Probabilistic Forecasts for Hierarchical Time Series

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Overview

Introduction

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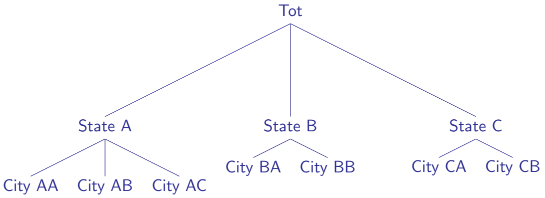
- **Example:** Electricity demand of a country can be disaggregated into demand of states, cities and households



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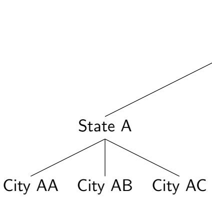

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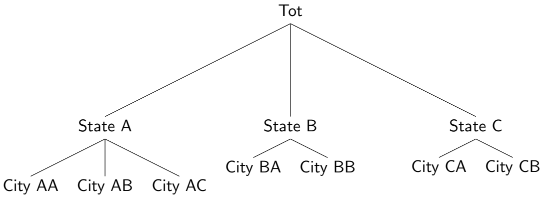

- Forecasts of these hierarchical time series should be *coherent*

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- Traditional methods are not reconciliation methods

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- Extending the "reconciliation" method into probabilistic framework
- **Objective:** Reconciliation of probabilistic forecasts for hierarchical time series such that they preserve the inherent properties of the hierarchical nature

Probabilistic forecast reconciliation for hierarchical time series

Notations in hierarchical time series and forecasting

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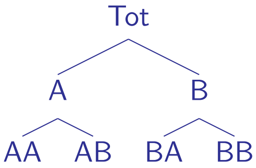

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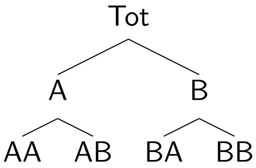

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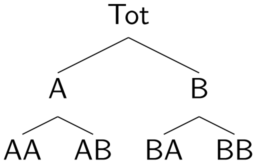

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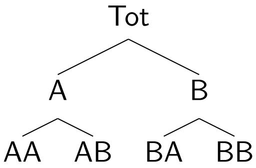

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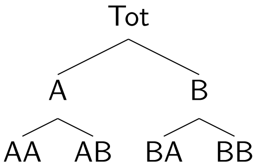

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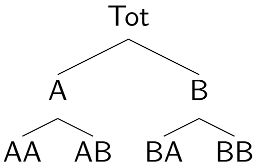

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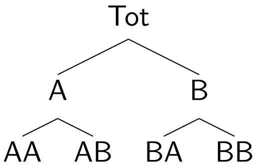

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$$\boldsymbol{S} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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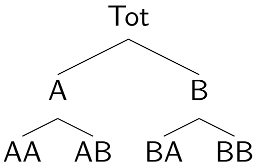

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- Due to the aggregation nature of the hierarchy we have,

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

Coherent forecasts

Coherent forecasts

Definition 1: Coherent subspace

Suppose an n-dimensional time series $\boldsymbol{y}_t \in \mathbb{R}^n$ is subject to the linear aggregation constraint $\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$ where $\boldsymbol{b}_t \in \mathbb{R}^m$ and $\boldsymbol{S}$ is an $n \times m$ constant matrix. Let $\mathbb{C}^m$ be an m-dimensional subspace of $\mathbb{R}^n$, where $\mathbb{C}^m \subset \mathbb{R}^n$. Then $\mathbb{C}^m$ is said to be a coherent space if it is spanned by the columns of $\boldsymbol{S}$.

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Suppose $\breve{\boldsymbol{y}}_{t+h|t} \in \mathbb{R}^n$ denotes point forecasts of each series in the hierarchy at time $t+h$. Then $\breve{\boldsymbol{y}}_{t+h|t}$ is *coherent* if $\breve{\boldsymbol{y}}_{t+h|t} \in \mathbb{C}^m$.

Coherent forecasts *cont.*

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Definition 3: Coherent probabilistic forecasts

Let $(\mathbb{R}^m, \mathscr{F}^m, \nu^m)$ be a probability triple where $\mathscr{F}^m$ is a σ-algebra on $\mathbb{R}^m$. Then, $(\mathbb{C}^m, \mathscr{F}_{\boldsymbol{S}}, \breve{\nu})$ is said to be a coherent probability triple iff

$$\breve{\nu}(\boldsymbol{S}(\boldsymbol{A})) = \nu^m(\boldsymbol{A}) \quad \forall \boldsymbol{A} \in \mathscr{F}^m,$$

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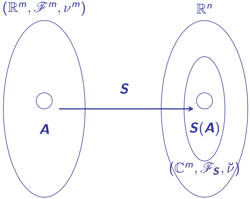

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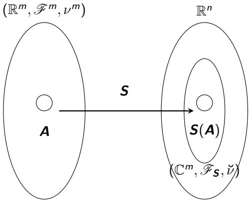

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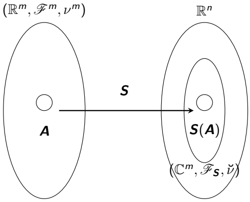

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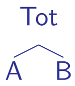

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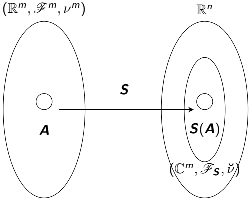

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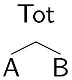

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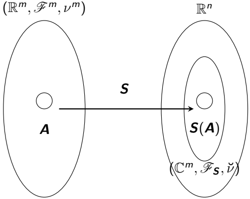

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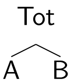

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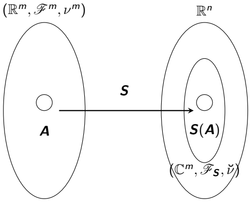

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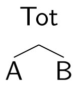

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$$f(\breve{\boldsymbol{y}}_{t+h}) = 0 \quad \forall \breve{\boldsymbol{y}}_{t+h} \in \mathbb{N}^{n-m}$$

Point forecast reconciliation

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Definition 4: Point forecast reconciliation

Let $\hat{\boldsymbol{y}}_{t+h} \in \mathbb{R}^n$ be any set of incoherent forecasts at time $t+h$, and let

$$\tilde{\boldsymbol{y}}_{t+h} = \boldsymbol{S} \circ \boldsymbol{g}(\hat{\boldsymbol{y}}_{t+h}),$$

where $\boldsymbol{g} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $\boldsymbol{S} \circ \boldsymbol{g}(\cdot)$ is a mapping of $\boldsymbol{g}(\cdot)$ onto $\mathbb{C}^m$. Then $\tilde{\boldsymbol{y}}_{t+h}$ is said to be "reconciled" if $\tilde{\boldsymbol{y}}_{t+h} \in \mathbb{C}^m$.

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- Definition 4 holds for linear as well as non-linear reconciliation
- Previous studies have only focused on linear case,

$$\boldsymbol{g}(\hat{\boldsymbol{y}}) = \boldsymbol{P}\hat{\boldsymbol{y}},$$

where $\boldsymbol{P}$ is a $m \times n$ matrix. Thus giving

$$\tilde{\boldsymbol{y}}_{t+h} = \boldsymbol{S}\boldsymbol{P}\hat{\boldsymbol{y}}_{t+h}$$

Point forecast reconciliation *cont..*

- Suppose $\boldsymbol{S} = \{\boldsymbol{s}_1, \dots, \boldsymbol{s}_m\}$ spans $\mathbb{C}^m$, $\boldsymbol{R} = \{\boldsymbol{r}_1, \dots, \boldsymbol{r}_{n-m}\}$ spans $\mathbb{N}^{n-m}$. Then $(\boldsymbol{S} \vdots \boldsymbol{R}) = \boldsymbol{B} = \{\boldsymbol{s}_1, \dots, \boldsymbol{s}_m, \boldsymbol{r}_1, \dots, \boldsymbol{r}_{n-m}\}$ spans $\mathbb{R}^n$.

Point forecast reconciliation *cont..*

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- Schematic in $\mathbb{R}^2$

Point forecast reconciliation *cont..*

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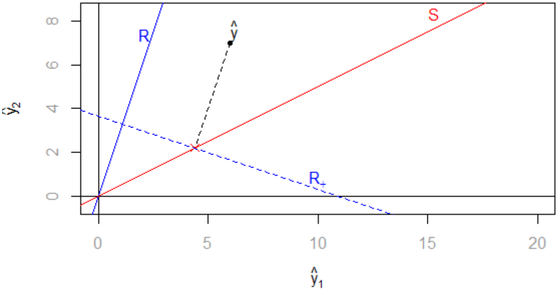
- Schematic in $\mathbb{R}^2$



Point forecast reconciliation *cont..*

- **Step 1: Change of coordinates (linear transformation step)**

Point forecast reconciliation *cont..*

- **Step 1: Change of coordinates (linear transformation step)**

  $(\tilde{\boldsymbol{b}}'_{t+h}, \boldsymbol{t}'_{t+h})'$: coordinates of $\hat{\boldsymbol{y}}_{t+h}$ with respect to $(\boldsymbol{S} \vdots \boldsymbol{R})$

Point forecast reconciliation *cont..*

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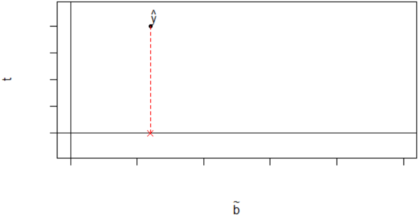

Point forecast reconciliation *cont..*

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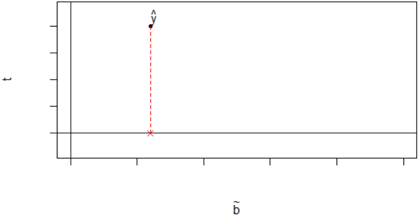

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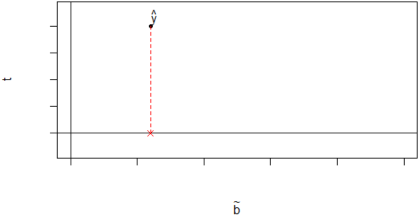

$$(\boldsymbol{S} \vdots \boldsymbol{R})(\tilde{\boldsymbol{b}}'_{t+h},\ \boldsymbol{t}'_{t+h})' = \hat{\boldsymbol{y}}_{t+h},$$

Point forecast reconciliation *cont..*

- **Step 1: Change of coordinates (linear transformation step)**

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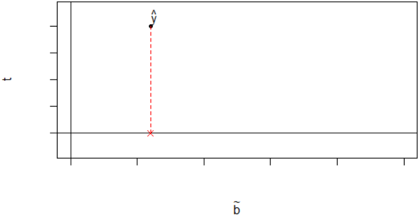

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Point forecast reconciliation *cont..*

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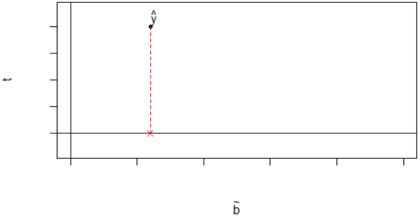

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Since $(\boldsymbol{S} \vdots \boldsymbol{R})^{-1} = \begin{pmatrix} (\boldsymbol{R}'_{\perp}\boldsymbol{S})^{-1}\boldsymbol{R}'_{\perp} \\ \cdots \\ (\boldsymbol{S}'_{\perp}\boldsymbol{R})^{-1}\boldsymbol{S}'_{\perp} \end{pmatrix}$ we have,

Point forecast reconciliation *cont..*

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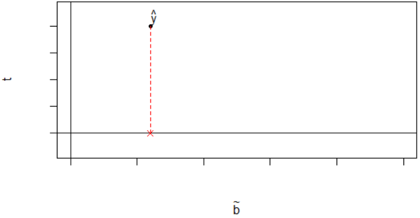

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$$\begin{pmatrix} \tilde{\boldsymbol{b}}_{t+h} \\ \cdots \\ \boldsymbol{t}_{t+h} \end{pmatrix} = \begin{pmatrix} (\boldsymbol{R}'_{\perp}\boldsymbol{S})^{-1}\boldsymbol{R}'_{\perp} \\ \cdots \\ (\boldsymbol{S}'_{\perp}\boldsymbol{R})^{-1}\boldsymbol{S}'_{\perp} \end{pmatrix} \hat{\boldsymbol{y}}_{t+h}.$$

Point forecast reconciliation *cont..*

- **Step 2: Ignoring the null coordinates**

$$\tilde{\boldsymbol{b}}_{t+h} = (\boldsymbol{R}'_{\perp}\boldsymbol{S})^{-1}\boldsymbol{R}'_{\perp}\hat{\boldsymbol{y}}_{t+h}. \tag{1}$$

Point forecast reconciliation *cont..*

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- **Step 3: Reconciled point forecasts for the whole hierarchy**

Point forecast reconciliation *cont..*

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- **Step 3: Reconciled point forecasts for the whole hierarchy**
  - Mapping $\tilde{\boldsymbol{b}}_{t+h} \in \mathbb{R}^m$ to $\mathbb{C}^m$ through $\boldsymbol{S}$ we get

Point forecast reconciliation *cont..*

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  - Mapping $\tilde{\boldsymbol{b}}_{t+h} \in \mathbb{R}^m$ to $\mathbb{C}^m$ through $\boldsymbol{S}$ we get

$$\tilde{\boldsymbol{y}}_{t+h} = \boldsymbol{S}(\boldsymbol{R}'_{\perp}\boldsymbol{S})^{-1}\boldsymbol{R}'_{\perp}\hat{\boldsymbol{y}}_{t+h}, \qquad \tilde{\boldsymbol{y}}_{t+h} \in \mathbb{C}^m \subset \mathbb{R}^n. \tag{2}$$

Point forecast reconciliation *cont..*

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- Previous definitions on point forecast reconciliation coincides if $\boldsymbol{P} = (\boldsymbol{R}'_{\perp}\boldsymbol{S})^{-1}\boldsymbol{R}'_{\perp}$

Point forecast reconciliation *cont..*

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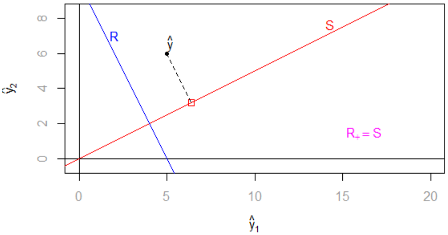
- Previous definitions on point forecast reconciliation coincides if $\boldsymbol{P} = (\boldsymbol{R}'_\perp \boldsymbol{S})^{-1} \boldsymbol{R}'_\perp$
- Further we find $\boldsymbol{R}_\perp$ s.t $(\boldsymbol{R}'_\perp \boldsymbol{S})^{-1} \boldsymbol{R}'_\perp \boldsymbol{S} = \boldsymbol{I}$. This coincides with the unbiased condition $\boldsymbol{SPS} = \boldsymbol{S}$ (Hyndman et al., 2011)

Point forecast reconciliation *cont..*

- OLS Reconciliation

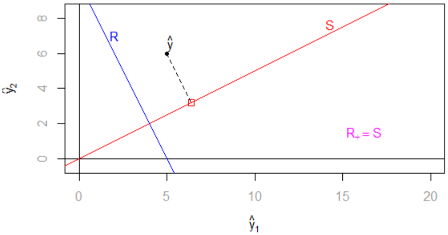
Point forecast reconciliation *cont..*

- OLS Reconciliation



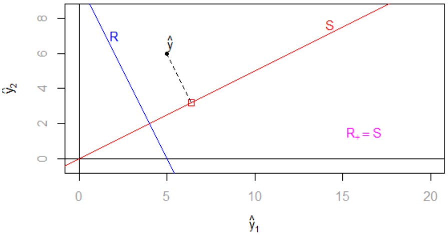
Point forecast reconciliation *cont..*

- OLS Reconciliation



Point forecast reconciliation *cont..*

- OLS Reconciliation



Method	Reconciled point forecasts	$R'_{\perp}$
OLS	$\tilde{\boldsymbol{y}}^{OLS}_{t+h} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'\hat{\boldsymbol{y}}_{t+h}$	$\boldsymbol{S}'$
MinT	$\tilde{\boldsymbol{y}}^{MinT}_{t+h} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}\hat{\boldsymbol{y}}_{t+h}$	$\boldsymbol{S}'\boldsymbol{W}_h^{-1}$

Probabilistic forecast reconciliation

Probabilistic forecast reconciliation

Definition 5: Probabilistic forecast reconciliation

Suppose $(\mathbb{R}^n, \mathscr{F}^n, \hat{\nu})$ is an incoherent probability triple and $(\mathbb{R}^m, \mathscr{F}^m, \nu^m)$ is a probability triple defined on $\mathbb{R}^m$. Let $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^m$. Then the probability measure on the reconciled bottom levels is such that

$$\nu^m(\boldsymbol{A}) = \hat{\nu}(\boldsymbol{g}^{-1}(\boldsymbol{A})), \quad \forall \quad \boldsymbol{A} \in \mathscr{F}^m.$$

Further the probability measure of the whole reconciled hierarchy is given by

$$\tilde{\nu}(\boldsymbol{S}(\boldsymbol{A})) = \hat{\nu}(\boldsymbol{g}^{-1}(\boldsymbol{A})) \qquad \forall \boldsymbol{A} \in \mathscr{F}^m,$$

where $\boldsymbol{S} : \mathbb{R}^m \to \mathbb{C}^m$, $\tilde{\nu}(\cdot)$ is the probability measure on the measure space $(\mathbb{C}^m, \mathscr{F}_S)$ and $\boldsymbol{g}^{-1}(\boldsymbol{A})$ is the pre-image of $\boldsymbol{A}$ in $\mathbb{R}^n$

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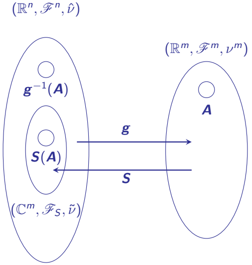
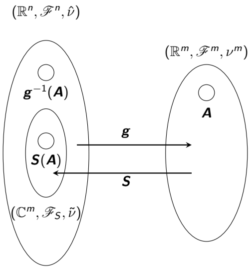
Probabilistic forecast reconciliation *cont..*

Probabilistic forecast reconciliation *cont..*

Probabilistic forecast reconciliation *cont..*



Probabilistic forecast reconciliation *cont..*



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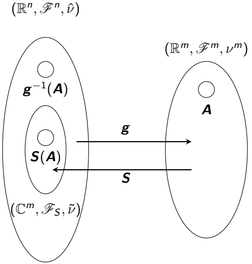
Probabilistic forecast reconciliation *cont..*



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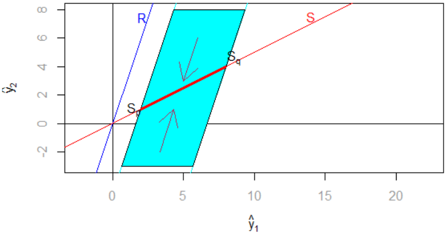
- Back to 2-dimensional schematic



Probabilistic forecast reconciliation *cont..*

- **Step 1: Change of coordinates** (linear transformation step)

Probabilistic forecast reconciliation *cont..*

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  Recall $(\tilde{\boldsymbol{b}}'_{t+h}, \boldsymbol{t}'_{t+h})'$: coordinates of $\hat{\boldsymbol{y}}_{t+h}$ with respect to $(\boldsymbol{S} \vdots \boldsymbol{R})$.

Probabilistic forecast reconciliation *cont..*

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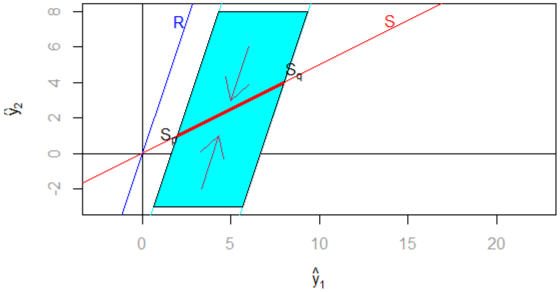

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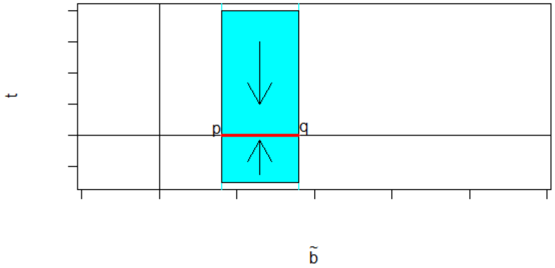

Probabilistic forecast reconciliation *cont..*

- Let $\hat{\boldsymbol{f}}(\cdot)$ be the probability density of $\hat{\boldsymbol{y}}_{t+h}$

Probabilistic forecast reconciliation *cont..*

- Let $\hat{\boldsymbol{f}}(\cdot)$ be the probability density of $\hat{\boldsymbol{y}}_{t+h}$
- Let $\boldsymbol{f}_{\boldsymbol{B}}(\cdot)$ be the density of the coordinates of $\hat{\boldsymbol{y}}_{t+h}$ with respect to basis $\boldsymbol{B}$

Probabilistic forecast reconciliation *cont..*

- Let $\hat{\boldsymbol{f}}(\cdot)$ be the probability density of $\hat{\boldsymbol{y}}_{t+h}$
- Let $\boldsymbol{f_B}(\cdot)$ be the density of the coordinates of $\hat{\boldsymbol{y}}_{t+h}$ with respect to basis $\boldsymbol{B}$
- Recall that $(\boldsymbol{S} \vdots \boldsymbol{R})(\tilde{\boldsymbol{b}}'_{t+h}, \ \boldsymbol{t}'_{t+h})' = \hat{\boldsymbol{y}}_{t+h}$, and $\hat{\boldsymbol{y}}_{t+h} = \boldsymbol{S}\tilde{\boldsymbol{b}}_{t+h} + \boldsymbol{R}\boldsymbol{t}_{t+h}$

Probabilistic forecast reconciliation *cont..*

- Let $\hat{\boldsymbol{f}}(\cdot)$ be the probability density of $\hat{\boldsymbol{y}}_{t+h}$
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  Then we have

$$\boldsymbol{f_B}(\tilde{\boldsymbol{b}}_{t+h}, \boldsymbol{t}_{t+h}) = \hat{\boldsymbol{f}}(\boldsymbol{S}\tilde{\boldsymbol{b}}_{t+h} + \boldsymbol{R}\boldsymbol{t}_{t+h}) \left|\boldsymbol{S} \vdots \boldsymbol{R}\right|$$

Probabilistic forecast reconciliation *cont..*

- Let $\hat{\boldsymbol{f}}(\cdot)$ be the probability density of $\hat{\boldsymbol{y}}_{t+h}$
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- **Step 2: Marginalizing over the null space**

Probabilistic forecast reconciliation *cont..*

- Let $\hat{\boldsymbol{f}}(\cdot)$ be the probability density of $\hat{\boldsymbol{y}}_{t+h}$
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- **Step 2: Marginalizing over the null space**
  i.e.,

$$\tilde{\boldsymbol{f}}_b(\tilde{\boldsymbol{b}}_{t+h}) = \int \hat{\boldsymbol{f}}(\boldsymbol{S}\tilde{\boldsymbol{b}}_{t+h} + \boldsymbol{R}\boldsymbol{t}_{t+h}) \left|\boldsymbol{S} \vdots \boldsymbol{R}\right| d\boldsymbol{t}_{t+h}$$

Probabilistic forecast reconciliation *cont..*

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- **Step 3: Reconciled density of the whole hierarchy**

$$\tilde{\boldsymbol{f}}(\tilde{\boldsymbol{y}}_{t+h}) = \tilde{\boldsymbol{f}}_b(\boldsymbol{S}\tilde{\boldsymbol{b}}_{t+h}), \quad [\tilde{\boldsymbol{y}}_{t+h} = \boldsymbol{S}\tilde{\boldsymbol{b}}_{t+h}]$$

Probabilistic forecasts reconciliation assuming Gaussianity

- Assuming Gaussianity, let an incoherent forecast distribution at time $T+h$ be given by

$$\mathscr{N}(\hat{\boldsymbol{y}}_{T+h}, \boldsymbol{W}_{T+h}) \overset{d}{\leftrightarrow} \hat{\boldsymbol{f}}(.)$$

where $\hat{\boldsymbol{y}}_{T+h}$ is the incoherent mean and $\boldsymbol{W}_{T+h} = E_{\boldsymbol{y}_{T+h}}[(\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h})(\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h})^T | \mathcal{I}_T]$ is incoherent variance

Probabilistic forecasts reconciliation assuming Gaussianity

- Assuming Gaussianity, let an incoherent forecast distribution at time $T+h$ be given by

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- Gaussian densities are closed under affine transformation and marginalization

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- Gaussian densities are closed under affine transformation and marginalization
- From the linear transformation

$$\boldsymbol{f_B}(\cdot) = \frac{\exp\left\{-\frac{1}{2}\left((\boldsymbol{S} \vdots \boldsymbol{R})\begin{pmatrix}\tilde{\boldsymbol{b}}_{T+h} \\ \boldsymbol{t}_{T+h}\end{pmatrix} - \hat{\boldsymbol{y}}_{T+h}\right)' \boldsymbol{W}_{\boldsymbol{T+h}}^{-1}\left((\boldsymbol{S} \vdots \boldsymbol{R})\begin{pmatrix}\tilde{\boldsymbol{b}}_{T+h} \\ \boldsymbol{t}_{T+h}\end{pmatrix} - \hat{\boldsymbol{y}}_{T+h}\right)\right\}}{(2\pi)^{\frac{n}{2}} \left|\boldsymbol{W}_{T+h}\right|^{\frac{1}{2}} \left|(\boldsymbol{S} \vdots \boldsymbol{R})^{-1}\right|}$$

Probabilistic forecasts reconciliation assuming Gaussianity

- Marginalizing over $\boldsymbol{t}_{T+h}$ A1

$$\tilde{\boldsymbol{f}}_b(\tilde{\boldsymbol{b}}_{T+h}) = \frac{\exp\left\{-\frac{1}{2}(\tilde{\boldsymbol{b}}_{T+h} - \boldsymbol{P}\hat{\boldsymbol{y}}_{T+h})'(\boldsymbol{P}\boldsymbol{W}_{\boldsymbol{T+h}}\boldsymbol{P}')^{-1}(\tilde{\boldsymbol{b}}_{T+h} - \boldsymbol{P}\hat{\boldsymbol{y}}_{T+h})\right\}}{(2\pi)^{\frac{n}{2}}\left|\boldsymbol{P}\boldsymbol{W}_{\boldsymbol{T+h}}\boldsymbol{P}'\right|^{\frac{1}{2}}}$$

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- Therefore the reconciled Gaussian distribution of the whole hierarchy is given by

$$\tilde{\boldsymbol{f}}(\tilde{\boldsymbol{y}}_{t+h}) = \tilde{\boldsymbol{f}}_b(\boldsymbol{S}\tilde{\boldsymbol{b}}_{T+h}) \overset{d}{\leftrightarrow} \mathscr{N}(\boldsymbol{S}\boldsymbol{P}\hat{\boldsymbol{y}}_{T+h},\ \boldsymbol{S}\boldsymbol{P}\boldsymbol{W}_{T+h}\boldsymbol{P}'\boldsymbol{S}'),$$

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where $\boldsymbol{P} = (\boldsymbol{R}_\perp'\boldsymbol{S})^{-1}\boldsymbol{R}_\perp'$

- **Goal**: Solve for $\boldsymbol{P}$ by minimizing a proper scoring rule

Probabilistic forecasts reconciliation assuming Gaussianity

- We use the Energy Score, which is a proper scoring rule

$$ES(\tilde{\boldsymbol{Y}}_{\boldsymbol{T+h}}, \boldsymbol{y}_{\boldsymbol{T+h}}) = E_{\boldsymbol{Y}_{T+h}}||\tilde{\boldsymbol{Y}}_{T+h} - \boldsymbol{y}_{T+h}||^{\alpha} - \frac{1}{2}E_{\boldsymbol{Y}_{T+h}}||\tilde{\boldsymbol{Y}}_{T+h} - \tilde{\boldsymbol{Y}}^{*}_{T+h}||^{\alpha},$$

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- There is no closed form expression for $ES(\tilde{\boldsymbol{Y}}_{\boldsymbol{T+h}}, \boldsymbol{y}_{\boldsymbol{T+h}})$ for $\alpha \in (0, 2)$ under the Gaussian predictive distribution

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- However in the upper limit of α, i.e. when $\alpha = 2$,

$$ES(\tilde{\boldsymbol{Y}}_{\boldsymbol{T+h}}, \boldsymbol{y}_{\boldsymbol{T+h}}) = ||\boldsymbol{SP}\hat{\boldsymbol{y}}_{T+h} - \boldsymbol{y}_{T+h}||^{2}$$

Probabilistic forecasts reconciliation assuming Gaussianity

- Then our objective function is

$$\arg\min_{\boldsymbol{P}} Tr\{E_{\boldsymbol{y}_{T+h}}[(\boldsymbol{y}_{T+h} - \boldsymbol{SP}\hat{\boldsymbol{y}}_{T+h})(\boldsymbol{y}_{T+h} - \boldsymbol{SP}\hat{\boldsymbol{y}}_{T+h})^T|\mathcal{I}_T]\},$$

where $\mathcal{I}_T = \{\boldsymbol{y}_1, ..., \boldsymbol{y}_T\}$

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- Assuming unbiasedness for coherent forecasts

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$$\underset{\boldsymbol{P}}{\arg\min}\ Tr\{\boldsymbol{SPW}_{T+h}\boldsymbol{P}^T\boldsymbol{S}^T\}$$

$$\text{Subject to} \quad \boldsymbol{SPS} = \boldsymbol{S}$$

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- Thus $\boldsymbol{R}'_{\perp} = \boldsymbol{S}'\boldsymbol{W}_{T+h}^{-1}$. This is referred to as MinT solution

Probabilistic forecasts in the Gaussian framework

$$\boldsymbol{R}_{\perp}' = \boldsymbol{S}'\boldsymbol{W}_{T+h}^{-1}$$

Probabilistic forecasts in the Gaussian framework

$\boldsymbol{R}_{\perp}' = \boldsymbol{S}'\boldsymbol{W}_{T+h}^{-1}$

- The optimally reconciled Gaussian forecast density is given by $\mathcal{N}[\boldsymbol{S}(\boldsymbol{R}_{\perp}'\boldsymbol{S})^{-1}\boldsymbol{R}_{\perp}'\hat{\boldsymbol{y}}_{T+h}, \boldsymbol{S}(\boldsymbol{R}_{\perp}'\boldsymbol{S})^{-1}\boldsymbol{R}_{\perp}'\boldsymbol{W}_{T+h}\boldsymbol{R}_{\perp}(\boldsymbol{R}_{\perp}'\boldsymbol{S})^{-1}\boldsymbol{S'}]$,

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- Different estimators of $\boldsymbol{W}_{T+h}$ yield different $\boldsymbol{R}_{\perp}'$ matrices

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Method	**Estimate of $\boldsymbol{W}_h$**	**Estimate of $\boldsymbol{R}'_{\perp}$**
OLS	$\boldsymbol{I}$	$\boldsymbol{S}'$
MinT(Sample)	$\hat{\boldsymbol{W}}^{sam}_{T+1}$	$\boldsymbol{S}'(\hat{\boldsymbol{W}}^{sam}_{T+1})^{-1}$
MinT(Shrink)	$\hat{\boldsymbol{W}}^{shr}_{T+1} = \tau \text{Diag}(\hat{\boldsymbol{W}}^{sam}_{T+1}) + (1-\tau)\hat{\boldsymbol{W}}^{sam}_{T+1}$ A2	$\boldsymbol{S}'(\hat{\boldsymbol{W}}^{shr}_{T+1})^{-1}$
MinT(WLS)	$\hat{\boldsymbol{W}}^{wls}_{T+1} = \text{Diag}(\hat{\boldsymbol{W}}^{shr}_{T+1})$	$\boldsymbol{S}'(\hat{\boldsymbol{W}}^{wls}_{T+1})^{-1}$

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OLS	$\boldsymbol{I}$	$\boldsymbol{S}'$
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$\boldsymbol{R}'_{\perp} = \boldsymbol{S}'\boldsymbol{W}_{T+h}^{-1}$

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OLS	$\boldsymbol{I}$	$\boldsymbol{S}'$
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MinT(WLS)	$\hat{\boldsymbol{W}}_{T+1}^{wls} = \text{Diag}(\hat{\boldsymbol{W}}_{T+1}^{shr})$	$\boldsymbol{S}'(\hat{\boldsymbol{W}}_{T+1}^{wls})^{-1}$

- Predictive ability of the reconciled Gaussian densities from these methods will be evaluated in a simulation setting

Evaluating probabilistic forecasts

Energy score (Gneiting et al., 2008)

$$\text{eS}(\breve{\boldsymbol{Y}}_{T+h}, \boldsymbol{y}_{T+h}) = \text{E}_{\breve{\boldsymbol{F}}}\|\breve{\boldsymbol{Y}}_{T+h} - \boldsymbol{y}_{T+h}\|^{\alpha} - \tfrac{1}{2}\text{E}_{\breve{\boldsymbol{F}}}\|\breve{\boldsymbol{Y}}_{T+h} - \breve{\boldsymbol{Y}}^{*}_{T+h}\|^{\alpha}, \quad \alpha \in (0, 2]$$

Log score (Gneiting and Raftery, 2007)

$$\text{LS}(\breve{\boldsymbol{F}}, \boldsymbol{y}_{T+h}) = -\log \breve{\boldsymbol{f}}(\boldsymbol{y}_{T+h})$$

Variogram score (**SCHEUERER˙2015**)

$$\text{VS}(\breve{\boldsymbol{F}}, \boldsymbol{y}_{T+h}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\left(|y_{T+h,i} - y_{T+h,j}|^{p} - \text{E}_{\breve{\boldsymbol{F}}}|\breve{Y}_{T+h,i} - \breve{Y}_{T+h,j}|^{p}\right)^{2}$$

CRPS (Gneiting and Raftery, 2007)

$$\text{CRPS}(\breve{F}_i, y_{T+h,i}) = \text{E}_{\breve{F}_i}|\breve{Y}_{T+h,i} - y_{T+h,i}| - \tfrac{1}{2}\text{E}_{\breve{F}_i}|\breve{Y}_{T+h,i} - \breve{Y}^{*}_{T+h,i}|$$

$\breve{\boldsymbol{Y}}_{T+h}$ and $\breve{\boldsymbol{Y}}^{*}_{T+h}$	:	Independent random vectors from the coherent forecast distribution $\breve{\boldsymbol{F}}$.
$\boldsymbol{y}_{T+h}$	:	Vector of realizations.
$\breve{Y}_{T+h,i}$ and $\breve{Y}_{T+h,j}$	:	ith and jth components of the vector $\breve{\boldsymbol{Y}}_{T+h}$
w_{ij}	:	Non-negative weights

Evaluating probabilistic forecasts *Cont..*

- To compare two competitive forecasting methods, we use Skill score (Gneiting and Raftery, 2007)

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Evaluating probabilistic forecasts *Cont..*

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- For a given forecasting method, evaluated by a particular scoring rule $S(\cdot)$, the skill score is given by

$$SS[S_B(\cdot)] = \frac{S_B(\boldsymbol{Y}, \boldsymbol{y})^{\text{ref}} - S_B(\check{\boldsymbol{Y}}, \boldsymbol{y})}{S_B(\boldsymbol{Y}, \boldsymbol{y})^{\text{ref}}} \times 100\%,$$

where

$S_B(\cdot)$	:	Average score over B samples
$S_B(\boldsymbol{Y}, \boldsymbol{y})^{\text{ref}}$	:	Average score of the reference forecasting methods

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- To compare two competitive forecasting methods, we use Skill score (Gneiting and Raftery, 2007)

- For a given forecasting method, evaluated by a particular scoring rule $S(\cdot)$, the skill score is given by

$$SS[S_B(\cdot)] = \frac{S_B(\boldsymbol{Y}, \boldsymbol{y})^{\text{ref}} - S_B(\check{\boldsymbol{Y}}, \boldsymbol{y})}{S_B(\boldsymbol{Y}, \boldsymbol{y})^{\text{ref}}} \times 100\%,$$

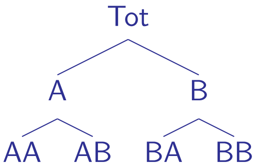
where

$S_B(\cdot)$	:	Average score over B samples
$S_B(\boldsymbol{Y}, \boldsymbol{y})^{\text{ref}}$	:	Average score of the reference forecasting methods

- $SS[S_B(\cdot)]$ gives the percentage improvement of the preferred forecasting method relative to the reference method

Monte-Carlo simulation

- **Data generating process** A3



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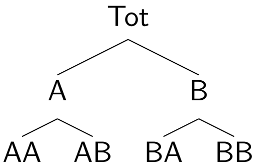

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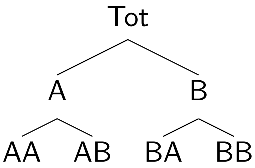

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Monte-Carlo simulation

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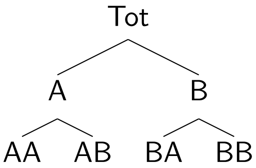

- DGP was designed such that we have noisier series in the bottom levels than in the aggregate levels
- **Simulation setup**
  - 501 observations were generated

Monte-Carlo simulation

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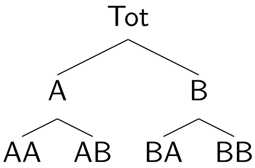

- DGP was designed such that we have noisier series in the bottom levels than in the aggregate levels
- **Simulation setup**
  - 501 observations were generated
  - Univariate ARIMA models were fitted to the first 500 observations and 1- step ahead incoherent forecasts were generated

Monte-Carlo simulation

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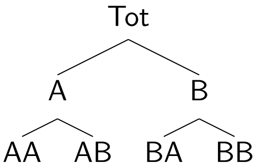

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  - 501 observations were generated
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  - Mean and variance forecasts of incoherent Gaussian densities were reconciled through different estimates of $\boldsymbol{R}'_{\perp}$

Monte-Carlo simulation

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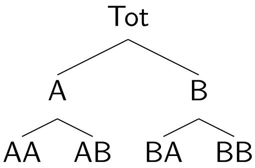

- DGP was designed such that we have noisier series in the bottom levels than in the aggregate levels
- **Simulation setup**
  - 501 observations were generated
  - Univariate ARIMA models were fitted to the first 500 observations and 1- step ahead incoherent forecasts were generated
  - Mean and variance forecasts of incoherent Gaussian densities were reconciled through different estimates of $\boldsymbol{R}'_{\perp}$
  - This process was replicated using 1000 different series from the same (DGP)

Monte-Carlo simulation *Cont..*

- Comparison of reconciled vs bottom-up Gaussian forecasts

Forecasting method	Energy score		Log score		Variogram score	
	Average score	Skill score (%)	Average score	Skill score (%)	Average score	Skill score (%)
MinT(Shrink)	7.47	10.11	11.34	6.44	3.05	4.69
MinT(Sample)	7.47	10.11	11.33	6.52	3.05	4.69
MinT(WLS)	7.91	4.81	12.64	−4.29	3.23	−0.94
OLS	10.14	−22.02	135.13	−1014.93	4.60	−43.75
Bottom up	8.31		12.12		3.20	

Note: *Positive(/negative) entry in "Skill score (%)" column represent the percentage increase(/decrease) of preferred forecasting method over Bottom-up method.*

Monte-Carlo simulation *Cont..*

- Comparison of coherent vs incoherent Gaussian forecasts

Forecasting method	Total CRPS	Total LogS	Series - A CRPS	Series - A LogS	Series - B CRPS	Series - B LogS
MinT(Shrink)	1.12	0.34	10.07	2.93	5.41	1.52
MinT(Sample)	1.12	0.34	10.07	2.93	5.41	1.52
MinT(WLS)	−2.61	−2.02	5.28	−4.40	2.70	−4.24
OLS	−38.06	−698.99	−24.70	−1368.33	−24.86	−1159.09
Bottom up	−89.55	−21.83	−8.87	−2.35	−9.46	−2.73
Incoherent	*2.68*	*2.97*	*4.17*	*3.41*	*3.70*	*3.30*

Forecasting method	Series - AA CRPS	Series - AA LogS	Series - AB CRPS	Series - AB LogS	Series - BA CRPS	Series - BA LogS	Series - BB CRPS	Series - BB LogS
MinT(Shrink)	8.71	2.71	10.57	3.04	5.95	1.86	7.91	2.46
MinT(Sample)	8.71	2.71	10.57	3.04	5.95	1.86	8.19	2.46
MinT(WLS)	5.54	0.30	5.96	0.30	2.43	−0.62	5.08	0.62
OLS	−22.43	−931.63	−22.49	−886.32	−26.01	−834.67	−23.45	−812.92
Incoherent	*3.79*	*3.32*	*3.69*	*3.29*	*3.46*	*3.23*	*3.54*	*3.25*

Note: *"Incoherent" row represents the average score for incoherent forecasts.*
Each entry above this row represent the percentage skill score with reference to the Incoherent forecasts.

Probabilistic forecasts in a non-parametric framework: A bootstrap approach

Probabilistic forecast reconciliation in a non-parametric framework: A bootstrap approach

1. Fit univariate models at each node using data up to time T

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Probabilistic forecast reconciliation in a non-parametric framework: A bootstrap approach

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2. Let $\Gamma_{(T\times n)} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_T)'$ be a matrix of in-sample residuals where $\boldsymbol{e}_t = \boldsymbol{y}_t - \hat{\boldsymbol{y}}_t$

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3. Let $\boldsymbol{E}^b_{(h\times n)} = (\boldsymbol{e}_{i+1}, \boldsymbol{e}_{i+2}, \dots, \boldsymbol{e}_{i+h})'$ be a block bootstrap sample of size h from $\boldsymbol{\Gamma}$

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3. Let $\boldsymbol{E}^b_{(h\times n)} = (\boldsymbol{e}_{i+1}, \boldsymbol{e}_{i+2}, \dots, \boldsymbol{e}_{i+h})'$ be a block bootstrap sample of size h from Γ
4. Generate h-step ahead sample paths from the fitted models incorporating $\boldsymbol{E}^b$. Denote these by $\boldsymbol{y}^b_{T+h}$
5. Repeat step 3-5 for J times
6. $\tilde{\boldsymbol{y}}^b_{T+h} = \boldsymbol{S}(\boldsymbol{R}'_\perp \boldsymbol{S})^{-1}\boldsymbol{R}'_\perp \boldsymbol{y}^b_{T+h}$,
   where $\tilde{\boldsymbol{y}}^b_{T+h}$ denotes h-step ahead coherent sample paths

Analyzing the predictive performance of probabilistic forecasts in a non-parametric framework

- **Simulation setup**

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  - 5000 reconciled future sample paths were generated for forecast horizons $h = 1, ..., 5$

Analyzing the predictive performance of probabilistic forecasts in a non-parametric framework

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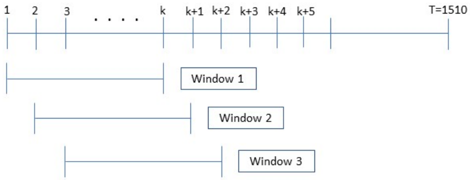

Analyzing the predictive performance of probabilistic forecasts in a non-parametric framework

- Comparison of reconciled vs bottom-up forecasts

Forecasting method	h=1 ES	h=1 VS	h=2 ES	h=2 VS	h=3 ES	h=3 VS	h=4 ES	h=4 VS	h=5 ES	h=5 VS
MinT(Shrink)	6.39	5.83	6.48	5.54	5.62	5.43	3.94	3.98	3.45	3.64
MinT(Sample)	6.39	5.83	6.35	5.19	5.29	4.89	3.47	3.09	3.04	2.73
MinT(WLS)	3.83	0.89	3.89	1.04	3.35	1.09	2.44	1.33	2.06	1.28
OLS	3.35	0	3.24	0.35	2.69	0.27	2.06	0.66	1.73	0.73
Bottom Up	*6.26*	*2.23*	*7.72*	*2.89*	*9.26*	*3.68*	*10.67*	*4.52*	*12.16*	*5.49*

Note: *"Bottom-up" row represent the average score for incoherent forecasts. Each entry above this row represent the percentage skill score with reference to the Bottom-up forecasts.*

Analyzing the predictive performance of probabilistic forecasts in a non-parametric framework *Cont..*

- Comparison of coherent vs incoherent forecasts

Forecasting method	Total	Series-A	Series-B	Series-AA	Series-AB	Series-BA	Series-BB
				h=1			
MinT(Shrink)	0.28	7.17	6.57	10.04	2.93	7.35	0.41
MinT(Sample)	0.29	7.10	6.45	10.04	3.02	7.22	0.32
MinT(WLS)	−1.78	6.46	5.80	5.17	−0.03	6.37	−0.68
OLS	−6.30	5.62	5.49	4.50	0.09	5.42	−1.59
Bottom Up	−69.68	−5.25	−4.51	0.00	0.00	0.00	0.00
Incoherent	*1.86*	*2.73*	*2.68*	*2.93*	*2.78*	*2.98*	*2.35*
				h=5			
MinT(Shrink)	0.63	6.30	0.67	7.56	0.35	1.63	−2.01
MinT(Sample)	0.56	6.39	0.58	7.15	0.02	1.29	−3.47
MinT(WLS)	1.24	5.45	0.62	5.39	−3.10	2.41	−3.36
OLS	1.16	5.07	1.20	4.64	−2.78	2.43	−5.33
Bottom Up	−9.44	−0.52	−1.42	0.00	0.00	0.00	0.00
Incoherent	*8.72*	*7.46*	*5.96*	*6.78*	*4.66*	*5.95*	*2.68*
Note:	*"Incoherent" row represent the average score for incoherent forecasts. Each entry above this row represent the percentage skill score with reference to the Incoherent forecasts.*						

Probabilistic forecasts for Australian domestic tourism flows

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- Domestic tourism in Australia can be disaggregated based on a geographical hierarchy

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Geographical hierarchical structure for Australia

Level	**No.Series per level**
Total (Australia)	1
Level-1 (States)	7
Level-2 (Zones)	27
Level-3 (Regions)	75

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- Reconciliation of point forecasts (Athanasopoulos, Ahmed, and Hyndman, 2009, Hyndman et al., 2011, Wickramasuriya, Athanasopoulos, and Hyndman, 2018)
- We use a non-parametric bootstrap approach to obtain reconciled probabilistic forecasts

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- We consider monthly "overnight trips" over the period January 1998 - December 2016

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  - Univariate ARIMA models were fitted using a rolling window of 100 observations

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  - Univariate ARIMA models were fitted using a rolling window of 100 observations
  - 5000 reconciled future sample paths were generated for forecast horizons $h = 1, ..., 5$

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  - Univariate ARIMA models were fitted using a rolling window of 100 observations
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  - The training window was rolled one observation ahead and the process repeated until we get 123 reconciled future sample paths

Probabilistic forecasts for Australian domestic tourism flow

- Comparison of reconciled vs. bottom-up probabilistic forecasts

Forecasting method	h=1 ES	h=1 VS	h=2 ES	h=2 VS	h=3 ES	h=3 VS	h=4 ES	h=4 VS	h=5 ES	h=5 VS
States										
MinT(Shrink)	19.58	27.95	14.65	23.12	12.05	21.25	7.76	14.09	9.50	15.22
MinT(WLS)	4.50	6.58	4.37	6.31	3.32	6.01	1.89	3.52	3.58	4.67
OLS	7.00	3.15	5.58	5.01	2.65	3.31	1.83	−0.66	3.58	2.01
Bottom Up	*345.63*	*271.05*	*348.68*	*281.63*	*355.98*	*290.47*	*367.41*	*300.25*	*364.81*	*299.75*
Zones										
MinT(Shrink)	13.11	19.30	10.94	19.48	9.40	18.24	6.99	15.10	7.62	15.26
MinT(WLS)	2.32	1.99	2.55	2.99	1.95	2.61	1.05	1.28	1.27	1.97
OLS	1.91	−5.82	1.58	−2.36	0.47	−1.78	0.25	−2.07	1.29	−1.57
Bottom Up	*228.70*	*2366.28*	*229.54*	*2440.32*	*234.09*	*2510.51*	*238.98*	*2556.95*	*237.98*	*2552.26*
Regions										
MinT(Shrink)	10.43	15.96	9.03	16.11	7.87	15.05	6.28	12.81	6.96	13.21
MinT(WLS)	1.54	1.43	1.75	2.16	1.30	1.83	0.68	0.95	0.97	1.46
OLS	1.37	−3.01	1.28	−0.77	0.49	−0.90	0.30	−1.08	1.14	−0.18
Bottom Up	*194.19*	*10956.41*	*193.50*	*11145.87*	*196.26*	*11355.30*	*198.25*	*11461.30*	*197.76*	*11438.31*

Note: *"Bottom-up" row represent the average score for incoherent forecasts. Each entry above this row represent the percentage skill score with reference to the Bottom-up forecasts.*

Probabilistic forecasts for Australian domestic tourism flow

- Comparison of coherent vs. incoherent probabilistic forecasts in total Australia and each states

Forecasting method	Australia	NSW	Victoria	Queensland	South Australia	Western Australia	Tasmania	Northern Territory
				h=1				
MinT(Shrink)	8.07	9.61	16.84	22.38	23.38	11.25	13.38	20.41
MinT(WLS)	−12.31	−7.54	2.64	4.95	1.12	1.31	6.93	2.55
OLS	1.39	2.96	5.14	7.68	1.58	−0.48	−13.04	−22.4
Bottom Up	−20.00	−16.22	−1.12	3.07	−4.64	−6.52	4.43	−0.96
Incoherent	*478.81*	*188.48*	*159.49*	*136.27*	*51.29*	*52.97*	*35.23*	*24.27*
				h=5				
MinT(Shrink)	1.72	6.65	11.96	6.30	14.61	3.66	16.54	24.58
MinT(WLS)	−6.83	−0.30	2.01	2.21	2.26	−1.77	1.40	6.45
OLS	−0.29	2.20	3.41	4.23	9.50	2.58	9.99	−4.52
Bottom Up	−9.07	−3.33	0.24	1.48	0.44	−7.35	−2.59	4.70
Incoherent	*552.45*	*217.09*	*170.55*	*145.01*	*54.99*	*59.59*	*38.76*	*33.87*

Note: *"Incoherent" row represent the average score for incoherent forecasts. Each entry above this row represent the percentage skill score with reference to the Incoherent forecasts.*

Probabilistic forecasts for Australian domestic tourism flow

- Comparison of coherent vs. incoherent probabilistic forecasts in Zones and Regions

Forecasting	Zones					Regions				
method	h=1	h=2	h=3	h=4	h=5	h=1	h=2	h=3	h=4	h=5
MinT(Shrink)	14.38	11.25	10.20	8.67	9.39	9.96	8.58	7.71	6.53	6.98
MinT(WLS)	2.88	2.15	1.85	1.76	1.75	1.54	1.73	1.42	0.82	0.94
OLS	1.79	0.55	−0.12	1.00	2.22	−0.11	−0.07	−0.51	−0.20	0.74
Bottom Up	0.21	−0.66	−0.43	0.35	0.26					
Incoherent	*31.75*	*31.87*	*32.71*	*33.73*	*33.49*	*14.36*	*14.45*	*14.70*	*14.87*	*14.81*

Note: *"Incoherent" row represent the average score for incoherent forecasts. Each entry above this row represent the percentage skill score with reference to the Incoherent forecasts.*

Probabilistic forecasts for Australian domestic tourism flow

- Comparison of coherent vs. incoherent mean forecasts

Forecasting	Australia					States				
method	h=1	h=2	h=3	h=4	h=5	h=1	h=2	h=3	h=4	h=5
MinT(Shrink)	11.09	5.56	4.61	2.17	1.51	16.64	11.28	12.19	6.39	8.10
MinT(WLS)	−9.90	−5.33	−2.17	−0.71	−5.48	0.08	0.26	1.31	−0.16	0.41
OLS	1.62	1.53	1.05	1.16	0.12	2.26	0.47	0.43	−0.22	3.06
Bottom Up	−18.88	−12.33	−7.45	−3.66	−8.23	−5.86	−4.64	−2.60	−2.64	−2.16
Incoherent	*669.21*	*697.51*	*744.54*	*812.17*	*764.24*	*128.71*	*132.00*	*138.92*	*142.92*	*142.07*
	Zones					Regions				
	h=1	h=2	h=3	h=4	h=5	h=1	h=2	h=3	h=4	h=5
MinT(Shrink)	14.64	11.15	10.71	8.58	9.16	10.87	8.81	8.05	6.71	6.88
MinT(WLS)	2.77	2.13	1.92	1.59	1.68	1.75	1.80	1.54	0.88	0.95
OLS	1.59	0.55	0.04	0.55	1.80	−0.07	−0.17	−0.53	−0.41	0.31
Bottom Up	−0.09	−0.90	−0.66	0.26	0.28					
Incoherent	*43.95*	*44.38*	*45.58*	*47.01*	*46.53*	*20.11*	*20.23*	*20.58*	*20.85*	*20.72*

Note: *"Incoherent" row represent the average MAE for incoherent forecasts. Each entry above this row represent the percentage improvement with reference to the Incoherent forecasts.*

Conclusions and future work

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  Step 1: Transform coordinates of incoherent probabilistic forecasts

  Step 2: Marginalize over the null space of the coherent subspace
- MinT not only produces optimal point forecasts, but also optimally reconcile Gaussian forecast densities

Conclusions

- We introduce a novel non-parametric bootstrap approach for producing reconciled probabilistic forecasts

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- We introduce a novel non-parametric bootstrap approach for producing reconciled probabilistic forecasts
- Results from a simulation study favored MinT for reconciling bootstrapped future paths
- Finally we apply the non-parametric approach to obtain coherent probabilistic forecasts for domestic tourism flow in Australia

Time plan

	Thesis Chapter	Task description	Time duration	Progress
2.	Literature Review	Writing the chapter.	May/2018 - July/2018	Incomplete
3.	Probabilistic forecast reconciliation for hierarchical time series	Completing the paper.	Mar/2018 - Apr/2018	90% Completed
4.	Probabilistic forecast reconciliation in non-parametric framework	Methodology. Simulation study. Providing theoretical foundation.	Nov/2016 - Feb/2017 Mar/2017 - June/2017 Apr/2018 - July/2018	Completed Completed Incomplete
5.	Application	Forecasting Australian domestic tourism flow. Forecasting Walmart sales.	Dec/2017 - Mar/2017 Oct/2017 - Mar/2019	Completed Incomplete

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Thank You!!

Appendix

A1

◂ A1

- By substituting the Gaussian distribution function for $\hat{\boldsymbol{f}}(\cdot)$ in $\boldsymbol{f_B}(\cdot)$ we get,

$$\boldsymbol{f_B}(\cdot) = \frac{1}{(2\pi)^{\frac{n}{2}} \left| \begin{pmatrix} \boldsymbol{P} \\ \boldsymbol{Q} \end{pmatrix} \boldsymbol{W}_{t+h} \begin{pmatrix} \boldsymbol{P} \\ \boldsymbol{Q} \end{pmatrix}' \right|^{\frac{1}{2}}} \exp\Bigg\{ -\frac{1}{2} \begin{pmatrix} \tilde{\boldsymbol{b}}_{t+h} - \boldsymbol{P}\hat{\boldsymbol{y}}_{t+h} \\ \boldsymbol{t}_{t+h} - \boldsymbol{Q}\hat{\boldsymbol{y}}_{t+h} \end{pmatrix}' \left[\begin{pmatrix} \boldsymbol{P} \\ \boldsymbol{Q} \end{pmatrix} \boldsymbol{W}_{t+h} \begin{pmatrix} \boldsymbol{P} \\ \boldsymbol{Q} \end{pmatrix}' \right]^{-1} \begin{pmatrix} \tilde{\boldsymbol{b}}_{t+h} - \boldsymbol{P}\hat{\boldsymbol{y}}_{t+h} \\ \boldsymbol{t}_{t+h} - \boldsymbol{Q}\hat{\boldsymbol{y}}_{t+h} \end{pmatrix} \Bigg\}.$$

where

$$(\boldsymbol{S} \vdots \boldsymbol{R})^{-1} = \begin{pmatrix} (\boldsymbol{R}'_{\perp}\boldsymbol{S})^{-1}\boldsymbol{R}'_{\perp} \\ \cdots \\ (\boldsymbol{S}'_{\perp}\boldsymbol{R})^{-1}\boldsymbol{S}'_{\perp} \end{pmatrix} = \begin{pmatrix} \boldsymbol{P} \\ \boldsymbol{Q} \end{pmatrix},$$

and $\boldsymbol{P} = (\boldsymbol{R}'_{\perp}\boldsymbol{S})^{-1}\boldsymbol{R}'_{\perp}$ and $\boldsymbol{Q} = (\boldsymbol{S}'_{\perp}\boldsymbol{R})^{-1}\boldsymbol{S}'_{\perp}$.

Appendix

- A2 Shrinkage estimator for 1-step ahead base forecast errors

$$\hat{\boldsymbol{\Sigma}}^{shr}_{T+1} = \tau \hat{\boldsymbol{\Sigma}}^{D}_{T+1} + (1-\tau)\hat{\boldsymbol{\Sigma}}_{T+1},$$

where $\hat{\boldsymbol{\Sigma}}^{D}_{T+1}$ is the diagonal matrix comprising diagonal entries of $\hat{\boldsymbol{\Sigma}}_{T+1}$ and

$$\tau = \frac{\sum_{i \neq j} \hat{Var}(\hat{r}_{ij})}{\sum_{i \neq j} \hat{r}^2_{ij}}$$

is a shrinkage parameter. $\hat{r}_{ij}$ is the *ij*-th element of sample correlation matrix. In this estimation, the off-diagonal elements of 1-step ahead sample covariance matrix will be shrunk to zero depending on the sparsity.

Monte-Carlo simulation

- **Data generating process** A3

Monte-Carlo simulation

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Monte-Carlo simulation

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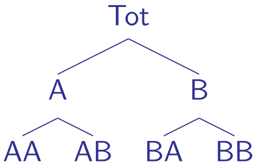

Monte-Carlo simulation

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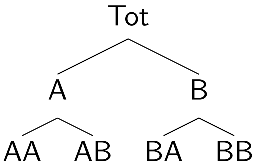

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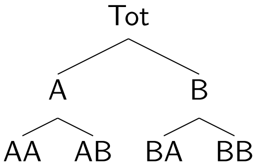
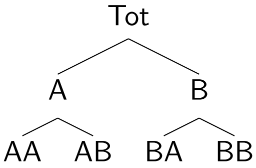


- $\{w_{AA,t}, w_{AB,t}, w_{BA,t}, w_{BB,t}\} \sim ARIMA(p, d, q)$

Monte-Carlo simulation

- **Data generating process** A3



- $\{w_{AA,t}, w_{AB,t}, w_{BA,t}, w_{BB,t}\} \sim ARIMA(p, d, q)$
- $p \in \{1, 2\}$ and $d \in \{0, 1\}$

Monte-Carlo simulation

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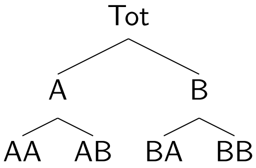

- $\{w_{AA,t}, w_{AB,t}, w_{BA,t}, w_{BB,t}\} \sim ARIMA(p, d, q)$
- $p \in \{1, 2\}$ and $d \in \{0, 1\}$
- $\{\epsilon_{AA,t}, \epsilon_{AB,t}, \epsilon_{BA,t}, \epsilon_{BB,t}\} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$

Monte-Carlo simulation

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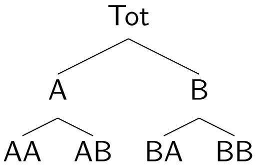

- $\{w_{AA,t}, w_{AB,t}, w_{BA,t}, w_{BB,t}\} \sim ARIMA(p, d, q)$
- $p \in \{1, 2\}$ and $d \in \{0, 1\}$
- $\{\epsilon_{AA,t}, \epsilon_{AB,t}, \epsilon_{BA,t}, \epsilon_{BB,t}\} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$
- Parameters for *AR* and *MA* components were randomly and uniformly generated from $[0.3, 0.5]$ and $[0.3, 0.7]$ respectively

Monte-Carlo simulation

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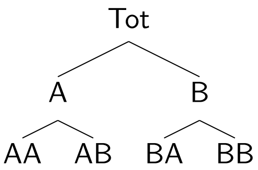

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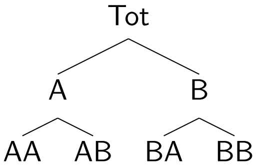

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- $\boldsymbol{y}_t$ are then generated as follows

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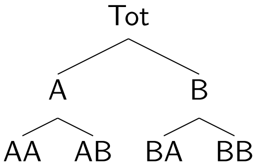

- $\{w_{AA,t}, w_{AB,t}, w_{BA,t}, w_{BB,t}\} \sim ARIMA(p, d, q)$
- $p \in \{1, 2\}$ and $d \in \{0, 1\}$
- $\{\epsilon_{AA,t}, \epsilon_{AB,t}, \epsilon_{BA,t}, \epsilon_{BB,t}\} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$
- Parameters for *AR* and *MA* components were randomly and uniformly generated from $[0.3, 0.5]$ and $[0.3, 0.7]$ respectively

- $\boldsymbol{y}_t$ are then generated as follows

Bottom level	**Aggregate level 1**	**Total**
$y_{AA,t} = w_{AA,t} + u_t - 0.5v_t$	$y_{A,t} = w_{AA,t} + w_{AB,t} - v_t$	$y_{Tot,t} = w_{AA,t} + w_{AB,t} + w_{BA,t} + w_{BB,t}$
$y_{AB,t} = w_{AB,t} - u_t - 0.5v_t$	$y_{B,t} = w_{BA,t} + w_{BB,t} + v_t$	
$y_{BA,t} = w_{BA,t} + u_t + 0.5v_t$		
$y_{BB,t} = w_{BB,t} - u_t + 0.5v_t$		

Monte-Carlo simulation *Cont..*

- To get less noisier series at aggregate levels, we choose $\mathbf{\Sigma}, \sigma_u^2$ and σ_v^2 such that,

$$\mathrm{Var}(\epsilon_{AA,t} + \epsilon_{AB,t} + \epsilon_{BA,t} + \epsilon_{BB,t}) \leq \mathrm{Var}(\epsilon_{AA,t} + \epsilon_{AB,t} - v_t) \leq \mathrm{Var}(\epsilon_{AA,t} + u_t - 0.5v_t),$$

Monte-Carlo simulation *Cont..*

- To get less noisier series at aggregate levels, we choose $\mathbf{\Sigma}, \sigma_u^2$ and σ_v^2 such that,

$$\mathrm{Var}(\epsilon_{AA,t} + \epsilon_{AB,t} + \epsilon_{BA,t} + \epsilon_{BB,t}) \leq \mathrm{Var}(\epsilon_{AA,t} + \epsilon_{AB,t} - v_t) \leq \mathrm{Var}(\epsilon_{AA,t} + u_t - 0.5v_t),$$

- Thus we choose, $\mathbf{\Sigma} = \begin{pmatrix} 5.0 & 3.1 & 0.6 & 0.4 \\ 3.1 & 4.0 & 0.9 & 1.4 \\ 0.6 & 0.9 & 2.0 & 1.8 \\ 0.4 & 1.4 & 1.8 & 3.0 \end{pmatrix}$, $\sigma_u^2 = 19$ and $\sigma_u^2 = 18$.

- For a possible finite sample of size B from the multivariate forecast density $\breve{\boldsymbol{F}}$, the variogram score is defined as,

$$\text{VS}(\breve{\boldsymbol{F}}, \boldsymbol{y}_{T+h}) =$$
$$\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\left(|y_{T+h,i} - y_{T+h,j}|^p - \frac{1}{B}\sum_{k=1}^{B}|\breve{Y}^k_{T+h,i} - \breve{Y}^k_{T+h,j}|^p\right)^2$$