Probabilistic Forecasts for Hierarchical Time Series

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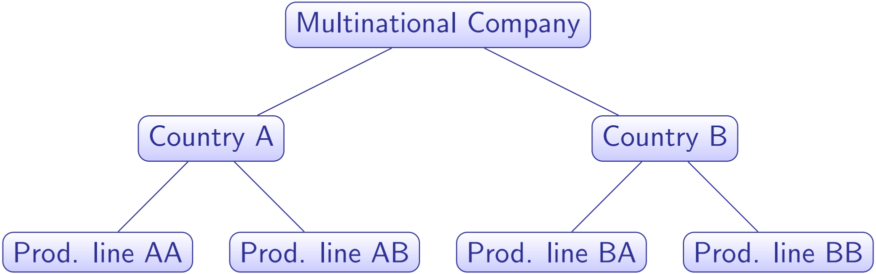
Overview

1. Project 1: Hierarchical Forecast Reconciliation: A Geometric View
2. Project 2: Probabilistic Forecasts for Hierarchical Time Series
   - Part 1: Definitions and A Parametric Approach
   - Part 2: A Non-parametric Bootstrap Approach
3. Project 3: Hierarchical forecasts for macroeconomic variables - An application to Australian GDP
4. Summary and time plan for completion

Project 1: Hierarchical Forecast Reconciliation: A Geometric View

Project 1: *Motivation and Contributions*

- **Example:**



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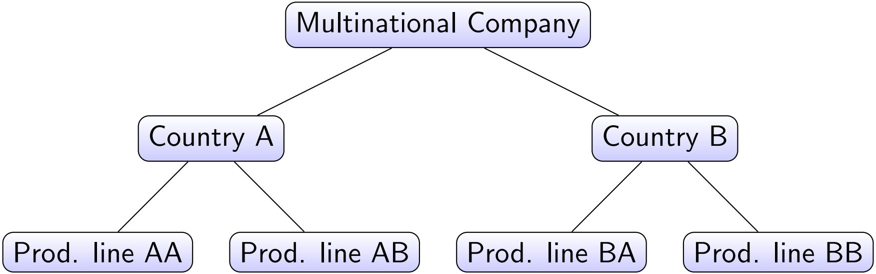

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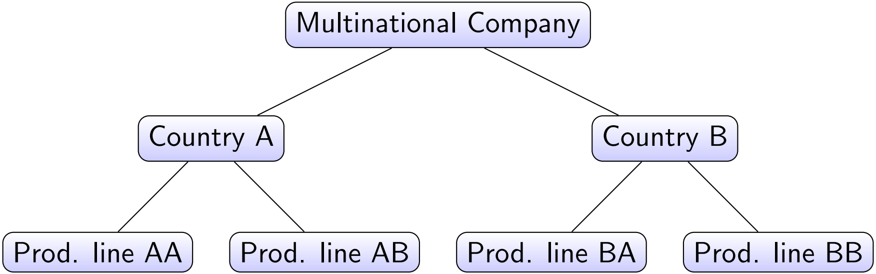

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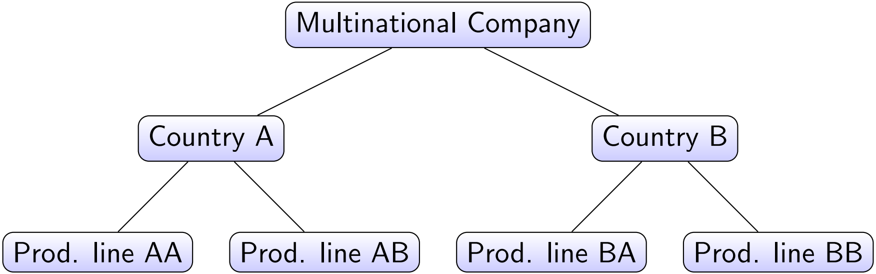

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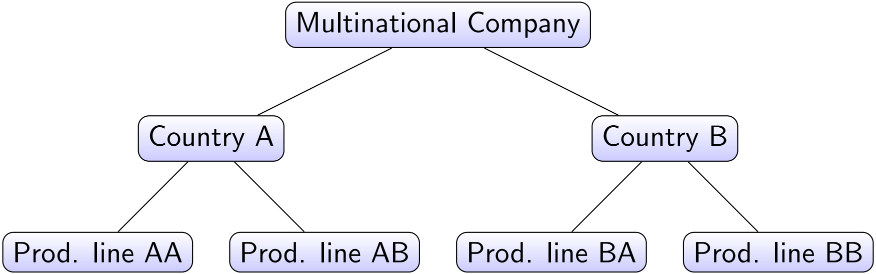

- **Hierarchical time series:** A collection of multiple time series that has an inherent aggregation structure.
- Forecasts should add up. We call these *coherent*.
- **Contribution:** Provide definitions for coherency and reconciliation of point forecasts in terms of geometric concepts.

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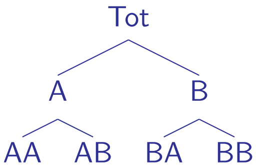

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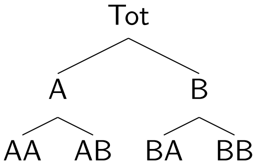

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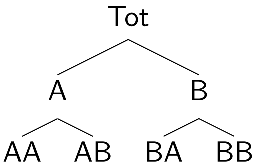

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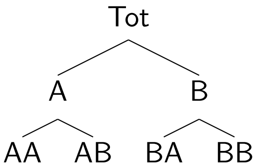

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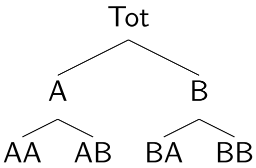

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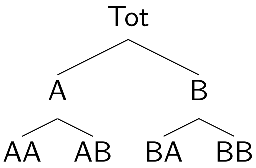

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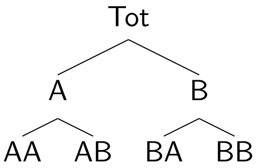

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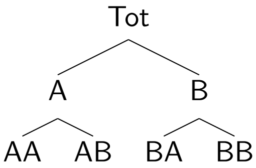

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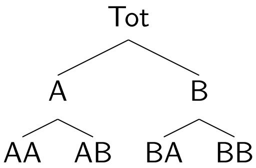

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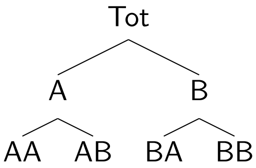

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Coherent subspace

The m-dimensional linear subspace $\mathfrak{s} \subset \mathbb{R}^n$ that is spanned by the columns of $\boldsymbol{S}$, i.e. $\mathfrak{s} = \text{span}(\boldsymbol{S})$, is defined as the *coherent space*.

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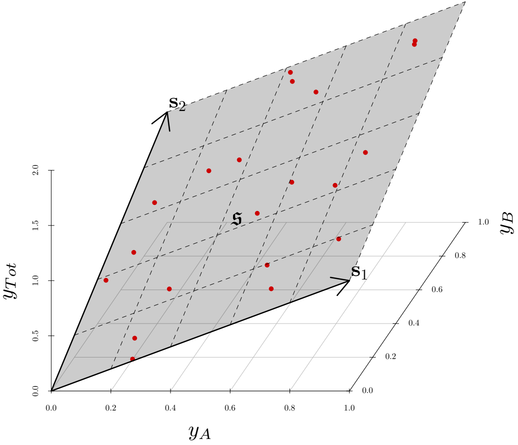

- Three dimensional hierarchy, $y_{Tot} = y_A + y_B$.
- $\vec{s}_1 = (1, 1, 0)'$ and $\vec{s}_2 = (1, 0, 1)'$ form a basis for $\mathfrak{s}$.

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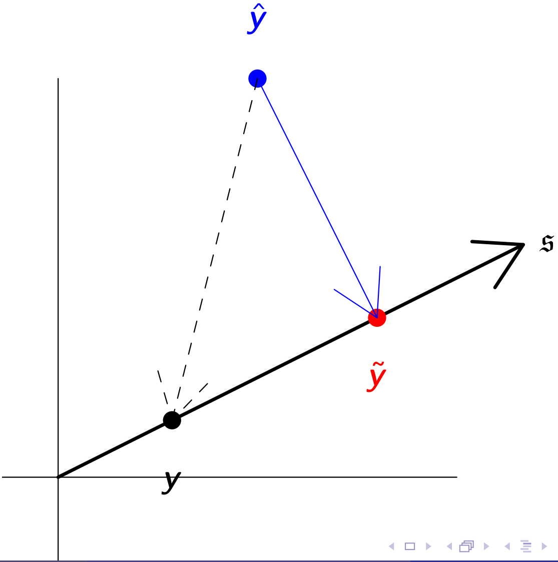

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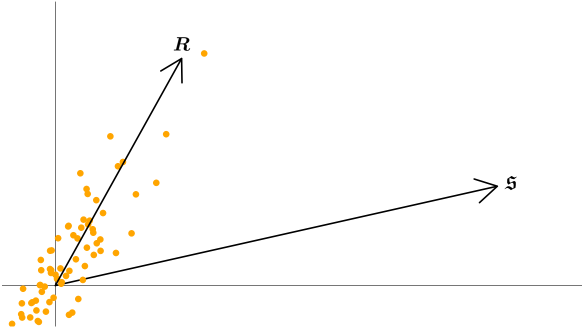

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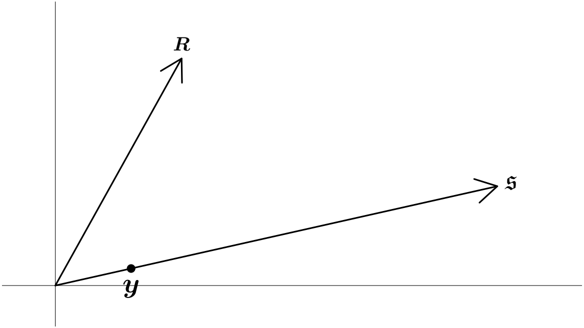

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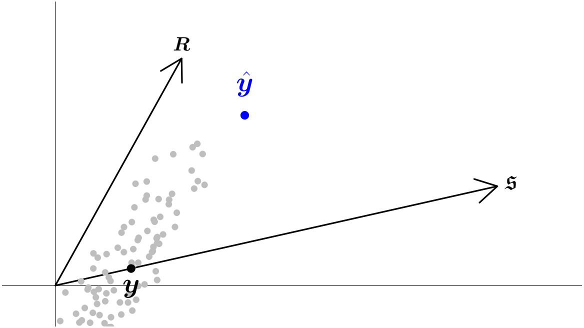

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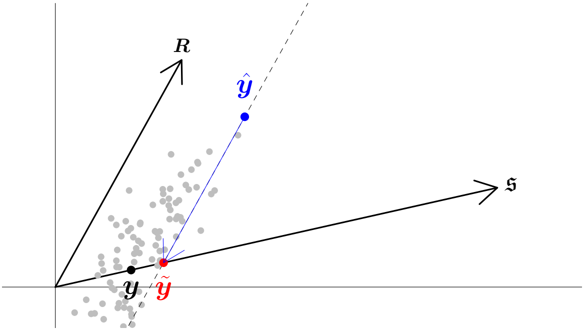

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- Probabilistic forecasts should reflect the inherent properties of real data. In particular,
  - ⋆ Aggregation structure
  - ⋆ Correlation structure
- Lack of attention
  - Ben Taieb et al. (2017)
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- **Contributions:**
  1. Provide definitions for coherency and reconciliation for probabilistic forecasts.

  2. A parametric framework for probabilistic forecast reconciliation.

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Project 2: *Coherent probabilistic forecasts*

Let $(\mathbb{R}^m, \mathcal{F}_{\mathbb{R}^m}, \nu)$ and $(\mathfrak{s}, \mathcal{F}_{\mathfrak{s}}, \mu)$ be probability triples on m-dimensional space and the coherent subspace respectively.

Definition

The probability measure μ is coherent if

$$\nu(\mathcal{B}) = \mu(s(\mathcal{B})) \quad \forall \mathcal{B} \in \mathcal{F}_{\mathbb{R}_m}$$

where $s(\mathcal{B})$ is the image of $\mathcal{B}$ under premultiplication by $\boldsymbol{S}$

Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then

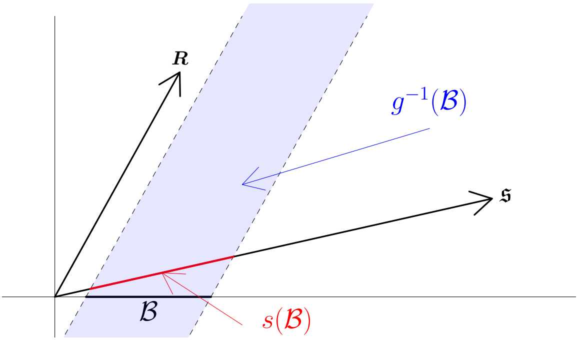
Definition

The probability triple $(\mathfrak{s}, \mathcal{F}_{\mathfrak{s}}, \tilde{\nu})$ reconciles the probability triple $(\mathbb{R}^n, \mathcal{F}_{\mathbb{R}^n}, \hat{\nu})$ with with respect to g iff

$$\tilde{\nu}(s(\mathcal{B})) = \nu(\mathcal{B}) = \hat{\nu}(g^{-1}(\mathcal{B})) \quad \forall \mathcal{B} \in \mathcal{F}_{\mathbb{R}_m}$$

where g^{-1} is the pre-image of g.

Project 2: *Geometry*



$$\tilde{\nu}(s(\mathcal{B})) = \hat{\nu}(g^{-1}(\mathcal{B})) \quad \forall \mathcal{B} \in \mathcal{F}_{\mathbb{R}_m}$$

Project 2: *Analytically*

If we have an unreconciled density the reconciled density can be obtained by linear transformations and marginalisation.

$$
\begin{aligned}
\Pr(\hat{\boldsymbol{y}} \in g^{-1}(\mathcal{B})) &= \int\limits_{g^{-1}(\mathcal{B})} f(\hat{\boldsymbol{y}}) d\hat{\boldsymbol{y}} \\
&= \int\limits_{\mathcal{B}} \int f(\boldsymbol{S}\tilde{\boldsymbol{b}} + \boldsymbol{R}\tilde{\boldsymbol{a}}) \left| \begin{pmatrix} \boldsymbol{S} & \boldsymbol{R} \end{pmatrix} \right| d\tilde{\boldsymbol{a}} d\tilde{\boldsymbol{b}} \\
&= \Pr(\tilde{\boldsymbol{b}} \in \mathcal{B})
\end{aligned}
$$

Project 2: *Assuming Gaussian distribution*

A1

- Let $\mathscr{N}(\hat{\boldsymbol{y}}_{T+h}, \boldsymbol{W}_{T+h})$ be an incoherent forecast distribution at time $T+h$ where $\hat{\boldsymbol{y}}_{T+h}$ is the incoherent mean and $\boldsymbol{W}_{T+h} = E_{\boldsymbol{y}_{T+h}}[(\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h})(\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h})^T|\mathcal{I}_T]$ is the incoherent variance

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- The reconciled Gaussian distribution is given by,

$$\mathscr{N}(\boldsymbol{SG}\hat{\boldsymbol{y}}_{T+h},\ \boldsymbol{SGW}_{T+h}\boldsymbol{G}'\boldsymbol{S}')$$

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- Simulation study evidence for improved predictive performance in reconciled Gaussian forecast distributions.

Project 2: *For elliptical distributions*

Consider the case where the base and true predictive distributions are elliptical.

Theorem

There exists a matrix $\boldsymbol{G}$ such that the true predictive distribution can be recovered by linear reconciliation.

This follows from the closure property of elliptical distributions under affine transformations and marginalisation.

Project 2: *Evaluation*

- Scoring rules can be used to evaluate probabilistic forecasts A2

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    - Log Score

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  - Coherent vs Coherent

Project 2: *Evaluation - Coherent vs Incoherent*

When using log score

Theorem

Let $f(\boldsymbol{y})$ be the true predictive density (on $\mathfrak{s}$) and LS be the (negatively-oriented) log score. Then there exists an unreconciled density $\hat{f}(\boldsymbol{y})$ on $\mathbb{R}^n$ such that

$$E_{\boldsymbol{y}}\left[LS(\hat{f}, \boldsymbol{y})\right] < E_{\boldsymbol{y}}\left[LS(f, \boldsymbol{y})\right]$$

The log score is not proper **in this context**.

Today's talk

- A non-parametric bootstrap approach for probabilistic forecast reconciliation.

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- Hierarchical forecasts for macroeconomic variables - An application to Australian GDP.

Part 2: A non-parametric bootstrap approach for probabilistic forecast reconciliation

Probabilistic forecast reconciliation: Non-parametric approach

- Often parametric densities are unavailable but we can simulate a sample from the predictive distribution.

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- Suppose $\hat{\boldsymbol{y}}_{T+h}^{[1]}, ..., \hat{\boldsymbol{y}}_{T+h}^{[J]}$ is a sample from the incoherent predictive distribution.

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- Suppose $\hat{\boldsymbol{y}}_{T+h}^{[1]}, \dots, \hat{\boldsymbol{y}}_{T+h}^{[J]}$ is a sample from the incoherent predictive distribution.
- Then setting $\tilde{\boldsymbol{y}}_{T+h}^{[j]} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h}^{[j]}$ produces a sample from the reconciled predictive distribution with respect to $\boldsymbol{G}$.

A Non-parametric approach

1. Fit univariate models at each node using data up to time T.

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6. Setting $\tilde{\mathbf{\Upsilon}}_{T+h} = \boldsymbol{SG}\hat{\mathbf{\Upsilon}}'_{T+h}$ produces a sample from the reconciled distribution.

Optimal reconciliation of future paths

- We propose to find an optimal $\boldsymbol{G}_h$ matrix by minimizing Energy score.

$$\underset{\boldsymbol{G}_h}{\operatorname{argmin}} \quad \mathrm{E}_{\boldsymbol{Q}}[eS(\tilde{\boldsymbol{F}}, \boldsymbol{y}_{T+h})], \quad \tilde{\boldsymbol{F}} := \tilde{\Upsilon}_{T+h} = \boldsymbol{S}\boldsymbol{G}_h \hat{\Upsilon}'_{T+h}$$

where,

$$\mathrm{eS}(\tilde{\boldsymbol{F}}, \boldsymbol{y}_{T+h}) = \mathrm{E}_{\tilde{\boldsymbol{F}}} \| \tilde{\boldsymbol{Y}}_{T+h} - \boldsymbol{y}_{T+h} \|^{\alpha} - \frac{1}{2} \mathrm{E}_{\tilde{\boldsymbol{F}}} \| \tilde{\boldsymbol{Y}}_{T+h} - \tilde{\boldsymbol{Y}}^{*}_{T+h} \|^{\alpha}, \quad \alpha \in (0, 2]$$

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Optimal reconciliation of future paths *Cont.*

- We impose the following structure to the $\boldsymbol{G}_h$ matrix

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$\boldsymbol{W}_h = \boldsymbol{R}_h'\boldsymbol{R}_h$ s.t $\boldsymbol{i}'\boldsymbol{W}_h\boldsymbol{i} = 1$ where $\boldsymbol{i} = (1, 0, .., 0)'$

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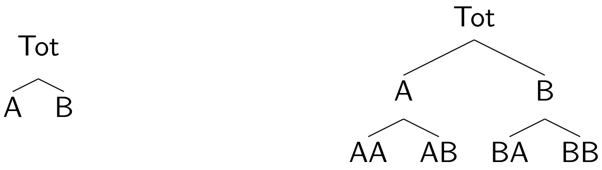
Method 4: Optimising $\boldsymbol{G}_h$ such that $\boldsymbol{G}_h\boldsymbol{S} = \boldsymbol{I}$

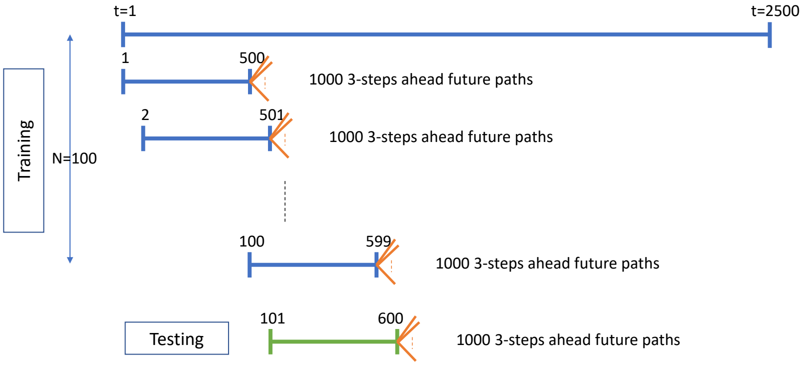
Monte-Carlo Simulation

- Data generating process



- DGP was designed such that we have much noisier series in the bottom level.

Monte-Carlo Simulation



Monte-Carlo Simulation *Cont.*

- Simulation setup:
  - First 2500 observations were generated.

Monte-Carlo Simulation *Cont.*

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  - Fit Univariate ARIMA models for a rolling window of 500 observations.

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  - This optimal $\boldsymbol{G}_h$ is then used to reconcile the incoherent future paths for the test set.
  - The Process was repeated 1000 times and average scores were calculated for the test set.

Monte-Carlo Simulation *Cont.*

Optimisation method	Hierarchy 1				Hierarchy 2			
	$h=1$		$h=3$		$h=1$		$h=3$	
	ES	VS	ES	VS	ES	VS	ES	VS
Method 1 - Optimising $\boldsymbol{W}$	2.48	0.11	2.75	0.11	5.36	1.21	5.83	1.38
Method 2 - Optimising $\boldsymbol{R}$	2.48	0.11	2.75	0.11	5.37	1.21	5.83	1.37
Method 3 - Optimising $\boldsymbol{R}$(Restricted)	2.48	0.11	2.75	0.11	5.37	1.21	5.83	1.37
Method 4 - Optimising $\boldsymbol{G}$	2.48	0.11	2.75	0.11	5.38	1.21	5.83	1.38

- Parameterisation does not matter

Monte-Carlo Simulation *Cont.*

- Comparison with point forecast reconciliation methods.

Reconciliation method	Hierarchy 1				Hierarchy 2			
	$h = 1$		$h = 3$		$h = 1$		$h = 3$	
	ES	VS	ES	VS	ES	VS	ES	VS
Optimal $\boldsymbol{G}$	2.48*	0.106	2.75*	0.106	5.36*	1.21*	5.83*	1.38*
MinT(Shrink)	2.47*	0.105	2.74*	0.105	5.33*	1.19*	5.77*	1.34*
WLS	2.46*	0.105	2.74*	0.105	5.43*	1.23	5.98*	1.40*
OLS	2.54*	0.105	2.80*	0.105	5.51*	1.23	5.98*	1.40*
Base	*2.67*	*0.105*	*2.94*	*0.105*	*5.71*	*1.28*	*6.27*	*1.49*

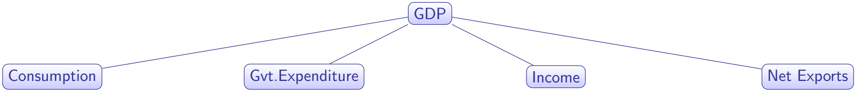
"*" indicates if the average score for a particular reconciliation method is significantly different from that of base forecasts.

- Reconciliation methods perform better than Base forecasts.

- MinT(Shrink) is at least as good as Optimal method. Thus going forward with MinT projection.

Project 3: Hierarchical forecasts for macroeconomic variables - An application to Australian GDP

Macroeconomic forecasting



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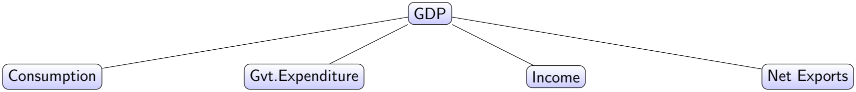

- Common forecasting approaches involves univariate methods or multivariate methods such as VAR.

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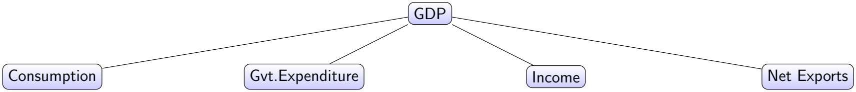

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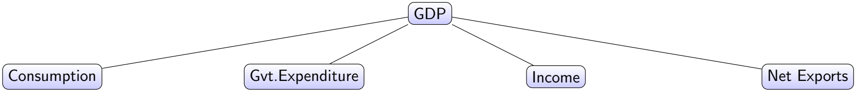

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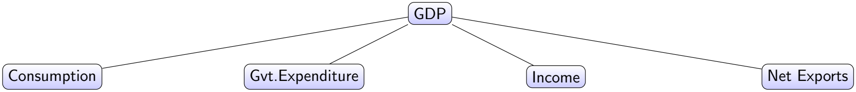

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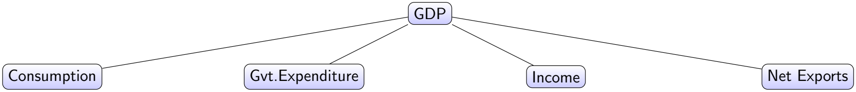

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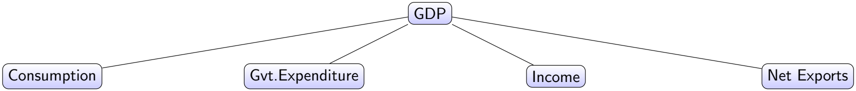

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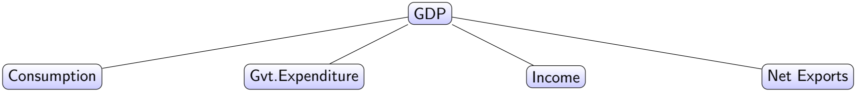

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- The era of big data led to the use of regularization and shrinkage methods - dynamic factor models, Lasso, Bayesian VARs.
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- This might fail to reflect the deterministic relationship between macroeconomic variables in the forecasts.

Macroeconomic forecasting

- Both aligned decision making and forecast accuracy are key concerns for economic agents and policy makers.

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- Related literature: Only one application on point forecasting for inflation (Capistrán, Constandse, and Ramos-Francia, 2010; Weiss, 2018)

- To the best of our knowledge we use hierarchical forecasting methods for point as well as probabilistic forecasts for the first time in macroeconomic literature.

Australian GDP : *Data structures*

- We consider Gross Domestic Product (GDP) of Australia with quarterly data spanning the period 1984:Q4–2018:Q3.

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Income approach

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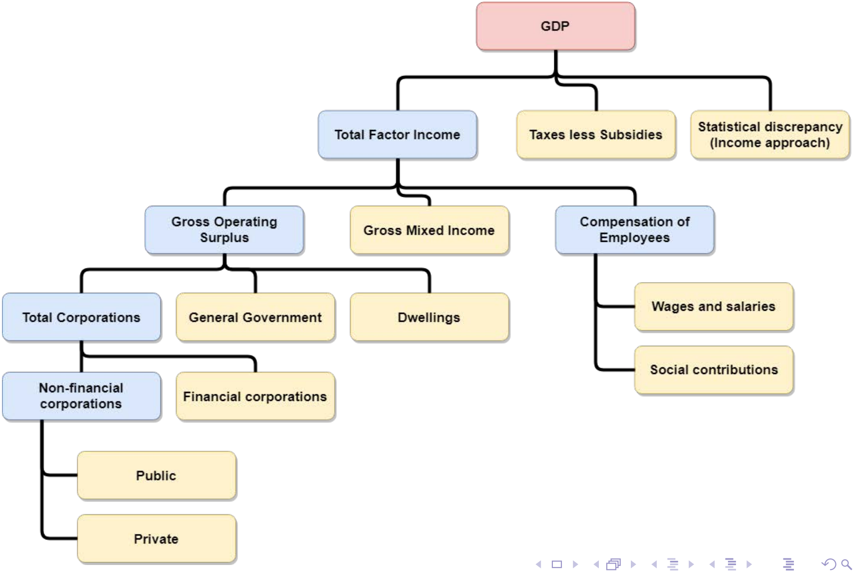
- The hierarchy has two levels of aggregation below the top-level, with a total of $n = 16$ series and $m = 10$ bottom level series.

Australian GDP : *Data structures - Income approach*



Australian GDP : *Data structures*

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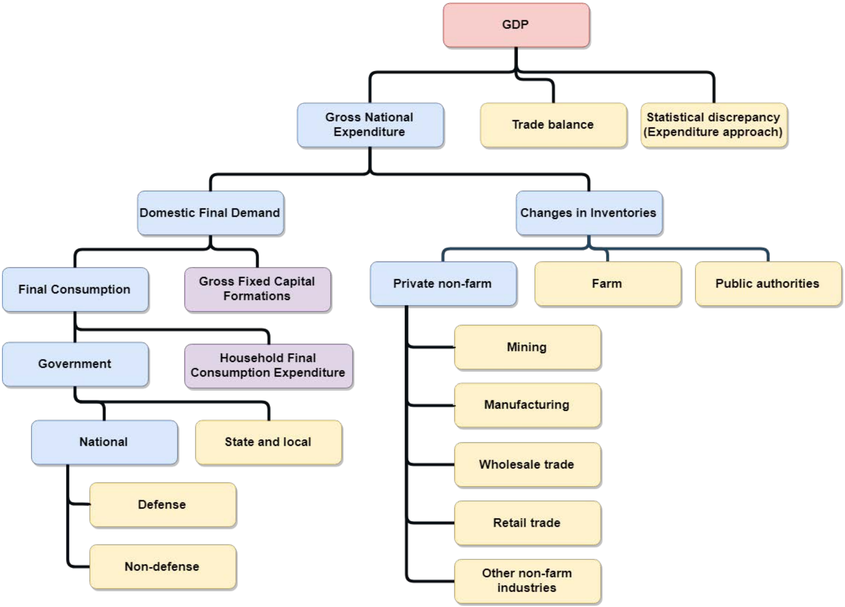
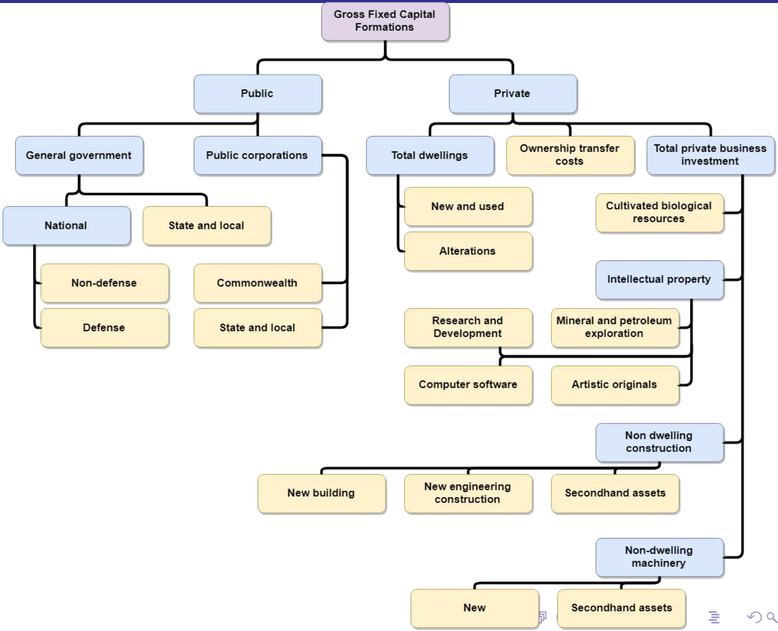
- The hierarchy has three levels of aggregation below the top-level, with a total of $n = 80$ series and $m = 53$ series at the bottom level.


GDP
Gross National Expenditure
Trade balance
Statistical discrepancy (Expenditure approach)
Domestic Final Demand
Changes in Inventories
Final Consumption
Gross Fixed Capital Formations
Private non-farm
Farm
Public authorities
Government
Household Final Consumption Expenditure
Mining
Manufacturing
National
State and local
Wholesale trade
Defense
Retail trade
Non-defense
Other non-farm industries

Australian GDP : *Data structures - Expenditure approach*



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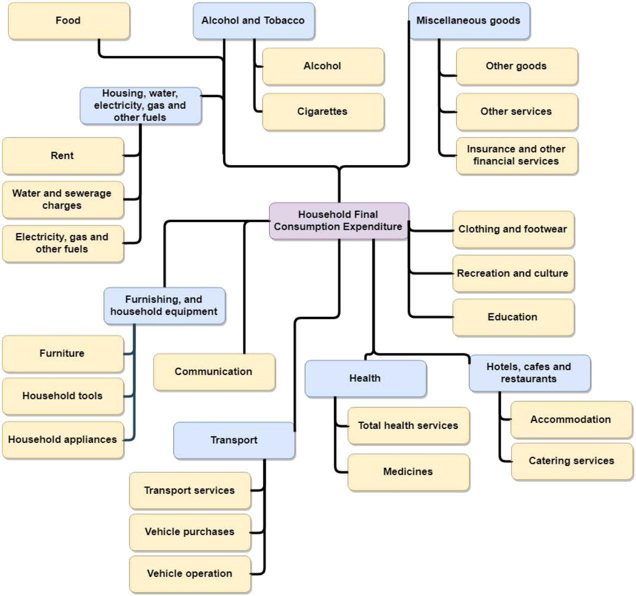
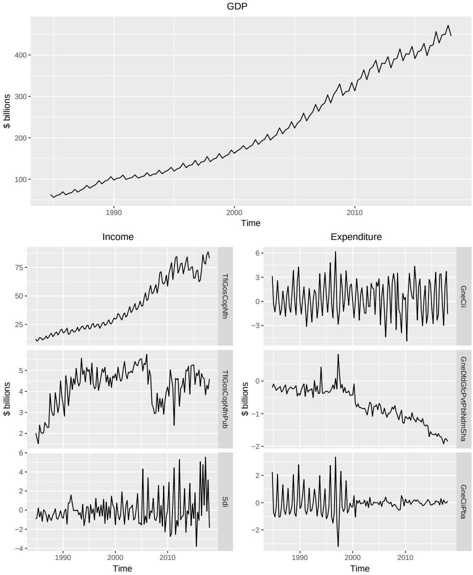


Australian GDP : *Time plots for different levels*



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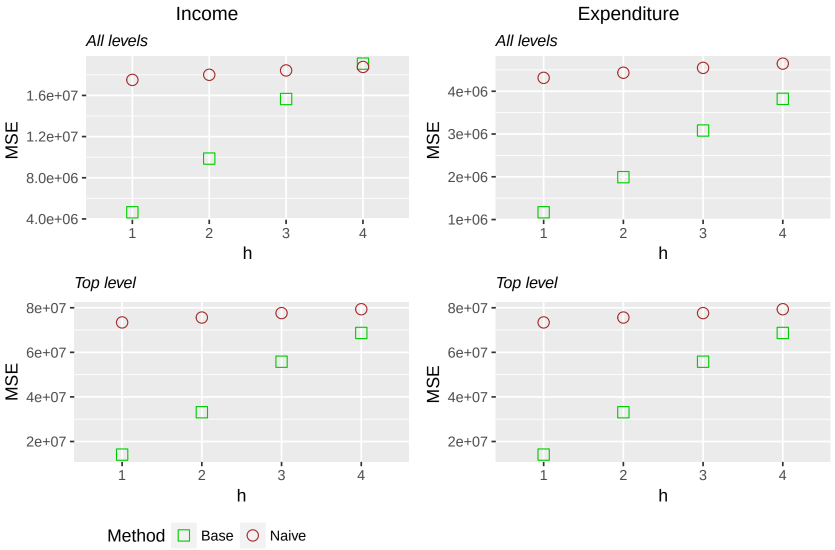
  - $h = 1, 2, 3, 4$ steps ahead forecasts were generated using the fitted models.

Point forecasts: *Base vs Seasonal Naive*



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\hat{\boldsymbol{W}}^{wls}_{T+1} &= \mathrm{Diag}(\hat{\boldsymbol{W}}^{shr}_{T+1})
\end{aligned}$$

Point forecasts: *Reconciliation*

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Method	$\boldsymbol{G}$
BU	$(\boldsymbol{0}_{m\times n-m} \;\; \boldsymbol{I}_{m\times m})$
OLS	$(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$
WLS	$\left(\boldsymbol{S}'\hat{\boldsymbol{W}}^{wls}_{T+1}\boldsymbol{S}\right)^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}^{wls}_{T+1}$
MinT(Shrink)	$\left(\boldsymbol{S}'\hat{\boldsymbol{W}}^{shr}_{T+1}\boldsymbol{S}\right)^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}^{shr}_{T+1}$

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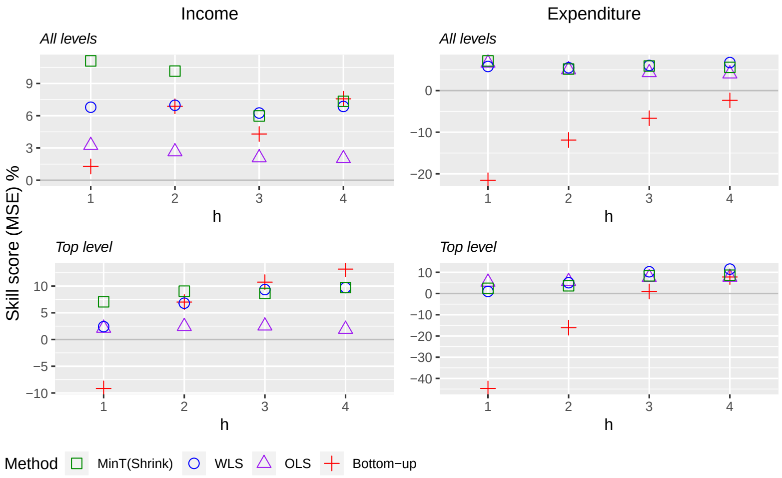
Reconciled Point forecasts - *Results*



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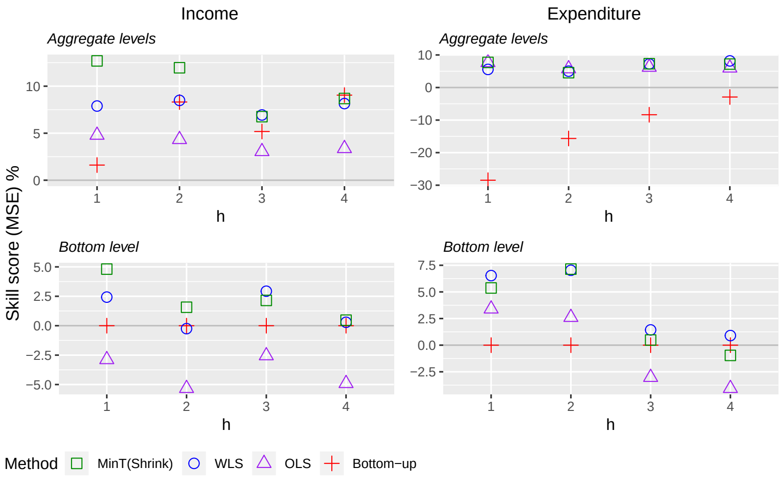

Probabilistic forecasts

- Gaussian approach :

$$\mathcal{N}(\boldsymbol{SG\hat{y}}_{T+h},\ \boldsymbol{SGW}_{T+h}\boldsymbol{G}'\boldsymbol{S}')$$

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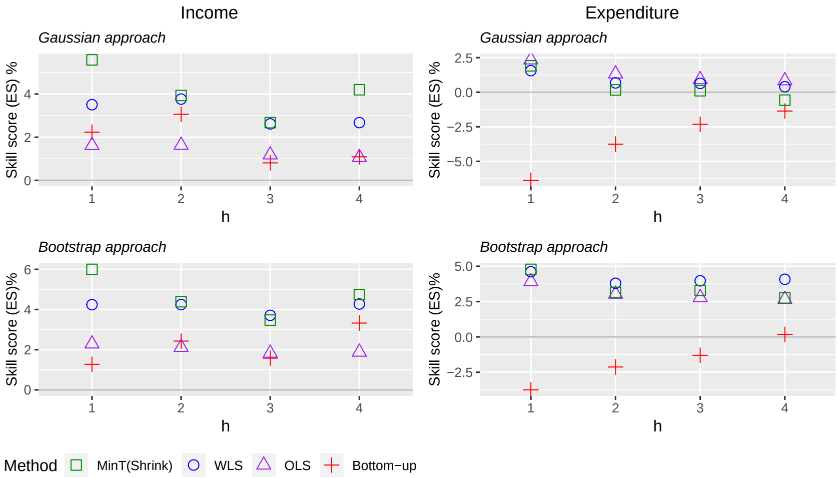
- Non-parametric Bootstrap approach :

$$\tilde{\Upsilon}_{T+h} = \boldsymbol{SG}\hat{\Upsilon}'_{T+h}$$

where, $\hat{\Upsilon}_{T+h} = (\hat{\boldsymbol{y}}^1_{T+h}, ..., \hat{\boldsymbol{y}}^B_{T+h})'$

Reconciled Probabilistic Forecasts



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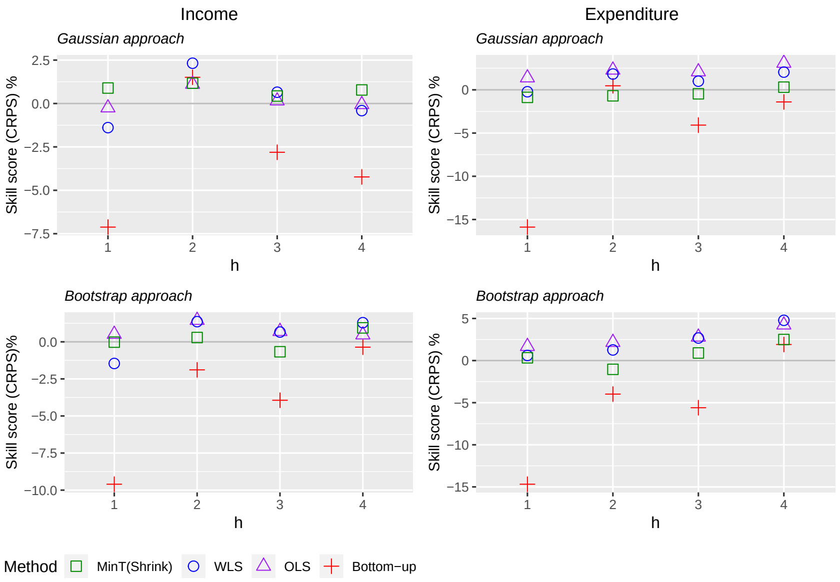

Summary and time plan for completion

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- We apply hierarchical forecast reconciliation methods to forecast Australian GDP in point as well as probabilistic framework.

Time plan for completion

	Thesis Chapter	Task description	Time duration	Progress
1 and 2.	Introduction and Background Review	Writing the chapter.	September/2019 - October/2019	40% complete
3.	Hierarchical forecast reconciliation in Geometric view	Bias correction and application	May/2019 - July/2019	75% Completed
4.	Probabilistic forecast reconciliation for hierarchical time series	Completing the paper.	June/2019 - August/2019	90% Completed
5.	Application	Forecasting Australian GDP		100% Completed

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Thank You!!

Appendix: Probabilistic forecasts reconciliation assuming Gaussianity

A1

- Assuming Gaussianity, let an incoherent forecast distribution at time $T+h$ be given by

$$\mathscr{N}(\hat{\boldsymbol{y}}_{T+h}, \boldsymbol{W}_{T+h}) \stackrel{d}{\leftrightarrow} \hat{\boldsymbol{f}}(.)$$

where $\hat{\boldsymbol{y}}_{T+h}$ is the incoherent mean and $\boldsymbol{W}_{T+h} = E_{\boldsymbol{y}_{T+h}}[(\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h})(\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h})^T | \mathcal{I}_T]$ is incoherent variance

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- Assuming Gaussianity, let an incoherent forecast distribution at time $T+h$ be given by

$$\mathscr{N}(\hat{\boldsymbol{y}}_{T+h}, \boldsymbol{W}_{T+h}) \stackrel{d}{\leftrightarrow} \hat{\boldsymbol{f}}(.)$$

  where $\hat{\boldsymbol{y}}_{T+h}$ is the incoherent mean and $\boldsymbol{W}_{T+h} = E_{\boldsymbol{y}_{T+h}}[(\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h})(\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h})^T | \mathcal{I}_T]$ is incoherent variance

- Gaussian densities are closed under affine transformation and marginalization

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- Gaussian densities are closed under affine transformation and marginalization
- From the linear transformation

$$\boldsymbol{f_B}(\cdot) = \frac{\exp\left\{-\frac{1}{2}\left((\boldsymbol{S} \vdots \boldsymbol{R})\begin{pmatrix}\tilde{\boldsymbol{b}}_{T+h} \\ \boldsymbol{a}_{T+h}\end{pmatrix} - \hat{\boldsymbol{y}}_{T+h}\right)' \boldsymbol{W}_{\boldsymbol{T+h}}^{-1}\left((\boldsymbol{S} \vdots \boldsymbol{R})\begin{pmatrix}\tilde{\boldsymbol{b}}_{T+h} \\ \boldsymbol{a}_{T+h}\end{pmatrix} - \hat{\boldsymbol{y}}_{T+h}\right)\right\}}{(2\pi)^{\frac{n}{2}} \left|\boldsymbol{W}_{T+h}\right|^{\frac{1}{2}} \left|(\boldsymbol{S} \vdots \boldsymbol{R})^{-1}\right|}$$

Appendix: Probabilistic forecasts reconciliation assuming Gaussianity

- Marginalizing over $\boldsymbol{t}_{T+h}$ A1

$$\tilde{\boldsymbol{f}}_b(\tilde{\boldsymbol{b}}_{T+h}) = \frac{\exp\left\{-\frac{1}{2}(\tilde{\boldsymbol{b}}_{T+h} - \boldsymbol{G}\hat{\boldsymbol{y}}_{T+h})'(\boldsymbol{G}\boldsymbol{W}_{\boldsymbol{T+h}}\boldsymbol{G}')^{-1}(\tilde{\boldsymbol{b}}_{T+h} - \boldsymbol{G}\hat{\boldsymbol{y}}_{T+h})\right\}}{(2\pi)^{\frac{n}{2}}\left|\boldsymbol{G}\boldsymbol{W}_{\boldsymbol{T+h}}\boldsymbol{G}'\right|^{\frac{1}{2}}}$$

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- Therefore the reconciled Gaussian distribution of the whole hierarchy is given by

$$\tilde{\boldsymbol{f}}(\tilde{\boldsymbol{y}}_{t+h}) = \tilde{\boldsymbol{f}}_b(\boldsymbol{S}\tilde{\boldsymbol{b}}_{T+h}) \overset{d}{\leftrightarrow} \mathcal{N}(\boldsymbol{S}\boldsymbol{P}\hat{\boldsymbol{y}}_{T+h},\ \boldsymbol{S}\boldsymbol{P}\boldsymbol{W}_{T+h}\boldsymbol{G}'\boldsymbol{S}'),$$

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- **Goal**: Solve for $\boldsymbol{G}$ by minimizing a proper scoring rule

Appendix: Probabilistic forecasts reconciliation assuming Gaussianity

- We use the Energy Score, which is a proper scoring rule

$$ES(\tilde{\boldsymbol{Y}}_{\boldsymbol{T+h}}, \boldsymbol{y}_{\boldsymbol{T+h}}) = E_{\boldsymbol{Y}_{T+h}}||\tilde{\boldsymbol{Y}}_{T+h} - \boldsymbol{y}_{T+h}||^{\alpha} - \frac{1}{2}E_{\boldsymbol{Y}_{T+h}}||\tilde{\boldsymbol{Y}}_{T+h} - \tilde{\boldsymbol{Y}}^{*}_{T+h}||^{\alpha},$$

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- However in the upper limit of α, i.e. when $\alpha = 2$,

  $$ES(\tilde{\boldsymbol{Y}}_{\boldsymbol{T+h}}, \boldsymbol{y}_{\boldsymbol{T+h}}) = ||\boldsymbol{SP}\hat{\boldsymbol{y}}_{T+h} - \boldsymbol{y}_{T+h}||^2$$

Appendix: Probabilistic forecasts reconciliation assuming Gaussianity

- Then our objective function is

$$\underset{\boldsymbol{G}}{\arg\min}\ Tr\{E_{\boldsymbol{y}_{T+h}}[(\boldsymbol{y}_{T+h} - \boldsymbol{SP}\hat{\boldsymbol{y}}_{T+h})(\boldsymbol{y}_{T+h} - \boldsymbol{SP}\hat{\boldsymbol{y}}_{T+h})^T|\mathcal{I}_T]\},$$

where $\mathcal{I}_T = \{\boldsymbol{y}_1, ..., \boldsymbol{y}_T\}$

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- Assuming unbiasedness for coherent forecasts

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- Thus $\boldsymbol{R}'_{\perp} = \boldsymbol{S}'\boldsymbol{W}_{T+h}^{-1}$. This is referred to as MinT solution

Appendix: Probabilistic forecasts in the Gaussian framework

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$\boldsymbol{R}'_{\perp} = \boldsymbol{S}'\boldsymbol{W}^{-1}_{T+h}$

- The optimally reconciled Gaussian forecast density is given by $\mathcal{N}[\boldsymbol{S}(\boldsymbol{R}'_{\perp}\boldsymbol{S})^{-1}\boldsymbol{R}'_{\perp}\hat{\boldsymbol{y}}_{T+h}, \boldsymbol{S}(\boldsymbol{R}'_{\perp}\boldsymbol{S})^{-1}\boldsymbol{R}'_{\perp}\boldsymbol{W}_{T+h}\boldsymbol{R}_{\perp}(\boldsymbol{R}'_{\perp}\boldsymbol{S})^{-1}\boldsymbol{S}'],$

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- Different estimators of $\boldsymbol{W}_{T+h}$ yield different $\boldsymbol{R}'_{\perp}$ matrices

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- Different estimators of $\boldsymbol{W}_{T+h}$ yield different $\boldsymbol{R}'_{\perp}$ matrices

Method	**Estimate of $\boldsymbol{W}_h$**	**Estimate of $\boldsymbol{R}'_{\perp}$**
OLS	$\boldsymbol{I}$	$\boldsymbol{S}'$
MinT(Sample)	$\hat{\boldsymbol{W}}^{sam}_{T+1}$	$\boldsymbol{S}'(\hat{\boldsymbol{W}}^{sam}_{T+1})^{-1}$
MinT(Shrink)	$\hat{\boldsymbol{W}}^{shr}_{T+1} = \tau\text{Diag}(\hat{\boldsymbol{W}}^{sam}_{T+1}) + (1-\tau)\hat{\boldsymbol{W}}^{sam}_{T+1}$ A2	$\boldsymbol{S}'(\hat{\boldsymbol{W}}^{shr}_{T+1})^{-1}$
MinT(WLS)	$\hat{\boldsymbol{W}}^{wls}_{T+1} = \text{Diag}(\hat{\boldsymbol{W}}^{shr}_{T+1})$	$\boldsymbol{S}'(\hat{\boldsymbol{W}}^{wls}_{T+1})^{-1}$

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- Different estimators of $\boldsymbol{W}_{T+h}$ yield different $\boldsymbol{R}'_{\perp}$ matrices

Method	**Estimate of $\boldsymbol{W}_h$**	**Estimate of $\boldsymbol{R}'_{\perp}$**
OLS	$\boldsymbol{I}$	$\boldsymbol{S}'$
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$\boldsymbol{R}'_{\perp} = \boldsymbol{S}'\boldsymbol{W}_{T+h}^{-1}$

- The optimally reconciled Gaussian forecast density is given by $\mathcal{N}[\boldsymbol{S}(\boldsymbol{R}'_{\perp}\boldsymbol{S})^{-1}\boldsymbol{R}'_{\perp}\hat{\boldsymbol{y}}_{T+h}, \boldsymbol{S}(\boldsymbol{R}'_{\perp}\boldsymbol{S})^{-1}\boldsymbol{R}'_{\perp}\boldsymbol{W}_{T+h}\boldsymbol{R}_{\perp}(\boldsymbol{R}'_{\perp}\boldsymbol{S})^{-1}\boldsymbol{S}'],$
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OLS	$\boldsymbol{I}$	$\boldsymbol{S}'$
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MinT(WLS)	$\hat{\boldsymbol{W}}^{wls}_{T+1} = \text{Diag}(\hat{\boldsymbol{W}}^{shr}_{T+1})$	$\boldsymbol{S}'(\hat{\boldsymbol{W}}^{wls}_{T+1})^{-1}$

- Predictive ability of the reconciled Gaussian densities from these methods will be evaluated in a simulation setting

Appendix: *Probabilistic forecasts evaluation*

A2

Energy score (Gneiting et al., 2008)

$$\text{eS}(\breve{\boldsymbol{Y}}_{T+h}, \boldsymbol{y}_{T+h}) = \text{E}_{\breve{\boldsymbol{F}}}\|\breve{\boldsymbol{Y}}_{T+h} - \boldsymbol{y}_{T+h}\|^{\alpha} - \tfrac{1}{2}\text{E}_{\breve{\boldsymbol{F}}}\|\breve{\boldsymbol{Y}}_{T+h} - \breve{\boldsymbol{Y}}^{*}_{T+h}\|^{\alpha}, \quad \alpha \in (0, 2]$$

Log score (Gneiting and Raftery, 2007)

$$\text{LS}(\breve{\boldsymbol{F}}, \boldsymbol{y}_{T+h}) = -\log \breve{\boldsymbol{f}}(\boldsymbol{y}_{T+h})$$

Variogram score (Scheuerer and Hamill, 2015)

$$\text{VS}(\breve{\boldsymbol{F}}, \boldsymbol{y}_{T+h}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\Big(|y_{T+h,i} - y_{T+h,j}|^{p} - \text{E}_{\breve{\boldsymbol{F}}}|\breve{Y}_{T+h,i} - \breve{Y}_{T+h,j}|^{p}\Big)^{2}$$

CRPS (Gneiting and Raftery, 2007)

$$\text{CRPS}(\breve{F}_i, y_{T+h,i}) = \text{E}_{\breve{F}_i}|\breve{Y}_{T+h,i} - y_{T+h,i}| - \tfrac{1}{2}\text{E}_{\breve{F}_i}|\breve{Y}_{T+h,i} - \breve{Y}^{*}_{T+h,i}|$$

$\breve{\boldsymbol{Y}}_{T+h}$ and $\breve{\boldsymbol{Y}}^{*}_{T+h}$	:	Independent random vectors from the coherent forecast distribution $\breve{\boldsymbol{F}}$.
$\boldsymbol{y}_{T+h}$	:	Vector of realizations.
$\breve{Y}_{T+h,i}$ and $\breve{Y}_{T+h,j}$	:	ith and jth components of the vector $\breve{\boldsymbol{Y}}_{T+h}$

Appendix

- A2 Shrinkage estimator for 1-step ahead base forecast errors

$$\hat{\boldsymbol{\Sigma}}^{shr}_{T+1} = \tau \hat{\boldsymbol{\Sigma}}^{D}_{T+1} + (1-\tau)\hat{\boldsymbol{\Sigma}}_{T+1},$$

where $\hat{\boldsymbol{\Sigma}}^{D}_{T+1}$ is the diagonal matrix comprising diagonal entries of $\hat{\boldsymbol{\Sigma}}_{T+1}$ and

$$\tau = \frac{\sum_{i \neq j} \hat{Var}(\hat{r}_{ij})}{\sum_{i \neq j} \hat{r}^2_{ij}}$$

is a shrinkage parameter. $\hat{r}_{ij}$ is the ij-th element of sample correlation matrix. In this estimation, the off-diagonal elements of 1-step ahead sample covariance matrix will be shrunk to zero depending on the sparsity.

Monte-Carlo simulation

- **Data generating process** A3

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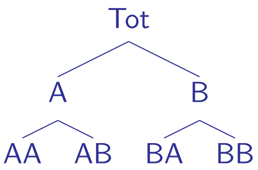
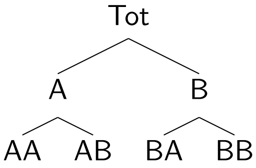


Monte-Carlo simulation

- **Data generating process** A3



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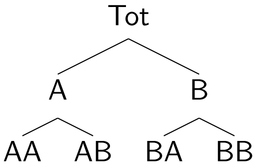
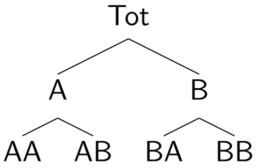


- $\{w_{AA,t}, w_{AB,t}, w_{BA,t}, w_{BB,t}\} \sim ARIMA(p, d, q)$

Monte-Carlo simulation

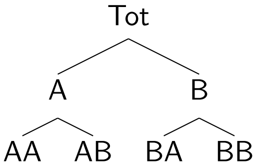
- **Data generating process** A3



- $\{w_{AA,t}, w_{AB,t}, w_{BA,t}, w_{BB,t}\} \sim ARIMA(p, d, q)$
- $p \in \{1, 2\}$ and $d \in \{0, 1\}$

Monte-Carlo simulation

- **Data generating process** A3



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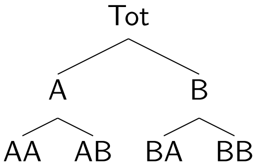
- $p \in \{1, 2\}$ and $d \in \{0, 1\}$
- $\{\epsilon_{AA,t}, \epsilon_{AB,t}, \epsilon_{BA,t}, \epsilon_{BB,t}\} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$

Monte-Carlo simulation

- **Data generating process** A3



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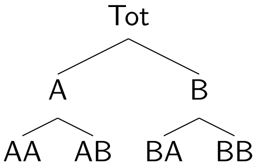
- Parameters for *AR* and *MA* components were randomly and uniformly generated from $[0.3, 0.5]$ and $[0.3, 0.7]$ respectively

Monte-Carlo simulation

- **Data generating process** A3



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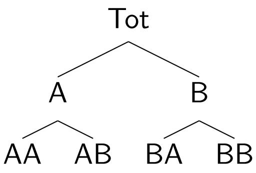
Monte-Carlo simulation

- **Data generating process** A3



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- $\boldsymbol{y}_t$ are then generated as follows

Monte-Carlo simulation

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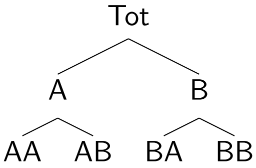

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- $\{\epsilon_{AA,t}, \epsilon_{AB,t}, \epsilon_{BA,t}, \epsilon_{BB,t}\} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$
- Parameters for *AR* and *MA* components were randomly and uniformly generated from $[0.3, 0.5]$ and $[0.3, 0.7]$ respectively

- $\boldsymbol{y}_t$ are then generated as follows

Bottom level	**Aggregate level 1**	**Total**
$y_{AA,t} = w_{AA,t} + u_t - 0.5v_t$	$y_{A,t} = w_{AA,t} + w_{AB,t} - v_t$	$y_{Tot,t} = w_{AA,t} + w_{AB,t} + w_{BA,t} + w_{BB,t}$
$y_{AB,t} = w_{AB,t} - u_t - 0.5v_t$	$y_{B,t} = w_{BA,t} + w_{BB,t} + v_t$	
$y_{BA,t} = w_{BA,t} + u_t + 0.5v_t$		
$y_{BB,t} = w_{BB,t} - u_t + 0.5v_t$		

Monte-Carlo simulation *Cont..*

- To get less noisier series at aggregate levels, we choose $\boldsymbol{\Sigma}, \sigma_u^2$ and σ_v^2 such that,

$$\mathsf{Var}(\epsilon_{AA,t} + \epsilon_{AB,t} + \epsilon_{BA,t} + \epsilon_{BB,t}) \leq \mathsf{Var}(\epsilon_{AA,t} + \epsilon_{AB,t} - v_t) \leq \mathsf{Var}(\epsilon_{AA,t} + u_t - 0.5v_t),$$

Monte-Carlo simulation *Cont..*

- To get less noisier series at aggregate levels, we choose $\mathbf{\Sigma}, \sigma_u^2$ and σ_v^2 such that,

$$\mathrm{Var}(\epsilon_{AA,t} + \epsilon_{AB,t} + \epsilon_{BA,t} + \epsilon_{BB,t}) \leq \mathrm{Var}(\epsilon_{AA,t} + \epsilon_{AB,t} - v_t) \leq \mathrm{Var}(\epsilon_{AA,t} + u_t - 0.5v_t),$$

- Thus we choose, $\mathbf{\Sigma} = \begin{pmatrix} 5.0 & 3.1 & 0.6 & 0.4 \\ 3.1 & 4.0 & 0.9 & 1.4 \\ 0.6 & 0.9 & 2.0 & 1.8 \\ 0.4 & 1.4 & 1.8 & 3.0 \end{pmatrix}$, $\sigma_u^2 = 19$ and $\sigma_u^2 = 18$.

Sample version of the scoring rules

- For a possible finite sample of size B from the multivariate forecast density $\breve{\boldsymbol{F}}$, the variogram score is defined as,

$$\mathrm{VS}(\breve{\boldsymbol{F}}, \boldsymbol{y}_{T+h}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\left(|y_{T+h,i} - y_{T+h,j}|^p - \frac{1}{B}\sum_{k=1}^{B}|\breve{Y}_{T+h,i}^{k} - \breve{Y}_{T+h,j}^{k}|^p\right)^2$$