ADVANCES IN THE ESTIMATION OF FRACTIONALLY INTEGRATED MODELS

Pre-submission Seminar

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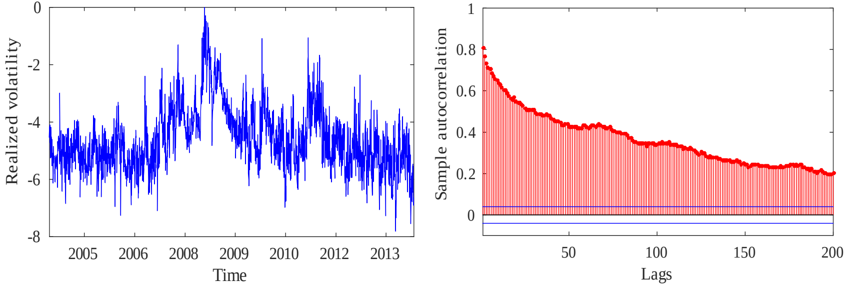
Outline

1. Fractionally integrated processes and related models
2. Parametric and semi-parametric estimation techniques
3. Overview of the thesis
4. Talk of the day:

   Mean estimation and mis-specification in fractionally integrated models
5. Progress to date
6. Future plans
7. Timetable for the completion of the thesis

Fractionally Integrated models [Granger and Joeyux, 1980]

Log transformed realized volatility measure of S&P500 return index



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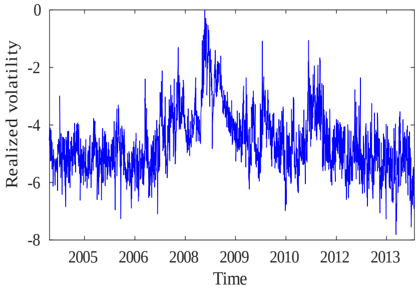

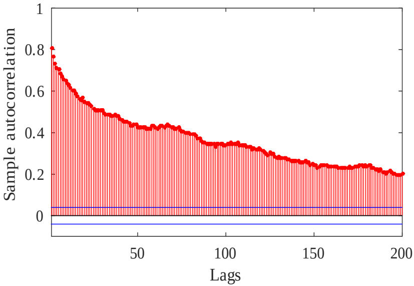

- Fractionally integrated autoregressive moving average (ARFIMA) model: For $|d| < 1/2$ and $\{\varepsilon_t\} \sim WN\left(0, \sigma_\varepsilon^2\right)$,

$$\phi(z)(1-z)^d\{y_t - \mu\} = \theta(z)\varepsilon_t, \quad t \in \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$$

- The spectral density of a fractional process:

$$f(\lambda) = \frac{\sigma_\varepsilon^2}{2\pi}|1 - e^{i\lambda}|^{-2d} g(\lambda), \quad \lambda \in [-\pi, \pi]$$

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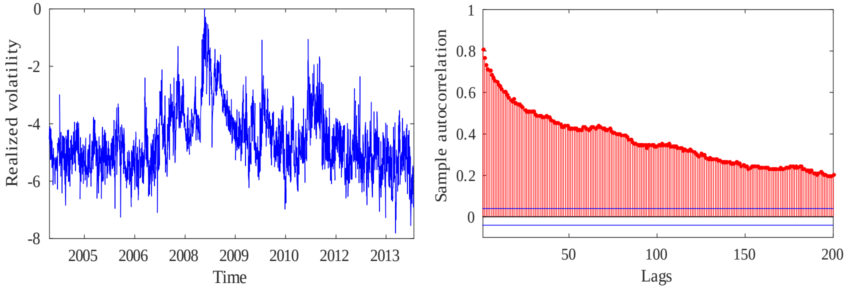

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- Key parameter: fractional differencing parameter d

Estimation techniques for d

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Semi-parametric methods

- Ignore the structure of the short memory dynamics and focus only on d
- E.g.: the log periodogram regression estimator [Geweke and Porter-Hudak, 1983] and the local Whittle estimator [Künsch, 1987, Robinson, 1995a,b]
- Limitation:
  - Finite sample bias [Agiakloglou, Newbold and Wohar, 1993]
  - loss of $\sqrt{n}-$ rate of convergence

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Parametric methods

- Require correct specification of the model for a given data generating process

  $\Rightarrow$ $\sqrt{n}-$ consistency and asymptotic normality
- E.g.: maximum likelihood, conditional sum of squares, Whittle estimation
- Limitation: Violation of standard asymptotic properties under mis-specification of the short-memory dynamics [Chen and Deo, 2006, Martin, Nadarajah and Poskitt, 2018]

Overview of the Thesis

Broad aims:

Project 1:

Optimal bias-correction of a semi-parametric estimator in stationary fractionally integrated models: A jackknife approach

Project 2:

Mean estimation and mis-specification in fractionally integrated models

Project 3:

Parametric estimation in mis-specified non-stationary fractionally integrated models

Project 1: Optimal bias-correction of a semi-parametric estimator in stationary fractionally integrated models: A jackknife approach

Motivation: reduction in bias $\Longrightarrow$ increase in variance $\Longrightarrow$ inefficiency

Objectives:

- Obtain a bias-corrected semi-parametric estimator
- Minimize the variance associated with the reduction in bias

Bias-correction focuses on:

The log-periodogram regression estimator [Geweke and Porter-Hudak, 1983]

Bias-correction technique: Jackknife

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$$\widehat{d}_{J,m}^{opt} = w_n^* \underbrace{\widehat{d}_n}_{\text{Full sample LPR}} - \sum_{i=1}^{m} w_{i,m}^* \underbrace{\widehat{d}_i}_{\text{subsample LPR}}$$

Project 1 (Completed)

- Construction of optimal jackknife log periodogram regression estimator:
  - Closed-form expressions are available for the covariance between the log-periodograms corresponding to full and sub-samples
  - periodograms are asymptotic chi-square random variables
- Established consistency and asymptotic normality - no loss in efficiency
- Optimal jackknife estimator outperforms the pre-filtered sieve bootstrap estimator and the generalized least-squares LPR estimator
- Require the knowledge of the data generating process

  $\Rightarrow$ infeasible in practice

  $\Rightarrow$ use an iterative procedure with some consistent estimates for the unknown parameters associated with the weights
- Illustrated performance of an iterative procedure via simulation and an empirical example
- Extended the results to non-Gaussian processes
- Nearly ready for submission!!

Related Literature on mis-specification in fractionally integrated models
[Chen and Deo (2006) and Martin, Nadarajah and Poskitt (2018)]

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- **Parametric estimation** of $\boldsymbol{\eta}$: $(\widehat{\boldsymbol{\eta}}_1^{(i)},\ i = 1, 2, 3, 4)$
  1. Frequency domain maximum likelihood (FML)
  2. Discretized version of exact Whittle (DWH)
  3. Time domain maximum likelihood (TML)
  4. Conditional sum of squares (CSS)

Frequency domain maximum likelihood (FML)

- Define $\boldsymbol{\beta} = \left(\boldsymbol{\theta}^\top, \boldsymbol{\phi}^\top\right)^\top$, $\boldsymbol{\eta} = \left(d, \boldsymbol{\beta}^\top\right)^\top$, $|d| < 0.5$

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- FML estimator of $\boldsymbol{\eta}$:

$$\widehat{\boldsymbol{\eta}}_1^{(1)} = \arg\min_{\boldsymbol{\eta}} Q_n^{(1)}(\boldsymbol{\eta}) = \frac{2\pi}{n} \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{I(\lambda_j)}{f_1(\boldsymbol{\eta}, \lambda_j)},$$

where

$$\begin{aligned} I(\lambda) &= \frac{1}{2\pi n} \left| \sum_{t=1}^{n} (y_t - \mu) \exp(-\imath \lambda t) \right|^2 \\ f_1(\boldsymbol{\eta}, \lambda) &= \frac{\sigma^2}{2\pi} \left| 1 - \exp(-\imath\lambda) \right|^{-2d} g_1(\boldsymbol{\beta}, \lambda) \\ \lambda_j &= 2\pi j / n, \quad j = 1, ..., \lfloor n/2 \rfloor, \end{aligned}$$

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- $Q_n^{(1)}(\boldsymbol{\eta}) \approx$ the negative of the exact Gaussian log-likelihood

- Limiting criterion function:

$$Q(\boldsymbol{\eta}) = \lim_{n\to\infty} E\left[Q_n^{(1)}(\boldsymbol{\eta})\right] = \int_0^{\pi} \frac{f_0(\lambda)}{f_1(\boldsymbol{\eta},\lambda)} d\lambda$$

where

$f_0(\lambda)$: True spectral density - ARFIMA(p_0, d_0, q_0)

$f_1(\boldsymbol{\eta},\lambda)$: Est. spectral density - ARFIMA(p, d, q)

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$$\boxed{\begin{array}{ccc} Q_n^{(1)}(\boldsymbol{\eta}) & \to^p & Q(\boldsymbol{\eta}) \\ \Downarrow & & \Downarrow \\ \widehat{\boldsymbol{\eta}}_1^{(1)} & \to^p & \underbrace{\boldsymbol{\eta}_1} \\ \multicolumn{3}{c}{\text{pseudo-true value of } \boldsymbol{\eta}} \end{array}}$$

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$$R_n\left(\widehat{\eta}_1^{(1)} - \eta_1 - \delta_n\right) \quad \to \quad ^D \underbrace{\mathcal{G}\left(d^*, \eta_1\right)}_{a\ centered\ process}$$

$d^* = d_0 - d_1$	T_n	$\mathcal{G}$	δ_n
> 0.25	$\frac{n^{1-2d^*}}{\log n}$	$Non-normal$	$non-zero$
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  $\Rightarrow\ \widehat{\eta}_1^{(i)} \rightarrow^{P} \eta_1$

  $\Rightarrow$ all four parametric estimators possess the same limiting distribution

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Main questions: Under mis-specification and when mean is unknown,

1. Do the estimators still converge to some limit?
2. Is the limit the same for all the estimators?
3. If the estimators are consistent for some value, does the limiting distribution obtained under the known mean case still apply?

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Assumption: For $\widehat{\mu}_n$, the estimator of mean μ_0, is such that

$$\widehat{\mu}_n = \mu_0 + o_p\left(n^{-1/2+d_0}\right)$$

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Estimators of η: $\widehat{\eta}_1^{(i)}$, $i = 1, 2, \mathbf{3}, \mathbf{4}, \mathbf{5}$

1. Frequency domain maximum likelihood (FML)
2. Discretized version of Whittle (DWH)
3. **Time domain maximum likelihood (TML)**
4. **Conditional sum of squares (CSS)**
5. **Exact Whittle (EWH)**

Alternative parametric estimation under mis-specification

Define $\boldsymbol{\psi} = \left(\sigma^2, \boldsymbol{\eta}^\top\right)^\top$

- DWH estimator of $\boldsymbol{\psi}$:

$$\widehat{\boldsymbol{\psi}}_1^{(2)} = \arg\min_{\boldsymbol{\psi}} Q_n^{(2)}\left(\boldsymbol{\psi}\right) = \frac{2\pi}{n}\sum_{j=1}^{\lfloor n/2 \rfloor} \log \frac{\sigma^2}{2\pi} f_1\left(\boldsymbol{\eta}, \lambda_j\right) + \frac{(2\pi)^2}{\sigma^2 n}\sum_{j=1}^{\lfloor n/2 \rfloor} \frac{I\left(\lambda_j\right)}{f_1\left(\boldsymbol{\eta}, \lambda_j\right)}$$

- TML estimator of $\boldsymbol{\psi}$:

$$\widehat{\boldsymbol{\psi}}_1^{(3)} = \arg\min_{\boldsymbol{\psi}} Q_n^{(3)}\left(\boldsymbol{\psi}\right) = \frac{1}{n}\log\left|\Sigma_{\boldsymbol{\eta}}\right| + \frac{1}{n}\left(\mathbf{y} - \widehat{\mu}\mathbf{1}\right)^\top \Sigma_{\boldsymbol{\eta}}^{-1}\left(\mathbf{y} - \widehat{\mu}\mathbf{1}\right)$$

- CSS estimator of $\boldsymbol{\eta}$:

$$\widehat{\boldsymbol{\eta}}_1^{(4)} = \arg\min_{\boldsymbol{\eta}} Q_n^{(4)}\left(\boldsymbol{\eta}\right) = \frac{1}{n}\sum_{t=1}^{n} e_t^2, \;\; e_t = \sum_{i=0}^{t-1} \tau_i\left(\boldsymbol{\eta}\right)\left\{y_{t-i} - \widehat{\mu}\right\}$$

- EWH estimator of $\boldsymbol{\psi}$:

$$\widehat{\boldsymbol{\psi}}_1^{(5)} = \arg\min_{\boldsymbol{\psi}} Q_n^{(5)}\left(\boldsymbol{\psi}\right) = \int_{-\pi}^{\pi}\left\{\log\left(\frac{\sigma^2}{2\pi} f_1\left(\boldsymbol{\eta}, \lambda\right)\right) + \frac{2\pi I\left(\lambda\right)}{\sigma^2 f_1\left(\boldsymbol{\eta}, \lambda\right)}\right\} d\lambda$$

Asymptotic theory for the parametric estimators under mis-specification

- Focus only on the estimator of the parameter vector $\boldsymbol{\eta}$

Theorem (Main statement)

Under certain regularity conditions,

(1) $\widehat{\boldsymbol{\eta}}_1^{(i)} \rightarrow^P \boldsymbol{\eta}_1$

where $\boldsymbol{\eta}_1 = \arg\min_{\boldsymbol{\eta}} Q(\boldsymbol{\eta})$

(2) $\lim_{n\to\infty} \left\| \widehat{\boldsymbol{\eta}}_1^{(i)} - \widehat{\boldsymbol{\eta}}_1^{(j)} \right\| = 0$, *almost surely for all five parametric estimators*

- **Limiting distribution**:

$$\sqrt{R_n}\left(\widehat{\boldsymbol{\eta}}_1 - \boldsymbol{\eta}_1 - \boldsymbol{\delta}_n\right) \rightarrow^D \mathcal{G}\left(d^*, \boldsymbol{\eta}_1\right)$$

Proved for the case: $\widehat{\mu}_n = \overline{Y}$

Limiting distribution [Chen and Deo (2006)]

<u>Case 1</u>: $d^* = d_0 - d_1 > 0.25$ (extreme mis-specification)

$$\frac{n^{1-2d^*}}{\log n}\left(\widehat{\eta}_1 - \eta_1 - \delta_n\right) \rightarrow^D \mathbf{B}^{-1}\left[\sum_{j=1}^{\infty} W_j, 0, \ldots, 0\right]^T,$$

where

$$W_j = \frac{(2\pi)^{1-2d^*} g_0(\eta_0, 0)}{j^{2d^*} g_1(\eta_1, 0)}\left[U_j^2 + V_j^2 - E_0\left(U_j^2 + V_j^2\right)\right],$$

where, $\{U_j, V_k\}$ are a sequence of zero mean Gaussian random variables with a specified covariance structure.

- Non-Gaussian limiting distribution
- Bias term $\delta_n : \delta_n \rightarrow \mathbf{0}$ as $n \rightarrow \infty$
- Slower rate of convergence than $\sqrt{n}$
- Increase in $(d_0 - d_1) \Rightarrow$ slower rate of convergence

Case 2: $d^* = d_0 - d_1 = 0.25$ (borderline mis-specification)

$$n^{1/2}\overline{\Lambda}^{-1/2}\left(\widehat{\boldsymbol{\eta}}_1 - \boldsymbol{\eta}_1\right) \to^D \mathbf{B}^{-1}\left(Z, 0, ..., 0\right)^T ,$$

where

$$\overline{\Lambda} = \frac{1}{n}\sum_{j=1}^{n/2}\left(\frac{f_0(\lambda_j)}{f_1(\boldsymbol{\eta}_1,\lambda_j)}\frac{\partial \log f_1(\boldsymbol{\eta}_1,\lambda_j)}{\partial d}\right)^2 ,$$

and Z is a standard normal random variable.

Case 3: $d^* = d_0 - d_1 < 0.25$ (mild mis-specification)

$$\sqrt{n}\left(\widehat{\boldsymbol{\eta}}_1 - \boldsymbol{\eta}_1\right) \to^D N(0, \Xi),$$

where $\Xi = \mathbf{B}^{-1}\Lambda\mathbf{B}^{-1}$, and

$$\Lambda = 2\pi\int_0^{\pi}\left(\frac{f_0(\lambda)}{f_1(\boldsymbol{\eta}_1,\lambda)}\right)^2\left(\frac{\partial \log f_1(\boldsymbol{\eta}_1,\lambda)}{\partial \boldsymbol{\eta}}\right)\left(\frac{\partial \log f_1(\boldsymbol{\eta}_1,\lambda)}{\partial \boldsymbol{\eta}}\right)^T d\lambda.$$

Simulation exercise

Example (1)

TDGP: ARFIMA$(0, d_0, 1)$, $d_0 = 0.2$ and $\theta_0 = \{-0.7, -0.444978, -0.3\}$
Mis-M: ARFIMA$(0, d, 0)$
$-$ Estimation of only parameter d.

- The values of θ_0 considered here correspond to the three cases on d^*.
  - $d^* > 0.25 \Rightarrow \theta_0 < -0.444978$
  - $d^* = 0.25 \Rightarrow \theta_0 = -0.444978$
  - $d^* < 0.25 \Rightarrow \theta_0 > -0.444978$
- TDGP: Gaussian

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Example (2)

TDGP: ARFIMA$(0, d_0, 1)$, $d_0 = 0.2$ and $\theta_0 = \{-0.7, -0.637014, -0.3\}$
Mis-M: ARFIMA$(1, d, 0)$
$-$ Estimation of the parameter d and the AR coefficient

Table: Estimates of the bias and RMSE for the FML, Whittle, EWH, TML and CSS estimators of d_1 Example 1 - TDGP: ARFIMA$(0, d_0, 1)$ vis-a-vis Mis-M: ARFIMA$(0, d, 0)$. Process mean $\mu = 0$, is known.

			FML		DWH		EWH		TML		CSS	
d^*	θ_0	n	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
0.372	-0.7	100	-0.277	0.297	-0.237	0.257	-0.177	0.207	-0.147	0.182	-0.121	0.171
		500	-0.142	0.151	-0.167	0.144	-0.098	0.111	-0.090	0.102	-0.074	0.086
		1000	-0.109	0.116	-0.086	0.103	-0.064	0.085	-0.062	0.079	-0.058	0.065
0.250	-0.44	100	-0.149	0.181	-0.128	0.166	-0.114	0.131	-0.079	0.111	-0.047	0.097
		500	-0.058	0.073	-0.044	0.053	-0.039	0.050	-0.020	0.041	-0.019	0.041
		1000	-0.040	0.051	-0.030	0.039	-0.022	0.040	-0.013	0.038	-0.013	0.027
0.174	-0.3	100	-0.104	0.144	-0.074	0.101	-0.055	0.090	-0.034	0.083	-0.024	0.063
		500	-0.034	0.053	-0.024	0.048	-0.022	0.041	-0.018	0.032	-0.008	0.022
		1000	-0.021	0.036	-0.011	0.027	-0.014	0.023	-0.007	0.020	-0.005	0.016

Table: Estimates of the bias and RMSE for the EWH, TML and CSS estimators of d_1 Example 1 - TDGP: ARFIMA$(0, d_0, 1)$ vis-a-vis Mis-M: ARFIMA$(0, d, 0)$. Process mean $\mu = 0$, is unknown.

		Sample mean						BLUE					
		EWH		TML		CSS		EWH		TML		CSS	
θ_0	n	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
-0.7	100	-0.274	0.294	-0.261	0.281	-0.252	0.272	-0.285	0.305	-0.274	0.296	-0.256	0.277
	500	-0.157	0.173	-0.139	0.168	-0.136	0.156	-0.168	0.180	-0.141	0.160	-0.136	0.156
	1000	-0.111	0.138	-0.106	0.124	-0.095	0.102	-0.127	0.145	-0.105	0.112	-0.092	0.109
-0.44	100	-0.143	0.175	-0.136	0.169	-0.130	0.169	-0.155	0.185	-0.149	0.172	-0.138	0.161
	500	-0.075	0.090	-0.056	0.088	-0.045	0.071	-0.077	0.099	-0.056	0.088	-0.046	0.071
	1000	-0.043	0.054	-0.038	0.050	-0.030	0.040	-0.058	0.062	-0.047	0.057	-0.038	0.050
-0.3	100	-0.094	0.144	-0.084	0.137	-0.074	0.116	-0.109	0.168	-0.095	0.146	-0.085	0.136
	500	-0.053	0.072	-0.034	0.067	-0.026	0.052	-0.053	0.077	-0.043	0.062	-0.033	0.052
	1000	-0.035	0.059	-0.021	0.035	-0.012	0.030	-0.025	0.039	-0.022	0.033	-0.012	0.025

Finite sampling distributions of the parametric estimators of pseudo-true value of d

Figure: Kernel density of $\frac{n^{1-2d^*}}{\log n}(\widehat{d}_1 - d_1 - \mu_n)$ for an $ARFIMA(0, d_0, 1)$ TDGP with $d_0 = 0.2$ and $\theta_0 = -0.7$, and an $ARFIMA(0, d, 0)$ Mis-M, $d^* > 0.25$.

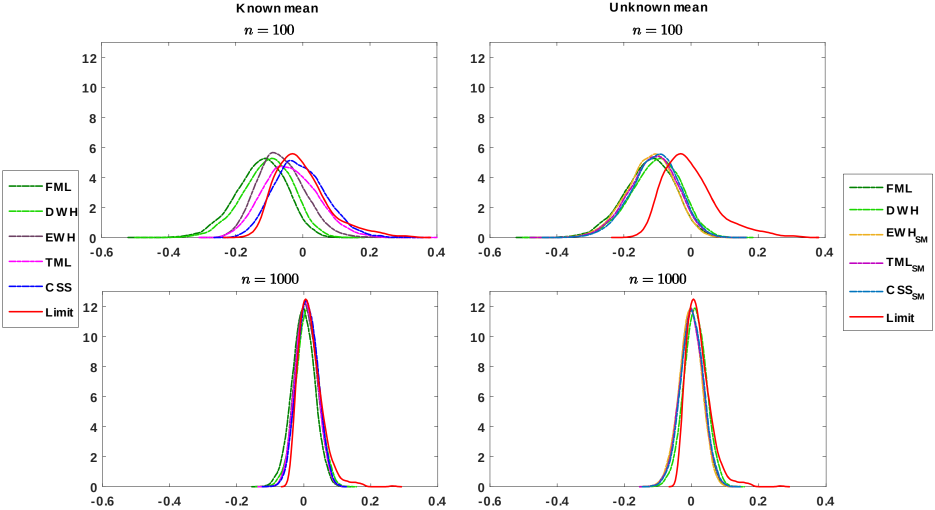


Figure: Kernel density of $n^{1/2}[\overline{\Lambda}_{dd}]^{-1/2}(\widehat{d}_1 - d_1)$ for an $ARFIMA(0, d_0, 1)$ TDGP with $d_0 = 0.2$ and $\theta_0 = -0.444978$, and an $ARFIMA(0, d, 0)$ Mis-M, $d^* = 0.25$.

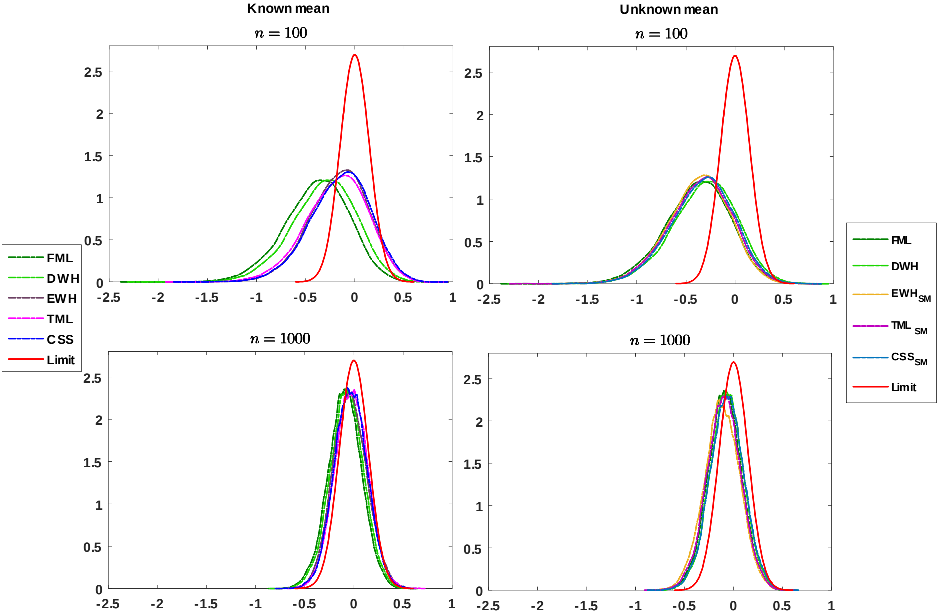
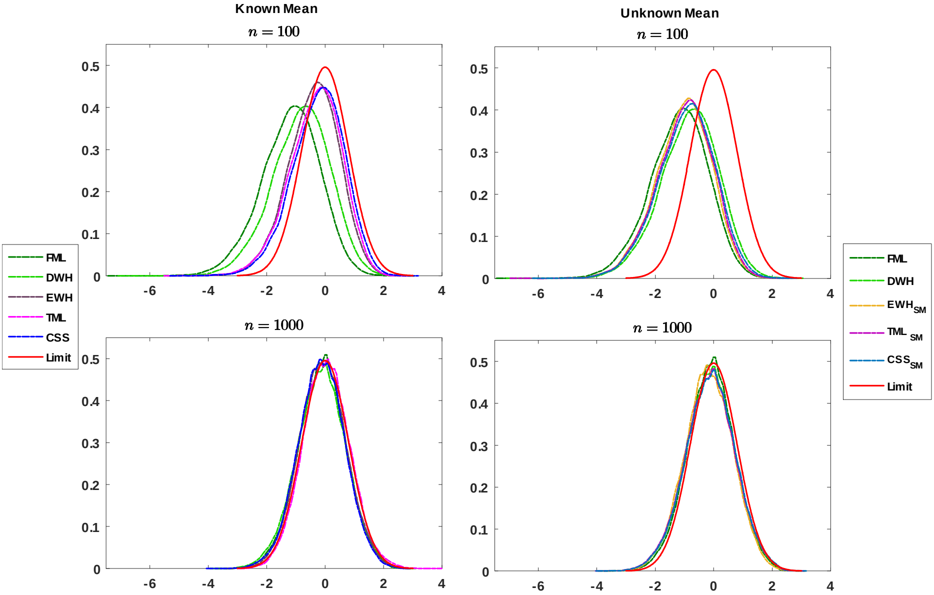


Figure: Kernel density of $\sqrt{n}(\widehat{d}_1 - d_1)$ for an ARFIMA$(0, d_0, 1)$ TDGP with $d_0 = 0.2$ and $\theta_0 = -0.3$ and an ARFIMA$(0, d, 0)$ Mis-M, $d^* < 0.25$.



Summary and future plans

- **Project 2:** *Parametric estimation of mis-specified fractionally integrated models: unknown mean*
  - Asymptotic properties of BLUE under mis-specification
  - Convergence of the five parametric estimators to the (same) pseudo-true parameter
  - Limiting distribution of the parametric estimators when the mean is estimated with sample mean
  - DWH estimator outperforms other estimators - bias and RMSE

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- **Project 2:** *Parametric estimation of mis-specified fractionally integrated models: unknown mean*
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  - Limiting distribution of the parametric estimators when the mean is estimated with sample mean
  - DWH estimator outperforms other estimators - bias and RMSE
- **Project 3:** *Parametric estimation in mis-specified non-stationary fractionally integrated models*
  - Study the behaviour of parametric estimators when $d \geq 1/2$
  - E.g.: treasury bills; interest rates
  - Obtain asymptotic distributions of the FML estimator
  - Simulation exercise

Progress to date

- **Project 1:** *Optimal bias-correction of a semi-parametric estimator in stationary fractionally integrated models: a jackknife approach*
  - Construction of optimal of jackknife log periodogram regression estimator of d
  - Consistency and asymptotic normality of the estimator proven
  - Simulation work conducted for Gaussian and some non-Gaussian processes
  - Extension of the results to non-Gaussian processes

- **Project 2:** *Parametric estimation of mis-specified fractionally integrated models: unknown mean*
  - Statistical properties established for BLUE of mean
  - Convergence results for the parametric estimators of dynamic parameters
  - Simulation study for Gaussian processes

Timetable for the completion of the thesis

- March 2018 – April 2018: Chapter 3
  - Completion of limiting distribution theory
  - Completion of the write-up of a paper on "Mean estimation and mis-specified fractionally integrated models"
  - Completion and submission of project 1
- May 2018 – October 2018: Chapter 4
  - Asymptotic results for the estimators of the mis-specified non-stationary fractionally integrated models
  - Write Chapter 5 depending on the results of Chapter 4
- November 2018 – February 2019:
  - Finalizing Chapters 3 – 5
  - Write-up the following chapters:
    - Chapter 1: Introduction
    - Chapter 6: General discussion and conclusions
  - Proof-reading and final editing of the thesis

THANK YOU!!

Time domain maximum likelihood estimation

The Gaussian log-likelihood function: writing $\Sigma_{\boldsymbol{\eta}} = \sigma^2 \mathbf{R}$.

$$Q_n^{(4)}(\mu, \sigma^2, \boldsymbol{\eta}) = -\frac{n}{2} \log\left(\sigma^2\right) - \frac{1}{2} \log|\mathbf{R}| - \frac{\sigma^2}{2} \left(\mathbf{y} - \mu \mathbf{1}\right)^\top \mathbf{R}^{-1} \left(\mathbf{y} - \mu \mathbf{1}\right)$$

Estimators of the static parameters:

$$\widehat{\mu} = \left(\mathbf{1}^\top \mathbf{R}^{-1} \mathbf{1}\right)^{-1} \mathbf{1}^\top \mathbf{R}^{-1} \mathbf{y} = \widehat{\mu}_{BLU}(\boldsymbol{\eta}) \quad \textit{and} \quad \widehat{\sigma}^2 = \frac{\left(\mathbf{y} - \widehat{\mu}\mathbf{1}\right)^\top \mathbf{R}^{-1} \left(\mathbf{y} - \widehat{\mu}\mathbf{1}\right)}{n}$$

The modified profile log-likelihood (MPL) function:

$$Q_n^{(4)}(\boldsymbol{\eta}) = -\frac{n}{2} \log n - \frac{n}{2} \log\left(\left(\mathbf{y} - \widehat{\mu}\mathbf{1}\right)^\top \mathbf{R}^{-1} \left(\mathbf{y} - \widehat{\mu}\mathbf{1}\right)\right) - \frac{1}{2} \log|\mathbf{R}| - \frac{n}{2}$$