Probabilistic Forecast Reconciliation

Puwasala Gamakumara, Anastasios Panagiotelis, George Athanasopoulos and Rob Hyndman

November 14, 2018

Motivating Examples

- Multiple time series $\rightarrow$ some series are aggregates of others.
- Gross Domestic Product

Motivating Examples

- Multiple time series $\rightarrow$ some series are aggregates of others.
- Gross Domestic Product
  - An aggregate of consumption, investment, government spending and trade balance
  - Further breakdown, e.g. consumption of food, rent, etc.

Motivating Examples

- Multiple time series $\rightarrow$ some series are aggregates of others.
- Gross Domestic Product
  - An aggregate of consumption, investment, government spending and trade balance
  - Further breakdown, e.g. consumption of food, rent, etc.
- Wind power
  - Forecasts required at a daily and hourly resolution.
  - Daily series is aggregate of 24 hourly series.

Motivating Examples

- Multiple time series $\rightarrow$ some series are aggregates of others.
- Gross Domestic Product
  - An aggregate of consumption, investment, government spending and trade balance
  - Further breakdown, e.g. consumption of food, rent, etc.
- Wind power
  - Forecasts required at a daily and hourly resolution.
  - Daily series is aggregate of 24 hourly series.
- Potentially need forecasts of all time series.

Incoherent Forecasts

- Potential approaches

Incoherent Forecasts

- Potential approaches
  - Multivariate models

Incoherent Forecasts

- Potential approaches
  - Multivariate models
  - Univariate models

Incoherent Forecasts

- Potential approaches
  - Multivariate models
  - Univariate models
  - Judgemental forecasts

Incoherent Forecasts

- Potential approaches
  - Multivariate models
  - Univariate models
  - Judgemental forecasts
- Forecasts do not respect aggregation structure

Incoherent Forecasts

- Potential approaches
  - Multivariate models
  - Univariate models
  - Judgemental forecasts
- Forecasts do not respect aggregation structure (**Incoherent**)

Incoherent Forecasts

- Potential approaches
  - Multivariate models
  - Univariate models
  - Judgemental forecasts
- Forecasts do not respect aggregation structure (**Incoherent**)
- Outcome *does* respect aggregation structure

Incoherent Forecasts

- Potential approaches
  - Multivariate models
  - Univariate models
  - Judgemental forecasts
- Forecasts do not respect aggregation structure (**Incoherent**)
- Outcome *does* respect aggregation structure (**Coherent**)

Incoherent Forecasts

- Potential approaches
  - Multivariate models
  - Univariate models
  - Judgemental forecasts
- Forecasts do not respect aggregation structure (**Incoherent**)
- Outcome *does* respect aggregation structure (**Coherent**)
- Motivation is aggregation but can be generalised to any linear constraints.

Reconciliation

- Begin with a vector of *base* forecasts that are incoherent.

Reconciliation

- Begin with a vector of *base* forecasts that are incoherent.
- Adjust these **ex post** to make them coherent.

Reconciliation

- Begin with a vector of *base* forecasts that are incoherent.
- Adjust these **ex post** to make them coherent.
- There are good solutions for point forecasting that:

Reconciliation

- Begin with a vector of *base* forecasts that are incoherent.
- Adjust these **ex post** to make them coherent.
- There are good solutions for point forecasting that:
  - Guarantee coherent forecasts.
  - Improve forecast accuracy overall.

Reconciliation

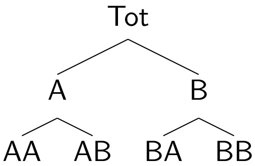
- Begin with a vector of *base* forecasts that are incoherent.
- Adjust these **ex post** to make them coherent.
- There are good solutions for point forecasting that:
  - Guarantee coherent forecasts.
  - Improve forecast accuracy overall.
- Generalisation to probabilistic forecasts is our contribution.

Reconciliation

- Begin with a vector of *base* forecasts that are incoherent.
- Adjust these **ex post** to make them coherent.
- There are good solutions for point forecasting that:
    - Guarantee coherent forecasts.
    - Improve forecast accuracy overall.
- Generalisation to probabilistic forecasts is our contribution.
- Getting there necessitates a rethink of the existing point forecasting literature.

A simple hierarchy

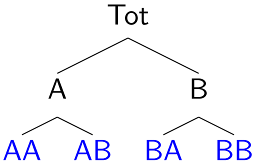
Consider a hierarchy given by



- Let n be the number of series, $\boldsymbol{y}_t$ be an n-vector of all series.

A simple hierarchy

Consider a hierarchy given by



- Let n be the number of series, $\boldsymbol{y}_t$ be an n-vector of all series.
- Let m be the number of bottom level series and $\boldsymbol{b}_t$ be an m-vector of the bottom level series.

The $\boldsymbol{S}$ matrix

Coherence holds when

$$\boldsymbol{y} = \boldsymbol{S}\boldsymbol{b}$$

The $n \times m$ matrix $\boldsymbol{S}$ defines the aggregation constraints, e.g.

$$\boldsymbol{S} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ & \boldsymbol{I}_{4\times 4} & & \end{pmatrix}$$

As a regression model

- Cast the problem as a regression model with base forecasts $\hat{\boldsymbol{y}}_{T+h}$ as the "dependent variable" and $\boldsymbol{S}$ as the "design matrix".

$$\hat{\boldsymbol{y}}_{T+h} = \boldsymbol{S}\beta_{T+h} + \boldsymbol{e}_{T+h}$$

- Initial approach (Athanasopoulos et al, 2009; Hyndman et al, 2011) was to fit by OLS yielding reconciled forecasts:

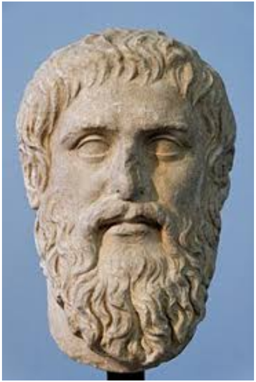
$$\tilde{\boldsymbol{y}}_{T+h} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'\hat{\boldsymbol{y}}_{T+h}$$

Generalisation

- Wherever we can use OLS we can use GLS

$$\tilde{\boldsymbol{y}}_{T+h} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}^{-1}\hat{\boldsymbol{y}}_{T+h}$$

- Diagonal $\boldsymbol{W}$ considered by Athanasopoulos et al (2017)
- MinT approach (Wickremasuriya et al, 2018) use a $\boldsymbol{W}$ that is an estimate of the *in-sample* forecast error covariance matrix.



ΑΓΕΩΜΤΕΤΡΗΤΟΣ ΜΗΔΕΙΣ ΕΙΣΙΤΩ

Those without knowledge of geometry may not enter.

Coherent Subspace

Definition

The **coherent subspace** is the m-dimensional linear subspace of $\mathbb{R}^n$ spanned by the columns of $\boldsymbol{S}$, i.e. $\mathfrak{s} = \mathrm{sp}(\boldsymbol{S})$

Instead of using bottom-level series a different combination of m **basis series** could be used (e.g. top and $m-1$ bottom). Although $\boldsymbol{S}$ would be different $\mathfrak{s}$ would be the same.

Coherent Point Forecast

Definition

A **coherent point forecast** is any forecast lying in the linear subspace $\mathfrak{s}$

Reconciled Point Forecast

Let $\hat{\boldsymbol{y}} \in \mathbb{R}^n$ be an incoherent forecast and $g(.)$ be a function $g : \mathbb{R}^n \to \mathbb{R}^m$.

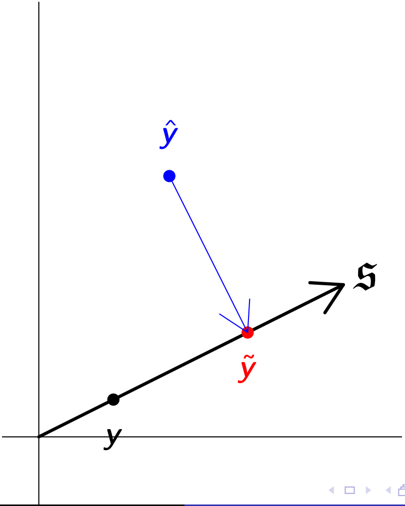
Definition

A **point forecast** $\tilde{\boldsymbol{y}}$ is reconciled with respect to $g(.)$ iff

$$\tilde{\boldsymbol{y}} = \boldsymbol{S}g(\hat{\boldsymbol{y}})$$

when $g(.)$ is linear it is easier to write $\tilde{\boldsymbol{y}} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}$

Geometry



Why reconciliation works

- The realised observation always lies on $\mathfrak{s}$.
- Orthogonal projections always get us 'closer' to all points in $\mathfrak{s}$ including the actual realisation.
- Ergo reconciliation reduces the error and not just in expectation.

Why reconciliation works

- The realised observation always lies on $\mathfrak{s}$.
- Orthogonal projections always get us ‘closer’ to all points in $\mathfrak{s}$ including the actual realisation.
- Ergo reconciliation reduces the error and not just in expectation.
- What about the MinT approach?

Finding a direction

- Consider the covariance matrix of $\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h}$.

Finding a direction

- Consider the covariance matrix of $\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h}$.
- This can be estimated using in-sample forecast errors.

$$\boldsymbol{W} = \sum_{t=1}^{T} (\boldsymbol{y}_t - \hat{\boldsymbol{y}}_t)(\boldsymbol{y}_t - \hat{\boldsymbol{y}}_t)'$$

Finding a direction

- Consider the covariance matrix of $\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h}$.
- This can be estimated using in-sample forecast errors.

$$\boldsymbol{W} = \sum_{t=1}^{T} (\boldsymbol{y}_t - \hat{\boldsymbol{y}}_t)(\boldsymbol{y}_t - \hat{\boldsymbol{y}}_t)'$$

- This provides information about the likely direction of an error.

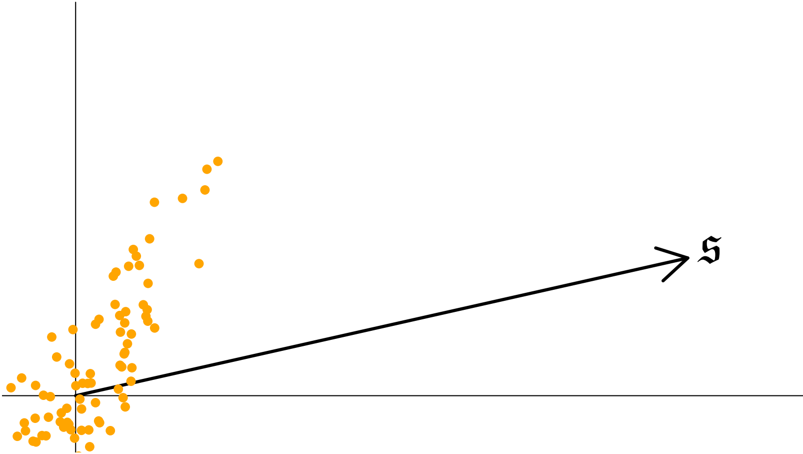
Finding a direction

- Consider the covariance matrix of $\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h}$.
- This can be estimated using in-sample forecast errors.

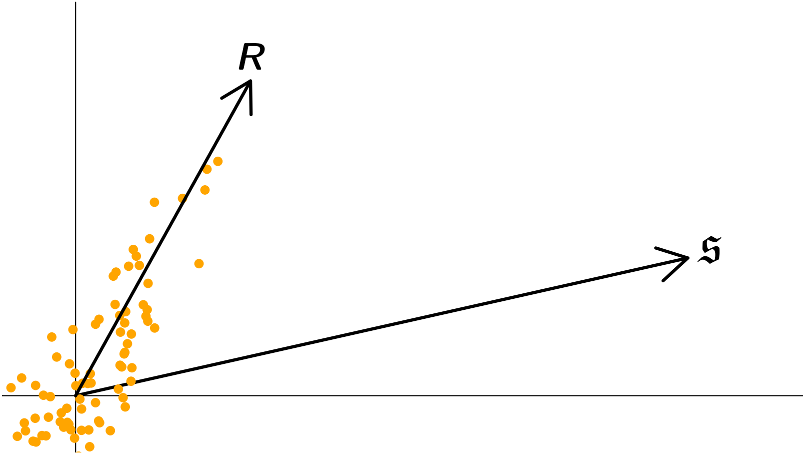
$$\boldsymbol{W} = \sum_{t=1}^{T} (\boldsymbol{y}_t - \hat{\boldsymbol{y}}_t)(\boldsymbol{y}_t - \hat{\boldsymbol{y}}_t)'$$

- This provides information about the likely direction of an error.
- Projecting along this direction is more likely to result in a reconciled forecast that is closer to the target.

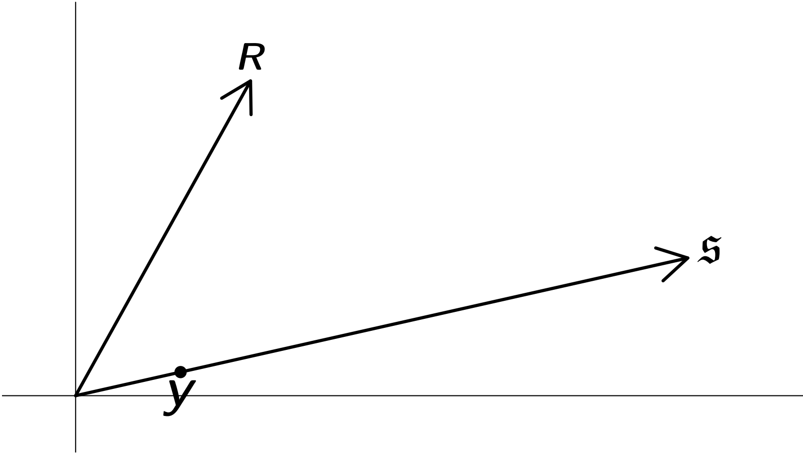
In-Sample errors



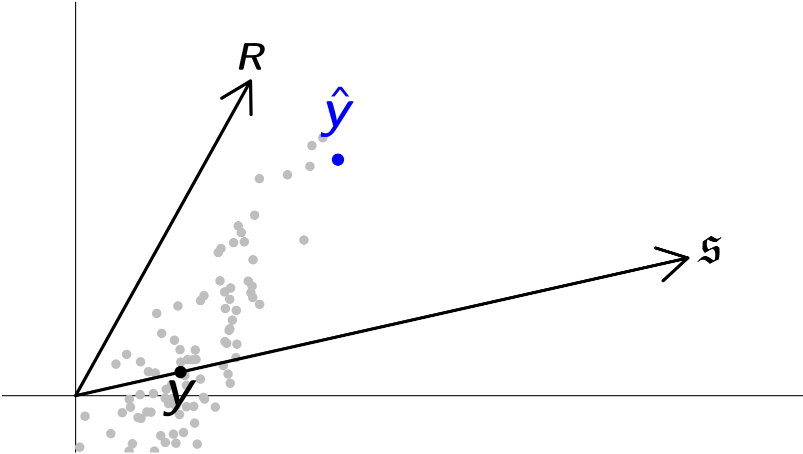
Most likely direction



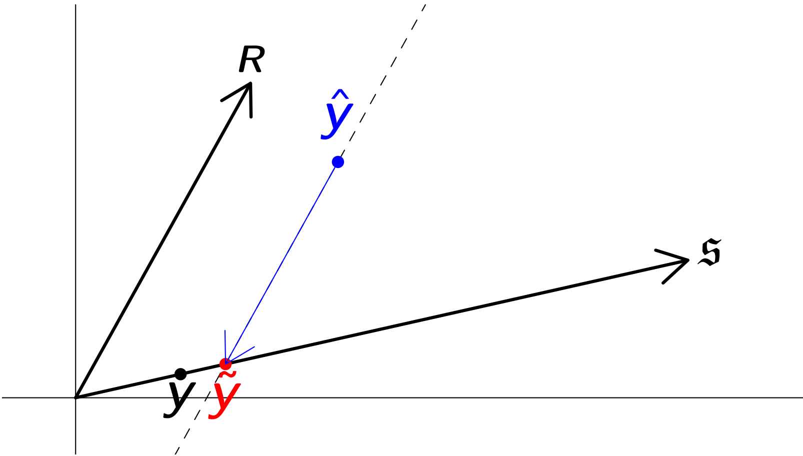
Geometry: Oblique Projection



Geometry: Oblique Projection



Geometry: Oblique Projection



Is this overkill?

- Is this geometric interpretation really necessary?

Is this overkill?

- Is this geometric interpretation really necessary?
- When we generalise to probabilistic forecasts the regression interpretation does not really fit.

Is this overkill?

- Is this geometric interpretation really necessary?
- When we generalise to probabilistic forecasts the regression interpretation does not really fit.
- Geometric ideas can easily be generalised.

Coherent Probabilistic Forecast

Let $(\mathbb{R}^m, \mathcal{F}_{\mathbb{R}^m}, \nu)$ and $(\mathfrak{s}, \mathcal{F}_{\mathfrak{s}}, \mu)$ be probability triples on m-dimensional space and the coherent subspace respectively.

Definition

The probability measure μ is coherent if

$$\nu(\mathcal{B}) = \mu(s(\mathcal{B})) \quad \forall \mathcal{B} \in \mathcal{F}_{\mathbb{R}_m}$$

where $s(\mathcal{B})$ is the image of $\mathcal{B}$ under premultiplication by $\boldsymbol{S}$

Reconciled Probabilistic Forecast

Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then

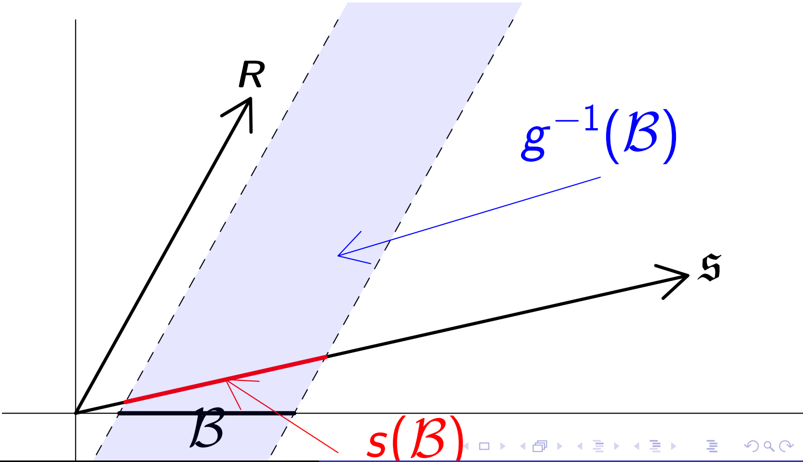
Definition

The probability triple $(\mathfrak{s}, \mathcal{F}_{\mathfrak{s}}, \tilde{\nu})$ reconciles the probability triple $(\mathbb{R}^n, \mathcal{F}_{\mathbb{R}^n}, \hat{\nu})$ with with respect to g iff

$$\tilde{\nu}(s(\mathcal{B})) = \nu(\mathcal{B}) = \hat{\nu}(g^{-1}(\mathcal{B})) \quad \forall \mathcal{B} \in \mathcal{F}_{\mathbb{R}_m}$$

where g^{-1} is the pre-image of g.

Geometry



Analytically

If we have an unreconciled density the reconciled density can be obtained by linear transformations and marginalisation.

$$
\begin{aligned}
\Pr(\tilde{\boldsymbol{b}} \in \mathcal{B}) &= \Pr(\hat{\boldsymbol{y}} \in g^{-1}(\mathcal{B})) \\
&= \int\limits_{g^{-1}(\mathcal{B})} f(\hat{\boldsymbol{y}}) d\hat{\boldsymbol{y}} \\
&= \int\limits_{\mathcal{B}} \int f(\boldsymbol{S}\tilde{\boldsymbol{b}} + \boldsymbol{R}\tilde{\boldsymbol{a}}) \left| \begin{pmatrix} \boldsymbol{S} & \boldsymbol{R} \end{pmatrix} \right| d\tilde{\boldsymbol{a}} d\tilde{\boldsymbol{b}}
\end{aligned}
$$

Elliptical distributions

Consider case where the base and true predictive distributions are elliptical.

Theorem

There exists a matrix $\boldsymbol{G}$ such that the true predictive distribution can be recovered by linear reconciliation.

This follows from the closure property of elliptical distributions under affine transformations and marginalisation.

With a sample

- Often densities are unavailable but we can simulate a sample from the predictive distribution.

With a sample

- Often densities are unavailable but we can simulate a sample from the predictive distribution.
- Suppose $\hat{\boldsymbol{y}}^{[1]}_{T+h},\ldots,\hat{\boldsymbol{y}}^{[J]}_{T+h}$ is a sample from the unreconciled probabilistic forecast.

With a sample

- Often densities are unavailable but we can simulate a sample from the predictive distribution.
- Suppose $\hat{\boldsymbol{y}}_{T+h}^{[1]}, \ldots, \hat{\boldsymbol{y}}_{T+h}^{[J]}$ is a sample from the unreconciled probabilistic forecast.
- Then setting $\tilde{\boldsymbol{y}}_{T+h}^{[j]} = \boldsymbol{SG}\hat{\boldsymbol{y}}_{T+h}^{[j]}$ produces a sample from the reconciled distribution with respect to g.

Multivariate Scores

- Scoring rules can be used to evaluate probabilistic forecasts

Multivariate Scores

- Scoring rules can be used to evaluate probabilistic forecasts
  - Log Score
  - Energy Score
  - Variogram score

Multivariate Scores

- Scoring rules can be used to evaluate probabilistic forecasts
    - Log Score
    - Energy Score
    - Variogram score
- We may want to compare
    - Coherent v Incoherent
    - Coherent v Coherent

Coherent v Incoherent

When using log score

Theorem

Let $f(\boldsymbol{y})$ be the true predictive density (on $\mathfrak{s}$) and LS be the (negatively-oriented) log score. Then there exists an unreconciled density $\hat{f}(\boldsymbol{y})$ on $\mathbb{R}^n$ such that

$$E_{\boldsymbol{y}}\left[LS(\hat{f},\boldsymbol{y})\right] < E_{\boldsymbol{y}}\left[LS(f,\boldsymbol{y})\right]$$

The log score is not proper **in this context**.

Simulations

- We have run lots of simulations.

Simulations

- We have run lots of simulations.
- The main takeaway messages are:

Simulations

- We have run lots of simulations.
- The main takeaway messages are:
  - Reconciliation is better than no reconciliation.

Simulations

- We have run lots of simulations.
- The main takeaway messages are:
  - Reconciliation is better than no reconciliation.
  - Bottom up does not do well.

Simulations

- We have run lots of simulations.
- The main takeaway messages are:
  - Reconciliation is better than no reconciliation.
  - Bottom up does not do well.
  - OLS (an orthogonal projection) does reasonably well.

Simulations

- We have run lots of simulations.
- The main takeaway messages are:
  - Reconciliation is better than no reconciliation.
  - Bottom up does not do well.
  - OLS (an orthogonal projection) does reasonably well.
  - MinT (an oblique projection) does best.

Looking ahead

- The optimal feasible reconciliation method remains an open question even for elliptical distributions.

Looking ahead

- The optimal feasible reconciliation method remains an open question even for elliptical distributions.
  - It is likely to depend on the specific score used.

Looking ahead

- The optimal feasible reconciliation method remains an open question even for elliptical distributions.
  - It is likely to depend on the specific score used.
- How should probabilistic reconciliation work for non-elliptical distributions.

Looking ahead

- The optimal feasible reconciliation method remains an open question even for elliptical distributions.
  - It is likely to depend on the specific score used.
- How should probabilistic reconciliation work for non-elliptical distributions.
- Are non-linear reconciliation methods worthwhile?

Looking ahead

- The optimal feasible reconciliation method remains an open question even for elliptical distributions.
  - It is likely to depend on the specific score used.
- How should probabilistic reconciliation work for non-elliptical distributions.
- Are non-linear reconciliation methods worthwhile?
- Further development of multivariate scoring rules.