Claims regarding parallel composition.

Consider the case of the parallel composition of d differentially private algorithms. Intuitively, since the input data of different algorithms are independent with each other, we could partition the input dataset into d independent datasets. The change of one element could only influence one of these independent datasets. Therefore, if every private algorithm is (ε, δ)-differentially private, then the composed private algorithm is (ε, δ)-differentially private. We give a formal proof as follows.

- **Theorem** Let M_i each provide (ε, δ)-differential privacy. Let D_i be arbitrary disjoint subsets of the input domain D. For any input dataset X, the sequence of $M_i(X \cap D_i)$ provides (ε, δ)-differential privacy.

- **Proof**. Let X, X' be neighboring datasets. Suppose that they are both divided into d subsets of disjoint data, where $X_i = X \cap D_i$ and $X'_i = X' \cap D_i$. Without loss of generality, X and X' are only different between X_1 and X'_1 for one element. For any $r_1 \subseteq Range(M_1)$, We have:

$$Pr[M_1(X_1) \in r_1] \leq e^{\varepsilon} Pr[M_1(X'_1) \in r_1] + \delta.$$

  For any $r \subseteq Range(M)$ and $r_i \subseteq Range(M_i)$, where M is the sequence of M_i, the probability of output from the sequence of $M(X)$ is

$$\begin{aligned}
Pr[M(X) \in r] &= \prod_{i=1}^{d} Pr[M_i(X_i) \in r_i] \\
&= \prod_{i=2}^{d} Pr[M_i(X_i) \in r_i] Pr[M_1(X_1) \in r_1] \\
&\leq \prod_{i=2}^{d} Pr[M_i(X_i) \in r_i](e^{\varepsilon} Pr[M_1(X'_1) \in r_1] + \delta) \\
&= e^{\varepsilon} Pr[M_1(X'_1) \in r_1] \prod_{i=2}^{d} Pr[M_i(X_i) \in r_i] + \delta \prod_{i=2}^{d} Pr[M_i(X_i) \in r_i] \\
&\leq e^{\varepsilon} Pr[M(X') \in r] + \delta,
\end{aligned}$$

  which completes the proof.