	−44.2491	−12.5531	4.1985	−84.6833	−43.3660	−14.8151	−2.6141	−79.4035	−38.5125	−11.4300	2.2691
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Bottom-up	−1.05	1.81	3.01	2.92	1.81	4.10	5.14	7.08	−6.37	−3.76	−2.32	−1.36	−2.25	−1.35	0.57	1.21
OLS	2.63	2.00	1.59	1.16	−0.41	−2.42	−3.94	−5.01	2.34	1.32	0.91	0.83	0.77	−0.27	−2.16	−2.43
WLS	3.93	3.66	2.84	1.91	3.89	3.38	3.06	2.37	1.56	0.67	0.64	0.39	4.55	3.89	2.87	2.76
MinT Shrink	4.37	4.02	2.88	1.85	5.00	2.88	2.23	0.73	1.91	0.17	0.11	−0.57	4.23	3.81	2.82	2.17

Table 8 Skill scores with respect to log score for multivariate Gaussian forecast distribution

| | ETS | | | | ARIMA | | | |
Method	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4
Bottom-up	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
OLS	0.12	2.92	6.09	9.12	0.53	4.69	9.20	13.10
WLS	−3.26	−5.20	−6.58	−6.95	−3.05	−4.56	−5.51	−5.90
MinT Shrink	−3.47	−5.80	−7.59	−8.22	−3.48	−5.33	−6.47	−7.06

Table 9 Skill scores for univariate Gaussian forecast distributions of individual series of the hierarchy

| | ETS | | | | | | | | ARIMA | | | | | | | |
| | CRPS | | | | LS | | | | CRPS | | | | LS | | | |
Method	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4
	Most aggregate level															
Benchmark	−130.00	−58.82	−12.94	−4.28	−5.53	11.24	29.09	36.32	−133.49	−49.65	−20.19	−8.14	−4.15	15.78	31.60	37.55
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
OLS	4.24	3.26	4.98	3.17	−1.44	−4.49	−5.58	−7.27	1.39	2.29	2.09	3.08	−2.14	−4.19	−5.14	−5.59
WLS	5.87	3.75	7.82	2.85	−2.85	−7.56	−7.39	−9.31	−0.22	1.83	1.00	2.05	−3.73	−5.77	−5.50	−5.18
MinT Shrink	5.72	2.98	7.43	4.21	−3.36	−9.06	−10.51	−12.34		−0.69	−0.46	0.31	−4.19	−8.94	−10.48	−11.27
	Average across all aggregate levels															
Benchmark	−70.80	−55.79	−22.73	−5.75	11.95	1.74	12.44	22.13	−66.95	−55.09	−23.68	−9.08	11.26	1.09	11.34	19.55
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
OLS	2.25	1.65	1.12	0.74	−1.35	−3.26	−4.78	−4.81	2.07	0.52	0.47	0.11	−0.68	−3.28	−4.07	−4.36
WLS	4.13	3.80	2.27	0.97	−11.85	−8.79	−13.68	−17.23	0.89	−0.62	0.29	−0.41	−12.82	−8.85	−12.79	−16.23
MinT Shrink	4.78	4.90	2.65	1.15	−12.91	−9.45	−15.31	−19.68	1.54	−0.96	0.19	−0.98	−14.56	−10.49	−15.12	−19.28
	Average across all bottom level															
Benchmark	−21.37	−6.39	−22.58	−19.45	21.03	36.61	17.12	12.00	−18.89	−3.17	−18.96	−16.45	27.52	45.46	18.07	14.67
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
OLS	−4.64	−6.43	−6.11	−7.67	12.98	21.02	3.57	6.95	−3.16	−3.09	−4.04	−4.66	21.14	33.84	4.57	10.33
WLS	−0.06	−1.30	−1.58	−3.08	−3.02	−3.76	−2.94	−2.91	0.44	0.13	−0.36	−0.65	−2.22	−2.34	−2.39	−2.07
MinT Shrink	0.64	−1.56	−2.05	−3.54	−4.12	−5.36	−4.20	−3.92	0.54	0.19	−0.79	−1.52	−3.60	−4.06	−3.84	−3.37
	Average across all levels															
Benchmark	−50.77	−35.31	−22.68	−9.74	17.94	27.04	15.46	16.15	−47.50	−33.24	−22.13	−11.32	22.25	34.67	15.72	16.57
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
OLS	−0.54	−1.70	−1.20	−1.71	8.03	14.31	0.51	2.22	−0.05	−1.00	−1.01	−1.33	13.99	24.84	1.52	4.62
WLS	2.43	1.69	1.03	−0.21	−6.14	−5.18	−6.82	−8.58	0.71	−0.30	0.08	−0.48	−5.63	−3.93	−6.09	−7.63
MinT Shrink	3.10	2.22	1.14	−0.22	−7.14	−6.50	−8.24	−10.19	1.14	−0.48	−0.13	−1.14	−7.14	−5.58	−7.81	−9.54

6.3 Probabilistic forecasting: Non-parametric bootstrap approach

6.3.1 Income approach

Table 10 Skill scores with respect to energy score and variogram score for multivariate forecast distribution

| | ETS | | | | | | | | ARIMA | | | | | | | |
| | ES | | | | VS | | | | ES | | | | VS | | | |
Method	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4
Benchmark	−103.31	−39.64	−8.52	3.84	−164.35	−52.47	−9.68	6.05	−95.27	−36.15	−5.41	4.93	−150.24	−41.37	0.33	14.06
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Bottom-up	0.44	1.11	−0.99	−2.29	1.38	3.24	2.69	2.65	2.21	3.09	0.98	1.26	4.13	6.66	3.67	5.20
OLS	2.35	2.25	1.77	1.85	0.72	−1.08	−0.74	−0.11	2.29	2.02	1.61	1.55	−0.91	−0.43	−0.14	−0.37
WLS	4.63	4.69	2.90	2.03	7.26	5.43	3.72	2.66	4.35	4.59	3.51	3.55	6.52	6.76	6.01	5.61
MinT Sample	5.52	0.75	−1.62	−2.32	8.63	−1.24	−8.06	−13.44	0.20	−3.64	−5.26	−1.72	−0.49	−0.43	−6.76	−4.58
MinT Shrink	6.34	5.16	3.18	2.27	9.84	6.06	3.34	1.05	6.32	4.61	3.34	4.88	8.79	9.20	6.34	6.79

Table 11 Skill scores for univariate forecast distributions of individual series of the hierarchy

| Method | ETS (CRPS) | | | | ARIMA (CRPS) | | | |
	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4
	Most aggregate level							
Benchmark	−139.03	−60.56	−17.03	−5.22	−144.75	−56.38	−22.61	−9.99
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Bottom-up	−6.01	−2.62	−2.92	−12.05	−7.99	1.29	−2.93	−4.41
OLS	−0.13	1.60	3.20	1.62	0.59	1.55	0.67	0.36
WLS	−0.95	3.20	3.80	−0.75	−0.63	2.79	1.46	0.27
MinT Sample	3.29	1.26	−0.09	−4.15	−4.68	−9.35	−9.40	−5.79
MinT Shrink	1.61	4.31	3.91	−0.74	1.28	1.43	1.04	1.28
	Average across all aggregate levels							
Benchmark	−116.05	−44.38	−8.96	3.46	−104.93	−41.23	−8.22	3.04
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
OLS	2.12	2.22	1.99	1.84	2.18	1.75	0.99	1.69
WLS	3.28	3.20	1.68	0.71	3.22	2.75	1.30	2.20
MinT Sample	4.02	−1.54	−2.53	−3.34	−0.69	−5.80	−7.97	−3.03
MinT Shrink	4.84	3.53	2.07	1.09	5.93	3.13	1.02	3.81
	Average across all bottom level							
Benchmark	−81.67	−32.82	−6.70	3.68	−78.46	−31.91	−6.22	3.38
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
OLS	−6.79	−8.46	−6.40	−4.31	−8.31	−10.13	−8.85	−8.94
WLS	1.91	1.17	0.80	0.74	0.05	0.17	0.68	0.65
MinT Sample	1.32	−2.31	−1.66	−3.82	−5.67	−7.72	−8.88	−6.51
MinT Shrink	3.91	2.19	1.99	1.30	0.36	−0.37	−0.04	1.06
	Average across all levels							
Benchmark	−104.77	−40.82	−8.29	3.52	−96.44	−38.38	−7.63	3.14
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
OLS	−0.81	−1.07	−0.48	0.05	−1.18	−1.88	−1.91	−1.40
WLS	2.83	2.58	1.42	0.72	2.20	1.96	1.12	1.75
MinT Sample	3.13	−1.78	−2.27	−3.48	−2.29	−6.38	−8.24	−4.04
MinT Shrink	4.53	3.11	2.05	1.15	4.14	2.06	0.71	3.01

6.3.2 Expenditure approach

Table 12 Skill scores with respect to energy score and variogram score for multivariate forecast distribution

| | ETS | | | | | | | | ARIMA | | | | | | | |
| | ES | | | | VS | | | | ES | | | | VS | | | |
Method	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4
Benchmark	−103.71	−50.21	−16.69	−1.06	−95.71	−45.37	−12.23	4.58	−93.65	−48.12	−17.52	−4.58	−84.33	−40.65	−11.04	2.55
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Bottom-up	−1.16	1.74	3.00	3.23	1.49	3.84	4.86	7.07	−5.24	−2.23	−0.85	0.23	−2.22	−1.27	1.00	1.54
OLS	4.42	3.79	3.35	2.90	0.44	−1.80	−3.56	−4.66	3.92	3.04	2.65	2.54	0.90	−0.28	−1.93	−2.19
WLS	6.58	6.36	5.37	4.40	5.09	4.29	3.66	2.91	4.25	3.50	3.31	3.24	5.18	4.26	3.53	3.31
MinT Shrink	6.73	6.31	4.89	4.05	6.12	3.63	2.66	1.28	4.34	2.62	2.42	1.84	4.82	4.11	3.28	2.54

Table 13 Skill scores for univariate forecast distributions of individual series of the hierarchy

| | ETS (CRPS) | | | | ARIMA (CRPS) | | | |
Method	h=1	h=2	h=3	h=4	h=1	h=2	h=3	h=4
Most aggregate level								
Benchmark	−145.58	−62.25	−15.91	−5.28	−146.33	−54.60	−22.52	−9.98
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Bottom-up	−9.78	−4.06	4.99	−0.41	−15.96	1.99	−2.61	−0.23
OLS	5.54	4.93	5.94	4.29	2.18	3.60	3.17	4.03
WLS	6.86	5.91	8.71	4.73	0.75	3.38	2.41	3.65
MinT Shrink	6.86	4.51	8.15	5.67	0.38	0.87	0.75	1.59
Average across all aggregate levels								
Benchmark	−78.02	−61.91	−26.48	−7.91	−73.77	−60.75	−26.93	−11.29
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
OLS	3.96	3.77	3.28	2.87	3.66	2.39	2.54	1.96
WLS	6.72	6.84	5.29	3.96	3.61	2.45	3.50	2.75
MinT Shrink	7.07	7.35	5.21	3.84	3.92	1.72	2.90	1.66
Average across all bottom level								
Benchmark	−21.66	−6.40	−23.56	−20.25	−19.81	−3.71	−20.12	−17.64
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
OLS	−4.76	−6.87	−6.02	−7.84	−3.47	−3.37	−3.99	−4.75
WLS	1.17	−0.43	−0.73	−2.24	1.18	0.81	0.43	0.07
MinT Shrink	1.68	−0.89	−1.40	−2.87	1.05	0.61	−0.25	−1.03
Average across all levels								
Benchmark	−55.07	−38.75	−25.54	−11.54	−51.91	−36.69	−24.68	−13.23
Base	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
OLS	0.41	−0.67	0.27	−0.28	0.78	−0.04	0.39	−0.08
WLS	4.46	3.81	3.35	2.14	2.63	1.76	2.49	1.93
MinT Shrink	4.87	3.91	3.08	1.87	2.76	1.25	1.86	0.84

References

Ben Taieb, S., Huser, R., Hyndman, R. J. & Genton, M. G. (2017), 'Forecasting uncertainty in electricity smart meter data by boosting additive quantile regression', *IEEE Transactions on Smart Grid* **7**(5), 2448–2455.

Dunn, D. M., Williams, W. H. & Dechaine, T. L. (1976), 'Aggregate Versus Sub-aggregate Models in Local Area Forecasting', *Journal of American Statistical Association* **71**(353), 68–71.

Gamakumara, P., Panagiotelis, A., Athanasopoulos, G. & Hyndman, R. J. (2018), Probabilisitic Forecasts in Hierarchical Time Series.

Gel, Y., Raftery, A. E. & Gneiting, T. (2004), 'Calibrated Probabilistic Mesoscale Weather Field Forecasting', *Journal of the American Statistical Association* **99**(July), 575–583.

Gneiting, T. & Katzfuss, M. (2014), 'Probabilistic Forecasting', *Annual Review of Statistics and Its Application* **1**, 125–151.

Gneiting, T. & Raftery, A. E. (2005), 'Weather forecasting with ensemble methods', *Science* **310.5746**, 248–249.

Gneiting, T. & Raftery, A. E. (2007), 'Strictly Proper Scoring Rules, Prediction, and Estimation', *Journal of the American Statistical Association* **102**(477), 359–378.

Gneiting, T., Raftery, A. E., Westveld, A. H. & Goldman, T. (2005), 'Calibrated Probabilistic Forecasting Using Ensemble Model Output Statistics and Minimum CRPS Estimation', *Monthly Weather Review* **133**(5), 1098–1118.

Gneiting, T., Stanberry, L. I., Grimit, E. P., Held, L. & Johnson, N. A. (2008), 'Assessing probabilistic forecasts of multivariate quantities, with an application to ensemble predictions of surface winds', *Test* **17**(2), 211–235.

Gross, C. W. & Sohl, J. E. (1990), 'Disaggregation methods to expedite product line forecasting', *Journal of Forecasting* **9**(3), 233–254.

Hyndman, R. J. & Athanasopoulos, G. (2018), *Forecasting: principles and practice*, OTexts.

Jeon, J., Panagiotelis, A. & Petropoulos, F. (2018), Reconciliation of probabilistic forecasts with an application to wind power.

Manzan, S. & Zerom, D. (2008), 'A bootstrap-based non-parametric forecast density', *International Journal of Forecasting* **24**(3), 535–550.

McSharry, P. E., Bouwman, S. & Bloemhof, G. (2005), 'Probabilistic forecasts of the magnitude and timing of peak electricity demand', *IEEE Transactions on Power Systems* **20**(2), 1166–1172.

Mirčetić, D., Nikoličić, S., Stojanović, D. & Maslarić, M. (2017), 'Modified top down approach for hierarchical forecasting in a beverage supply chain', *Transportation Research Procedia* **22**, 193–202.

Pinson, P., Madsen, H., Papaefthymiou, G. & Klöckl, B. (2009), 'From Probabilistic Forecasts to Wind Power Production', *Wind Energy* **12**(1), 51–62.

Schäfer, J. & Strimmer, K. (2005), 'A Shrinkage Approach to Large-Scale Covariance Matrix Estimation and Implications for Functional Genomics', *Statistical Applications in Genetics and Molecular Biology* **4**(1).

URL: *https://www.degruyter.com/view/j/sagmb.2005.4.issue-1/sagmb.2005.4.1.1175/sagmb.2005.4.1.1175.xml*

Scheuerer, M. & Hamill, T. M. (2015), 'Variogram-Based Proper Scoring Rules for Probabilistic Forecasts of Multivariate Quantities *', *Monthly Weather Review* **143**(4), 1321–1334.

Shang, H. L. (2017), 'Reconciling Forecasts of Infant Mortality Rates at National and Sub-National Levels: Grouped Time-Series Methods', *Population Research and Policy Review* **36**(1), 55–84.

Taieb, S. B., Taylor, J. W. & Hyndman, R. J. (2017), 'Hierarchical Probabilistic Forecasting of Electricity Demand with Smart Meter Data', pp. 1–30.

Vilar, J. A. & Vilar, J. A. (2013), 'Time series clustering based on nonparametric multidimensional forecast densities', *Electronic Journal of Statistics* **7**(1), 1019–1046.