

# Forecast reconciliation: A geometric view with new insights on bias correction

Anastasios Panagiotelis

Department of Econometrics and Business Statistics,  
Monash University, VIC 3145, Australia.

Email: Anastasios.Panagiotelis@monash.edu

Puwasala Gamakumara

Department of Econometrics and Business Statistics,  
Monash University, VIC 3800, Australia.

Email: Puwasala.Gamakumara@monash.edu

George Athanasopoulos\*

Department of Econometrics and Business Statistics,  
Monash University, VIC 3145, Australia.

Email: George.Athanasopoulos@monash.edu

Rob J Hyndman

Department of Econometrics and Business Statistics,  
Monash University, VIC 3800, Australia.

Email: Rob.Hyndman@monash.edu

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\*Corresponding Author. The authors gratefully acknowledge the support of Australian Research Council Grant DP140103220. We also thank Professor Mervyn Silvapulle for valuable comments.

## Abstract

A geometric interpretation is developed for so-called *reconciliation* methodologies used to forecast time series that adhere to known linear constraints. In particular, a general framework is established nesting many existing popular reconciliation methods within the class of *projections*. This interpretation facilitates the derivation of novel results that explain why and how reconciliation via projection is guaranteed to improve forecast accuracy with respect to a specific class of loss functions. The result is also demonstrated empirically. The geometric interpretation is further used to provide a new proof that forecast reconciliation results in unbiased forecasts provided the initial *base* forecasts are also unbiased. Approaches for dealing with biased base forecasts are proposed and explored in an extensive empirical study on Australian tourism flows. Overall, the method of bias-correcting before carrying out reconciliation is shown to outperform alternatives that only bias-correct or only reconcile forecasts.

# 1 Introduction

The past decade has seen rapid development in methodologies for forecasting time series that follow a hierarchical aggregation structure. Of particular prominence have been *forecast reconciliation* methods involving two steps: first separate forecasts are produced for all series, then these are adjusted to ensure coherence with aggregation constraints. Forecast reconciliation has mostly been formulated using a regression model [that admits a generalised least squares \(GLS\) solution](#), see Hyndman et al. (2011) and Wickramasuriya, Athanasopoulos & Hyndman (2019) for examples. [Alternatively, Van Erven & Cugliari \(2015\) and Nystrup et al. \(2019\) arrive at a GLS solution by formulating the problem as an optimisation problem to minimise weighted squared errors between coherent and incoherent forecasts.](#) The regression setup can be counter-intuitive since a vector comprised of forecasts from different time series models is also assumed to be the dependent variable in a regression model. In this paper, we eschew a regression interpretation in favour of a novel, geometric understanding of forecast reconciliation. This allows us to develop novel proofs and a clearer understanding of the interplay between forecast bias and reconciliation methods.

Multivariate time series following an aggregation structure arise in many sectors such as retail, energy, insurance, health and welfare and economics (see for example Karim & Maldonado 2019, Ben Taieb et al. 2017, Nystrup et al. 2019, Almeida et al. 2016, Jeon et al. 2019, Mahkya et al. 2017, Li & Tang 2019, Shang & Hyndman 2017, Athanasopoulos et al. 2019). Forecasts of these series should adhere to aggregation constraints to ensure aligned decision making. Earlier studies achieved this by only forecasting a single level of the hierarchy and then either aggregating in a bottom-up fashion (Dunn et al. 1976) or disaggregating in a top-down fashion (Gross & Sohl 1990, Athanasopoulos et al. 2009). For reviews of these approaches, including a discussion of their advantages and disadvantages, see Schwarzkopf et al. (1988), Kahn (1998), Lapide (1998), Fliedner (2001).

In contrast to these methods, Hyndman et al. (2011) proposed forecasting all series in the hierarchy, referring to these as *base* forecasts. Since base forecasts were produced independently they were not guaranteed to adhere to aggregation constraints and could thus be

improved via further adjustment. A framework was proposed whereby the aggregation constraints were expressed in a regression model for the base forecasts. The predicted values from this model were guaranteed to adhere to the linear constraints by construction and could thus be used as a new set of forecasts. This approach and later modifications have subsequently been shown to outperform bottom-up and top-down approaches in a variety of empirical settings (see for example Athanasopoulos et al. 2009, 2017, Wickramasuriya, Athanasopoulos & Hyndman 2019, among others). Some theoretical insight into the performance of forecast reconciliation methods has been provided by Van Erven & Cugliari (2015) and Wickramasuriya, Athanasopoulos & Hyndman (2019). Both papers provide a proof that reconciliation is guaranteed to improve base forecasts. The latter paper also proposes a particular version of reconciliation known as the Minimum Trace (MinT) method. This is optimal in the sense of minimising the trace of the reconciled forecast error covariance matrix under the assumption that the base forecasts are unbiased.

Our main contribution is to propose a geometric interpretation of the entire hierarchical forecasting problem. In this setting, we show that reconciled forecasts have a number of attractive properties when they are obtained via projections. We believe that this is clearer and more intuitive than explanations based on regression modelling. In addition to casting existing results in a new light, the geometric interpretation also allows us to derive three new important results.

First, our approach makes it clear that the defining characteristic of so-called *hierarchical time series* is not aggregation but linear constraints. As a result forecast reconciliation can be applied in contexts where there are no clear candidates of *bottom level* series, an insight that is not apparent when the problem is viewed through the lens of regression modelling. Second, we provide a new proof that reconciled forecasts dominate unreconciled forecasts which makes explicit the link between a reconciliation method and a loss function. We believe that this link is lacking in previous work that attempts to establish similar results, in particular Van Erven & Cugliari (2015) and Wickramasuriya, Athanasopoulos & Hyndman (2019). Furthermore, unlike Van Erven & Cugliari (2015) and Wickramasuriya, Athanasopoulos & Hyndman (2019), our proof does not require an assumption

about convexity that may not hold in general. Third, we prove that reconciliation using certain projection matrices guarantees unbiased reconciled forecasts provided the base forecasts are also unbiased. A natural question that arises is what to do in the case of biased reconciled forecasts. Rather than addressing this issue by considering matrices that are not projections, we propose to bias-correct before reconciliation. This is evaluated in an extensive empirical study where we find that even when bias correction fails, the extent of the problem is mitigated by reconciling forecasts.

The remainder of this paper is structured as follows. Section 2 deals with the concept of coherence and defines hierarchical time series in a way that does not depend on any notion of bottom-level series. Section 3 defines forecast reconciliation in terms of projections and includes a proof that reconciled forecasts dominate base forecasts with respect to a specific loss function. In Section 4 we prove the unbiasedness preserving property of reconciliation via certain projection matrices and propose methods for bias correction. In Section 5 we conduct an extensive empirical application to domestic tourism flow in Australia with two objectives; first to demonstrate the theorems discussed in Section 3, second to evaluate the methods for bias correction discussed in Section 4. Section 6 concludes with some discussion and thoughts on the future research directions [for](#) forecast reconciliation.

## 2 Coherent forecasts

### 2.1 Notation and preliminaries

We briefly define the concept of a *hierarchical time series* in a fashion similar to Athanasopoulos et al. (2019), Hyndman & Athanasopoulos (2018) and others, before elaborating on some of the limitations of this understanding. A *hierarchical time series* is a collection of  $n$  variables indexed by time, where some variables are aggregates of other variables. We let  $\mathbf{y}_t \in \mathbb{R}^n$  be a vector comprising observations of all variables in the hierarchy at time  $t$ . The *bottom-level series* are defined as those  $m$  variables that cannot be formed as aggregates of other variables; we let  $\mathbf{b}_t \in \mathbb{R}^m$  be a vector comprised of observations of all bottom-level series at time  $t$ . The hierarchical structure of the data implies that the following holds for

all  $t$ :

$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t,$$

where  $\mathbf{S}$  is an  $n \times m$  constant matrix that encodes the aggregation constraints.

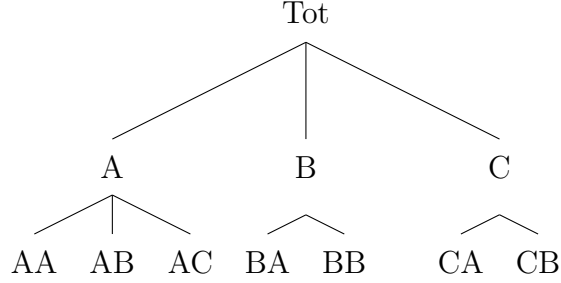


Figure 1: An example of a two level hierarchical structure.

To clarify these concepts, consider the example of the hierarchy in Figure 1. For this hierarchy,  $n = 11$ ,  $\mathbf{y}_t = [y_{Tot,t}, y_{A,t}, y_{B,t}, y_{C,t}, y_{AA,t}, y_{AB,t}, y_{AC,t}, y_{BA,t}, y_{BB,t}, y_{CA,t}, y_{CB,t}]'$ ,  $m = 7$ ,  $\mathbf{b}_t = [y_{AA,t}, y_{AB,t}, y_{AC,t}, y_{BA,t}, y_{BB,t}, y_{CA,t}, y_{CB,t}]'$  and

$$\mathbf{S} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ & & & \mathbf{I}_7 \end{pmatrix},$$

where  $\mathbf{I}_7$  is the  $7 \times 7$  identity matrix.

While such a definition is completely serviceable, it obscures the full generality of the literature on so-called hierarchical time series. In fact, concepts such as coherence and reconciliation, defined in full below, require the data to have only two important characteristics: the first is that they are multivariate, the second is that they adhere to linear constraints.

## 2.2 Coherence

The property that data adhere to some linear constraints is referred to as *coherence*. We now provide definitions aimed at providing geometric intuition of hierarchical time series.

**Definition 2.1** (Coherent subspace). The  $m$ -dimensional linear subspace  $\mathfrak{s} \subset \mathbb{R}^n$  for which some linear constraints hold for all  $\mathbf{y} \in \mathfrak{s}$  is defined as the *coherent subspace*.

To further illustrate, Figure 2 depicts the simplest three variable hierarchy where  $y_{Tot,t} = y_{A,t} + y_{B,t}$ . The coherent subspace is depicted as a grey 2-dimensional plane within 3-dimensional space; i.e.  $m = 2$  and  $n = 3$ . It is worth noting that the coherent subspace is spanned by the columns of  $\mathbf{S}$ ; i.e.  $\mathfrak{s} = \text{span}(\mathbf{S})$ . In Figure 2, these columns are  $\vec{s}_1 = (1, 1, 0)'$  and  $\vec{s}_2 = (1, 0, 1)'$ . It is equally important to recognise that the hierarchy could also have been defined in terms of  $y_{Tot,t}$  and  $y_{A,t}$  rather than the bottom-level series,  $y_{A,t}$  and  $y_{B,t}$ . In this case the corresponding ‘ $\mathbf{S}$  matrix’ would have columns  $(1, 0, 1)'$  and  $(0, 1, -1)'$ . However, while there are multiple ways to define an  $\mathbf{S}$  matrix, in all cases the columns will span the same coherent subspace, which is unique.

**Definition 2.2** (Hierarchical Time Series). A hierarchical time series is an  $n$ -dimensional multivariate time series such that all observed values  $\mathbf{y}_1, \dots, \mathbf{y}_T$  and all future values  $\mathbf{y}_{T+1}, \mathbf{y}_{T+2}, \dots$  lie in the coherent subspace, i.e.,  $\mathbf{y}_t \in \mathfrak{s} \quad \forall t$ .

Despite the common use of the term *hierarchical time series*, it should be clear from the definition that the data need not necessarily follow a hierarchy. Also notable by its absence in the above definition is any reference to *aggregation*. In some ways, terms such as *hierarchical* and *aggregation* can be misleading since the literature has covered instances that cannot be depicted in a similar fashion to Figure 1 and/or do not involve aggregation. Examples include, temporal hierarchies which involve grouped structures (see Athanasopoulos et al. 2017), overlapping temporal hierarchies (see Jeon et al. 2019), applications for which the difference rather than the aggregate is of interest (see Li & Tang 2019), or structures that involve both cross-sectional and temporal dimensions referred to as cross-temporal structures (see Kourentzes & Athanasopoulos 2019). Finally, although Definition 2.2 makes



Figure 2: Depiction of a three dimensional hierarchy with  $y_{\text{Tot}} = y_A + y_B$ . The gray coloured two dimensional plane depicts the coherent subspace  $\mathfrak{s}$  where  $\vec{s}_1 = (1, 1, 0)'$  and  $\vec{s}_2 = (1, 0, 1)'$  are basis vectors that span  $\mathfrak{s}$ . The red points in  $\mathfrak{s}$  represent realisations or coherent forecasts

reference to time series, this definition can be easily generalised to any vector-valued data for which some linear constraints are known to hold for all realisations.

**Definition 2.3** (Coherent Point Forecasts). Let  $\check{\mathbf{y}}_{t+h|t} \in \mathbb{R}^n$  be a vector of point forecasts of all series in the hierarchy where the subscript  $t+h|h$  implies that the forecast is made as time  $t$  for a period  $h$  steps into the future. Then  $\check{\mathbf{y}}_{t+h|t}$  is *coherent* if  $\check{\mathbf{y}}_{t+h|t} \in \mathfrak{s}$ .

Without any loss of generality, the above definition could also be applied to prediction for multivariate data in general, rather than just forecasting of time series.



Much of the early literature that dealt with the problem of forecasting hierarchical time series (see Gross & Sohl 1990, and references therein) produced forecasts at a single level of the hierarchy in the first stage. Subsequently forecasts for all series were recovered through aggregation, disaggregation according to historical or forecast proportions, or some combination of both. Consequently, incoherent forecasts were not a problem in these earlier papers.

Forecasting a single level of the hierarchy did not echo common practice within many industries. In many organisations different departments or ‘silos’ each produced their own forecasts, often with their own information sets and judgemental adjustments.<sup>1</sup> This approach does have several advantages over only forecasting a single level. First, there is no loss of information since all levels and series are modelled. Second, modelling higher level series often identifies features such as trend and seasonality that cannot be detected in noisy disaggregate data. However, when forecasts are produced independently at all levels, forecasts are likely to be incoherent.<sup>2</sup> While the problem of incoherent forecasts can be addressed by some multivariate approaches, including state space models, these can not always be generalised to models with more complicated features or scaled up to high-dimensional problems. Instead, the solution is to make an adjustment, *ex post* of base forecasting, that ensures coherence, a process known as *forecast reconciliation*

### 3 Forecast reconciliation

The concept of forecast reconciliation is predicated on there being an  $n$ -vector of forecasts that are incoherent. We call these *base forecasts* and denote them as  $\hat{\mathbf{y}}_{t+h|t}$ . In the sequel, this subscript will be dropped at times for ease of exposition. In the most general terms, reconciliation can be defined as follows.

**Definition 3.1** (Reconciled forecasts). Let  $\psi$  be a mapping,  $\psi : \mathbb{R}^n \rightarrow \mathfrak{s}$ . The point forecast  $\tilde{\mathbf{y}}_{t+h|t} = \psi(\hat{\mathbf{y}}_{t+h|t})$  “reconciles” a base forecast  $\hat{\mathbf{y}}_{t+h|t}$  with respect to the mapping

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<sup>1</sup>Chase (2013) discusses silos and the importance of information and data sharing across an organisation.

<sup>2</sup>There are some special cases of using simple approaches such as naïve, which extrapolate the coherent nature of the data to the forecasts.

$\psi(\cdot)$

All reconciliation methods that we are aware of consider a linear mapping for  $\psi$ , which involves pre-multiplying base forecasts by an  $n \times n$  matrix that has  $\mathfrak{s}$  as its image. One way to achieve this is with a matrix  $\mathbf{SG}$ , where  $\mathbf{G}$  is an  $m \times n$  matrix (some authors use  $\mathbf{P}$  in place of  $\mathbf{G}$ ). This facilitates an interpretation of reconciliation as a two-step process. In the first step, base forecasts  $\hat{\mathbf{y}}_{t+h|t}$  are combined to form a new set of bottom-level forecasts. In the second step, these are mapped to a full vector of coherent forecasts via pre-multiplication by  $\mathbf{S}$ .

Although pre-multiplying base forecasts by  $\mathbf{SG}$  will result in coherent forecasts, a number of desirable properties arise when  $\mathbf{SG}$  has the specific structure of a *projection* matrix onto  $\mathfrak{s}$ . In general a projection matrix is defined via its idempotence property, i.e.  $(\mathbf{SG})^2 = \mathbf{SG}$ . We also rely on another property of projection matrices, namely that any vector lying in the image of a projection is mapped to itself by that projection (see Lemma 2.4 in Rao 1974, for a proof). In our context this implies that for any  $\mathbf{v} \in \mathfrak{s}$ ,  $\mathbf{SG}\mathbf{v} = \mathbf{v}$ .

We begin by considering the special case of an orthogonal projection whereby  $\mathbf{G} = (\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'$ . This is equivalent to so called OLS reconciliation as introduced by Hyndman et al. (2011). We refrain from any discussion of regression models focusing instead on geometric interpretations. [Nonetheless](#), the connection between OLS and orthogonal projection should be clear in the context of regression modelling predicted values from OLS are obtained via an orthogonal projection of the response onto the span of the regressors.

### 3.1 Orthogonal projection

In this section we discuss two sensible properties that can be achieved by reconciliation via orthogonal projection.

- The first is that reconciliation should adjust the base forecasts as little as possible; i.e. the base and reconciled forecasts should be ‘close’. [This property is particularly emphasised by Van Erven & Cugliari \(2015\) and Nystrup et al. \(2019\)](#)
- The second is that reconciliation in some sense should improve forecast accuracy, or

more loosely, that the reconciled forecast should be ‘closer’ to the realised value targeted by the forecast. This property is particularly emphasised by Wickramasuriya, Athanasopoulos & Hyndman (2019).

To address the first of these properties we make the concept of closeness more concrete, by considering the Euclidean distance between the base forecast  $\hat{\mathbf{y}}$  and the reconciled forecast  $\tilde{\mathbf{y}}$ . A property of an orthogonal projection is that the distance between  $\hat{\mathbf{y}}$  and  $\tilde{\mathbf{y}}$  is minimal over any possible  $\tilde{\mathbf{y}} \in \mathfrak{s}$ . In this sense reconciliation via orthogonal projection leads to the smallest possible adjustments of the base forecasts.

The property that reconciliation should improve forecasts was touched upon in Section 2.3 of Wickramasuriya, Athanasopoulos & Hyndman (2019). The discussion in that paper focuses on the case of MinT. Here we provide a new explicit proof of that result. We do so first in the case of an orthogonal projection where the geometric intuition of the proof is clear and then generalise the result to reconciliation using any projection matrix in Section 3.2.

Consider the Euclidean distance between the target and a forecast. This is equivalent to the square root of the sum of squared forecast errors over the entire hierarchy. Let  $\mathbf{y}_{t+h}$  be the realisation of the data generating process at time  $t+h$ . The following theorem shows that reconciliation never increases the sum of squared errors of point forecasts.

**Theorem 3.1** (Distance reducing property). *If  $\tilde{\mathbf{y}}_{t+h|t} = \mathbf{S}\mathbf{G}\hat{\mathbf{y}}_{t+h|t}$ , where  $\mathbf{G}$  is such that  $\mathbf{S}\mathbf{G}$  is an orthogonal projection (in the Euclidean sense) onto  $\mathfrak{s}$  and let  $\|\mathbf{v}\|$  be the  $L_2$  norm (in the Euclidean sense) of vector  $\mathbf{v}$  then:*

$$\|(\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t})\| \leq \|(\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t})\|.$$

*Proof.* Since  $\mathbf{y}_{t+h|t}, \tilde{\mathbf{y}}_{t+h|t} \in \mathfrak{s}$  and since the projection is orthogonal, by Pythagoras’ theorem

$$\|(\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t})\|^2 = \|(\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t})\|^2 + \|(\tilde{\mathbf{y}}_{t+h|t} - \hat{\mathbf{y}}_{t+h|t})\|^2.$$

Since  $\|(\tilde{\mathbf{y}}_{t+h|t} - \hat{\mathbf{y}}_{t+h|t})\|^2 \geq 0$  this implies,

$$\|(\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t})\|^2 \geq \|(\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t})\|^2,$$

with equality only holding when  $\tilde{\mathbf{y}}_{t+h|t} = \hat{\mathbf{y}}_{t+h|t}$ . Taking the square root of both sides proves the desired result.  $\square$

The simple geometric intuition behind the proof is demonstrated in Figure 3. In this schematic, the coherent subspace is depicted as a black arrow, and the base forecast  $\hat{\mathbf{y}}$  is shown as a blue dot. Since  $\hat{\mathbf{y}}$  is incoherent,  $\hat{\mathbf{y}}_{t+h|t} \notin \mathfrak{s}$  and in this case the inequality is strict. Reconciliation is an orthogonal projection from  $\hat{\mathbf{y}}$  to the coherent subspace yielding the reconciled forecast  $\tilde{\mathbf{y}}$  shown in red. Finally, the target of the forecast  $\mathbf{y}$  is displayed as a black point, and although its exact location is unknown to the forecaster, it is known that it will lie somewhere along the coherent subspace.

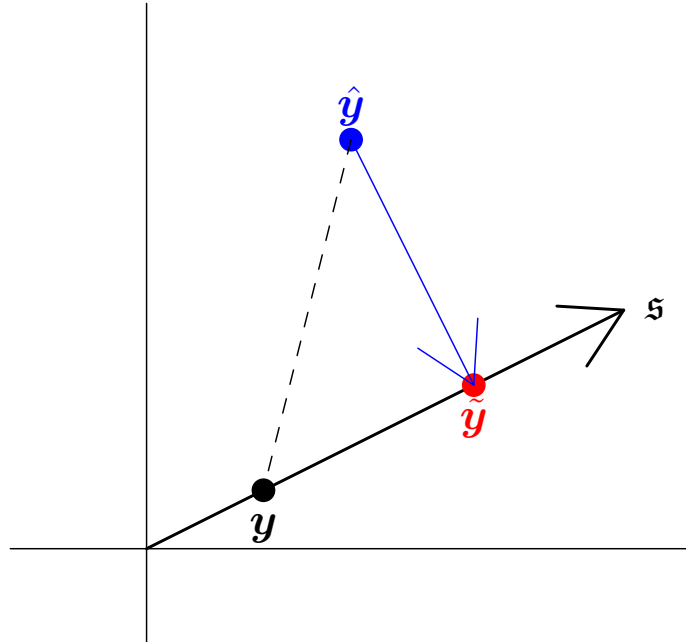


Figure 3: Orthogonal projection of  $\hat{\mathbf{y}}$  onto  $\mathfrak{s}$  yielding the reconciled forecast  $\tilde{\mathbf{y}}$ .

Figure 3 clearly shows that  $\hat{\mathbf{y}}$ ,  $\tilde{\mathbf{y}}$  and  $\mathbf{y}$  form a right angled triangle with  $\tilde{\mathbf{y}}$  at the right-angled vertex. In this triangle the line between  $\mathbf{y}$  and  $\hat{\mathbf{y}}$  is the hypotenuse and therefore must be longer than the distance between  $\mathbf{y}$  and  $\tilde{\mathbf{y}}$ . Therefore reconciliation is guaranteed to reduce the squared error of the forecast.

Theorem 3.1 is in some ways more powerful than perhaps previously understood. Cru-

cially, the result is not a result that requires taking expectations. This distance reducing property will hold for any realisation and any forecast and not just on average. Nothing needs to be assumed about the statistical properties of the data generating process or the process by which forecasts are made.

In other ways, Theorem 3.1 is weaker than perhaps often understood. First, when improvements in forecast accuracy are discussed in the context of the theorem, this refers to a very specific measure of forecast accuracy. In particular, this measure is the square root of the sum of squared forecast errors of *all* variables in the hierarchy. Consequently, while forecast improvement is guaranteed for the hierarchy overall, reconciliation can lead to less accurate forecasts for individual series. Second, although orthogonal projections are guaranteed to improve on base forecasts, they are not necessarily the projection that leads to the greatest improvement in forecast accuracy. Therefore, referring to reconciliation via orthogonal projections as ‘optimal’ is somewhat misleading since it does not have the optimality properties of some oblique projections, in particular MinT. It is to oblique projections that we now turn our attention.

## 3.2 Oblique Projections

One justification for using an orthogonal projection is that it leads to improved forecast accuracy in terms of the square root of the sum of squared errors of *all* variables in the hierarchy. A clear shortcoming of this measure of forecast accuracy is that forecast errors in all series should not necessarily be treated equally. For example, in hierarchies, top-level series tend to have a much larger scale than bottom-level series. Even when two series are on a similar scale, series that are more predictable or less variable will tend to be downweighted by simply aggregating squared errors. An even more sophisticated understanding may take the correlation between series into account. All of these considerations lead towards reconciliation of the form  $\tilde{\mathbf{y}} = \mathbf{S}(\mathbf{S}'\mathbf{W}^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}^{-1}\hat{\mathbf{y}}$ , where  $\mathbf{W}$  is a symmetric matrix. Generally, it is assumed that  $\mathbf{W}$  is invertible, otherwise a pseudo inverse can be used.

It should be noted that  $\mathbf{S}(\mathbf{S}'\mathbf{W}^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}^{-1}$  is an oblique, rather than an orthogonal projection matrix in the usual Euclidean geometry. [Alternatively](#), this matrix can be

considered to be an orthogonal projection for a different geometry defined by the norm  $\|\mathbf{v}\|_{\mathbf{W}^{-1}} = \mathbf{v}'\mathbf{W}^{-1}\mathbf{v}$ , referred to as the generalised Euclidean geometry with respect to  $\mathbf{W}^{-1}$ . One way to understand this geometry is that it is the same as Euclidean geometry when all vectors are first transformed by pre-multiplying by  $\mathbf{W}^{-1/2}$ . This leads to a transformed  $\mathbf{S}$  matrix  $\mathbf{S}^* = \mathbf{W}^{-1/2}\mathbf{S}$  and transformed  $\hat{\mathbf{y}}$  and  $\tilde{\mathbf{y}}$  vectors  $\hat{\mathbf{y}}^* = \mathbf{W}^{-1/2}\hat{\mathbf{y}}$  and  $\tilde{\mathbf{y}}^* = \mathbf{W}^{-1/2}\tilde{\mathbf{y}}$ . The transformed reconciled forecast results from an orthogonal projection in the transformed space since

$$\begin{aligned}\tilde{\mathbf{y}}^* &= \mathbf{W}^{-1/2}\tilde{\mathbf{y}} \\ &= \mathbf{W}^{-1/2}\mathbf{S}(\mathbf{S}'\mathbf{W}^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}^{-1}\hat{\mathbf{y}} \\ &= \mathbf{S}^*(\mathbf{S}^{*\prime}\mathbf{S}^*)^{-1}\mathbf{S}^{*\prime}\hat{\mathbf{y}}^*.\end{aligned}$$

Thinking of the problem in terms of a geometry defined by the norm  $\mathbf{v}'\mathbf{W}^{-1}\mathbf{v}$  is also quite instructive when it comes to thinking about the connection between distances and loss functions. In the generalised Euclidean geometry, the distance between the reconciled forecast and the realisation is given by  $(\mathbf{y} - \hat{\mathbf{y}})'\mathbf{W}^{-1}(\mathbf{y} - \hat{\mathbf{y}})$ . For diagonal  $\mathbf{W}^{-1}$ , this is equivalent to a weighted sum of squared error loss function and when  $\mathbf{W}$  is a covariance matrix, this is equivalent to a Mahalanobis distance. Therefore Theorem 3.1 can easily be generalised as follows.

**Theorem 3.2** (General distance reducing property). *If  $\tilde{\mathbf{y}}_{t+h|t} = \mathbf{S}\mathbf{G}\hat{\mathbf{y}}_{t+h|t}$ , where  $\mathbf{G}$  is such that  $\mathbf{S}\mathbf{G}$  is an orthogonal projection (in the generalised Euclidean sense) onto  $\mathfrak{s}$  then:*

$$\|(\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t})\|_{\mathbf{W}^{-1}} \leq \|(\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t})\|_{\mathbf{W}^{-1}}.$$

*Proof.* The proof is identical to the proof for Theorem 3.1 but relies on the Generalised Pythagorean Theorem (applicable to Generalised Euclidean space) rather than the Pythagorean Theorem.  $\square$

The implication of Theorem 3.2 is that if the objective function is some weighted sum of squared errors, or a Mahalanobis distance, then the projection matrix  $\mathbf{S}(\mathbf{S}'\mathbf{W}^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}^{-1}$  is guaranteed to improve forecast accuracy over base forecasts, for an appropriately selected  $\mathbf{W}$ .

Note here that we rely here on the Generalised Pythagorean Theorem (which involves an equality). In contrast, Wickramasuriya, Athanasopoulos & Hyndman (2019) follow Van Erven & Cugliari (2015) in stating their result in terms of the Generalised Pythagorean Inequality. The proof of Wickramasuriya, Athanasopoulos & Hyndman (2019) requires an assumption about convexity so that the angle between the base forecast and coherent subspace must be greater than 90 degrees. The proof we have provided here requires no such assumption, since this may not hold for an arbitrary  $\mathbf{W}$ . As such the statement from Wickramasuriya, Athanasopoulos & Hyndman (2019) that “*MinT reconciled forecasts are at least as good as the incoherent forecasts*” should be qualified — this is true only with respect to a loss function that depends on  $\mathbf{W}$ . If Euclidean distance (or mean squared error) is used, there will be realisations where the MinT estimator does not improve forecast accuracy relative to base forecasts. This will be demonstrated using a real data set in the empirical study in Section 5.2.

### 3.3 MinT

While the properties discussed so far hold for any projection matrix, the MinT method of Wickramasuriya, Athanasopoulos & Hyndman (2019) has an additional optimality property. Wickramasuriya, Athanasopoulos & Hyndman (2019) show that for unbiased base forecasts, the trace of the forecast error covariance matrix of reconciled forecasts is minimised by the oblique projection  $\mathbf{S}(\mathbf{S}'\mathbf{\Sigma}^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{\Sigma}^{-1}$ . Here,  $\mathbf{\Sigma}$  is the forecast error covariance matrix with errors come from using the base forecasts. Although the base forecast error covariance matrix is unknown, it can be estimated using in-sample errors.

Figure 4 provides geometrical intuition into the MinT method. Suppose that the orange points in panel (a) represent in-sample forecast errors. These provide information on the most likely direction of large deviations from the coherent subspace  $\mathfrak{s}$ . This direction is denoted by  $\mathbf{R}$ . Panel (b) shows a target value of  $\mathbf{y}$ , while the grey points indicate possible values for the base forecasts (the base forecasts are of course stochastic). One possible value of the forecast is depicted in blue as  $\hat{\mathbf{y}}$ . An oblique projection of the blue point back along the direction of  $\mathbf{R}$ , yields a reconciled forecast closer to the target, especially compared

to an orthogonal projection. Panel (c) shows the orthogonal projection of every potential base forecast onto the coherent subspace. Panel (d) depicts an oblique projection along  $\mathbf{R}$  for all the gray points. The oblique projection yields reconciled forecasts tightly packed near the target  $\mathbf{y}$ . In this sense, the oblique MinT projection minimises the forecast error variance of the reconciled forecasts. In contrast to the result in Theorem 3.2, this property is a statistical property in the sense that MinT is optimal in expectation.

### 3.4 Generalised MinT

As discussed in the previous section, the MinT method has a motivation that is quite separate from the considerations of Theorem 3.2. In spite of this, it is also true that Theorem 3.2 guarantees that MinT will always improve upon base forecasts when the loss function is based on the norm  $\|\cdot\|_{\Sigma^{-1}}$ . A similar insight has already been made by Van Erven & Cugliari (2015) and will be verified in our empirical example in Section 5. For this reason there is the potential to conflate the choice of loss function, with the desire to attain the attractive statistical properties of trace minimisation. We now propose a generalisation of the MinT method that uncouples these two ideas. Our proposed method targets more general loss functions than the sum of squared errors (in a similar vein to Van Erven & Cugliari (2015) and Nystrup et al. (2019)), while exploiting the correlation structure of forecast errors (in a similar vein to Wickramasuriya, Athanasopoulos & Hyndman (2019)).

Consider a loss function based on  $\|\cdot\|_{\mathbf{W}^{-1}}$  where the choice of  $\mathbf{W}$  is determined by the context of the forecasting problem, rather than by statistical considerations. For instance, our empirical example in Section 5 considers the number of tourists visiting different regions in Australia. Here, a sensible loss function may weight tourist numbers by the average spend of tourists in each region. Theorem 3.2 provides a guarantee to improve upon base forecasts with respect to the loss function based on  $\mathbf{W}$ . This is an orthogonal projection in the space transformed after pre-multiplying by  $\mathbf{W}^{-1/2}$ .

In the case where  $\mathbf{W} = \mathbf{I}$ , it has been established that MinT reconciliation often outperform OLS reconciliation in a statistical sense. Therefore we propose a generalisation of MinT that maintains trace minimising properties but also accounts for user-defined



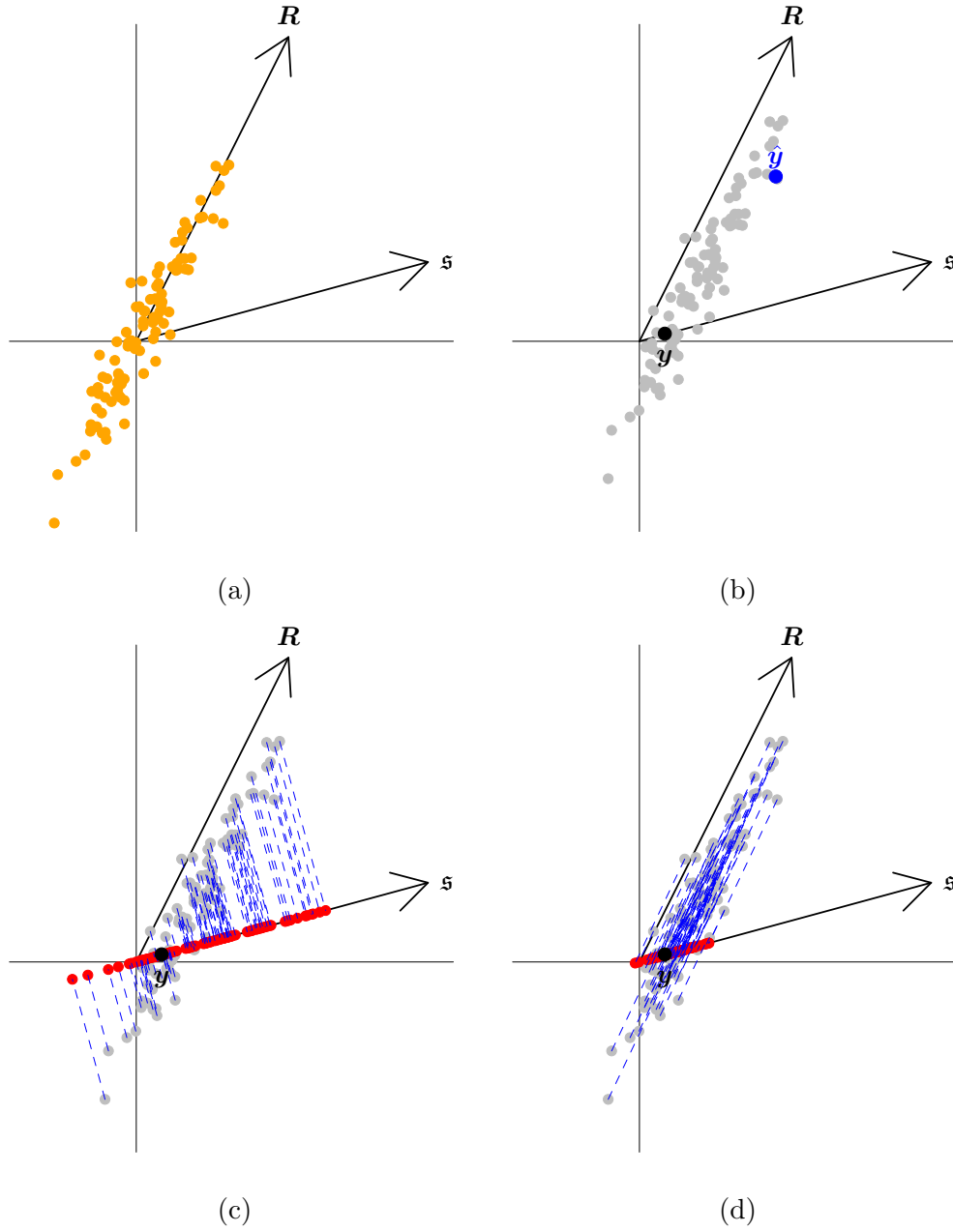


Figure 4: A schematic representation of orthogonal and oblique reconciliations. The orange points in (a) represent in-sample errors and  $\mathbf{R}$  shows the most likely direction of deviations from the coherent subspace  $\mathbf{s}$ . Grey points in (b) indicate potential base forecasts while the blue dot  $\hat{\mathbf{y}}$  represents one such realisation. The black dot  $\mathbf{y}$  denotes the (unknown) target of the forecast. (c) shows the orthogonal projection of all potential base forecasts onto the coherent subspace while (d) shows an oblique projection.

loss functions. The aim is to minimise the trace of the variance covariance matrix of the *transformed* forecast errors rather than the original errors. The projection achieves this is  $\mathbf{S}\mathbf{G}^{\text{GMinT}}$ , where

$$\mathbf{G}^{\text{GMinT}} = (\mathbf{S}'\mathbf{W}^{-1/2}\mathbf{\Sigma}^{-1}\mathbf{W}^{-1/2}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}^{-1/2}\mathbf{\Sigma}^{-1}\mathbf{W}^{-1/2}.$$

We note that some of solutions to the reconciliation problem proposed by Nystrup et al. (2019), for example *series GLASSO scaling* and *series shrinkage scaling* are special cases of this method where  $\mathbf{W}$  is diagonal. What we seek to make clear here is the connection between  $\mathbf{W}^{-1/2}$  (which need not be diagonal) and the loss function. In particular, Generalised MinT allows the choice of loss function to be decoupled from the choice of an oblique projection with attractive statistical properties.

As an important caveat, it should be noted that rank deficient choices of  $\mathbf{W}$  will in most cases lead to no solution. For instance, it is not uncommon in the hierarchical forecasting literature that the overall aggregate is of greatest interest. This may suggest a loss function that places a weight of one on forecast errors of the overall aggregate and a weight of zero on errors in all other series. In this case,  $\mathbf{S}'\mathbf{W}^{-1/2}\mathbf{\Sigma}^{-1}\mathbf{W}^{-1/2}\mathbf{S}$  is rank deficient and cannot be inverted. To be able to accrue the benefits of reconciliation by ‘borrowing strength’ from other series, the loss function must penalise errors in those series.

## 4 Bias in forecast reconciliation

Before turning our attention to the issue of bias itself it is important to state a desirable property that any reconciliation method should have. That is if base forecasts are already coherent then reconciliation should not change the forecasts. As stated in Section 3, this property holds only when  $\mathbf{S}\mathbf{G}$  is a projection matrix. As a corollary, reconciling using an arbitrary  $\mathbf{G}$ , may in fact change an already coherent forecast.

The property that projections map all vectors in the coherent subspace onto themselves is also useful in proving the unbiasedness preserving property of reconciliation of Wickramasuriya, Athanasopoulos & Hyndman (2019). Before restating this proof using a clear geometric interpretation, we discuss in a precise fashion what is meant by unbiasedness.

Suppose that the target of a point forecast is  $\boldsymbol{\mu}_{t+h|t} := E(\mathbf{y}_{t+h} \mid \mathbf{y}_1, \dots, \mathbf{y}_t)$  where the expectation is taken over the predictive density. Our point forecast can be thought of as an estimate of this quantity. The forecast is random due to uncertainty in the training sample and it is with respect to this uncertainty that unbiasedness is defined. Specifically, the point forecast will be unbiased if  $E_{1:t}(\hat{\mathbf{y}}_{t+h|t}) = \boldsymbol{\mu}_{t+h|t}$ , where the subscript  $1:t$  denotes an expectation taken over the training sample.

**Theorem 4.1** (Unbiasedness preserving property). *For unbiased  $\hat{\mathbf{y}}_{t+h|t}$ , the reconciled point forecast is also an unbiased prediction as long as  $\mathbf{SG}$  is a projection onto  $\mathfrak{s}$ .*

*Proof.* The expected value of the reconciled forecast is given by

$$E_{1:t}(\tilde{\mathbf{y}}_{t+h|t}) = E_{1:t}(\mathbf{SG}\hat{\mathbf{y}}_{t+h|t}) = \mathbf{SG}E_{1:t}(\hat{\mathbf{y}}_{t+h|t}) = \mathbf{SG}\boldsymbol{\mu}_{t+h|t}.$$

Since  $\boldsymbol{\mu}_{t+h|t}$  is an expectation taken with respect to the degenerate predictive density it must lie in  $\mathfrak{s}$ . We have already established that when  $\mathbf{SG}$  is a projection onto  $\mathfrak{s}$  then it maps all vectors in  $\mathfrak{s}$  onto themselves. As such  $\mathbf{SG}\boldsymbol{\mu}_{t+h|t} = \boldsymbol{\mu}_{t+h|t}$  when  $\mathbf{SG}$  is a projection matrix.  $\square$

The above result holds when the projection  $\mathbf{SG}$  has the coherent subspace  $\mathfrak{s}$  as its image and not for all projection matrices in general. To describe this more explicitly suppose  $\mathbf{SG}$  has as its image  $\mathfrak{L}$  which is itself a lower dimensional linear subspace of  $\mathfrak{s}$ , i.e.,  $\mathfrak{L} \subset \mathfrak{s}$ . Then for  $\{\boldsymbol{\mu}_{t+h|t} : \boldsymbol{\mu}_{t+h|t} \in \mathfrak{s}, \boldsymbol{\mu}_{t+h|t} \notin \mathfrak{L}\}$ ,  $\mathbf{SG}\boldsymbol{\mu}_{t+h|t} \neq \boldsymbol{\mu}_{t+h|t}$ . This is depicted in Figure 5 where  $\boldsymbol{\mu}$  is projected to a point  $\boldsymbol{\mu}^*$  in  $\mathfrak{L}$ . In this case, the expectation of reconciled forecast will be  $\boldsymbol{\mu}^*$  rather than  $\boldsymbol{\mu}$  and hence biased.

This result has implications in practice. The top-down method (Gross & Sohl 1990) has

$$\mathbf{G} = \begin{pmatrix} \mathbf{p} & \mathbf{0}_{(m \times n-1)} \end{pmatrix},$$

where  $\mathbf{p} = (p_1, \dots, p_m)'$  is an  $m$ -dimensional vector consisting a set of proportions used to disaggregate the top-level forecast. In this case it can be verified that  $\mathbf{SG}$  is idempotent, i.e.,  $\mathbf{SGSG} = \mathbf{SG}$  and therefore  $\mathbf{SG}$  is a projection matrix. However, the image of this projection is not an  $m$ -dimensional subspace but a 1-dimensional subspace. As such, top-down reconciliation produces biased forecasts even when the base forecasts are unbiased.

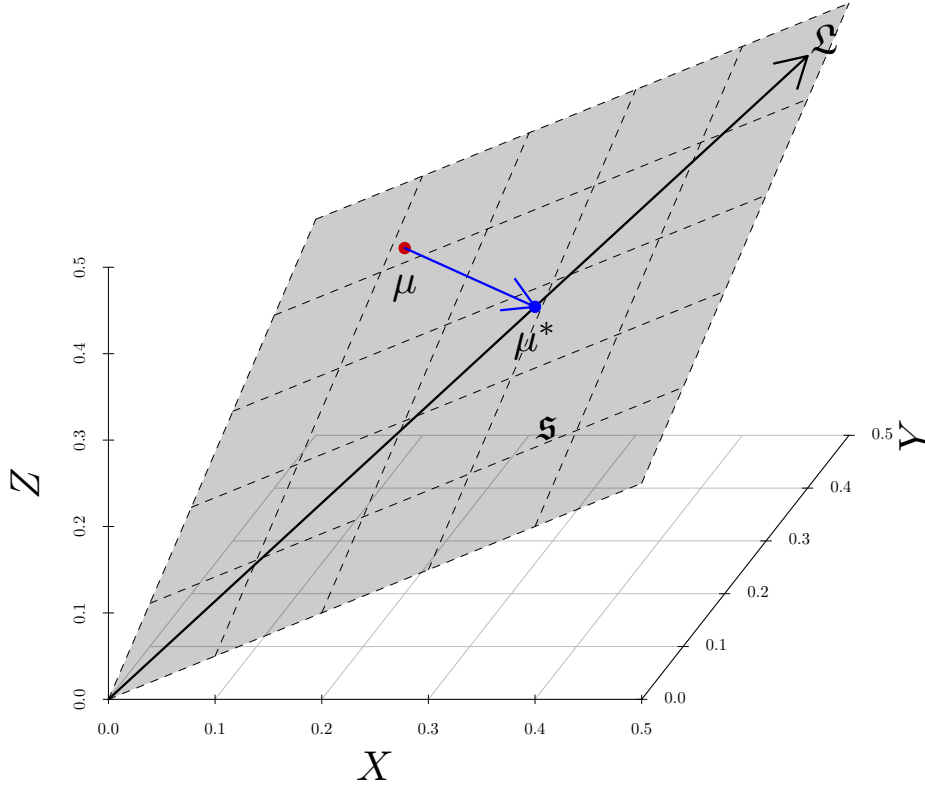


Figure 5:  $\mathcal{L}$  is a linear subspace of the coherent subspace  $\mathfrak{s}$ . If a projection is onto  $\mathcal{L}$  instead of  $\mathfrak{s}$ , then  $\mu \in \mathfrak{s}$  will be moved to  $\mu^* \in \mathcal{L}$ .

Finally, it is often stated that an assumption required to prove the unbiasedness preserving property is that  $\mathbf{S}\mathbf{G}\mathbf{S} = \mathbf{S}$  or alternatively that  $\mathbf{G}\mathbf{S} = \mathbf{I}$ . Both of these conditions are equivalent to assuming that  $\mathbf{S}\mathbf{G}$  is a projection matrix (see Section A.1 in Appendix A for a proof). [Despite this connection](#), problems arise when viewing the preservation of unbiasedness through the prism of imposing the constraint  $\mathbf{G}\mathbf{S} = \mathbf{I}$ . This thinking suggests that a way to deal with biased forecasts is to select  $\mathbf{G}$  in an unconstrained manner. Equipped with a geometric understanding of the problem, we would advise against this approach. The constraint  $\mathbf{G}\mathbf{S} = \mathbf{I}$  is not just about bias. Dropping the constraint compromises all of the attractive properties of projections. It also opens the door to reconciliation methods that change already coherent base forecasts, which suggests an increase in the

variability of the forecasts. This seems particularly perverse when the motivation for using a biased method in the first place is to reduce variance.

## 4.1 Bias correction

Our own solution to dealing with biased forecasts is to bias correct *before* reconciliation. In many cases the method for bias correction will be context specific. For instance, in our empirical study in Section 5 we consider a scenario where data are modelled after taking either a log transformation or a Box-Cox transformation. Since linear constraints that hold on the original scale do not hold for the transformed series, back-transforming to the original scale is necessary. Since this step induces a bias we propose to bias correct after this back-transformation step, and ultimately to reconcile the back-transformed, bias-corrected forecasts. In the well-known case of the Box-Cox transformation a number of bias correction methods exist based on Taylor expansions.

Alternatively, a more general purpose approach to bias correction is to simply estimate the bias by taking the sample mean of  $\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t}$  for all  $t + h$  in the training sample. This can then be subtracted from future forecasts. As stated in the discussion of MinT, in-sample errors are already used to estimate the optimal direction of projection. As such it may be possible to use the same errors to bias correct. Geometrically, the intuition is simple. In panel (a) of Figure 4, the orange points are centered around the origin as would be expected from an unbiased forecast. If forecasts are biased, then errors should simply be translated until they are centered at the origin. Nonetheless there are also a number of pitfalls to such an approach. First, for the very construction we consider, where bias is induced by taking a log or Box Cox transformation, bias should be corrected by a multiplicative rather than an additive factor. Second, if in-sample errors are non-stationary due to model misspecification or structural breaks, then the proposed method for bias correction may break down.

## 5 Empirical study

Using an empirical application to forecast Australian domestic tourism flows, we illustrate the usefulness of projection-based reconciliation in practice. Previous studies have found that reconciliation improves point forecast accuracy in domestic tourism flows for Australia (see for example Athanasopoulos et al. 2009, Hyndman et al. 2011, Wickramasuriya, Athanasopoulos & Hyndman 2019). Our motivation in this study is twofold. First, we demonstrate the implications of Theorem 3.1 by comparing reconciled and base forecasts. In contrast to previous studies, we consider individual periods rather than computing averages over a rolling window. Second, we demonstrate how the bias correction methods discussed in the previous section along with the projection-based reconciliation help to improve forecast accuracy.

### 5.1 Data

We consider “overnight trips” across Australia as a measure of domestic tourism flows. The data are provided by the National Visitor Survey and are collected through telephone interviews from an annual sample of 120,000 Australian residents aged 15 years or more. We disaggregate tourism flows into 7 states, 27 zones and 75 regions forming a natural geographical hierarchy that is of interest to tourism operators and policy makers amongst others. Hence, there are 110 series across the hierarchy with 75 bottom-level series. More information about the series and the geographical hierarchy is presented in Table 3 in Appendix B. The data span the period January 1998 to December 2017, which gives a total of 240 observations per series.

Figure 6 shows time, sub-series and seasonal plots of the aggregate overnight trips. As is usual with tourism data, these show a strong seasonal pattern with peaks observed every January corresponding to the summer vacation season in Australia. There are also some lower peaks observed in April, July and October corresponding to school term breaks. On the other hand, the month with the least overnight trips is February indicating that people travel least for the month following their summer vacation. The time plot also shows a pronounced upward trend starting from around 2010 to the end of the sample, with flows

being fairly flat from the beginning of the sample and a slight downward trend during 2004–2010.

The top panel of Figure 7 shows time plots for the seven states, hence the first level of the geographical hierarchy. The panels below show some selected series from the second-level zones and the bottom-level regions. The plots display the diversity of time series features, within but also between levels. For example, noticeable at the first level is the asynchronous seasonal pattern between the Northern Territory and the other states. For the Northern Territory the high tourist season occurs during June–August with July being the peak, while the low season is during December–February. This reflects the tropical climate of the Northern Territory, with Australians mostly visiting the North during its dry winter-season rather than the wet summer season. Noticeable as we move to the lower levels is the variation in the signal-to-noise ratio, with the regional bottom-level series being much noisier compared to the series from levels above. This of course highlights the importance of modelling series at all levels without any loss of valuable information. We should note here that we observed an anomalous (extremely high) observation for ‘Adelaide Hills’ for December 2002. We replaced this observation with the average overnight trips on December 2001 and December 2003 for the same destination.

## 5.2 Comparison to Base Forecasts

To demonstrate the implications of Theorem 3.1 we consider the improvement of different reconciliation methods over base forecasts. For each series the ARIMA model minimising AICc is chosen using the `auto.arima()` function in the `forecast` package. Using these fitted models, base forecasts are produced for  $h = 1$  to 12-steps ahead for each series in the hierarchy. This is first carried out with a training sample of 100 observations, i.e., Jan-1998 to Apr-2006. The training window is then rolled forward one observation at a time until the end of the sample. This generates 140 1-step-ahead, 139 2-steps-ahead through to 129 12-steps-ahead forecasts available for forecast evaluation.

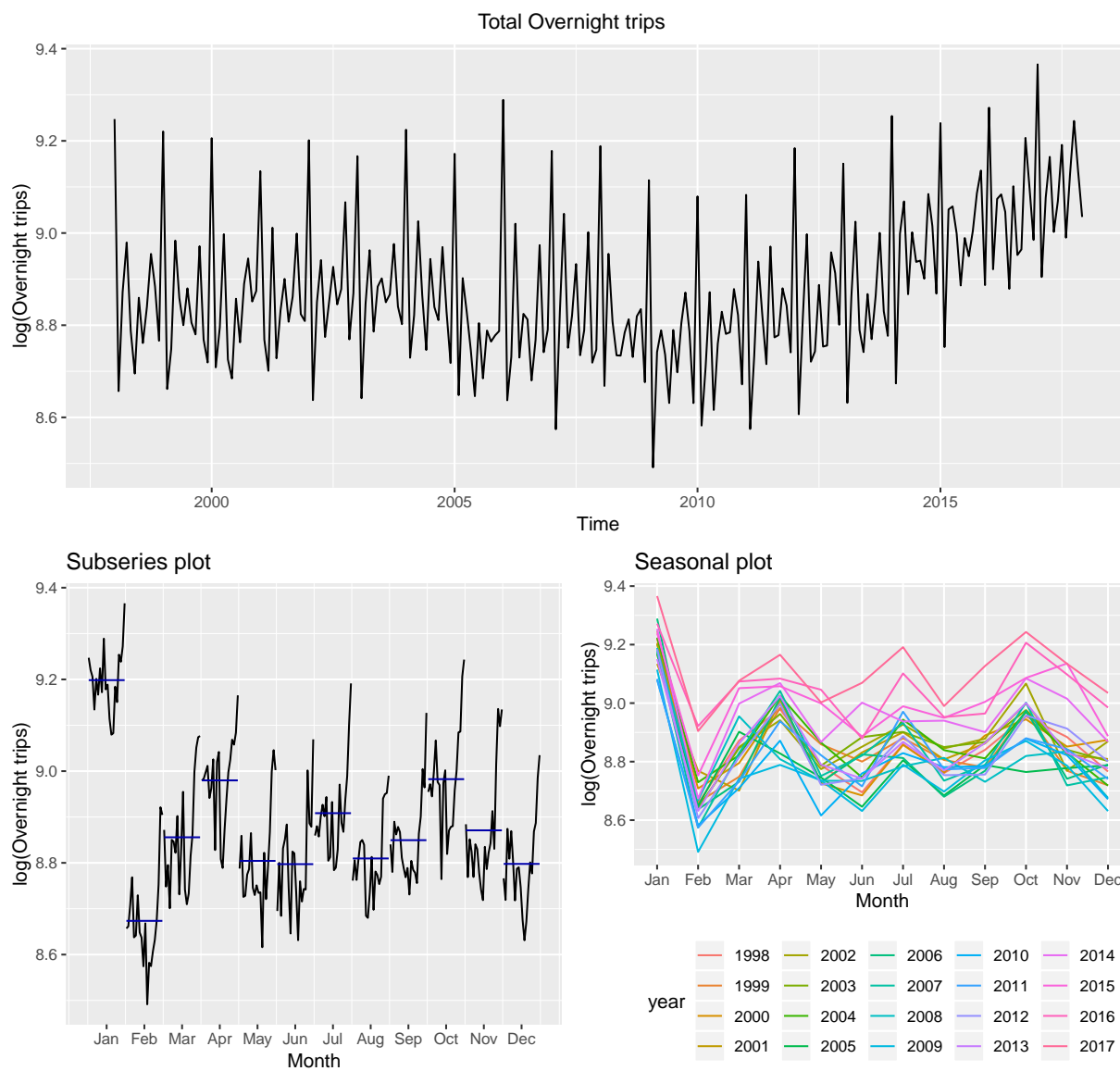


Figure 6: Total domestic overnight trips (in logs) for Australia from January 1998 to December 2017. The top-panel shows a time plot; the bottom-left panel a sub-series plot for each month; the bottom-right panel shows a seasonal plot coloured by year.



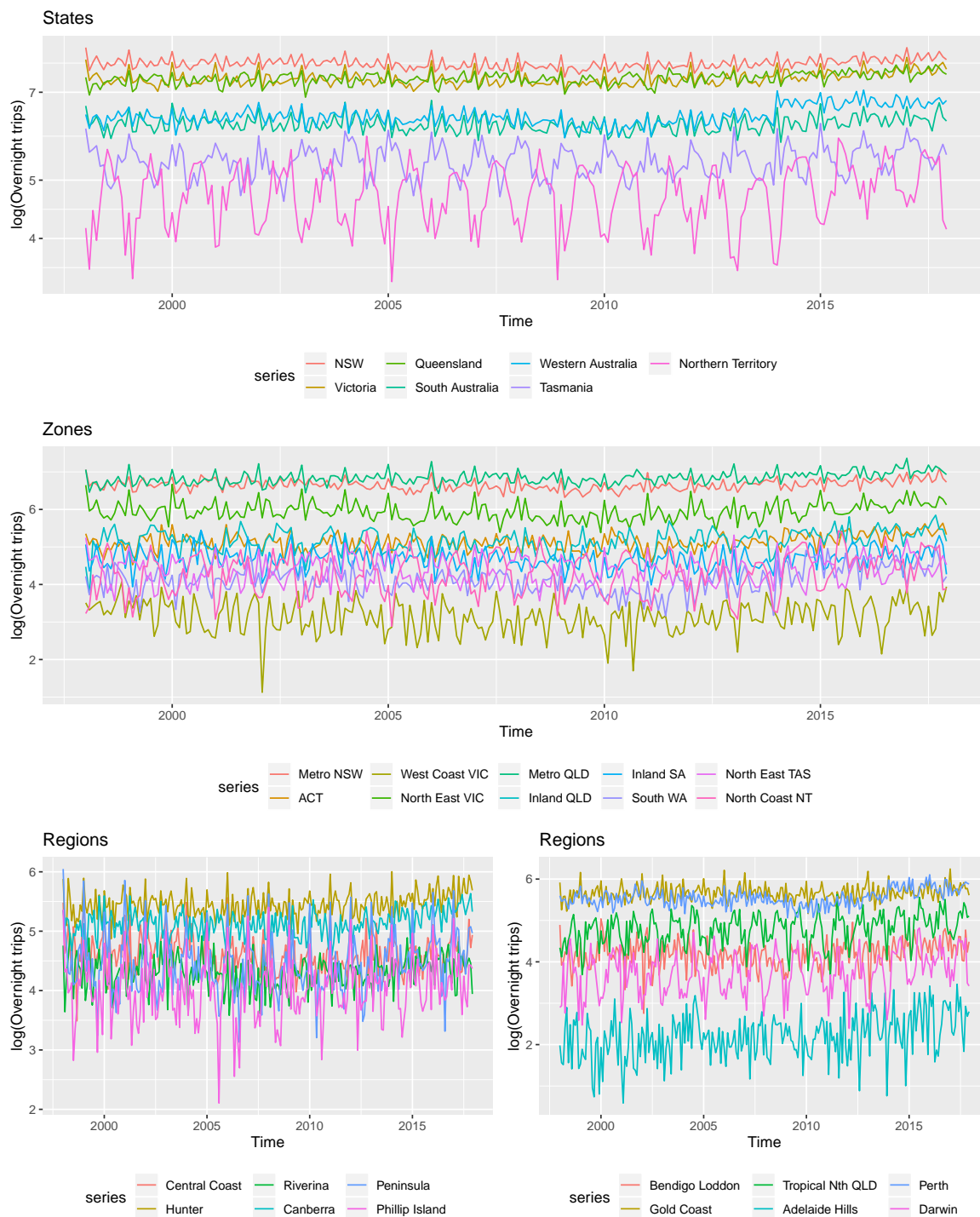


Figure 7: Time plot of overnight trips for some selected series from different disaggregate levels of the hierarchy. All values are presented in log scale. To avoid impact from the zero values we added a constant 1 to all observations

For each replication and for method  $q$ , the total squared error (TSE), is computed as

$$\text{TSE}_t^q = \sum_{i=1}^n (y_{i,t} - \tilde{y}_{i,t}^{(q)})^2,$$

where  $\tilde{y}^{(q)}$  is the reconciled forecast using method  $q$  for series  $i$  and replication  $t$ . This corresponds to the loss function described in Section 3.1. We also consider a loss function based on a generalised Euclidean distance of the form

$$\text{WSE}_t^q = \sum_{i=1}^n w_i (y_{i,t} - \tilde{y}_{i,t}^{(q)})^2,$$

where  $\tilde{y}^{(q)}$  is the reconciled forecast using method  $q$  for series  $i$  and replication  $t$ .

After obtaining the base forecasts these are reconciled using four different projection methods: OLS, WLS, MinT and GMinT. For MinT and GMinT, the shrinkage estimator of Schäfer & Strimmer (2005) is used to estimate the covariance matrix of one step ahead forecast errors. That is  $\Sigma = \tau \text{diag}(\hat{\Sigma}) + (1 - \tau)\hat{\Sigma}$  where  $\hat{\Sigma}$  is the sample estimate of the variance covariance matrix of the in-sample, one-step ahead forecast errors and,

$$\tau = \frac{\sum_{i \neq j} \text{Var}(\hat{\sigma}_{ij})}{\sum_{i \neq j} \hat{\sigma}_{ij}^2},$$

where  $\sigma_{ij}$  denotes the  $(i, j)$ th element of  $\hat{\Sigma}^{\text{sam}}$ . For WLS and GMinT, variance scaling is used, i.e.,  $\mathbf{W}^{\text{WLS}} = \text{diag}(\hat{\Sigma})$  CHANGE DEPENDING ON FINAL CHOICE OF W. For both loss function we compute the difference between base forecasts and reconciled forecasts, where positive values indicate that reconciled forecasts are more accurate.

The boxplots in Figure 8 summarise the distribution of this measure over each rolling window. We only present the results for  $h = 1$ , but the results and conclusions that follow are almost identical for the other longer forecast horizons. We do not present these here to save space but they are available upon request.

As the theory predicts, OLS reconciliation always leads to an improvement relative to base forecasts. The entire boxplot for OLS reconciliation is above zero. This is not the case for MinT and WLS which for some windows generate forecasts less accurate than the base forecasts. The boxplots also demonstrate that OLS reconciliation is more stable than MinT and WLS reconciliation, a result not entirely unsurprising since the latter two

methods require estimation of a  $\mathbf{W}$  matrix which in practice is not trivial. Nonetheless when averaging over the entire rolling window the MinT estimator performs best on average, which is again precisely what the theorem proved in Wickramasuriya, Athanasopoulos & Hyndman (2019) would suggest. The average MSE values across the entire rolling window of 140 replications are shown in Table 1.

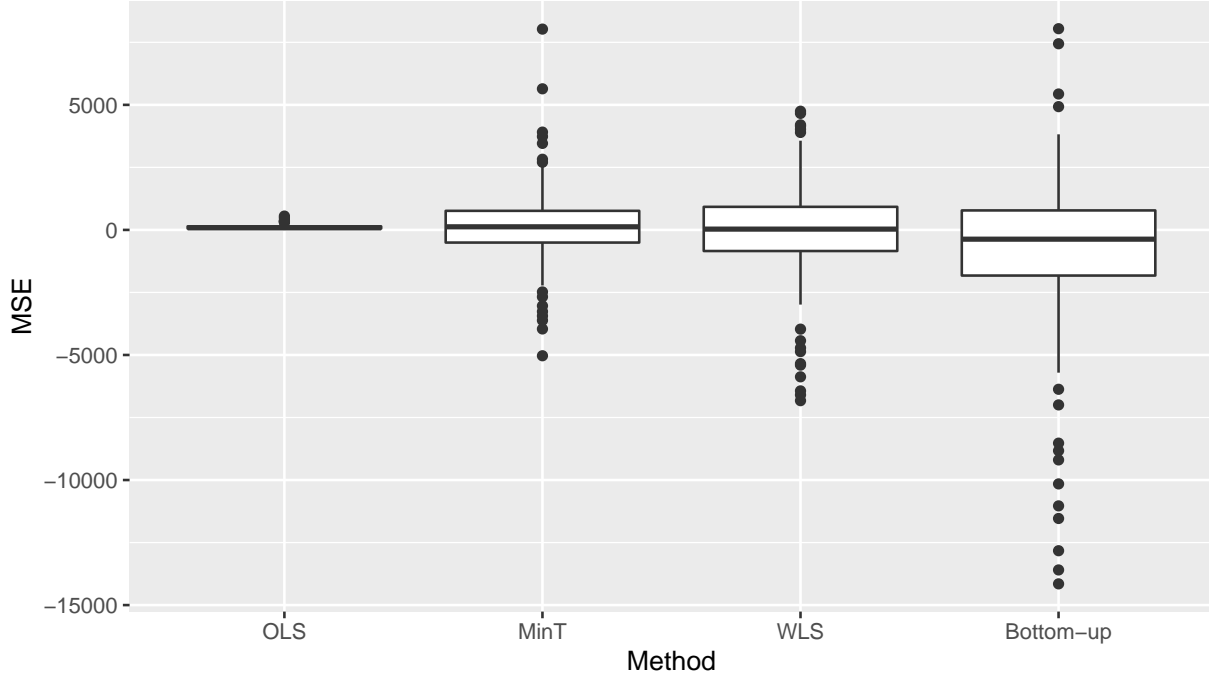


Figure 8: MSE differences between base forecasts and reconciled forecasts using OLS, MinT and WLS, and also the bottom-up approach for  $h = 1$ . A positive value indicates that the coherent forecasts are an improvement over the base forecasts.

Table 1: Average  $\text{MSE}(\times 10^3)$  over all 140 rolling window replications for  $h = 1$ .

|     | Base | OLS  | MinT        | WLS  | Bottom-up |
|-----|------|------|-------------|------|-----------|
| MSE | 4.31 | 4.20 | <b>4.16</b> | 4.48 | 5.28      |

### 5.3 Transformations and bias adjustment

We first transform each series in the hierarchy using two types of transformations. Namely, we perform a log-transformation and also the more general Box-Cox transformation. A Box-Cox transformation is defined as,

$$w_t = \begin{cases} \log(y_t) & \text{if } \lambda = 0; \\ \frac{y_t^\lambda - 1}{\lambda} & \text{otherwise.} \end{cases}$$

We first set  $\lambda = 0$  and hence consider only a log transformation. For the second more general Box-Cox transformation we select  $\lambda$  using the “Guerrero” method (Guerrero 1993) implemented in the `BoxCox.lambda()` function in the `forecast` package in R (Hyndman et al. 2019). In order to avoid extreme and volatile transformations we restrict  $\lambda \in (-0.5, 2)$ . As zero observations exist in some of the bottom-level series, before transforming we add a constant (more specifically 1) to each series. This overcomes the challenge of undefined transformed values for zero observations when we specifically implement the log transformation or when  $\lambda$  is selected to be zero by the “Guerrero” method. The constant is subtracted from the final forecasts.

After transformation we fit univariate ARIMA models to each transformed series. The `auto.arima()` function in the `forecast` package is used to choose the best model that minimises the AICc. Using the fitted models, forecasts are produced for  $h = 1$  to 12-steps ahead for each series in the hierarchy. The same rolling window described in Section 5.2 is used here as well.

The forecasts are then back-transformed by simply reversing the Box-Cox transformation using,

$$\hat{y}_{t+h|t} = \begin{cases} \exp(\hat{w}_{t+h|t}) & \text{if } \lambda = 0, \\ (\lambda \hat{w}_{t+h|t} + 1)^{1/\lambda} & \text{otherwise.} \end{cases} \quad (1)$$

These back-transformed forecasts are potentially biased as they are not the mean of the forecast distribution but the median (assuming that the distribution of the transformed space is symmetric). Hence, the reconciled forecasts that follow from these forecasts will also be biased. We refer to these as “Biased” base forecasts in the results that follow.

This is the exact scenario we want to demonstrate in this study and we next move to our proposed solution of bias correcting the base forecasts before reconciling for which we explore two scenarios.

Using a Taylor series expansion (Guerrero 1993), the back-transformed mean of the forecast distribution for a Box-Cox transformation is given by

$$\hat{y}_{t+h|t} = \begin{cases} \exp(\hat{w}_{t+h|t}) \left[ 1 + \frac{\sigma_h^2}{2} \right] & \text{if } \lambda = 0, \\ (\lambda \hat{w}_{t+h|t} + 1)^{1/\lambda} \left[ 1 + \frac{\sigma_h^2(1-\lambda)}{2(\lambda \hat{w}_{t+h|t} + 1)^2} \right] & \text{if } \lambda \neq 0, \end{cases} \quad (2)$$

where  $\hat{w}_{t+h|t}$  is the  $h$ -step-ahead forecast from the Box-Cox transformed series and  $\sigma_h^2$  is the variance of  $\hat{w}_{t+h|t}$ . Using the mean of the forecast distribution returns bias-adjusted base forecasts compared to the simple back-transformation of Eq. (1). We refer to this as “Method-1” in the results that follow. The second scenario of bias adjustment we explore is using the in-sample forecast error mean of the biased forecasts to adjust the out-of-sample forecasts. We refer to this as “Method-2” in the results that follow.

Using the three sets of base forecasts from each of the two transformations, we generate coherent forecasts implementing OLS, MinT and WLS reconciliation projections, and also the bottom-up approach and compare the results for when the base forecasts are biased and bias-adjusted, i.e., unbiased. For each replication and for method  $q$ , the total error (TE) and total squared error (TSE), are computed as

$$\text{TE}_t^q = \sum_{i=1}^n (y_{i,t} - \tilde{y}_{i,t}^{(q)}) \quad \text{and} \quad \text{TSE}_t^q = \sum_{i=1}^n (y_{i,t} - \tilde{y}_{i,t}^{(q)})^2,$$

where  $\tilde{y}^{(q)}$  is the reconciled forecast using method  $q$  for series  $i$  and replication  $t$ . The total error gives an indication of bias, while the total squared error is connected with Euclidean distance. Table 2 reports the relative mean total error (RMTE) and relative mean total error (RMTSE) defined as

$$\text{RMTE}^q = \frac{\sum_{t=1}^{140} \text{TE}_t^q}{\sum_{t=1}^{140} \text{TE}_t^{\text{Base}}} \quad \text{and} \quad \text{RMTSE}^q = \frac{\sum_{t=1}^{140} \text{TSE}_t^q}{\sum_{t=1}^{140} \text{TSE}_t^{\text{Base}}},$$

where  $\text{TE}_t^{\text{Base}}$  and  $\text{TSE}_t^{\text{Base}}$  are the total error and total squared error of the biased base forecasts at replication  $t$ . Stars indicate that the results are significantly different from the

biased base forecasts using a DETAILS OF TEST HERE. Results are shown for 1-step-ahead forecasts and as was the case with Section 5.2, the conclusions that follow extend almost identically to longer forecast horizons. These are provided in an online supplement.

Recall that reconciliation approaches via projections preserve unbiasedness in the reconciled forecasts iff the base forecasts are unbiased. Hence, the two columns labelled “Biased”

Table 2: Average RMTE and RMTSE of 1-step-ahead point forecasts from log and Box-Cox transformed series. Biased denotes forecasts from simply reversing the transformation via Eq. (1). Unbiased(Method-1) performs bias adjustment via a Taylor series expansion as shown in Eq. (2) whereas Unbiased(Method-2) bias adjusts by subtracting the in-sample forecast error mean. Stars indicate significant difference from biased base forecast according to a TEST DETAILS HERE.

| Method    | Log Transformation |                    |                    | Box-Cox Transformation |                    |                    |
|-----------|--------------------|--------------------|--------------------|------------------------|--------------------|--------------------|
|           | Biased             | Unbiased           | Unbiased           | Biased                 | Unbiased           | Unbiased           |
|           |                    | (Method-1)         | (Method-2)         |                        | (Method-1)         | (Method-2)         |
| ME        |                    |                    |                    |                        |                    |                    |
| Base      | 1.00               | 0.53* <sup>†</sup> | 1.40* <sup>†</sup> | 1.00                   | 0.63* <sup>†</sup> | 1.37* <sup>†</sup> |
| OLS       | 0.51*              | 0.39*              | 0.66               | 0.54*                  | 0.46*              | 0.74               |
| MinT      | 0.74               | 0.51* <sup>†</sup> | 1.10 <sup>†</sup>  | 0.75                   | 0.61*              | 1.03 <sup>†</sup>  |
| WLS       | 1.21               | 0.58* <sup>†</sup> | 1.74* <sup>†</sup> | 1.20                   | 0.69* <sup>†</sup> | 1.67* <sup>†</sup> |
| Bottom-up | 1.76*              | 0.64* <sup>†</sup> | 2.72* <sup>†</sup> | 1.73                   | 0.75* <sup>†</sup> | 2.57* <sup>†</sup> |
| MSE       |                    |                    |                    |                        |                    |                    |
| Base      | 1.00               | 0.99               | 1.01               | 1.00                   | 0.98               | 1.04               |
| OLS       | 0.97*              | 0.96*              | 0.98*              | 0.97*                  | 0.96*              | 1.01               |
| MinT      | 0.97*              | 0.93*              | 1.03               | 0.93*                  | 0.91*              | 0.99               |
| WLS       | 1.08               | 0.98               | 1.21               | 1.04                   | 0.96               | 1.14               |
| Bottom-up | 1.42               | 1.18               | 1.80               | 1.35                   | 1.16               | 1.63               |

contain results for biased base but also reconciled forecasts. Using Method-1 for first bias adjusting the base forecasts and then reconciling, results in forecast improvements for all methods for both the log and the Box-Cox transformation induced bias. In this case MinT returns the lowest MSE across all forecasts. Notice that the “Box-Cox biased” base forecasts carry a higher bias compared to the “log biased” base forecasts. Even for these MinT reconciliation of bias adjusted base forecasts generates very accurate forecasts, almost as accurate as the forecasts that come from the less biased log transformation.

In contrast to the results from using Method-1 for bias adjusting before reconciliation, using Method-2 has an adverse effect on the forecast accuracy of the reconciled forecasts. In this case the reconciled unbiased forecasts are less accurate than the biased reconciled forecasts. This reflects the fact and sends the warning that implementing inappropriate bias adjustment, in this case using an additive rather than a multiplicative factor, will hinder forecast accuracy and extra care must be taken in this bias adjustment procedure.

## 6 Conclusions

Defining concepts such as coherence and reconciliation in geometric terms provides new insights into hierarchical forecasting methods. We have also provided evidence that bias correction before reconciliation improves forecast accuracy compared to approaches that do not bias correct and/or do not use reconciliation. Our intention in proposing a geometric interpretation is also to provoke research into new areas. We now discuss three such possibilities.

First, it should be possible to extend the concept of coherence [to examples where the coherent space is not a linear plane in  \$\mathbb{R}^n\$](#) . This includes the case where in addition to aggregation constraints, forecasts are also constrained to be non-negative. We note the work of Wickramasuriya, Turlach & Hyndman (2019) as an attempt to address this issue. [Another possibility is](#) non-linear constraints where the coherent space may need to be defined by a manifold. Although much more challenging, it is still possible to define reconciled forecasts in terms of projections onto a manifold. Second, since we have established that the concept of bottom-level series is not crucial in forecast reconciliation, an open question

is whether it may be better to construct base forecasts of linear combinations of the time series rather than the time series themselves. Finally, the geometric interpretations of hierarchical forecast reconciliation facilitates an extension into a probabilistic framework. The latter two are issues we investigate in separate papers.



# A Appendix

## A.1 Proof $SGS = S$ implies $SG$ is a projection

Here we establish that if  $SG$  is a projection onto the linear subspace spanned by  $S$  then  $SGS = S$ . We also prove that the converse holds, namely that if the condition  $SGS = S$  holds then  $SG$  must be a projection onto the linear subspace spanned by  $S$ .

To establish the first statement, let  $s_j$  be the  $j$ th column of  $S$ . Since by definition,  $s_j$  lies in  $\mathfrak{s}$ , it must hold that  $SGs_j = s_j$ . Stacking these vectors horizontally

$$\begin{aligned} SGS &= \left( SGs_1, \quad SGs_2, \quad \cdots \quad SGs_m \right) \\ &= \left( s_1, \quad s_2, \quad \cdots \quad s_m \right) \\ &= S. \end{aligned}$$

To establish the converse it suffices to postmultiply the condition  $SGS = S$  by  $G$ . This yields  $SGSG = SG$ , which in turn implies idempotence since  $(SG)^2 = SG$ .

## B Australian Tourism Data

Table 3: Geographical hierarchy of Australian tourism flow

| <b>Level 0 - Total</b>   |     |                    | <i>Regions cont.</i>          | <i>Regions cont.</i>              |
|--------------------------|-----|--------------------|-------------------------------|-----------------------------------|
| 1                        | Tot | Australia          | 37 AAB Central Coast          | 76 CBD Mackay                     |
| <b>Level 1 - States</b>  |     |                    | 38 ABA Hunter                 | 77 CCA Whitsundays                |
| 2                        | A   | NSW                | 39 ABB North Coast NSW        | 78 CCB Northern                   |
| 3                        | B   | Victoria           | 40 ACA South Coast            | 79 CCC Tropical North Queensland  |
| 4                        | C   | Queensland         | 41 ADA Snowy Mountains        | 80 CDA Darling Downs              |
| 5                        | D   | South Australia    | 42 ADB Capital Country        | 81 CDB Outback                    |
| 6                        | E   | Western Australia  | 43 ADC The Murray             | 82 DAA Adelaide                   |
| 7                        | F   | Tasmania           | 44 ADD Riverina               | 83 DAB Barossa                    |
| 8                        | G   | Northern Territory | 45 AEA Central NSW            | 84 DAC Adelaide Hills             |
| <b>Level 2 - Zones</b>   |     |                    | 46 AEB New England North West | 85 DBA Limestone Coast            |
| 9                        | AA  | Metro NSW          | 47 AEC Outback NSW            | 86 DBB Fleurieu Peninsula         |
| 10                       | AB  | North Coast NSW    | 48 AED Blue Mountains         | 87 DBC Kangaroo Island            |
| 11                       | AC  | South Coast NSW    | 49 AFA Canberra               | 88 DCA Murraylands                |
| 12                       | AD  | South NSW          | 50 BAA Melbourne              | 89 DCB Riverland                  |
| 13                       | AE  | North NSW          | 51 BAB Peninsula              | 90 DCC Clare Valley               |
| 14                       | AC  | ACT                | 52 BAC Geelong                | 91 DCD Flinders Range and Outback |
| 15                       | BA  | Metro VIC          | 53 BBA Western                | 92 DDA Eyre Peninsula             |
| 16                       | BB  | West Coast VIC     | 54 BCA Lakes                  | 93 DDB Yorke Peninsula            |
| 17                       | BC  | East Coast VIC     | 55 BCB Grippsland             | 94 EAA Australia's Coral Coast    |
| 18                       | BC  | North East VIC     | 56 BCD Phillip Island         | 95 EAB Experience Perth           |
| 19                       | BD  | North West VIC     | 57 BDA Central Murray         | 96 EAC Australia's South West     |
| 20                       | CA  | Metro QLD          | 58 BDB Goulburn               | 97 EBA Australia's North West     |
| 21                       | CB  | Central Coast QLD  | 59 BDC High Country           | 98 ECA Australia's Golden Outback |
| 22                       | CC  | North Coast QLD    | 60 BDD Melbourne East         | 99 FAA Hobert and South           |
| 23                       | CD  | Inland QLD         | 61 BDE Upper Yarra            | 100 FBA East Coast                |
| 24                       | DA  | Metro SA           | 62 BDF Murray East            | 101 FBB Launceston, Tamar & North |
| 25                       | DB  | South Coast SA     | 63 BEA Wimmera+Mallee         | 102 FCA North West                |
| 26                       | DC  | Inland SA          | 64 BEB Western Grampians      | 103 FCB Wilderness West           |
| 27                       | DD  | West Coast SA      | 65 BEC Bendigo Loddon         | 104 GAA Darwin                    |
| 28                       | EA  | West Coast WA      | 66 BED Macedon                | 105 GAB Kakadu Arnhem             |
| 29                       | EB  | North WA           | 67 BEE Spa Country            | 106 GAC Katherine Daly            |
| 30                       | EC  | South WA           | 68 BEF Ballarat               | 107 GBA Barkly                    |
| 31                       | FA  | South TAS          | 69 BEG Central Highlands      | 108 GBB Lasseter                  |
| 32                       | FB  | North East TAS     | 70 CAA Gold Coast             | 109 GBC Alice Springs             |
| 33                       | FC  | North West TAS     | 71 CAB Brisbane               | 110 GBD MacDonnell                |
| 34                       | GA  | North Coast NT     | 72 CAC Sunshine Coast         |                                   |
| 35                       | GB  | Central NT         | 73 CBA Central Queensland     |                                   |
| <b>Level 2 - Regions</b> |     |                    | 74 CBB Bundaberg              |                                   |
| 36                       | AAA | Sydney             | 75 CBC Fraser Coast           |                                   |

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