Probabilistic Forecasts for Hierarchical Time Series

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Department of Econometrics and Business Statistics

Monash University

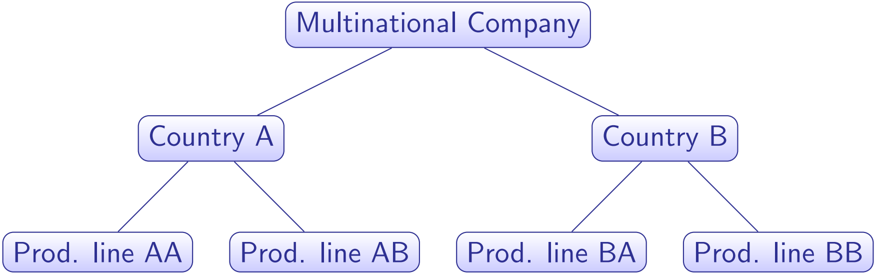
May 23, 2019

Overview

Project 1: Hierarchical Forecast Reconciliation: A Geometric View

Project 1: *Motivation and Objective*

- **Example:**



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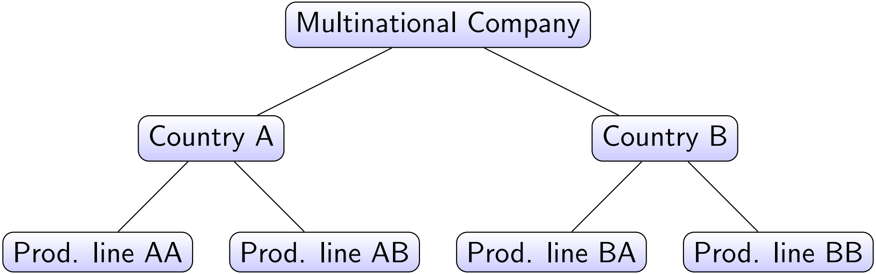

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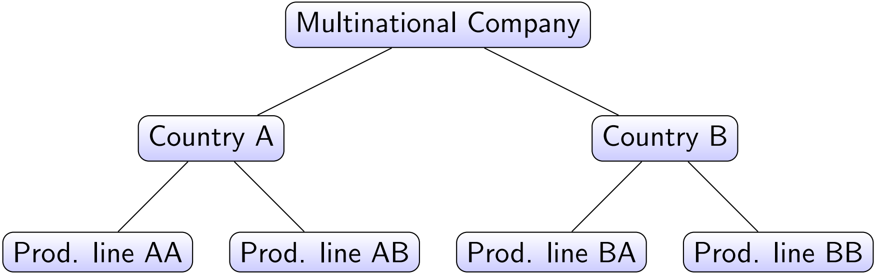

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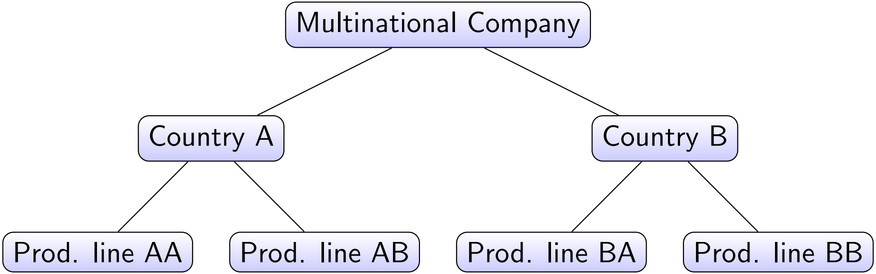

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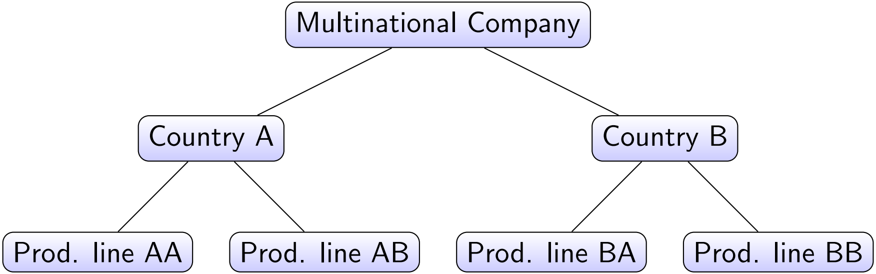

- **Hierarchical time series:** A collection of multiple time series that has an inherent aggregation structure.
- Forecasts should add up. We call these *coherent*.
- **Objective:** Defining coherency and reconciliation of point forecasts in terms of geometric concepts.

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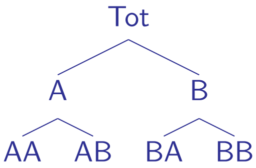

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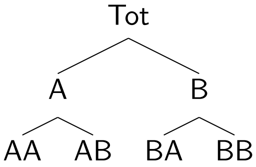

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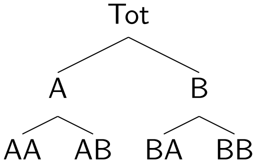

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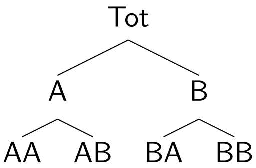

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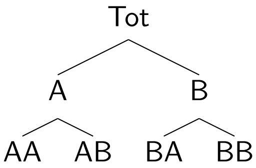

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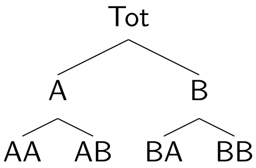

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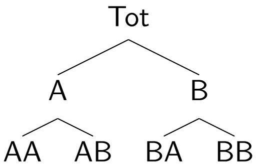

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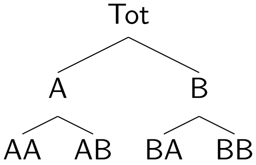

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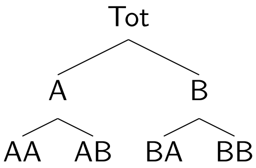

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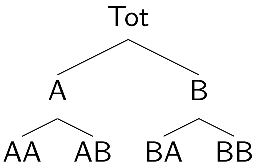

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Coherent subspace

The m-dimensional linear subspace $\mathfrak{s} \subset \mathbb{R}^n$ that is spanned by the columns of $\boldsymbol{S}$, i.e. $\mathfrak{s} = \text{span}(\boldsymbol{S})$, is defined as the *coherent space*.

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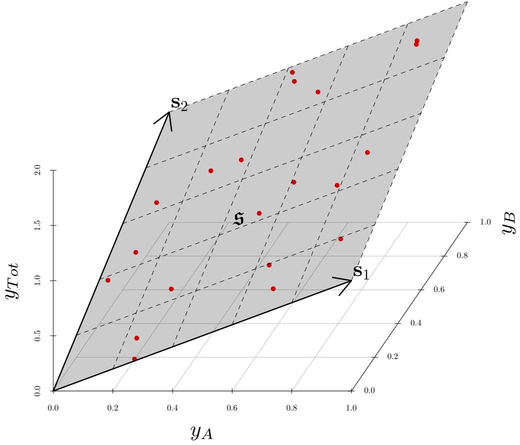

- Three dimensional hierarchy, $y_{Tot} = y_A + y_B$.
- $\vec{s}_1 = (1, 1, 0)'$ and $\vec{s}_2 = (1, 0, 1)'$ form a basis for $\mathfrak{s}$.

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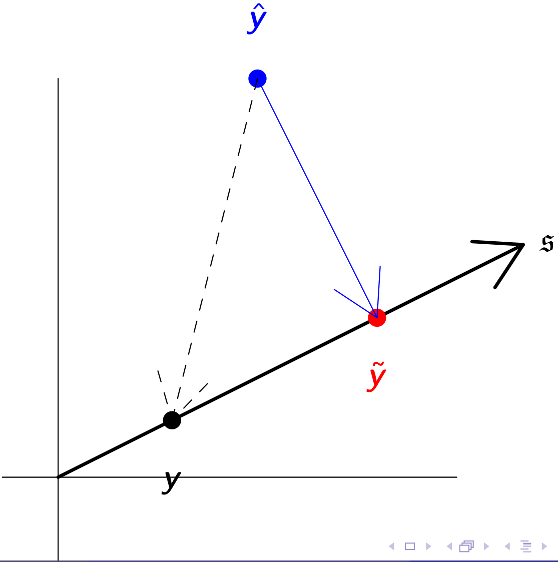

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  - Projecting along this direction is more likely to result in reconciled forecasts that are closer to the target.

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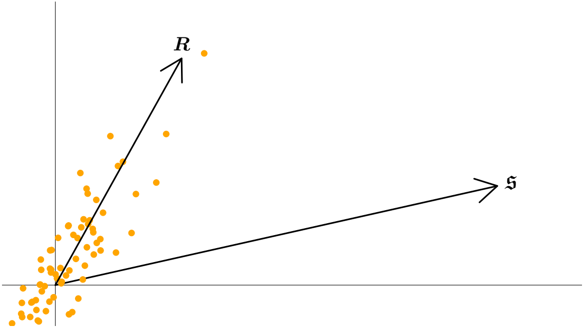

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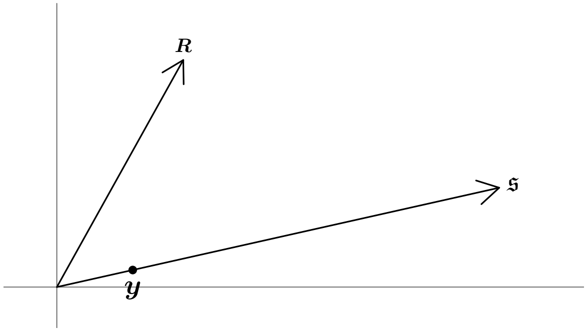

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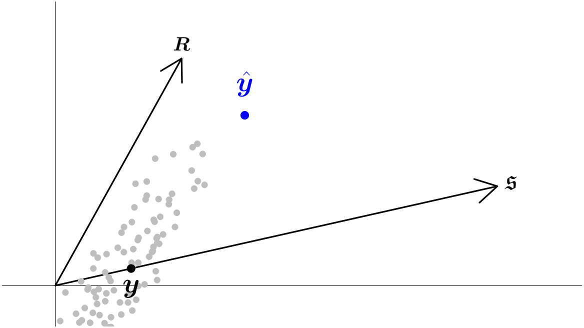

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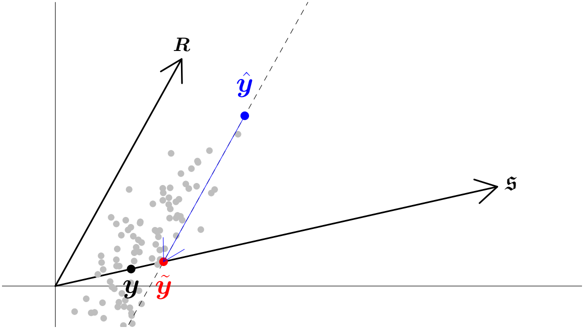

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Part 1: Definitions and A Parametric Approach

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  2. Probabilistic forecast reconciliation in a parametric framework.

Project 2: *Motivation and Objectives*

- Extending "reconciliation" into a probabilistic framework.

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Project 2: *Coherent probabilistic forecasts*

Let $(\mathbb{R}^m, \mathcal{F}_{\mathbb{R}^m}, \nu)$ and $(\mathfrak{s}, \mathcal{F}_{\mathfrak{s}}, \mu)$ be probability triples on m-dimensional space and the coherent subspace respectively.

Definition

The probability measure μ is coherent if

$$\nu(\mathcal{B}) = \mu(s(\mathcal{B})) \quad \forall \mathcal{B} \in \mathcal{F}_{\mathbb{R}_m}$$

where $s(\mathcal{B})$ is the image of $\mathcal{B}$ under premultiplication by $\boldsymbol{S}$

Project 2: *Reconciled Probabilistic Forecast*

Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then

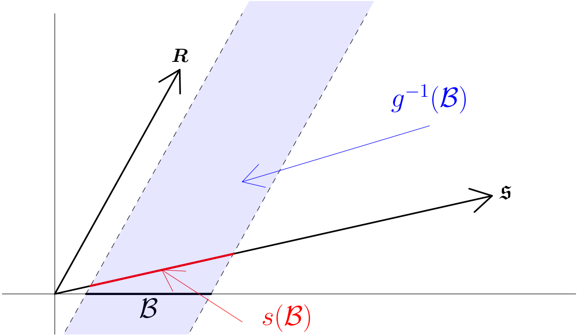
Definition

The probability triple $(\mathfrak{s}, \mathcal{F}_{\mathfrak{s}}, \tilde{\nu})$ reconciles the probability triple $(\mathbb{R}^n, \mathcal{F}_{\mathbb{R}^n}, \hat{\nu})$ with with respect to g iff

$$\tilde{\nu}(s(\mathcal{B})) = \nu(\mathcal{B}) = \hat{\nu}(g^{-1}(\mathcal{B})) \quad \forall \mathcal{B} \in \mathcal{F}_{\mathbb{R}_m}$$

where g^{-1} is the pre-image of g.

Project 2: *Geometry*



$$\tilde{\nu}(s(\mathcal{B})) = \hat{\nu}(g^{-1}(\mathcal{B})) \quad \forall \mathcal{B} \in \mathcal{F}_{\mathbb{R}_m}$$

Project 2: *Analytically*

If we have an unreconciled density the reconciled density can be obtained by linear transformations and marginalisation.

$$
\begin{aligned}
\Pr(\hat{\boldsymbol{y}} \in g^{-1}(\mathcal{B})) &= \int\limits_{g^{-1}(\mathcal{B})} f(\hat{\boldsymbol{y}}) d\hat{\boldsymbol{y}} \\
&= \int\limits_{\mathcal{B}} \int f(\boldsymbol{S}\tilde{\boldsymbol{b}} + \boldsymbol{R}\tilde{\boldsymbol{a}}) |\left(\boldsymbol{S} \ \ \boldsymbol{R}\right)| d\tilde{\boldsymbol{a}} d\tilde{\boldsymbol{b}} \\
&= \Pr(\tilde{\boldsymbol{b}} \in \mathcal{B})
\end{aligned}
$$

Project 2: *Assuming Gaussian distribution*

- Let $\mathscr{N}(\hat{\boldsymbol{y}}_{T+h}, \boldsymbol{W}_{T+h})$ be an incoherent forecast distribution at time $T+h$ where $\hat{\boldsymbol{y}}_{T+h}$ is the incoherent mean and $\boldsymbol{W}_{T+h} = E_{\boldsymbol{y}_{T+h}}[(\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h})(\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h})^T | \mathcal{I}_T]$ is the incoherent variance

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- Simulation study evidence for improved predictive performance in reconciled Gaussian forecast distributions.

Project 2: *For elliptical distributions*

Consider the case where the base and true predictive distributions are elliptical.

Theorem

There exists a matrix $\boldsymbol{G}$ such that the true predictive distribution can be recovered by linear reconciliation.

This follows from the closure property of elliptical distributions under affine transformations and marginalisation.

Today's talk

- A non-parametric bootstrap approach for probabilistic forecast reconciliation.

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- Hierarchical forecasts for macroeconomic variables - An application to Australian GDP.

Part 2: A non-parametric bootstrap approach for probabilistic forecast reconciliation

Probabilistic forecast reconciliation: Non-parametric approach

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- Often parametric densities are unavailable but we can simulate a sample from the predictive distribution.
- Suppose $\hat{\boldsymbol{y}}_{T+h}^{[1]}, \ldots, \hat{\boldsymbol{y}}_{T+h}^{[J]}$ is a sample from the incoherent predictive distribution.
- Then setting $\tilde{\boldsymbol{y}}_{T+h}^{[j]} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h}^{[j]}$ produces a sample from the reconciled predictive distribution with respect to $\boldsymbol{G}$.

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6. Setting $\tilde{\mathbf{\Upsilon}}_{T+h} = \boldsymbol{SG}\hat{\mathbf{\Upsilon}}'_{T+h}$ produces a sample from the reconciled distribution.

Optimal reconciliation of future paths

- We propose to find an optimal $\boldsymbol{G}_h$ matrix by minimizing Energy score.

$$\underset{\boldsymbol{G}_h}{\text{argmin}} \quad \mathrm{E}_{\boldsymbol{Q}}[eS(\tilde{\boldsymbol{F}}, \boldsymbol{y}_{T+h})], \quad \tilde{\boldsymbol{F}} := \tilde{\Upsilon}_{T+h} = \boldsymbol{S}\boldsymbol{G}_h \hat{\Upsilon}'_{T+h}$$

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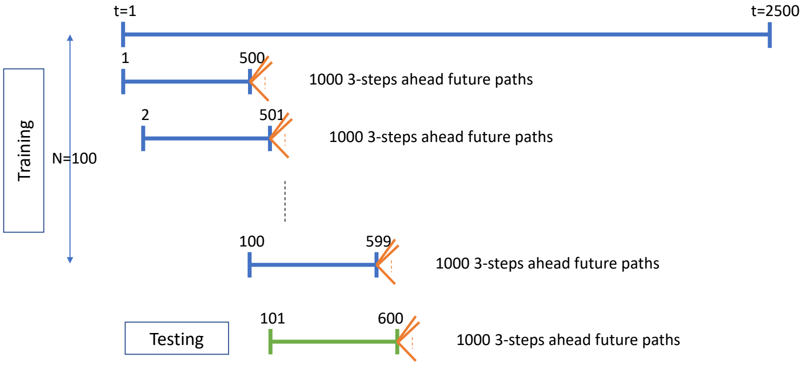
Method 4: Optimising $\boldsymbol{G}_h$ such that $\boldsymbol{G}_h\boldsymbol{S} = \boldsymbol{I}$

Monte-Carlo Simulation

- Data generating process

- DGP was designed such that we have much noisier series in the bottom level.

Monte-Carlo Simulation



Monte-Carlo Simulation *Cont.*

- Simulation setup:
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  - This optimal $\boldsymbol{G}_h$ is then used to reconcile the incoherent future paths for the test set.

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  - This optimal $\boldsymbol{G}_h$ is then used to reconcile the incoherent future paths for the test set.
  - The Process was repeated 1000 times and average scores were calculated for the test set.

Monte-Carlo Simulation *Cont.*

Optimisation method	Hierarchy 1				Hierarchy 2			
	$h = 1$		$h = 3$		$h = 1$		$h = 3$	
	ES	VS	ES	VS	ES	VS	ES	VS
Method 1 - Optimising $\boldsymbol{W}$	2.48	0.11	2.75	0.11	5.36	1.21	5.83	1.38
Method 2 - Optimising $\boldsymbol{R}$	2.48	0.11	2.75	0.11	5.37	1.21	5.83	1.37
Method 3 - Optimising $\boldsymbol{R}$(Restricted)	2.48	0.11	2.75	0.11	5.37	1.21	5.83	1.37
Method 4 - Optimising $\boldsymbol{G}$	2.48	0.11	2.75	0.11	5.38	1.21	5.83	1.38

- Parameterisation does not matter

Monte-Carlo Simulation *Cont.*

- Comparison with point forecast reconciliation methods.

Reconciliation method	Hierarchy 1				Hierarchy 2			
	$h = 1$		$h = 3$		$h = 1$		$h = 3$	
	ES	VS	ES	VS	ES	VS	ES	VS
Optimal $\boldsymbol{G}$	2.48*	0.106	2.75*	0.106	5.36*	1.21*	5.83*	1.38*
MinT(Shrink)	2.47*	0.105	2.74*	0.105	5.33*	1.19*	5.77*	1.34*
WLS	2.46*	0.105	2.74*	0.105	5.43*	1.23	5.98*	1.40*
OLS	2.54*	0.105	2.80*	0.105	5.51*	1.23	5.98*	1.40*
Base	*2.67*	*0.105*	*2.94*	*0.105*	*5.71*	*1.28*	*6.27*	*1.49*

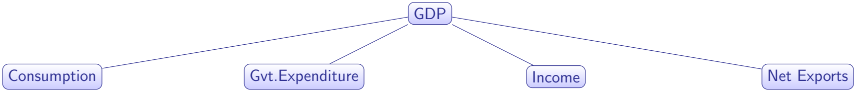
"*" indicates if the average score for a particular reconciliation method is significantly different from that of base forecasts.

- Reconciliation methods perform better than Base forecasts.

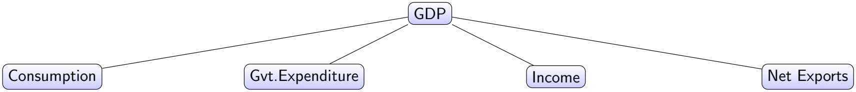
- MinT(Shrink) is at least as good as Optimal method. Thus going forward with MinT projection.

Project 3: Hierarchical forecasts for macroeconomic variables - An application to Australian GDP

Macroeconomic forecasting

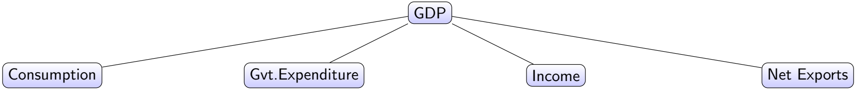


Macroeconomic forecasting



- Common forecasting approaches involves univariate methods or multivariate methods such as VAR.

Macroeconomic forecasting



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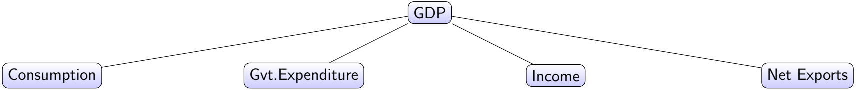

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- The era of big data led to the use of regularization and shrinkage methods - dynamic factor models, Lasso, Bayesian VARs.

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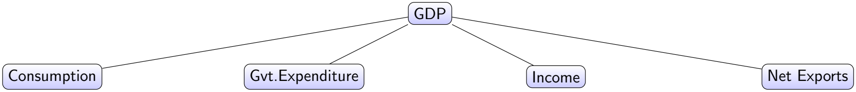

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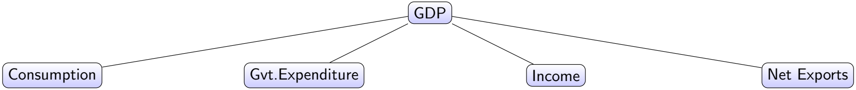

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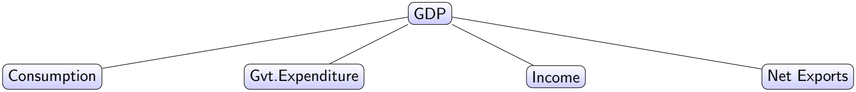

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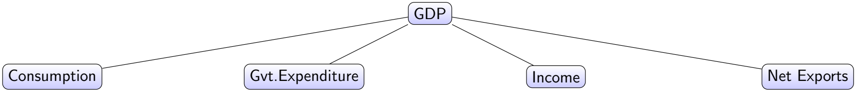

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- This might fail to reflect the deterministic relationship between macroeconomic variables in the forecasts.

Macroeconomic forecasting

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- Thus we propose to use hierarchical forecasting methods in macroeconomic forecasts.
- Related literature: Only one application on point forecasting for inflation (Capistrán, Constandse, and Ramos-Francia, 2010; Weiss, 2018)
- To the best of our knowledge we use hierarchical forecasting methods for point as well as probabilistic forecasts for the first time in macroeconomic literature.

Australian GDP : *Data structures*

- We consider Gross Domestic Product (GDP) of Australia with quarterly data spanning the period 1984:Q4–2018:Q3.

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Income approach

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\begin{aligned}
GDP = {} & Gross\ operating\ surplus + Gross\ mixed\ income \\
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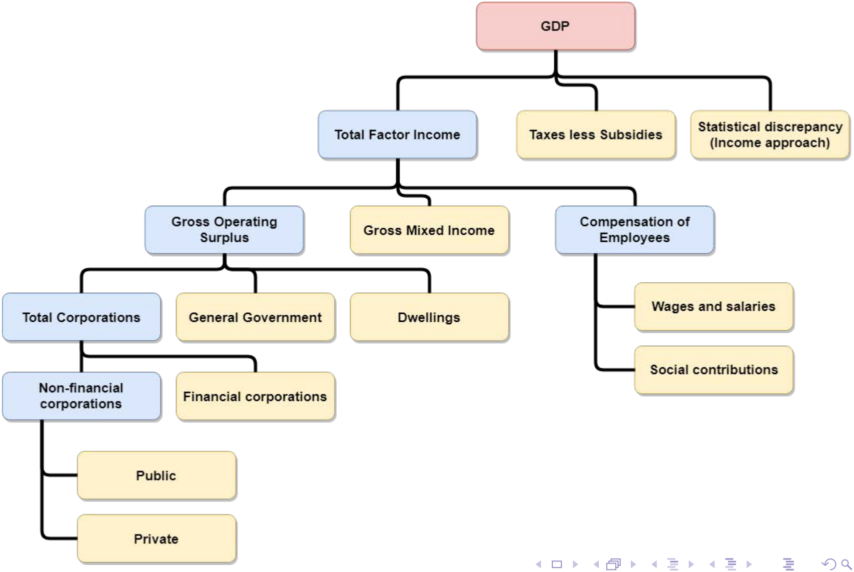
- The hierarchy has two levels of aggregation below the top-level, with a total of $n = 16$ series and $m = 10$ bottom level series.

Australian GDP : *Data structures - Income approach*



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Expenditure approach

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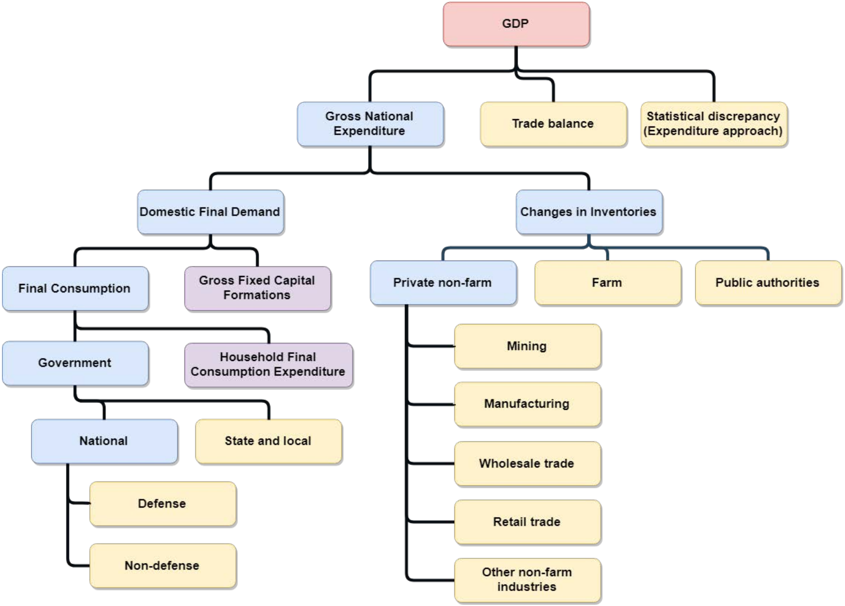
- The hierarchy has three levels of aggregation below the top-level, with a total of $n = 80$ series and $m = 53$ series at the bottom level.

Australian GDP : *Data structures - Expenditure approach*



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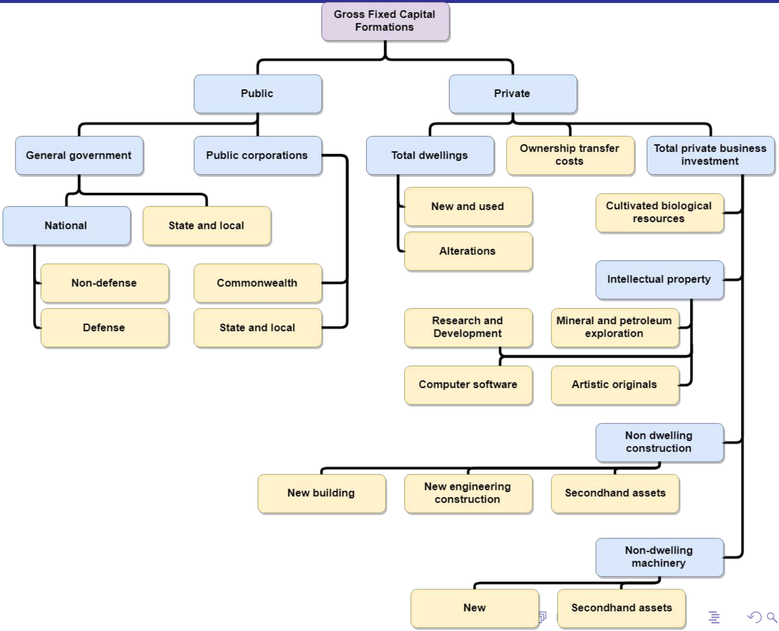

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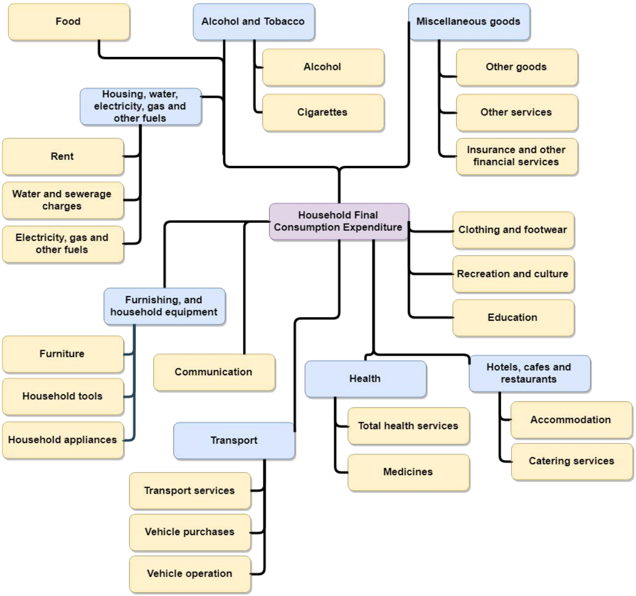
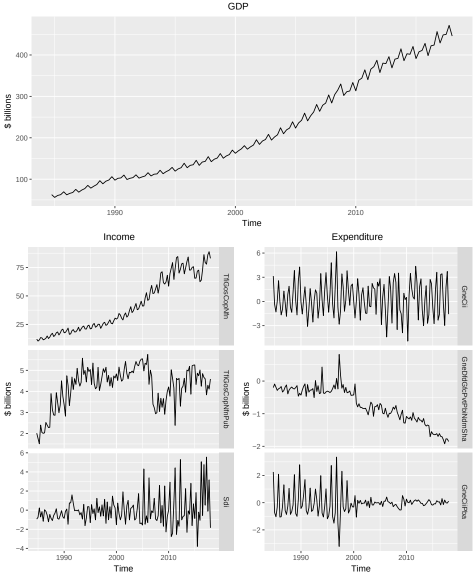


Australian GDP : *Time plots for different levels*



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- **Base forecasting models:**
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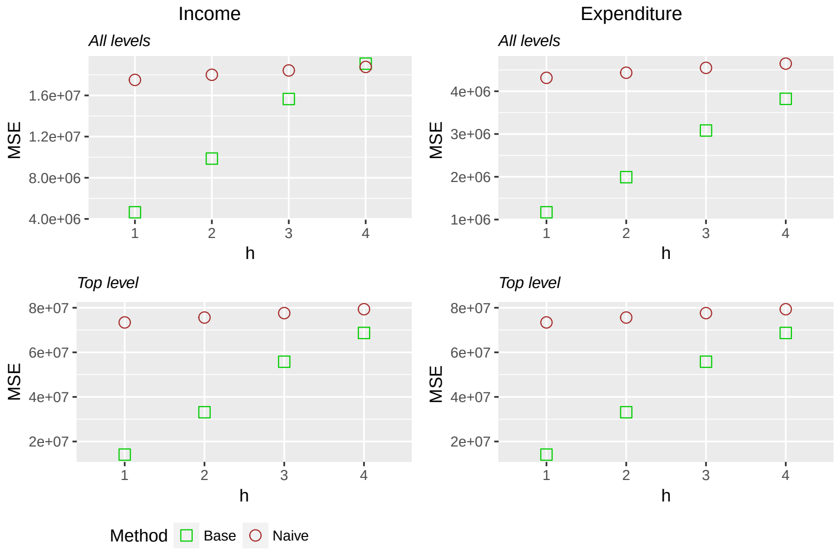
  - $h = 1, 2, 3, 4$ steps ahead forecasts were generated using the fitted models.

Point forecasts: *Base vs Seasonal Naive*



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Reconciled forecasts are given by,

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Point forecasts: *Reconciliation*

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Method	$\boldsymbol{G}$
BU	$(\boldsymbol{0}_{m\times n-m}\ \ \boldsymbol{I}_{m\times m})$
OLS	$(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$
WLS	$\left(\boldsymbol{S}'\hat{\boldsymbol{W}}^{wls}_{T+1}\boldsymbol{S}\right)^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}^{wls}_{T+1}$
MinT(Shrink)	$\left(\boldsymbol{S}'\hat{\boldsymbol{W}}^{shr}_{T+1}\boldsymbol{S}\right)^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}^{shr}_{T+1}$

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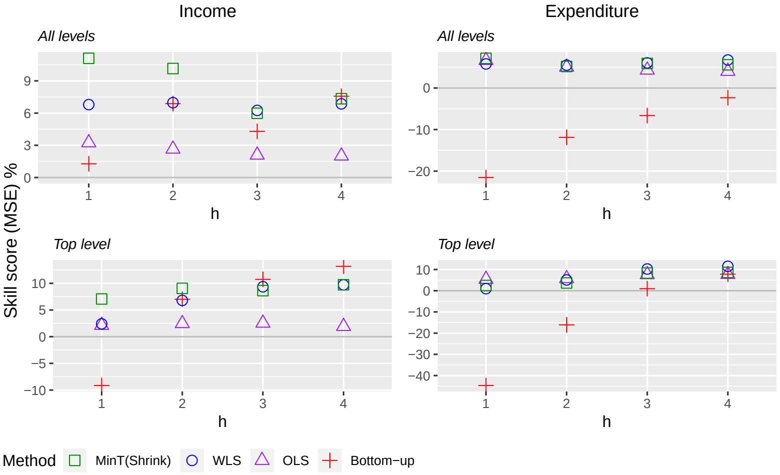
Reconciled Point forecasts - *Results*



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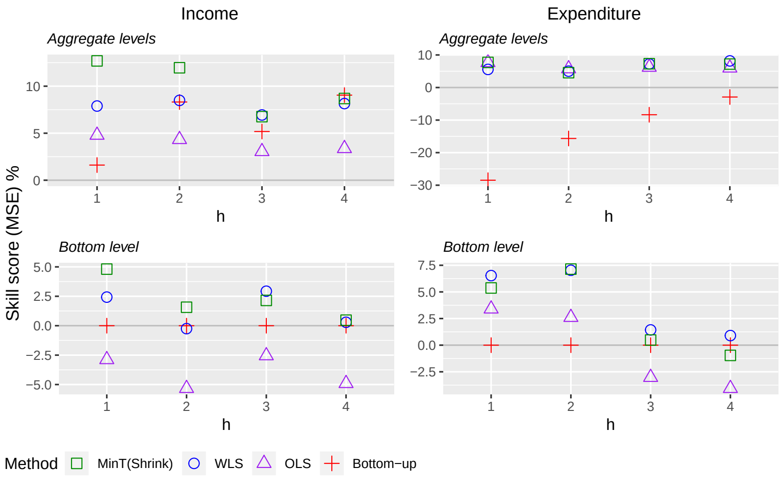

Probabilistic forecasts

- Gaussian approach :

$$\mathcal{N}(\boldsymbol{SG}\hat{\boldsymbol{y}}_{T+h},\ \boldsymbol{SGW}_{T+h}\boldsymbol{G}'\boldsymbol{S}')$$

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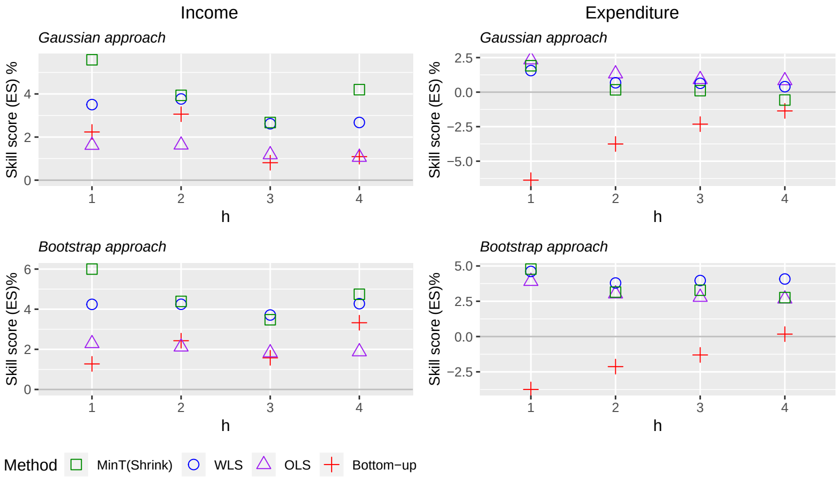
- Non-parametric Bootstrap approach :

$$\tilde{\Upsilon}_{T+h} = \boldsymbol{SG}\hat{\Upsilon}'_{T+h}$$

where, $\hat{\Upsilon}_{T+h} = (\hat{\boldsymbol{y}}^1_{T+h}, ..., \hat{\boldsymbol{y}}^B_{T+h})'$

Reconciled Probabilistic Forecasts



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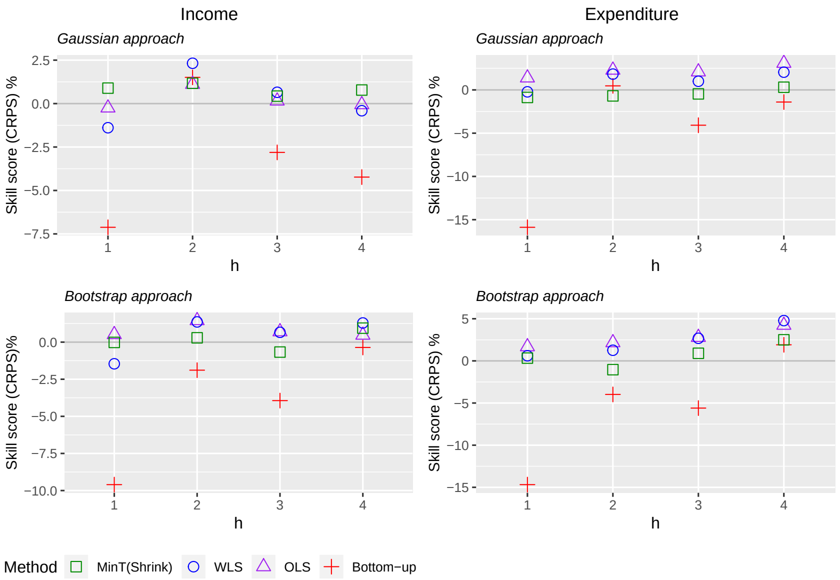

Summary and time plan for completion

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- Simulation study provides evidence that the optimal reconciliation with respect to energy score is equivalent to reconciling each sample path via MinT approach.

- We apply hierarchical forecast reconciliation methods to forecast Australian GDP in point as well as probabilistic framework.

Time plan for completion

	Thesis Chapter	Task description	Time duration	Progress
1 and 2.	Introduction and Background Review	Writing the chapter.	September/2019 - October/2019	40% complete
3.	Hierarchical forecast reconciliation in Geometric view	Bias correction and application	May/2019 - July/2019	75% Completed
4.	Probabilistic forecast reconciliation for hierarchical time series	Completing the paper.	June/2019 - August/2019	90% Completed
5.	Application	Forecasting Australian GDP		100% Completed

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Thank You!!

Project 2: *Probabilistic forecasts evaluation*

A1

Energy score (Gneiting et al., 2008)

$$\text{eS}(\breve{\boldsymbol{Y}}_{T+h}, \boldsymbol{y}_{T+h}) = \text{E}_{\breve{\boldsymbol{F}}}\|\breve{\boldsymbol{Y}}_{T+h} - \boldsymbol{y}_{T+h}\|^{\alpha} - \frac{1}{2}\text{E}_{\breve{\boldsymbol{F}}}\|\breve{\boldsymbol{Y}}_{T+h} - \breve{\boldsymbol{Y}}^{*}_{T+h}\|^{\alpha}, \quad \alpha \in (0,2]$$

Log score (Gneiting and Raftery, 2007)

$$\text{LS}(\breve{\boldsymbol{F}}, \boldsymbol{y}_{T+h}) = -\log \breve{\boldsymbol{f}}(\boldsymbol{y}_{T+h})$$

Variogram score (Scheuerer and Hamill, 2015)

$$\text{VS}(\breve{\boldsymbol{F}}, \boldsymbol{y}_{T+h}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\left(|y_{T+h,i} - y_{T+h,j}|^{p} - \text{E}_{\breve{\boldsymbol{F}}}|\breve{Y}_{T+h,i} - \breve{Y}_{T+h,j}|^{p}\right)^{2}$$

CRPS (Gneiting and Raftery, 2007)

$$\text{CRPS}(\breve{F}_i, y_{T+h,i}) = \text{E}_{\breve{F}_i}|\breve{Y}_{T+h,i} - y_{T+h,i}| - \frac{1}{2}\text{E}_{\breve{F}_i}|\breve{Y}_{T+h,i} - \breve{Y}^{*}_{T+h,i}|$$

$\breve{\boldsymbol{Y}}_{T+h}$ and $\breve{\boldsymbol{Y}}^{*}_{T+h}$	:	Independent random vectors from the coherent forecast distribution $\breve{\boldsymbol{F}}$.
$\boldsymbol{y}_{T+h}$	:	Vector of realizations.
$\breve{Y}_{T+h,i}$ and $\breve{Y}_{T+h,j}$	:	ith and jth components of the vector $\breve{\boldsymbol{Y}}_{T+h}$

- A2 Shrinkage estimator for 1-step ahead base forecast errors

$$\hat{\boldsymbol{\Sigma}}^{shr}_{T+1} = \tau \hat{\boldsymbol{\Sigma}}^{D}_{T+1} + (1-\tau)\hat{\boldsymbol{\Sigma}}_{T+1},$$

where $\hat{\boldsymbol{\Sigma}}^{D}_{T+1}$ is the diagonal matrix comprising diagonal entries of $\hat{\boldsymbol{\Sigma}}_{T+1}$ and

$$\tau = \frac{\sum_{i \neq j} \hat{Var}(\hat{r}_{ij})}{\sum_{i \neq j} \hat{r}^2_{ij}}$$

is a shrinkage parameter. $\hat{r}_{ij}$ is the *ij*-th element of sample correlation matrix. In this estimation, the off-diagonal elements of 1-step ahead sample covariance matrix will be shrunk to zero depending on the sparsity.

Monte-Carlo simulation

- **Data generating process** A3

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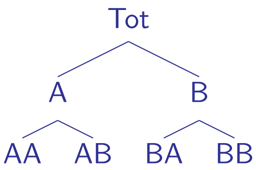

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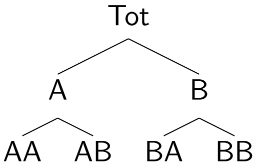
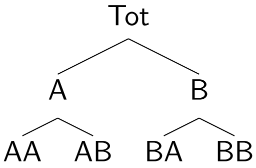


Monte-Carlo simulation

- **Data generating process** A3



- $\{w_{AA,t}, w_{AB,t}, w_{BA,t}, w_{BB,t}\} \sim ARIMA(p, d, q)$

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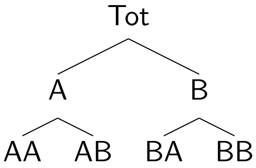

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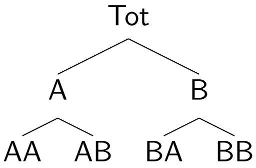

- $\{w_{AA,t}, w_{AB,t}, w_{BA,t}, w_{BB,t}\} \sim ARIMA(p, d, q)$
- $p \in \{1, 2\}$ and $d \in \{0, 1\}$
- $\{\epsilon_{AA,t}, \epsilon_{AB,t}, \epsilon_{BA,t}, \epsilon_{BB,t}\} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$

Monte-Carlo simulation

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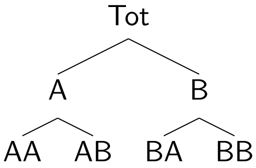

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- Parameters for *AR* and *MA* components were randomly and uniformly generated from $[0.3, 0.5]$ and $[0.3, 0.7]$ respectively

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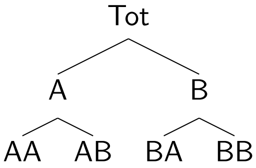

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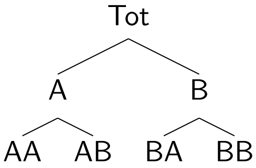

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- $\boldsymbol{y}_t$ are then generated as follows

Monte-Carlo simulation

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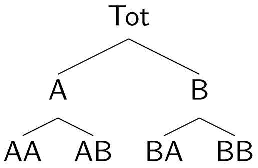

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- $\boldsymbol{y}_t$ are then generated as follows

Bottom level	**Aggregate level 1**	**Total**
$y_{AA,t} = w_{AA,t} + u_t - 0.5v_t$	$y_{A,t} = w_{AA,t} + w_{AB,t} - v_t$	$y_{Tot,t} = w_{AA,t} + w_{AB,t} + w_{BA,t} + w_{BB,t}$
$y_{AB,t} = w_{AB,t} - u_t - 0.5v_t$	$y_{B,t} = w_{BA,t} + w_{BB,t} + v_t$	
$y_{BA,t} = w_{BA,t} + u_t + 0.5v_t$		
$y_{BB,t} = w_{BB,t} - u_t + 0.5v_t$		

Monte-Carlo simulation *Cont..*

- To get less noisier series at aggregate levels, we choose $\boldsymbol{\Sigma}, \sigma_u^2$ and σ_v^2 such that,

$$\text{Var}(\epsilon_{AA,t} + \epsilon_{AB,t} + \epsilon_{BA,t} + \epsilon_{BB,t}) \leq \text{Var}(\epsilon_{AA,t} + \epsilon_{AB,t} - v_t) \leq \text{Var}(\epsilon_{AA,t} + u_t - 0.5v_t),$$

Monte-Carlo simulation *Cont..*

- To get less noisier series at aggregate levels, we choose $\mathbf{\Sigma}, \sigma_u^2$ and σ_v^2 such that,

$$\text{Var}(\epsilon_{AA,t} + \epsilon_{AB,t} + \epsilon_{BA,t} + \epsilon_{BB,t}) \leq \text{Var}(\epsilon_{AA,t} + \epsilon_{AB,t} - v_t) \leq \text{Var}(\epsilon_{AA,t} + u_t - 0.5v_t),$$

- Thus we choose, $\mathbf{\Sigma} = \begin{pmatrix} 5.0 & 3.1 & 0.6 & 0.4 \\ 3.1 & 4.0 & 0.9 & 1.4 \\ 0.6 & 0.9 & 2.0 & 1.8 \\ 0.4 & 1.4 & 1.8 & 3.0 \end{pmatrix}$, $\sigma_u^2 = 19$ and $\sigma_u^2 = 18$.

Sample version of the scoring rules

- For a possible finite sample of size B from the multivariate forecast density $\breve{\boldsymbol{F}}$, the variogram score is defined as,

$$\mathrm{VS}(\breve{\boldsymbol{F}}, \boldsymbol{y}_{T+h}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\left(|y_{T+h,i} - y_{T+h,j}|^p - \frac{1}{B}\sum_{k=1}^{B}|\breve{Y}_{T+h,i}^{k} - \breve{Y}_{T+h,j}^{k}|^p\right)^2$$