# 1 SIMO Systems

Remarks

- In SIMO Systems only coding and diversity gains can be exploited
- To realize these gains diversity combining has to be performed
- Diversity combining schemes vary in complexity and performance
- There are <u>many</u> diversity combining schemes. Here we consider:
    - Maximal ratio combining (MRC)
    - Equal gain combining (EGC)
    - Selection combining (SC)
- Diversity combining problem
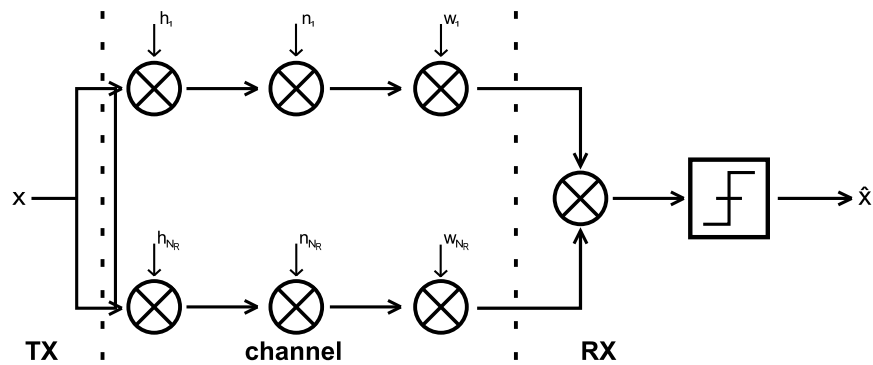
- how to choose combining weights w_n?
- what performance (e.g. error rate, outage probability) is achieved?
- what <u>diversity</u> and coding/combining gain is achieved?
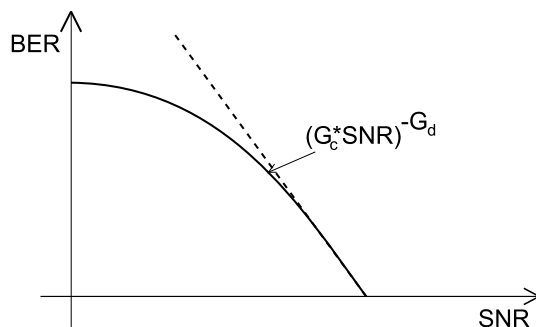
- G_c : Coding gain
- G_d : Diversity gain

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1.1 Preliminaries

Consider an equivalent system:

$$y = hx + n;$$

$$\mathcal{E}\{|x^2|\} = \epsilon_s; \qquad\qquad \mathcal{E}\{|n^2|\} = \sigma_n^2; \qquad\qquad \mathcal{E}\{|h|^2\} = 1$$

- Instantaneous SNR: $y_t = \frac{\epsilon_s}{\sigma_n^2} \times |h|^2$
- Average SNR: $\bar{y}_t = \mathcal{E}\{y_t\} = \frac{\epsilon_s}{\sigma_n^2}$

Bit and Symbol Error Rate

- The Bit and Symbol Error Rate of many modulation schemes can be expressed for given y_t as:

$$P_e(y_t) = aQ\{\sqrt{by_t}\}$$

where:

- $Q(x) = \frac{1}{\sqrt{2\pi}} \times \int_x^\infty e^{-\frac{t^2}{2}} \, dt$
- $P_e(y_t)$ may be exact or approximation
- BPSK: exact with $a = 1, b = 2$
- M-ary QAM: tight approximation with $a = 4\left(1 - \frac{1}{\sqrt{M}}\right), b = \frac{3}{M-1}$

$\left(Einschub : Gray - Code : BER = \frac{1}{\log_2 M} \times SER\right)$

- Alternative representation of Q - function:

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2 sin^2 \theta}} \, d\theta$$

$\rightarrow$ Integral limits are fixed and don't depend on integration variables!

- Average error probability

$$P_e = \mathcal{E}\{P_e(y_t)\} = \int_0^\infty aQ(\sqrt{bx})p_{\gamma_t}(x) \, dx$$

  - Integral may be difficult to solve analytically
  - Integral has infinite support $\rightarrow$ numerical evaluation difficult

- Using alternate representation of Q-function we get:

$$P_e = \int_0^\infty \frac{a}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{bx}{2 sin^2 \theta}} p_{\gamma_t}(x) \, d\theta \, dx$$

$$= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} \int_0^\infty p_{\gamma_t}(x) e^{-\frac{b}{2 sin^2 \theta}} \, dx \, d\theta \qquad\qquad = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\left(\frac{b}{2 sin^2 \theta}\right) \, d\theta$$

where:

- $M_{\gamma_t}(s) = \int_0^\infty p_{\gamma_t}(x)e^{-sx}\, dx$ is the Laplace transform of p_{γ_t}

- $M_{\gamma_t}(-s)$ is the so called Moment Generation Function (MGF) of p_{γ_t}

- Here, we will also refer to $M_{\gamma_t}(s)$ as MGF

- $M_{\gamma_t}(s)$ is sometimes easier to obtain than p_{γ_t}

- The above integral can be easily evaluated numerically because of the finite integral limits

Outage probability

- The outage probability is the probability that the channel cannot support a certain rate, R, i.e. (where γ_T is the threshold SNR):

$$C = \log_2(1 + \gamma_t) < R \quad \leftrightarrow \quad \gamma_t < 2^R - 1 \triangleq \gamma_T$$
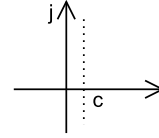
Thus, the outage probability is given by:

$$P_{out} = P_0 \gamma_t < \gamma - T = \int_0^{\gamma_T} p_{\gamma_t}(x)\, dx$$

- Using the inverse Laplace Transform

$$p_{\gamma_t}(x) = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s)e^{sx}\, dx$$

where $c > 0$ is a small constant that lies in the region of convergence of the integral, we obtain:

- 1.

$$P_{out} = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) \int_0^{\gamma_T} e^{sx}\, dx\, ds = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s)e^{\gamma_T s}\, \frac{ds}{s}$$

$\big($lower integral limit is 0 since $p_{\gamma_t}(0) = 0\big)$

- and 2.:

$$p_{\gamma_t}(x) = \int_0^x p_{\gamma_t}(t)\, dt = 0$$

$$\text{for } x = 0 \text{ note: } p_{\gamma_t}(x) \xleftarrow[transform]{Laplace} \frac{1}{s} M_{\gamma_t}(s)$$

General combining scheme

$$y = \left(\sum_{n=1}^{N_R} h_n w_n \right) x + \sum_{n=1}^{N_R} w_n n_n$$

$$\gamma_t = \frac{\epsilon_s \left| \sum_{n=1}^{N_R} h_n w_n \right|^2}{\sigma_n^2 \sum_{N=1}^{N_R} |w_n|^2}$$

where w_n depends on the particular combining scheme.

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1.2 MRC

- what weight w_n maximize γ_t?

  - Cauchy-Schwarz inequality

$$\left|\sum_{n=1}^{N_R} h_n w_n\right|^2 \leq \sum_{n=1}^{N_R} |h_n|^2 \cdot \sum_{n=1}^{N_R} |w_n|^2$$

  where equality holds if and only if $w_n = c \cdot h_n^*$ for some non-zero constant c.

  - for $w_n = h_n^*$, we obtain

$$\gamma_t = \frac{\epsilon_s}{\sigma_n^2} \cdot \frac{\left(\sum_{n=1}^{N_R} |h_n|^2\right)^2}{\sum_{n=1}^{N_R} |h_n|^2} = \frac{\epsilon_s}{\sigma_n^2} \sum_{n=1}^{N_R} |h_n|^2$$

  - $w_n = h_n^* \forall n$ are the MRC combining weights.

- For performance analysis we assume <u>independent identically distributed (IID)</u> Rayleigh fading

$$\rightarrow \mathcal{E}\{|h_n|^2\} = 1; \quad \bar{\gamma} = \frac{\epsilon_s}{\sigma_n^2}; \quad \gamma_n = \frac{\epsilon_s}{\sigma_n^2}|h_n|^2$$

$$p_\gamma(x) = \frac{1}{\bar{\gamma}}e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0$$

$$M_\gamma(s) = \frac{1}{1 + s\bar{\gamma}}$$

- Error rate

$$\gamma_t = \sum_{n=1}^{N_R} \gamma_n$$

$\rightarrow$ sum of IID random variables (r.v.s.)

$$M_{\gamma_t}(s) = \left(M_\gamma(s)\right)^{N_R} = \frac{1}{(1 + s\bar{\gamma})^{N_R}} = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{1}{(s + \frac{1}{\bar{\gamma}})^{N_R}}$$
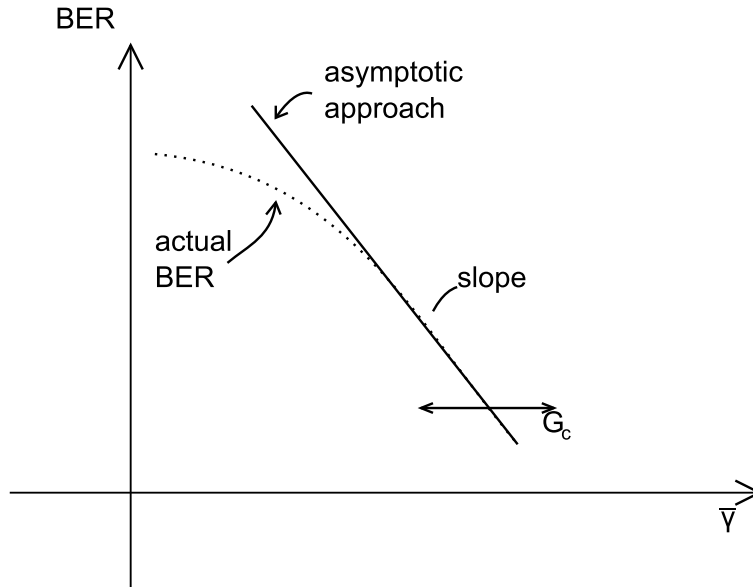
inverse Laplace-transform (from tables)

$$p_{\gamma_t}(x) = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{x^{N_R - 1}}{(N_R - 1)!}e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0$$

- Direct approach

$$p_e = \int_0^\infty a \cdot Q\left(\sqrt{ax}\right)p_{\gamma_t}(x)\,dx = a\left(\frac{1 - \mu}{2}\right)^{N_R} \cdot \sum_{n=0}^{N_R - 1} \binom{N_R - 1 + n}{n}\left(\frac{1 + \mu}{2}\right)^n$$

where $\mu = \sqrt{\dfrac{b\bar{\gamma}}{2 + b\bar{\gamma}}}$

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- MGF approach

$$p_e = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\left(\frac{b}{2\sin^2\theta}\right) d\theta$$

$$= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\bar{\gamma}^{N_R}\left(\frac{b}{\sin^2\theta} + \frac{1}{\bar{\gamma}}\right)^{N_R}} d\theta \quad \text{(numerisch berechnen!)}$$

- high SNR: $\bar{\gamma} \to \infty \Longleftrightarrow \frac{1}{\bar{\gamma}} \to 0$

$$p_e = \frac{a}{\pi} \cdot \frac{1}{\bar{\gamma}^{N_R}} \cdot \left(\frac{2}{b}\right)^{N_R} \int_0^{\frac{\pi}{2}} \sin^{2N_R}\theta \, d\theta$$

$$\text{(from MGF approach: } \int_0^{\frac{\pi}{2}} \sin^{2N_R}\theta \, d\theta = \frac{\pi}{2^{N_R+1}} \cdot \binom{2N_R}{N_R}\text{)}$$

$$= \frac{a}{2^{N_R+1} \cdot b^{N_R}} \begin{pmatrix} 2N_R & N_R \end{pmatrix} \frac{1}{\bar{\gamma}^{N_R}} \quad \text{as } \bar{\gamma} \to \infty$$

$$\stackrel{!}{=} \left(\frac{1}{G_c \bar{\gamma}}\right)$$

where: Diversity gain: $G_d = N_R$

Combining/Coding gain: $G_c = 2b\left(\frac{a}{2}\binom{2N_R}{N_R}\right)^{-\frac{1}{N_R}}$

- MRC exploits the maximal possible diversity
- Diversity gain is not affected by correlation as the branches are not <u>fully</u> correlated
- Diversity gain depends on fading distribution

Outage probability

$$P_{out} = \int_0^{\gamma_T} p_{\gamma_t}(x) \, dx = \frac{1}{\bar{\gamma}^{N_R}} \int_0^{\gamma_T} \frac{x^{N_R-1}}{(N_R-1)!} e^{-\frac{x}{\bar{\gamma}}} \, dx$$

$$= 1 - e^{-\frac{\gamma_T}{\bar{\gamma}}} \cdot \sum_{n=1}^{N_R} \frac{\left(\frac{\gamma_T}{\bar{\gamma}}\right)^n}{(n-1)!}$$

- Approximation (Taylor series): $\bar{\gamma} \to \infty$: $-e^{-\frac{x}{\bar{\gamma}}} = 1 - \frac{x}{\bar{\gamma}} + O(\frac{1}{\bar{\gamma}})$ where a function $f(x)$ is $O(x)$ if $\lim_{x \to \infty} \frac{f(x)}{x} = 0$.