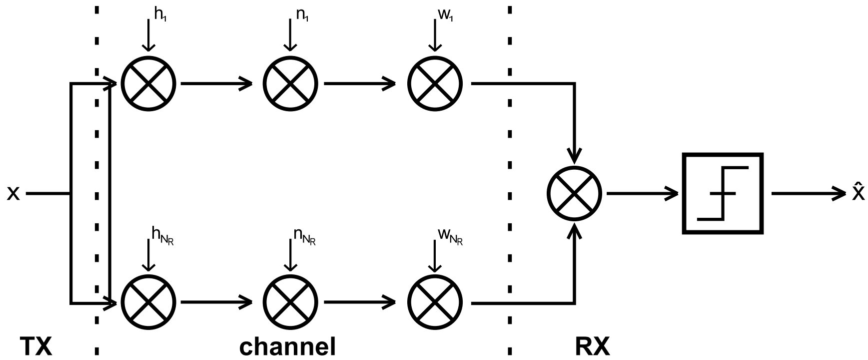
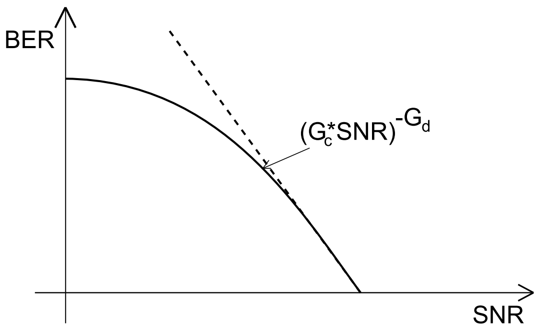
1 SIMO Systems

Remarks

- In SIMO Systems only coding and diversity gains can be exploited (no multiplexing gains)
- To realize these gains diversity combining has to be performed
- Diversity combining schemes vary in complexity and performance
- There are many diversity combining schemes. Here we consider:
  - Maximal ratio combining (MRC)
  - Equal gain combining (EGC)
  - Selection combining (SC)
- Diversity combining problem



  - how to choose combining weights w_n?
  - what performance (e.g. error rate, outage probability) is achieved?
  - what diversity and coding/combining gain is achieved?



- G_c : Coding gain
- G_d : Diversity gain

1.1 Preliminaries

Consider an equivalent system:

$$y = hx + n;$$
$$\mathcal{E}\{|x^2|\} = E_s; \qquad \mathcal{E}\{|n^2|\} = \sigma_n^2; \qquad \mathcal{E}\{|h|^2\} = 1$$

- Instantaneous SNR: $\gamma_t = \frac{E_s}{\sigma_n^2} \cdot |h|^2$
- Average SNR: $\bar{\gamma}_t = \mathcal{E}\{\gamma_t\} = \frac{E_s}{\sigma_n^2}$

Bit and Symbol Error Rate

- The Bit and Symbol Error Rate of many modulation schemes can be expressed for given γ_t as:

$$P_e(\gamma_t) = aQ\{\sqrt{b\gamma_t}\}$$

where:

- $Q(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_x^\infty e^{-\frac{t^2}{2}}\ dt$
- $P_e(\gamma_t)$ may be exact result or approximation
- BPSK: exact with $a = 1, b = 2$
- M-ary QAM: tight approximation with $a = 4\big(1 - \frac{1}{\sqrt{M}}\big), b = \frac{3}{M-1}$

$\big(Einschub : Gray - Code : BER = \frac{1}{\log_2 M} \cdot SER\big)$

- Alternative representation of Q - function:

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2sin^2\theta}}\ d\theta$$

  $\rightarrow$ Integral limits are fixed and do not depend on integration variables!

- Average error probability

$$P_e = \mathcal{E}\big\{P_e(\gamma_t)\big\} = \int_0^\infty aQ\big(\sqrt{bx}\big)p_{\gamma_t}(x)\ \mathrm{d}x$$

  - Integral may be difficult to solve analytically
  - Integral has infinite support $\rightarrow$ numerical evaluation difficult

- Using alternative representation of Q-function we get:

$$P_e = \int_0^\infty \frac{a}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{bx}{2sin^2\theta}} p_{\gamma_t}(x)\ d\theta\ dx$$
$$= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} \int_0^\infty p_{\gamma_t}(x) e^{-\frac{b}{2sin^2\theta}}\ dx\ d\theta \qquad = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\big(\frac{b}{2sin^2\theta}\big)\ d\theta$$

  where:

- $M_{\gamma_t}(s) = \int_0^\infty p_{\gamma_t}(x)e^{-sx}\ dx$ is the Laplace transform of p_{γ_t}
- $M_{\gamma_t}(-s)$ is the so called Moment Generation Function (MGF) of p_{γ_t}
- Here, we will also refer to $M_{\gamma_t}(s)$ as MGF
- $M_{\gamma_t}(s)$ is sometimes easier to obtain than p_{γ_t}
- The above integral can be easily evaluated numerically because of the finite integral limits

Outage probability

- The outage probability is the probability that the channel cannot support a certain rate, R, i.e. (where γ_T is the threshold SNR):

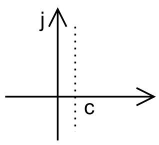
$$C = \log_2(1+\gamma_t) < R \quad \leftrightarrow \quad \gamma_t < 2^R - 1 \triangleq \gamma_T$$

Thus, the outage probability is given by:

$$P_{out} = P_0\gamma_t < \gamma - T \ = \int_0^{\gamma_T} p_{\gamma_t}(x)\ dx$$

- Using the inverse Laplace Transform

$$p_{\gamma_t}(x) = \frac{1}{2\pi j}\int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s)e^{sx}\ dx$$



where $c > 0$ is a small constant that lies in the region of convergence of the integral, we obtain:

- 1.

$$P_{out} = \frac{1}{2\pi j}\int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s)\int_0^{\gamma_T} e^{sx}\ dx\ ds = \frac{1}{2\pi j}\int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s)e^{\gamma_T s}\ \frac{ds}{s}$$

$\big($lower integral limit is 0 since $p_{\gamma_t}(0) = 0\big)$

- and 2.:

$$p_{\gamma_t}(x) = \int_0^x p_{\gamma_t}(t)\ dt = 0$$

$$\text{for } x = 0 \text{ note: } p_{\gamma_t}(x) \xleftrightarrow[transform]{Laplace} \frac{1}{s}M_{\gamma_t}(s)$$

General combining scheme

$$y = \Big(\sum_{n=1}^{N_R} h_n w_n\Big)x + \sum_{n=1}^{N_R} w_n n_n$$

$$\gamma_t = \frac{\epsilon_s\left|\sum_{n=1}^{N_R} h_n w_n\right|^2}{\sigma_n^2\sum_{N=1}^{N_R}|w_n|^2}$$

where w_n depends on the particular combining scheme.

1.2 MRC (Maximum Ratio Combining)

- what weight w_n maximize γ_t?
  - Cauchy-Schwarz inequality

$$\Big|\sum_{n=1}^{N_R} h_n w_n\Big|^2 \leq \sum_{n=1}^{N_R} |h_n|^2 \cdot \sum_{n=1}^{N_R} |w_n|^2$$

    where equality holds if and only if $w_n = c \cdot h_n^*$ for some non-zero constant c.
  - for $w_n = h_n^*$, we obtain

$$\gamma_t = \frac{\epsilon_s}{\sigma_n^2} \cdot \frac{\left(\sum_{n=1}^{N_R} |h_n|^2\right)^2}{\sum_{n=1}^{N_R} |h_n|^2} = \frac{\epsilon_s}{\sigma_n^2} \sum_{n=1}^{N_R} |h_n|^2$$

  - $w_n = h_n^* \forall n$ are the MRC combining weights.
- For performance analysis we assume independent identically distributed (IID) Rayleigh fading

$$\rightarrow \mathcal{E}\{|h_n|^2\} = 1; \quad \bar{\gamma} = \frac{\epsilon_s}{\sigma_n^2}; \quad \gamma_n = \frac{\epsilon_s}{\sigma_n^2}|h_n|^2$$

$$p_\gamma(x) = \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0$$

$$M_\gamma(s) = \frac{1}{1 + s\bar{\gamma}}$$

- Error rate

$$\gamma_t = \sum_{n=1}^{N_R} \gamma_n$$

  $\rightarrow$ sum of IID random variables (r.v.s.)

$$M_{\gamma_t}(s) = \Big(M_\gamma(s)\Big)^{N_R} = \frac{1}{(1 + s\bar{\gamma})^{N_R}} = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{1}{(s + \frac{1}{\bar{\gamma}})^{N_R}}$$

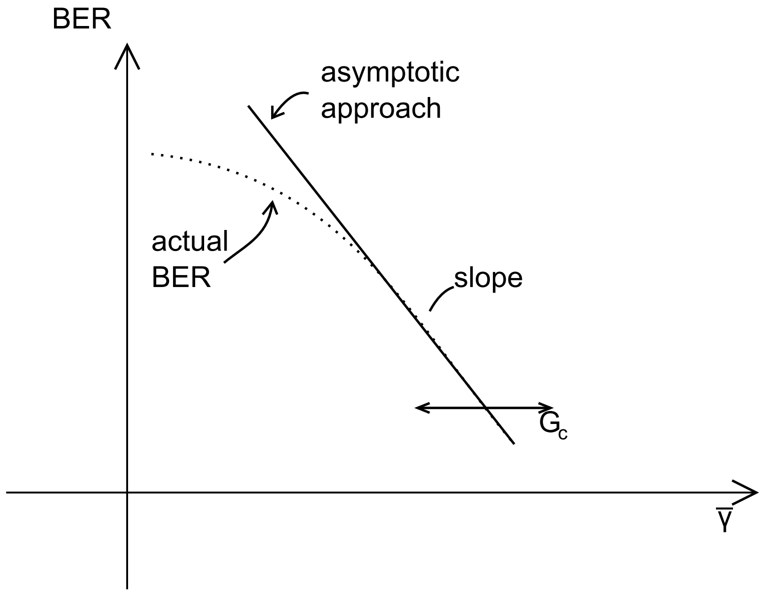
  inverse Laplace-transform (from tables)

$$p_{\gamma_t}(x) = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{x^{N_R - 1}}{(N_R - 1)!} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0$$

- Direct approach

$$p_e = \int_0^\infty a \cdot Q(\sqrt{ax}) p_{\gamma_t}(x)\, dx = a\Big(\frac{1-\mu}{2}\Big)^{N_R} \cdot \sum_{n=0}^{N_R - 1} \binom{N_R - 1 + n}{n} \Big(\frac{1+\mu}{2}\Big)^n$$

  where $\mu = \sqrt{\dfrac{b\bar{\gamma}}{2 + b\bar{\gamma}}}$



- MGF approach

$$p_e = \frac{a}{\pi}\int_0^{\frac{\pi}{2}} M_{\gamma_t}\Big(\frac{b}{2\sin^2\theta}\Big)\, d\theta$$
$$= \frac{a}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{\bar{\gamma}^{N_R}\big(\frac{b}{\sin^2\theta}+\frac{1}{\bar{\gamma}}\big)^{N_R}}\, d\theta \quad \text{(numerisch berechnen!)}$$

- high SNR: $\bar{\gamma}\to\infty \iff \frac{1}{\bar{\gamma}}\to 0$

$$p_e = \frac{a}{\pi}\cdot\frac{1}{\bar{\gamma}^{N_R}}\cdot\Big(\frac{2}{b}\Big)^{N_R}\int_0^{\frac{\pi}{2}}\sin^{2N_R}\theta\, d\theta$$

(from MGF approach: $\int_0^{\frac{\pi}{2}}\sin^{2N_R}\theta\, d\theta = \frac{\pi}{2^{N_R+1}}\cdot\binom{2N_R}{N_R})$

$$= \frac{a}{2^{N_R+1}\cdot b^{N_R}}\begin{pmatrix}2N_R & N_R\end{pmatrix}\frac{1}{\bar{\gamma}^{N_R}} \quad \text{as } \bar{\gamma}\to\infty$$
$$\overset{!}{=}\Big(\frac{1}{G_c\bar{\gamma}}\Big)$$

where: Diversity gain: $G_d = N_R$

Combining/Coding gain: $G_c = 2b\Big(\frac{a}{2}\binom{2N_R}{N_R}\Big)^{-\frac{1}{N_R}}$

- MRC exploits the maximal possible diversity
- Diversity gain is not affected by correlation as the branches are not <u>fully</u> correlated
- Diversity gain depends on fading distribution

Outage probability

$$P_{out} = \int_0^{\gamma_T} p_{\gamma_t}(x)\ dx = \frac{1}{\bar{\gamma}^{N_R}} \int_0^{\gamma_T} \frac{x^{N_R-1}}{(N_R-1)!} e^{-\frac{x}{\bar{\gamma}}}\ dx$$
$$= 1 - e^{-\frac{\gamma_T}{\bar{\gamma}}} \cdot \sum_{n=1}^{N_R} \frac{\left(\frac{\gamma_T}{\bar{\gamma}}\right)^n}{(n-1)!}$$

- Approximation (Taylor series): $\bar{\gamma} \to \infty$: $-e^{-\frac{x}{\bar{\gamma}}} = 1 - \frac{x}{\bar{\gamma}} + O(\frac{1}{\bar{\gamma}})$ where a function $f(x)$ is $O(x)$ if $\lim\limits_{x\to\infty} \frac{f(x)}{x} = 0$.

$$\Rightarrow P_{out} = \frac{1}{\gamma^{N_R}} \int_0^{\gamma_T} \frac{x^{N_R-1}}{(N_R-1)!} \left(1 - \frac{x}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right)\right)$$

- Diversity and coding gain can also be defined for P_{out}

1.3 EGC (Equal Gain Combining)

Combining Weights

- For MRC, both, the amplitudes and phases of the channel gains $h_n = |h_n| e^{j\varphi_n}$ have to be known (or estimated in practice)
- In EGC it is assumed that only the phases are known and weights $w_n = e^{-j\varphi_n}$ are used.

$$\Rightarrow \gamma_t = \frac{E_s}{\sigma_n^2} \frac{\left|\sum\limits_{n=1}^{N_R} |h_n| e^{j\varphi_n} e^{-j\varphi_n}\right|^2}{\sum\limits_{n=1}^{N_R} |e^{-j\varphi_n}|^2} = \frac{E_s}{\sigma_n^2} \frac{1}{N_R} \left(\sum_{n=1}^{N_R} |h_n|\right)^2$$
$$= \frac{1}{N_R} \left(\sum_{n=1}^{N_R} \sqrt{\gamma_n}\right)^2 ;\ \text{with } \gamma_n = \frac{E_s}{\sigma_n^2} |h_n|^2$$

Performance Analysis

- IID case
  $\Rightarrow \sqrt{\gamma_n}$ is Rayleigh distributed
  $\Rightarrow$ Exact analysis is much more difficult than for MRC $\Rightarrow$ see book by Simon & Alouini p.341
- Approximate result

$$P_e = \frac{a}{2} \left[1 - \sqrt{\frac{2b\bar{\gamma}}{5 + 2b\bar{\gamma}}} \sum_{n=0}^{N_R-1} \frac{\binom{2n}{n}}{4^n (1 + \frac{2}{5} b\bar{\gamma})^n} \right]$$

- high SNR
  $\Rightarrow$ use high SNR analysis of Wang & Giannakis, 2003
  $\Rightarrow$ at high SNR, only pdf of γ_n around 0 is relevant for performance

$$\Rightarrow \overset{\text{Rayleigh}}{p_\gamma(x)} = \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}} \overset{\text{Taylor Serie}}{=} \frac{1}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right) \text{ as } x \to 0$$

- need pdf γ_t: (γ_n bekannt, $\to$ ges.: Wurzel, etc.)
  (cumulative distribution function of $\sqrt{\gamma}$ ($\overset{\text{i.i.d}}{=} \sqrt{\gamma_n}$) (cdf))

$$P_{\sqrt{\gamma}}(x) = \Pr\left\{\sqrt{\gamma} \le x\right\} = \Pr\left\{\gamma \le x^2\right\} = P_\gamma(x^2) = \text{cdf of } \gamma$$
$$\to p_{\sqrt{\gamma}}(x) = \frac{d}{dx} P_{\sqrt{\gamma}}(x) = 2x \cdot p_\gamma(x^2) = \frac{2x}{\bar{\gamma}} + O\big(\frac{1}{\bar{\gamma}}\big)$$

- Laplace Transformation to MGF

$$\to M_{\sqrt{\gamma}}(s) = \mathcal{L}\left\{p_{\sqrt{\gamma}}(x)\right\} = \frac{2}{\bar{\gamma}} \cdot \frac{1}{s^2} + O\big(\frac{1}{\bar{\gamma}}\big)$$
$$\sqrt{\gamma_t} = \sum_{n=1}^{N_R} \frac{\sqrt{\gamma_n}}{N_R}$$
$$M_{\sqrt{\gamma_t}}(s) = \mathcal{E}\Big\{\exp(-s\sqrt{\gamma_t})\Big\} = \mathcal{E}\Big\{\exp(-\frac{s}{\sqrt{N_R}} \cdot \sum_{n=1}^{N_R} \sqrt{\gamma_n})\Big\} = \Big(\mathcal{E}\Big\{\exp(-\frac{s}{\sqrt{N_R}} \cdot \sqrt{\gamma_n}\Big\}\Big)^{N_R}$$
$$= \Big(M_{\sqrt{\gamma}}(\frac{s}{\sqrt{N_R}})\Big)^{N_R} = \Big(\frac{2}{\bar{\gamma}} \cdot \frac{N_R}{s^2}\Big)^{N_R} + O\Big(\frac{1}{\bar{\gamma}^{N_R}}\Big)$$

- inverse Laplace Transform

$$p_{\sqrt{\gamma_t}}(x) = \mathcal{L}^{-1}\Big\{M_{\sqrt{\gamma_t}}(s)\Big\} = \Big(\frac{2N_R}{\bar{\gamma}}\Big)^{N_R} \cdot \frac{x^{2N_R-1}}{(2N_R-1)!} + O\Big(\frac{1}{\bar{\gamma}^{N_R}}\Big)$$
$$P_{\gamma_t}(x) = \Pr\left\{\gamma_t \le x\right\} = \Pr\left\{\sqrt{\gamma_t} \le \sqrt{x}\right\} = P_{\sqrt{\gamma}_t}(\sqrt{x}) \to \text{cdf of } \sqrt{\gamma_t}$$
$$p_{\gamma_t}(x) = \frac{d}{dx} P_{\gamma_t}(x) = \frac{1}{2\sqrt{x}} \cdot p_{\gamma_t}(\sqrt{x}) = \frac{1}{2}\Big(\frac{2N_R}{\bar{\gamma}}\Big)^{N_R} \cdot \frac{x^{N_R-1}}{(2N_R-1)!} + O\big(\bar{\gamma}^{-N_R}\big)$$
$$\to M_{\gamma_t}(s) = \mathcal{L}\left\{p_{\gamma_t}(x)\right\} = \frac{1}{2}\Big(\frac{2N_R}{\bar{\gamma}}\Big)^{N_R} \cdot \frac{(N_R-1)!}{(2N_R-1)!\; b^{N_R}} + O\big(\bar{\gamma}^{-N_R}\big)$$

- Error Probability:

$$
\begin{aligned}
P_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\left(\frac{b}{2\sin^2(\theta)}\right) \mathrm{d}\theta \\
&= \frac{a}{\pi} \frac{1}{2} \left(\frac{2N_R}{\bar{\gamma}}\right)^{N_R} \frac{(N_R-1)!}{(2N_R-1)!} \frac{2^{N_R}}{b^{N_R}} \underbrace{\int_0^{\frac{\pi}{2}} \sin^{2N_R}(\theta)\mathrm{d}\theta}_{\frac{\pi}{2^{2N_R+1}}\binom{2N_R}{N_R} = \frac{\pi(2N_R)!}{2^{2N_R+1}(N_R!)^2}} + O\left(\frac{1}{\bar{\gamma}^{N_R}}\right) \\
&= \frac{aN_R^{N_R}}{2b^{N_R}N_R!} \frac{1}{\bar{\gamma}^{N_R}} + O\left(\frac{1}{\bar{\gamma}^{N_R}}\right) \stackrel{!}{=} \left(\frac{1}{G_c}\right)^{G_d}
\end{aligned}
$$

$\Longrightarrow$ Diversity gain: $G_d = N_R$

$\Longrightarrow$ Combining gain: $G_c = \frac{b}{N_R}\left(\frac{2N_R!}{a}\right)^{\frac{1}{N_R}}$

vergleiche auch Blatt mit Kurven III und IV

A similar asymptotic analysis can be conducted for the outage probability.

1.4 SC (Selection Combining)

Combining weights

- only the strongest branch is chosen
- strongest branch: $\hat{n} = \underset{n}{\operatorname{argmax}} \gamma_n \longrightarrow \gamma_t = \gamma_{\hat{n}}$
- only on RF receiver chain required $\rightarrow$ saves hardware complexity

Performance analysis

- cdf of: γ_t

$$
\begin{aligned}
P_{\gamma_t}(x) &= \Pr\{\gamma_{\hat{n}} \le x\} = \Pr\{\gamma_1 \le x \cap \gamma_2 \le x \cap \dots \gamma_{N_R} \le x\} \\
&\stackrel{(IID)}{=} \Big(\Pr\{\gamma_n \le x\}\Big)^{N_R} = \Big(P_\gamma(x)\Big)^{N_R}
\end{aligned}
$$

- pdf:

$$
\begin{aligned}
p_{\gamma_t}(x) &= \frac{d}{dx} P_{\gamma_t}(x) = N_R\big(P_\gamma(x)\big)^{N_R-1} \cdot p_\gamma(x) \\
\text{where:} \qquad p_{\gamma_t}(x) &= \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}}; \quad x \ge 0 \\
P_\gamma(x) &= \int_0^x p_\gamma(x)\ dx = 1 - e^{-\frac{x}{\bar{\gamma}}}; \quad x \ge 0 \\
\rightarrow p_{\gamma_t}(x) &= \frac{N_R}{\bar{\gamma}}\big(1 - e^{-\frac{x}{\bar{\gamma}}}\big)^{N_R-1} e^{-\frac{x}{\bar{\gamma}}}; \quad x \ge 0
\end{aligned}
$$

Error probability

- direct approach → closed-form solution possible
- MGF approach
  - Binomial expansion

$$
\begin{aligned}
p_{\gamma_t}(x) &= \frac{N_R}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} 1^{N_R-1-n} \left(-e^{-\frac{x}{\bar{\gamma}}}\right)^n \\
&= \frac{N_R}{\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} \cdot (-1)^n e^{-\frac{x(n+1)}{\bar{\gamma}}}; \quad x \geq 0
\end{aligned}
$$

  - MGF

$$
M_{\gamma_t}(s) = \frac{N_R}{\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} (-1)^n \frac{1}{s + \frac{n+1}{\bar{\gamma}}}
$$

  -

$$
P_e = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\Big(\frac{b}{2\sin^2\theta}\Big)\, d\theta = \frac{aN_R}{\pi\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} (-1)^n \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\frac{b}{2\sin^2\theta} + \frac{n+1}{\bar{\gamma}}}
$$

→ can be evaluated numerically

  - high SNR approach ⇒ $\bar{\gamma} \to \infty$

$$
\begin{aligned}
p_{\gamma_t} &= \frac{N_R}{\bar{\gamma}} \left[1 - \exp\left(-\frac{x}{\bar{\gamma}}\right)\right]^{N_R-1} \exp\left(-\frac{x}{\bar{\gamma}}\right) \\
&\overset{\bar{\gamma}\to\infty}{=} \frac{N_R}{\bar{\gamma}} \left[1 - \left(1 - \frac{x}{\bar{\gamma}} + O\left(\bar{\gamma}^{-1}\right)\right)\right]^{N_R-1} \left(1 - \frac{x}{\bar{\gamma}} + O\left(\bar{\gamma}^{-1}\right)\right) i \\
&= \frac{N_R}{\bar{\gamma}^{N_R}} x^{N_R-1} + o\left(\bar{\gamma}^{-N_R}\right)
\end{aligned}
$$

  - MGF:

$$
M_{\gamma_t}(s) = \frac{N_R}{\bar{\gamma}^{N_R}} \frac{(N_R-1)!}{s^{N_R}} + O\left(\bar{\gamma}^{-N_R}\right)
$$

$$
\begin{aligned}
\Big[\to P_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\left(\frac{b}{2\sin^2(\theta)}\right) \mathrm{d}\theta\Big] \\
&= \frac{a(2N_R)!}{b^{N_R} 2^{N_R+1} N_R!} \frac{1}{\bar{\gamma}^{N_R}} + O(\bar{\gamma}^{-N_R})
\end{aligned}
$$

$\Longrightarrow$ Diversity gain: $G_d = N_R$

$\Longrightarrow$ Combining gain: $G_c = 2b\left(\frac{2N_R!}{a(2N_R)!}\right)^{\frac{1}{N_R}}$

– Outage Probability

$$P_{out} = \Pr\{\gamma_{\hat{n}} \leq \gamma_T\} = P_{\gamma_{\hat{n}}}(\gamma_T) = \left[1 - \exp\left(-\frac{\gamma_T}{\bar{\gamma}}\right)\right]^{N_R}$$

$$\text{high SNR: } P_{out} = \left(\frac{\gamma_T}{\bar{\gamma}}\right)^{N_R} + O\left(\bar{\gamma}^{-N_R}\right)$$

1.5 Comparison

- Diversity Gain:
  MRC, EGC and SC all achieve the maximum possible diversity gain of $G_d = N_R$
- Combining Gain:
  The combining gains of MRC, EGC and SC are different
  – MRC/EGC:

$$\frac{G_C^{EGC}}{G_C^{MRC}} = \frac{\frac{1}{2b}\left(\frac{a}{2}\binom{2N_R}{N_R}\right)^{\frac{1}{N_R}}}{\frac{N_R}{b}\left(\frac{a}{2}\frac{1}{N_R!}\right)^{\frac{1}{N_R}}} = \frac{[(2N_R)!]^{\frac{1}{N_R}}}{2N_R(N_R)^{\frac{1}{N_R}}} \leq 1$$

(independent of a or b which are modulation parameters, only depends on number of antennas)

$$N_R \gg 1: \qquad N_R! \approx \sqrt{2\pi}e^{-N_R}N_R^{N_R+\frac{1}{2}} \qquad (Stirling)$$

$$\left.\frac{G_C^{EGC}}{G_C^{MRC}}\right|_{N_R\gg 1} = \frac{\left(\sqrt{2\pi}e^{-2N_R}(2N_R)^{2N_R+\frac{1}{2}}\right)^{\frac{1}{N_R}}}{2N_R\left(\sqrt{2\pi}e^{-N_R}N_R^{N_R+\frac{1}{2}}\right)^{\frac{1}{N_R}}} = \frac{2\cdot 2^{\frac{1}{2N_R}}}{2} \overset{N_R\to\infty}{\to} \frac{2}{e} \equiv -1.3\text{dB}$$

  – MRC/SC:

$$\frac{G_C^{SC}}{G_C^{MRC}} = \frac{2b\left(\frac{a}{2}\binom{2N_R}{N_R}\right)^{\frac{1}{N_R}}}{2b\left(\frac{a}{2}\frac{(2N_R)!}{N_R!}\right)^{\frac{1}{N_R}}} = \frac{1}{(N_R!)^{\frac{1}{N_R}}} \leq 1$$

$$\left.\frac{G_C^{SC}}{G_C^{MRC}}\right|_{N_R\gg 1} = \frac{1}{\sqrt{2\pi}^{\frac{1}{N_R}}e^{-1}N_R^{1+\frac{1}{2N_R}}} N_R \overset{\to}{\to} \infty \frac{e}{N_R}$$

$\to$ loss increases with N_R

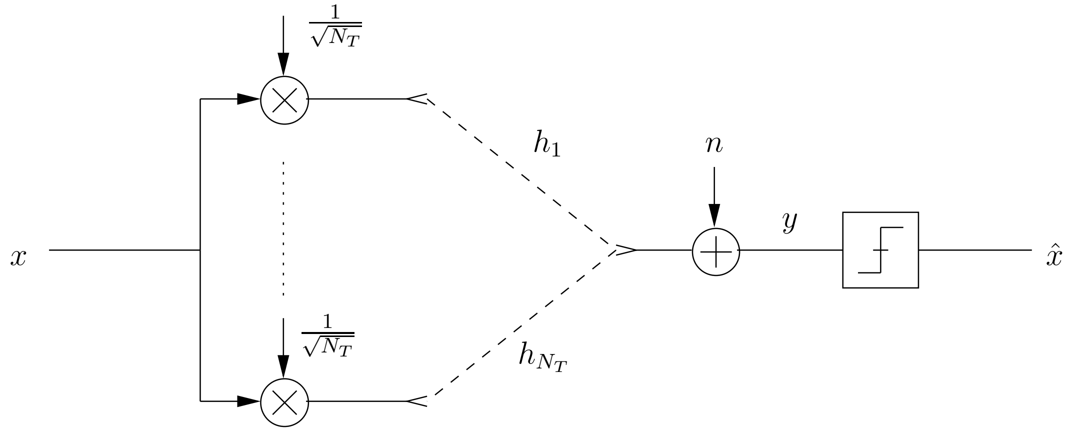
2 MISO Systems

Remarks

- Similar to SIMO systems, in MISO systems only coding and diversity gains can be obtained.
- To realize these gains, a careful transmitter design is necessary
- System design depends on whether or not channel state information (**CSI**) is available at transmitter

2.1 Naive Approach

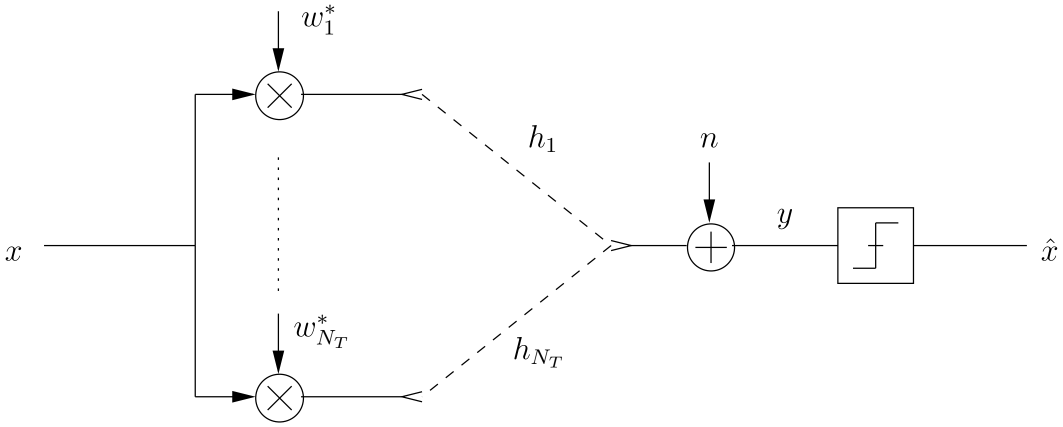
- Assume we simply send the same signal over all N_T transmit antennas



- Transmit power: $\mathcal{E}\left\{\left|\frac{1}{\sqrt{N_T}}x\right|^2 +, \ldots, \left|\frac{1}{\sqrt{N_T}}x\right|^2\right\} = \mathcal{E}\left\{N_T \frac{1}{N_T}|x|^2\right\} = E_s$
- Received signal: $y = \frac{1}{\sqrt{N_T}} \sum_{n=1}^{N_T} h_n \cdot x + n$
- Rayleigh fading: h_n are zero mean complex gaussian random variables
  $\rightarrow h$ is also zero mean complex gaussian
- i.i.d.:
  - $\mathcal{E}\{|h_n|^2\} = 1 \; \forall n$
  - $\mathcal{E}\{|h|^2\} = \frac{1}{N_T}\mathcal{E}\left\{\left|\sum_{n=1}^{N_T} h_n\right|^2\right\} = \frac{1}{N_T}\mathcal{E}\left\{\sum_{n=1}^{N_T} |h_n|^2\right\} = 1$
  - statistical properties of h are independent of N_T
  - the multiple transmit antennas have no benefit at all
  - <u>more sophisticated transmitter designs necessary</u>

2.2 Full CSI Available at the Transmitter

- $h_n, n \in \{1, \dots, N_T\}$ is known at the transmitter
- Perform "precoding" (beamforming) with coefficients w_n



- Transmit Power: Two constraints maybe considered
  - Average transmit power constraint

$$P_{av} = \mathcal{E}\left\{\sum_{n=1}^{N_T} |w_n^* x|^2\right\} = \sum_{n=1}^{N_T} |w_n|^2 \underbrace{\mathcal{E}\{|x|^2\}}_{E_s} = \mathcal{E}_s \Rightarrow \sum_{n=1}^{N_T} |w_n|^2 = 1$$

  - Power constraint for each transmit antenna

$$\to |w_n| = \frac{1}{\sqrt{N_T}} \qquad \to P_{av} = E_s$$

- Recveived signal: $y = \underbrace{\sum_{n=1}^{N_T} w_n^* h_n}_{h} x + n$ (equivalent SISO channel)

Maximum Ratio Transmission (MRT)

- we have only the average power constraint: $\sum_{n=1}^{N_T} |w_n|^2 = 1$
- SNR: $\gamma_t = \frac{E_s |h|^2}{\sigma_n^2} = \frac{\mathcal{E}_s \left| \sum_{n=1}^{N_T} w_n^* \cdot h_n \right|^2}{\sigma_n^2}$
- Maximize SNR under constraint $\sum_{n=1}^{N_T} |w_n|^2 = 1$

- constraint optimization problem → Lagrange method

$$L = \frac{E_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} w_n^* \cdot h_n \right|^2 + \lambda \left(\sum_{n=1}^{N_T} |w_n|^2 - 1 \right); \quad \text{where: } \lambda = \text{Lagrange Multiplier}$$

⇒ Wirtinger Kalkül: treat z and z^* as independent variables for differentiation:

$$\frac{\partial z^*}{\partial z} = 0; \quad \frac{\partial |z|^2}{\partial z} = \frac{\partial z \cdot z^*}{\partial z} = z^*$$
$$\frac{\partial x^2}{\partial x} = 2x; \quad \frac{\partial (z^*)^2}{\partial z^*} = 2 \cdot z^*; \frac{\partial |z|^2}{\partial z} = z^*$$

$$\frac{\partial L}{\partial w_m^*} = \frac{\epsilon_s}{\sigma_n^2} \left(\sum_{n=1}^{N_T} w_n^* \cdot h_n \right)^* h_m + \lambda w_m$$

$$\rightarrow w_m = \underbrace{\frac{\epsilon_s}{\sigma_n^2 \cdot \lambda} \left(\sum_{n=1}^{N_T} w_n^* h_n \right)^*}_{\text{const., independent of } m := c} h_m$$

$$\rightarrow w_m = c \cdot h_m$$

$$\rightarrow \sum_{n=1}^{N_T} |w_n|^2 = 1 \rightarrow c^2 = \frac{1}{\sum\limits_{n=1}^{N_T} |h_n|^2}$$

$$\rightarrow w_n = \frac{h_n}{\sqrt{\sum\limits_{n=1}^{N_T} |h_n|^2}} \equiv \text{MRT gains}$$

$$\rightarrow \text{SNR} = \frac{E_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} \frac{|h_n|^2}{\sqrt{\sum_{n=1}^{N_T} |h_m|^2}} \right|^2 = \frac{\epsilon_s}{\sigma_n^2} \sum_{n=1}^{N_T} |h_n|^2$$

⇒ same SNR as for maximum ration combining (MRC)

⇒ MRT with N_T transmit antennas achieves the same performance as MRC with N_T receive antennas

⇒ MRT/MRC can be extended to $N_T \times N_R$ MIMO systems

→ has the same performance as MRC with $N_T \cdot N_R$ receive antennas and one transmit antenna

Equal Gain Transmission (EGT)

- we employ gains: $w_n = \frac{1}{\sqrt{N_T}} \cdot \frac{h_n}{|h_m|} \rightarrow |w_n| = \frac{1}{\sqrt{N_T}}$

- SNR:

$$\begin{aligned}
\gamma_t &= \frac{E_s}{\sigma_n^2}\left|\sum_{n=1}^{N_T} w_n^* h_n\right|^2 \\
&= \frac{E_s}{\sigma_n^2}\left|\sum_{n=1}^{N_T} \frac{1}{\sqrt{N_T}\cdot\frac{|h_n|^2}{|h_n|}}\right|^2 = \frac{1}{N_T}\cdot\frac{\mathcal{E}_s}{\sigma_n^2}\left|\sum_{n=1}^{N_T}|h_n|\right|^2 \\
\gamma_n &= \frac{E_s}{\sigma_n^2}|h_n|^2 \\
&\text{same SNR as for EGC} \rightarrow \gamma_t = \frac{1}{N_T}\left|\sum_{n=1} N_T\sqrt{\gamma_n}\right|^2
\end{aligned}$$

$\rightarrow$ EGC with N_T transmit antennas achieves the same performance as EGC with N_T receive antennas

Transmit Antennas Selection

- select antenna with maximum channel gain for transmission:

$$w_n = \begin{cases} \frac{h_n}{|h_n|}, & \text{if } n = \hat{n} \\ 0, & \text{otherwise} \end{cases} \text{ where } \hat{n} = \underset{n}{\operatorname{argmax}}|h_n|$$

- antenna selection with N_T transmit antennas achieves the same performance as *Selection Combining* with N_T receive antennas

2.3 No CSI at Transmitter - Space-Time-Coding

- $h_n, n \in \{1, \ldots, N_T\}$, is only known at the receiver
- "Space-time-coding" has to be employed to realize diversity gain
- $T \times N_T$ matrics $\mathbf{X}$ are transmitted in T symbol intervals over N_T antennas
- $\mathbf{X}$ is drawn from a matrix alphabet $\mathcal{X}$
- Example:

$$\mathbf{X} = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,N_T} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,N_T} \\ \vdots & \vdots & \ddots & \vdots \\ x_{T,1} & x_{T,2} & \cdots & x_{T,N_T} \end{pmatrix}$$

- We distinguish:
  - Space-time-block-codes (STBCs)
    $\rightarrow$ $\mathbf{X}$ is obtained by mapping K scalar symbols $s_k,\ k = 1, \ldots, K$ from a scalar alphabet $\mathcal{A}$ to matrix $\mathbf{X}$

- Space-time-trellis-codes (STTCs)
  $\rightarrow$ $\mathbf{X}$ is obtained from scalar symbols s_k through a trellis encoding process.
  [see: Tarokh, Seshadri, Calderbank: Space-time-codes for high datarate wireless communication: Performance criterions and coder construction; IEEE Trans. Inf. Theory 1998]
- here: We concentrate on space-time-block-codes (STBCs), but many results can be easily extended to space-time-trellis-codes

- STBCs:
  - K M-ary scalar symbols (e.g. M-PSK symbols) are mapped to STBC matrices $\mathbf{X}$
    $\mathbf{S} = [s_1, \ldots, s_K] \rightarrow \mathbf{X}$
    $s_k \in \mathcal{A} \rightarrow x \in \mathcal{X}$ with $|\mathcal{X}| = M^K$
  - Example: "Alamouti"-Code

$$\mathbf{X} = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix}$$

    [Alamouti: A simple transmit diversity technique for wireless communication, IEEE JSAC 1998]

2.3.1 Optimal Detection

- Signal model:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_T \end{pmatrix} = \mathbf{X} \begin{pmatrix} h_1 \\ \vdots \\ h_{N_T} \end{pmatrix} + \begin{pmatrix} n_1 \\ \vdots \\ n_T \end{pmatrix}$$

$$\mathbf{y} = \mathbf{X} \cdot \mathbf{h} + \mathbf{n}$$

- Optimal detection - ML-detection
  - $\mathbf{h}$ is known at receiver
  - $\mathbf{n}$ is AWGN with $\mathcal{E}\{\mathbf{n} \cdot \mathbf{n^H}\} = \sigma_{\mathbf{n}}^2 \cdots \mathbf{I}_{T \times T}$
  - $p(\mathbf{y}|\mathbf{x})$

$$\begin{aligned} &= \frac{1}{\pi^T |\sigma_n^2 \mathbf{I}_{T\times T}|} \exp\left(-(\mathbf{y} - \mathbf{xh})^H (\sigma_n^2 \mathbf{I}_{T\times T})^{-1} (\mathbf{y} - \mathbf{xh})\right) \\ &= \frac{1}{\pi^T \sigma_n^{2T}} \exp\left(-\frac{1}{\sigma_n^2} (\mathbf{y} - \mathbf{xh})^H (\mathbf{y} - \mathbf{xh})\right) = \frac{1}{\pi^T \sigma_n^{2T}} \exp\left(||\mathbf{y} - \mathbf{xh}||^2\right) \end{aligned}$$

    $\rightarrow$ the optimal estimate $\hat{\mathbf{X}}$ or equivalently the optimal estimate $\hat{\mathbf{s}}$ can be obtained as

$$\hat{\mathbf{s}} = \underset{\mathbf{s} \in \mathcal{A}^K}{\operatorname{argmax}}\, p(\mathbf{y}|\mathbf{x}) = \underset{\mathbf{s} \in \mathcal{A}^K}{\operatorname{argmin}} ||\mathbf{y} - \mathbf{hx}||^2$$

  - Disadvantage: In general, metric $||\mathbf{y} - \mathbf{hx}||^2$ has to be calculated M^K times
    $\rightarrow$ complexity increases exponentially with K

2.3.2 Types of STBCs

- Orthogonal STBCs (OSTBCs)
  - OSTBCs are a special class of STBCs which allow independent detection of each s_k → only $K \cdot M$ metrics have to be evaluated
  - Rate STBCs: $R_{STBC} = \frac{K}{T}$
  - Examples:
    - Alamouti Code $(K = 2, T = 2) \to R_{STBC} = 1$

$$\mathbf{X} = \frac{1}{\sqrt{2}} \underbrace{\begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix}}_{\longleftrightarrow N_T} \updownarrow T$$

→ only "full rate OSTBC for complex s_k

    - $N_T = 3, K = 3, T = 4$

$$\mathbf{X} = \frac{1}{\sqrt{3}} \begin{pmatrix} s_1 & s_2 & s_3 \\ -s_2^* & s_1^* & 0 \\ s_3^* & 0 & -s_3^* \\ 0 & -s_3^* & s_2^* \end{pmatrix} \to R_{STBC} = \frac{K}{T} = \frac{3}{4}$$

  - Orthogonality: $\mathbf{X}^H \mathbf{X} = const \cdot \mathbf{I}_{N_T \times N_T}$
  - Independent detection of s_1 & s_2 for Alamouti Code

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

$$\to \underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{\tilde{y}} = \frac{1}{\sqrt{2}} \underbrace{\begin{pmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{pmatrix}}_{\tilde{F}} \underbrace{\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}}_{s} + \underbrace{\begin{pmatrix} n_1 \\ n_2^* \end{pmatrix}}_{\tilde{n}}$$

$\Big($ Anmerkung: <u>nur</u> $\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$ gewünscht, nicht: $s_1^*, s_2^* \Big)$

$$\mathbf{F}^H \mathbf{F} = \frac{1}{2} \begin{pmatrix} h_1^* & h_2 \\ h_2^* & -h_1 \end{pmatrix} \begin{pmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{pmatrix} = \frac{1}{2} \begin{pmatrix} |h_1|^2 + |h_2|^2 & 0 \\ 0 & |h_1|^2 + |h_2|^2 \end{pmatrix}$$

→ $\frac{\sqrt{2}}{\sqrt{|h_1|^2+|h_2|^2}} \cdot \mathbf{F}$ is unitary matrix

→ $\frac{2}{|h_1|^2+|h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{y}} = \mathbf{s} + \frac{2}{|h_1|^2+|h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{n}}$
$\big(\frac{2}{|h_1|^2+|h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{n}}$ is AWGN vector with covariance matrix $\frac{2\sigma_n^2}{|h_1|^2+|h_2|^2} \cdot \mathbf{I}_{T \times T}\big)$

→ ML decision: $\hat{\mathbf{s}} = \underset{\mathbf{s}}{\operatorname{argmin}} \left|\left| \frac{2}{|h_1|^2+|h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{y}} - \mathbf{s} \right|\right|^2$

→ independent ML decoding

$$\hat{s}_1 = \underset{s_1}{\operatorname{argmin}} \left| s_1 - \frac{h_1^* y_1 + h_2 y_2^*}{\frac{1}{\sqrt{2}}(|h_1|^2 + |h_2|^2)} \right|$$

$$\hat{s}_2 = \underset{s_2}{\operatorname{argmin}} \left| s_2 - \frac{h_1^* y_1 - h_2 y_2^*}{\frac{1}{\sqrt{2}}(|h_1|^2 + |h_2|^2)} \right|$$

- independent decoding property can be proved for all OSTBCs
- low complexity is at the expense of a rate-loss compared to other STBCs for $N_T > 2$
  - → Frequenzhopping
  - → keine Kanalinformation aus vorher empfangenen Symbolen möglich ⇒ Kanal ändert sich ständig: nur Entscheidung, ob Rauschen oder Signal + Rauschen

- Performance Analysis of Alamouti Code
  - Decision-variables after combining

$$r_1 = \sqrt{2}\frac{h_1^* y_1 + h_2 y_2^*}{|h_1|^2 + |h_2|^2}$$
$$r_2 = \sqrt{2}\frac{h_1^* y_1 - h_2 y_2^*}{|h_1|^2 + |h_2|^2}$$

because of symmetry it suffices to consider r_1

$$\begin{aligned} r_1 &= \sqrt{2}\frac{h_1^*\big(\frac{1}{\sqrt{2}}s_1 h_1 + \frac{1}{\sqrt{2}}h_2 s_2 + n_1\big) + h_2\big(-\frac{1}{\sqrt{2}}h_2 s_1^* + \frac{1}{\sqrt{2}}h_1 s_2^* + n_2\big)^*}{|h_1|^2 + |h_2|^2} \\ &= \sqrt{2}\frac{\frac{1}{\sqrt{2}}\big(|h_1|^2 + |h_2|^2\big)s_1 + h_1^* n_1 + h_2 n_2^*}{|h_1|^2 + |h_2|^2} \\ &= 1 \cdot s_1 + n_{eq} \end{aligned}$$

where

$$n_{eq} = \sqrt{2}\ \frac{h_1^* n_1 + h_2 n_2^*}{|h_1|^2 + |h_2|^2}$$
$$\text{SNR} \to \gamma_t = \frac{E_s \cdot 1^2}{\sigma_{eq}^2} \quad \text{with} \quad \mathcal{E}\big\{|s_1|^2\big\} = \mathcal{E}_s$$
$$\sigma_{eq}^2 = 2\frac{|h_1|^2\sigma_n^2 + |h_2|^2\sigma_{eq}}{\big(|h_1|^2 + |h_2|^2\big)^2} = \frac{2\sigma_n^2}{|h_1|^2 + |h_2|^2}$$

→ $\gamma_t = \frac{1}{2}\frac{E_s}{\sigma_n}\big(|h_1|^2 + |h_2|^2\big)$

→ $\text{SNR}_{\text{Alamouti}} = \frac{1}{2}\text{SNR}_{\text{MRC}} = \frac{1}{2}\text{SNR}_{\text{MRT}}$

→ Alamouti code has diversity gain $G_d = 2$

→ Transmission with Alamouti STBC requires 3dB higher SNR to achieve same performance as MRT → 3dB loss in coding gain G_c

→ Lack of CSI knowledge at transmitter "costs" 3dB in power efficiency

→ General:

  - · OSTBCs achieve a diversity gain of $G_d = N_T$ if only one receive antenna is available
  - · if N_R receive antennas are available, MRC can be used at the receiver to yield a diversity gain of $\underline{G_d = N_T N_R}$

- Other STBCs:

- Quasi orthogonal STBCs
  - higher rate than OSTBCs
  - only subset of symbols have to be decoded jointly
  - Example: $K = N_T = T = 4$

$$\mathbf{X} = \frac{1}{2}\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^* & s_1^* & -s_4^* & s_3^* \\ -s_3^* & -s_4^* & s_1^* & s_2^* \\ s_4 & -s_3 & -s_2 & s_1 \end{pmatrix}$$

  - Anmerkung 1: $\mathbf{X}$ ist ähnlich zu Alamouti Code
  - Anmerkung 2: $\mathbf{X}^H\mathbf{X}$: viele Nicht-diagonal Elemente sind Null; die, die ungleich Null sind, zeigen, welche Symbole gemeinsam entschlüsselt werden müssen
- Golden Code for $N_T = N_R = 2$: achieves a rate of $R_{STBC} = 2$ and full diversity of $G_d = N_T, N_R = 4$
- Differential STBCs: $\mathbf{X}_k = \mathbf{X}_{k-1} \cdot \mathbf{D}_k$. $\mathbf{X}_k$ is transmitted, $\mathbf{D}_k$ is transmitted
- Linear dispersion codes: designed to achieve high mutual information
- noncoherent STBCs (On-Off-Keying)

2.3.3 Space Time Code Design

Given:

- Code $\mathscr{X} = \{\mathbf{X}_1, \dots, \mathbf{X}_{|\mathscr{X}|}\}$
- Channel: IID Rayleigh-fading:
  - $h_n \sim \mathcal{CN}(0,1); \quad n \in \{1, 2, \dots, N_T\}$
  - AWGN $n \sim \mathcal{CN}(0, \sigma_n^2)$

Problem: How should we design codebook $\mathscr{X}$?

- Need to derive error rate for general codebooks $\mathscr{X}$!
  - Codeword error rate

$$P_e = \frac{1}{|\mathscr{X}|}\sum_{i=1}^{|\mathscr{X}|} \Pr\{\mathbf{x}_i \neq \hat{\mathbf{x}}_i\}$$

    where $\hat{\mathbf{x}}_i$ is the detected codeword and we assume that all codewords are equally likely

  Problem: $\Pr\{\mathbf{x}_i \neq \hat{\mathbf{x}}_i\}$ is not tractable in general

- Use union bound to upper bound $\Pr\{\mathbf{x}_i \neq \hat{\mathbf{x}}_i\}$ as upper sum over pairwise error probabilities(PEP) $\Pr\{\mathbf{x}_i \to \mathbf{x}_j\}$ where it is assumed that $\mathbf{x}_i$ was transmitted and $\mathbf{x}_i$ and $\mathbf{x}_j$ are the only codewords in the codebook

$$\boxed{P_e \leq \frac{1}{|\mathscr{X}|} \sum_{i=1}^{|\mathscr{X}|} \sum_{j=1}^{|\mathscr{X}|} \Pr\{\mathbf{x}_i \to \mathbf{x}_j\} \text{ where } j \neq i}$$

Calculation of PEPs

Recall: $\hat{\mathbf{x}} = \underset{\mathbf{x}\in\mathscr{X}}{\operatorname{argmin}} ||\mathbf{y} - \mathbf{x}\mathbf{h}||^2$

Now, $\mathbf{x}_i$ and $\mathbf{x}_j$ are the only alternatives and an error is made if $||\mathbf{y} - \mathbf{x}_i\mathbf{h}||^2 > ||\mathbf{y} - \mathbf{x}_j\mathbf{h}||^2$ since $\mathbf{x}_i$ was sent but $\mathbf{x}_j$ was detected

$$\to ||\mathbf{x}_i\mathbf{h} + \mathbf{n} - \mathbf{x}_i\mathbf{h}||^2 > ||\mathbf{x}_i\mathbf{h} + \mathbf{n} - \mathbf{x}_j\mathbf{h}||^2$$

$$||\mathbf{n}|| > ||(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h} + \mathbf{n}||^2$$

$$\to ||\mathbf{n}|| > \underbrace{\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h}}_{\Delta} + \mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{n} + \mathbf{n}^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h} + ||\mathbf{n}||^2$$

$$\to \underbrace{-\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)^H\mathbf{n} - \mathbf{n}^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h}}_{z} > \Delta$$

for given $\mathbf{h}$, z is a gaussian random variable

$$\sigma_z^2 = \mathcal{E}\{|z|^2\} = \mathcal{E}\{2\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)\overbrace{\mathbf{n}\mathbf{n}^H}^{\sigma_n^2\mathbf{I}}(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h} + 2\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)^H\overbrace{\mathbf{n}\mathbf{n}^T}^{=0}(\mathbf{x}_i - \mathbf{x}_j)^*\mathbf{h}^*\}$$
$$= 2\sigma_n^2\Delta + 0$$

$$\Pr\{\mathbf{x}_i \to \mathbf{x}_j\} = \int_{\Delta}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_z}\exp\left(-\frac{z^2}{2\sigma_z^2}\right)\mathrm{d}z,\ t = \frac{z}{\sigma_z}$$
$$= \frac{1}{\sqrt{2\pi}}\int_{\frac{\Delta}{\sigma_z}}^{\infty} e^{-\frac{t^2}{2}}\mathrm{d}t = Q\left(\frac{\Delta}{\sigma_z}\right) = Q\left(\frac{\Delta}{\sqrt{2\sigma_n^2\Delta}}\right)$$
$$= Q\left(\sqrt{\frac{\Delta}{2\sigma_n^2}}\right)$$

- $\Pr\{\mathbf{x}_i \to \mathbf{x}_j\} = \mathcal{E}\left\{Q\left(\sqrt{\frac{\Delta}{2\sigma_n^2}}\right)\right\}$
  - to avoid cumbersome Q-function we use Chernoff bound:

$$\boxed{Q(x) \leq \frac{1}{2}e^{-\frac{x^2}{2}}}$$

$$\Pr\{\mathbf{x}_i \to \mathbf{x}_j\} \leq \frac{1}{2}\mathcal{E}_h\left\{\exp\left(-\frac{\mathbf{h}^H\mathbf{Q}\mathbf{h}}{4\sigma_n^2}\right)\right\}$$
$$\text{where } \mathbf{Q} = (\mathbf{x}_i - \mathbf{x}_j)^H(\mathbf{x}_i - \mathbf{x}_j)$$

- Eigendecomposition: $\mathbf{Q} = \mathbf{U}^H \boldsymbol{\Lambda} \mathbf{U}$ with $\boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_r, 0, \ldots, 0\}$ $\qquad r = \mathrm{rank}\{\mathbf{Q}\}$

- Elements $\mathbf{h}$ are i.i.d. Gaussian
  - $\underline{\beta} = \mathbf{Uh}$ has also i.i.d. Gaussian random variables as elements since $\mathbf{U}$ is unitary matrix
  - $\mathbf{h}^H\mathbf{Q}\mathbf{h} = \underbrace{\mathbf{h}^H\mathbf{U}^H}_{\underline{\beta}^*}\boldsymbol{\Lambda}\underbrace{\mathbf{Uh}}_{\underline{\beta}} = \sum_{i=1}^{r} \lambda_i |\beta_i|^2$ with $\underline{\beta} = [\beta_1, \ldots, \beta_{N_T}]$

$$\begin{aligned}
\Pr\{\mathbf{x}_i \to \mathbf{x}_j\} &= \frac{1}{2}\mathcal{E}_{\underline{\beta}}\left\{\exp\left(-\frac{\sum_{i=1}^{r}\lambda_i|\beta_i|^2}{4\sigma_n^2}\right)\right\} \\
&= \frac{1}{2}\mathcal{E}_{\underline{\beta}}\left\{\prod_{i=1}^{r} e^{-\frac{\lambda_i}{4\sigma_n^2}|\beta_i|^2}\right\} \\
&= \frac{1}{2}\prod_{i=1}^{r}\mathcal{E}_{\beta_i}\left\{e^{-\frac{\lambda_i}{4\sigma_n^2}|\beta_i|^2}\right\} \\
&= \frac{1}{2}\prod_{i=1}^{r}\mathcal{E}_{|\beta_i|^2}\left\{e^{-\frac{\lambda_i}{4\sigma_n^2}|\beta_i|^2}\right\} \mathrel{\widehat{=}} \text{MGF of exponentially distributed variable } \alpha_i = |\beta_i|^2
\end{aligned}$$

$\to P_{\alpha_i}(x) = e^{-x},\ x \geq 0$

$$\begin{aligned}
\to \Pr\{\mathbf{x}_i \to \mathbf{x}_j\} &\leq \frac{1}{2}\prod_{i=1}^{r}\frac{1}{1+\frac{\lambda_i}{4\sigma_n^2}} \\
&\leq \prod_{i=1}^{r}\frac{1}{\frac{\lambda_i}{4\sigma_n^2}} = 2^{2r-1}\frac{1}{\prod_{i=1}^{r}\lambda_i}\left(\underbrace{\frac{1}{\sigma_n^2}}_{\widehat{=}SNR}\right)^{-r}
\end{aligned}$$

- upper bound on P_e:

$$\begin{aligned}
\lambda_n(i,j) &= n\text{th eigenvalue of } (\mathbf{x}_i - \mathbf{x}_j)^H(\mathbf{x}_i - \mathbf{x}_j) \\
r(i,j) &= \text{ rank of } (\mathbf{x}_i - \mathbf{x}_j)^H(\mathbf{x}_i - \mathbf{x}_j)
\end{aligned}$$

$$\to P_e \leq \frac{1}{|\mathscr{X}|}\sum_{i=1}^{|\mathscr{X}|}\sum_{j=1}^{|\mathscr{X}|} 2^{2r(i,j)-1}\frac{1}{\prod_{n=1}^{r(i,j)}\lambda_n(i,j)}\left(\frac{1}{\sigma_n^2}\right)^{-r(i,j)}$$

- generally loose bound but offers significant insight for code design

Two criteria:

Rank criterion: The diversity gain of a ST code is given by

$G_d = \min\limits_{i,j}(r(i,j)) = \min\limits_{i,j} \operatorname{rank}\left((\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j)\right)$

$\rightarrow$ Design code such that minimum rank of all possible matrices $(\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j)$ is maximized

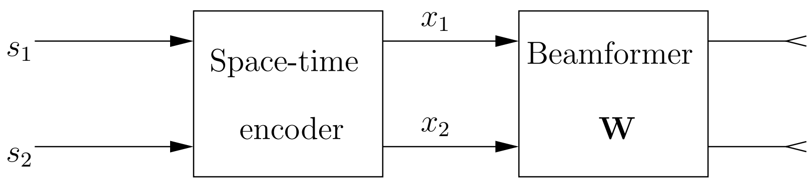
$$T \updownarrow \overset{\overset{N_T}{\longleftrightarrow}}{\mathbf{X}_i} \Rightarrow r(i,j) = N_T \qquad \forall i \neq j$$

Determinant criterion: To maximize the coding gain among all codes with $r(i,j) = N_T$, we need to maximize $\max \min\limits_{i,j} \prod\limits_{n=1}^{N_T} \lambda_n(i,j) = \max \min\limits_{i,j} \left|(\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j)\right| \qquad \forall i \neq j$

- Rank and determinant criterion can be used for the search for good space-time block codes and space-time trellis codes. These two criteria were first derived by Tarokh, et. al. 1998.
- diversity increases to $N_T N_R$ if N_K receive antennas are available
- Example: see Bäro, Bauch, Hansmann: Improved codes for space-time trellis coded modulation. IEEE Comm. Letters, 2000.

2.4 Partial or Imperfect CSI at the Transmitter

- In practice, the CSI cannot be perfect. Channel estimation, quantization and noisy feedback channels introduce errors.
- If the system is optimized for perfect CSI (*e.g.* using MRT or EGT), the performance for imperfect CSI may be worse than for a system designed for no CSI(*e.g.* space-time coding)
- In this case, it is advantageous to use a hybrid approach and combine beamforming and space-time coding.



- $\mathbf{W}$ is the beamforming matrix which depends on the reliability of the CSI
- CSI is modeled as

$$\hat{h}_i = \rho h_i + \sqrt{1-\rho^2} e_i$$

where:

- $\hat{h}_i$ is the CSI estimate
- ρ is the correlation between $\hat{h}_i$ and h_i

  - e_i is the CSI error modeled as AWGN

  extreme cases:

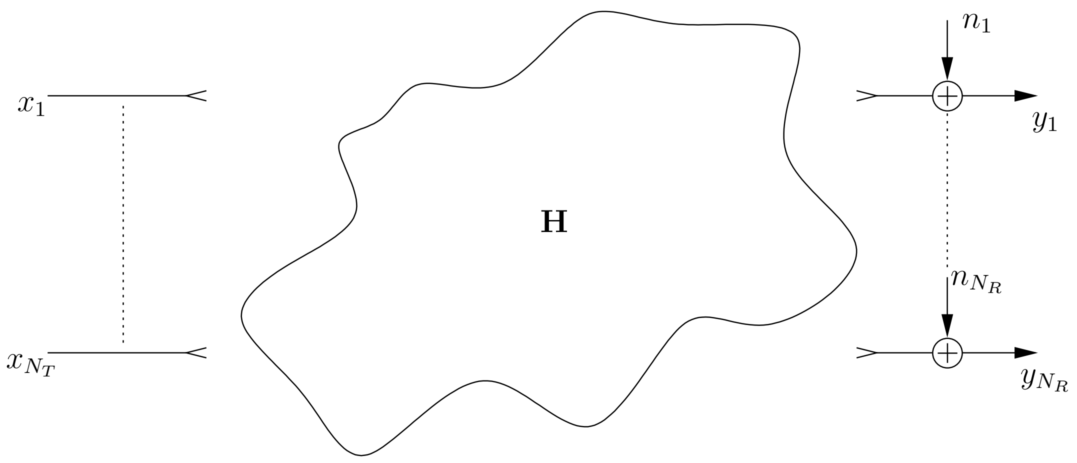
  - $\rho = 0$: $\hat{h}_i$ independent of $h_i \rightarrow$ no CSI ($\mathbf{W} = \mathbf{I}$)
  - $\rho = 1$: $\hat{h}_i = h_i \rightarrow$ perfect CSI ($\mathbf{W}$ performs MRT)

- $\mathbf{W}$ can be optimized under the assumptions for given ρ and $\hat{h}_i$
  $\rightarrow$ see for details: Jöngren, Skorglund and Ottersten: "Combining Beamforming and Orthogonal Space-time Block Coding", IEEE on IT, 2002.

3 MIMO Systems without CSI at the transmitter

- We consider $N_T \times N_R$ MIMO system and assume that the channel matrix $\mathbf{H}$ is not known at the transmitter
  $\rightarrow$ no CSI at the transmitter (CSIT)
- signal model:

$$N_R \updownarrow \mathbf{y} = N_R \updownarrow \overset{N_T}{\overleftrightarrow{\mathbf{H}}}\, \mathbf{x} \updownarrow N_T + \mathbf{n} \updownarrow N_R$$



- x_n are M-ary i.i.d. scalar symbols taken *e.g* from an M-PSK or M-QAM symbol alphabet $\mathscr{A}$
- This scheme is often called "spatial multiplexing"
- We transmit N_T symbols per symbol interval
  $\rightarrow$ rate $R = \log_2(M) \cdot N_T$ for uncoded transmission
- Problem: How to detect $\mathbf{x}$ at the receiver considering
  - performance
  - complexity

3.1 Optimum Detection

- Elements of $\mathbf{n}$ are gaussian random variables with variance σ_n^2
- $\mathbf{H}$ is known at the receiver

$$
\begin{aligned}
p(\mathbf{y}|\mathbf{x}) &= \frac{1}{\pi^{N_R}\sigma_n^2 \mathbf{I}_{N_R\times N_R}} \exp\left(-(\mathbf{y}-\mathbf{H}\mathbf{x})^H(\sigma_n^2\mathbf{I}_{N_R\times N_R})^{-1}(\mathbf{y}-\mathbf{H}\mathbf{x})\right) \\
&= \frac{1}{\pi^{N_R}\sigma_n^{2N_R}} \exp\left(-\frac{1}{\sigma_n^2}||\mathbf{y}-\mathbf{x}\mathbf{H}||^2\right)
\end{aligned}
$$

- ML-Detection

$$
\hat{x} = \underset{\mathbf{x}\in\mathscr{A}^{N_T}}{\operatorname{argmin}} ||\mathbf{y}-\mathbf{x}\mathbf{H}||^2 = \underset{\mathbf{x}\in\mathscr{A}^{N_T}}{\operatorname{argmax}} \quad p(\mathbf{y}|\mathbf{x})
$$

  $\rightarrow$ M^{N_T} metric calculations $\rightarrow$ complexity is exponential in N_T!!

  $\rightarrow$ in general too complex in practice

- Performance
  - consider worst case pairwise error probability (PEP) to evaluate <u>diversity gain</u>
  - PEP $\rightarrow$ x_i is transmitted but $x_j \neq x_i$ is detected
    this happens if $||\mathbf{y}-\mathbf{H}\mathbf{x}_i||^2 > ||\mathbf{y}-\mathbf{H}\mathbf{x}_j||^2$
    $\rightarrow$ $||\mathbf{n}||^2 > ||\mathbf{H}(\mathbf{x}_i-\mathbf{x}_j)+\mathbf{n}||^2$
  - the "worst case" is if $\mathbf{x}_i$ & $\mathbf{x}_j$ differ only in one element *i.e.*,

$$
\mathbf{x}_i - \mathbf{x}_j = (x_{ni}-x_{nj}) = \begin{bmatrix} 0\\0\\0\\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{“1” in position } n
$$

    where $\mathbf{x}_i = [x_{1i}, x_{2i}, \ldots, x_{N_T i}]$
  - $||\mathbf{n}||^2 > || \underbrace{\mathbf{h}_n}_{n\text{th column of } \mathbf{H}} \underbrace{(x_{ni}-x_{nj})}_{\Delta x_n(i,j)} + \mathbf{n}||^2$
  - $||\mathbf{n}||^2 > \mathbf{h}_n^H \mathbf{n}\Delta x_n^*(i,j) + \mathbf{n}^H\mathbf{h}_n\Delta x_n(i,j) + ||\mathbf{n}||^2 + ||\mathbf{h}_n||^2 - |\Delta x_n(i,j)|^2$

$$
||\mathbf{h}_n||^2|\Delta x_n(i,j)|^2 < \underbrace{-\mathbf{h}_n^H\mathbf{n}\Delta x_n(i,j) - \mathbf{n}^H\mathbf{h}_n\Delta x_n(i,j)}_{\text{Gaussian random variable with variance } \sigma_{eq}^2 = 2\sigma_n^2|\Delta x_n(i,j)|^2||\mathbf{h}_n||^2}
$$

  - $\Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j | \mathbf{H}\} = Q\left(\sqrt{\frac{||\mathbf{h}_n||^2|\Delta x_n(i,j)|^2}{2\sigma_n^2}}\right)$

- $\Pr\{\mathbf{x}_i \to \mathbf{x}_j\} = \mathcal{E}\left\{Q\left(\sqrt{\frac{||\mathbf{h}_n||^2|\Delta x_n(i,j)|^2}{2\sigma_n^2}}\right)\right\}$
  $\to$ use same approach as for space-time code design to get diversity order
  or : SNR is

$$\gamma_t = \frac{||\mathbf{h}_n||^2|\Delta x_n(i,j)|^2}{2\sigma_n^2} = \frac{|\Delta x_n(i,j)|^2}{2\sigma_n^2}(|h_{1n}|^2 + |h_{2n}^2 + \cdots + |h_{N_R n}|^2)$$

- same form as SNR of MRC with N_R receive antennas
- diversity gain of spatial multiplexing wit ML-decoding is

$$G_d = N_R$$

- diversity of N_T transmit antennas is not exploited with spatial multiplexing
- to exploit this additional gain, coding across space is required (at the expense of rate)
  (Hier gehören die detection performance kurven für BPSK hin)

3.2 Linear Receivers

- How can we avoid the complexity associated wit the joint detection of the elements of $\mathbf{x}$?
- Idea: Employ linear filter (matrix) to seperate the elements of $\mathbf{x}$
- Requires: $N_T \leq N_R$
- We form

$$\mathbf{r} = N_T \updownarrow \overset{\overset{N_R}{\longleftrightarrow}}{\mathbf{F}}\, \mathbf{y} = [r_1, \dots, r_{N_T}]^T$$

where $\mathbf{F}$ is the filter matrix and $\mathbf{y}$ is the received vector

such that x_n can be obtained from

$$\hat{x}_n = \underset{x_n \in \mathscr{A}}{\operatorname{argmin}} |r_i - x_n|^2 \qquad\qquad \text{where } \mathbf{F} \in \mathbb{C}^{N_T \times N_R}$$

- Two popular design criteria for $\mathbf{F}$
  - Zero-forcing (ZF) criterion
  - minimum mean squared error (MMSE) crtiterion

3.2.1 ZF Detection

$$\mathbf{r} = \mathbf{F}\mathbf{y} = \mathbf{F}(\mathbf{H}\mathbf{x} + \mathbf{n}) = \mathbf{F}\mathbf{H}\mathbf{x} + \mathbf{F}\mathbf{n}$$

ZF $\leftrightarrow$ we require $\mathbf{F}\mathbf{H} = \mathbf{I}_{N_T \times N_T}$

- noise covariance matrix

$$\Phi_{ee} = \mathcal{E}\{\mathbf{Fn}(\mathbf{Fn})^H\} = \sigma_n^2 \mathbf{F}\mathbf{F}^H$$

- $N_T = N_R \;\rightarrow \mathbf{FH} = \mathbf{I}_{N_T \times N_T} \;\rightarrow \mathbf{F} = \mathbf{H}^{-1}$ assuming $\mathbf{H}$ is invertible
- $\rightarrow N_T \leq N_R \rightarrow$ which one of the many $\mathbf{F}$ that yield $\mathbf{FH} = \mathbf{I}_{N_T \times N_T}$?
- choose $\mathbf{F}$ that leads to the smallest noise enhancement
- optimal $\mathbf{F}$ is the solution to the following problem:

$$\begin{aligned} &\min_{\mathbf{F}} \operatorname{tr}\{\sigma_n^2 \mathbf{F}\mathbf{F}^H\} \\ &\text{s.t } \mathbf{FH} = \mathbf{I}_{N_T \times N_T} \end{aligned}$$

the constraint is equivalent to $\operatorname{tr}\{(\mathbf{FH} - \mathbf{I})(\mathbf{FH} - \mathbf{I})^H\} = 0$

Lagrangian:

$$\begin{aligned} L(\mathbf{F}) &= \operatorname{tr}\{\sigma_n^2 \mathbf{F}\mathbf{F}^H\} + \lambda \operatorname{tr}\{\mathbf{F}\mathbf{H}\mathbf{H}^H\mathbf{F} - \mathbf{FH} - \mathbf{H}^H\mathbf{F}^H + \mathbf{I}\} \\ &= \sigma_n^2 \operatorname{tr}\{\mathbf{F}\mathbf{F}^H\} + \lambda \operatorname{tr}\{\mathbf{F}\mathbf{H}\mathbf{H}^H\mathbf{F}^H\} - \lambda \operatorname{tr}\{\mathbf{FH}\} - \lambda \operatorname{tr}\{\mathbf{H}^H\mathbf{F}^H\} + \lambda N_T \end{aligned}$$

- use rules for complex matrix differentiation in Table IV in paper by Hjörunges & Gesbert

$$\begin{aligned} \frac{\delta L(\mathbf{F})}{\delta \mathbf{F}^*} &= \sigma_n^2 \mathbf{F} + \lambda \mathbf{F}\mathbf{H}\mathbf{H}^H - \lambda \mathbf{H}^H = 0 \\ &\rightarrow \mathbf{F}(\sigma_n^2 \mathbf{I} + \lambda \mathbf{H}\mathbf{H}^H) = \lambda \mathbf{H}^H \\ &\rightarrow \mathbf{F} = \lambda \mathbf{H}^H (\sigma_n^2 \mathbf{I} + \lambda \mathbf{H}\mathbf{H}^H)^{-1} \end{aligned}$$

use matrix inversion lemma

$$(\mathbf{A} + \mathbf{UBV})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{B}^{-1} + \mathbf{V}\mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}\mathbf{A}^{-1}$$

$$\begin{aligned} \rightarrow \mathbf{F} &= \lambda \mathbf{H}^H \left[\frac{1}{\sigma_n^2}\mathbf{I} - \frac{1}{\sigma_n^2}\mathbf{H} \left[\frac{1}{\lambda}\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H} \right]^{-1} \mathbf{H}^H \frac{1}{\sigma_n^2} \right] \\ &= \frac{\lambda}{\sigma_n^2} \left[\overbrace{\left(\tfrac{1}{\lambda}\mathbf{I} + \tfrac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H}\right)\left(\tfrac{1}{\lambda}\mathbf{I} + \tfrac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H}\right)^{-1}}^{\mathbf{I}} - \frac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H} \left[\frac{1}{\lambda}\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H} \right] \right] \mathbf{H}^H \\ &= \frac{\lambda}{\sigma_n^2} \left[\frac{1}{\lambda}\mathbf{I} + \frac{1}{\lambda}\mathbf{H}^H\mathbf{H} - \frac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H} \right] \left(\frac{1}{\lambda}\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H} \right)^{-1} \mathbf{H}^H \\ &= \frac{1}{\sigma_n^2} \left(\frac{1}{\lambda}\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H} \right) \mathbf{H}^H \end{aligned}$$

- How to choose λ

$$\mathbf{FH} = \frac{1}{\sigma_n^2} \left(\frac{1}{\lambda}\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H} \right)^{-1} \mathbf{H}^H\mathbf{H} = \mathbf{I}$$

$$\frac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H} = \frac{1}{\lambda}\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H}$$

$$\Rightarrow \lambda \rightarrow \infty$$

$$\Rightarrow \boxed{\mathbf{F} = (\mathbf{H}^H\mathbf{H})^{-1}} \widehat{=} \text{ Moore-Penrose pseudoinverse}$$

noise covariance:

$$\Phi_{ee} = \sigma_n^2 \mathbf{F}\mathbf{F}^H = \sigma_n^2 (\mathbf{H}^H\mathbf{H})^{-1} \underbrace{\mathbf{H}^H\mathbf{H}(\mathbf{H}\mathbf{H})^{-1}}_{\mathbf{I}} = \sigma_n^2 (\mathbf{H}^H\mathbf{H})^{-1}$$

Φ_{ee} is not in general a diagonal matrix

- effective noise $\mathbf{Fn}$ is spatially correlated
- "equalization of channel leads to coloring of noise

- Interpretation:
  we have

$$\mathbf{FHx} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_{N_T} \end{bmatrix} \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \dots & \mathbf{h}_{N_T} \end{bmatrix} \mathbf{x} = \mathbf{x}$$

  - $\mathbf{f}_i\mathbf{h}_i = 1 \quad \mathbf{f}_i\mathbf{h}_j = 0 \quad \forall i \neq j$
  - $\mathbf{f}_i^T$ is orthogonal to $\begin{bmatrix} \mathbf{h}_1 & \dots \mathbf{h}_{i-1} & \mathbf{h}_{i+1} & \dots & \mathbf{h}_{N_T} \end{bmatrix} \updownarrow N_R$
  - $\mathbf{f}_i^T$ is confined to an $N_R - (N_T - 1)$ dimensional subspace of the N_R dimensional space spanned by $\mathbf{H}$

- Diversity gain
  - e.g. SISO model: $r_i = \mathbf{f}_i \cdot \mathbf{h}_i \mathbf{x}_i + \mathbf{f}_i \cdot \mathbf{n}_i$

    $\rightarrow$ $\text{SNR}_{\text{eq}} = \frac{\epsilon_s |\mathbf{f}_i\mathbf{h}_i|^2}{\sigma_n^2 \|\mathbf{f}_i\|^2} = \frac{\epsilon_s}{\sigma_n^2} \left|\tilde{\mathbf{f}}_i \cdot \mathbf{h}_i\right|$, where $\mathbf{f}_i = \alpha \mathbf{f}_i$ with $\|\tilde{\mathbf{f}}_i\|^2 = 1$

    we can represent $\mathbf{f}_i$ as: $\tilde{\mathbf{f}}_i^T = \alpha \mathbf{M}\boldsymbol{\beta}$,

    where: $\mathbf{M} \in \mathbb{C}^{N_R \times (N_R - N_T + 1)}$ and $\boldsymbol{\beta} \in \mathbb{C}^{(N_R - N_T + 1) \times 1}$ $\quad \widehat{=}$ basis of subspace

    $\rightarrow$ $\tilde{\mathbf{f}}_i \mathbf{M}_i = \boldsymbol{\beta}^T \mathbf{M}_i^T \mathbf{M}_i$,

    where: $\mathbf{M}^H\mathbf{M} = \tilde{\mathbf{M}}_i = \mathbf{I}$ and $\tilde{\mathbf{M}}_i \rightarrow \mathcal{CN}(0, \sigma_n^2 \mathbf{I}_{(N_R - N_T + 1)}$

    (since rows of $\mathbf{M}^T$ are orthogonal)

$$\text{SNR}_{\text{eq}} = \frac{\epsilon_s}{\sigma_n^2} \alpha^2 \left| \sum_{j=1}^{N_R - N_T + 1} \boldsymbol{\beta}_{ji} \tilde{\mathbf{h}}_j i \right|^2 ; \tilde{\mathbf{h}}_i = \begin{pmatrix} \tilde{h}_{1i}, & \tilde{h}_{2i}, & \dots, & \tilde{h}_{N_R - N_T + 1} \end{pmatrix}^T$$

$$\boldsymbol{\beta}_i = \begin{pmatrix} \beta_{1i}, & \dots, & \beta_{N_R - N_T + 1, i} \end{pmatrix}^T$$

    $\rightarrow$ SNR_{eq} includes only $N_R - N_T + 1$ independent Gaussian RV

    $\rightarrow$ diversity gain is limited to: $\underline{G_d = N_R - N_T + 1}$

Example:

$$N_T = N_R = 3$$
$$G_d^{ZF} = 1 \text{ but } G_d^{ML} = N_R = 3$$

$\rightarrow$ huge performance loss because of linear ZF

3.2.2 MMSE detection

- ZF criterion may be too strict and leads to noise enhancement
  - $\rightarrow$ maybe it is better to allow some interferences between signals but reduce noise enhancement
  - $\rightarrow$ What is the optimal trade-off between interference and noise?
  - $\rightarrow$ MMSE criterion
- MMSE criterion
  - error signal: $\mathbf{e} = \mathbf{F}\mathbf{y} - \mathbf{x}$
  - total error variance: $\sigma_e^2 = \mathcal{E}\big\{\|\mathbf{e}\|^2\big\} = \mathcal{E}\big\{\mathrm{tr}\{\mathbf{e}\mathbf{e}^H\}\big\} = \mathrm{tr}\big\{\mathcal{E}\{\mathbf{e}\mathbf{e}^H\}\big\} = \mathrm{tr}\{\Phi_{ee}\}$
  - Φ_{ee}: error covariance matrix
  - optimal filter: $\mathbf{F}_{\mathrm{opt}} = \underset{\mathbf{F}}{\mathrm{argmin}}\,\mathrm{tr}\{\Phi_{ee}\}$
- Deviation of $\mathbf{F}_{\mathrm{opt}}$
  - $\Phi_{ee} = \mathcal{E}\{\mathbf{e}\mathbf{e}^H\} = \mathcal{E}\{(\mathbf{F}\mathbf{y}-\mathbf{x})(\mathbf{F}\mathbf{y}-\mathbf{x})^H\} = \mathbf{F}\cdot\mathbf{\Phi}_{yy}\cdot\mathbf{F}^H - \mathbf{F}\cdot\mathbf{\Phi}_{yx} - \mathbf{\Phi}_{xx}\cdot\mathbf{\Phi}_{xy}\cdot\mathbf{F}^H + \mathbf{\Phi}_{xx}$

    with:

$$\mathbf{\Phi}_{yy} = \mathcal{E}\{\mathbf{y}\mathbf{y}^H\} = \mathcal{E}\{(\mathbf{H}\mathbf{x}+\mathbf{n})(\mathbf{H}\mathbf{x}+\mathbf{n})^H\} = \epsilon_s \cdot \mathbf{H}\mathbf{H}^H + \sigma_n^2 \cdot \mathbf{I}_{N_R \times N_T}$$
$$\mathbf{\Phi}_{yx} = \mathcal{E}\{\mathbf{y}\mathbf{x}^H\} = \mathcal{E}\{(\mathbf{H}\mathbf{x}+\mathbf{n})\cdot\mathbf{x}^H\} = \epsilon_s \cdot \mathbf{H}^H = \mathbf{\Phi}_{xy}^H$$
$$\mathbf{\Phi}_{xx} = \epsilon_s \cdot \mathbf{I}_{N_T \times N_R}$$

  - $\mathbf{F}_{\mathrm{opt}} \quad \rightarrow \quad \frac{d}{d\mathbf{F}^*}\Big(\mathrm{tr}\{\mathbf{F}\mathbf{\Phi}_{yy}\mathbf{F}^H\} - \mathrm{tr}\{\mathbf{F}\mathbf{\Phi}_{yx}\} - \mathrm{tr}\{\mathbf{\Phi}_{xy}\mathbf{F}^H\} + \mathrm{tr}\{\mathbf{\Phi}_{xx}\}\Big) \overset{!}{=} 0$

    with Table IV in paper by Hjoranges & Gesbert:

    $\Rightarrow\ \mathbf{F}\cdot\mathbf{\Phi}_{yy} - \mathbf{\Phi}_{xy} = 0$

    $\Rightarrow\ \mathbf{F}_{\mathrm{opt}} = \mathbf{\Phi}_{xy}\cdot\mathbf{\Phi}_{yy}^{-1} = \epsilon_s\mathbf{H}^H\big(\epsilon_s\mathbf{H}\mathbf{H}^H + \sigma_n^2\mathbf{I}\big)^{-1}$

    $= \text{(Matrix inversion Lemma)} =$

    $= \big(\mathbf{H}^H\mathbf{H} + \frac{\sigma_n^2}{\epsilon_s}\mathbf{I}\big)^{-1}\cdot\mathbf{H}^H$
  - Comparison:

$$\mathbf{F}_{\mathrm{MMSE}} = \Big(\mathbf{H}^H\mathbf{H} + \frac{\sigma_n^2}{\epsilon_s}\mathbf{I}\Big)^{-1}\cdot\mathbf{H}^H \xrightarrow[\frac{\sigma_n^2}{\epsilon}\to 0]{} \big(\mathbf{H}^H\cdot\mathbf{H}\big)^{-1}\mathbf{H}^H = \mathbf{F}_{ZF}$$
$$\xrightarrow[\frac{\sigma_n^2}{\epsilon_s}\to\infty]{} \frac{\epsilon_s}{\sigma_n^2}\cdot\mathbf{H}^H = \mathbf{F}_{MF} \quad \widehat{=}\text{matched filter}$$

$\Rightarrow$ For high SNR, $\frac{\epsilon_s}{\sigma_n^2}$, the MMSE filter approaches the ZF-Filter, for low SNR, it approaches the matched filter.

$\rightarrow$ MMSE receiver yields the same diversity gain as the ZF receiver

$G_d^{MMSE} = G_d^{ZF} = N_R - N_T + 1 \leq G_d^{ML} = N_R$

– End-to-End Channel: $\mathbf{K} = \mathbf{FH} = \left(\mathbf{H}^H\mathbf{H} + \frac{\sigma_n^2}{\epsilon_s} \cdot \mathbf{I}\right)^{-1} \cdot \mathbf{H}^H \cdot \mathbf{H} \neq$ diagonal matrix

$\Rightarrow$ crosstalk/interference between elements $\mathbf{x}$ in received signal after filtering $\mathbf{r}$.

elements of $\mathbf{K}$: $K_{l,n}$

– Covariance for $\mathbf{F}_{\text{opt}}$

$$
\begin{aligned}
\mathbf{\Phi}_{ee} &= \mathbf{\Phi}_{xy} \cdot \overbrace{\mathbf{\Phi}_{yy}^{-1} \cdot \mathbf{\Phi}_{yy} \cdot \mathbf{\Phi}_{yy}^{-1}}^{\mathbf{\Phi}_{yy}^{-1}} \cdot \mathbf{\Phi}_{xy}^H - \mathbf{\Phi}_{xy} \cdot \mathbf{\Phi}_{yy}^{-1} \cdot \mathbf{\Phi}_{yx} - \mathbf{\Phi}_{xy} \cdot \mathbf{\Phi}_{yy}^{-1} \cdot \mathbf{\Phi}_{xy}^H + \mathbf{\Phi}_{xx} \\
&= \mathbf{\Phi}_{xx} - \overbrace{\mathbf{\Phi}_{xy} \cdot \mathbf{\Phi}_{yy}^{-1}}^{\mathbf{F}_{\text{opt}}} \cdot \mathbf{\Phi}_{yx} \\
&= \epsilon_s \mathbf{I} - \epsilon_s \left(\mathbf{H}^H\mathbf{H} + \frac{\sigma_n^2}{\epsilon_s} \cdot \mathbf{I}\right)^{-1} \cdot \mathbf{H}^H\mathbf{H} = \text{(Matrix inversion Lemma)} \\
&= \sigma_n^2 \left(\mathbf{H}^H\mathbf{H} + \frac{\sigma_n^2}{\epsilon_s} \cdot \mathbf{I}\right)^{-1} \\
\mathbf{\Phi}_{ee} &= \epsilon_s (\mathbf{I} - \mathbf{K})
\end{aligned}
$$

$\rightarrow$ $0 \leq K_{m,m} \leq 1$ since main diagonal elements of $\mathbf{\Phi}_{ee}$ are $0 \leq [\mathbf{\Phi}_{ee}]_{m,m} \leq \epsilon_s$

siehe auch Abbildung 1

3.2.3 SNR (biased vs. unbiased)

a) biased SNR

$$\text{SNR}_{\text{bias,m}} = \frac{\epsilon_s}{[\mathbf{\Phi}_{ee}]_{mm}} = \frac{\epsilon_s}{\epsilon_s(1 - K_{mm})} = \frac{1}{1 - K_{mm}}, \quad 1 \leq m \leq 4$$

Anmerkung: $\mathbf{K} = \mathbf{F}_{\text{opt}} \cdot \mathbf{H} \rightarrow$ SNR $= 1$ *falls* $\mathbf{K} =$ zeros() $\Rightarrow$ woher SNR $= 1$ *bei keiner Uebertragung?* $\Rightarrow$ *nicht vorteilhaft: siehe: b) unbiased SNR* but:

- $\text{SNR}_{\text{bias,m}}$ does not represent the actual SNR since the main diagonal elements of $\mathbf{K}$ are smaller than 1
- $\mathbf{r} = \mathbf{FHx} + \mathbf{Fn} = \mathbf{Kx} + \underbrace{\mathbf{Fn}}_{\tilde{\mathbf{n}} = [\tilde{n}_1, \ldots, \tilde{n}_{N_T}]^T}$
- $r_m = \underbrace{K_{mm}}_{<1} x_m + \sum\limits_{\substack{n=1 \\ n \neq m}}^{N_T} K_{mn} x_n + \tilde{n}_m$

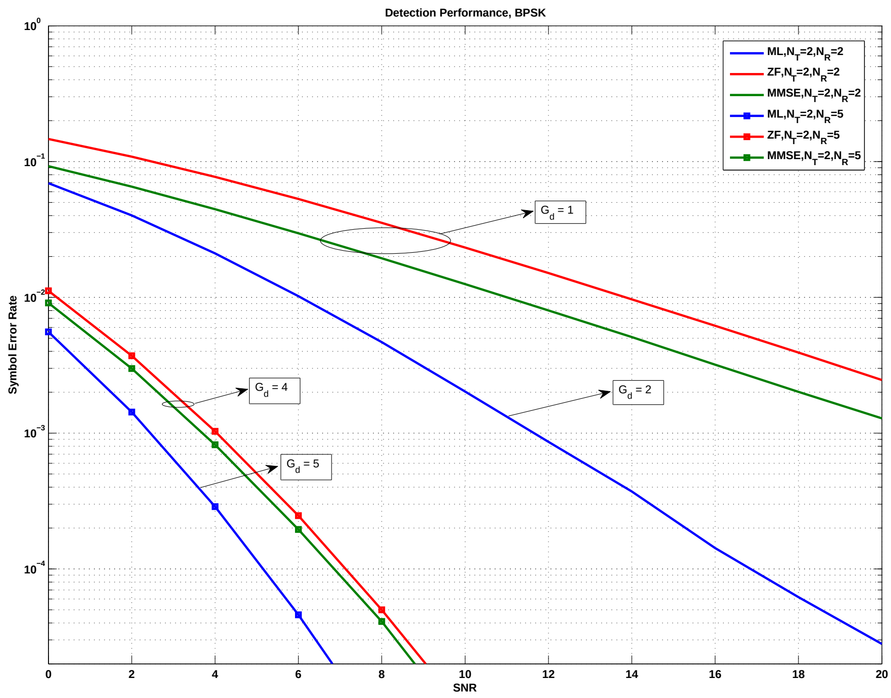


Figure 1: MMSE

b) unbiased SNR

- remove bias: $r'_m = \dfrac{r_m}{K_{mm}} = x_m + \underbrace{\dfrac{\tilde{e}_m}{K_{mm}}}_{e'_m}$
- SNR?
- scaling matrix: $\mathbf{C} = \operatorname{diag}\left\{\frac{1}{K_{11}}, \frac{1}{K_{22}}, \ldots, \frac{1}{K_{N_T N_T}}\right\}$
- $\mathbf{r}' = \mathbf{C}\mathbf{r} \rightarrow \mathbf{e}' = [e'_1, \ldots, e'_{N_T}]^T = \mathbf{r}' - \mathbf{x} = \mathbf{C}\mathbf{r} - \mathbf{x} = \mathbf{C}\mathbf{F}\mathbf{y} - \mathbf{x}$
- $\mathbf{\Phi}_{e'e'} = \mathcal{E}\left\{(\mathbf{C}\mathbf{F}\mathbf{y} - \mathbf{x})(\mathbf{C}\mathbf{F}\mathbf{y} - \mathbf{x})^H\right\} = \mathbf{C}\mathbf{F}\mathbf{\Phi}_{yy}\mathbf{F}^H\mathbf{C}^H - \mathbf{C}\mathbf{F}\mathbf{\Phi}_{yx} - \mathbf{\Phi}_{xy}\mathbf{F}^H\mathbf{C}^H + \mathbf{\Phi}_{xx}$
- $\mathbf{F} = \mathbf{F}_{\text{opt}} \rightarrow \mathbf{F}_{\text{opt}}\mathbf{\Phi}_{yy} = \mathbf{\Phi}_{xy} = \epsilon_s \cdot \mathbf{H}^H$

$\rightarrow$ $\mathbf{\Phi}_{e'e'} = \epsilon_s \mathbf{C} \underbrace{\mathbf{H}^H\mathbf{F}^H}_{\mathbf{K}^H} \mathbf{C}^H - \epsilon_s \mathbf{C} \underbrace{\mathbf{F}\mathbf{H}}_{\mathbf{K}} - \epsilon_s \underbrace{\mathbf{H}^H\mathbf{F}^H}_{\mathbf{K}^H} \mathbf{C} + \epsilon_s \mathbf{I}$

$= \epsilon_s\left[\mathbf{I} + (\mathbf{C} - \mathbf{I})\mathbf{K}^H\mathbf{C}^H - \mathbf{C}\mathbf{K}\right]$

Anmerkung: nur Hauptdiagonalelemente interessieren, da diese die Varianz darstellen

$\rightarrow$ maindiagonal elements of $\mathbf{\Phi}_{e'e'}$ = variances of e'_m $= \epsilon_s\left(1 + \left(\frac{1}{K_{mm}} - 1\right)K_{mm}\frac{1}{K_{mm}} - \frac{1}{K_{mm}}K_{mm}\right) = \epsilon_s\frac{1-K_{mm}}{K_{mm}}$

→ vgl. Abbildung 1

→ $SNR_{unbiased} = \dfrac{E_s}{[\mathbf{\Phi}_{e'e'}]_{mm}} = \dfrac{E_s}{E_s \dfrac{1-K_{mm}}{K_{mm}}} = \dfrac{K_{mm}}{1-K_{mm}}, \quad 1 \leq m \leq N_T$

→ the SNR after bias removal is by "1" smaller than the biased SNR → general result valid for any type of MMSE estimation

3.3 Decision - Feadback Equalization (Detection)

- Also known as:
  - BLAST (Bell Laboratories space-time system)
  - successive interference cancellation
- Problem of linear receiver: Noise enhancementbecause of linear filtering → nonlinear filtering processing necessary

3.3.1 Basic Idea

- Recall (linear filter): $\mathbf{FH} = \mathbf{I}$ for linear ZF receiver → ith row of $\mathbf{F}, \mathbf{f}_i$, is orthogonal to the jth column of $\mathbf{H}, \mathbf{h}_j$, if $i \neq j$ (if $i = j$: inner product $= 1$)
- we can detect x_i, based on $r_i = \mathbf{f}_i\mathbf{y}$
- Once we have detected x_i, we can substract its contribution from $\mathbf{y}$: $\mathbf{y}_1 = \mathbf{y} - \mathbf{h}_i\hat{x}_i$ ($\hat{x}_i$ is detected symbol, we assume for now, $\hat{x}_i = x_i$)
- $\mathbf{y}_1$ can be expressed as $\mathbf{y}_1 = \mathbf{H}_i\mathbf{x}_i + \mathbf{n}$ where

$$\begin{aligned}\mathbf{H}_i &= \big[\mathbf{h}_1, \ldots, \mathbf{h}_{i-1}, \mathbf{h}_{i+1}, \ldots, \mathbf{h}_{N_T}\big]\\ \mathbf{x}_i &= \big[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{N_T}\big]\end{aligned}$$

  → we have reduced the number of signal streams to $N_T - 1$ (N_R bleibt gleich)

- apply now linear ZF filter for symbol to detected next, e.g. x_j, where $j \in \big\{1, \ldots, i-1, i+1, \ldots, N_T\big\}$

→ $\mathbf{r}_j = \mathbf{f}_j\mathbf{y}_1$ where $\mathbf{f}_j$ is the ZF filter for $\mathbf{H}_1$

- substract contribution of x_j from $\mathbf{y}_1$: $\mathbf{y}_2 = \mathbf{y}_1 - \mathbf{h}_j x_j$
- substract until last symbol is detected
- Blockdiagram see figure2
- Observations:
  - The order in which the x_1 are detected can be freely chosen and effects the performance → $N_T!$ possible orders → cannot explore all of them
  - Practical approach: Select in each step that x_i for which the noise variance enhancement is minimum, i.e. which has the smallest $\mathcal{E}\Big\{|\mathbf{f}_i\mathbf{n}|^2\Big\} = \sigma_n^2 |||\mathbf{f}_i|||^2$

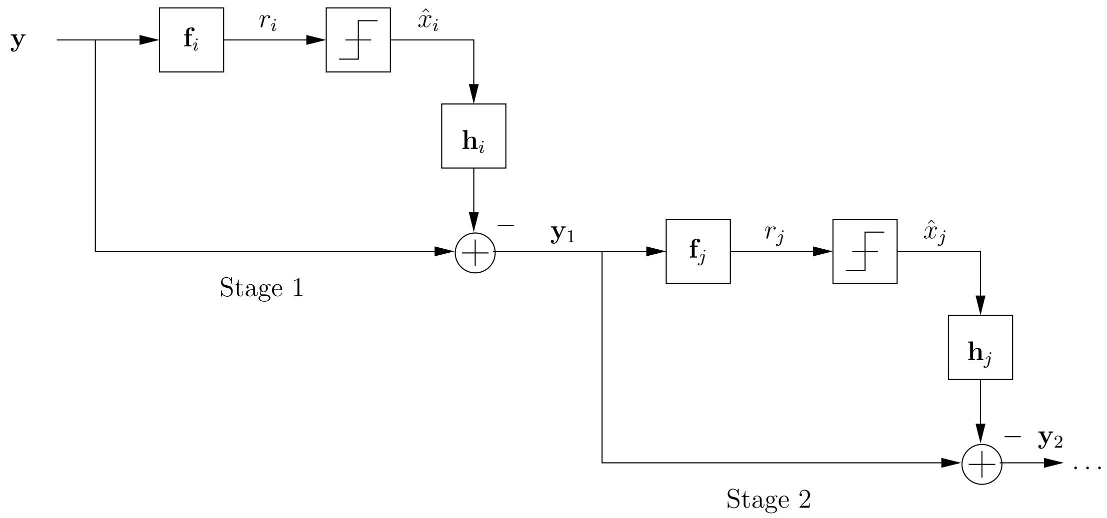


Figure 2: DFE Blockdiagram

- Diversity order:
  - stage 1: $G_d^1 = N_R - N_T + 1$
  - stage 2: $G_d^2 = G_d^1 + 1 = N_R - N_T + 2$

    $\vdots$
  - stage N_T: $G_d^{N_T} = N_R$
  - overall: $\underline{G_d = N_R - N_T + 1}$
  - *(Anmerkung: der schlechteste Fall dominiert (Stage 1), weitere koennen nur schwer beeinflussen)*

3.3.2 ZF - DFE - Matrix Model

- Signal model: $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} = \underbrace{\mathbf{H}\mathbf{P}}_{\tilde{\mathbf{H}}} \cdot \underbrace{\mathbf{P}^{-1}\mathbf{x}}_{\tilde{\mathbf{x}}} + \mathbf{n}$ with permutation matrix $\mathbf{P}$
- $\mathbf{P}$ has one „1“ per column and row, all other elements are „0“

$\rightarrow$ can change the detection order to maximize performance

- note: $\mathbf{P}^T = \mathbf{P}^{-1}$

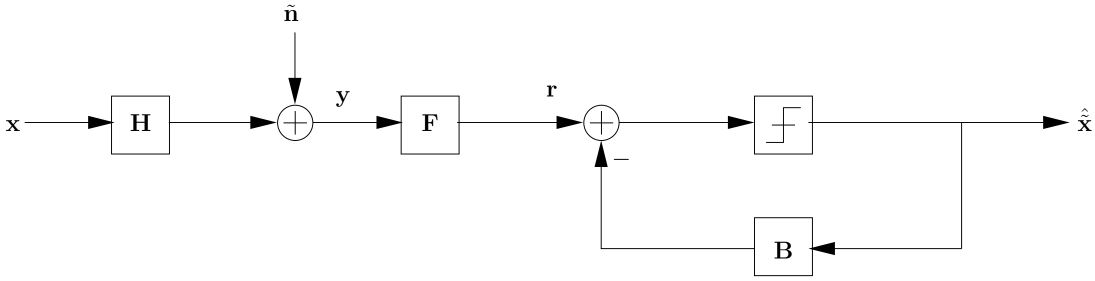


Figure 3: ZF-DFE Matrix Model

- Example:

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} [r] \rightarrow \mathbf{P}^{-1} = \mathbf{P}^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\tilde{\mathbf{x}} = \mathbf{P}^T \cdot \mathbf{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}$$

- Blockdiagram see figure 3
- DFE Filters:
  - $\mathbf{F}$ feedworward filter
  - $\mathbf{B}$ feedback filter
- Filter calculation
  - Cholesky factorization: $\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} = \mathbf{L}^H\mathbf{D}\mathbf{L}$ with diagonal matrix $\mathbf{D}$ and lower triangular matrix $\mathbf{L}$ (maindiagonal elements of $\mathbf{L}$ are "1")

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ l_{21} & 1 & \ddots & \ddots & \vdots \\ l_{31} & l_{32} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \cdots & 1 \end{bmatrix}$$

  - $\mathbf{F} = \mathbf{D}^{-1}\mathbf{L}^{-H}\tilde{\mathbf{H}}^H$

$$\rightarrow \mathbf{r} = \mathbf{F}\mathbf{y} = \underbrace{\mathbf{D}^{-1}\mathbf{L}^{-H}\underbrace{\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}}_{\mathbf{L}^H\mathbf{D}\mathbf{L}}}_{\mathbf{L}}\tilde{\mathbf{x}} + \underbrace{\mathbf{D}^{-1}\mathbf{L}^{-H}\tilde{\mathbf{H}}^H\mathbf{n}}_{\tilde{\mathbf{n}}}$$

– $\mathbf{B} = \mathbf{L} - \mathbf{I}$ = lower triangular matrix with maindiagonal elements “0”

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ l_{21} & 0 & \ddots & \ddots & \vdots \\ l_{31} & l_{32} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

- Interpretation:
  – after feedforward filter

$$\mathbf{r} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ l_{21} & 1 & \ddots & \ddots & \vdots \\ l_{31} & l_{32} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \cdots & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_{N_T} \end{bmatrix} + \tilde{\mathbf{n}}$$

- from feedback filter

$$\tilde{\mathbf{r}} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ l_{21} & 0 & \ddots & \ddots & \vdots \\ l_{31} & l_{32} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} \hat{\tilde{x}}_1 \\ \hat{\tilde{x}}_2 \\ \vdots \\ \hat{\tilde{x}}_{N_T} \end{bmatrix} \quad \hat{\tilde{x}}_n : \text{decision on } \tilde{x}_n$$

$$\rightarrow \mathbf{r} - \tilde{\mathbf{r}} = \begin{bmatrix} \tilde{x}_1 + 0 \\ l_{21}(\tilde{x}_1 - \hat{\tilde{x}}_1) + \tilde{x}_2 + 0 \\ l_{31}(\tilde{x}_1 - \hat{\tilde{x}}_2) + l_{32}(\tilde{x}_2 - \hat{\tilde{x}}_2) + \tilde{x}_3 + 0 \\ \vdots \end{bmatrix} + \tilde{\mathbf{n}}$$

MMSE-DFE

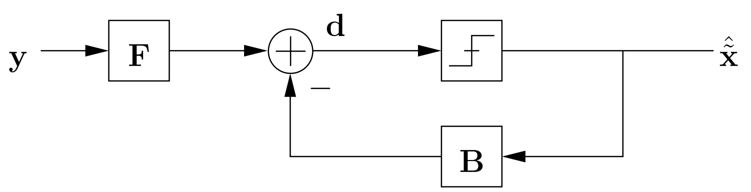


Figure 4: MMSE-DFE Blockdiagram

- $\mathbf{B} = \mathbf{L} - \mathbf{I}$ has to be lower triangular matrx with all-zero main diagonal elements because of causality.

- $\mathbf{d} = \mathbf{F}\mathbf{y} - \mathbf{B}\hat{\tilde{\mathbf{x}}} \quad \rightarrow \mathbf{e} = \mathbf{d} - \hat{\mathbf{x}} = \mathbf{F}\mathbf{y} - \underbrace{(\mathbf{B}+\mathbf{I})}_{\mathbf{L}}\tilde{\mathbf{x}}$

  where we assume $\hat{\tilde{\mathbf{x}}} = \tilde{\mathbf{x}}$ for filter optimization

- Optimal $\mathbf{F}$ for given $\mathbf{L}$

$$\sigma_e^2 = \operatorname{tr}\left\{\underbrace{\mathcal{E}\{\mathbf{e}\mathbf{e}^H\}}_{\mathbf{\Phi}_{ee}}\right\} = \operatorname{tr}\left\{\mathbf{F}\mathbf{\Phi}_{yy}\mathbf{F}^H\right\} - \operatorname{tr}\left\{\mathbf{F}\mathbf{\Phi}_{y\tilde{x}}\mathbf{L}^H\right\} - \operatorname{tr}\left\{\mathbf{L}\mathbf{\Phi}_{\tilde{x}y}\mathbf{F}^H\right\} + \operatorname{tr}\left\{\mathbf{L}\mathbf{\Phi}_{\tilde{x}\tilde{x}}\mathbf{L}^H\right\}$$

$$\frac{\delta}{\delta\mathbf{F}^*}\sigma_e^2 = \mathbf{F}\mathbf{\Phi}_{yy} - \mathbf{L}\mathbf{\Phi}_{\tilde{x}y} = 0 \quad \rightarrow \mathbf{F} = \mathbf{L}\mathbf{\Phi}_{\tilde{x}y}\mathbf{\Phi}_{yy}^{-1}$$

where

$$\mathbf{\Phi}_{yy} = \mathcal{E}_s\tilde{\mathbf{H}}\tilde{\mathbf{H}}^H + \sigma_n^2\mathbf{I}$$
$$\mathbf{\Phi}_{\tilde{x}y} = \mathcal{E}_s\tilde{\mathbf{H}}^H$$

$$\rightarrow \mathbf{F} = \mathbf{L}\tilde{\mathbf{H}}^H(\tilde{\mathbf{H}}\tilde{\mathbf{H}}^h + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I})^{-1} = \mathbf{L}\,\overbrace{(\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I})^{-1}\tilde{\mathbf{H}}^H}^{\text{MMSE Linear Equalizer}}$$

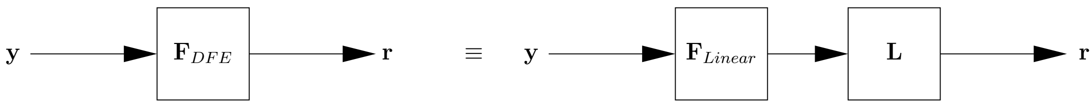


Figure 5: MMSE-DFE Equivalence

- Optimal $\mathbf{L}$:

$$\begin{aligned}\mathbf{\Phi}_{ee} &= \mathbf{L}\mathbf{\Phi}_{\tilde{x}y}\mathbf{\Phi}_{yy}^{-1}\mathbf{\Phi}_{yy}\mathbf{\Phi}_{yy}^{-1}\mathbf{\Phi}_{\tilde{x}y}^H\mathbf{L}^H - \mathbf{L}\mathbf{\Phi}_{xy}\mathbf{\Phi}_{yy}^{-1}\mathbf{\Phi}_{yz}\mathbf{L}^H - \mathbf{L}\mathbf{\Phi}_{\tilde{x}y}\mathbf{\Phi}_{yy}^{-1}\mathbf{\Phi}_{\tilde{x}y}^H\mathbf{H}^H + \mathbf{L}\mathbf{\Phi}_{\tilde{x}\tilde{x}}\mathbf{L}^H \\ &= \mathbf{L}(\mathbf{\Phi}_{\tilde{x}\tilde{x}} - \mathbf{\Phi}_{\tilde{x}y}\mathbf{\Phi}_{yy}^{-1}\mathbf{\Phi}_{\tilde{x}y}^H)\mathbf{L}^H\end{aligned}$$

  = MMSE covariance matrix for a linear MMSE filter

  $\rightarrow$ $\mathbf{\Phi}_{ee} = \sigma_n^2\mathbf{L}(\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I})^{-1}\mathbf{L}^H$

  $\rightarrow$ need lower triangular matrix $\mathbf{L}$ which minimizes $\operatorname{tr}\{\mathbf{\Phi}_{ee}\}$

  $\rightarrow$ the optimum $\mathbf{L}$ whitenes $\mathbf{\Phi}_{ee} \rightarrow \mathbf{\Phi}_{ee}$ becomes diagonal matrix i.e. $\mathbf{L}$ exploits the correlation after linear MMSE filtering to reduce noise variance!

  $\rightarrow$ $\mathbf{L}$ is obtained via Cholesky factorization

$$\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I} = \mathbf{L}\mathbf{D}\mathbf{L}^H$$

  $\rightarrow$ $\mathbf{\Phi}_{ee} = \sigma_n^2\mathbf{L}(\mathbf{L}\mathbf{D}\mathbf{L}^H)^{-1}\mathbf{L} = \sigma_n^2\mathbf{L}\mathbf{L}^{-1}\mathbf{D}^{-1}\mathbf{L}^{-H}\mathbf{L} = \sigma_n^2\mathbf{D}^{-1} \widehat{=}$ diagonal matrix

- Summary of MMSE calculation
  - $\mathbf{B} = \mathbf{L} - \mathbf{I}$ where $\mathbf{H}^H\mathbf{H} + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I} = \mathbf{LDL}^H$
  - $\mathbf{F} = \mathbf{L}\underbrace{(\mathbf{H}^H\mathbf{H} + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I})^{-1}\tilde{\mathbf{H}}}_{\mathbf{F}_{Linear}}$
  - $\mathbf{L}$ is a prediction error filter, which whitenes the error signal $\mathbf{e}$
  - SNR:

$$\mathbf{D} = \mathrm{diag}\{\xi_1, \dots, \xi_{N_T}\}$$
$$\mathbf{\Phi}_{ee} = \mathrm{diag}\{\frac{\sigma_n^2}{\xi_1}, \dots, \frac{\sigma_n^2}{\xi_{N_T}}\}$$

  - end-to-end channel

$$\begin{aligned}
\mathbf{K} &= \mathbf{F}\tilde{\mathbf{H}} \\
&= \underbrace{\mathbf{L}(\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I})^{-1}\mathbf{L}^H}_{\frac{1}{\sigma_n^2}\mathbf{\Phi}_{ee}}\mathbf{L}^{-H}\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} \\
&= \frac{1}{\sigma_n^2}\mathbf{\Phi}_{ee}\mathbf{L}^{-H}(\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I} - \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I})\mathbf{L}^{-1}\mathbf{L} \\
&= \frac{1}{\sigma_n^2}[\underbrace{\mathbf{L}^{-H}(\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I})\mathbf{L}^{-1}}_{\sigma_n^2\mathbf{\Phi}_{ee}^{-1}}\mathbf{L} - \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{L}^{-H}] \\
&= \mathbf{L} - \frac{1}{\mathcal{E}_s}\mathbf{\Phi}_{ee}\mathbf{L}^{-H}
\end{aligned}$$

    $\rightarrow$ main diagonal of $\mathbf{K}$:

$$K_{mm} = 1 - \frac{\sigma_n^2}{\mathcal{E}_s\xi_m} < 1$$

  - biased SNR

$$SNR_{m,bias} = \frac{\mathcal{E}_S}{[\mathbf{\Phi}_{ee}]_{mm}} = \frac{\mathcal{E}_s}{\sigma_n^2}\xi_m = \frac{1}{1 - K_{mm}}, \quad 1 \le m \le N_T$$

  - unbiased SNR

$$SNR_{m,unbiased} = SNR_{m,bias} - 1 = \frac{1}{1 - K_{mm}} - 1 = \frac{K_{mm}}{1 - K_{mm}}, \quad 1 \le m \le N_T$$

3.3.3 Sphere Decoding

- Linear and DFE receivers cannot approach performance of ML - detector

- ML - detector: $\hat{\mathbf{x}} = \underset{\mathbf{x}\in A^{N_T}}{\operatorname{argmin}} ||\mathbf{y} - \mathbf{H}\mathbf{x}||^2$

$\rightarrow$ high complexity for brute force search

Main Idea

- can we search ML metric in a "smarter" way, akin to the smart search of the viterbi algorithm for problems with a trellis structure
- we need to find a way to prune/dismiss non - ML sequences/vectores early on

$\rightarrow$ this is accomplished by sphere decoding

Step 1 Bring metric into a suitable form:

- real represantation: $||\mathbf{y} - \mathbf{H}\mathbf{x}||^2 = ||\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\mathbf{x}}||^2$
- where:

$$\begin{aligned}
\tilde{\mathbf{y}} &= \begin{bmatrix} \mathrm{Re}\{\mathbf{y}^T\} & \mathrm{Im}\{\mathbf{y}^T\} \end{bmatrix}^T \\
\tilde{\mathbf{x}} &= \begin{bmatrix} \mathrm{Re}\{\mathbf{x}^T\} & \mathrm{Im}\{\mathbf{x}^T\} \end{bmatrix}^T \\
\tilde{\mathbf{H}} &= \begin{bmatrix} \mathrm{Re}\{\mathbf{H}\} & -\mathrm{Im}\{\mathbf{H}\} \\ \mathrm{Im}\{\mathbf{H}\} & \mathrm{Re}\{\mathbf{H}\} \end{bmatrix}^T
\end{aligned}$$

$\rightarrow$ QAM: $\tilde{\mathbf{x}}$ defines points in an $2N_T$ dimensional lattice

- QL decomposition: $\tilde{\mathbf{H}} = \mathbf{Q}\mathbf{L}$ with
  - orthogonal matrix $\mathbf{Q} \in \mathbb{R}^{2N_T \times 2N_T}$; $\quad \mathbf{Q}\mathbf{Q}^T = \mathbf{I}$
  - and lower triangular matrix $\mathbf{L} \in \mathbb{R}^{2N_T \times 2N_T}$

$$\begin{aligned}
||\tilde{\mathbf{y}} - \mathbf{Q}\mathbf{L}\tilde{\mathbf{x}}||^2 &= ||\mathbf{Q}\mathbf{Q}^T(\tilde{\mathbf{y}} - \mathbf{Q}\mathbf{L}\tilde{\mathbf{x}})||^2 + ||(\mathbf{I} - \mathbf{Q}\mathbf{Q}^T)(\tilde{\mathbf{y}} - \mathbf{Q}\mathbf{L}\tilde{\mathbf{x}})||^2 \\
&= ||\mathbf{Q}(\mathbf{Q}^T\tilde{\mathbf{y}} - \mathbf{L}\tilde{\mathbf{x}})||^2 + ||(\mathbf{I} - \mathbf{Q}\mathbf{Q}^T)\tilde{\mathbf{y}} - \underbrace{(\mathbf{Q} - \mathbf{Q})}_{=0}\mathbf{L}\tilde{\mathbf{x}}||^2 \\
&= ||\mathbf{Q}^T\tilde{\mathbf{y}} - \mathbf{L}\tilde{\mathbf{x}}||^2 + \underbrace{||(\mathbf{I} - \mathbf{Q}\mathbf{Q}^T)\tilde{\mathbf{y}}||^2}_{\text{unabhängig von } \tilde{\mathbf{x}}}
\end{aligned}$$

$\Rightarrow$ $\underset{\tilde{\mathbf{x}}\in A^{2N_T}}{\operatorname{argmin}} ||\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\mathbf{x}}||^2 = \underset{\tilde{\mathbf{x}}\in A^{2N_T}}{\operatorname{argmin}} ||\underbrace{\mathbf{Q}^T\tilde{\mathbf{y}}}_{\bar{\mathbf{y}}} - \mathbf{L}\tilde{\mathbf{x}}||^2$
where A^{2N_T} contains all M^{N_T} possible vectors $\tilde{\mathbf{x}}$

- Observation:
  - with

$$\mathbf{L} = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ l_{2N_T\,1} & \cdots & & l_{2N_T\,2N_T} \end{bmatrix}$$

$$\bar{\mathbf{y}} = \begin{bmatrix} \bar{y}_1 & \cdots & \bar{y}_{2N_T} \end{bmatrix}^T ; \quad \tilde{\mathbf{x}} = \begin{bmatrix} \tilde{x}_1 & \cdots & \tilde{x}_{2N_T} \end{bmatrix}$$

– we have

$$||\bar{\mathbf{y}} - \mathbf{L}\tilde{\mathbf{x}}||^2 = (\bar{y} - l_{11}\tilde{x}_1)^2 + (\bar{y}_2 - l_{21}\tilde{x}_1 - l_{22}\tilde{x}_2)^2 + (\bar{y}_3 - l_{31}\tilde{x}_1 - l_{32}\tilde{x}_2 - l_{33}\tilde{x}_3)^2 + \dots \quad (1)$$

$\rightarrow$ the n-th term in the above sum contains only $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$

Step 2 Sphere decoding algorithm

Define:

$$d(\tilde{\mathbf{x}}) = \sum_{n=1}^{2N_T} f_n(\tilde{\mathbf{x}}_n) \ ; \quad \text{with } f_n(\tilde{\mathbf{x}}_n) = \big(\bar{y}_n - \sum_{i=1}^{n} l_{ni}\tilde{x}_i\big)^2 \ ; \quad \tilde{\mathbf{x}}_n = \begin{bmatrix}\tilde{x}_1 & \dots & \tilde{x}_n\end{bmatrix}^T$$

$$d_n(\tilde{x}_n) = \sum_{m=1}^{n} f_n(\tilde{\mathbf{x}}_m)$$

Main Idea:

- Assume we know that $d(\tilde{\mathbf{x}}) \leq R$ holds for some $\tilde{\mathbf{x}}$, where R is the so called "sphere radius" $\rightarrow$ any $\tilde{\mathbf{x}}$ with $d(\tilde{\mathbf{x}}) \geq R$ cannot be the ML solution and can be discarded
- since $d_{n+1}(\tilde{\mathbf{x}}'_n, \tilde{x}_{n+1}) \geq d_n(\tilde{\mathbf{x}}'_n)$, we can easily discard $\mathbf{x}'_n$ and all possible $\tilde{\mathbf{x}} = \begin{bmatrix}\tilde{\mathbf{x}}'_n & \tilde{x}_{n+1} & \tilde{x}_{n+2} & \dots & \tilde{x}_{2N_T}\end{bmatrix}^T$ if we find $d_n(\tilde{\mathbf{x}}'_n) > R$
  $\Rightarrow$ we can exclude many possible vectors $\tilde{\mathbf{x}}$ without evaluating any metrics for them
- How to find a suitable R?
  - Initial R: $R = d(\tilde{\mathbf{x}}_{\text{subopt}})$ where $\tilde{\mathbf{x}}_{\text{subopt}}$ was obtained with some suboptimum receiver
  - R is updated as $R = R_{\text{new}} = d(\tilde{\mathbf{x}}_{\text{cand}})$ where $\tilde{\mathbf{x}}_{\text{cand}}$ is an $\tilde{\mathbf{x}}$ which yields $d(\tilde{\mathbf{x}}_{\text{cand}}) < R_{\text{old}} = R$
- Use tree structure to represent all possible $\tilde{\mathbf{x}}$

  Example: BPSK, $N_T = 2$