Probabilistic Forecast Reconciliation

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Motivating Examples

- Gross Domestic Product
  - An aggregate of consumption, investment, government spending and trade balance
  - Further breakdown, e.g. consumption of food, rent, etc.
- Retail sales
  - Aggregate of different product category sales
  - Further breakdown into individual stock keeping units.
- Potentially need forecasts of all time series.

Incoherent Forecasts

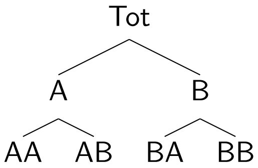
- Potential approaches
  - Univariate models
  - Multivariate models
  - Judgemental forecasts
- Forecasts may not respect aggregation structure. (**Incoherent**)
- Outcome *will* respect aggregation structure. (**Coherent**)
- Motivation is aggregation but can be generalised to any linear constraints.

Reconciliation

- Begin with a vector of forecasts that are incoherent.
- Adjust these **ex post** to make them coherent.
- There are good solutions for point forecasting that:
  - Guarantee coherent forecasts.
  - Improve forecast accuracy overall.
- Generalisation to probabilistic forecasts is our contribution.
- Getting there necessitates a rethink of the existing point forecasting literature.

Hierarchical and Grouped Time Series

- Collections of time series are often characterised by aggregation constraints
  - Cross-Sectionally
  - Temporally
- **Coherent** forecasts respect such constraints.
- Independently produced forecasts are generally incoherent.



Forecast Reconciliation

- Forecast reconciliation involves
  1. Producing incoherent **base** forecasts for all series in an $n \times 1$ vector $\hat{\boldsymbol{y}}$
  2. Adjusting base forecasts to obtain coherent **reconciled** forecasts in an $n \times 1$ vector $\tilde{\boldsymbol{y}}$
- Why do we care?
  1. Aligned decision making.
  2. Improved forecast accuracy

Reconciliation in two steps

- Many reconciliation methods involve two steps
  1. Pre-multiply $\hat{\boldsymbol{y}}$ by a $m \times n$ matrix $\boldsymbol{G}$ to obtain **bottom** level series $\boldsymbol{b} = \boldsymbol{G}\hat{\boldsymbol{y}}$
  2. Pre-multiply $\boldsymbol{b}$ by a $n \times m$ matrix $\boldsymbol{S}$ to obtain $\tilde{\boldsymbol{y}}$, i.e. $\tilde{\boldsymbol{y}} = \boldsymbol{S}\boldsymbol{b}$
- The matrix $\boldsymbol{S}$ defines the aggregation constraints, e.g.

$$\boldsymbol{S} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ & \boldsymbol{I}_{4\times 4} & & \end{pmatrix}$$

- Choice of $\boldsymbol{G}$ defines reconciliation method, e.g. OLS: $\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$ and Bottom Up: $\boldsymbol{G} = (\boldsymbol{0}_{m\times n-m}\ \boldsymbol{I}_{m\times m})$

Coherent Subspace

Definition

The **coherent subspace** is the linear subspace spanned by the columns of $\boldsymbol{S}$, i.e. $\mathfrak{s} = \text{sp}(\boldsymbol{S})$

Instead of using bottom-level series a different combination of m **basis series** could be used (e.g. top and $m-1$ bottom). Although $\boldsymbol{S}$ would be different $\mathfrak{s}$ would be the same.

Coherent Point Forecast

Definition

A **coherent point forecast** is any forecast lying in the linear subspace $\mathfrak{s}$

Reconciled Point Forecast

Let $\hat{\boldsymbol{y}} \in \mathbb{R}^n$ be an incoherent forecast and $g(.)$ be a function $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

Definition

A **point forecast** $\tilde{\boldsymbol{y}}$ is reconciled with respect to $g(.)$ iff

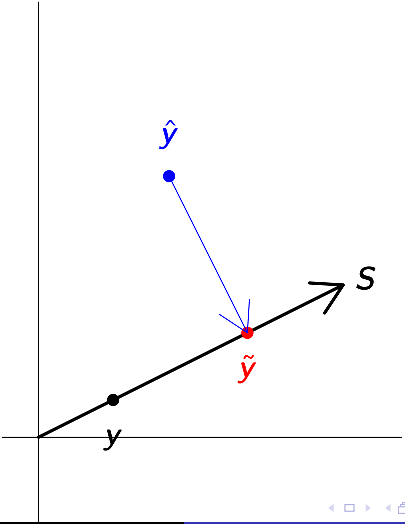
$$\tilde{\boldsymbol{y}} = \boldsymbol{S}g(\hat{\boldsymbol{y}})$$

when $g(.)$ is linear it is easier to write $\tilde{\boldsymbol{y}} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}$

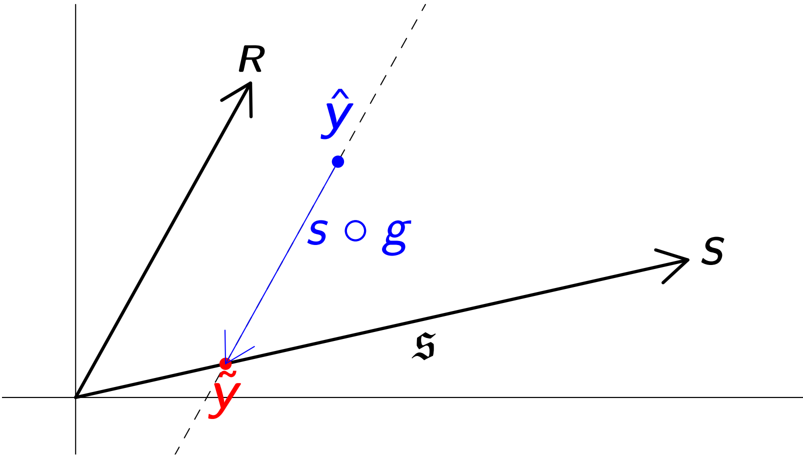
Special Case: Projection

- An important special case is where $\boldsymbol{SG}$ is a projection.
  - $\boldsymbol{SG}$ is symmetric
  - $\boldsymbol{SG}$ is idempotent
- Let $\boldsymbol{v} \in \mathfrak{s}$
  - $\boldsymbol{SGv}$ will also lie in $\mathfrak{s}$.
  - $\boldsymbol{SGv} = \boldsymbol{v}$ only when $\boldsymbol{SG}$ is a projection.

Geometry



Geometry: Oblique Projection



Projections preserve unbiasedness

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be an unbiased forecast that is $E_{1:t}(\hat{\boldsymbol{y}}_{t+h|t}) = \boldsymbol{\mu}_{t+h|t}$ where $\boldsymbol{\mu}_{t+h|t} = E(\boldsymbol{y}_{t+h} \mid \boldsymbol{y}_1, \ldots, \boldsymbol{y}_t)$

Theorem

The reconciled forecast $\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{t+h|t}$ will also be unbiased iff $\boldsymbol{S}\boldsymbol{G}$ is a projection.

Previously, this was often stated as an assumption that $\boldsymbol{S}\boldsymbol{G}\boldsymbol{S} = \boldsymbol{S}$.

Proof

Very easy proof

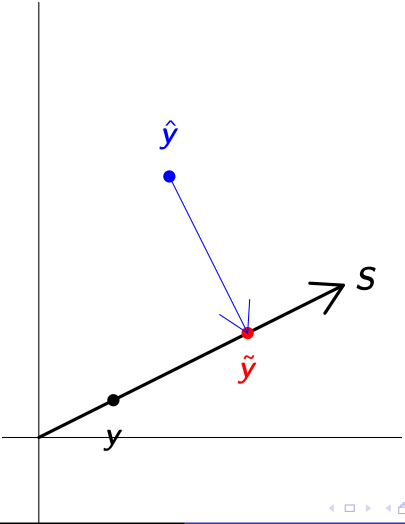
$$
\begin{aligned}
E_{1:t}(\tilde{\boldsymbol{y}}_{t+h|t}) &= E_{1:t}(\boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{t+h|t}) \\
&= \boldsymbol{S}\boldsymbol{G}\,E_{1:t}(\hat{\boldsymbol{y}}_{t+h|t}) \\
&= \boldsymbol{S}\boldsymbol{G}\boldsymbol{\mu}_{t+h|t} \\
&= \boldsymbol{\mu}_{t+h|t}
\end{aligned}
$$

The last equality does not hold for $\boldsymbol{G}$ in general but does hold when $\boldsymbol{S}\boldsymbol{G}$ is a projection.

Why reconciliation works

- The realised observation always lies on $\mathfrak{s}$.
- Orthogonal projections always get us ‘closer’ to all points in $\mathfrak{s}$ including the actual realisation.
- Ergo reconciliation reduces the error and not just in expectation.
- Oblique projections have the same property for non-Euclidean distance.

Geometry



Coherent Probabilistic Forecast

Let $(\mathbb{R}^m, \mathcal{F}_{\mathbb{R}^m}, \nu)$ and $(\mathfrak{s}, \mathcal{F}_{\mathfrak{s}}, \mu)$ be probability triples on m-dimensional space and the coherent subspace respectively.

Definition

The probability measure ν is coherent if

$$\nu(\mathcal{B}) = \mu(s(\mathcal{B})) \quad \forall \mathcal{B} \in \mathcal{F}_{\mathbb{R}_m}$$

where $s(\mathcal{B})$ is the image of $\mathcal{B}$ under premultiplication by $\boldsymbol{S}$

Reconciled Probabilistic Forecast

Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function. Then

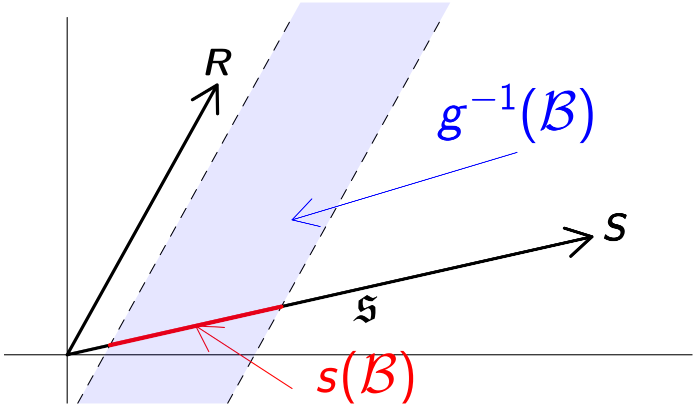
Definition

The probability triple $(\mathfrak{s}, \mathcal{F}_\mathfrak{s}, \tilde{\nu})$ reconciles the probability triple $(\mathbb{R}^n, \mathcal{F}_{\mathbb{R}^n}, \hat{\nu})$ with with respect to g iff

$$\tilde{\nu}(s(\mathcal{B})) = \nu(\mathcal{B}) = \hat{\nu}(g^{-1}(\mathcal{B})) \quad \forall \mathcal{B} \in \mathcal{F}_{\mathbb{R}_m}$$

where g^{-1} is the pre-image of g.

Geometry



Analytically

If we have an unreconciled density the reconciled density can be obtained by linear transformations and marginalisation.

$$\begin{aligned}
\Pr(\tilde{\boldsymbol{b}} \in \mathcal{B}) &= \Pr(\hat{\boldsymbol{y}} \in g^{-1}(\mathcal{B})) \\
&= \int\limits_{g^{-1}(\mathcal{B})} f(\hat{\boldsymbol{y}}) d\hat{\boldsymbol{y}} \\
&= \int\limits_{\mathcal{B}} \int f(\boldsymbol{S}\tilde{\boldsymbol{b}} + \boldsymbol{R}\tilde{\boldsymbol{a}}) \left| \begin{pmatrix} \boldsymbol{S} & \boldsymbol{R} \end{pmatrix} \right| d\tilde{\boldsymbol{a}} d\tilde{\boldsymbol{b}}
\end{aligned}$$

Elliptical distributions

Let the true predictive densities be elliptical.

Theorem

There exists a function $g(.)$ such that the true predictive distribution can be recovered by linear reconciliation as long as the unreconciled probabilistic forecast comes from the correct elliptical class.

This follows from the closure property of elliptical distributions under affine transformations and marginalisation. Conditions are also derived for when this $g(.)$ is also a projection.

With a sample

- Often densities are unavailable but we can simulate a sample from the predictive distribution.
- Suppose $\hat{\boldsymbol{y}}^{[1]}, \ldots, \hat{\boldsymbol{y}}^{[J]}$ is a sample from the unreconciled probabilistic forecast.
- Then setting $\tilde{\boldsymbol{y}}^{[j]} = s \circ g(\hat{\boldsymbol{y}}^{[j]}) = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}^{[j]}$ produces a sample from the reconciled distribution with respect to g.

Univariate v Multivariate Scores

Scoring rules can be used to evaluate probabilistic forecasts

- Univariate
  - Log Score
  - Continuous Rank Probability Score
- Multivariate
  - Log Score
  - Energy Score

These may be computed using densities or a sample.

Approaches

1. Use a summary of all univariate scores.
2. Make comparisons on the joint distribution of bottom level series only.
3. Make comparisons using the full joint distribution

There are pitfalls to the third approach.

Coherent v Incoherent

When comparing reconciled and unreconciled probabilistic forecasts on the basis of log score

Theorem

Let $f(\boldsymbol{y})$ be the true predictive density (on $\mathfrak{s}$) and LS be the (negatively-oriented) log score. Then there exists an unreconciled density $\hat{f}(\boldsymbol{y})$ on $\mathbb{R}^n$ such that

$$E_{\boldsymbol{y}}\left[LS(\hat{f},\boldsymbol{y})\right] < E_{\boldsymbol{y}}\left[LS(f,\boldsymbol{y})\right]$$

The log score is not proper **in this context**.

Reconciled v Reconciled

- For two reconciled probabilistic forecasts log score can be used.
- Comparisons can be made on the basis of bottom level series (or any basis series).
- By the definition of coherence $\log(f(\boldsymbol{b})) = \log(f(\boldsymbol{Sb})\boldsymbol{J})$
- The Jacobian does not affect the ordering of log score.

Energy score

- Using bottom level series only is a bad idea for energy score.
- Energy score is invariant to orthogonal transformation but not affine transformations.
- Since $\boldsymbol{S}$ is not a rotation the ranking of different methods based on the full hierarchy may differ from the ranking based on bottom level series only.

Simulations

- The main takeaway messages are:
  - Reconciliation is better than no reconciliation.
  - Bottom up does not do well.
  - OLS (an orthogonal projection) does poorly.
  - MinT (an oblique projection) does best.

Looking ahead

- The optimal feasible reconciliation method remains an open question even for elliptical distributions.
  - It is likely to depend on the specific score used.
- Are non-linear reconciliation methods worthwhile?
- How should probabilistic reconciliation work for non-elliptical distributions.
- Further development of multivariate scoring rules.