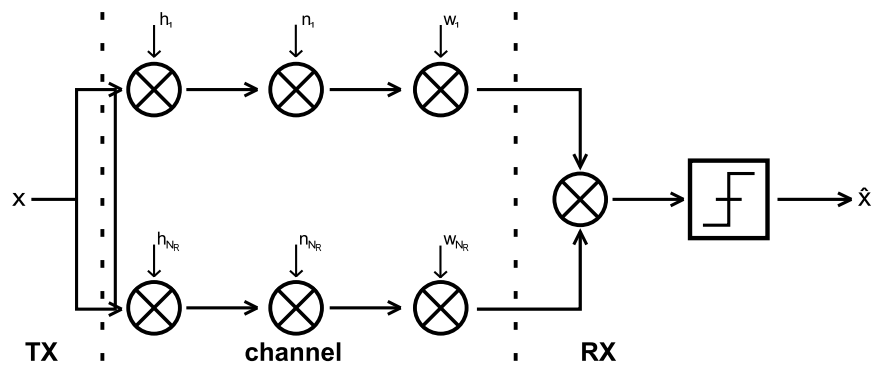


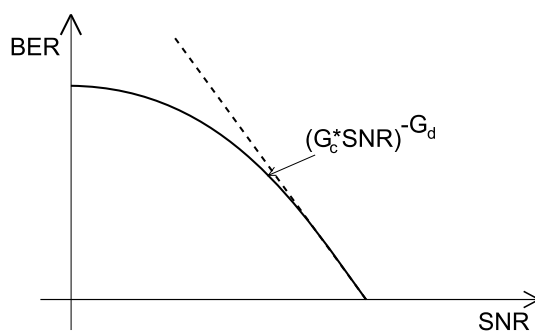
1 SIMO Systems

Remarks

- In SIMO Systems only coding and diversity gains can be exploited (no multiplexing gains)
- To realize these gains diversity combining has to be performed
- Diversity combining schemes vary in complexity and performance
- There are many diversity combining schemes. Here we consider:
 - Maximal ratio combining (MRC)
 - Equal gain combining (EGC)
 - Selection combining (SC)
- Diversity combining problem



- how to choose combining weights w_n ?
- what performance (e.g. error rate, outage probability) is achieved?
- what diversity and coding/combining gain is achieved?



- G_c : Coding gain
- G_d : Diversity gain

1.1 Preliminaries

Consider an equivalent system:

$$y = hx + n; \quad \mathcal{E}\{|x|^2\} = \epsilon_s; \quad \mathcal{E}\{|n|^2\} = \sigma_n^2; \quad \mathcal{E}\{|h|^2\} = 1$$

- Instantaneous SNR: $\gamma_t = \frac{\epsilon_s}{\sigma_n^2} \times |h|^2$
- Average SNR: $\bar{\gamma}_t = \mathcal{E}\{\gamma_t\} = \frac{\epsilon_s}{\sigma_n^2}$

Bit and Symbol Error Rate

- The Bit and Symbol Error Rate of many modulation schemes can be expressed for given γ_t as:

$$P_e(\gamma_t) = aQ\{\sqrt{b\gamma_t}\}$$

where:

- $Q(x) = \frac{1}{\sqrt{2\pi}} \times \int_x^\infty e^{-\frac{t^2}{2}} dt$
- $P_e(\gamma_t)$ may be exact result or approximation
- BPSK: exact with $a = 1, b = 2$
- M-ary QAM: tight approximation with $a = 4(1 - \frac{1}{\sqrt{M}}), b = \frac{3}{M-1}$

(*Einschub* : Gray – Code : $BER = \frac{1}{\log_2 M} \times SER$)

- Alternative representation of Q - function:

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}} d\theta$$

→ Integral limits are fixed and do not depend on integration variables!

- Average error probability

$$P_e = \mathcal{E}\{P_e(\gamma_t)\} = \int_0^\infty aQ(\sqrt{bx})p_{\gamma_t}(x) dx$$

- Integral may be difficult to solve analytically
- Integral has infinite support → numerical evaluation difficult
- Using alternative representation of Q-function we get:

$$\begin{aligned} P_e &= \int_0^\infty \frac{a}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{bx}{2\sin^2\theta}} p_{\gamma_t}(x) d\theta dx \\ &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} \int_0^\infty p_{\gamma_t}(x) e^{-\frac{bx}{2\sin^2\theta}} dx d\theta = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\left(\frac{b}{2\sin^2\theta}\right) d\theta \end{aligned}$$

where:

- $M_{\gamma_t}(s) = \int_0^\infty p_{\gamma_t}(x) e^{-sx} dx$ is the Laplace transform of p_{γ_t}
- $M_{\gamma_t}(-s)$ is the so called Moment Generation Function (MGF) of p_{γ_t}
- Here, we will also refer to $M_{\gamma_t}(s)$ as MGF
- $M_{\gamma_t}(s)$ is sometimes easier to obtain than p_{γ_t}
- The above integral can be easily evaluated numerically because of the finite integral limits

Outage probability

- The outage probability is the probability that the channel cannot support a certain rate, R , i.e. (where γ_T is the threshold SNR):

$$C = \log_2(1 + \gamma_t) < R \quad \leftrightarrow \quad \gamma_t < 2^R - 1 \triangleq \gamma_T$$

Thus, the outage probability is given by:

$$P_{out} = P_{\gamma_t < \gamma_T} = \int_0^{\gamma_T} p_{\gamma_t}(x) dx$$

- Using the inverse Laplace Transform

$$p_{\gamma_t}(x) = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) e^{sx} dx$$

where $c > 0$ is a small constant that lies in the region of convergence of the integral, we obtain:



- 1.

$$P_{out} = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) \int_0^{\gamma_T} e^{sx} dx ds = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) e^{\gamma_T s} \frac{ds}{s}$$

(lower integral limit is 0 since $p_{\gamma_t}(0) = 0$)

- and 2.:

$$p_{\gamma_t}(x) = \int_0^x p_{\gamma_t}(t) dt = 0$$

$$\text{for } x = 0 \text{ note: } p_{\gamma_t}(x) \xleftrightarrow[\text{transform}]{\text{Laplace}} \frac{1}{s} M_{\gamma_t}(s)$$

General combining scheme

$$y = \left(\sum_{n=1}^{N_R} h_n w_n \right) x + \sum_{n=1}^{N_R} w_n n_n$$

$$\gamma_t = \frac{\epsilon_s \left| \sum_{n=1}^{N_R} h_n w_n \right|^2}{\sigma_n^2 \sum_{n=1}^{N_R} |w_n|^2}$$

where w_n depends on the particular combining scheme.

1.2 MRC (Maximum Ratio Combining)

- what weight w_n maximize γ_t ?
 - Cauchy-Schwarz inequality

$$\left| \sum_{n=1}^{N_R} h_n w_n \right|^2 \leq \sum_{n=1}^{N_R} |h_n|^2 \cdot \sum_{n=1}^{N_R} |w_n|^2$$

where equality holds if and only if $w_n = c \cdot h_n^*$ for some non-zero constant c .

- for $w_n = h_n^*$, we obtain

$$\gamma_t = \frac{\epsilon_s}{\sigma_n^2} \cdot \frac{\left(\sum_{n=1}^{N_R} |h_n|^2 \right)^2}{\sum_{n=1}^{N_R} |h_n|^2} = \frac{\epsilon_s}{\sigma_n^2} \sum_{n=1}^{N_R} |h_n|^2$$

- $w_n = h_n^* \forall n$ are the MRC combining weights.
- For performance analysis we assume independent identically distributed (IID) Rayleigh fading

$$\begin{aligned} \rightarrow \mathcal{E}\{|h_n|^2\} &= 1; \quad \bar{\gamma} = \frac{\epsilon_s}{\sigma_n^2}; \quad \gamma_n = \frac{\epsilon_s}{\sigma_n^2} |h_n|^2 \\ p_\gamma(x) &= \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0 \\ M_\gamma(s) &= \frac{1}{1 + s\bar{\gamma}} \end{aligned}$$

- Error rate

$$\gamma_t = \sum_{n=1}^{N_R} \gamma_n$$

\rightarrow sum of IID random variables (r.v.s.)

$$M_{\gamma_t}(s) = \left(M_\gamma(s) \right)^{N_R} = \frac{1}{(1 + s\bar{\gamma})^{N_R}} = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{1}{\left(s + \frac{1}{\bar{\gamma}}\right)^{N_R}}$$

inverse Laplace-transform (from tables)

$$p_{\gamma_t}(x) = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{x^{N_R-1}}{(N_R-1)!} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0$$

- Direct approach

$$p_e = \int_0^\infty a \cdot Q(\sqrt{ax}) p_{\gamma_t}(x) dx = a \left(\frac{1-\mu}{2} \right)^{N_R} \cdot \sum_{n=0}^{N_R-1} \binom{N_R-1+n}{n} \left(\frac{1+\mu}{2} \right)^n$$

$$\text{where } \mu = \sqrt{\frac{b\bar{\gamma}}{2 + b\bar{\gamma}}}$$



- MGF approach

$$\begin{aligned}
 p_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t} \left(\frac{b}{2 \sin^2 \theta} \right) d\theta \\
 &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\bar{\gamma}^{N_R} \left(\frac{b}{\sin^2 \theta} + \frac{1}{\bar{\gamma}} \right)^{N_R}} d\theta \quad (\text{numerisch berechnen!})
 \end{aligned}$$

- high SNR: $\bar{\gamma} \rightarrow \infty \iff \frac{1}{\bar{\gamma}} \rightarrow 0$

$$\begin{aligned}
 p_e &= \frac{a}{\pi} \cdot \frac{1}{\bar{\gamma}^{N_R}} \cdot \left(\frac{2}{b} \right)^{N_R} \int_0^{\frac{\pi}{2}} \sin^{2N_R} \theta d\theta \\
 (\text{from MGF approach: } \int_0^{\frac{\pi}{2}} \sin^{2N_R} \theta d\theta &= \frac{\pi}{2^{N_R+1}} \cdot \binom{2N_R}{N_R}) \\
 &= \frac{a}{2^{N_R+1} \cdot b^{N_R}} \binom{2N_R}{N_R} \frac{1}{\bar{\gamma}^{N_R}} \quad \text{as } \bar{\gamma} \rightarrow \infty \\
 &\stackrel{!}{=} \left(\frac{1}{G_c \bar{\gamma}} \right)
 \end{aligned}$$

where: Diversity gain: $G_d = N_R$

$$\text{Combining/Coding gain: } G_c = 2b \left(\frac{a}{2} \binom{2N_R}{N_R} \right)^{-\frac{1}{N_R}}$$

- MRC exploits the maximal possible diversity
- Diversity gain is not affected by correlation as the branches are not fully correlated
- Diversity gain depends on fading distribution

Outage probability

$$\begin{aligned} P_{out} &= \int_0^{\gamma_T} p_{\gamma_t}(x) dx = \frac{1}{\bar{\gamma}^{N_R}} \int_0^{\gamma_T} \frac{x^{N_R-1}}{(N_R-1)!} e^{-\frac{x}{\bar{\gamma}}} dx \\ &= 1 - e^{-\frac{\gamma_T}{\bar{\gamma}}} \cdot \sum_{n=1}^{N_R} \frac{\left(\frac{\gamma_T}{\bar{\gamma}}\right)^n}{(n-1)!} \end{aligned}$$

- Approximation (Taylor series): $\bar{\gamma} \rightarrow \infty : -e^{-\frac{x}{\bar{\gamma}}} = 1 - \frac{x}{\bar{\gamma}} + O(\frac{1}{\bar{\gamma}})$ where a function $f(x)$ is $O(x)$ if $\lim_{x \rightarrow \infty} \frac{f(x)}{x} = 0$.

$$\Rightarrow P_{out} = \frac{1}{\gamma^{N_R}} \int_0^{\gamma_T} \frac{x^{N_R-1}}{(N_R-1)!} \left(1 - \frac{x}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right)\right) dx$$

- Diversity and coding gain can also be defined for P_{out}

1.3 EGC (Equal Gain Combining)

Combining Weights

- For MRC, both, the amplitudes and phases of the channel gains $h_n = |h_n|e^{j\varphi_n}$ have to be known (or estimated in practice)
- In EGC it is assumed that only the phases are known and weights $w_n = e^{-j\varphi_n}$ are used.

$$\begin{aligned} \Rightarrow \gamma_t &= \frac{\mathcal{E}_s}{\sigma_n^2} \frac{\left| \sum_{n=1}^{N_R} |h_n| e^{j\varphi_n} e^{-j\varphi_n} \right|^2}{\sum_{n=1}^{N_R} |e^{-j\varphi_n}|^2} \\ &= \frac{\mathcal{E}_s}{\sigma_n^2} \frac{1}{N_R} \left(\sum_{n=1}^{N_R} |h_n| \right)^2 \\ &= \frac{1}{N_R} \left(\sum_{n=1}^{N_R} \sqrt{\gamma_n} \right)^2 ; \text{ with } \gamma_n = \frac{\mathcal{E}_s}{\sigma_n^2} |h_n|^2 \end{aligned}$$

Performance Analysis

- i.i.d. case
 $\Rightarrow \sqrt{\gamma_n}$ is Rayleigh distributed
 \Rightarrow Exact analysis is much more difficult than for MRC \Rightarrow see book by Simon&Alouini p.341
- Approximate result

$$P_e = \frac{a}{2} \left[1 - \sqrt{\frac{2b\bar{\gamma}}{5+2b\bar{\gamma}}} \sum_{n=0}^{N_R-1} \frac{\binom{2n}{n}}{4^n (1 + \frac{2}{5}b\bar{\gamma})^n} \right]$$

- high SNR
 \Rightarrow use high SNR analysis of Wang&Giannakis, 2003
 \Rightarrow at high SNR, only pdf of γ_n around 0 is relevant for performance

$$\stackrel{\text{Rayleigh}}{\Rightarrow} p_{\gamma}(x) = \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}} \stackrel{\text{Taylor Serie}}{=} \frac{1}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right) \text{ as } x \rightarrow 0$$