Probabilistic Forecasts for Hierarchical Time Series

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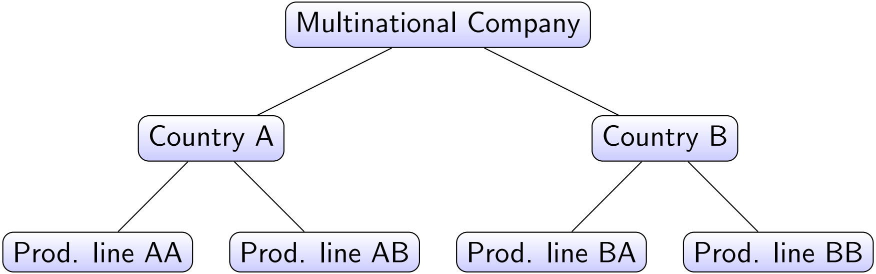
Overview

1. Project 1: Hierarchical Forecast Reconciliation: ~~in~~ a Geometric View
2. Project 2: Probabilistic Forecasts for Hierarchical Time Series
   - Part 1: Definitions and a Parametric Approach
   - Part 2: A non-parametric bootstrap approach ~~for probabilistic forecasts reconciliation~~
3. Hierarchical forecasts ~~in~~ for macroeconomic variables - An application to Australian GDP ~~forecasts~~
4. Summary and time plan for completion

Project 1: Hierarchical Forecast Reconciliation in Geometric View

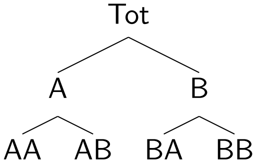
Project 1: Motivation

- **Example:**



- **Hierarchical time series:** A collection of multiple time series that has an inherent aggregation structure.
- Forecasts should add up. We call ~~it~~ these *coherent*,
- **Objective:** Defining ~~the~~ coherency and reconciliation of point forecasts in terms of geometric~~al~~ concepts.

Project 1: *Notations and Preliminaries*



$$\boldsymbol{y}_t = [y_{Tot,t}, y_{A,t}, y_{B,t}, y_{AA,t}, y_{AB,t}, y_{BA,t}, y_{BB,t}]^T$$

$$\boldsymbol{b}_t = [y_{AA,t}, y_{AB,t}, y_{BA,t}, y_{BB,t}]^T$$

$$m = 4$$

$$n = 7$$

$$\boldsymbol{S} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ & \boldsymbol{I}_{4\times 4} & & \end{pmatrix}$$

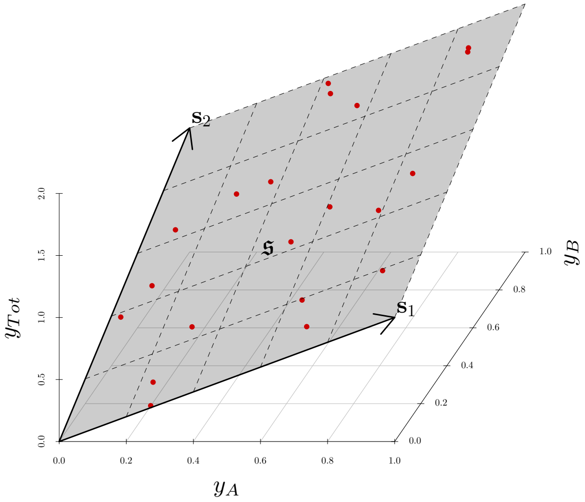
- Due to the aggregation nature of the hierarchy we have,

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

Coherent subspace

The m-dimensional linear subspace $\mathfrak{s} \subset \mathbb{R}^n$ that is spanned by the columns of $\boldsymbol{S}$, i.e. $\mathfrak{s} = \text{span}(\boldsymbol{S})$, is defined as the *coherent space*.

Project 1: *Coherent forecasts*



Project 1: *Point forecast reconciliation*

- Let $\hat{\boldsymbol{y}} \in \mathbb{R}^n$ be an incoherent forecast and $g(.)$ be a function $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

Definition

~~A~~ **Point forecasts** $\tilde{\boldsymbol{y}}$ ~~is~~ are reconciled with respect to $g(.)$ iff

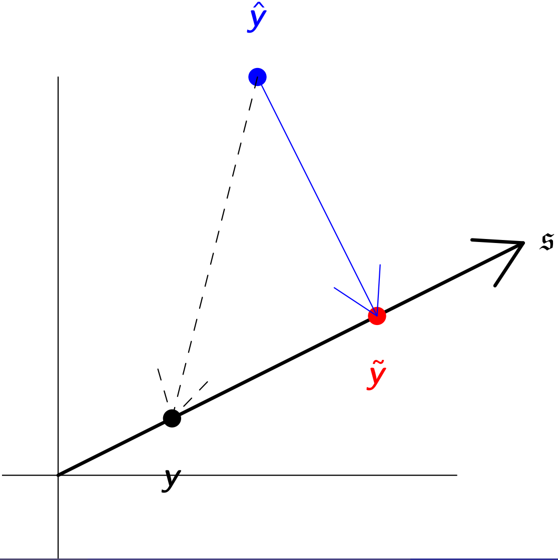
$$\tilde{\boldsymbol{y}} = \boldsymbol{S}g(\hat{\boldsymbol{y}})$$

- If $g(.)$ is linear,

$$\tilde{\boldsymbol{y}}_{T+h} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h}$$

Project 1: *Point forecast reconciliation - OLS*

$\boldsymbol{SG} = \boldsymbol{S}\left(\boldsymbol{S}'\boldsymbol{S}\right)^{-1}\boldsymbol{S}'$ (Hyndman et al., 2011)



Project 1: *Point forecast reconciliation - MinT*

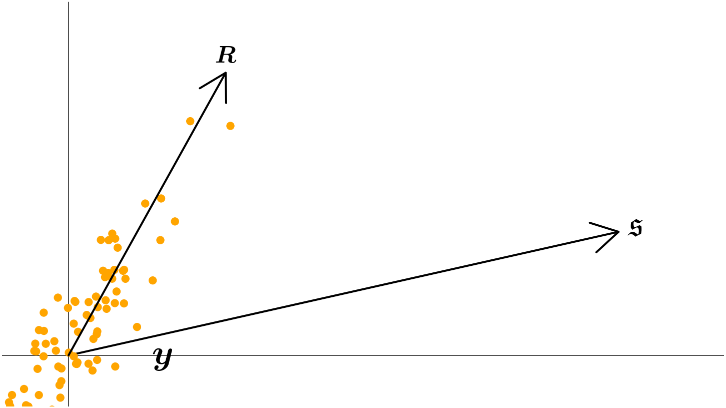
- Minimises the trace of mean squared reconciled forecast errors (Wickramasuriya, Athanasopoulos, and Hyndman, 2018).
- **Geometry**
  - Consider the covariance matrix of $\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h}$.
  - This can be estimated using in-sample forecast errors.

$$\boldsymbol{W} = \sum_{t=1}^{T} (\boldsymbol{y}_t - \hat{\boldsymbol{y}}_t)(\boldsymbol{y}_t - \hat{\boldsymbol{y}}_t)'$$

  - This provides information about the likely direction of an error.
  - Projecting along this direction is more likely to result in ~~a~~ reconciled forecasts that ~~is~~ are closer to the target.

Project 1: *Point forecast reconciliation - MinT*

$$\boldsymbol{SG} = \boldsymbol{S}\left(\boldsymbol{S}'\boldsymbol{W}_{T+h}^{-1}\boldsymbol{S}\right)^{-1}\boldsymbol{S}'\boldsymbol{W}_{T+h}^{-1}$$



Project 1: *Point forecast reconciliation - MinT*

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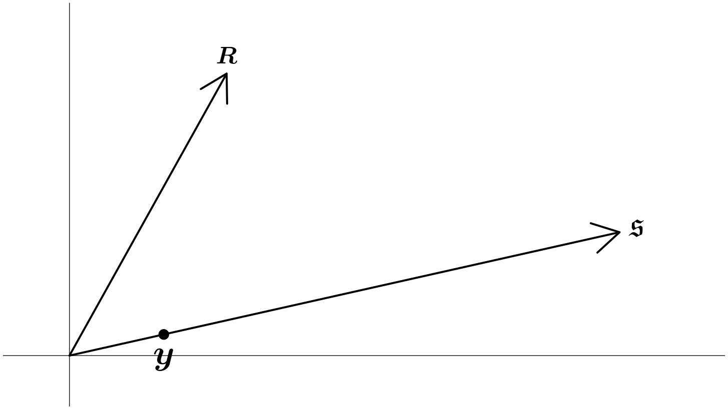

Project 1: *Point forecast reconciliation - MinT*

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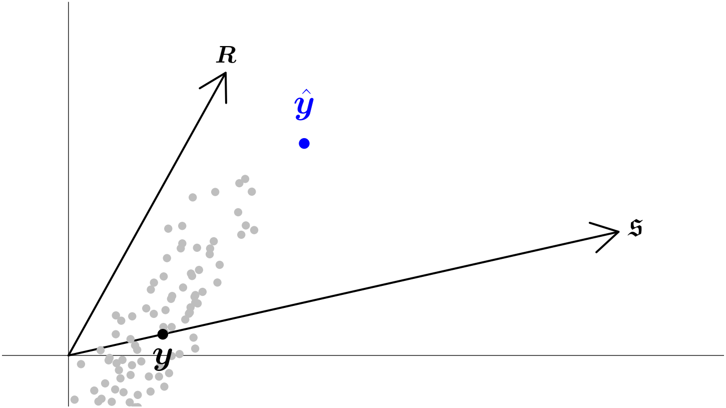

Project 1: *Point forecast reconciliation - MinT*

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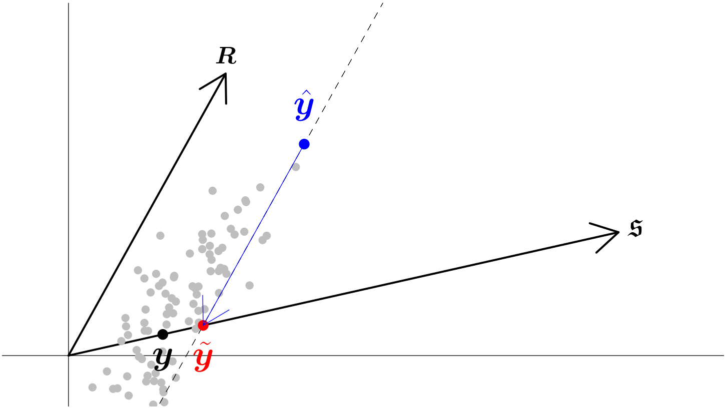


Project 1: *Why projections?*

- Projections preserve ~~the~~ unbiasedness.

  base?

- What if the incoherent forecasts are biased?

- Can we bias correct and proceed with projections?

  We investigate this further.

Project 2: Probabilistic Forecasts for Hierarchical Time Series

Part 1: Definitions and Parametric Approach

Project 2: *Motivation and Objectives*

- Extending ~~the~~ "reconciliation" ~~method~~ into a probabilistic framework

- Probabilistic forecasts should reflect the inherent properties of real data. In particular,
  - ⋆ Aggregation structure
  - ⋆ Correlation structure
- Existing literature
  - (Ben Taieb et al., 2017)
  - (Jeon, Panagiotelis, and Petropoulos, 2018)

- **Objectives:**
  1. Defining coherency and reconciliation for probabilistic forecasts

  2. Probabilistic forecast reconciliation in ~~the~~ a parametric framework

  3. Probabilistic forecast reconciliation in ~~the~~ a non-parametric framework

Project 2: *Coherent probabilistic forecasts*

Let $(\mathbb{R}^m, \mathcal{F}_{\mathbb{R}^m}, \nu)$ and $(\mathfrak{s}, \mathcal{F}_{\mathfrak{s}}, \mu)$ be probability triples on m-dimensional space and the coherent subspace respectively.

Definition

The probability measure μ is coherent if

$$\nu(\mathcal{B}) = \mu(s(\mathcal{B})) \quad \forall \mathcal{B} \in \mathcal{F}_{\mathbb{R}_m}$$

where $s(\mathcal{B})$ is the image of $\mathcal{B}$ under premultiplication by $\boldsymbol{S}$

Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then

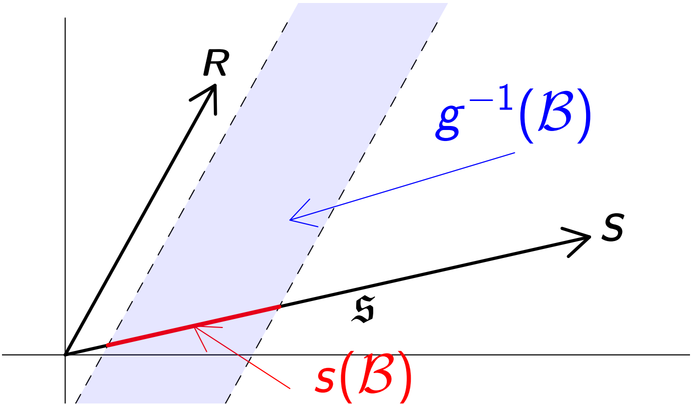
Definition

The probability triple $(\mathfrak{s}, \mathcal{F}_{\mathfrak{s}}, \tilde{\nu})$ reconciles the probability triple $(\mathbb{R}^n, \mathcal{F}_{\mathbb{R}^n}, \hat{\nu})$ with with respect to g iff

$$\tilde{\nu}(s(\mathcal{B})) = \nu(\mathcal{B}) = \hat{\nu}(g^{-1}(\mathcal{B})) \quad \forall \mathcal{B} \in \mathcal{F}_{\mathbb{R}_m}$$

where g^{-1} is the pre-image of g.

Project 2: *Geometry*



Project 2: *Analytically*

If we have an unreconciled density the reconciled density can be obtained by linear transformations and marginalisation.

$$\begin{aligned}
\Pr(\tilde{\boldsymbol{b}} \in \mathcal{B}) &= \Pr(\hat{\boldsymbol{y}} \in g^{-1}(\mathcal{B})) \\
&= \int\limits_{g^{-1}(\mathcal{B})} f(\hat{\boldsymbol{y}}) d\hat{\boldsymbol{y}} \\
&= \int\limits_{\mathcal{B}} \int f(\boldsymbol{S}\tilde{\boldsymbol{b}} + \boldsymbol{R}\tilde{\boldsymbol{a}}) |\left(\boldsymbol{S} \; \boldsymbol{R}\right)| d\tilde{\boldsymbol{a}} d\tilde{\boldsymbol{b}}
\end{aligned}$$

Project 2: *For elliptical distributions*

Consider the case where the base and true predictive distributions are elliptical.

Theorem

There exists a matrix $\boldsymbol{G}$ *such that the true predictive distribution can be recovered by linear reconciliation.*

This follows from the closure property of elliptical distributions under affine transformations and marginalisation.

Project 2: *Assuming Gaussian distribution*

- Let $\mathcal{N}(\hat{\boldsymbol{y}}_{T+h}, \boldsymbol{W}_{T+h})$ be an incoherent forecast distribution at time $T+h$ where $\hat{\boldsymbol{y}}_{T+h}$ is the incoherent mean and $\boldsymbol{W}_{T+h} = E_{\boldsymbol{y}_{T+h}}[(\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h})(\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h})^T | \mathcal{I}_T]$ is the incoherent variance
- The reconciled Gaussian distribution is given by,

$$\mathcal{N}(\boldsymbol{SG}\hat{\boldsymbol{y}}_{T+h}, \ \boldsymbol{SGW}_{T+h}\boldsymbol{G}'\boldsymbol{S}')$$

- $\boldsymbol{G} = \left(\boldsymbol{S}'\boldsymbol{W}_{T+h}^{-1}\boldsymbol{S}\right)^{-1}\boldsymbol{S}'\boldsymbol{W}_{T+h}^{-1}$ minimizes the energy score in the limiting case A1
- Simulation study evident for improved predictive performance in reconciled Gaussian forecast distributions

evidence?

Today's talk

- A non-parametric bootstrap approach for probabilistic ~~forecasts~~ forecast reconciliation

- Hierarchical forecasts ~~in~~ for macroeconomic variables - An application to Australian GDP ~~forecasts~~

Part 2: A non-parametric bootstrap approach for probabilistic forecasts reconciliation

Probabilistic forecast reconciliation: Non-parametric approach

? unsuitable?

- Often parametric densities are unavailable but we can simulate a sample from the predictive distribution
- Suppose $\hat{\boldsymbol{y}}_{T+h}^{[1]}, \ldots, \hat{\boldsymbol{y}}_{T+h}^{[J]}$ is a sample from the incoherent predictive distribution
- Then setting $\tilde{\boldsymbol{y}}_{T+h}^{[j]} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h}^{[j]}$ produces a sample from the reconciled predictive distribution with respect to $\boldsymbol{G}$

Probabilistic forecast reconciliation: Non-parametric approach

1. Fit univariate models at each node using data up to time T
2. Let $\boldsymbol{\Gamma}_{(T\times n)} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_T)'$ be a matrix of in-sample residuals where $\boldsymbol{e}_t = \boldsymbol{y}_t - \hat{\boldsymbol{y}}_t$
3. Let $\boldsymbol{\Gamma}^b_{(H\times n)} = (\boldsymbol{e}^b_1, ..., \boldsymbol{e}^b_H)'$ be a block bootstrap sample of size h from $\boldsymbol{\Gamma}$ and repeat this for $b = 1, ..., B$.
4. Generate h-step ahead sample paths from the fitted models incorporating $\boldsymbol{\Gamma}^b$. Denote these by $\hat{\boldsymbol{y}}^b_{T+h}$, for $h = 1, ..., H$.
5. Repeat step 4 for $b = 1, ..., B$ times
6. Setting $\tilde{\boldsymbol{y}}^b_{T+h,j} = \boldsymbol{SG}\hat{\boldsymbol{y}}^b_{T+h,j}$ produces a sample from the reconciled distribution

Optimal reconciliation of future paths

- We propose to find an optimal $\boldsymbol{G}$ matrix by minimizing Energy score

$$\underset{\boldsymbol{G}_h}{\text{argmin}} \quad \mathrm{E}_{\boldsymbol{Q}}[eS(\boldsymbol{S}\boldsymbol{G}_h \hat{\boldsymbol{\Upsilon}}'_{T+h}, \boldsymbol{y}_{T+h})],$$

where,

$$\mathrm{eS}(\tilde{\boldsymbol{\Upsilon}}_{T+h}, \boldsymbol{y}_{T+h}) = \mathrm{E}_{\tilde{\boldsymbol{F}}}\|\tilde{\boldsymbol{Y}}_{T+h} - \boldsymbol{y}_{T+h}\|^{\alpha} - \frac{1}{2}\mathrm{E}_{\tilde{\boldsymbol{F}}}\|\tilde{\boldsymbol{Y}}_{T+h} - \tilde{\boldsymbol{Y}}^{*}_{T+h}\|^{\alpha}, \quad \alpha \in (0,2]$$

- Monte-Carlo approximation to the above objective function is,

$$\underset{\boldsymbol{G}}{\text{argmin}} \quad \sum_{i=1}^{N}\left\{\frac{1}{B}\sum_{j=1}^{B}||\boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}^{b}_{T+h,i,j} - \boldsymbol{y}_{T+h}|| - \frac{1}{2(B-1)}\sum_{j=1}^{B-1}||\boldsymbol{S}\boldsymbol{G}(\hat{\boldsymbol{y}}^{b}_{T+h,i,j} - \hat{\boldsymbol{y}}^{b}_{T+h,i,j+1})||\right\}$$

Optimal reconciliation of future paths *Cont.*

- We impose the following structure to the $\boldsymbol{G}$ matrix

$$\boldsymbol{G} = \left(\boldsymbol{S}'\boldsymbol{W}\boldsymbol{S}\right)^{-1}\boldsymbol{S}'\boldsymbol{W} \tag{1}$$

- We propose four methods to optimise $\boldsymbol{G}$

Method 1: Optimising $\boldsymbol{W}$

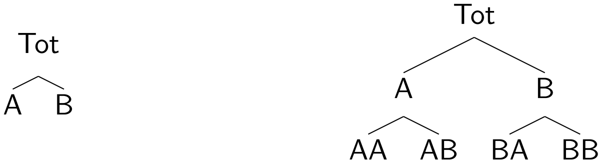
Method 2: Optimising cholesky decomposition of $\boldsymbol{W}$
$\boldsymbol{W} = \boldsymbol{R}'\boldsymbol{R}$ where $\boldsymbol{R}$ is an upper triangular matrix

Method 3: Optimising cholesky of $\boldsymbol{W}$ - restricted for scaling
$\boldsymbol{W} = \boldsymbol{R}'\boldsymbol{R}$ s.t $\boldsymbol{i}'\boldsymbol{W}\boldsymbol{i} = 1$ where $\boldsymbol{i} = (1, 0, .., 0)'$

Method 4: Optimising $\boldsymbol{G}$ such that $\boldsymbol{G}\boldsymbol{S} = \boldsymbol{I}$

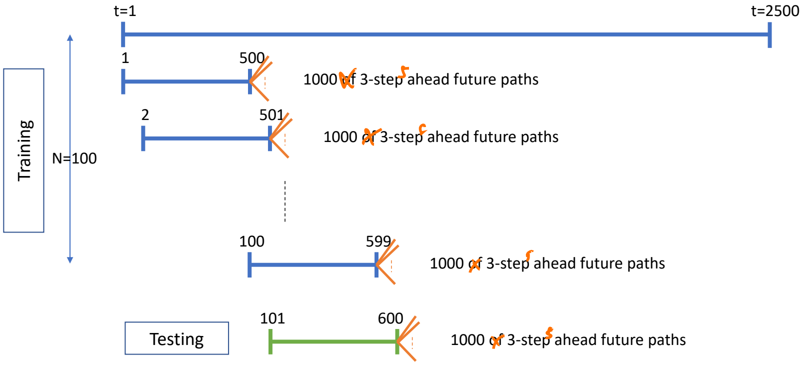
Monte-Carlo Simulation

- Data generating process



- DGP was designed such that we have much noisier series in the bottom level

Monte-Carlo Simulation



Monte-Carlo Simulation *Cont.*

- Simulation setup:
  - First 2500 observations were generated
  - Fit Univariate ARIMA models for a rolling window of 500 observations
  - $B = 1000$ of $h = 1, 2, 3$ steps-ahead bootstrap future paths generated
  - Training window is rolled one observation ahead and process was repeated until ~~we get~~ $N = 100$ ~~of~~ incoherent, $h = 1, 2, 3$ steps-ahead future paths were generated
  - We find the optimal $\boldsymbol{G}_h$ for $h = 1, 2, 3$ that reconciles h-step-ahead future paths giving minimal average Energy score
  - This optimal $\boldsymbol{G}_h$ is then used to reconcile the incoherent future paths ~~obtained~~ for the test set
  - The Process was repeated ~~for~~ 1000 times and average scores were calculated for the test set

Monte-Carlo Simulation *Cont.*

Optimisation method	Hierarchy 1				Hierarchy 2			
	$h = 1$		$h = 3$		$h = 1$		$h = 3$	
	ES	VS	ES	VS	ES	VS	ES	VS
Method 1 - Optimising $\boldsymbol{W}$	2.48	0.11	2.75	0.11	5.36	1.21	5.83	1.38
Method 2 - Optimising $\boldsymbol{R}$	2.48	0.11	2.75	0.11	5.37	1.21	5.83	1.37
Method 3 - Optimising $\boldsymbol{R}$(Restricted)	2.48	0.11	2.75	0.11	5.37	1.21	5.83	1.37
Method 4 - Optimising $\boldsymbol{G}$	2.48	0.11	2.75	0.11	5.38	1.21	5.83	1.38

- Parameterisation does not matter

Monte-Carlo Simulation *Cont.*

- Comparison with point forecast reconciliation methods

Reconciliation method	Hierarchy 1				Hierarchy 2			
	$h=1$		$h=3$		$h=1$		$h=3$	
	ES	VS	ES	VS	ES	VS	ES	VS
Optimal $\boldsymbol{G}$	2.48*	0.106	2.75*	0.106	5.36*	1.21*	5.83*	1.38*
MinT(Shrink)	2.47*	0.105	2.74*	0.105	5.33*	1.19*	5.77*	1.34*
WLS	2.46*	0.105	2.74*	0.105	5.43*	1.23	5.98*	1.40*
OLS	2.54*	0.105	2.80*	0.105	5.51*	1.23	5.98*	1.40*
Base	*2.67*	*0.105*	*2.94*	*0.105*	*5.71*	*1.28*	*6.27*	*1.49*

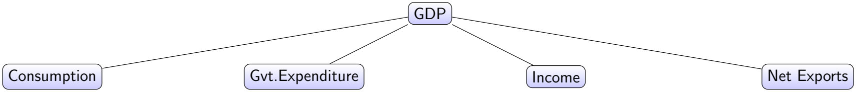
* Add definition

- Reconciliation methods perform better than Base forecasts

- MinT(Shrink) is at least as good as Optimal method. Thus going forward with MinT projection

Hierarchical forecasts ~~in~~ for macroeconomic variables - An application to Australian GDP ~~forecasts~~

Macroeconomic forecasting



- Common forecasting approaches involves univariate methods of multivariate methods such as VAR.
- The era of big data led to the use of regularization and shrinkage methods - dynamic factor models, Lasso, Bayesian VARs
- The predictors in these methods commonly include the components of the variables of interest.
- This might fail to reflect the deterministic relationship between macroeconomic variables in the forecasts.

Macroeconomic forecasting

- Both aligned decision making and forecast accuracy are key concerns for economic agents and policy makers
- Thus we propose to use hierarchical forecasting methods in macroeconomic forecasts.
- Related literature: Only one application on point forecasting for inflation (Capistrán, Constandse, and Ramos-Francia, 2010; Weiss, 2018)
- To the best of our knowledge we use hierarchical forecasting methods for point as well as probabilistic forecasts for the first time in macroeconomic literature

Australian GDP : *Data structures*

- We consider Gross Domestic Product (GDP) of Australia with quarterly data spanning the period 1984:Q4–2018:Q3
- The Australian Bureau of Statistics (ABS) measures GDP using three main approaches - Production, Income and Expenditure

  The final GDP figure is obtained as an average of these three figures.
- We restrict our attention to nominal, seasonally unadjusted data
- Thus we concentrate on the Income and Expenditure approaches

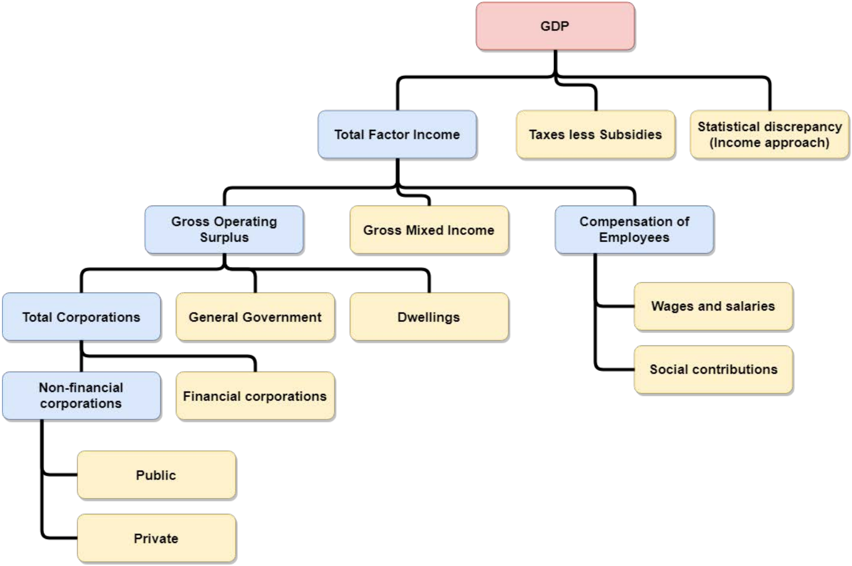
Australian GDP : *Data structures*

Income approach

$$
\begin{aligned}
GDP = {} & \textit{Gross operating surplus} + \textit{Gross mixed income} \\
& + \textit{Compensation of employees} \\
& + \textit{Taxes less subsidies on production and imports} \\
& + \textit{Statistical discrepancy (I)}.
\end{aligned}
$$

- The hierarchy has two levels of aggregation below the top-level, with a total of $n = 16$ series and $m = 10$ bottom level series

Australian GDP : *Data structures - Income approach*



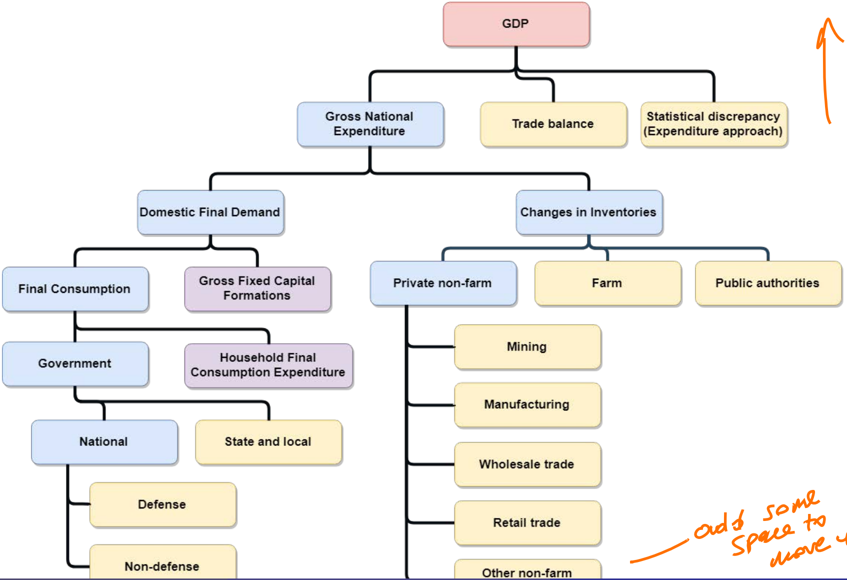
Australian GDP : *Data structures*

Expenditure approach

$$GDP = \textit{Final consumption expenditure} + \textit{Gross fixed capital formation} \\ + \textit{Changes in inventories} + \textit{Trade balance} \\ + \textit{Statistical discrepancy (E)}.$$

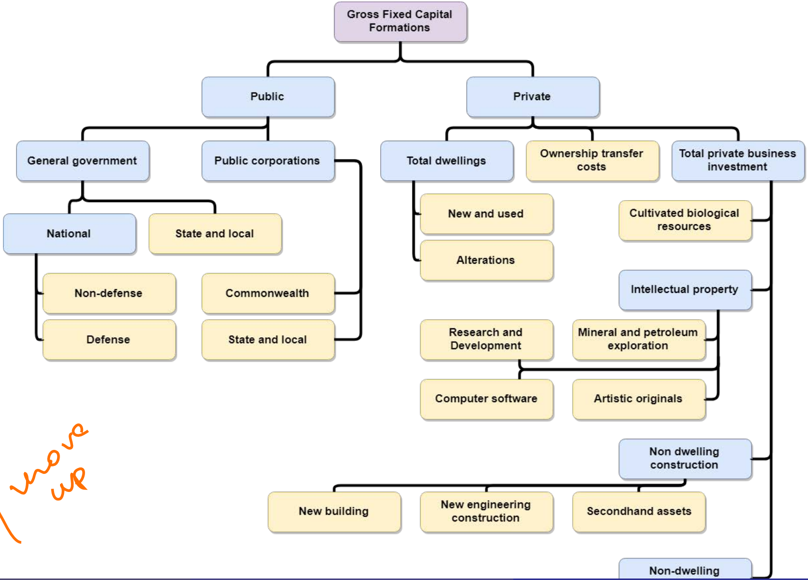
- The hierarchy has three levels of aggregation below the top-level, with a total of $n = 80$ series and $m = 53$ series at the bottom level.

Australian GDP : *Data structures - Expenditure approach*



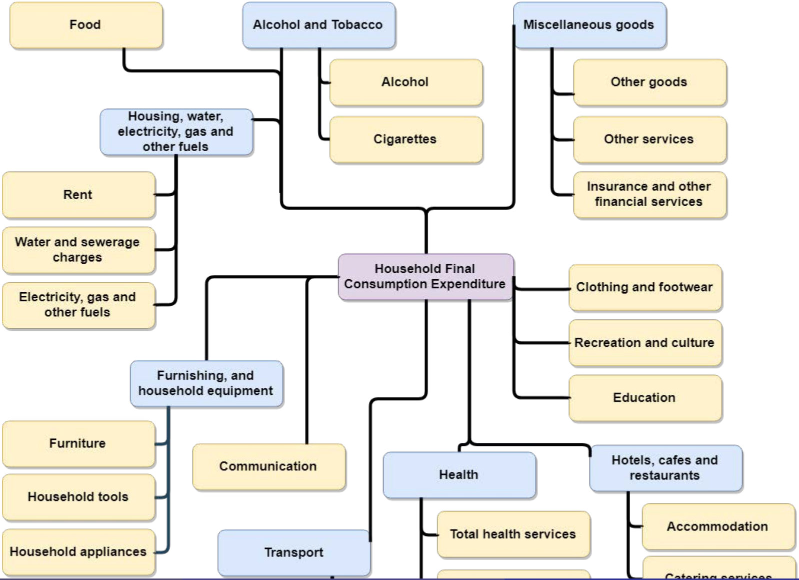
add some space to move up

Australian GDP : *Data structures - Expenditure approach*



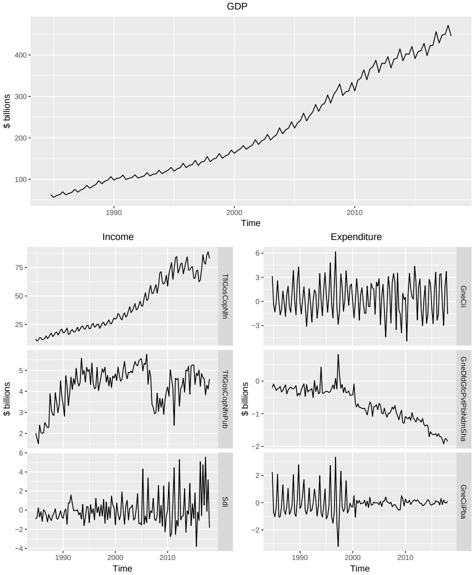
move up

Australian GDP : *Data structures - Expenditure approach*



move up

Australian GDP : *Time plots for different levels*



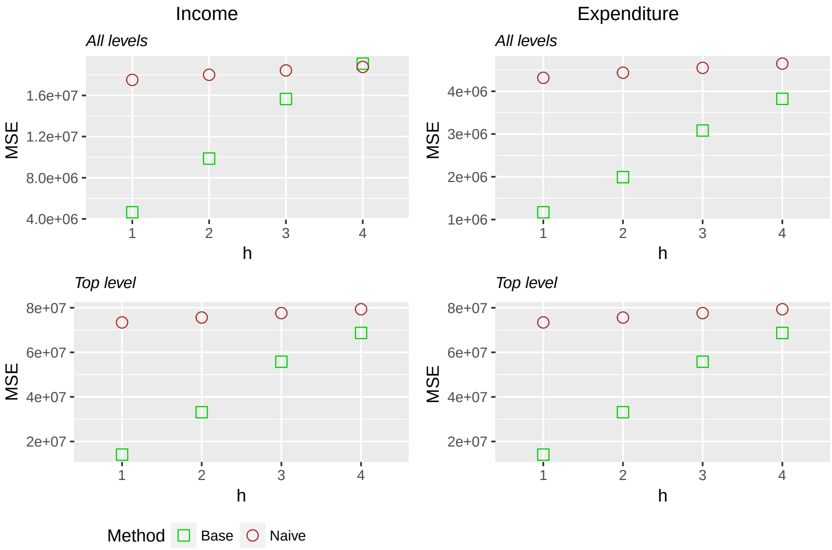
Methodology

- **Analysis set up:**
  - First training sample is set from 1984:Q4 to 1994:Q3 and forecasts produces for four quarters ahead (1994:Q4 to 1995:Q3)
  - Then the training window is expanded by one quarter at a time
  - This leads to 94 1-step-ahead, 93 2-steps-ahead, 92 3-steps-ahead and 91 4-steps-ahead forecasts available for evaluation.

- **Base forecasting models:**
  - Univariate ARIMA and ETS models were fitted for each training set
  - Four step ahead forecasts were generated using the fitted models

One to four

Point forecasts: *Base vs Naive*



Point forecasts: *Reconciliation*

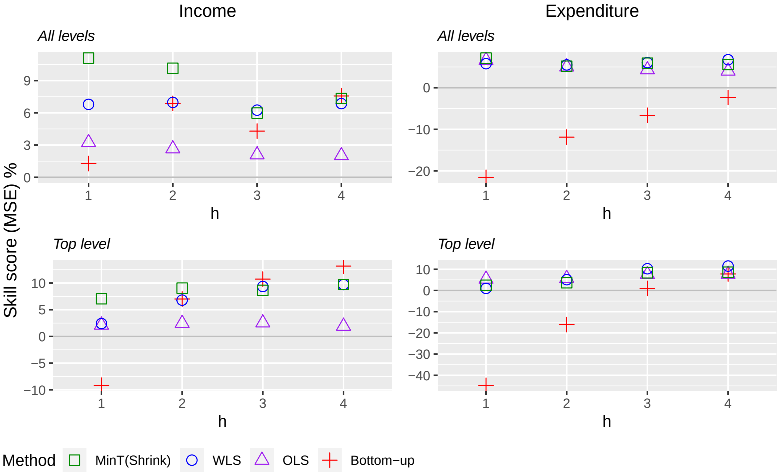
Reconciled forecasts are given by,

$$\tilde{\boldsymbol{y}}_{T+h} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h}$$

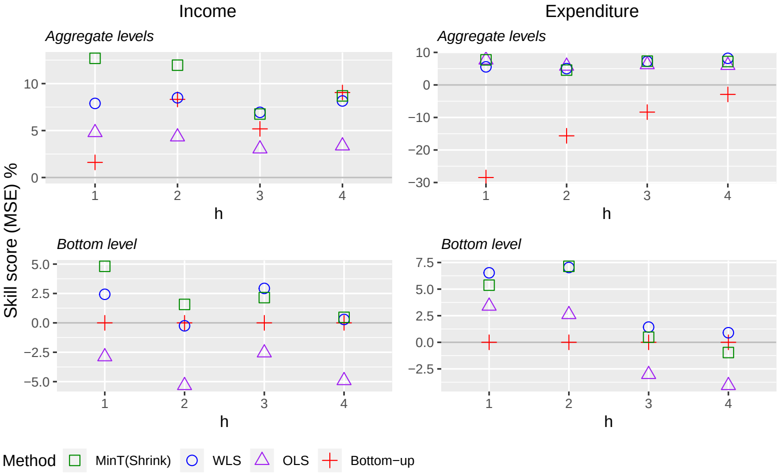
Method	$\boldsymbol{G}$
BU	$(\boldsymbol{0}_{m\times n-m} \ \boldsymbol{I}_{m\times m})$
OLS	$(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$
WLS	$\left(\boldsymbol{S}'\hat{\boldsymbol{W}}^{wls}_{T+1}\boldsymbol{S}\right)^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}^{wls}_{T+1}$
MinT(Shrink)	$\left(\boldsymbol{S}'\hat{\boldsymbol{W}}^{shr}_{T+1}\boldsymbol{S}\right)^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}^{shr}_{T+1}$

$$\begin{aligned}
\hat{\boldsymbol{W}}^{shr}_{T+1} &= \tau\text{Diag}(\hat{\boldsymbol{W}}^{sam}_{T+1}) + (1-\tau)\hat{\boldsymbol{W}}^{sam}_{T+1} \\
\hat{\boldsymbol{W}}^{wls}_{T+1} &= \text{Diag}(\hat{\boldsymbol{W}}^{shr}_{T+1})
\end{aligned}$$

Reconciled Point forecasts - *Results*



Reconciled Point forecasts - *Results*



Probabilistic forecasts

- Gaussian approach :

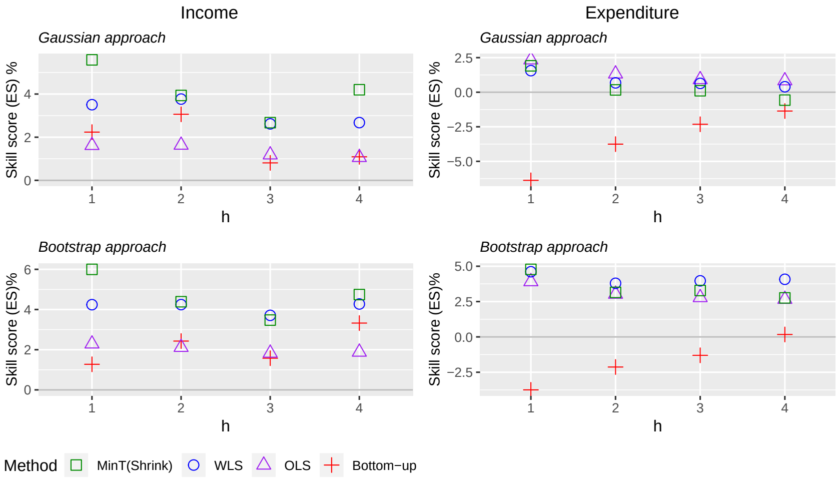
$$\mathcal{N}(\boldsymbol{SG}\hat{\boldsymbol{y}}_{T+h},\ \boldsymbol{SGW}_{T+h}\boldsymbol{G}'\boldsymbol{S}')$$

- Non-parametric Bootstrap approach :

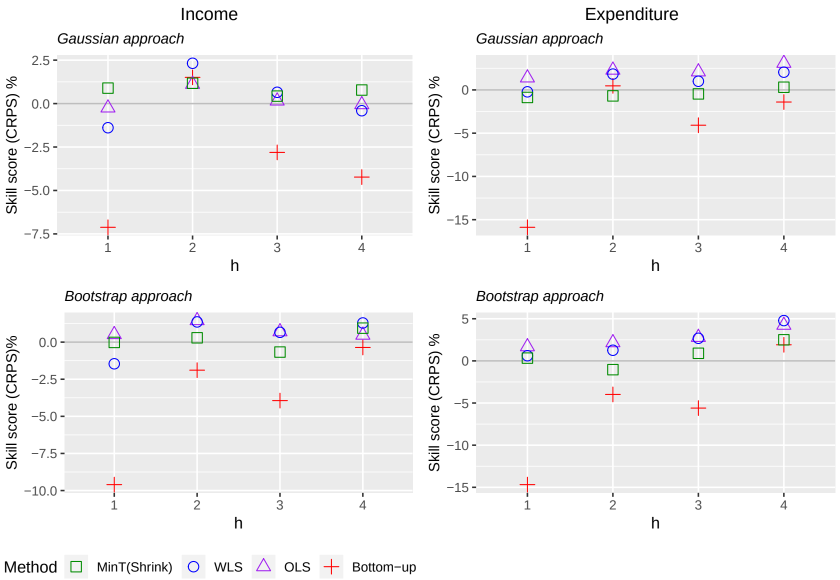
$$\tilde{\Upsilon}_{T+h} = \boldsymbol{SG}\hat{\Upsilon}'_{T+h}$$

where, $\hat{\Upsilon}_{T+h} = (\hat{\boldsymbol{y}}^1_{T+h}, ..., \hat{\boldsymbol{y}}^B_{T+h})'$

Reconciled Probabilistic Forecasts



Reconciled Probabilistic Forecasts



Summary and time plan for completion

Summary

- We define point and probabilistic forecast reconciliation in geometric terms.
- We propose a parametric approach for probabilistic forecast reconciliation.
- We introduce a novel non-parametric bootstrap approach for producing reconciled probabilistic forecasts.

provides

- Simulation study evident that the optimal reconciliation with respect to energy score is equivalent to reconciling each sample path via MinT approach.
- We apply hierarchical forecast reconciliation methods to forecast Australian GDP in point as well as probabilistic framework.

Time plan for completion

	Thesis Chapter	Task description	Time duration	Progress
1 and 2.	Introduction and Background Review	Writing the chapter.	September/2019 - October/2019	40% complete
3.	Hierarchical forecast reconciliation in Geometric view	Bias correction and application	May/2019 - July/2019	75% Completed
4.	Probabilistic forecast reconciliation for hierarchical time series	Completing the paper.	June/2019 - August/2019	90% Completed
5.	Application	Forecasting Australian GDP		100% Completed

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Thank You!!

Project 2: *Probabilistic forecasts evaluation*

A1

Energy score (Gneiting et al., 2008)

$$\mathrm{eS}(\check{\boldsymbol{Y}}_{T+h}, \boldsymbol{y}_{T+h}) = \mathrm{E}_{\check{\boldsymbol{F}}}\|\check{\boldsymbol{Y}}_{T+h} - \boldsymbol{y}_{T+h}\|^{\alpha} - \tfrac{1}{2}\mathrm{E}_{\check{\boldsymbol{F}}}\|\check{\boldsymbol{Y}}_{T+h} - \check{\boldsymbol{Y}}^{*}_{T+h}\|^{\alpha}, \quad \alpha \in (0,2]$$

Log score (Gneiting and Raftery, 2007)

$$\mathrm{LS}(\check{\boldsymbol{F}}, \boldsymbol{y}_{T+h}) = -\log \check{\boldsymbol{f}}(\boldsymbol{y}_{T+h})$$

Variogram score (Scheuerer and Hamill, 2015)

$$\mathrm{VS}(\check{\boldsymbol{F}}, \boldsymbol{y}_{T+h}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\Big(|y_{T+h,i} - y_{T+h,j}|^{p} - \mathrm{E}_{\check{\boldsymbol{F}}}|\check{Y}_{T+h,i} - \check{Y}_{T+h,j}|^{p}\Big)^{2}$$

CRPS (Gneiting and Raftery, 2007)

$$\mathrm{CRPS}(\check{F}_i, y_{T+h,i}) = \mathrm{E}_{\check{F}_i}|\check{Y}_{T+h,i} - y_{T+h,i}| - \tfrac{1}{2}\mathrm{E}_{\check{F}_i}|\check{Y}_{T+h,i} - \check{Y}^{*}_{T+h,i}|$$

$\check{\boldsymbol{Y}}_{T+h}$ and $\check{\boldsymbol{Y}}^{*}_{T+h}$	:	Independent random vectors from the coherent forecast distribution $\check{\boldsymbol{F}}$.
$\boldsymbol{y}_{T+h}$	:	Vector of realizations.
$\check{Y}_{T+h,i}$ and $\check{Y}_{T+h,j}$	:	ith and jth components of the vector $\check{\boldsymbol{Y}}_{T+h}$

Appendix

- A2 Shrinkage estimator for 1-step ahead base forecast errors

$$\hat{\boldsymbol{\Sigma}}^{shr}_{T+1} = \tau \hat{\boldsymbol{\Sigma}}^{D}_{T+1} + (1-\tau)\hat{\boldsymbol{\Sigma}}_{T+1},$$

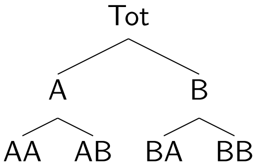
where $\hat{\boldsymbol{\Sigma}}^{D}_{T+1}$ is the diagonal matrix comprising diagonal entries of $\hat{\boldsymbol{\Sigma}}_{T+1}$ and

$$\tau = \frac{\sum_{i \neq j} \hat{Var}(\hat{r}_{ij})}{\sum_{i \neq j} \hat{r}^2_{ij}}$$

is a shrinkage parameter. $\hat{r}_{ij}$ is the ij-th element of sample correlation matrix. In this estimation, the off-diagonal elements of 1-step ahead sample covariance matrix will be shrunk to zero depending on the sparsity.

Monte-Carlo simulation

- **Data generating process** A3



- $\{w_{AA,t}, w_{AB,t}, w_{BA,t}, w_{BB,t}\} \sim ARIMA(p, d, q)$
- $p \in \{1, 2\}$ and $d \in \{0, 1\}$
- $\{\epsilon_{AA,t}, \epsilon_{AB,t}, \epsilon_{BA,t}, \epsilon_{BB,t}\} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$
- Parameters for *AR* and *MA* components were randomly and uniformly generated from $[0.3, 0.5]$ and $[0.3, 0.7]$ respectively

- $\boldsymbol{y}_t$ are then generated as follows

Bottom level	**Aggregate level 1**	**Total**
$y_{AA,t} = w_{AA,t} + u_t - 0.5v_t$	$y_{A,t} = w_{AA,t} + w_{AB,t} - v_t$	$y_{Tot,t} = w_{AA,t} + w_{AB,t} + w_{BA,t} + w_{BB,t}$
$y_{AB,t} = w_{AB,t} - u_t - 0.5v_t$	$y_{B,t} = w_{BA,t} + w_{BB,t} + v_t$	
$y_{BA,t} = w_{BA,t} + u_t + 0.5v_t$		
$y_{BB,t} = w_{BB,t} - u_t + 0.5v_t$		

Monte-Carlo simulation *Cont..*

- To get less noisier series at aggregate levels, we choose $\mathbf{\Sigma}, \sigma_u^2$ and σ_v^2 such that,

$$\text{Var}(\epsilon_{AA,t} + \epsilon_{AB,t} + \epsilon_{BA,t} + \epsilon_{BB,t}) \leq \text{Var}(\epsilon_{AA,t} + \epsilon_{AB,t} - v_t) \leq \text{Var}(\epsilon_{AA,t} + u_t - 0.5v_t),$$

- Thus we choose, $\mathbf{\Sigma} = \begin{pmatrix} 5.0 & 3.1 & 0.6 & 0.4 \\ 3.1 & 4.0 & 0.9 & 1.4 \\ 0.6 & 0.9 & 2.0 & 1.8 \\ 0.4 & 1.4 & 1.8 & 3.0 \end{pmatrix}$, $\sigma_u^2 = 19$ and $\sigma_u^2 = 18$.

Sample version of the scoring rules

- For a possible finite sample of size B from the multivariate forecast density $\breve{\boldsymbol{F}}$, the variogram score is defined as,

$$\mathrm{VS}(\breve{\boldsymbol{F}}, \boldsymbol{y}_{T+h}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\left(|y_{T+h,i} - y_{T+h,j}|^p - \frac{1}{B}\sum_{k=1}^{B}|\breve{Y}^k_{T+h,i} - \breve{Y}^k_{T+h,j}|^p\right)^2$$