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Author(s): C. Radhakrishna Rao
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Projectors, Generalized Inverses and the BLUE's

By C. RADHAKRISHNA RAO

Indian Statistical Institute and Indiana University

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SUMMARY

It is well known that in the Gauss–Markov model $(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \sigma^2 \mathbf{V})$ with $|\mathbf{V}| \neq 0$, the BLUE (best linear unbiased estimator) of $\mathbf{X}\boldsymbol{\beta}$ is $\mathbf{Y}_1$, the orthogonal projection of $\mathbf{Y}$ on $\mathscr{M}(\mathbf{X})$, the space spanned by the columns of $\mathbf{X}$, with inner product defined as $(\mathbf{x}, \mathbf{y}) = \mathbf{x}'\mathbf{V}^{-1}\mathbf{y}$. A quadratic function of $\mathbf{Y}_2$, the projection of $\mathbf{Y}$ on the orthogonal complement of $\mathscr{M}(\mathbf{X})$, provides an estimate of σ^2. It may be seen that $\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2$. When $\mathbf{V}$ is singular, the inner product definition as in non-singular case is not possible. In this paper a suitable theory of projection operators is developed for the case $|\mathbf{V}| = 0$, and a decomposition $\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2$ is obtained such that $\mathbf{Y}_1$ is the BLUE of $\mathbf{X}\boldsymbol{\beta}$ and a quadratic function of $\mathbf{Y}_2$ is the MINQUE (Minimum Norm Quadratic Unbiased Estimator) of σ^2 in the sense of Rao (1972).

Keywords: GAUSS–MARKOV MODEL; BLUE; MINQUE; PROJECTION OPERATOR; G-INVERSE; SINGULAR DISPERSION MATRIX

1. INTRODUCTION

WE consider the G.G.M. (General Gauss–Markov) linear model, $(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{V})$, where $E(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta}$, $D(\mathbf{Y}) = \sigma^2 \mathbf{V}$, E and D being expectation and dispersion (variance–covariance) operators. $\mathbf{X}$ and $\mathbf{V}$ are known matrices of orders $n \times m$ and $n \times n$ respectively, one or both of which may be deficient in rank.

Various computational procedures for the estimation of $\mathbf{X}\boldsymbol{\beta}$ and σ^2, the unknown parameters in the G.G.M. model, have been given by the author in a series of papers. The reader is referred specially to Rao and Mitra (1971, pp. 148–150) and the unified theory developed in Rao (1973a, pp. 294–302) and the references given in Rao (1973b). The principal tool used in these papers is the generalized inverse of a matrix whose usefulness in the estimation of parameters in linear models was demonstrated in Rao (1962).

The object of the present paper is to express some of the earlier results in a geometrical language, specially using a suitably defined projection operator as a main tool. Such a discussion may be of interest in generalizing the results to random variables defined in abstract spaces, and also tie up estimation procedures with the computation of projection operators.

In the case of non-singular $\mathbf{V}$, we consider the Euclidean space E^n with the inner product defined as $(\mathbf{x}, \mathbf{y}) = \mathbf{x}'\mathbf{V}^{-1}\mathbf{y}$. Let $\mathbf{Y}_1$ be the orthogonal projection of the observation vector $\mathbf{Y}$ on the subspace $\mathscr{M}(\mathbf{X})$ and $\mathbf{Y}_2$ the orthogonal projection on the subspace orthogonal to $\mathscr{M}(\mathbf{X})$. Then we have the decomposition $\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2$. Kolmogorov (1946, 1967), who was perhaps the first to introduce geometrical concepts in linear estimation, showed that $\mathbf{Y}_1$ is the BLUE of $\mathbf{X}\boldsymbol{\beta}$. It is easily shown that $\mathbf{Y}_2'\mathbf{A}\mathbf{Y}_2$ for a suitable choice of $\mathbf{A}$ is the MINQUE (defined in Rao, 1972) of σ^2.

Thus when $|\mathbf{V}|\neq 0$, the decomposition of $\mathbf{Y}$ in terms of its projections on orthogonal subspaces spanning the *whole space* plays an important role in the estimation of unknown parameters.

The decomposition of $\mathbf{Y}=\mathbf{Y}_1+\mathbf{Y}_2$ was explicitly stressed by Kolmogorov (1946, 1967) although it was known to earlier writers in the context of regression theory.

When $|\mathbf{V}|=0$, the inner product definition does not hold, and naturally the question arises as to how a decomposition of $\mathbf{Y}$ can be obtained to serve the same purpose as in the case of $|\mathbf{V}|\neq 0$. The object of the present paper is to obtain such a decomposition by defining projectors suitably when one or both of $\mathbf{V}$ and $\mathbf{X}$ are deficient in rank. Let z be a matrix of maximum rank such that $z'x=0$.

The main result of the paper is as follows: The unique resolution $\mathbf{Y}=\mathbf{Y}_1+\mathbf{Y}_2$, where $\mathbf{Y}_1\in\mathscr{M}(\mathbf{X})$ and $\mathbf{Y}_2\in\mathscr{M}(\mathbf{VZ})$, provides the desired decomposition. $\mathbf{Y}_1$ is the projection of $\mathbf{Y}$ on the subspace $\mathscr{M}(\mathbf{X})$ parallel to (or along) the disjoint subspace $\mathscr{M}(\mathbf{VZ})$ and is the BLUE of $\mathbf{X}\boldsymbol{\beta}$, and a quadratic function of $\mathbf{Y}_2$ is the MINQUE of σ^2. Note that $\mathscr{M}(\mathbf{X})\oplus\mathscr{M}(\mathbf{VZ})$ may not be equal to E^n, the entire Euclidean space of n dimensions, but $\mathbf{Y}\in\mathscr{M}(\mathbf{X})\oplus\mathscr{M}(\mathbf{VZ})$ and can therefore be resolved along $\mathscr{M}(\mathbf{X})$ and $\mathscr{M}(\mathbf{VZ})$. Explicit expressions are obtained for the projection operator $\mathbf{P}$ such that $\mathbf{PY}=\mathbf{Y}_1$.

2. PRELIMINARY RESULTS

The following notations and operations on a matrix $\mathbf{A}$ are used throughout.

$\mathbf{A}'$ = Transpose of $\mathbf{A}$. $R(\mathbf{A})$ = Rank of $\mathbf{A}$.

$\mathscr{M}(\mathbf{A})$ = Linear manifold generated by the columns of $\mathbf{A}$.

$\mathbf{A}^-$ = Any g-inverse of $\mathbf{A}$ in the sense of Rao (1962), i.e. such that $\mathbf{AA}^-\mathbf{A}=\mathbf{A}$.

$(\mathbf{A}:\mathbf{B})$ = Matrix obtained by adjoining the columns of $\mathbf{B}$ to those of $\mathbf{A}$.

$\mathbf{A}^\perp$ = A matrix of maximum rank such that $\mathbf{A}'\mathbf{A}^\perp=\mathbf{0}$.

$$\mathscr{M}(\mathbf{A})\oplus\mathscr{M}(\mathbf{B})=\{\mathbf{x}+\mathbf{y}:\ \mathbf{x}\in\mathscr{M}(\mathbf{A}),\mathbf{y}\in\mathscr{M}(\mathbf{B})\},\text{ where }\mathscr{M}(\mathbf{A})\cap\mathscr{M}(\mathbf{B})=\{\mathbf{0}\}.$$

Two matrices $\mathbf{A}$ and $\mathbf{B}$ may be said to be disjoint if the spaces $\mathscr{M}(\mathbf{A})$ and $\mathscr{M}(\mathbf{B})$ are disjoint, i.e. $\mathscr{M}(\mathbf{A})\cap\mathscr{M}(\mathbf{B})=\{\mathbf{0}\}$.

The following lemmas are used in later sections:

Lemma 2.1. Let $\mathbf{X}$ be an $n\times m$ matrix, $\mathbf{V}$ be an n.n.d. (non-negative definite) matrix of order n, and $\mathbf{Z}=\mathbf{X}^\perp$. Then:

$$\text{(i) } \mathbf{X} \text{ and } \mathbf{VZ} \text{ are disjoint matrices.} \tag{2.1}$$

$$\text{(ii) } \mathscr{M}(\mathbf{V}:\mathbf{X})=\mathscr{M}(\mathbf{VZ}:\mathbf{X}). \tag{2.2}$$

Proof. Suppose $\mathbf{VZb}=\mathbf{Xa}$ for some vectors $\mathbf{a}$ and $\mathbf{b}$. Then

$$\mathbf{Z}'\mathbf{VZb}=\mathbf{Z}'\mathbf{Xa}=\mathbf{0}\Rightarrow\mathbf{b}'\mathbf{Z}'\mathbf{VZb}=0$$

or $\mathbf{VZb}=\mathbf{0}$, which shows that $\mathbf{VZ}$ and $\mathbf{X}$ are disjoint. (i) is proved, then (ii) follows from (i).

Lemma 2.2. Let $\mathbf{U}$ be any symmetric matrix such that $R(\mathbf{T}=\mathbf{V}+\mathbf{XUX}')=R(\mathbf{V}:\mathbf{X})$, and $\mathbf{T}^-$ be a g-inverse of $\mathbf{T}$. Then:

$$\mathbf{TT}^-\mathbf{V}=\mathbf{V},\quad \mathbf{TT}^-\mathbf{X}=\mathbf{X}. \tag{2.3}$$

Proof. The results follow by using the definition of $\mathbf{T}^-$, observing that $\mathscr{M}(\mathbf{V}) \subset \mathscr{M}(\mathbf{T})$, i.e. $\mathbf{V} = \mathbf{T}\boldsymbol{\Lambda}_1$ for some $\boldsymbol{\Lambda}_1$ and similarly $\mathbf{X} = \mathbf{T}\boldsymbol{\Lambda}_2$ for some $\boldsymbol{\Lambda}_2$.

Lemma 2.3. Let $(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{V})$ be a G.G.M. model. Then:

$$\mathbf{Y} \in \mathscr{M}(\mathbf{V} : \mathbf{X}) = \mathscr{M}(\mathbf{VZ} : \mathbf{X}) \quad \text{with probability 1.} \tag{2.4}$$

Proof. Let $\mathbf{L}$ be a vector such that $\mathbf{L}'\mathbf{V} = \mathbf{0}, \mathbf{L}'\mathbf{X} = \mathbf{0}$. This implies that $E(\mathbf{L}'\mathbf{Y}) = 0$, $V(\mathbf{L}'\mathbf{Y}) = 0$, in which case $\mathbf{L}'\mathbf{Y} = 0$ with probability 1. Hence we have the result (2.4).

We consider two definitions of projection operators which are useful in the discussion of statistical problems.

Definition 1. Let $\mathbf{A}$ and $\mathbf{B}$ be disjoint matrices each with the same number n of rows. Any vector $\mathbf{Y} \in \mathscr{M}(\mathbf{A} : \mathbf{B})$ has the unique resolution:

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2, \quad \mathbf{Y}_1 \in \mathscr{M}(\mathbf{A}), \quad \mathbf{Y}_2 \in \mathscr{M}(\mathbf{B}). \tag{2.5}$$

Then $\mathbf{P}_{\mathbf{A}|\mathbf{B}}$ is said to be a projector onto $\mathscr{M}(\mathbf{A})$ parallel to (or along) $\mathscr{M}(\mathbf{B})$ iff

$$\mathbf{P}_{\mathbf{A}|\mathbf{B}}\mathbf{Y} = \mathbf{Y}_1 \quad \text{for all } \mathbf{Y} \in \mathscr{M}(\mathbf{A} : \mathbf{B}). \tag{2.6}$$

Definition 1 is well known when $\mathscr{M}(\mathbf{A}) \oplus \mathscr{M}(\mathbf{B}) = E^n$ (the Euclidean space of n dimensions) The same concept will be extended to the case when $\mathscr{M}(\mathbf{A}) \oplus \mathscr{M}(\mathbf{B})$ is a subspace of E^n. The properties of $\mathbf{P}_{\mathbf{A}|\mathbf{B}}$ are examined in the following lemmas:

Lemma 2.4. A necessary and sufficient condition for an operator $\mathbf{P}_{\mathbf{A}|\mathbf{B}}$ to satisfy Definition 1 above is

$$\mathbf{P}_{\mathbf{A}|\mathbf{B}}\mathbf{A} = \mathbf{A}, \quad \mathbf{P}_{\mathbf{A}|\mathbf{B}}\mathbf{B} = \mathbf{0}. \tag{2.7}$$

Proof. Consider $\mathbf{Y} = \mathbf{Aa} + \mathbf{Bb}$ where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary Then, by definition

$$\mathbf{P}_{\mathbf{A}|\mathbf{B}}(\mathbf{Aa} + \mathbf{Bb}) = \mathbf{Aa} \quad \text{for all } \mathbf{a} \text{ and } \mathbf{b} \Leftrightarrow (2.7).$$

Lemma 2.5. $\mathbf{P}_{\mathbf{A}|\mathbf{B}}$ is idempotent and unique if $\mathscr{M}(\mathbf{A}) + \mathscr{M}(\mathbf{B}) = E^n$.

The result is well known. However, it is important to note that idempotency and uniqueness may not hold if $\mathscr{M}(\mathbf{A}) \oplus \mathscr{M}(\mathbf{B})$ is only a subspace of E^n.

For example, let $\mathbf{A}'$ and $\mathbf{B}'$ be the unit vectors $(1 \quad 0 \quad 0)$ and $(0 \quad 1 \quad 0)$ respectively. Then the matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix}$$

for arbitrary values of a, b, c satisfies Definition 1 and is a projector onto $\mathscr{M}(\mathbf{A})$ parallel to $\mathscr{M}(\mathbf{B})$. But $\mathbf{P}$ is neither unique nor idempotent, although an idempotent choice exists, e.g. by choosing $a = b = c = 0$.

Lemma 2.6. One representation of $\mathbf{P}_{\mathbf{A}|\mathbf{B}}$ is

$$\mathbf{P}_{\mathbf{A}|\mathbf{B}} = \mathbf{A}(\mathbf{C}'\mathbf{A})^-\mathbf{C}', \quad \mathbf{C} = \mathbf{B}^\perp. \tag{2.8}$$

Proof. From (2.7), $\mathbf{P}_{\mathbf{A}|\mathbf{B}}\mathbf{B} = \mathbf{0} \Leftrightarrow \mathbf{P}_{\mathbf{A}|\mathbf{B}} = \mathbf{KC}'$ for some $\mathbf{K}$. Substituting in $\mathbf{P}_{\mathbf{A}|\mathbf{B}}\mathbf{A} = \mathbf{A}, \mathbf{KC}'\mathbf{A} = \mathbf{A}$ or $\mathbf{K} = \mathbf{A}(\mathbf{C}'\mathbf{A})^-$, which establishes (2.8). To obtain the general representation of $\mathbf{P}_{\mathbf{A}|\mathbf{B}}$, we have only to add to (2 8) a general solution of the equation $\mathbf{XA} = \mathbf{0}, \mathbf{XB} = \mathbf{0}$.

Note 1. Let $\mathbf{B} = \mathbf{A}^{\perp}$, i.e. $\mathbf{A}$ and $\mathbf{B}$ are orthogonal complements spanning the whole space in which case $\mathbf{P}_{\mathbf{A}|\mathbf{B}}$ is called the orthogonal projector onto $\mathscr{M}(\mathbf{A})$ under the inner product $(\mathbf{x}, \mathbf{y}) = \mathbf{x}'\mathbf{y}$ and is represented by $\mathbf{P}_{\mathbf{AI}}$. It is seen from (2.8) that

$$\mathbf{P}_{\mathbf{AI}} = \mathbf{A}(\mathbf{A}'\mathbf{A})^{-}\mathbf{A}' \tag{2.9}$$

which is symmetric, idempotent and unique for any choice of the g-inverse of $\mathbf{A}'\mathbf{A}$. Further, if $\mathbf{C}$ is any other matrix such that $\mathscr{M}(\mathbf{A}) = \mathscr{M}(\mathbf{C})$, then

$$\mathbf{P}_{\mathbf{AI}} = \mathbf{A}(\mathbf{A}'\mathbf{A})^{-}\mathbf{A}' = \mathbf{C}(\mathbf{C}'\mathbf{C})^{-}\mathbf{C}' = \mathbf{P}_{\mathbf{CI}}. \tag{2.10}$$

Note 2. Let $\mathbf{B}$ be a matrix of maximum rank such that $\mathbf{A}'\mathbf{\Sigma}\mathbf{B} = \mathbf{0}$, where $\mathbf{\Sigma}$ is a positive definite (p.d.) matrix. Then $(\mathbf{B}^{\perp})' = \mathbf{A}'\mathbf{\Sigma}$ so that, from (2.8)

$$\mathbf{P}_{\mathbf{A}|\mathbf{B}} = \mathbf{A}(\mathbf{A}'\mathbf{\Sigma}\mathbf{A})^{-}\mathbf{A}'\mathbf{\Sigma} = \mathbf{P}_{\mathbf{A}\mathbf{\Sigma}}. \tag{2.11}$$

It is seen that $\mathbf{P}_{\mathbf{A}|\mathbf{B}}$ is idempotent, $\mathbf{\Sigma}\mathbf{P}_{\mathbf{A}|\mathbf{B}}$ is symmetric and $\mathbf{P}_{\mathbf{A}|\mathbf{B}}$ is unique for any choice of the g-inverse of $\mathbf{A}'\mathbf{\Sigma}\mathbf{A}$. The operator $\mathbf{P}_{\mathbf{A}\mathbf{\Sigma}}$ as defined in (2.11) is the orthogonal projector onto $\mathscr{M}(\mathbf{A})$ under the inner product $(\mathbf{x}, \mathbf{y}) = \mathbf{x}'\mathbf{\Sigma}\mathbf{y}$. In (2.11) $\mathbf{A}$ can be replaced by any matrix whose columns span the same subspace.

Note 3. Let $\mathbf{A}$ and $\mathbf{B}$ be such that $\mathscr{M}(\mathbf{A}) \oplus \mathscr{M}(\mathbf{B}) = E^n$. Then

$$\mathbf{P}_{\mathbf{A}|\mathbf{B}} = \mathbf{A}(\mathbf{A}'\mathbf{Q}_{\mathbf{BI}}\mathbf{A})^{-}\mathbf{A}'\mathbf{Q}_{\mathbf{BI}}, \tag{2.12}$$

where $\mathbf{Q}_{\mathbf{BI}} = \mathbf{I} - \mathbf{P}_{\mathbf{BI}}$. Further, if the columns of $\mathbf{C}$ form a basis of $\mathscr{M}(\mathbf{A})$, then $\mathbf{C}'\mathbf{Q}_{\mathbf{BI}}\mathbf{C}$ is non-singular and

$$\mathbf{P}_{\mathbf{A}|\mathbf{B}} = \mathbf{C}(\mathbf{C}'\mathbf{Q}_{\mathbf{BI}}\mathbf{C})^{-1}\mathbf{C}'\mathbf{Q}_{\mathbf{BI}} \tag{2.13}$$

which is the expression given by Afriat (1957).

Let us write the decomposition (2.5) more explicitly as

$$\mathbf{Y} = \mathbf{A}\mathbf{a} + \mathbf{B}\mathbf{b}, \quad \mathbf{Y} \in \mathscr{M}(\mathbf{A} : \mathbf{B}) \tag{2.14}$$

and suppose that $\mathbf{a}$ has the representation $\mathbf{a} = \mathbf{G}\mathbf{Y}$. Then

$$\left.\begin{aligned} &\mathbf{AGY} = \mathbf{AG}(\mathbf{Aa} + \mathbf{Bb}) = \mathbf{Aa} \quad \text{for all } \mathbf{a} \text{ and } \mathbf{b}, \\ &\Leftrightarrow \mathbf{AGA} = \mathbf{A}, \quad \mathbf{AGB} = \mathbf{0}, \end{aligned}\right\} \tag{2.15}$$

which shows that $\mathbf{G}$ is a g-inverse of $\mathbf{A}$ with the constraint $\mathscr{M}(\mathbf{G}'\mathbf{A}') \subset \mathscr{M}(\mathbf{B}^{\perp})$. We represent such a g-inverse by $\mathbf{G}_{\mathbf{A}|\mathbf{B}}$, a g-inverse of $\mathbf{A}$ constrained by $\mathbf{B}$. Note that

$$\mathbf{P}_{\mathbf{A}|\mathbf{B}} = \mathbf{A}\mathbf{G}_{\mathbf{A}|\mathbf{B}}. \tag{2.16}$$

Lemma 2.7. One choice of $\mathbf{G}$ satisfying (2.15) is

$$\mathbf{G}_{\mathbf{A}|\mathbf{B}} = (\mathbf{C}'\mathbf{A})^{-}\mathbf{C}', \quad \mathbf{C} = \mathbf{B}^{\perp}. \tag{2.17}$$

Proof. The result is established by verification. A general solution is obtained by adding to (2.17) a general solution of the equation $\mathbf{AXA} = 0$ and $\mathbf{AXB} = \mathbf{0}$.

Definition 2. Let $\mathbf{M}$ be an n.n.d. matrix. A matrix $\mathbf{P}_{\mathbf{AM}}$ is called a projector into $\mathscr{M}(\mathbf{A})$ with respect to the norm or semi-norm defined by $\|\mathbf{x}\| = (\mathbf{x}'\mathbf{M}\mathbf{x})^{\frac{1}{2}}$, $\mathbf{x} \in E^n$ iff the following two conditions hold.

$$\mathbf{P}_{\mathbf{AM}}\mathbf{Y} \in \mathscr{M}(\mathbf{A}), \quad \forall\, \mathbf{Y} \in E^n, \tag{2.18}$$

$$\|\mathbf{Y} - \mathbf{P}_{\mathbf{AM}}\mathbf{Y}\| \leqslant \|\mathbf{Y} - \mathbf{AX}\|, \quad \forall\, \mathbf{X} \in E^m, \quad \mathbf{Y} \in E^n. \tag{2.19}$$

The above definition was introduced by Rao and Mitra (1971) and further examined by Mitra and Rao (1973).

Lemma 2.8. $\mathbf{P_{AM}}$ satisfies the conditions (2.18) and (2.19) of Definition 2 iff

$$\text{(i)}\ \mathcal{M}(\mathbf{P_{AM}}) \subset \mathcal{M}(\mathbf{A}), \tag{2.20}$$

$$\text{(ii)}\ \mathbf{P'_{AM}MA} = \mathbf{MA}. \tag{2.21}$$

Proof. Condition (i) is equivalent to (2.18) and condition (ii) to (2.19); then the statement of the lemma follows. As a consequence of (i) and (ii) we have

$$\text{(iii)}\ \mathbf{P'_{AM}M} = \mathbf{MP_{AM}}.$$

Note that $\mathbf{P_{AM}}$ need not be idempotent or unique unless $\mathbf{M}$ is p.d. We denote by $\{\mathbf{P_{AM}}\}$, the class of all matrices $\mathbf{P_{AM}}$ satisfying the conditions (2.20) and (2.21).

Lemma 2.9. One representation of $\mathbf{P_{AM}}$ is

$$\mathbf{P_{AM}} = \mathbf{A(A'MA)^{-}A'M}. \tag{2.22}$$

Proof. The result is established by checking that $\mathbf{P_{AM}}$ as defined in (2.22) satisfies the conditions (2.20) and (2.21).

Lemma 2.10. Let $\mathbf{M}$ and $\mathbf{A}$ be as in Definition 2 and denote $\mathbf{A}^{\perp}$ by $\mathbf{H}$. Further, let $\mathbf{T} = \mathbf{M}+\mathbf{AUA'}$ where $\mathbf{U}$ is any symmetric matrix such that $R(\mathbf{T}) = R(\mathbf{M} : \mathbf{A})$ and $\mathbf{T}$ is n.n.d. Denote by $\mathbf{T}^-$ any n.n.d. g-inverse of $\mathbf{T}$. Then:

$$\text{(i)}\quad \{\mathbf{P'_{AM}}\} = \{\mathbf{P_{MA|H}}\}, \tag{2.23}$$

$$\text{(ii)}\quad \mathbf{P'_{HM}}+\mathbf{P_{A|MH}} \equiv \mathbf{I} \quad \text{in } \mathcal{M}(\mathbf{M} : \mathbf{A}), \tag{2.24}$$

$$\text{(iii)}\quad \mathbf{P_{A|MH}} \equiv \mathbf{P_{AT^-}} \quad \text{in } \mathcal{M}(\mathbf{M} : \mathbf{A}). \tag{2.25}$$

Proof. Result (i) follows from the definitions of the projection operators involved.

To prove (ii), observe that $\mathcal{M}(\mathbf{A})$ and $\mathcal{M}(\mathbf{MH})$ are disjoint and $\mathcal{M}(\mathbf{M} : \mathbf{A}) = \mathcal{M}(\mathbf{A} : \mathbf{MH})$. Consequently,

$$\mathbf{P_{A|MH}}+\mathbf{P_{MH|A}} = \mathbf{I} \quad \text{in } \mathcal{M}(\mathbf{M} : \mathbf{A}). \tag{2.26}$$

Now using (2.23),

$$\mathbf{P_{MH|A}} = \mathbf{P'_{HM}}. \tag{2.27}$$

Substituting (2.27) in (2.26), the result (ii) is established.

Since by definition (Lemma 2.4)

$$\mathbf{P_{A|MH}A} = \mathbf{A}, \quad \mathbf{P_{A|MH}MH} = \mathbf{0} \tag{2.28}$$

the result (iii) is established if in (2.28), $\mathbf{P_{A|MH}}$ is replaced by $\mathbf{P_{AT^-}}$. Consider

$$\mathbf{P'_{AT^-}T^-A} = \mathbf{T^-A} \Leftrightarrow \mathbf{T^-P_{AT^-}A} = \mathbf{T^-A}. \tag{2.29}$$

Multiplying both sides of the last equality in (2.29) by $\mathbf{T}$ we have

$$\mathbf{TT^-P_{AT^-}A} = \mathbf{TT^-A} \Rightarrow \mathbf{P_{AT^-}A} = \mathbf{A} \tag{2.30}$$

since both $\mathbf{P_{AT^-}}$ and $\mathbf{A}$ are contained in the range of $\mathbf{T}$. Now

$$\begin{aligned}\mathbf{P'_{AT^-}T^-} = \mathbf{T^-P_{AT^-}} &\Rightarrow \mathbf{TP'_{AT^-}T^-} = \mathbf{P_{AT^-}}\\ &\Rightarrow \mathbf{TP'_{AT^-}T^-TH} = \mathbf{P_{AT^-}TH} = \mathbf{P_{AT^-}MH}.\end{aligned} \tag{2.31}$$

But

$$\mathbf{TP}'_{\mathbf{AT}^-}\mathbf{T}^-\mathbf{TH} = \mathbf{TP}'_{\mathbf{AT}^-}\mathbf{H} = \mathbf{0} \tag{2.32}$$

using (2.19). Hence the last expression in (2.31) is a null matrix. This together with (2.30) proves the result (iii).

3. THE GENERAL GAUSS–MARKOV MODEL

Consider the G.G.M. model

$$(\mathbf{Y}, \mathbf{X\beta}, \sigma^2\mathbf{V}), \quad E(\mathbf{Y}) = \mathbf{X\beta}, \quad D(\mathbf{Y}) = \sigma^2\mathbf{V}, \tag{3.1}$$

where $\mathbf{V}$ may be singular and $\mathbf{X}$ may be deficient in rank. We say that a linear function $\mathbf{L}'_*\mathbf{Y}$ is the BLUE of the linear parametric function $\mathbf{p}'\boldsymbol{\beta}$ if $\mathbf{L}'_*\mathbf{X} = \mathbf{p}'$ and

$$\mathbf{L}'_*\mathbf{VL}_* \leqslant \mathbf{L}'\mathbf{VL} \quad \forall\, \mathbf{L} \ni \mathbf{L}'\mathbf{X} = \mathbf{p}'.$$

For a wider definition of the BLUE called BLUE (W) which is relevant when $\mathbf{V}$ is singular, the reader is referred to Rao (1973b). The main results are given in the following theorems, where we denote $\mathbf{Z} = \mathbf{X}^\perp$.

Theorem 3.1. Let $\mathbf{L}_0$ be any given vector such that $\mathbf{L}'_0\mathbf{X} = \mathbf{p}'$. Then $\mathbf{L}'_*\mathbf{Y}$ is the BLUE of $\mathbf{p}'\boldsymbol{\beta}$ where

$$\mathbf{L}_* = (\mathbf{I} - \mathbf{P}_{\mathbf{ZV}})\mathbf{L}_0 = (\mathbf{I} - \mathbf{P}'_{\mathbf{VZ}|\mathbf{X}})\mathbf{L}_0. \tag{3.2}$$

Proof. A general solution of $\mathbf{L}'\mathbf{X} = \mathbf{p}'$ is $\mathbf{L}_0 - \mathbf{Z\lambda}$ where $\boldsymbol{\lambda}$ is arbitrary. Then by definition $\|\mathbf{L}_0 - \mathbf{Z\lambda}\|_\mathbf{V} \geqslant \|\mathbf{L}_0 - \mathbf{P}_{\mathbf{ZV}}\mathbf{L}_0\|_\mathbf{V}$ which proves the first part of the equality in (3.2) and the second part follows by applying (2.23).

Corollary 3.1. Let $\boldsymbol{\Lambda}_0$ be $n \times k$ matrix such that $\boldsymbol{\Lambda}'_0\mathbf{X} = \boldsymbol{\pi}'$. Then $\boldsymbol{\Lambda}'_*\mathbf{Y}$ is the BLUE of $\boldsymbol{\pi}'\boldsymbol{\beta}$ (k parametric functions) where

$$\boldsymbol{\Lambda}_* = (\mathbf{I} - \mathbf{P}_{\mathbf{ZV}})\boldsymbol{\Lambda}_0 = (\mathbf{I} - \mathbf{P}'_{\mathbf{VZ}|\mathbf{X}})\boldsymbol{\Lambda}_0. \tag{3.3}$$

Theorem 3.2. The BLUE of $\mathbf{X\beta}$ can be expressed in the alternative forms

$$\text{(a)} \quad (\mathbf{I} - \mathbf{P}'_{\mathbf{ZV}})\mathbf{Y}, \tag{3.4}$$

$$\text{(b)} \quad \mathbf{P}_{\mathbf{X}|\mathbf{VZ}}\mathbf{Y} = (\mathbf{I} - \mathbf{P}_{\mathbf{VZ}|\mathbf{X}})\mathbf{Y}, \tag{3.5}$$

$$\text{(c)} \quad \mathbf{P}_{\mathbf{XT}^-}\mathbf{Y}. \tag{3.6}$$

In (c), $\mathbf{T}^-$ is an n.n.d. g-inverse of $\mathbf{T} = \mathbf{V} + \mathbf{XUX}'$ for any symmetric $\mathbf{U}$ such that $\mathbf{T}$ is n.n.d. and $R(\mathbf{T}) = R(\mathbf{V} : \mathbf{X})$.

Proof. The result (a) follows by choosing $\boldsymbol{\Lambda}_0 = \mathbf{I}$ in the Corollary 3.1. The equivalence of (a) with (b) is a consequence of the equality (2.23) observing that $\mathbf{Y} \in \mathscr{M}(\mathbf{V} : \mathbf{X}) = \mathscr{M}(\mathbf{X} : \mathbf{VZ})$. Again (c) is a consequence of (2.25).

Corresponding to expressions (3.4)–(3.6), we have the decompositions

$$\text{(a)}' \quad \mathbf{Y} = (\mathbf{I} - \mathbf{P}'_{\mathbf{ZV}})\mathbf{Y} + \mathbf{P}'_{\mathbf{ZV}}\mathbf{Y}, \tag{3.7}$$

$$\text{(b)}' \quad \mathbf{Y} = \mathbf{P}_{\mathbf{X}|\mathbf{VZ}}\mathbf{Y} + \mathbf{P}_{\mathbf{VZ}|\mathbf{X}}\mathbf{Y}, \tag{3.8}$$

$$\text{(c)}' \quad \mathbf{Y} = \mathbf{P}_{\mathbf{XT}^-}\mathbf{Y} + (\mathbf{I} - \mathbf{P}_{\mathbf{XT}^-})\mathbf{Y}, \tag{3.9}$$

where the first term on the right-hand side of each is the BLUE of $\mathbf{X\beta}$ and the second term is the residual vector providing the MINQUE of σ^2.

Theorem 3.3. The BLUE of an estimable parametric function $\mathbf{p}'\boldsymbol{\beta}$ (i.e. $\exists$ an $\mathbf{L} \ni \mathbf{L}'\mathbf{X} = \mathbf{p}'$) is $\mathbf{p}'\hat{\boldsymbol{\beta}}$ where

$$\hat{\boldsymbol{\beta}} = \mathbf{G}_{\mathbf{X}|\mathbf{VZ}}\mathbf{Y}, \tag{3.10}$$

where $\mathbf{G}_{\mathbf{X}|\mathbf{VZ}}$ is a g-inverse of $\mathbf{X}$ constrained by $\mathbf{VZ}$ as defined in (2.15).

Proof. From (3.5)

$$\begin{aligned}\text{the BLUE of } \mathbf{X}\boldsymbol{\beta} = \mathbf{P}_{\mathbf{X}|\mathbf{VZ}}\mathbf{Y} \Rightarrow \text{the BLUE of } \mathbf{L}'\mathbf{X}\boldsymbol{\beta} &= \mathbf{L}'\mathbf{P}_{\mathbf{X}|\mathbf{VZ}}\mathbf{Y} \\ &= \mathbf{L}'\mathbf{X}\mathbf{G}_{\mathbf{X}|\mathbf{VZ}}\mathbf{Y} = \mathbf{p}'\hat{\boldsymbol{\beta}}\end{aligned}$$

using the relationship (2.16).

Theorem 3.4. The MINQUE estimate of $f\sigma^2$, $f = R(\mathbf{V} : \mathbf{X}) - R(\mathbf{X})$, can be written in the alternate forms

$$\text{(a)}\quad \mathbf{Y}'\mathbf{Z}(\mathbf{Z}'\mathbf{V}\mathbf{Z})^-\mathbf{Z}\mathbf{Y}, \tag{3.11}$$

$$\text{(b)}\quad \mathbf{Y}'\mathbf{P}'_{\mathbf{VZ}|\mathbf{X}}\mathbf{V}^-\mathbf{P}_{\mathbf{VZ}|\mathbf{X}}\mathbf{Y}, \tag{3.12}$$

$$\text{(c)}\quad \mathbf{Y}'\mathbf{T}^-(\mathbf{I} - \mathbf{P}_{\mathbf{X}\mathbf{T}^-})\mathbf{Y}. \tag{3.13}$$

Proof. The MINQUE estimator of $f\sigma^2$ as defined by Rao (1972) is $\mathbf{Y}'\mathbf{Z}(\mathbf{Z}'\mathbf{V}\mathbf{Z})^-\mathbf{Z}\mathbf{Y}$ which is expression (a). The equivalence of (b) with (a) is easily established. Since $\mathbf{Y} \in \mathscr{M}(\mathbf{T})$, (c) is established if

$$\begin{aligned}\mathbf{T}\mathbf{Z}(\mathbf{Z}'\mathbf{V}\mathbf{Z})^-\mathbf{Z}'\mathbf{T} &= \mathbf{T}\mathbf{T}^-(\mathbf{I} - \mathbf{P}_{\mathbf{X}\mathbf{T}^-})\mathbf{T} \\ &= \mathbf{T}\mathbf{T}^-\mathbf{P}_{\mathbf{VZ}|\mathbf{X}}\mathbf{T} \quad \text{applying (2.25).}\end{aligned} \tag{3.14}$$

Writing $\mathbf{T} = \mathbf{VZE} + \mathbf{XF}$, the left-hand side of (3.14) is

$$\mathbf{T}\mathbf{Z}(\mathbf{Z}'\mathbf{V}\mathbf{Z})^-\mathbf{Z}'\mathbf{VZE} = \mathbf{VZ}(\mathbf{Z}'\mathbf{V}\mathbf{Z})^-\mathbf{Z}'\mathbf{VZE} = \mathbf{VZE}. \tag{3.15}$$

The right-hand side of (3.14) is

$$\mathbf{T}\mathbf{T}^-\mathbf{P}_{\mathbf{VZ}|\mathbf{X}}(\mathbf{VZE} + \mathbf{XF}) = \mathbf{T}\mathbf{T}^-\mathbf{VZE} = \mathbf{VZE}, \tag{3.16}$$

and the equality of (3.15) and (3.16) proves the result (c).

Note that when $\mathbf{V}$ is non-singular, expression (c) can be written as

$$\mathbf{Y}'\mathbf{V}^{-1}(\mathbf{I} - \mathbf{P}_{\mathbf{X}\mathbf{V}^{-1}})\mathbf{Y} \tag{3.17}$$

which is the familiar expression for estimating $f\sigma^2$.

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