Probabilistic Hierarchical Reconciliation via a Non-parametric Bootstrap Approach

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with
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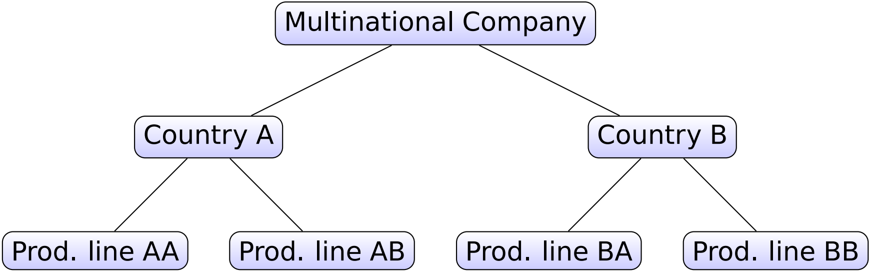
Introduction

Introduction

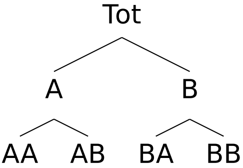
- **Example**: Forecasting the revenue of a large organisation



- **Hierarchical time series:** A collection of multiple time series that has an inherent aggregation structure
- Forecasts should add up. We call it *coherent*
- Why coherent forecasts?

Preliminaries

Notations



$$\boldsymbol{y}_t = [y_{Tot,t}, y_{A,t}, y_{B,t}, y_{AA,t}, y_{AB,t}, y_{BA,t}, y_{BB,t}]^T$$

$$\boldsymbol{b}_t = [y_{AA,t}, y_{AB,t}, y_{BA,t}, y_{BB,t}]^T$$

$$m = 4$$

$$n = 7$$

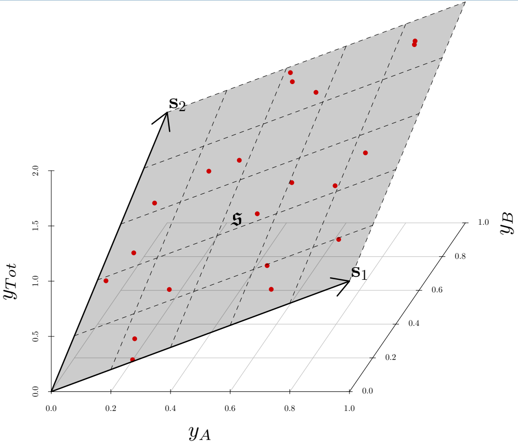
$$\boldsymbol{S} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Due to the aggregation nature of the hierarchy we have,

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

- *Coherent subspace:* $\mathfrak{s} = Span(\boldsymbol{S}), \quad \mathfrak{s} \subset \mathbb{R}^n$

Preliminaries: *Coherent forecasts*



- Three dimensional hierarchy, $y_{Tot} = y_A + y_B$.
- $\vec{s}_1 = (1, 1, 0)'$ and $\vec{s}_2 = (1, 0, 1)'$ form a basis for $\mathfrak{s}$.

Preliminaries: *Point forecasts reconciliation*

- Let $\hat{\boldsymbol{y}} \in \mathbb{R}^n$ be an incoherent forecast. Then, the reconciled forecasts $\tilde{\boldsymbol{y}}_{T+h}$ are given by,

$$\tilde{\boldsymbol{y}}_{T+h} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h}$$

Method	$\boldsymbol{G}$
BU	$\left(\boldsymbol{0}_{m \times n-m} \ \boldsymbol{I}_{m \times m}\right)$
OLS	$\left(\boldsymbol{S}'\boldsymbol{S}\right)^{-1}\boldsymbol{S}'$
WLS	$\left(\boldsymbol{S}'\hat{\boldsymbol{W}}^{wls}_{T+1}\boldsymbol{S}\right)^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}^{wls}_{T+1}$
MinT(Shrink)	$\left(\boldsymbol{S}'\hat{\boldsymbol{W}}^{shr}_{T+1}\boldsymbol{S}\right)^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}^{shr}_{T+1}$

$$\begin{aligned} \hat{\boldsymbol{W}}^{shr}_{T+1} &= \tau \mathrm{Diag}(\hat{\boldsymbol{W}}^{sam}_{T+1}) + (1-\tau)\hat{\boldsymbol{W}}^{sam}_{T+1} \\ \hat{\boldsymbol{W}}^{wls}_{T+1} &= \mathrm{Diag}(\hat{\boldsymbol{W}}^{shr}_{T+1}) \end{aligned}$$

Probabilistic forecasts for hierarchical time series

Motivation

- Lack of attention in probabilistic forecasts
  - Ben Taieb et al. (2017)
  - Jeon, Panagiotelis, and Petropoulos (2018)

- Probabilistic forecasts should reflect the inherent properties of real data. In particular,
  - ⋆ Aggregation structure
  - ⋆ Correlation structure

- Extending the “reconciliation" method into probabilistic framework

Probabilistic forecast reconciliation

- Often parametric densities are unavailable but we can simulate a sample from the predictive distribution.
- Suppose $\hat{\boldsymbol{y}}_{T+h}^{[1]}, \ldots, \hat{\boldsymbol{y}}_{T+h}^{[J]}$ is a sample from the incoherent predictive distribution.
- Then setting $\tilde{\boldsymbol{y}}_{T+h}^{[j]} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h}^{[j]}$ produces a sample from the reconciled predictive distribution with respect to $\boldsymbol{G}$.

Non-parametric bootstrap approach

1. Fit univariate models at each node using data up to time T.
2. Let $\boldsymbol{\Gamma}_{(T\times n)} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_T)'$ be a matrix of in-sample residuals where $\boldsymbol{e}_t = \boldsymbol{y}_t - \hat{\boldsymbol{y}}_t$.
3. Let $\boldsymbol{\Gamma}^b_{(H\times n)} = (\boldsymbol{e}^b_1, ..., \boldsymbol{e}^b_H)'$ be a block bootstrap sample of size H from $\boldsymbol{\Gamma}$.
4. Generate h-step ahead sample paths from the fitted models incorporating $\boldsymbol{\Gamma}^b$. Denote these by $\hat{\boldsymbol{y}}^b_{T+h}$, for $h = 1, ..., H$.
5. Repeat step 3 and 4 for $b = 1, ..., B$ times. Denote these as $\hat{\boldsymbol{\Upsilon}}_{T+h} = (\hat{\boldsymbol{y}}^1_{T+h}, ..., \hat{\boldsymbol{y}}^B_{T+h})'$ for all h.
6. Setting $\tilde{\boldsymbol{\Upsilon}}_{T+h} = \boldsymbol{SG}\hat{\boldsymbol{\Upsilon}}'_{T+h}$ produces a sample from the reconciled distribution.

Optimal reconciliation of future paths

- We propose to find an optimal $\boldsymbol{G}_h$ matrix by minimizing Energy score.

$$\underset{\boldsymbol{G}_h}{\operatorname{argmin}} \quad \mathrm{E}_{\boldsymbol{Q}}[eS(\tilde{\boldsymbol{F}}, \boldsymbol{y}_{T+h})], \quad \tilde{\boldsymbol{F}} := \tilde{\Upsilon}_{T+h} = \boldsymbol{S}\boldsymbol{G}_h \hat{\Upsilon}'_{T+h}$$

where,

$$\mathrm{eS}(\tilde{\boldsymbol{F}}, \boldsymbol{y}_{T+h}) = \mathrm{E}_{\tilde{\boldsymbol{F}}} \|\tilde{\boldsymbol{Y}}_{T+h} - \boldsymbol{y}_{T+h}\|^{\alpha} - \frac{1}{2}\mathrm{E}_{\tilde{\boldsymbol{F}}} \|\tilde{\boldsymbol{Y}}_{T+h} - \tilde{\boldsymbol{Y}}^{*}_{T+h}\|^{\alpha},$$

$$\alpha \in (0, 2]$$

Optimal reconciliation of future paths

- Monte-Carlo approximation to the objective function is,

$$\underset{\boldsymbol{G}_h}{\operatorname{argmin}} \sum_{i=1}^{N} \Bigg\{ \frac{1}{B} \sum_{b=1}^{B} ||\boldsymbol{S}\boldsymbol{G}_h \hat{\boldsymbol{y}}_{T+h,i}^{b} - \boldsymbol{y}_{T+h}|| - \frac{1}{2(B-1)} \sum_{b=1}^{B-1} ||\boldsymbol{S}\boldsymbol{G}_h(\hat{\boldsymbol{y}}_{T+h,i}^{b} - \hat{\boldsymbol{y}}_{T+h,i,}^{b+1})|| \Bigg\}$$

Optimal reconciliation of future paths *Cont.*

- We impose the following structure to the $\boldsymbol{G}_h$ matrix

$$\boldsymbol{G}_h = \left(\boldsymbol{S}'\boldsymbol{W}_h\boldsymbol{S}\right)^{-1}\boldsymbol{S}'\boldsymbol{W}_h \qquad (1)$$

- We propose four methods to optimise $\boldsymbol{G}_h$

Method 1: Optimising $\boldsymbol{W}_h$

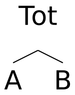
Method 2: Optimising Cholesky decomposition of $\boldsymbol{W}_h$
$\boldsymbol{W}_h = \boldsymbol{R_h'R_h}$ where $\boldsymbol{R}_h$ is an upper triangular matrix

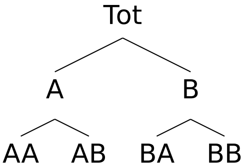
Method 3: Optimising Cholesky of $\boldsymbol{W}_h$ - restricted for scaling
$\boldsymbol{W}_h = \boldsymbol{R_h'R_h}$ s.t $\boldsymbol{i'W_hi} = 1$ where $\boldsymbol{i} = (1, 0, .., 0)'$

Method 4: Optimising $\boldsymbol{G}_h$ such that $\boldsymbol{G}_h\boldsymbol{S} = \boldsymbol{I}$

Monte-Carlo Simulation

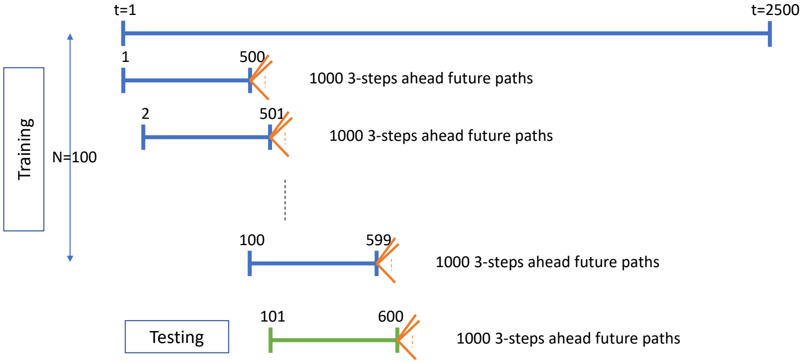
- Data generating process





- DGP was designed such that we have much noisier series in the bottom level.

Monte-Carlo Simulation



Monte-Carlo Simulation *Cont.*

Optimisation method	Hierarchy 1				Hierarchy 2			
	$h = 1$		$h = 3$		$h = 1$		$h = 3$	
	ES	VS	ES	VS	ES	VS	ES	VS
Method 1	2.48	0.11	2.75	0.11	5.36	1.21	5.83	1.38
Method 2	2.48	0.11	2.75	0.11	5.37	1.21	5.83	1.37
Method 3	2.48	0.11	2.75	0.11	5.37	1.21	5.83	1.37
Method 4	2.48	0.11	2.75	0.11	5.38	1.21	5.83	1.38

- Parameterisation does not matter

Monte-Carlo Simulation *Cont.*

- Comparison with point forecast reconciliation methods.

Reconciliation method	Hierarchy 1				Hierarchy 2			
	$h=1$		$h=3$		$h=1$		$h=3$	
	ES	VS	ES	VS	ES	VS	ES	VS
Optimal $\boldsymbol{G}$	2.48*	0.106	2.75*	0.106	5.36*	1.21*	5.83*	1.38*
MinT(Shrink)	2.47*	0.105	2.74*	0.105	5.33*	1.19*	5.77*	1.34*
WLS	2.46*	0.105	2.74*	0.105	5.43*	1.23	5.98*	1.40*
OLS	2.54*	0.105	2.80*	0.105	5.51*	1.23	5.98*	1.40*
Base	*2.67*	*0.105*	*2.94*	*0.105*	*5.71*	*1.28*	*6.27*	*1.49*

"*" indicates if the average score for a particular reconciliation method is significantly different from that of base forecasts.

- Reconciliation methods perform better than Base forecasts.

- MinT(Shrink) is at least as good as Optimal method. Thus going forward with MinT projection.

Forecasting Australian domestic tourism flows

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Geographical hierarchical structure for Australia

Level	**No.Series per level**
Total (Australia)	1
Level-1 (States)	7
Level-2 (Zones)	27
Level-3 (Regions)	75

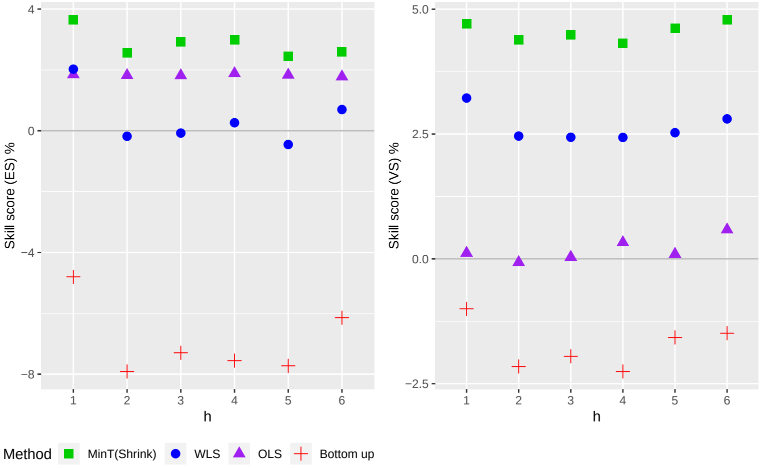
- Data: monthly “overnight trips" over the period January 1998 - December 2017
- Source of data: National Visitor Survey (NVS) [Tourism Research Australia]

Forecasting Australian domestic tourism flows

- **Analysis set up:**
  - First training sample is set from 1998:Jan to 2006:Apr
  - Univariate ARIMA models were fitted for each series in the hierarchy.
  - Reconciled probabilistic forecasts were produced for six months ahead (2006:May to 2006:Oct)
  - Then the training window is rolled by one month ahead at a time.
  - This leads to 140 1-step-ahead, 139 2-steps-ahead, through 135 6-step-ahead forecasts available for evaluation.

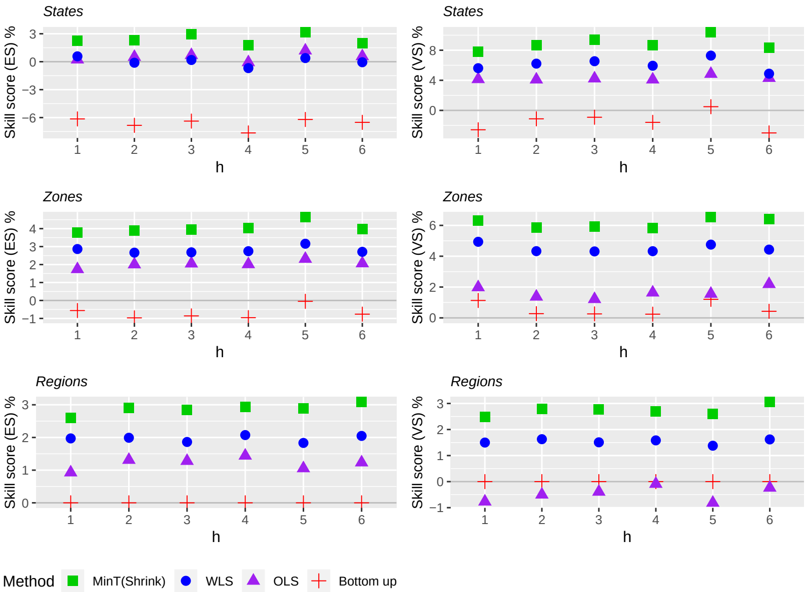
Results

Overall probabilistic forecast performance



Results

Probabilistic forecast performance for different levels



Conclusions

Conclusions

- We introduce a novel non-parametric bootstrap approach for producing reconciled probabilistic forecasts
- Simulation study evident that the optimal reconciliation with respect to energy score is equivalent to reconciling each sample path via MinT approach
- We apply this non-parametric bootstrap approach to obtain coherent probabilistic forecasts for domestic tourism flow in Australia

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THANK YOU!

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Appendix

Appendix: *Probabilistic forecasts evaluation*

A2

Energy score (Gneiting et al., 2008)

$$\mathrm{eS}(\check{\boldsymbol{Y}}_{T+h}, \boldsymbol{y}_{T+h}) = \mathrm{E}_{\check{\boldsymbol{F}}}\|\check{\boldsymbol{Y}}_{T+h} - \boldsymbol{y}_{T+h}\|^{\alpha} - \frac{1}{2}\mathrm{E}_{\check{\boldsymbol{F}}}\|\check{\boldsymbol{Y}}_{T+h} - \check{\boldsymbol{Y}}^{*}_{T+h}\|^{\alpha}, \quad \alpha \in (0,2]$$

Log score (Gneiting and Raftery, 2007)

$$\mathrm{LS}(\check{\boldsymbol{F}}, \boldsymbol{y}_{T+h}) = -\log \check{\boldsymbol{f}}(\boldsymbol{y}_{T+h})$$

Variogram score (Scheuerer and Hamill, 2015)

$$\mathrm{VS}(\check{\boldsymbol{F}}, \boldsymbol{y}_{T+h}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\Big(|y_{T+h,i} - y_{T+h,j}|^{p} - \mathrm{E}_{\check{\boldsymbol{F}}}|\check{Y}_{T+h,i} - \check{Y}_{T+h,j}|^{p}\Big)^{2}$$

CRPS (Gneiting and Raftery, 2007)

$$\mathrm{CRPS}(\check{F}_i, y_{T+h,i}) = \mathrm{E}_{\check{F}_i}|\check{Y}_{T+h,i} - y_{T+h,i}| - \frac{1}{2}\mathrm{E}_{\check{F}_i}|\check{Y}_{T+h,i} - \check{Y}^{*}_{T+h,i}|$$

$\check{\boldsymbol{Y}}_{T+h}$ and $\check{\boldsymbol{Y}}^{*}_{T+h}$: Independent random vectors from the coherent forecast distribution $\check{\boldsymbol{F}}$.

$\boldsymbol{y}_{T+h}$: Vector of realizations.