

MIMO Skript - Wintersemester 2013

Kapitel 2

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2 Point-to-Point MIMO Systems

2.1 MIMO Channel Capacity

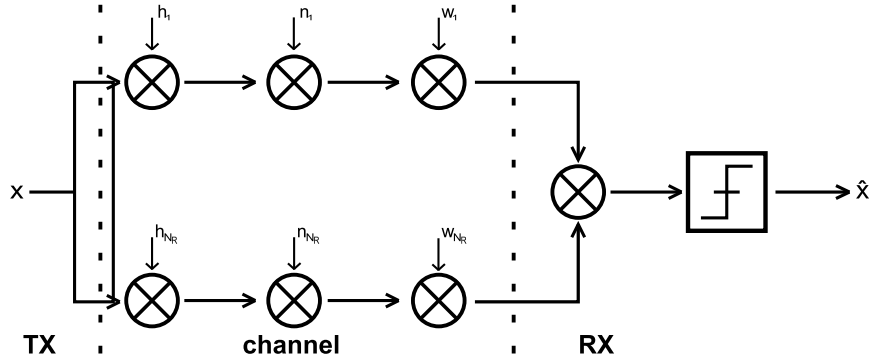
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2.2 SIMO Systems

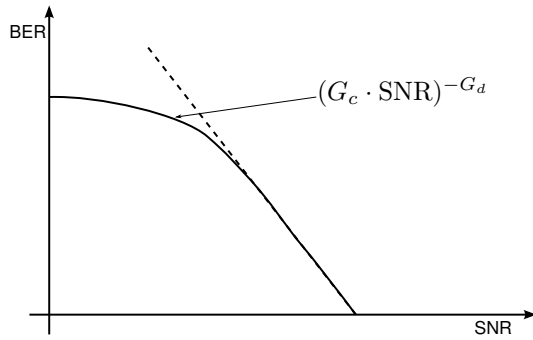
Remarks

- In SIMO Systems only coding and diversity gains can be exploited (no multiplexing gains)
- To realize these gains diversity combining has to be performed

- Diversity combining schemes vary in complexity and performance
- There are many diversity combining schemes. Here we consider:
 - Maximal ratio combining (MRC)
 - Equal gain combining (EGC)
 - Selection combining (SC)
- Diversity combining problem



- how to choose combining weights w_n ?
- what performance (e.g. error rate, outage probability) is achieved?
- what diversity and coding/combining gain is achieved?



- G_c : Coding gain
- G_d : Diversity gain

2.2.1 Preliminaries

Consider an equivalent system:

$$y = hx + n;$$

$$\mathcal{E}\{|x|^2\} = \mathcal{E}_s; \quad \mathcal{E}\{|n|^2\} = \sigma_n^2; \quad \mathcal{E}\{|h|^2\} = 1$$

- Instantaneous SNR: $\gamma_t = \frac{\mathcal{E}_s}{\sigma_n^2} \cdot |h|^2$
- Average SNR: $\bar{\gamma}_t = \mathcal{E}\{\gamma_t\} = \frac{\mathcal{E}_s}{\sigma_n^2}$

Bit and Symbol Error Rate

- The Bit and Symbol Error Rate of many modulation schemes can be expressed for given γ_t as:

$$P_e(\gamma_t) = aQ(\sqrt{b\gamma_t})$$

where:

- $Q(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_x^\infty e^{-\frac{t^2}{2}} dt$
- $P_e(\gamma_t)$ may be exact result or approximation
- BPSK: exact with $a = 1, b = 2$
- M-ary QAM: tight approximation with $a = 4(1 - \frac{1}{\sqrt{M}}), b = \frac{3}{M-1}$

(Einschub : Gray – Code : $BER = \frac{1}{\log_2 M} \cdot SER$)

- Alternative representation of Q - function:

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}} d\theta$$

→ Integral limits are fixed and do not depend on integration variables!

- Average error probability

$$P_e = \mathcal{E}\{P_e(\gamma_t)\} = \int_0^\infty aQ(\sqrt{bx})p_{\gamma_t}(x) dx$$

- Integral may be difficult to solve analytically
- Integral has infinite support → numerical evaluation difficult
- Using alternative representation of Q-function we get:

$$\begin{aligned} P_e &= \int_0^\infty \frac{a}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{bx}{2\sin^2\theta}} p_{\gamma_t}(x) d\theta dx \\ &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} \int_0^\infty p_{\gamma_t}(x) e^{-\frac{b}{2\sin^2\theta}x} dx d\theta = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\left(\frac{b}{2\sin^2\theta}\right) d\theta \end{aligned}$$

where:

- $M_{\gamma_t}(s) = \int_0^\infty p_{\gamma_t}(x)e^{-sx} dx$ is the Laplace transform of p_{γ_t}
- $M_{\gamma_t}(-s)$ is the so called Moment Generation Function (MGF) of p_{γ_t}
- Here, we will also refer to $M_{\gamma_t}(s)$ as MGF
- $M_{\gamma_t}(s)$ is sometimes easier to obtain than p_{γ_t}
- The above integral can be easily evaluated numerically because of the finite integral limits

Outage probability

- The outage probability is the probability that the channel cannot support a certain rate, R , i.e. (where γ_T is the threshold SNR):

$$C = \log_2(1 + \gamma_t) < R \quad \leftrightarrow \quad \gamma_t < 2^R - 1 \triangleq \gamma_T$$

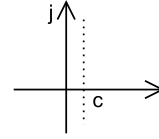
Thus, the outage probability is given by:

$$P_{out} = P_0\{\gamma_t < \gamma_T\} = \int_0^{\gamma_T} p_{\gamma_t}(x) dx$$

- Using the inverse Laplace Transform

$$p_{\gamma_t}(x) = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) e^{sx} dx$$

where $c > 0$ is a small constant that lies in the region of convergence of the integral, we obtain:



– 1.

$$P_{out} = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) \int_0^{\gamma_T} e^{sx} dx ds = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) e^{\gamma_T s} \frac{ds}{s}$$

(lower integral limit is 0 since $p_{\gamma_t}(0) = 0$)

– and 2.:

$$p_{\gamma_t}(x) = \int_0^x p_{\gamma_t}(t) dt = 0$$

$$\text{for } x = 0 \text{ note: } p_{\gamma_t}(x) \xleftrightarrow[\text{transform}]{\text{Laplace}} \frac{1}{s} M_{\gamma_t}(s)$$

General combining scheme

$$y = \left(\sum_{n=1}^{N_R} h_n w_n \right) x + \sum_{n=1}^{N_R} w_n n_n$$

$$\gamma_t = \frac{\mathcal{E}_s \left| \sum_{n=1}^{N_R} h_n w_n \right|^2}{\sigma_n^2 \sum_{n=1}^{N_R} |w_n|^2}$$

where w_n depends on the particular combining scheme.

2.2.2 MRC (Maximum Ratio Combining)

- what weight w_n maximize γ_t ?

- Cauchy-Schwarz inequality

$$\left| \sum_{n=1}^{N_R} h_n w_n \right|^2 \leq \sum_{n=1}^{N_R} |h_n|^2 \cdot \sum_{n=1}^{N_R} |w_n|^2$$

where equality holds if and only if $w_n = c \cdot h_n^*$ for some non-zero constant c .

- for $w_n = h_n^*$, we obtain

$$\gamma_t = \frac{\mathcal{E}_s}{\sigma_n^2} \cdot \frac{\left(\sum_{n=1}^{N_R} |h_n|^2 \right)^2}{\sum_{n=1}^{N_R} |h_n|^2} = \frac{\epsilon_s}{\sigma_n^2} \sum_{n=1}^{N_R} |h_n|^2$$

- $w_n = h_n^* \forall n$ are the MRC combining weights.

- For performance analysis we assume independent identically distributed (IID) Rayleigh fading

$$\rightarrow \mathcal{E}\{|h_n|^2\} = 1; \quad \bar{\gamma} = \frac{\mathcal{E}_s}{\sigma_n^2}; \quad \gamma_n = \frac{\mathcal{E}_s}{\sigma_n^2} |h_n|^2$$

$$p_\gamma(x) = \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0$$

$$M_\gamma(s) = \frac{1}{1 + s\bar{\gamma}}$$

- Error rate

$$\gamma_t = \sum_{n=1}^{N_R} \gamma_n$$

\rightarrow sum of IID random variables (r.v.s.)

$$M_{\gamma_t}(s) = \left(M_\gamma(s) \right)^{N_R} = \frac{1}{(1 + s\bar{\gamma})^{N_R}} = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{1}{\left(s + \frac{1}{\bar{\gamma}}\right)^{N_R}}$$

inverse Laplace-transform (from tables)

$$p_{\gamma_t}(x) = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{x^{N_R-1}}{(N_R-1)!} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0$$

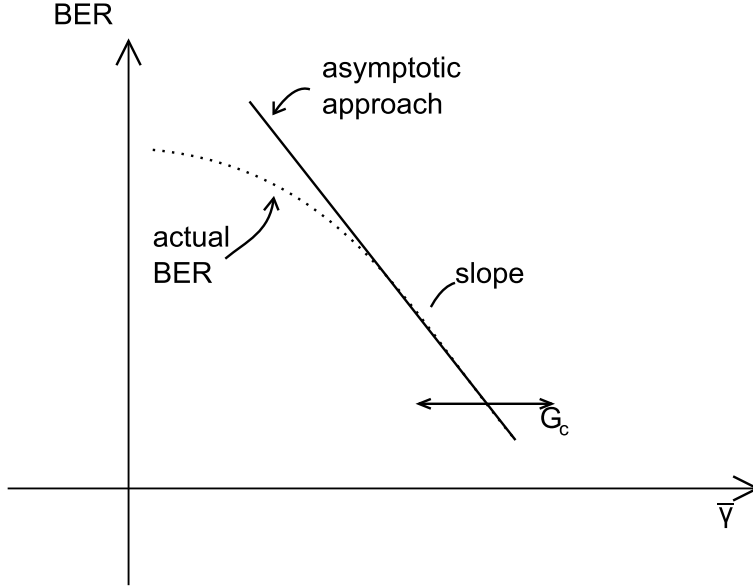
- Direct approach

$$p_e = \int_0^\infty a \cdot Q(\sqrt{ax}) p_{\gamma_t}(x) dx = a \left(\frac{1-\mu}{2} \right)^{N_R} \cdot \sum_{n=0}^{N_R-1} \binom{N_R-1+n}{n} \left(\frac{1+\mu}{2} \right)^n$$

$$\text{where } \mu = \sqrt{\frac{b\bar{\gamma}}{2 + b\bar{\gamma}}}$$

- MGF approach

$$\begin{aligned} p_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t} \left(\frac{b}{2 \sin^2 \theta} \right) d\theta \\ &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\bar{\gamma}^{N_R} \left(\frac{b}{\sin^2 \theta} + \frac{1}{\bar{\gamma}} \right)^{N_R}} d\theta \quad (\text{numerisch berechnen!}) \end{aligned}$$



- high SNR: $\bar{\gamma} \rightarrow \infty \iff \frac{1}{\bar{\gamma}} \rightarrow 0$

$$p_e = \frac{a}{\pi} \cdot \frac{1}{\bar{\gamma}^{N_R}} \cdot \left(\frac{2}{b}\right)^{N_R} \int_0^{\frac{\pi}{2}} \sin^{2N_R} \theta \, d\theta$$

(from MGF approach: $\int_0^{\frac{\pi}{2}} \sin^{2N_R} \theta \, d\theta = \frac{\pi}{2^{N_R+1}} \cdot \binom{2N_R}{N_R}$)

$$= \frac{a}{2^{N_R+1} \cdot b^{N_R}} (2N_R - N_R) \frac{1}{\bar{\gamma}^{N_R}} \quad \text{as } \bar{\gamma} \rightarrow \infty$$

$$\stackrel{!}{=} \left(\frac{1}{G_c \bar{\gamma}}\right)$$

where: Diversity gain: $G_d = N_R$

Combining/Coding gain: $G_c = 2b \left(\frac{a}{2} \binom{2N_R}{N_R}\right)^{-\frac{1}{N_R}}$

- MRC exploits the maximal possible diversity
- Diversity gain is not affected by correlation as the branches are not fully correlated
- Diversity gain depends on fading distribution

Outage probability

$$P_{out} = \int_0^{\gamma_T} p_{\gamma_t}(x) \, dx = \frac{1}{\bar{\gamma}^{N_R}} \int_0^{\gamma_T} \frac{x^{N_R-1}}{(N_R-1)!} e^{-\frac{x}{\bar{\gamma}}} \, dx$$

$$= 1 - e^{-\frac{\gamma_T}{\bar{\gamma}}} \cdot \sum_{n=1}^{N_R} \frac{\left(\frac{\gamma_T}{\bar{\gamma}}\right)^n}{(n-1)!}$$

- Approximation (Taylor series): $\bar{\gamma} \rightarrow \infty : -e^{-\frac{x}{\bar{\gamma}}} = 1 - \frac{x}{\bar{\gamma}} + O(\frac{1}{\bar{\gamma}})$ where a function $f(x)$ is $O(x)$ if $\lim_{x \rightarrow \infty} \frac{f(x)}{x} = 0$.

$$\Rightarrow P_{out} = \frac{1}{\gamma^{N_R}} \int_0^{\gamma^T} \frac{x^{N_R-1}}{(N_R-1)!} \left(1 - \frac{x}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right)\right)$$

- Diversity and coding gain can also be defined for P_{out}

2.2.3 EGC (Equal Gain Combining)

Combining Weights

- For MRC, both, the amplitudes and phases of the channel gains $h_n = |h_n|e^{j\varphi_n}$ have to be known (or estimated in practice)
- In EGC it is assumed that only the phases are known and weights $w_n = e^{-j\varphi_n}$ are used.

$$\begin{aligned} \Rightarrow \gamma_t &= \frac{E_s}{\sigma_n^2} \frac{\left| \sum_{n=1}^{N_R} |h_n| e^{j\varphi_n} e^{-j\varphi_n} \right|^2}{\sum_{n=1}^{N_R} |e^{-j\varphi_n}|^2} = \frac{E_s}{\sigma_n^2} \frac{1}{N_R} \left(\sum_{n=1}^{N_R} |h_n| \right)^2 \\ &= \frac{1}{N_R} \left(\sum_{n=1}^{N_R} \sqrt{\gamma_n} \right)^2 ; \text{ with } \gamma_n = \frac{E_s}{\sigma_n^2} |h_n|^2 \end{aligned}$$

Performance Analysis

- IID case
 - $\Rightarrow \sqrt{\gamma_n}$ is Rayleigh distributed
 - \Rightarrow Exact analysis is much more difficult than for MRC \Rightarrow see book by Simon & Alouini p.341
- Approximate result

$$P_e = \frac{a}{2} \left[1 - \sqrt{\frac{2b\bar{\gamma}}{5+2b\bar{\gamma}}} \sum_{n=0}^{N_R-1} \frac{\binom{2n}{n}}{4^n (1 + \frac{2}{5}b\bar{\gamma})^n} \right]$$

- high SNR
 - \Rightarrow use high SNR analysis of Wang & Giannakis, 2003
 - \Rightarrow at high SNR, only pdf of γ_n around 0 is relevant for performance

$$\Rightarrow p_{\gamma}^{\text{Rayleigh}}(x) = \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}} \stackrel{\text{Taylor Serie}}{=} \frac{1}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right) \text{ as } x \rightarrow 0$$

- need pdf γ_t : (γ_n bekannt, \rightarrow ges.: Wurzel, etc.)
(cumulative distribution function of $\sqrt{\gamma}$ ($\stackrel{\text{i.i.d.}}{=} \sqrt{\gamma_n}$) (cdf))

$$P_{\sqrt{\gamma}}(x) = \Pr\{\sqrt{\gamma} \leq x\} = \Pr\{\gamma \leq x^2\} = P_{\gamma}(x^2) = \text{cdf of } \gamma$$

$$\rightarrow p_{\sqrt{\gamma}}(x) = \frac{d}{dx} P_{\sqrt{\gamma}}(x) = 2x \cdot p_{\gamma}(x^2) = \frac{2x}{\gamma} + O\left(\frac{1}{\gamma}\right)$$

- Laplace Transformation to MGF

$$\rightarrow M_{\sqrt{\gamma}}(s) = \mathcal{L}\{p_{\sqrt{\gamma}}(x)\} = \frac{2}{\gamma} \cdot \frac{1}{s^2} + O\left(\frac{1}{\gamma}\right)$$

$$\sqrt{\gamma_t} = \sum_{n=1}^{N_R} \frac{\sqrt{\gamma_n}}{N_R}$$

$$M_{\sqrt{\gamma_t}}(s) = \mathcal{E}\left\{\exp(-s\sqrt{\gamma_t})\right\} = \mathcal{E}\left\{\exp\left(-\frac{s}{\sqrt{N_R}} \cdot \sum_{n=1}^{N_R} \sqrt{\gamma_n}\right)\right\} = \left(\mathcal{E}\left\{\exp\left(-\frac{s}{\sqrt{N_R}} \cdot \sqrt{\gamma_n}\right)\right\}\right)^{N_R}$$

$$= \left(M_{\sqrt{\gamma}}\left(\frac{s}{\sqrt{N_R}}\right)\right)^{N_R} = \left(\frac{2}{\gamma} \cdot \frac{N_R}{s^2}\right)^{N_R} + O\left(\frac{1}{\gamma^{N_R}}\right)$$

- inverse Laplace Transform

$$p_{\sqrt{\gamma_t}}(x) = \mathcal{L}^{-1}\left\{M_{\sqrt{\gamma_t}}(s)\right\} = \left(\frac{2N_R}{\gamma}\right)^{N_R} \cdot \frac{x^{2N_R-1}}{(2N_R-1)!} + O\left(\frac{1}{\gamma^{N_R}}\right)$$

$$P_{\gamma_t}(x) = \Pr\{\gamma_t \leq x\} = \Pr\{\sqrt{\gamma_t} \leq \sqrt{x}\} = P_{\sqrt{\gamma_t}}(\sqrt{x}) \rightarrow \text{cdf of } \sqrt{\gamma_t}$$

$$p_{\gamma_t}(x) = \frac{d}{dx} P_{\gamma_t}(x) = \frac{1}{2\sqrt{x}} \cdot p_{\gamma_t}(\sqrt{x}) = \frac{1}{2} \left(\frac{2N_R}{\gamma}\right)^{N_R} \cdot \frac{x^{N_R-1}}{(2N_R-1)!} + O(\gamma^{-N_R})$$

$$\rightarrow M_{\gamma_t}(s) = \mathcal{L}\{p_{\gamma_t}(x)\} = \frac{1}{2} \left(\frac{2N_R}{\gamma}\right)^{N_R} \cdot \frac{(N_R-1)!}{(2N_R-1)!} \frac{1}{b^{N_R}} + O(\gamma^{-N_R})$$

- Error Probability:

$$P_e = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t} \left(\frac{b}{2 \sin^2(\theta)} \right) d\theta$$

$$= \frac{a}{\pi} \frac{1}{2} \left(\frac{2N_R}{\gamma} \right)^{N_R} \frac{(N_R-1)!}{(2N_R-1)!} \frac{2^{N_R}}{b^{N_R}} \underbrace{\int_0^{\frac{\pi}{2}} \sin^{2N_R}(\theta) d\theta}_{\frac{2^{\frac{\pi}{2}N_R+1}}{2^{2N_R+1}} \frac{(2N_R)!}{(N_R!)^2}} + O\left(\frac{1}{\gamma^{N_R}}\right)$$

$$= \frac{aN_R^{N_R}}{2b^{N_R}N_R!} \frac{1}{\gamma^{N_R}} + O\left(\frac{1}{\gamma^{N_R}}\right) \stackrel{!}{=} \left(\frac{1}{G_c}\right)^{G_d}$$

$$\implies \text{Diversity gain: } G_d = N_R$$

$$\implies \text{Combining gain: } G_c = \frac{b}{N_R} \left(\frac{2N_R!}{a} \right)^{\frac{1}{N_R}}$$

vergleiche auch Blatt mit Kurven III und IV

A similar asymptotic analysis can be conducted for the outage probability.

2.2.4 SC (Selection Combining)

Combining weights

- only the strongest branch is chosen
- strongest branch: $\hat{n} = \underset{n}{\operatorname{argmax}} \gamma_n \longrightarrow \gamma_t = \gamma_{\hat{n}}$
- only on RF receiver chain required \rightarrow saves hardware complexity

Performance analysis

- cdf of: γ_t

$$\begin{aligned} P_{\gamma_t}(x) &= \Pr\{\gamma_{\hat{n}} \leq x\} = \Pr\{\gamma_1 \leq x \cap \gamma_2 \leq x \cap \dots \cap \gamma_{N_R} \leq x\} \\ &\stackrel{(IID)}{=} \left(\Pr\{\gamma_n \leq x\} \right)^{N_R} = \left(P_{\gamma}(x) \right)^{N_R} \end{aligned}$$

- pdf:

$$\begin{aligned} p_{\gamma_t}(x) &= \frac{d}{dx} P_{\gamma_t}(x) = N_R \left(P_{\gamma}(x) \right)^{N_R-1} \cdot p_{\gamma}(x) \\ \text{where: } p_{\gamma_t}(x) &= \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0 \\ P_{\gamma}(x) &= \int_0^x p_{\gamma}(x) dx = 1 - e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0 \\ \rightarrow p_{\gamma_t}(x) &= \frac{N_R}{\bar{\gamma}} \left(1 - e^{-\frac{x}{\bar{\gamma}}} \right)^{N_R-1} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0 \end{aligned}$$

Error probability

- direct approach \rightarrow closed-form solution possible
- MGF approach
 - Binomial expansion

$$\begin{aligned} p_{\gamma_t}(x) &= \frac{N_R}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} 1^{N_R-1-n} \left(-e^{-\frac{x}{\bar{\gamma}}} \right)^n \\ &= \frac{N_R}{\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} \cdot (-1)^n e^{-\frac{x(n+1)}{\bar{\gamma}}}; \quad x \geq 0 \end{aligned}$$

– MGF

$$M_{\gamma_t}(s) = \frac{N_R}{\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} (-1)^n \frac{1}{s + \frac{n+1}{\bar{\gamma}}}$$

–

$$P_e = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t} \left(\frac{b}{2 \sin^2 \theta} \right) d\theta = \frac{aN_R}{\pi \bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} (-1)^n \int_0^{\frac{\pi}{2}} \frac{d\theta}{\frac{b}{2 \sin^2 \theta} + \frac{n+1}{\bar{\gamma}}}$$

→ can be evaluated numerically

– high SNR approach $\Rightarrow \bar{\gamma} \rightarrow \infty$

$$\begin{aligned} p_{\gamma_t} &= \frac{N_R}{\bar{\gamma}} \left[1 - \exp \left(-\frac{x}{\bar{\gamma}} \right) \right]^{N_R-1} \exp \left(-\frac{x}{\bar{\gamma}} \right) \\ &\stackrel{\bar{\gamma} \rightarrow \infty}{=} \frac{N_R}{\bar{\gamma}} \left[1 - \left(1 - \frac{x}{\bar{\gamma}} + O(\bar{\gamma}^{-1}) \right) \right]^{N_R-1} \left(1 - \frac{x}{\bar{\gamma}} + O(\bar{\gamma}^{-1}) \right) \\ &= \frac{N_R}{\bar{\gamma}^{N_R}} x^{N_R-1} + O(\bar{\gamma}^{-N_R}) \end{aligned}$$

– MGF:

$$\begin{aligned} M_{\gamma_t}(s) &= \frac{N_R}{\bar{\gamma}^{N_R}} \frac{(N_R-1)!}{s^{N_R}} + O(\bar{\gamma}^{-N_R}) \\ \left[\rightarrow P_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t} \left(\frac{b}{2 \sin^2(\theta)} \right) d\theta \right] \\ &= \frac{a(2N_R)!}{b^{N_R} 2^{N_R+1} N_R!} \frac{1}{\bar{\gamma}^{N_R}} + O(\bar{\gamma}^{-N_R}) \end{aligned}$$

\Rightarrow Diversity gain: $G_d = N_R$

\Rightarrow Combining gain: $G_c = 2b \left(\frac{2N_R!}{a(2N_R)!} \right)^{\frac{1}{N_R}}$

Outage Probability

$$P_{out} = \Pr\{\gamma_{\hat{n}} \leq \gamma_T\} = P_{\gamma_{\hat{n}}}(\gamma_T) = \left[1 - \exp \left(-\frac{\gamma_T}{\bar{\gamma}} \right) \right]^{N_R}$$

$$\text{high SNR: } P_{out} = \left(\frac{\gamma_T}{\bar{\gamma}} \right)^{N_R} + O(\bar{\gamma}^{-N_R})$$

2.2.5 Comparison

- Diversity Gain:
MRC, EGC and SC all achieve the maximum possible diversity gain of $G_d = N_R$

- Combining Gain:

The combining gains of MRC, EGC and SC are different

– MRC/EGC:

$$\frac{G_C^{EGC}}{G_C^{MRC}} = \frac{\frac{1}{2b} \left(\frac{a}{2} \binom{2N_R}{N_R} \right)^{\frac{1}{N_R}}}{\frac{N_R}{b} \left(\frac{a}{2} \frac{1}{N_R!} \right)^{\frac{1}{N_R}}} = \frac{[(2N_R)!]^{\frac{1}{N_R}}}{2N_R (N_R)^{\frac{1}{N_R}}} \leq 1$$

(independent of a or b which are modulation parameters, only depends on number of antennas)

$$N_R \gg 1 : \quad N_R! \approx \sqrt{2\pi} e^{-N_R} N_R^{N_R + \frac{1}{2}} \quad (Stirling)$$

$$\left. \frac{G_C^{EGC}}{G_C^{MRC}} \right|_{N_R \gg 1} = \frac{\left(\sqrt{2\pi} e^{-2N_R} (2N_R)^{2N_R + \frac{1}{2}} \right)^{\frac{1}{N_R}}}{2N_R \left(\sqrt{2\pi} e^{-N_R} N_R^{N_R + \frac{1}{2}} \right)^{\frac{1}{N_R}}} = \frac{2 \cdot 2^{\frac{1}{2N_R}}}{2} \xrightarrow{N_R \rightarrow \infty} \frac{2}{e} \equiv -1.3\text{dB}$$

– MRC/SC:

$$\frac{G_C^{SC}}{G_C^{MRC}} = \frac{2b \left(\frac{a}{2} \binom{2N_R}{N_R} \right)^{\frac{1}{N_R}}}{2b \left(\frac{a}{2} \frac{(2N_R)!}{N_R!} \right)^{\frac{1}{N_R}}} = \frac{1}{(N_R!)^{\frac{1}{N_R}}} \leq 1$$

$$\left. \frac{G_C^{SC}}{G_C^{MRC}} \right|_{N_R \gg 1} = \frac{1}{\sqrt{2\pi}^{\frac{1}{N_R}} e^{-1} N_R^{1 + \frac{1}{2N_R}}} N_R \xrightarrow{} \infty \frac{e}{N_R}$$

→ loss increases with N_R

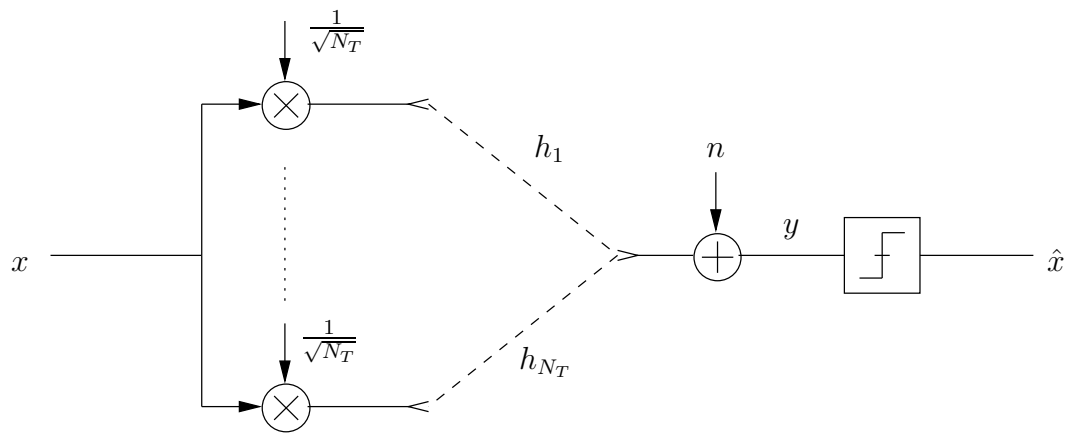
2.3 MISO Systems

Remarks

- Similar to SIMO systems, in MISO systems only coding and diversity gains can be obtained.
- To realize these gains, a careful transmitter design is necessary
- System design depends on whether or not channel state information (**CSI**) is available at transmitter

2.3.1 Naive Approach

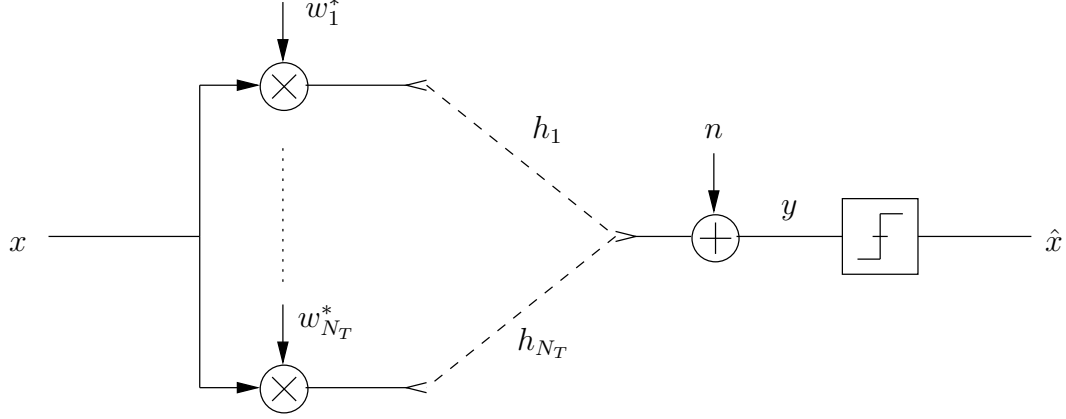
- Assume we simply send the same signal over all N_T transmit antennas



- Transmit power: $\mathcal{E} \left\{ \left| \frac{1}{\sqrt{N_T}} x \right|^2 + \dots + \left| \frac{1}{\sqrt{N_T}} x \right|^2 \right\} = \mathcal{E} \left\{ N_T \frac{1}{N_T} |x|^2 \right\} = E_s$
- Received signal: $y = \frac{1}{\sqrt{N_T}} \sum_{n=1}^{N_T} h_n \cdot x + n$
- Rayleigh fading: h_n are zero mean complex gaussian random variables
 $\rightarrow h$ is also zero mean complex gaussian
- i.i.d.:
 - $\mathcal{E}\{|h_n|^2\} = 1 \forall n$
 - $\mathcal{E}\{|h|^2\} = \frac{1}{N_T} \mathcal{E} \left\{ \left| \sum_{n=1}^{N_T} h_n \right|^2 \right\} = \frac{1}{N_T} \mathcal{E} \left\{ \sum_{n=1}^{N_T} |h_n|^2 \right\} = 1$
 - statistical properties of h are independent of N_T
 - the multiple transmit antennas have no benefit at all
 - more sophisticated transmitter designs necessary

2.3.2 Full CSI Available at the Transmitter

- $h_n, n \in \{1, \dots, N_T\}$ is known at the transmitter
- Perform “precoding” (beamforming) with coefficients w_n



- Transmit Power: Two constraints may be considered
 - Average transmit power constraint

$$P_{av} = \mathcal{E} \left\{ \sum_{n=1}^{N_T} |w_n^* x|^2 \right\} = \sum_{n=1}^{N_T} |w_n|^2 \underbrace{\mathcal{E}\{|x|^2\}}_{E_s} = \mathcal{E}_s \Rightarrow \sum_{n=1}^{N_T} |w_n|^2 = 1$$

- Power constraint for each transmit antenna

$$\rightarrow |w_n| = \frac{1}{\sqrt{N_T}} \quad \rightarrow P_{av} = E_s$$

- Received signal: $y = \underbrace{\sum_{n=1}^{N_T} w_n^* h_n x}_h + n$ (equivalent SISO channel)

Maximum Ratio Transmission (MRT)

- we have only the average power constraint: $\sum_{n=1}^{N_T} |w_n|^2 = 1$

$$\bullet \text{ SNR: } \gamma_t = \frac{E_s |h|^2}{\sigma_n^2} = \frac{\mathcal{E}_s \left| \sum_{n=1}^{N_T} w_n^* h_n \right|^2}{\sigma_n^2}$$

- Maximize SNR under constraint $\sum_{n=1}^{N_T} |w_n|^2 = 1$

- constraint optimization problem \rightarrow Lagrange method

$$L = \frac{E_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} w_n^* \cdot h_n \right|^2 + \lambda \left(\sum_{n=1}^{N_T} |w_n|^2 - 1 \right); \quad \text{where: } \lambda = \text{Lagrange Multiplier}$$

\Rightarrow Wirtinger Kalkül: treat z and z^* as independent variables for differentiation:

$$\begin{aligned} \frac{\partial z^*}{\partial z} &= 0; & \frac{\partial |z|^2}{\partial z} &= \frac{\partial z \cdot z^*}{\partial z} = z^* \\ \frac{\partial x^2}{\partial x} &= 2x; & \frac{\partial (z^*)^2}{\partial z^*} &= 2 \cdot z^*; & \frac{\partial |z|^2}{\partial z^*} &= z^* \end{aligned}$$

$$\frac{\partial L}{\partial w_m^*} = \frac{\epsilon_s}{\sigma_n^2} \left(\sum_{n=1}^{N_T} w_n^* \cdot h_n \right)^* h_m + \lambda w_m$$

$$\rightarrow w_m = \frac{\frac{\epsilon_s}{\sigma_n^2} \cdot \lambda}{\text{const., independent of } m} \left(\sum_{n=1}^{N_T} w_n^* h_n \right)^* h_m$$

$$\rightarrow w_m = c \cdot h_m$$

$$\rightarrow \sum_{n=1}^{N_T} |w_n|^2 = 1 \rightarrow c^2 = \frac{1}{\sum_{n=1}^{N_T} |h_n|^2}$$

$$\rightarrow w_n = \frac{h_n}{\sqrt{\sum_{n=1}^{N_T} |h_n|^2}} \equiv \text{MRT gains}$$

$$\rightarrow \text{SNR} = \frac{E_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} \frac{|h_n|^2}{\sqrt{\sum_{n=1}^{N_T} |h_n|^2}} \right|^2 = \frac{\epsilon_s}{\sigma_n^2} \sum_{n=1}^{N_T} |h_n|^2$$

\Rightarrow same SNR as for maximum ratio combining (MRC)

\Rightarrow MRT with N_T transmit antennas achieves the same performance as MRC with N_T receive antennas

\Rightarrow MRT/MRC can be extended to $N_T \times N_R$ MIMO systems

\rightarrow has the same performance as MRC with $N_T \cdot N_R$ receive antennas and one transmit antenna

Equal Gain Transmission (EGT)

- we employ gains: $w_n = \frac{1}{\sqrt{N_T}} \cdot \frac{h_n}{|h_m|} \rightarrow |w_n| = \frac{1}{\sqrt{N_T}}$

- SNR:

$$\begin{aligned}
\gamma_t &= \frac{E_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} w_n^* h_n \right|^2 \\
&= \frac{E_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} \frac{1}{\sqrt{N_T} \cdot \frac{|h_n|^2}{|h_n|}} \right|^2 = \frac{1}{N_T} \cdot \frac{E_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} |h_n| \right|^2 \\
\gamma_n &= \frac{E_s}{\sigma_n^2} |h_n|^2 \\
\text{same SNR as for EGC} &\rightarrow \gamma_t = \frac{1}{N_T} \left| \sum_{n=1}^{N_T} \sqrt{\gamma_n} \right|^2
\end{aligned}$$

→ EGC with N_T transmit antennas achieves the same performance as EGC with N_T receive antennas

Transmit Antennas Selection

- select antenna with maximum channel gain for transmission:

$$w_n = \begin{cases} \frac{h_n}{|h_n|}, & \text{if } n = \hat{n} \\ 0, & \text{otherwise} \end{cases} \quad \text{where } \hat{n} = \underset{n}{\operatorname{argmax}} |h_n|$$

- antenna selection with N_T transmit antennas achieves the same performance as *Selection Combining* with N_T receive antennas

2.3.3 No CSI at Transmitter - Space-Time-Coding

- $h_n, n \in \{1, \dots, N_T\}$, is only known at the receiver
- “Space-time-coding” has to be employed to realize diversity gain
- $T \times N_T$ matrices \mathbf{X} are transmitted in T symbol intervals over N_T antennas
- \mathbf{X} is drawn from a matrix alphabet \mathcal{X}
- Example:

$$\mathbf{X} = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,N_T} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,N_T} \\ \vdots & \vdots & \ddots & \vdots \\ x_{T,1} & x_{T,2} & \cdots & x_{T,N_T} \end{pmatrix}$$

- We distinguish:
 - Space-time-block-codes (STBCs)
 - \mathbf{X} is obtained by mapping K scalar symbols $s_k, k = 1, \dots, K$ from a scalar alphabet \mathcal{A} to matrix \mathbf{X}

- Space-time-trellis-codes (STTCs)
 - \mathbf{X} is obtained from scalar symbols s_k through a trellis encoding process.
 - [see: Tarokh, Seshadri, Calderbank: Space-time-codes for high data rate wireless communication: Performance criteria and coder construction; IEEE Trans. Inf. Theory 1998]
- here: We concentrate on space-time-block-codes (STBCs), but many results can be easily extended to space-time-trellis-codes
- STBCs:
 - K M -ary scalar symbols (e.g. M -PSK symbols) are mapped to STBC matrices \mathbf{X}

$$\mathbf{S} = [s_1, \dots, s_K] \rightarrow \mathbf{X}$$

$$s_k \in \mathcal{A} \rightarrow x \in \mathcal{X} \text{ with } |\mathcal{X}| = M^K$$
 - Example: “Alamouti”-Code

$$\mathbf{X} = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix}$$

[Alamouti: A simple transmit diversity technique for wireless communication, IEEE JSAC 1998]

Optimal Detection

- Signal model:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_T \end{pmatrix} = \mathbf{X} \begin{pmatrix} h_1 \\ \vdots \\ h_{N_T} \end{pmatrix} + \begin{pmatrix} n_1 \\ \vdots \\ n_T \end{pmatrix}$$

$$\mathbf{y} = \mathbf{X} \cdot \mathbf{h} + \mathbf{n}$$

- Optimal detection - ML-detection
 - \mathbf{h} is known at receiver
 - \mathbf{n} is AWGN with $\mathcal{E}\{\mathbf{n} \cdot \mathbf{n}^H\} = \sigma_n^2 \cdots \mathbf{I}_{T \times T}$
 - $p(\mathbf{y}|\mathbf{x})$

$$\begin{aligned} &= \frac{1}{\pi^T |\sigma_n^2 \mathbf{I}_{T \times T}|} \exp \left(-(\mathbf{y} - \mathbf{xh})^H (\sigma_n^2 \mathbf{I}_{T \times T})^{-1} (\mathbf{y} - \mathbf{xh}) \right) \\ &= \frac{1}{\pi^T \sigma_n^{2T}} \exp \left(-\frac{1}{\sigma_n^2} (\mathbf{y} - \mathbf{xh})^H (\mathbf{y} - \mathbf{xh}) \right) = \frac{1}{\pi^T \sigma_n^{2T}} \exp (||\mathbf{y} - \mathbf{xh}||^2) \end{aligned}$$

→ the optimal estimate $\hat{\mathbf{X}}$ or equivalently the optimal estimate $\hat{\mathbf{s}}$ can be obtained as

$$\hat{\mathbf{s}} = \underset{\mathbf{s} \in \mathcal{A}^K}{\operatorname{argmax}} p(\mathbf{y}|\mathbf{x}) = \underset{\mathbf{s} \in \mathcal{A}^K}{\operatorname{argmin}} ||\mathbf{y} - \mathbf{hx}||^2$$

- Disadvantage: In general, metric $||\mathbf{y} - \mathbf{hx}||^2$ has to be calculated M^K times
 - complexity increases exponentially with K

Types of STBCs

- Orthogonal STBCs (OSTBCs)

- OSTBCs are a special class of STBCs which allow independent detection of each $s_k \rightarrow$ only $K \cdot M$ metrics have to be evaluated

- Rate STBCs: $R_{STBC} = \frac{K}{T}$

- Examples:

- * Alamouti Code ($K = 2, T = 2$) $\rightarrow R_{STBC} = 1$

$$\mathbf{X} = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix} \begin{matrix} \uparrow T \\ \leftarrow N_T \end{matrix}$$

\rightarrow only “full rate” OSTBC for complex s_k

- * $N_T = 3, K = 3, T = 4$

$$\mathbf{X} = \frac{1}{\sqrt{3}} \begin{pmatrix} s_1 & s_2 & s_3 \\ -s_2^* & s_1^* & 0 \\ s_3^* & 0 & -s_3^* \\ 0 & -s_3^* & s_2^* \end{pmatrix} \rightarrow R_{STBC} = \frac{K}{T} = \frac{3}{4}$$

- Orthogonality: $\mathbf{X}^H \mathbf{X} = \text{const} \cdot \mathbf{I}_{N_T \times N_T}$

- Independent detection of s_1 & s_2 for Alamouti Code

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \\ \rightarrow \underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{\tilde{\mathbf{y}}} &= \frac{1}{\sqrt{2}} \underbrace{\begin{pmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{pmatrix}}_{\tilde{\mathbf{F}}} \underbrace{\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}}_{\mathbf{s}} + \underbrace{\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}}_{\tilde{\mathbf{n}}} \end{aligned}$$

(Anmerkung: nur $\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$ gewünscht, nicht: s_1^*, s_2^*)

$$\mathbf{F}^H \mathbf{F} = \frac{1}{2} \begin{pmatrix} h_1^* & h_2 \\ h_2^* & -h_1 \end{pmatrix} \begin{pmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{pmatrix} = \frac{1}{2} \begin{pmatrix} |h_1|^2 + |h_2|^2 & 0 \\ 0 & |h_1|^2 + |h_2|^2 \end{pmatrix}$$

$\rightarrow \frac{\sqrt{2}}{\sqrt{|h_1|^2 + |h_2|^2}} \cdot \mathbf{F}$ is unitary matrix

$\rightarrow \frac{2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{y}} = \mathbf{s} + \frac{2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{n}}$

($\frac{2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{n}}$ is AWGN vector with covariance matrix $\frac{2\sigma_n^2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{I}_{T \times T}$)

\rightarrow ML decision: $\hat{\mathbf{s}} = \underset{\mathbf{s}}{\text{argmin}} \left\| \frac{2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{y}} - \mathbf{s} \right\|^2$

\rightarrow independent ML decoding

$$\begin{aligned} \hat{s}_1 &= \underset{s_1}{\text{argmin}} \left| s_1 - \frac{h_1^* y_1 + h_2 y_2^*}{\frac{1}{\sqrt{2}}(|h_1|^2 + |h_2|^2)} \right| \\ \hat{s}_2 &= \underset{s_2}{\text{argmin}} \left| s_2 - \frac{h_1^* y_1 - h_2 y_2^*}{\frac{1}{\sqrt{2}}(|h_1|^2 + |h_2|^2)} \right| \end{aligned}$$

- independent decoding property can be proved for all OSTBCs
- low complexity is at the expense of a rate-loss compared to other STBCs for $N_T > 2$
 - Frequenzzhopping
 - keine Kanalinformation aus vorher empfangenen Symbolen möglich \Rightarrow Kanal ändert sich ständig: nur Entscheidung, ob Rauschen oder Signal + Rauschen
- Performance Analysis of Alamouti Code
 - Decision-variables after combining

$$r_1 = \sqrt{2} \frac{h_1^* y_1 + h_2 y_2^*}{|h_1|^2 + |h_2|^2}$$

$$r_2 = \sqrt{2} \frac{h_1^* y_1 - h_2 y_2^*}{|h_1|^2 + |h_2|^2}$$

because of symmetry it suffices to consider r_1

$$r_1 = \sqrt{2} \frac{h_1^* \left(\frac{1}{\sqrt{2}} s_1 h_1 + \frac{1}{\sqrt{2}} h_2 s_2 + n_1 \right) + h_2 \left(-\frac{1}{\sqrt{2}} h_2 s_1^* + \frac{1}{\sqrt{2}} h_1 s_2^* + n_2 \right)^*}{|h_1|^2 + |h_2|^2}$$

$$= \sqrt{2} \frac{\frac{1}{\sqrt{2}} (|h_1|^2 + |h_2|^2) s_1 + h_1^* n_1 + h_2 n_2^*}{|h_1|^2 + |h_2|^2}$$

$$= 1 \cdot s_1 + n_{eq}$$

where

$$n_{eq} = \sqrt{2} \frac{h_1^* n_1 + h_2 n_2^*}{|h_1|^2 + |h_2|^2}$$

$$\text{SNR} \rightarrow \gamma_t = \frac{E_s \cdot 1^2}{\sigma_{eq}^2} \quad \text{with} \quad \mathcal{E}\{|s_1|^2\} = \mathcal{E}_s$$

$$\sigma_{eq}^2 = 2 \frac{|h_1|^2 \sigma_n^2 + |h_2|^2 \sigma_{eq}^2}{(|h_1|^2 + |h_2|^2)^2} = \frac{2\sigma_n^2}{|h_1|^2 + |h_2|^2}$$

- $\gamma_t = \frac{1}{2} \frac{E_s}{\sigma_n^2} (|h_1|^2 + |h_2|^2)$
- $\text{SNR}_{\text{Alamouti}} = \frac{1}{2} \text{SNR}_{\text{MRC}} = \frac{1}{2} \text{SNR}_{\text{MRT}}$
- Alamouti code has diversity gain $G_d = 2$
- Transmission with Alamouti STBC requires 3dB higher SNR to achieve same performance as MRT → 3dB loss in coding gain G_c
- Lack of CSI knowledge at transmitter “costs” 3dB in power efficiency
- General:
 - OSTBCs achieve a diversity gain of $G_d = N_T$ if only one receive antenna is available
 - if N_R receive antennas are available, MRC can be used at the receiver to yield a diversity gain of $G_d = N_T N_R$

- Other STBCs:

- Quasi orthogonal STBCs
 - * higher rate than OSTBCs
 - * only subset of symbols have to be decoded jointly
 - * Example: $K = N_T = T = 4$

$$\mathbf{X} = \frac{1}{2} \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^* & s_1^* & -s_4^* & s_3^* \\ -s_3^* & -s_4^* & s_1^* & s_2^* \\ s_4 & -s_3 & -s_2 & s_1 \end{pmatrix}$$

- * Anmerkung 1: \mathbf{X} ist ähnlich zu Alamouti Code
- * Anmerkung 2: $\mathbf{X}^H \mathbf{X}$: viele Nicht-diagonal Elemente sind Null; die, die ungleich Null sind, zeigen, welche Symbole gemeinsam entschlüsselt werden müssen
- Golden Code for $N_T = N_R = 2$: achieves a rate of $R_{STBC} = 2$ and full diversity of $G_d = N_T, N_R = 4$
- Differential STBCs: $\mathbf{X}_k = \mathbf{X}_{k-1} \cdot \mathbf{D}_k$. \mathbf{X}_k is transmitted, \mathbf{D}_k is transmitted
- Linear dispersion codes: designed to achieve high mutual information
- noncoherent STBCs (On-Off-Keying)

Space Time Code Design Given:

- Code $\mathcal{X} = \{\mathbf{X}_1, \dots, \mathbf{X}_{|\mathcal{X}|}\}$
- Channel: IID Rayleigh-fading:
 - $h_n \sim \mathcal{CN}(0, 1)$; $n \in \{1, 2, \dots, N_T\}$
 - AWGN $n \sim \mathcal{CN}(0, \sigma_n^2)$

Problem: How should we design codebook \mathcal{X} ?

- Need to derive error rate for general codebooks \mathcal{X} !
 - Codeword error rate

$$P_e = \frac{1}{|\mathcal{X}|} \sum_{i=1}^{|\mathcal{X}|} \Pr\{\mathbf{x}_i \neq \hat{\mathbf{x}}_i\}$$

where $\hat{\mathbf{x}}_i$ is the detected codeword and we assume that all codewords are equally likely

Problem: $\Pr\{\mathbf{x}_i \neq \hat{\mathbf{x}}_i\}$ is not tractable in general

- Use union bound to upper bound $\Pr\{\mathbf{x}_i \neq \hat{\mathbf{x}}_i\}$ as upper sum over pairwise error probabilities (PEP) $\Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\}$ where it is assumed that \mathbf{x}_i was transmitted and \mathbf{x}_i and \mathbf{x}_j are the only codewords in the codebook

$$P_e \leq \frac{1}{|\mathcal{X}|} \sum_{i=1}^{|\mathcal{X}|} \sum_{j=1}^{|\mathcal{X}|} \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} \text{ where } j \neq i$$

Calculation of PEPs

Recall: $\hat{\mathbf{x}} = \underset{\mathbf{x} \in \mathcal{X}}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{x}\mathbf{h}\|^2$

Now, \mathbf{x}_i and \mathbf{x}_j are the only alternatives and an error is made if $\|\mathbf{y} - \mathbf{x}_i\mathbf{h}\|^2 > \|\mathbf{y} - \mathbf{x}_j\mathbf{h}\|^2$ since \mathbf{x}_i was sent but \mathbf{x}_j was detected

$$\begin{aligned} \rightarrow \|\mathbf{x}_i\mathbf{h} + \mathbf{n} - \mathbf{x}_j\mathbf{h}\|^2 &> \|\mathbf{x}_i\mathbf{h} + \mathbf{n} - \mathbf{x}_j\mathbf{h}\|^2 \\ \|\mathbf{n}\| &> \|(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h} + \mathbf{n}\|^2 \\ \rightarrow \|\mathbf{n}\| &> \underbrace{\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h}}_{\Delta} + \mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{n} + \mathbf{n}^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h} + \|\mathbf{n}\|^2 \\ &\rightarrow \underbrace{-\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)^H\mathbf{n} - \mathbf{n}^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h}}_z > \Delta \end{aligned}$$

for given \mathbf{h} , z is a gaussian random variable

$$\begin{aligned} \sigma_z^2 &= \mathcal{E}\{|z|^2\} = \mathcal{E}\{2\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j) \underbrace{\mathbf{n}\mathbf{n}^H}_{\sigma_n^2 \mathbf{I}}(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h} + 2\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)^H \underbrace{\mathbf{n}\mathbf{n}^T}_{=0}(\mathbf{x}_i - \mathbf{x}_j)^*\mathbf{h}^*\} \\ &= 2\sigma_n^2 \Delta + 0 \end{aligned}$$

$$\begin{aligned} \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} &= \int_{\Delta}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{z^2}{2\sigma_z^2}\right) dz, \quad t = \frac{z}{\sigma_z} \\ &= \frac{1}{\sqrt{2\pi}} \int_{\frac{\Delta}{\sigma_z}}^{\infty} e^{-\frac{t^2}{2}} dt = Q\left(\frac{\Delta}{\sigma_z}\right) = Q\left(\frac{\Delta}{\sqrt{2\sigma_n^2 \Delta}}\right) \\ &= Q\left(\sqrt{\frac{\Delta}{2\sigma_n^2}}\right) \end{aligned}$$

- $\Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} = \mathcal{E}\left\{Q\left(\sqrt{\frac{\Delta}{2\sigma_n^2}}\right)\right\}$
 – to avoid cumbersome Q-function we use Chernoff bound:

$$\boxed{Q(x) \leq \frac{1}{2}e^{-\frac{x^2}{2}}}$$

$$\begin{aligned} \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} &\leq \frac{1}{2}\mathcal{E}_h\left\{\exp\left(-\frac{\mathbf{h}^H\mathbf{Q}\mathbf{h}}{4\sigma_n^2}\right)\right\} \\ \text{where } \mathbf{Q} &= (\mathbf{x}_i - \mathbf{x}_j)^H(\mathbf{x}_i - \mathbf{x}_j) \end{aligned}$$

- Eigendecomposition: $\mathbf{Q} = \mathbf{U}^H \mathbf{\Lambda} \mathbf{U}$ with $\mathbf{\Lambda} = \operatorname{diag}\{\lambda_1, \dots, \lambda_r, 0, \dots, 0\}$ $r = \operatorname{rank}\{\mathbf{Q}\}$
- Elements \mathbf{h} are i.i.d. Gaussian

– $\underline{\beta} = \mathbf{U}\mathbf{h}$ has also i.i.d. Gaussian random variables as elements since \mathbf{U} is unitary matrix

– $\mathbf{h}^H \mathbf{Q} \mathbf{h} = \underbrace{\mathbf{h}^H \mathbf{U}^H}_{\underline{\beta}^*} \underbrace{\mathbf{\Lambda} \mathbf{U} \mathbf{h}}_{\underline{\beta}} = \sum_{i=1}^r \lambda_i |\beta_i|^2$ with $\underline{\beta} = [\beta_1, \dots, \beta_{N_T}]$

$$\begin{aligned} \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} &= \frac{1}{2} \mathcal{E}_{\underline{\beta}} \left\{ \exp \left(-\frac{\sum_{i=1}^r \lambda_i |\beta_i|^2}{4\sigma_n^2} \right) \right\} \\ &= \frac{1}{2} \mathcal{E}_{\underline{\beta}} \left\{ \prod_{i=1}^r e^{-\frac{\lambda_i}{4\sigma_n^2} |\beta_i|^2} \right\} \\ &= \frac{1}{2} \prod_{i=1}^r \mathcal{E}_{\beta_i} \left\{ e^{-\frac{\lambda_i}{4\sigma_n^2} |\beta_i|^2} \right\} \\ &= \frac{1}{2} \prod_{i=1}^r \mathcal{E}_{|\beta_i|^2} \left\{ e^{-\frac{\lambda_i}{4\sigma_n^2} |\beta_i|^2} \right\} \triangleq \text{MGF of exponentially distributed variable } \alpha_i = |\beta_i|^2 \end{aligned}$$

$$\rightarrow P_{\alpha_i}(x) = e^{-x}, \quad x \geq 0$$

$$\begin{aligned} \rightarrow \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} &\leq \frac{1}{2} \prod_{i=1}^r \frac{1}{1 + \frac{\lambda_i}{4\sigma_n^2}} \\ &\leq \prod_{i=1}^r \frac{1}{\frac{\lambda_i}{4\sigma_n^2}} = 2^{2r-1} \frac{1}{\prod_{i=1}^r \lambda_i} \left(\underbrace{\frac{1}{\sigma_n^2}}_{\triangleq SNR} \right)^{-r} \end{aligned}$$

- upper bound on P_e :

$$\begin{aligned} \lambda_n(i, j) &= n\text{th eigenvalue of } (\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j) \\ r(i, j) &= \text{rank of } (\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j) \end{aligned}$$

$$\rightarrow P_e \leq \frac{1}{|\mathcal{X}|} \sum_{i=1}^{|\mathcal{X}|} \sum_{j=1}^{|\mathcal{X}|} 2^{2r(i,j)-1} \frac{1}{\prod_{n=1}^{r(i,j)} \lambda_n(i, j)} \left(\frac{1}{\sigma_n^2} \right)^{-r(i,j)}$$

- generally loose bound but offers significant insight for code design

Two criteria:

Rank criterion: The diversity gain of a ST code is given by

$$G_d = \min_{i,j} (r(i,j)) = \min_{i,j} \text{rank}((\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j))$$

→ Design code such that minimum rank of all possible matrices $(\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j)$ is maximized

$$T \overset{N_T}{\updownarrow} \mathbf{X}_i \Rightarrow r(i,j) = N_T \quad \forall i \neq j$$

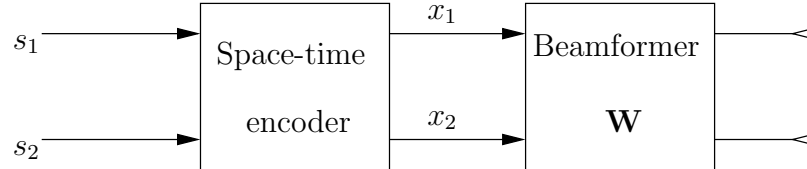
Determinant criterion: To maximize the coding gain among all codes with $r(i,j) = N_T$,

$$\text{we need to maximize } \max_{i,j} \min_{n=1}^{N_T} \lambda_n(i,j) = \max_{i,j} \min_{n=1}^{N_T} |(\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j)| \quad \forall i \neq j$$

- Rank and determinant criterion can be used for the search for good space-time block codes and space-time trellis codes. These two criteria were first derived by Tarokh, et. al. 1998.
- diversity increases to $N_T N_R$ if N_K receive antennas are available
- Example: see B  ro, Bauch, Hansmann: Improved codes for space-time trellis coded modulation. IEEE Comm. Letters, 2000.

2.3.4 Partial or Imperfect CSI at the Transmitter

- In practice, the CSI cannot be perfect. Channel estimation, quantization and noisy feedback channels introduce errors.
- If the system is optimized for perfect CSI (*e.g.* using MRT or EGT), the performance for imperfect CSI may be worse than for a system designed for no CSI (*e.g.* space-time coding)
- In this case, it is advantageous to use a hybrid approach and combine beamforming and space-time coding.



- \mathbf{W} is the beamforming matrix which depends on the reliability of the CSI
- CSI is modeled as

$$\hat{h}_i = \rho h_i + \sqrt{1 - \rho^2} e_i$$

where:

- \hat{h}_i is the CSI estimate
- ρ is the correlation between \hat{h}_i and h_i
- e_i is the CSI error modeled as AWGN

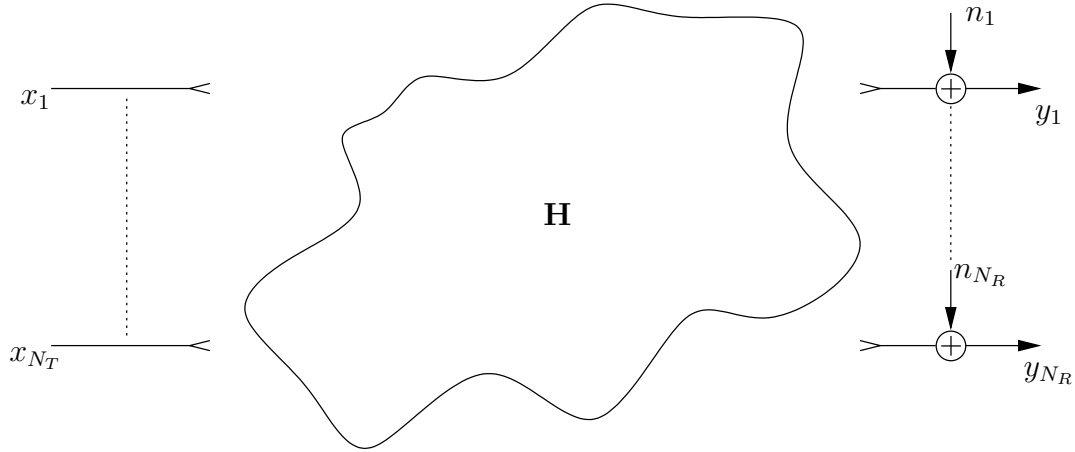
extreme cases:

- $\rho = 0$: \hat{h}_i independent of $h_i \rightarrow$ no CSI ($\mathbf{W} = \mathbf{I}$)
- $\rho = 1$: $\hat{h}_i = h_i \rightarrow$ perfect CSI (\mathbf{W} performs MRT)
- \mathbf{W} can be optimized under the assumptions for given ρ and \hat{h}_i
 \rightarrow see for details: Jöngren, Skorglund and Ottersten: "Combining Beamforming and Orthogonal Space-time Block Coding", IEEE on IT, 2002.

2.4 MIMO Systems without CSI at the transmitter

- We consider $N_T \times N_R$ MIMO system and assume that the channel matrix \mathbf{H} is not known at the transmitter
 \rightarrow no CSI at the transmitter (CSIT)
- signal model:

$$N_R \updownarrow \mathbf{y} = N_R \updownarrow \overset{N_T}{\overleftrightarrow{\mathbf{H}}} \mathbf{x} \updownarrow N_T + \mathbf{n} \updownarrow N_R$$



- x_n are M -ary i.i.d. scalar symbols taken *e.g* from an M -PSK or M -QAM symbol alphabet \mathcal{A}
- This scheme is often called “spatial multiplexing”
- We transmit N_T symbols per symbol interval
 \rightarrow rate $R = \log_2(M) \cdot N_T$ for uncoded transmission
- Problem: How to detect \mathbf{x} at the receiver considering
 - performance
 - complexity

2.4.1 Optimum Detection

- Elements of \mathbf{n} are gaussian random variables with variance σ_n^2
- \mathbf{H} is known at the receiver

$$\begin{aligned} p(\mathbf{y}|\mathbf{x}) &= \frac{1}{\pi^{N_R} \sigma_n^2 \mathbf{I}_{N_R \times N_R}} \exp \left(-(\mathbf{y} - \mathbf{H}\mathbf{x})^H (\sigma_n^2 \mathbf{I}_{N_R \times N_R})^{-1} (\mathbf{y} - \mathbf{H}\mathbf{x}) \right) \\ &= \frac{1}{\pi^{N_R} \sigma_n^2} \exp \left(-\frac{1}{\sigma_n^2} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2 \right) \end{aligned}$$

- ML-Detection

$$\hat{x} = \underset{\mathbf{x} \in \mathcal{A}^{N_T}}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2 = \underset{\mathbf{x} \in \mathcal{A}^{N_T}}{\operatorname{argmax}} p(\mathbf{y}|\mathbf{x})$$

→ M^{N_T} metric calculations → complexity is exponential in N_T !!

→ in general too complex in practice

- Performance
 - consider worst case pairwise error probability (PEP) to evaluate diversity gain
 - PEP → x_i is transmitted but $x_j \neq x_i$ is detected
 this happens if $\|\mathbf{y} - \mathbf{H}\mathbf{x}_i\|^2 > \|\mathbf{y} - \mathbf{H}\mathbf{x}_j\|^2$
 → $\|\mathbf{n}\|^2 > \|\mathbf{H}(\mathbf{x}_i - \mathbf{x}_j) + \mathbf{n}\|^2$
 - the “worst case” is if \mathbf{x}_i & \mathbf{x}_j differ only in one element *i.e.*,

$$\mathbf{x}_i - \mathbf{x}_j = (x_{ni} - x_{nj}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{“1” in position } n$$

where $\mathbf{x}_i = [x_{1i}, x_{2i}, \dots, x_{N_T i}]$

- $\|\mathbf{n}\|^2 > \left\| \underbrace{\mathbf{h}_n}_{\text{nth column of } \mathbf{H}} \underbrace{(x_{ni} - x_{nj})}_{\Delta x_n(i,j)} + \mathbf{n} \right\|^2$
- $\|\mathbf{n}\|^2 > \mathbf{h}_n^H \mathbf{n} \Delta x_n^*(i, j) + \mathbf{n}^H \mathbf{h}_n \Delta x_n(i, j) + \|\mathbf{n}\|^2 + \|\mathbf{h}_n\|^2 - |\Delta x_n(i, j)|^2$

$$\|\mathbf{h}_n\|^2 |\Delta x_n(i, j)|^2 < \underbrace{-\mathbf{h}_n^H \mathbf{n} \Delta x_n(i, j) - \mathbf{n}^H \mathbf{h}_n \Delta x_n(i, j)}_{\text{Gaussian random variable with variance } \sigma_{eq}^2 = 2\sigma_n^2 |\Delta x_n(i, j)|^2 \|\mathbf{h}_n\|^2}$$
- $\Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j | \mathbf{H}\} = Q \left(\sqrt{\frac{\|\mathbf{h}_n\|^2 |\Delta x_n(i, j)|^2}{2\sigma_n^2}} \right)$

- $\Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} = \mathcal{E} \left\{ Q \left(\sqrt{\frac{\|\mathbf{h}_n\|^2 |\Delta x_n(i,j)|^2}{2\sigma_n^2}} \right) \right\}$
 → use same approach as for space-time code design to get diversity order
 or : SNR is

$$\gamma_t = \frac{\|\mathbf{h}_n\|^2 |\Delta x_n(i,j)|^2}{2\sigma_n^2} = \frac{|\Delta x_n(i,j)|^2}{2\sigma_n^2} (|h_{1n}|^2 + |h_{2n}|^2 + \dots + |h_{N_R n}|^2)$$

- same form as SNR of MRC with N_R receive antennas
- diversity gain of spatial multiplexing with ML-decoding is

$$G_d = N_R$$

- diversity of N_T transmit antennas is not exploited with spatial multiplexing
- to exploit this additional gain, coding across space is required (at the expense of rate)
 (Hier gehören die detection performance kurven für BPSK hin)

2.4.2 Linear Receivers

- How can we avoid the complexity associated with the joint detection of the elements of \mathbf{x} ?
- Idea: Employ linear filter (matrix) to separate the elements of \mathbf{x}
- Requires: $N_T \leq N_R$
- We form

$$\mathbf{r} = N_T \overset{N_R}{\updownarrow} \mathbf{F} \mathbf{y} = [r_1, \dots, r_{N_T}]^T$$

where \mathbf{F} is the filter matrix and \mathbf{y} is the received vector

such that x_n can be obtained from

$$\hat{x}_n = \underset{x_n \in \mathcal{A}}{\operatorname{argmin}} |r_i - x_n|^2 \quad \text{where } \mathbf{F} \in \mathbb{C}^{N_T \times N_R}$$

- Two popular design criteria for \mathbf{F}
 - Zero-forcing (ZF) criterion
 - minimum mean squared error (MMSE) criterion

ZF Detection

$$\mathbf{r} = \mathbf{F}\mathbf{y} = \mathbf{F}(\mathbf{H}\mathbf{x} + \mathbf{n}) = \mathbf{F}\mathbf{H}\mathbf{x} + \mathbf{F}\mathbf{n}$$

ZF \leftrightarrow we require $\mathbf{F}\mathbf{H} = \mathbf{I}_{N_T \times N_T}$

- noise covariance matrix

$$\Phi_{ee} = \mathcal{E}\{\mathbf{F}\mathbf{n}(\mathbf{F}\mathbf{n})^H\} = \sigma_n^2 \mathbf{F}\mathbf{F}^H$$

- $N_T = N_R \rightarrow \mathbf{F}\mathbf{H} = \mathbf{I}_{N_T \times N_T} \rightarrow \mathbf{F} = \mathbf{H}^{-1}$ assuming \mathbf{H} is invertible
- $\rightarrow N_T \leq N_R \rightarrow$ which one of the many \mathbf{F} that yield $\mathbf{F}\mathbf{H} = \mathbf{I}_{N_T \times N_T}$?
- choose \mathbf{F} that leads to the smallest noise enhancement
- optimal \mathbf{F} is the solution to the following problem:

$$\min_{\mathbf{F}} \text{tr}\{\sigma_n^2 \mathbf{F}\mathbf{F}^H\}$$

$$\text{s.t } \mathbf{F}\mathbf{H} = \mathbf{I}_{N_T \times N_T}$$

$$\text{the constraint is equivalent to } \text{tr}\{(\mathbf{F}\mathbf{H} - \mathbf{I})(\mathbf{F}\mathbf{H} - \mathbf{I})^H\} = 0$$

Lagrangian:

$$\begin{aligned} L(\mathbf{F}) &= \text{tr}\{\sigma_n^2 \mathbf{F}\mathbf{F}^H\} + \lambda \text{tr}\{\mathbf{F}\mathbf{H}\mathbf{H}^H \mathbf{F} - \mathbf{F}\mathbf{H} - \mathbf{H}^H \mathbf{F}^H + \mathbf{I}\} \\ &= \sigma_n^2 \text{tr}\{\mathbf{F}\mathbf{F}^H\} + \lambda \text{tr}\{\mathbf{F}\mathbf{H}\mathbf{H}^H \mathbf{F}^H\} - \lambda \text{tr}\{\mathbf{F}\mathbf{H}\} - \lambda \text{tr}\{\mathbf{H}^H \mathbf{F}^H\} + \lambda N_T \end{aligned}$$

- use rules for complex matrix differentiation in Table IV in paper by Hjørungnes & Gesbert

$$\begin{aligned} \frac{\delta L(\mathbf{F})}{\delta \mathbf{F}^*} &= \sigma_n^2 \mathbf{F} + \lambda \mathbf{F}\mathbf{H}\mathbf{H}^H - \lambda \mathbf{H}^H = 0 \\ &\rightarrow \mathbf{F}(\sigma_n^2 \mathbf{I} + \lambda \mathbf{H}\mathbf{H}^H) = \lambda \mathbf{H}^H \\ &\rightarrow \mathbf{F} = \lambda \mathbf{H}^H (\sigma_n^2 \mathbf{I} + \lambda \mathbf{H}\mathbf{H}^H)^{-1} \end{aligned}$$

use matrix inversion lemma

$$(\mathbf{A} + \mathbf{U}\mathbf{B}\mathbf{V})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{B}^{-1} + \mathbf{V}\mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}\mathbf{A}^{-1}$$

$$\begin{aligned} \rightarrow \mathbf{F} &= \lambda \mathbf{H}^H \left[\frac{1}{\sigma_n^2} \mathbf{I} - \frac{1}{\sigma_n^2} \mathbf{H} \left[\frac{1}{\lambda} \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \right]^{-1} \mathbf{H}^H \frac{1}{\sigma_n^2} \right] \\ &= \frac{\lambda}{\sigma_n^2} \left[\begin{array}{c} \mathbf{I} \\ \left(\left(\frac{1}{\lambda} \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \right) \left(\frac{1}{\lambda} \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \right)^{-1} - \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \left[\frac{1}{\lambda} \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \right] \right) \mathbf{H}^H \end{array} \right] \mathbf{H}^H \\ &= \frac{\lambda}{\sigma_n^2} \left[\frac{1}{\lambda} \mathbf{I} + \frac{1}{\lambda} \mathbf{H}^H \mathbf{H} - \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \right] \left(\frac{1}{\lambda} \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \right)^{-1} \mathbf{H}^H \\ &= \frac{1}{\sigma_n^2} \left(\frac{1}{\lambda} \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \right) \mathbf{H}^H \end{aligned}$$

- How to choose λ

$$\begin{aligned} \mathbf{F}\mathbf{H} &= \frac{1}{\sigma_n^2} \left(\frac{1}{\lambda} \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \right)^{-1} \mathbf{H}^H \mathbf{H} = \mathbf{I} \\ \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} &= \frac{1}{\lambda} \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \\ &\Rightarrow \lambda \rightarrow \infty \end{aligned}$$

$$\Rightarrow \boxed{\mathbf{F} = (\mathbf{H}^H \mathbf{H})^{-1}} \hat{=} \text{Moore-Penrose pseudoinverse}$$

noise covariance:

$$\Phi_{ee} = \sigma_n^2 \mathbf{F} \mathbf{F}^H = \sigma_n^2 (\mathbf{H}^H \mathbf{H})^{-1} \underbrace{\mathbf{H}^H \mathbf{H} (\mathbf{H} \mathbf{H})^{-1}}_{\mathbf{I}} = \sigma_n^2 (\mathbf{H}^H \mathbf{H})^{-1}$$

Φ_{ee} is not in general a diagonal matrix

- effective noise $\mathbf{F} \mathbf{n}$ is spatially correlated
- "equalization of channel leads to coloring of noise"

- Interpretation:
we have

$$\mathbf{F} \mathbf{H} \mathbf{x} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_{N_T} \end{bmatrix} \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \dots & \mathbf{h}_{N_T} \end{bmatrix} \mathbf{x} = \mathbf{x}$$

- $\mathbf{f}_i \mathbf{h}_i = 1 \quad \mathbf{f}_i \mathbf{h}_j = 0 \quad \forall i \neq j$
- \mathbf{f}_i^T is orthogonal to $[\mathbf{h}_1 \quad \dots \quad \mathbf{h}_{i-1} \quad \mathbf{h}_{i+1} \quad \dots \quad \mathbf{h}_{N_T}] \uparrow N_R$
- \mathbf{f}_i^T is confined to an $N_R - (N_T - 1)$ dimensional subspace of the N_R dimensional space spanned by \mathbf{H}

- Diversity gain

- e.g. SISO model: $r_i = \mathbf{f}_i \cdot \mathbf{h}_i \mathbf{x}_i + \mathbf{f}_i \cdot \mathbf{n}_i$

$$\rightarrow \text{SNR}_{\text{eq}} = \frac{\epsilon_s |\mathbf{f}_i \mathbf{h}_i|^2}{\sigma_n^2 \|\mathbf{f}_i\|^2} = \frac{\epsilon_s}{\sigma_n^2} \left| \tilde{\mathbf{f}}_i \cdot \mathbf{h}_i \right|, \quad \text{where } \mathbf{f}_i = \alpha \tilde{\mathbf{f}}_i \text{ with } \|\tilde{\mathbf{f}}_i\|^2 = 1$$

we can represent \mathbf{f}_i as: $\tilde{\mathbf{f}}_i^T = \alpha \mathbf{M} \boldsymbol{\beta}$,

where: $\mathbf{M} \in \mathbb{C}^{N_R \times (N_R - N_T + 1)}$ and $\boldsymbol{\beta} \in \mathbb{C}^{(N_R - N_T + 1) \times 1} \hat{=} \text{basis of subspace}$

$$\rightarrow \tilde{\mathbf{f}}_i \mathbf{M}_i = \boldsymbol{\beta}^T \mathbf{M}_i^T \mathbf{M}_i,$$

where: $\mathbf{M}^H \mathbf{M} = \tilde{\mathbf{M}} = \mathbf{I}$ and $\tilde{\mathbf{M}}_i \rightarrow \mathcal{CN}(0, \sigma_n^2 \mathbf{I}_{(N_R - N_T + 1)})$

(since rows of \mathbf{M}^T are orthogonal)

$$\text{SNR}_{\text{eq}} = \frac{\epsilon_s}{\sigma_n^2} \alpha^2 \left| \sum_{j=1}^{N_R - N_T + 1} \beta_{ji} \tilde{\mathbf{h}}_j \right|^2; \tilde{\mathbf{h}}_i = (\tilde{h}_{1i}, \quad \tilde{h}_{2i}, \quad \dots, \quad \tilde{h}_{N_R - N_T + 1, i})^T$$

$$\boldsymbol{\beta}_i = (\beta_{1i}, \quad \dots, \quad \beta_{N_R - N_T + 1, i})^T$$

$\rightarrow \text{SNR}_{\text{eq}}$ includes only $N_R - N_T + 1$ independent Gaussian RV

\rightarrow diversity gain is limited to: $G_d = N_R - N_T + 1$

Example:

$$N_T = N_R = 3$$

$$G_d^{ZF} = 1 \text{ but } G_d^{ML} = N_R = 3$$

→ huge performance loss because of linear ZF

MMSE detection

- ZF criterion may be too strict and leads to noise enhancement
 - maybe it is better to allow some interferences between signals but reduce noise enhancement
 - What is the optimal trade-off between interference and noise?
 - MMSE criterion
- MMSE criterion
 - error signal: $\mathbf{e} = \mathbf{F}\mathbf{y} - \mathbf{x}$
 - total error variance: $\sigma_e^2 = \mathcal{E}\{\|\mathbf{e}\|^2\} = \mathcal{E}\{\text{tr}\{\mathbf{e}\mathbf{e}^H\}\} = \text{tr}\{\mathcal{E}\{\mathbf{e}\mathbf{e}^H\}\} = \text{tr}\{\Phi_{ee}\}$
 - Φ_{ee} : error covariance matrix
 - optimal filter: $\mathbf{F}_{\text{opt}} = \underset{\mathbf{F}}{\text{argmin}} \text{tr}\{\Phi_{ee}\}$

- Deviation of \mathbf{F}_{opt}

$$- \Phi_{ee} = \mathcal{E}\{\mathbf{e}\mathbf{e}^H\} = \mathcal{E}\{(\mathbf{F}\mathbf{y} - \mathbf{x})(\mathbf{F}\mathbf{y} - \mathbf{x})^H\} = \mathbf{F} \cdot \Phi_{yy} \cdot \mathbf{F}^H - \mathbf{F} \cdot \Phi_{yx} - \Phi_{xx} \cdot \Phi_{xy} \cdot \mathbf{F}^H + \Phi_{xx}$$

with:

$$\Phi_{yy} = \mathcal{E}\{\mathbf{y}\mathbf{y}^H\} = \mathcal{E}\{(\mathbf{H}\mathbf{x} + \mathbf{n})(\mathbf{H}\mathbf{x} + \mathbf{n})^H\} = \epsilon_s \cdot \mathbf{H}\mathbf{H}^H + \sigma_n^2 \cdot \mathbf{I}_{N_R \times N_T}$$

$$\Phi_{yx} = \mathcal{E}\{\mathbf{y}\mathbf{x}^H\} = \mathcal{E}\{(\mathbf{H}\mathbf{x} + \mathbf{n}) \cdot \mathbf{x}^H\} = \epsilon_s \cdot \mathbf{H}^H = \Phi_{xy}^H$$

$$\Phi_{xx} = \epsilon_s \cdot \mathbf{I}_{N_T \times N_R}$$

$$- \mathbf{F}_{\text{opt}} \rightarrow \frac{d}{d\mathbf{F}^*} \left(\text{tr}\{\mathbf{F}\Phi_{yy}\mathbf{F}^H\} - \text{tr}\{\mathbf{F}\Phi_{yx}\} - \text{tr}\{\Phi_{xy}\mathbf{F}^H\} + \text{tr}\{\Phi_{xx}\} \right) \stackrel{!}{=} 0$$

with Table IV in paper by Hjørnager & Gesbert:

$$\Rightarrow \mathbf{F} \cdot \Phi_{yy} - \Phi_{xy} = 0$$

$$\Rightarrow \mathbf{F}_{\text{opt}} = \Phi_{xy} \cdot \Phi_{yy}^{-1} = \epsilon_s \mathbf{H}^H (\epsilon_s \mathbf{H}\mathbf{H}^H + \sigma_n^2 \mathbf{I})^{-1}$$

$$= (\text{Matrix inversion Lemma}) =$$

$$= \left(\mathbf{H}^H \mathbf{H} + \frac{\sigma_n^2}{\epsilon_s} \mathbf{I} \right)^{-1} \cdot \mathbf{H}^H$$

- Comparison:

$$\mathbf{F}_{\text{MMSE}} = \left(\mathbf{H}^H \mathbf{H} + \frac{\sigma_n^2}{\epsilon_s} \mathbf{I} \right)^{-1} \cdot \mathbf{H}^H \xrightarrow[\frac{\sigma_n^2}{\epsilon_s} \rightarrow 0]{} (\mathbf{H}^H \cdot \mathbf{H})^{-1} \mathbf{H}^H = \mathbf{F}_{ZF}$$

$$\xrightarrow[\frac{\sigma_n^2}{\epsilon_s} \rightarrow \infty]{\frac{\epsilon_s}{\sigma_n^2}} \frac{\epsilon_s}{\sigma_n^2} \cdot \mathbf{H}^H = \mathbf{F}_{MF} \hat{=} \text{matched filter}$$

⇒ For high SNR, $\frac{\epsilon_s}{\sigma_n^2}$, the MMSE filter approaches the ZF-Filter, for low SNR, it approaches the matched filter.

→ MMSE receiver yields the same diversity gain as the ZF receiver

$$G_d^{MMSE} = G_d^{ZF} = N_R - N_T + 1 \leq G_d^{ML} = N_R$$

– End-to-End Channel: $\mathbf{K} = \mathbf{F}\mathbf{H} = (\mathbf{H}^H\mathbf{H} + \frac{\sigma_n^2}{\epsilon_s} \cdot \mathbf{I})^{-1} \cdot \mathbf{H}^H \cdot \mathbf{H} \neq$ diagonal matrix

⇒ crosstalk/interference between elements \mathbf{x} in received signal after filtering \mathbf{r} .

elements of $\mathbf{K} : K_{l,n}$

– Covariance for \mathbf{F}_{opt}

$$\begin{aligned} \Phi_{ee} &= \Phi_{xy} \cdot \overbrace{\Phi_{yy}^{-1} \cdot \Phi_{yy} \cdot \Phi_{yy}^{-1}}^{\Phi_{yy}^{-1}} \cdot \Phi_{xy}^H - \Phi_{xy} \cdot \Phi_{yy}^{-1} \cdot \Phi_{yx} - \Phi_{xy} \cdot \Phi_{yy}^{-1} \cdot \Phi_{xy}^H + \Phi_{xx} \\ &= \Phi_{xx} - \overbrace{\Phi_{xy} \cdot \Phi_{yy}^{-1} \cdot \Phi_{yx}}^{\mathbf{F}_{\text{opt}}} \\ &= \epsilon_s \mathbf{I} - \epsilon_s (\mathbf{H}^H \mathbf{H} + \frac{\sigma_n^2}{\epsilon_s} \cdot \mathbf{I})^{-1} \cdot \mathbf{H}^H \mathbf{H} = (\text{Matrix inversion Lemma}) \\ &= \sigma_n^2 (\mathbf{H}^H \mathbf{H} + \frac{\sigma_n^2}{\epsilon_s} \cdot \mathbf{I})^{-1} \\ \Phi_{ee} &= \epsilon_s (\mathbf{I} - \mathbf{K}) \end{aligned}$$

→ $0 \leq K_{m,m} \leq 1$ since main diagonal elements of Φ_{ee} are $0 \leq [\Phi_{ee}]_{m,m} \leq \epsilon_s$

siehe auch Abbildung 1

SNR (biased vs. unbiased)

a) biased SNR

$$\text{SNR}_{\text{bias},m} = \frac{\epsilon_s}{[\Phi_{ee}]_{mm}} = \frac{\epsilon_s}{\epsilon_s(1 - K_{mm})} = \frac{1}{1 - K_{mm}}, \quad 1 \leq m \leq 4$$

Anmerkung: $\mathbf{K} = \mathbf{F}_{\text{opt}} \cdot \mathbf{H} \rightarrow \text{SNR} = 1$ falls $\mathbf{K} = \text{zeros}() \Rightarrow$ woher $\text{SNR} = 1$ bei keiner Uebertragung? \Rightarrow nicht vorteilhaft: siehe: b) unbiased SNR but:

- $\text{SNR}_{\text{bias},m}$ does not represent the actual SNR since the main diagonal elements of \mathbf{K} are smaller than 1

- $\mathbf{r} = \mathbf{F}\mathbf{H}\mathbf{x} + \mathbf{F}\mathbf{n} = \mathbf{K}\mathbf{x} + \underbrace{\mathbf{F}\mathbf{n}}_{\tilde{\mathbf{n}}=[\tilde{n}_1, \dots, \tilde{n}_{N_T}]^T}$

- $r_m = \underbrace{K_{mm}}_{<1} x_m + \sum_{\substack{n=1 \\ n \neq m}}^{N_T} K_{mn} x_n + \tilde{n}_m$

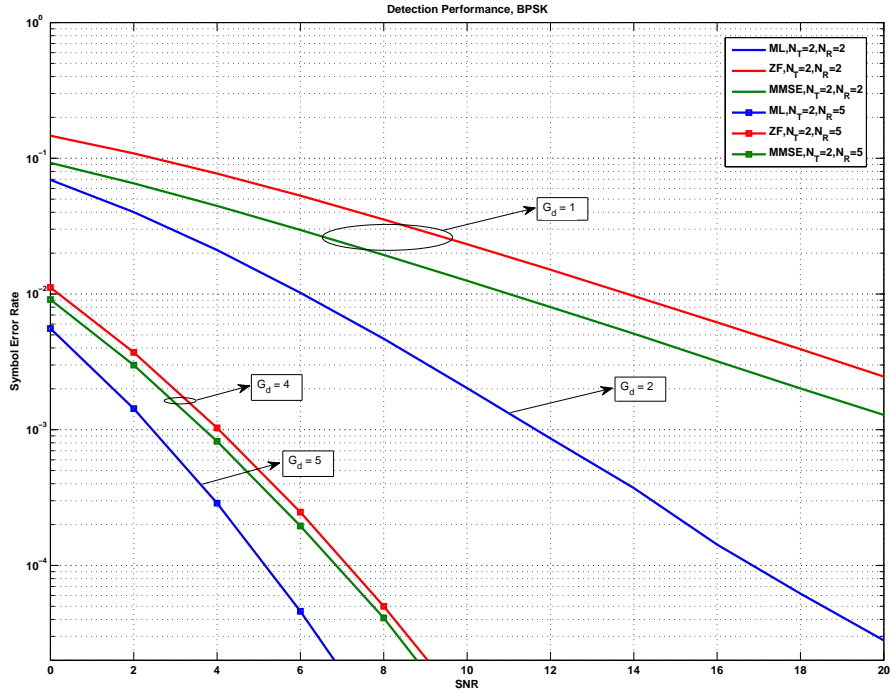


Figure 1: MMSE

b) unbiased SNR

- remove bias: $r'_m = \frac{r_m}{K_{mm}} = x_m + \underbrace{\frac{\tilde{e}_m}{K_{mm}}}_{e'_m}$

- SNR?

- scaling matrix: $\mathbf{C} = \text{diag}\left\{\frac{1}{K_{11}}, \frac{1}{K_{22}}, \dots, \frac{1}{K_{N_T N_T}}\right\}$

- $\mathbf{r}' = \mathbf{C}\mathbf{r} \rightarrow \mathbf{e}' = [e'_1, \dots, e'_{N_T}]^T = \mathbf{r}' - \mathbf{x} = \mathbf{C}\mathbf{r} - \mathbf{x} = \mathbf{C}\mathbf{F}\mathbf{y} - \mathbf{x}$

- $\Phi_{e'e'} = \mathcal{E}\left\{(\mathbf{C}\mathbf{F}\mathbf{y} - \mathbf{x})(\mathbf{C}\mathbf{F}\mathbf{y} - \mathbf{x})^H\right\} = \mathbf{C}\mathbf{F}\Phi_{yy}\mathbf{F}^H\mathbf{C}^H - \mathbf{C}\mathbf{F}\Phi_{yx} - \Phi_{xy}\mathbf{F}^H\mathbf{C}^H + \Phi_{xx}$

- $\mathbf{F} = \mathbf{F}_{\text{opt}} \rightarrow \mathbf{F}_{\text{opt}}\Phi_{yy} = \Phi_{xy} = \epsilon_s \cdot \mathbf{H}^H$

$$\begin{aligned} \rightarrow \Phi_{e'e'} &= \epsilon_s \underbrace{\mathbf{C}\mathbf{H}^H\mathbf{F}^H}_{\mathbf{K}^H} \mathbf{C}^H - \epsilon_s \underbrace{\mathbf{C}\mathbf{F}\mathbf{H}}_{\mathbf{K}} - \epsilon_s \underbrace{\mathbf{H}^H\mathbf{F}^H}_{\mathbf{K}^H} \mathbf{C} + \epsilon_s \mathbf{I} \\ &= \epsilon_s [\mathbf{I} + (\mathbf{C} - \mathbf{I})\mathbf{K}^H\mathbf{C}^H - \mathbf{C}\mathbf{K}] \end{aligned}$$

Anmerkung: nur Hauptdiagonalelemente interessieren, da diese die Varianz darstellen

$$\begin{aligned} \rightarrow \text{maindiagonal elements of } \Phi_{e'e'} &= \text{variances of } e'_m = \epsilon_s \left(1 + \left(\frac{1}{K_{mm}} - 1\right)K_{mm}\frac{1}{K_{mm}} - \frac{1}{K_{mm}}K_{mm}\right) \\ &= \epsilon_s \frac{1-K_{mm}}{K_{mm}} \end{aligned}$$

→ vgl. Abbildung 1

$$\rightarrow SNR_{unbiased} = \frac{E_s}{[\Phi_{e'e'}]_{mm}} = \frac{E_s}{E_s \frac{1 - K_{mm}}{K_{mm}}} = \frac{K_{mm}}{1 - K_{mm}}, \quad 1 \leq m \leq N_T$$

→ the SNR after bias removal is by “1” smaller than the biased SNR → general result valid for any type of MMSE estimation

2.4.3 Decision - Feedback Equalization (Detection)

- Also known as:
 - BLAST (Bell Laboratories space-time system)
 - successive interference cancellation
- Problem of linear receiver: Noise enhancement because of linear filtering → nonlinear filtering processing necessary

Basic Idea

- Recall (linear filter): $\mathbf{F}\mathbf{H} = \mathbf{I}$ for linear ZF receiver → i th row of \mathbf{F} , \mathbf{f}_i , is orthogonal to the j th column of \mathbf{H} , \mathbf{h}_j , if $i \neq j$ (if $i = j$: inner product = 1)
- we can detect x_i , based on $r_i = \mathbf{f}_i \mathbf{y}$
- Once we have detected x_i , we can subtract its contribution from \mathbf{y} : $\mathbf{y}_1 = \mathbf{y} - \mathbf{h}_i \hat{x}_i$ (\hat{x}_i is detected symbol, we assume for now, $\hat{x}_i = x_i$)
- \mathbf{y}_1 can be expressed as $\mathbf{y}_1 = \mathbf{H}_i \mathbf{x}_i + \mathbf{n}$ where

$$\mathbf{H}_i = [\mathbf{h}_1, \dots, \mathbf{h}_{i-1}, \mathbf{h}_{i+1}, \dots, \mathbf{h}_{N_T}]$$

$$\mathbf{x}_i = [x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{N_T}]$$

→ we have reduced the number of signal streams to $N_T - 1$ (N_R bleibt gleich)

- apply now linear ZF filter for symbol to detected next, e.g. x_j , where $j \in \{1, \dots, i - 1, i + 1, \dots, N_T\}$

→ $\mathbf{r}_j = \mathbf{f}_j \mathbf{y}_1$ where \mathbf{f}_j is the ZF filter for \mathbf{H}_1

- subtract contribution of x_j from \mathbf{y}_1 : $\mathbf{y}_2 = \mathbf{y}_1 - \mathbf{h}_j x_j$
- subtract until last symbol is detected
- Blockdiagram see figure2
- Observations:
 - The order in which the x_1 are detected can be freely chosen and effects the performance → $N_T!$ possible orders → cannot explore all of them
 - Practical approach: Select in each step that x_i for which the noise variance enhancement is minimum, i.e. which has the smallest $\mathcal{E}\{|\mathbf{f}_i \mathbf{n}|^2\} = \sigma_n^2 \|\mathbf{f}_i\|^2$
- Diversity order:

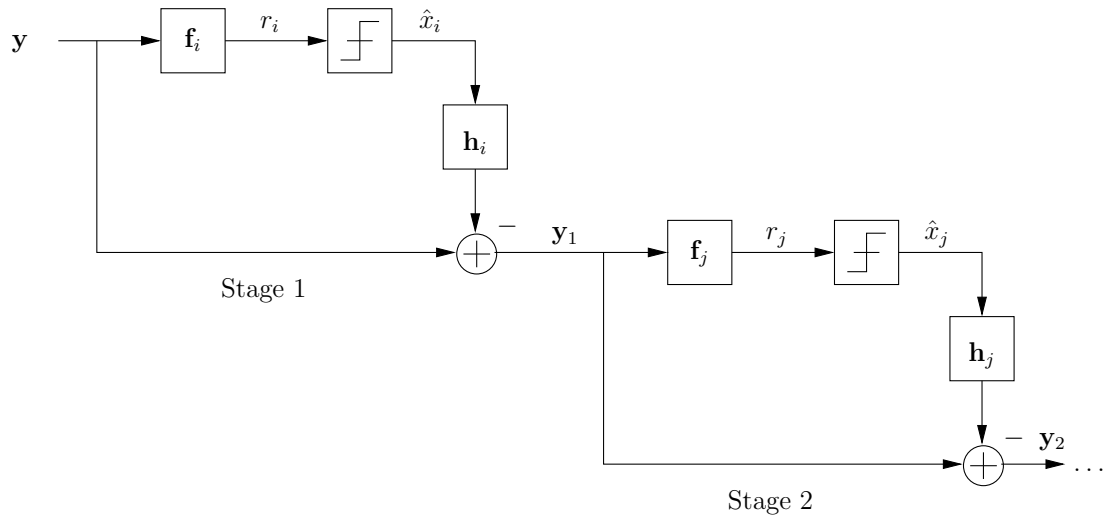


Figure 2: DFE Blockdiagram

- stage 1: $G_d^1 = N_R - N_T + 1$
- stage 2: $G_d^2 = G_d^1 + 1 = N_R - N_T + 2$
- \vdots
- stage N_T : $G_d^{N_T} = N_R$
- overall: $G_d = N_R - N_T + 1$
- (Anmerkung: der schlechteste Fall dominiert (Stage 1), weitere koennen nur schwer beeinflussen)

ZF - DFE - Matrix Model

- Signal model: $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} = \underbrace{\mathbf{H}\mathbf{P}}_{\tilde{\mathbf{H}}} \cdot \underbrace{\mathbf{P}^{-1}\mathbf{x}}_{\tilde{\mathbf{x}}} + \mathbf{n}$ with permutation matrix \mathbf{P}
- \mathbf{P} has one „1“ per column and row, all other elements are „0“
- can change the detection order to maximize performance
- note: $\mathbf{P}^T = \mathbf{P}^{-1}$
- Example:

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} [r] \rightarrow \mathbf{P}^{-1} = \mathbf{P}^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\tilde{\mathbf{x}} = \mathbf{P}^T \cdot \mathbf{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}$$

- Blockdiagram see figure 3

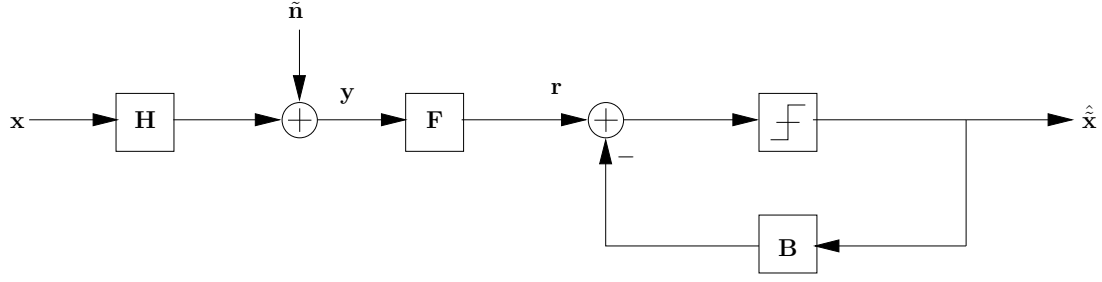


Figure 3: ZF-DFE Matrix Model

- DFE Filters:
 - \mathbf{F} feedforward filter
 - \mathbf{B} feedback filter
- Filter calculation
 - Cholesky factorization: $\tilde{\mathbf{H}}^H \tilde{\mathbf{H}} = \mathbf{L}^H \mathbf{D} \mathbf{L}$ with diagonal matrix \mathbf{D} and lower triangular matrix \mathbf{L} (maindiagonal elements of \mathbf{L} are “1”)

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ l_{21} & 1 & \ddots & \ddots & \vdots \\ l_{31} & l_{32} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \cdots & 1 \end{bmatrix}$$

$$- \mathbf{F} = \mathbf{D}^{-1} \mathbf{L}^{-H} \tilde{\mathbf{H}}^H$$

$$\rightarrow \mathbf{r} = \mathbf{F} \mathbf{y} = \underbrace{\mathbf{D}^{-1} \mathbf{L}^{-H} \tilde{\mathbf{H}}^H \tilde{\mathbf{H}}}_{\mathbf{L}} \tilde{\mathbf{x}} + \underbrace{\mathbf{D}^{-1} \mathbf{L}^{-H} \tilde{\mathbf{H}}^H}_{\tilde{\mathbf{n}}} \mathbf{n}$$

$$- \mathbf{B} = \mathbf{L} - \mathbf{I} = \text{lower triangular matrix with maindiagonal elements "0"}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ l_{21} & 0 & \ddots & \ddots & \vdots \\ l_{31} & l_{32} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

- Interpretation:

– after feedforward filter

$$\mathbf{r} = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ l_{21} & 1 & \ddots & \ddots & \vdots \\ l_{31} & l_{32} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & \dots & \dots & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_{N_T} \end{bmatrix} + \tilde{\mathbf{n}}$$

• from feedback filter

$$\tilde{\mathbf{r}} = \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ l_{21} & 0 & \ddots & \ddots & \vdots \\ l_{31} & l_{32} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & \dots & \dots & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_{N_T} \end{bmatrix} \quad \hat{x}_n : \text{decision on } \tilde{x}_n$$

$$\rightarrow \mathbf{r} - \tilde{\mathbf{r}} = \begin{bmatrix} \tilde{x}_1 + 0 \\ l_{21}(\tilde{x}_1 - \hat{x}_1) + \tilde{x}_2 + 0 \\ l_{31}(\tilde{x}_1 - \hat{x}_2) + l_{32}(\tilde{x}_2 - \hat{x}_2) + \tilde{x}_3 + 0 \\ \vdots \end{bmatrix} + \tilde{\mathbf{n}}$$

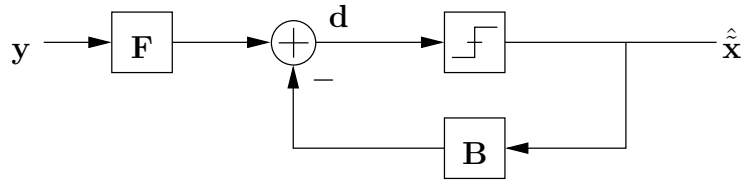


Figure 4: MMSE-DFE Blockdiagram

MMSE-DFE

- $\mathbf{B} = \mathbf{L} - \mathbf{I}$ has to be lower triangular matrix with all-zero main diagonal elements because of causality.

- $\mathbf{d} = \mathbf{F}\mathbf{y} - \mathbf{B}\hat{\mathbf{x}} \rightarrow \mathbf{e} = \mathbf{d} - \hat{\mathbf{x}} = \mathbf{F}\mathbf{y} - \underbrace{(\mathbf{B} + \mathbf{I})}_{\mathbf{L}} \hat{\mathbf{x}}$

where we assume $\hat{\hat{\mathbf{x}}} = \hat{\mathbf{x}}$ for filter optimization

- Optimal \mathbf{F} for given \mathbf{L}

$$\sigma_e^2 = \text{tr} \left\{ \underbrace{\mathcal{E}\{\mathbf{e}\mathbf{e}^H\}}_{\Phi_{ee}} \right\} = \text{tr} \{ \mathbf{F} \Phi_{yy} \mathbf{F}^H \} - \text{tr} \{ \mathbf{F} \Phi_{y\hat{x}} \mathbf{L}^H \} - \text{tr} \{ \mathbf{L} \Phi_{\hat{x}y} \mathbf{F}^H \} + \text{tr} \{ \mathbf{L} \Phi_{\hat{x}\hat{x}} \mathbf{L}^H \}$$

$$\frac{\delta}{\delta \mathbf{F}^*} \sigma_e^2 = \mathbf{F} \Phi_{yy} - \mathbf{L} \Phi_{\tilde{x}y} = 0 \quad \rightarrow \mathbf{F} = \mathbf{L} \Phi_{\tilde{x}y} \Phi_{yy}^{-1}$$

where

$$\begin{aligned} \Phi_{yy} &= \mathcal{E}_s \tilde{\mathbf{H}} \tilde{\mathbf{H}}^H + \sigma_n^2 \mathbf{I} \\ \Phi_{\tilde{x}y} &= \mathcal{E}_s \tilde{\mathbf{H}}^H \end{aligned}$$

$$\rightarrow \mathbf{F} = \mathbf{L} \tilde{\mathbf{H}}^H (\tilde{\mathbf{H}} \tilde{\mathbf{H}}^H + \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I})^{-1} = \mathbf{L} \overbrace{(\tilde{\mathbf{H}}^H \tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I})^{-1} \tilde{\mathbf{H}}^H}^{\text{MMSE Linear Equalizer}}$$

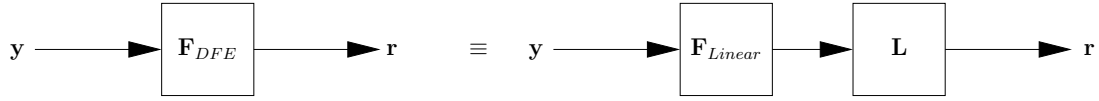


Figure 5: MMSE-DFE Equivalence

- Optimal \mathbf{L} :

$$\begin{aligned} \Phi_{ee} &= \mathbf{L} \Phi_{\tilde{x}y} \Phi_{yy}^{-1} \Phi_{yy} \Phi_{yy}^{-1} \Phi_{\tilde{x}y}^H \mathbf{L}^H - \mathbf{L} \Phi_{\tilde{x}y} \Phi_{yy}^{-1} \Phi_{y\tilde{x}} \mathbf{L}^H - \mathbf{L} \Phi_{\tilde{x}y} \Phi_{yy}^{-1} \Phi_{\tilde{x}y}^H \mathbf{H}^H + \mathbf{L} \Phi_{\tilde{x}\tilde{x}} \mathbf{L}^H \\ &= \mathbf{L} (\Phi_{\tilde{x}\tilde{x}} - \Phi_{\tilde{x}y} \Phi_{yy}^{-1} \Phi_{\tilde{x}y}^H) \mathbf{L}^H \end{aligned}$$

= MMSE covariance matrix for a linear MMSE filter

$$\rightarrow \Phi_{ee} = \sigma_n^2 \mathbf{L} (\tilde{\mathbf{H}}^H \tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I})^{-1} \mathbf{L}^H$$

\rightarrow need lower triangular matrix \mathbf{L} which minimizes $\text{tr}\{\Phi_{ee}\}$

\rightarrow the optimum \mathbf{L} whitens $\Phi_{ee} \rightarrow \Phi_{ee}$ becomes diagonal matrix i.e. \mathbf{L} exploits the correlation after linear MMSE filtering to reduce noise variance!

$\rightarrow \mathbf{L}$ is obtained via Cholesky factorization

$$\tilde{\mathbf{H}}^H \tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I} = \mathbf{L} \mathbf{D} \mathbf{L}^H$$

$$\rightarrow \Phi_{ee} = \sigma_n^2 \mathbf{L} (\mathbf{L} \mathbf{D} \mathbf{L}^H)^{-1} \mathbf{L} = \sigma_n^2 \mathbf{L} \mathbf{L}^{-1} \mathbf{D}^{-1} \mathbf{L}^{-H} \mathbf{L} = \sigma_n^2 \mathbf{D}^{-1} \hat{=} \text{diagonal matrix}$$

- Summary of MMSE calculation

$$- \mathbf{B} = \mathbf{L} - \mathbf{I} \text{ where } \mathbf{H}^H \mathbf{H} + \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I} = \mathbf{L} \mathbf{D} \mathbf{L}^H$$

$$- \mathbf{F} = \mathbf{L} \underbrace{(\mathbf{H}^H \mathbf{H} + \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I})^{-1} \tilde{\mathbf{H}}}_{\mathbf{F}_{Linear}}$$

- \mathbf{L} is a prediction error filter, which whitens the error signal \mathbf{e}

– SNR:

$$\mathbf{D} = \text{diag}\{\xi_1, \dots, \xi_{N_T}\}$$

$$\mathbf{\Phi}_{ee} = \text{diag}\left\{\frac{\sigma_n^2}{\xi_1}, \dots, \frac{\sigma_n^2}{\xi_{N_T}}\right\}$$

– end-to-end channel

$$\begin{aligned}\mathbf{K} &= \mathbf{F}\tilde{\mathbf{H}} \\ &= \underbrace{\mathbf{L}(\tilde{\mathbf{H}}^H \tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I})^{-1} \mathbf{L}^H \mathbf{L}^{-H} \tilde{\mathbf{H}}^H \tilde{\mathbf{H}}}_{\frac{1}{\sigma_n^2} \mathbf{\Phi}_{ee}} \\ &= \frac{1}{\sigma_n^2} \mathbf{\Phi}_{ee} \mathbf{L}^{-H} (\tilde{\mathbf{H}}^H \tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I} - \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I}) \mathbf{L}^{-1} \mathbf{L} \\ &= \frac{1}{\sigma_n^2} [\underbrace{\mathbf{L}^{-H} (\tilde{\mathbf{H}}^H \tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I}) \mathbf{L}^{-1} \mathbf{L}}_{\sigma_n^2 \mathbf{\Phi}_{ee}^{-1}} - \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{L}^{-H}] \\ &= \mathbf{L} - \frac{1}{\mathcal{E}_s} \mathbf{\Phi}_{ee} \mathbf{L}^{-H}\end{aligned}$$

→ main diagonal of \mathbf{K} :

$$K_{mm} = 1 - \frac{\sigma_n^2}{\mathcal{E}_s \xi_m} < 1$$

– biased SNR

$$SNR_{m,bias} = \frac{\mathcal{E}_S}{[\mathbf{\Phi}_{ee}]_{mm}} = \frac{\mathcal{E}_s}{\sigma_n^2} \xi_m = \frac{1}{1 - K_{mm}}, \quad 1 \leq m \leq N_T$$

– unbiased SNR

$$SNR_{m,unbiased} = SNR_{m,bias} - 1 = \frac{1}{1 - K_{mm}} - 1 = \frac{K_{mm}}{1 - K_{mm}}, \quad 1 \leq m \leq N_T$$

2.4.4 Sphere Decoding

- Linear and DFE receivers cannot approach performance of ML - detector
- ML - detector: $\hat{\mathbf{x}} = \underset{\mathbf{x} \in A^{N_T}}{\text{argmin}} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2$

→ high complexity for brute force search

Main Idea

- can we search ML metric in a “smarter” way, akin to the smart search of the viterbi algorithm for problems with a trellis structure
- we need to find a way to prune/dismiss non-ML sequences/vectores early on

→ this is accomplished by sphere decoding

Step 1 Bring metric into a suitable form:

- real representation: $\|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2 = \|\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\mathbf{x}}\|^2$
- where:

$$\begin{aligned}\tilde{\mathbf{y}} &= [\operatorname{Re}\{\mathbf{y}^T\} \quad \operatorname{Im}\{\mathbf{y}^T\}]^T \\ \tilde{\mathbf{x}} &= [\operatorname{Re}\{\mathbf{x}^T\} \quad \operatorname{Im}\{\mathbf{x}^T\}]^T \\ \tilde{\mathbf{H}} &= \begin{bmatrix} \operatorname{Re}\{\mathbf{H}\} & -\operatorname{Im}\{\mathbf{H}\} \\ \operatorname{Im}\{\mathbf{H}\} & \operatorname{Re}\{\mathbf{H}\} \end{bmatrix}^T\end{aligned}$$

→ QAM: $\tilde{\mathbf{x}}$ defines points in an $2N_T$ dimensional lattice

- QL decomposition: $\tilde{\mathbf{H}} = \mathbf{Q}\mathbf{L}$ with
 - orthogonal matrix $\mathbf{Q} \in \mathbb{R}^{2N_T \times 2N_T}$; $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$
 - and lower triangular matrix $\mathbf{L} \in \mathbb{R}^{2N_T \times 2N_T}$

$$\begin{aligned}\|\tilde{\mathbf{y}} - \mathbf{Q}\mathbf{L}\tilde{\mathbf{x}}\|^2 &= \|\mathbf{Q}\mathbf{Q}^T(\tilde{\mathbf{y}} - \mathbf{Q}\mathbf{L}\tilde{\mathbf{x}})\|^2 + \|(\mathbf{I} - \mathbf{Q}\mathbf{Q}^T)(\tilde{\mathbf{y}} - \mathbf{Q}\mathbf{L}\tilde{\mathbf{x}})\|^2 \\ &= \|\mathbf{Q}(\mathbf{Q}^T\tilde{\mathbf{y}} - \mathbf{L}\tilde{\mathbf{x}})\|^2 + \|(\mathbf{I} - \mathbf{Q}\mathbf{Q}^T)\tilde{\mathbf{y}} - \underbrace{(\mathbf{Q} - \mathbf{Q})\mathbf{L}\tilde{\mathbf{x}}}_{=0}\|^2 \\ &= \|\mathbf{Q}^T\tilde{\mathbf{y}} - \mathbf{L}\tilde{\mathbf{x}}\|^2 + \underbrace{\|(\mathbf{I} - \mathbf{Q}\mathbf{Q}^T)\tilde{\mathbf{y}}\|^2}_{\text{unabhängig von } \tilde{\mathbf{x}}}\end{aligned}$$

$$\Rightarrow \operatorname{argmin}_{\tilde{\mathbf{x}} \in A^{2N_T}} \|\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\mathbf{x}}\|^2 = \operatorname{argmin}_{\tilde{\mathbf{x}} \in A^{2N_T}} \underbrace{\|\mathbf{Q}^T\tilde{\mathbf{y}} - \mathbf{L}\tilde{\mathbf{x}}\|^2}_{\tilde{\mathbf{y}}}$$

where A^{2N_T} contains all M^{N_T} possible vectors $\tilde{\mathbf{x}}$

- Observation:
 - with

$$\begin{aligned}\mathbf{L} &= \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ l_{2N_T-1} & \cdots & & l_{2N_T-2N_T} \end{bmatrix} \\ \bar{\mathbf{y}} &= [\bar{y}_1 \quad \cdots \quad \bar{y}_{2N_T}]^T; \quad \tilde{\mathbf{x}} = [\tilde{x}_1 \quad \cdots \quad \tilde{x}_{2N_T}]\end{aligned}$$

- we have

$$\|\bar{\mathbf{y}} - \mathbf{L}\tilde{\mathbf{x}}\|^2 = (\bar{y} - l_{11}\tilde{x}_1)^2 + (\bar{y}_2 - l_{21}\tilde{x}_1 - l_{22}\tilde{x}_2)^2 + (\bar{y}_3 - l_{31}\tilde{x}_1 - l_{32}\tilde{x}_2 - l_{33}\tilde{x}_3)^2 + \dots \quad (1)$$

→ the n -th term in the above sum contains only $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$

Step 2 Sphere decoding algorithm**Define:**

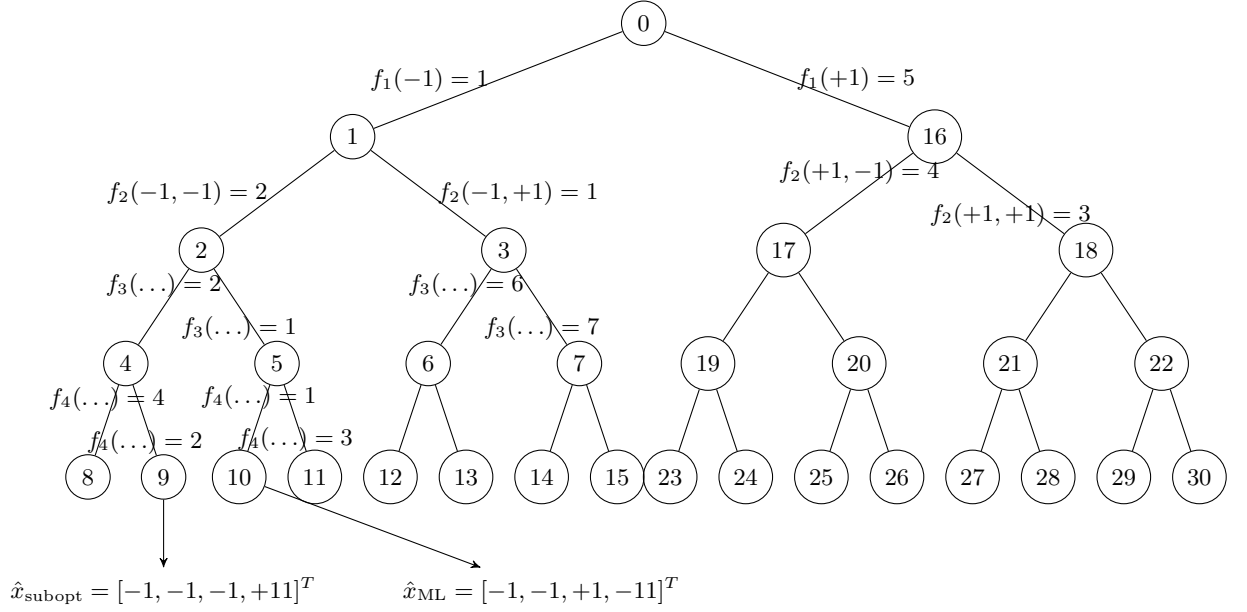
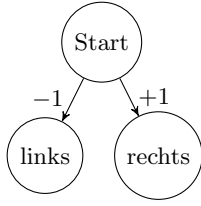
$$d(\tilde{\mathbf{x}}) = \sum_{n=1}^{2N_T} f_n(\tilde{\mathbf{x}}_n) ; \quad \text{with } f_n(\tilde{\mathbf{x}}_n) = \left(\bar{y}_n - \sum_{i=1}^n l_{ni} \tilde{x}_i \right)^2 ; \quad \tilde{\mathbf{x}}_n = [\tilde{x}_1 \quad \dots \quad \tilde{x}_n]^T$$

$$d_n(\tilde{x}_n) = \sum_{m=1}^n f_n(\tilde{\mathbf{x}}_m)$$

Main Idea:

- Assume we know that $d(\tilde{\mathbf{x}}) \leq R$ holds for some $\tilde{\mathbf{x}}$, where R is the so called “sphere radius” \rightarrow any $\tilde{\mathbf{x}}$ with $d(\tilde{\mathbf{x}}) \geq R$ cannot be the ML solution and can be discarded
- since $d_{n+1}(\tilde{\mathbf{x}}'_n, \tilde{x}_{n+1}) \geq d_n(\tilde{\mathbf{x}}'_n)$, we can easily discard \mathbf{x}'_n and all possible $\tilde{\mathbf{x}} = [\tilde{\mathbf{x}}'_n \quad \tilde{x}_{n+1} \quad \tilde{x}_{n+2} \quad \dots \quad \tilde{x}_{2N_T}]^T$ if we find $d_n(\tilde{\mathbf{x}}'_n) > R$
 \Rightarrow we can exclude many possible vectors $\tilde{\mathbf{x}}$ without evaluating any metrics for them
- How to find a suitable R ?
 - Initial R : $R = d(\tilde{\mathbf{x}}_{\text{subopt}})$ where $\tilde{\mathbf{x}}_{\text{subopt}}$ was obtained with some suboptimum receiver
 - R is updated as $R = R_{\text{new}} = d(\tilde{\mathbf{x}}_{\text{cand}})$ where $\tilde{\mathbf{x}}_{\text{cand}}$ is an $\tilde{\mathbf{x}}$ which yields $d(\tilde{\mathbf{x}}_{\text{cand}}) < R_{\text{old}} = R$
- Use tree structure to represent all possible $\tilde{\mathbf{x}}$

Example: BPSK = 2, $N_T = 2$



- Assume $\tilde{\mathbf{x}}_{\text{subopt}} = [-1 \ -1 \ -1 \ 1]^T$
 $\Rightarrow d(\tilde{\mathbf{x}}_{\text{subopt}}) = 1 + 2 + 2 + 2 = 7 = R$
- For $\tilde{x}_1 = 1$ we find $d_2(1, 1) = 5 + 3 = 8 > R$ and $d(1, -1) = 5 + 4 > R$
 \Rightarrow we don't have to calculate metrics for remaining branches, i.e., for nodes 19, ..., 31
- For $\tilde{x}_1 = -1$ and $\tilde{x}_2 = 1$, we find that $d_3(-1, 1, -1) = 8 > R$ and $d_3(-1, 1, 1) = 9 > R$
 \Rightarrow remaining branches emerging from nodes 6 and 7 can be discarded
- For the ML solution we find $d(-1, -1, 1, -1) = 5 < R$
- Different strategies exist, regarding in which order the nodes in the tree (points in the lattice) are investigated
 - Pohst strategy
 - Schnorr - Enchner strategy
 - rich literature on lattice decoding algorithms
- Complexity of sphere decoding
 - worst-case complexity is still exponential in N_T
 - (for practical case:) for sufficiently high SNR, the average complexity is only polynomial in $N_T \Rightarrow$ efficient ML decoding

- sphere decoding has found application in many fields:
 - ML detection in MIMO and multiuser systems
 - precoding
 - source coding
 - multiple symbol differential detection