

# MIMO Skript - Wintersemester 2013

## Kapitel 2

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## 2 Point - to - Point MIMO Systems

### 2.1 MIMO Channel Capacity

Inhalt: s. Skript in VL ausgeteilt

### 2.2 SIMO Systems

#### Remarks

- In SIMO Systems only coding and diversity gains can be exploited (no multiplexing gains)
- To realize these gains diversity combining has to be performed

- Diversity combining schemes vary in complexity and performance
- There are many diversity combining schemes. Here we consider:
  - Maximal ratio combining (MRC)
  - Equal gain combining (EGC)
  - Selection combining (SC)
- Diversity combining problem

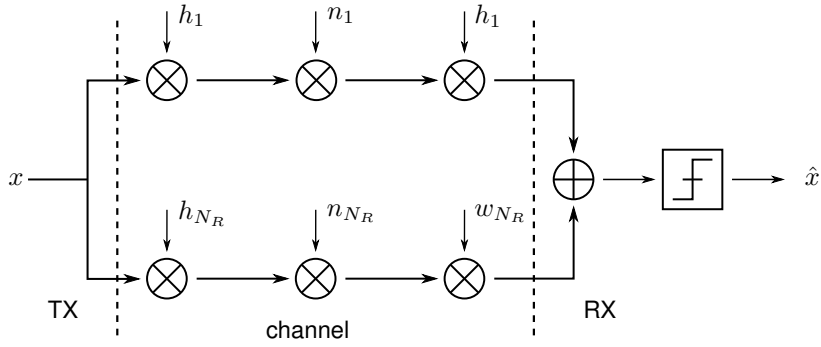


Figure 1: Block Diagramm for SIMO

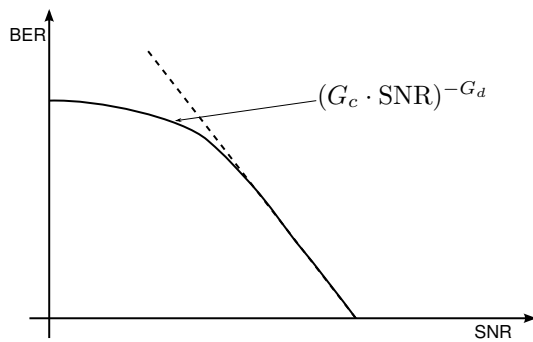
- how to choose combining weights  $w_n$ ?
- what performance (e.g. error rate, outage probability) is achieved?
- what diversity and coding/combining gain is achieved?

### 2.2.1 Preliminaries

Consider an equivalent system:

$$y = hx + n;$$

$$\mathcal{E}\{|x|^2\} = \mathcal{E}_s; \quad \mathcal{E}\{|n|^2\} = \sigma_n^2; \quad \mathcal{E}\{|h|^2\} = 1$$



- $G_c$  : Coding gain
- $G_d$  : Diversity gain

Figure 2: Exemplary BER for SIMO

- Instantaneous SNR:  $\gamma_t = \frac{\mathcal{E}_s}{\sigma_n^2} \cdot |h|^2$
- Average SNR:  $\bar{\gamma}_t = \mathcal{E}\{\gamma_t\} = \frac{\mathcal{E}_s}{\sigma_n^2}$

### Bit and Symbol Error Rate

- The Bit and Symbol Error Rate of many modulation schemes can be expressed for given  $\gamma_t$  as:

$$P_e(\gamma_t) = aQ(\sqrt{b\gamma_t})$$

where:

- $Q(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_x^\infty e^{-\frac{t^2}{2}} dt$
- $P_e(\gamma_t)$  may be exact result or approximation
- BPSK: exact with  $a = 1, b = 2$
- M-ary QAM: tight approximation with  $a = 4(1 - \frac{1}{\sqrt{M}}), b = \frac{3}{M-1}$

(*Einschub : Gray - Code :  $BER = \frac{1}{\log_2 M} \cdot SER$* )

- Alternative representation of Q - function:

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2 \sin^2 \theta}} d\theta$$

→ Integral limits are fixed and do not depend on integration variables!

- Average error probability

$$P_e = \mathcal{E}\{P_e(\gamma_t)\} = \int_0^\infty aQ(\sqrt{bx})p_{\gamma_t}(x) dx$$

- Integral may be difficult to solve analytically
- Integral has infinite support → numerical evaluation difficult

- Using alternative representation of Q-function we get:

$$\begin{aligned} P_e &= \int_0^\infty \frac{a}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{bx}{2 \sin^2 \theta}} p_{\gamma_t}(x) d\theta dx \\ &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} \int_0^\infty p_{\gamma_t}(x) e^{-\frac{bx}{2 \sin^2 \theta}} dx d\theta = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\left(\frac{b}{2 \sin^2 \theta}\right) d\theta \end{aligned}$$

where:

- $M_{\gamma_t}(s) = \int_0^\infty p_{\gamma_t}(x)e^{-sx} dx$  is the Laplace transform of  $p_{\gamma_t}$
- $M_{\gamma_t}(-s)$  is the so called Moment Generation Function (MGF) of  $p_{\gamma_t}$
- Here, we will also refer to  $M_{\gamma_t}(s)$  as MGF
- $M_{\gamma_t}(s)$  is sometimes easier to obtain than  $p_{\gamma_t}$
- The above integral can be easily evaluated numerically because of the finite integral limits

### Outage probability

- The outage probability is the probability that the channel cannot support a certain rate,  $R$ , i.e. (where  $\gamma_T$  is the threshold SNR):

$$C = \log_2(1 + \gamma_t) < R \quad \leftrightarrow \quad \gamma_t < 2^R - 1 \triangleq \gamma_T$$

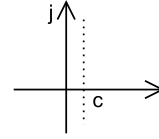
Thus, the outage probability is given by:

$$P_{out} = P_0\{\gamma_t < \gamma_T\} = \int_0^{\gamma_T} p_{\gamma_t}(x) dx$$

- Using the inverse Laplace Transform

$$p_{\gamma_t}(x) = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) e^{sx} dx$$

where  $c > 0$  is a small constant that lies in the region of convergence of the integral, we obtain:



– 1.

$$P_{out} = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) \int_0^{\gamma_T} e^{sx} dx ds = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) e^{\gamma_T s} \frac{ds}{s}$$

(lower integral limit is 0 since  $p_{\gamma_t}(0) = 0$ )

– and 2.:

$$p_{\gamma_t}(x) = \int_0^x p_{\gamma_t}(t) dt = 0$$

$$\text{for } x = 0 \text{ note: } p_{\gamma_t}(x) \xleftrightarrow[\text{transform}]{\text{Laplace}} \frac{1}{s} M_{\gamma_t}(s)$$

### General combining scheme

$$y = \left( \sum_{n=1}^{N_R} h_n w_n \right) x + \sum_{n=1}^{N_R} w_n n_n$$

$$\gamma_t = \frac{\mathcal{E}_s \left| \sum_{n=1}^{N_R} h_n w_n \right|^2}{\sigma_n^2 \sum_{n=1}^{N_R} |w_n|^2}$$

where  $w_n$  depends on the particular combining scheme.

#### 2.2.2 MRC (Maximum Ratio Combining)

- what weight  $w_n$  maximize  $\gamma_t$ ?

- Cauchy-Schwarz inequality

$$\left| \sum_{n=1}^{N_R} h_n w_n \right|^2 \leq \sum_{n=1}^{N_R} |h_n|^2 \cdot \sum_{n=1}^{N_R} |w_n|^2$$

where equality holds if and only if  $w_n = c \cdot h_n^*$  for some non-zero constant  $c$ .

- for  $w_n = h_n^*$ , we obtain

$$\gamma_t = \frac{\mathcal{E}_s}{\sigma_n^2} \cdot \frac{\left( \sum_{n=1}^{N_R} |h_n|^2 \right)^2}{\sum_{n=1}^{N_R} |h_n|^2} = \frac{\mathcal{E}_s}{\sigma_n^2} \sum_{n=1}^{N_R} |h_n|^2$$

- $w_n = h_n^* \forall n$  are the MRC combining weights.

- For performance analysis we assume independent identically distributed (IID) Rayleigh fading

$$\rightarrow \mathcal{E}\{|h_n|^2\} = 1; \quad \bar{\gamma} = \frac{\mathcal{E}_s}{\sigma_n^2}; \quad \gamma_n = \frac{\mathcal{E}_s}{\sigma_n^2} |h_n|^2$$

$$p_\gamma(x) = \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0$$

$$M_\gamma(s) = \frac{1}{1 + s\bar{\gamma}}$$

- Error rate

$$\gamma_t = \sum_{n=1}^{N_R} \gamma_n$$

$\rightarrow$  sum of IID random variables (r.v.s.)

$$M_{\gamma_t}(s) = \left( M_\gamma(s) \right)^{N_R} = \frac{1}{(1 + s\bar{\gamma})^{N_R}} = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{1}{\left(s + \frac{1}{\bar{\gamma}}\right)^{N_R}}$$

inverse Laplace-transform (from tables)

$$p_{\gamma_t}(x) = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{x^{N_R-1}}{(N_R-1)!} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0$$

- Direct approach

$$P_e = \int_0^\infty a \cdot Q(\sqrt{ax}) p_{\gamma_t}(x) dx = a \left( \frac{1-\mu}{2} \right)^{N_R} \cdot \sum_{n=0}^{N_R-1} \binom{N_R-1+n}{n} \left( \frac{1+\mu}{2} \right)^n$$

$$\text{where } \mu = \sqrt{\frac{b\bar{\gamma}}{2 + b\bar{\gamma}}}$$

- MGF approach

$$\begin{aligned} P_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t} \left( \frac{b}{2 \sin^2 \theta} \right) d\theta \\ &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\bar{\gamma}^{N_R} \left( \frac{b}{\sin^2 \theta} + \frac{1}{\bar{\gamma}} \right)^{N_R}} d\theta \quad (\text{numerisch berechnen!}) \end{aligned}$$

- with high SNR:  $\bar{\gamma} \rightarrow \infty \iff \frac{1}{\bar{\gamma}} \rightarrow 0$  and with MGF approach this leads to Average Error Probability  $P_e$ :

$$\begin{aligned}
P_e &= \frac{a}{\pi} \cdot \frac{1}{\bar{\gamma}^{N_R}} \cdot \left(\frac{2}{b}\right)^{N_R} \int_0^{\frac{\pi}{2}} \sin^{2N_R} \theta \, d\theta \\
&= \frac{a}{2^{N_R+1} \cdot b^{N_R}} \binom{2N_R}{N_R} \frac{1}{\bar{\gamma}^{N_R}} \quad \text{as } \bar{\gamma} \rightarrow \infty \\
&\stackrel{!}{=} \left(\frac{1}{G_c \bar{\gamma}}\right)
\end{aligned}$$

where:

- $\int_0^{\frac{\pi}{2}} \sin^{2N_R} \theta \, d\theta = \frac{\pi}{2^{N_R+1}} \cdot \binom{2N_R}{N_R}$
- Diversity gain:  $G_d = N_R$
- Combining/Coding gain:  $G_c = 2b \left(\frac{a}{2} \binom{2N_R}{N_R}\right)^{-\frac{1}{N_R}}$

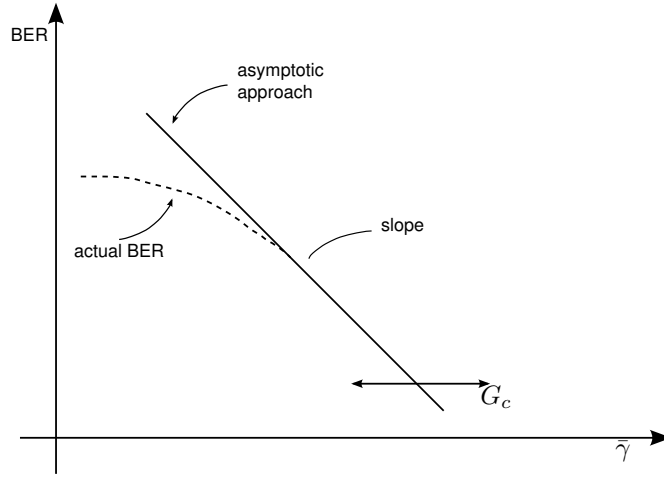


Figure 3: BER for average SNR  $\bar{\gamma}_t$

- MRC exploits the maximal possible diversity
- Diversity gain is not affected by correlation as the branches are not fully correlated
- Diversity gain depends on fading distribution

### Outage probability

$$\begin{aligned}
P_{out} &= \int_0^{\gamma_T} p_{\gamma_t}(x) \, dx = \frac{1}{\bar{\gamma}^{N_R}} \int_0^{\gamma_T} \frac{x^{N_R-1}}{(N_R-1)!} e^{-\frac{x}{\bar{\gamma}}} \, dx \\
&= 1 - e^{-\frac{\gamma_T}{\bar{\gamma}}} \cdot \sum_{n=1}^{N_R} \frac{\left(\frac{\gamma_T}{\bar{\gamma}}\right)^n}{(n-1)!}
\end{aligned}$$

- Approximation (Taylor series):  $\bar{\gamma} \rightarrow \infty : -e^{-\frac{x}{\bar{\gamma}}} = 1 - \frac{x}{\bar{\gamma}} + O(\frac{1}{\bar{\gamma}})$  where a function  $f(x)$  is  $O(x)$  if  $\lim_{x \rightarrow \infty} \frac{f(x)}{x} = 0$ .

$$\Rightarrow P_{out} = \frac{1}{\gamma^{N_R}} \int_0^{\gamma^T} \frac{x^{N_R-1}}{(N_R-1)!} \left(1 - \frac{x}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right)\right)$$

- Diversity and coding gain can also be defined for  $P_{out}$

### 2.2.3 EGC (Equal Gain Combining)

#### Combining Weights

- For MRC, both, the amplitudes and phases of the channel gains  $h_n = |h_n|e^{j\varphi_n}$  have to be known (or estimated in practice)
- In EGC it is assumed that only the phases are known and weights  $w_n = e^{-j\varphi_n}$  are used.

$$\begin{aligned} \Rightarrow \gamma_t &= \frac{E_s}{\sigma_n^2} \frac{\left| \sum_{n=1}^{N_R} |h_n| e^{j\varphi_n} e^{-j\varphi_n} \right|^2}{\sum_{n=1}^{N_R} |e^{-j\varphi_n}|^2} = \frac{E_s}{\sigma_n^2} \frac{1}{N_R} \left( \sum_{n=1}^{N_R} |h_n| \right)^2 \\ &= \frac{1}{N_R} \left( \sum_{n=1}^{N_R} \sqrt{\gamma_n} \right)^2 ; \text{ with } \gamma_n = \frac{E_s}{\sigma_n^2} |h_n|^2 \end{aligned}$$

#### Performance Analysis

- IID case  
 $\Rightarrow \sqrt{\gamma_n}$  is Rayleigh distributed  
 $\Rightarrow$  Exact analysis is much more difficult than for MRC  $\Rightarrow$  see book by Simon & Alouini p.341
- Approximate result

$$P_e = \frac{a}{2} \left[ 1 - \sqrt{\frac{2b\bar{\gamma}}{5+2b\bar{\gamma}}} \sum_{n=0}^{N_R-1} \frac{\binom{2n}{n}}{4^n (1 + \frac{2}{5}b\bar{\gamma})^n} \right]$$

- high SNR  
 $\Rightarrow$  use high SNR analysis of Wang & Giannakis, 2003  
 $\Rightarrow$  at high SNR, only pdf of  $\gamma_n$  around 0 is relevant for performance

$$\Rightarrow p_{\gamma}^{\text{Rayleigh}}(x) = \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}} \stackrel{\text{Taylor Serie}}{=} \frac{1}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right) \text{ as } x \rightarrow 0$$

- need pdf  $\gamma_t$ : ( $\gamma_n$  bekannt,  $\rightarrow$  ges.: Wurzel, etc.)  
(cumulative distribution function of  $\sqrt{\gamma}$  ( $\stackrel{\text{i.i.d.}}{=} \sqrt{\gamma_n}$ ) (cdf))

$$P_{\sqrt{\gamma}}(x) = \Pr\{\sqrt{\gamma} \leq x\} = \Pr\{\gamma \leq x^2\} = P_{\gamma}(x^2) = \text{cdf of } \gamma$$

$$\rightarrow p_{\sqrt{\gamma}}(x) = \frac{d}{dx} P_{\sqrt{\gamma}}(x) = 2x \cdot p_{\gamma}(x^2) = \frac{2x}{\gamma} + O\left(\frac{1}{\gamma}\right)$$

- Laplace Transformation to MGF

$$\rightarrow M_{\sqrt{\gamma}}(s) = \mathcal{L}\{p_{\sqrt{\gamma}}(x)\} = \frac{2}{\gamma} \cdot \frac{1}{s^2} + O\left(\frac{1}{\gamma}\right)$$

$$\sqrt{\gamma_t} = \sum_{n=1}^{N_R} \frac{\sqrt{\gamma_n}}{N_R}$$

$$M_{\sqrt{\gamma_t}}(s) = \mathcal{E}\left\{\exp(-s\sqrt{\gamma_t})\right\} = \mathcal{E}\left\{\exp\left(-\frac{s}{\sqrt{N_R}} \cdot \sum_{n=1}^{N_R} \sqrt{\gamma_n}\right)\right\} = \left(\mathcal{E}\left\{\exp\left(-\frac{s}{\sqrt{N_R}} \cdot \sqrt{\gamma_n}\right)\right\}\right)^{N_R}$$

$$= \left(M_{\sqrt{\gamma}}\left(\frac{s}{\sqrt{N_R}}\right)\right)^{N_R} = \left(\frac{2}{\gamma} \cdot \frac{N_R}{s^2}\right)^{N_R} + O\left(\frac{1}{\gamma^{N_R}}\right)$$

- inverse Laplace Transform

$$p_{\sqrt{\gamma_t}}(x) = \mathcal{L}^{-1}\left\{M_{\sqrt{\gamma_t}}(s)\right\} = \left(\frac{2N_R}{\gamma}\right)^{N_R} \cdot \frac{x^{2N_R-1}}{(2N_R-1)!} + O\left(\frac{1}{\gamma^{N_R}}\right)$$

$$P_{\gamma_t}(x) = \Pr\{\gamma_t \leq x\} = \Pr\{\sqrt{\gamma_t} \leq \sqrt{x}\} = P_{\sqrt{\gamma_t}}(\sqrt{x}) \rightarrow \text{cdf of } \sqrt{\gamma_t}$$

$$p_{\gamma_t}(x) = \frac{d}{dx} P_{\gamma_t}(x) = \frac{1}{2\sqrt{x}} \cdot p_{\gamma_t}(\sqrt{x}) = \frac{1}{2} \left(\frac{2N_R}{\gamma}\right)^{N_R} \cdot \frac{x^{N_R-1}}{(2N_R-1)!} + O(\gamma^{-N_R})$$

$$\rightarrow M_{\gamma_t}(s) = \mathcal{L}\{p_{\gamma_t}(x)\} = \frac{1}{2} \left(\frac{2N_R}{\gamma}\right)^{N_R} \cdot \frac{(N_R-1)!}{(2N_R-1)!} \frac{1}{b^{N_R}} + O(\gamma^{-N_R})$$

- Error Probability:

$$P_e = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t} \left( \frac{b}{2 \sin^2(\theta)} \right) d\theta$$

$$= \frac{a}{\pi} \frac{1}{2} \left( \frac{2N_R}{\gamma} \right)^{N_R} \frac{(N_R-1)!}{(2N_R-1)!} \frac{2^{N_R}}{b^{N_R}} \underbrace{\int_0^{\frac{\pi}{2}} \sin^{2N_R}(\theta) d\theta}_{\frac{2^{\frac{\pi}{2}N_R+1}}{2^{2N_R+1}} \frac{(2N_R)!}{(N_R!)^2}} + O\left(\frac{1}{\gamma^{N_R}}\right)$$

$$= \frac{aN_R^{N_R}}{2b^{N_R}N_R!} \frac{1}{\gamma^{N_R}} + O\left(\frac{1}{\gamma^{N_R}}\right) \stackrel{!}{=} \left(\frac{1}{G_c}\right)^{G_d}$$

$$\implies \text{Diversity gain: } G_d = N_R$$



$$\implies \text{Combining gain: } G_c = \frac{b}{N_R} \left( \frac{2N_R!}{a} \right)^{\frac{1}{N_R}}$$

vergleiche auch Blatt mit Kurven III und IV

A similar asymptotic analysis can be conducted for the outage probability.

## 2.2.4 SC (Selection Combining)

### Combining weights

- only the strongest branch is chosen
- strongest branch:  $\hat{n} = \underset{n}{\operatorname{argmax}} \gamma_n \longrightarrow \gamma_t = \gamma_{\hat{n}}$
- only on RF receiver chain required  $\rightarrow$  saves hardware complexity

### Performance analysis

- cdf of:  $\gamma_t$

$$\begin{aligned} P_{\gamma_t}(x) &= \Pr\{\gamma_{\hat{n}} \leq x\} = \Pr\{\gamma_1 \leq x \cap \gamma_2 \leq x \cap \dots \cap \gamma_{N_R} \leq x\} \\ &\stackrel{(IID)}{=} \left( \Pr\{\gamma_n \leq x\} \right)^{N_R} = \left( P_{\gamma}(x) \right)^{N_R} \end{aligned}$$

- pdf:

$$\begin{aligned} p_{\gamma_t}(x) &= \frac{d}{dx} P_{\gamma_t}(x) = N_R \left( P_{\gamma}(x) \right)^{N_R-1} \cdot p_{\gamma}(x) \\ \text{where: } p_{\gamma_t}(x) &= \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0 \\ P_{\gamma}(x) &= \int_0^x p_{\gamma}(x) dx = 1 - e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0 \\ \rightarrow p_{\gamma_t}(x) &= \frac{N_R}{\bar{\gamma}} \left( 1 - e^{-\frac{x}{\bar{\gamma}}} \right)^{N_R-1} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0 \end{aligned}$$

### Error probability

- direct approach  $\rightarrow$  closed-form solution possible
- MGF approach
  - Binomial expansion

$$\begin{aligned} p_{\gamma_t}(x) &= \frac{N_R}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} 1^{N_R-1-n} \left( -e^{-\frac{x}{\bar{\gamma}}} \right)^n \\ &= \frac{N_R}{\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} \cdot (-1)^n e^{-\frac{x(n+1)}{\bar{\gamma}}}; \quad x \geq 0 \end{aligned}$$

– MGF

$$M_{\gamma_t}(s) = \frac{N_R}{\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} (-1)^n \frac{1}{s + \frac{n+1}{\bar{\gamma}}}$$

–

$$P_e = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t} \left( \frac{b}{2 \sin^2 \theta} \right) d\theta = \frac{aN_R}{\pi \bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} (-1)^n \int_0^{\frac{\pi}{2}} \frac{d\theta}{\frac{b}{2 \sin^2 \theta} + \frac{n+1}{\bar{\gamma}}}$$

→ can be evaluated numerically

– high SNR approach  $\Rightarrow \bar{\gamma} \rightarrow \infty$

$$\begin{aligned} p_{\gamma_t} &= \frac{N_R}{\bar{\gamma}} \left[ 1 - \exp \left( -\frac{x}{\bar{\gamma}} \right) \right]^{N_R-1} \exp \left( -\frac{x}{\bar{\gamma}} \right) \\ &\stackrel{\bar{\gamma} \rightarrow \infty}{=} \frac{N_R}{\bar{\gamma}} \left[ 1 - \left( 1 - \frac{x}{\bar{\gamma}} + O(\bar{\gamma}^{-1}) \right) \right]^{N_R-1} \left( 1 - \frac{x}{\bar{\gamma}} + O(\bar{\gamma}^{-1}) \right) \\ &= \frac{N_R}{\bar{\gamma}^{N_R}} x^{N_R-1} + O(\bar{\gamma}^{-N_R}) \end{aligned}$$

– MGF:

$$\begin{aligned} M_{\gamma_t}(s) &= \frac{N_R}{\bar{\gamma}^{N_R}} \frac{(N_R-1)!}{s^{N_R}} + O(\bar{\gamma}^{-N_R}) \\ \left[ \rightarrow P_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t} \left( \frac{b}{2 \sin^2(\theta)} \right) d\theta \right] \\ &= \frac{a(2N_R)!}{b^{N_R} 2^{N_R+1} N_R!} \frac{1}{\bar{\gamma}^{N_R}} + O(\bar{\gamma}^{-N_R}) \end{aligned}$$

$\Rightarrow$  Diversity gain:  $G_d = N_R$

$\Rightarrow$  Combining gain:  $G_c = 2b \left( \frac{2N_R!}{a(2N_R)!} \right)^{\frac{1}{N_R}}$

## Outage Probability

$$P_{out} = \Pr\{\gamma_{\hat{n}} \leq \gamma_T\} = P_{\gamma_{\hat{n}}}(\gamma_T) = \left[ 1 - \exp \left( -\frac{\gamma_T}{\bar{\gamma}} \right) \right]^{N_R}$$

$$\text{high SNR: } P_{out} = \left( \frac{\gamma_T}{\bar{\gamma}} \right)^{N_R} + O(\bar{\gamma}^{-N_R})$$

### 2.2.5 Comparison

- Diversity Gain:  
MRC, EGC and SC all achieve the maximum possible diversity gain of  $G_d = N_R$

- Combining Gain:

The combining gains of MRC, EGC and SC are different

– MRC/EGC:

$$\frac{G_C^{EGC}}{G_C^{MRC}} = \frac{\frac{1}{2b} \left( \frac{a}{2} \binom{2N_R}{N_R} \right)^{\frac{1}{N_R}}}{\frac{N_R}{b} \left( \frac{a}{2} \frac{1}{N_R!} \right)^{\frac{1}{N_R}}} = \frac{[(2N_R)!]^{\frac{1}{N_R}}}{2N_R (N_R)^{\frac{1}{N_R}}} \leq 1$$

(independent of a or b which are modulation parameters, only depends on number of antennas)

$$N_R \gg 1 : \quad N_R! \approx \sqrt{2\pi} e^{-N_R} N_R^{N_R + \frac{1}{2}} \quad (Stirling)$$

$$\left. \frac{G_C^{EGC}}{G_C^{MRC}} \right|_{N_R \gg 1} = \frac{\left( \sqrt{2\pi} e^{-2N_R} (2N_R)^{2N_R + \frac{1}{2}} \right)^{\frac{1}{N_R}}}{2N_R \left( \sqrt{2\pi} e^{-N_R} N_R^{N_R + \frac{1}{2}} \right)^{\frac{1}{N_R}}} = \frac{2 \cdot 2^{\frac{1}{2N_R}}}{2} \xrightarrow{N_R \rightarrow \infty} \frac{2}{e} \equiv -1.3\text{dB}$$

– MRC/SC:

$$\frac{G_C^{SC}}{G_C^{MRC}} = \frac{2b \left( \frac{a}{2} \binom{2N_R}{N_R} \right)^{\frac{1}{N_R}}}{2b \left( \frac{a}{2} \frac{(2N_R)!}{N_R!} \right)^{\frac{1}{N_R}}} = \frac{1}{(N_R!)^{\frac{1}{N_R}}} \leq 1$$

$$\left. \frac{G_C^{SC}}{G_C^{MRC}} \right|_{N_R \gg 1} = \frac{1}{\sqrt{2\pi}^{\frac{1}{N_R}} e^{-1} N_R^{1 + \frac{1}{2N_R}}} N_R \xrightarrow{} \infty \frac{e}{N_R}$$

→ loss increases with  $N_R$

## 2.3 MISO Systems

### Remarks

- Similar to SIMO systems, in MISO systems only coding and diversity gains can be obtained.
- To realize these gains, a careful transmitter design is necessary
- System design depends on whether or not channel state information (**CSI**) is available at transmitter

### 2.3.1 Naive Approach

- Assume we simply send the same signal over all  $N_T$  transmit antennas

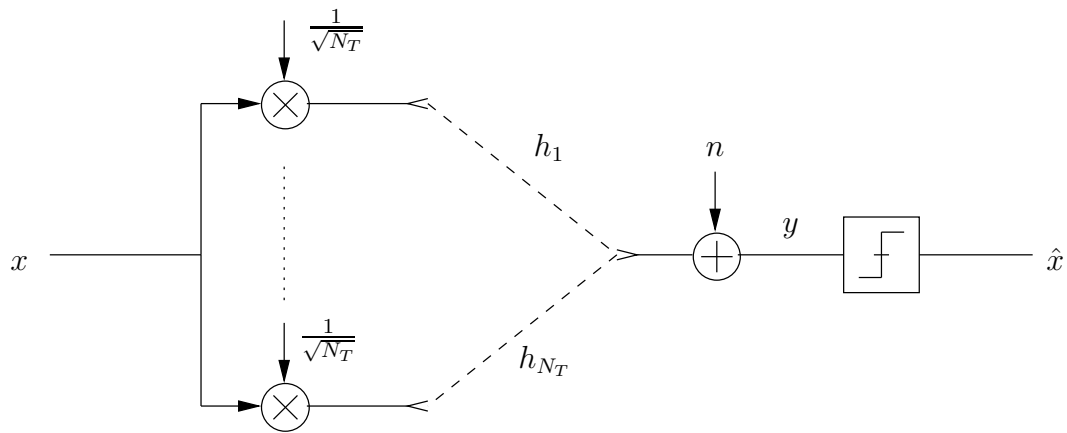


Figure 4: Block Diagramm of naive MISO

- Transmit power:  $\mathcal{E} \left\{ \left| \frac{1}{\sqrt{N_T}} x \right|^2 + \dots + \left| \frac{1}{\sqrt{N_T}} x \right|^2 \right\} = \mathcal{E} \left\{ N_T \frac{1}{N_T} |x|^2 \right\} = E_s$
- Received signal:  $y = \frac{1}{\sqrt{N_T}} \sum_{n=1}^{N_T} h_n \cdot x + n$
- Rayleigh fading:  $h_n$  are zero mean complex gaussian random variables  
 $\rightarrow h$  is also zero mean complex gaussian
- i.i.d.:
  - $\mathcal{E}\{|h_n|^2\} = 1 \ \forall n$
  - $\mathcal{E}\{|h|^2\} = \frac{1}{N_T} \mathcal{E} \left\{ \left| \sum_{n=1}^{N_T} h_n \right|^2 \right\} = \frac{1}{N_T} \mathcal{E} \left\{ \sum_{n=1}^{N_T} |h_n|^2 \right\} = 1$
  - statistical properties of  $h$  are independent of  $N_T$
  - the multiple transmit antennas have no benefit at all
  - more sophisticated transmitter designs necessary

### 2.3.2 Full CSI Available at the Transmitter

- $h_n, n \in \{1, \dots, N_T\}$  is known at the transmitter
- Perform “precoding” (beamforming) with coefficients  $w_n$

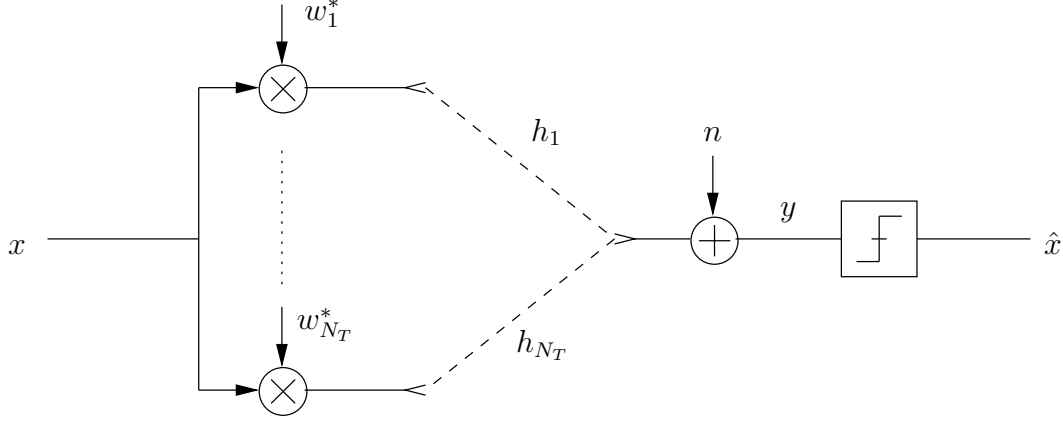


Figure 5: Block Diagramm of MISO with CSI

- Transmit Power: Two constraints maybe considered
  - Average transmit power constraint

$$P_{av} = \mathcal{E} \left\{ \sum_{n=1}^{N_T} |w_n^* x|^2 \right\} = \sum_{n=1}^{N_T} |w_n|^2 \underbrace{\mathcal{E}\{|x|^2\}}_{E_s} = \mathcal{E}_s \Rightarrow \sum_{n=1}^{N_T} |w_n|^2 = 1$$

- Power constraint for each transmit antenna

$$\rightarrow |w_n| = \frac{1}{\sqrt{N_T}} \quad \rightarrow P_{av} = E_s$$

- Received signal:  $y = \underbrace{\sum_{n=1}^{N_T} w_n^* h_n}_{h} x + n$  (equivalent SISO channel)

### Maximum Ratio Transmission (MRT)

- we have only the average power constraint:  $\sum_{n=1}^{N_T} |w_n|^2 = 1$

$$\bullet \text{ SNR: } \gamma_t = \frac{E_s |h|^2}{\sigma_n^2} = \frac{\mathcal{E}_s \left| \sum_{n=1}^{N_T} w_n^* \cdot h_n \right|^2}{\sigma_n^2}$$

- Maximize SNR under constraint  $\sum_{n=1}^{N_T} |w_n|^2 = 1$

- constraint optimization problem  $\rightarrow$  Lagrange method

$$L = \frac{E_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} w_n^* \cdot h_n \right|^2 + \lambda \left( \sum_{n=1}^{N_T} |w_n|^2 - 1 \right); \quad \text{where: } \lambda = \text{Lagrange Multiplier}$$

$\Rightarrow$  Wirtinger Kalkül: treat  $z$  and  $z^*$  as independent variables for differentiation:

$$\begin{aligned} \frac{\partial z^*}{\partial z} &= 0; & \frac{\partial |z|^2}{\partial z} &= \frac{\partial z \cdot z^*}{\partial z} = z^* \\ \frac{\partial x^2}{\partial x} &= 2x; & \frac{\partial (z^*)^2}{\partial z^*} &= 2 \cdot z^*; & \frac{\partial |z|^2}{\partial z^*} &= z^* \end{aligned}$$

$$\frac{\partial L}{\partial w_m^*} = \frac{\epsilon_s}{\sigma_n^2} \left( \sum_{n=1}^{N_T} w_n^* \cdot h_n \right)^* h_m + \lambda w_m$$

$$\rightarrow w_m = \frac{\frac{\epsilon_s}{\sigma_n^2} \cdot \lambda}{\text{const., independent of } m} \left( \sum_{n=1}^{N_T} w_n^* h_n \right)^* h_m$$

$$\rightarrow w_m = c \cdot h_m$$

$$\rightarrow \sum_{n=1}^{N_T} |w_n|^2 = 1 \rightarrow c^2 = \frac{1}{\sum_{n=1}^{N_T} |h_n|^2}$$

$$\rightarrow w_n = \frac{h_n}{\sqrt{\sum_{n=1}^{N_T} |h_n|^2}} \equiv \text{MRT gains}$$

$$\rightarrow \text{SNR} = \frac{E_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} \frac{|h_n|^2}{\sqrt{\sum_{n=1}^{N_T} |h_n|^2}} \right|^2 = \frac{\epsilon_s}{\sigma_n^2} \sum_{n=1}^{N_T} |h_n|^2$$

$\Rightarrow$  same SNR as for maximum ratio combining (MRC)

$\Rightarrow$  MRT with  $N_T$  transmit antennas achieves the same performance as MRC with  $N_T$  receive antennas

$\Rightarrow$  MRT/MRC can be extended to  $N_T \times N_R$  MIMO systems

$\rightarrow$  has the same performance as MRC with  $N_T \cdot N_R$  receive antennas and one transmit antenna

### Equal Gain Transmission (EGT)

- we employ gains:  $w_n = \frac{1}{\sqrt{N_T}} \cdot \frac{h_n}{|h_m|} \rightarrow |w_n| = \frac{1}{\sqrt{N_T}}$

- SNR:

$$\begin{aligned}
\gamma_t &= \frac{E_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} w_n^* h_n \right|^2 \\
&= \frac{E_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} \frac{1}{\sqrt{N_T} \cdot \frac{|h_n|^2}{|h_n|}} \right|^2 = \frac{1}{N_T} \cdot \frac{E_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} |h_n| \right|^2 \\
\gamma_n &= \frac{E_s}{\sigma_n^2} |h_n|^2 \\
\text{same SNR as for EGC} &\rightarrow \gamma_t = \frac{1}{N_T} \left| \sum_{n=1}^{N_T} \sqrt{\gamma_n} \right|^2
\end{aligned}$$

→ EGC with  $N_T$  transmit antennas achieves the same performance as EGC with  $N_T$  receive antennas

### Transmit Antennas Selection

- select antenna with maximum channel gain for transmission:

$$w_n = \begin{cases} \frac{h_n}{|h_n|}, & \text{if } n = \hat{n} \\ 0, & \text{otherwise} \end{cases} \quad \text{where } \hat{n} = \underset{n}{\operatorname{argmax}} |h_n|$$

- antenna selection with  $N_T$  transmit antennas achieves the same performance as *Selection Combining* with  $N_T$  receive antennas

### 2.3.3 No CSI at Transmitter - Space-Time-Coding

- $h_n, n \in \{1, \dots, N_T\}$ , is only known at the receiver
- “Space-time-coding” has to be employed to realize diversity gain
- $T \times N_T$  matrices  $\mathbf{X}$  are transmitted in  $T$  symbol intervals over  $N_T$  antennas
- $\mathbf{X}$  is drawn from a matrix alphabet  $\mathcal{X}$
- Example:

$$\mathbf{X} = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,N_T} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,N_T} \\ \vdots & \vdots & \ddots & \vdots \\ x_{T,1} & x_{T,2} & \cdots & x_{T,N_T} \end{pmatrix}$$

- We distinguish:
  - Space-time-block-codes (STBCs)
    - $\mathbf{X}$  is obtained by mapping  $K$  scalar symbols  $s_k, k = 1, \dots, K$  from a scalar alphabet  $\mathcal{A}$  to matrix  $\mathbf{X}$

- Space-time-trellis-codes (STTCs)
  - $\mathbf{X}$  is obtained from scalar symbols  $s_k$  through a trellis encoding process.
  - [see: Tarokh, Seshadri, Calderbank: Space-time-codes for high data-rate wireless communication: Performance criteria and coder construction; IEEE Trans. Inf. Theory 1998]
- here: We concentrate on space-time-block-codes (STBCs), but many results can be easily extended to space-time-trellis-codes
- STBCs:
  - $K$   $M$ -ary scalar symbols (e.g.  $M$ -PSK symbols) are mapped to STBC matrices  $\mathbf{X}$ 
 $\mathbf{S} = [s_1, \dots, s_K] \rightarrow \mathbf{X}$ 
 $s_k \in \mathcal{A} \rightarrow x \in \mathcal{X}$  with  $|\mathcal{X}| = M^K$
  - Example: “Alamouti”-Code

$$\mathbf{X} = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix}$$

[Alamouti: A simple transmit diversity technique for wireless communication, IEEE JSAC 1998]

## Optimal Detection

- Signal model:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_T \end{pmatrix} = \mathbf{X} \begin{pmatrix} h_1 \\ \vdots \\ h_{N_T} \end{pmatrix} + \begin{pmatrix} n_1 \\ \vdots \\ n_T \end{pmatrix}$$

$$\mathbf{y} = \mathbf{X} \cdot \mathbf{h} + \mathbf{n}$$

- Optimal detection - ML-detection

- $\mathbf{h}$  is known at receiver
- $\mathbf{n}$  is AWGN with  $\mathcal{E}\{\mathbf{n} \cdot \mathbf{n}^H\} = \sigma_n^2 \cdots \mathbf{I}_{T \times T}$
- $p(\mathbf{y}|\mathbf{x})$

$$\begin{aligned} &= \frac{1}{\pi^T |\sigma_n^2 \mathbf{I}_{T \times T}|} \exp \left( -(\mathbf{y} - \mathbf{xh})^H (\sigma_n^2 \mathbf{I}_{T \times T})^{-1} (\mathbf{y} - \mathbf{xh}) \right) \\ &= \frac{1}{\pi^T \sigma_n^{2T}} \exp \left( -\frac{1}{\sigma_n^2} (\mathbf{y} - \mathbf{xh})^H (\mathbf{y} - \mathbf{xh}) \right) = \frac{1}{\pi^T \sigma_n^{2T}} \exp (||\mathbf{y} - \mathbf{xh}||^2) \end{aligned}$$

→ the optimal estimate  $\hat{\mathbf{X}}$  or equivalently the optimal estimate  $\hat{\mathbf{s}}$  can be obtained as

$$\hat{\mathbf{s}} = \underset{\mathbf{s} \in \mathcal{A}^K}{\operatorname{argmax}} p(\mathbf{y}|\mathbf{x}) = \underset{\mathbf{s} \in \mathcal{A}^K}{\operatorname{argmin}} ||\mathbf{y} - \mathbf{hx}||^2$$

- Disadvantage: In general, metric  $||\mathbf{y} - \mathbf{hx}||^2$  has to be calculated  $M^K$  times
  - complexity increases exponentially with  $K$



## Types of STBCs

- Orthogonal STBCs (OSTBCs)

- OSTBCs are a special class of STBCs which allow independent detection of each  $s_k \rightarrow$  only  $K \cdot M$  metrics have to be evaluated

- Rate STBCs:  $R_{STBC} = \frac{K}{T}$

- Examples:

- \* Alamouti Code ( $K = 2, T = 2$ )  $\rightarrow R_{STBC} = 1$

$$\mathbf{X} = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix} \begin{matrix} \uparrow T \\ \longleftrightarrow N_T \end{matrix}$$

$\rightarrow$  only “full rate” OSTBC for complex  $s_k$

- \*  $N_T = 3, K = 3, T = 4$

$$\mathbf{X} = \frac{1}{\sqrt{3}} \begin{pmatrix} s_1 & s_2 & s_3 \\ -s_2^* & s_1^* & 0 \\ s_3^* & 0 & -s_3^* \\ 0 & -s_3^* & s_2^* \end{pmatrix} \rightarrow R_{STBC} = \frac{K}{T} = \frac{3}{4}$$

- Orthogonality:  $\mathbf{X}^H \mathbf{X} = \text{const} \cdot \mathbf{I}_{N_T \times N_T}$

- Independent detection of  $s_1$  &  $s_2$  for Alamouti Code

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \\ \rightarrow \underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{\tilde{\mathbf{y}}} &= \frac{1}{\sqrt{2}} \underbrace{\begin{pmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{pmatrix}}_{\tilde{\mathbf{F}}} \underbrace{\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}}_{\mathbf{s}} + \underbrace{\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}}_{\tilde{\mathbf{n}}} \end{aligned}$$

(Anmerkung: nur  $\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$  gewünscht, nicht:  $s_1^*, s_2^*$ )

$$\mathbf{F}^H \mathbf{F} = \frac{1}{2} \begin{pmatrix} h_1^* & h_2 \\ h_2^* & -h_1 \end{pmatrix} \begin{pmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{pmatrix} = \frac{1}{2} \begin{pmatrix} |h_1|^2 + |h_2|^2 & 0 \\ 0 & |h_1|^2 + |h_2|^2 \end{pmatrix}$$

$\rightarrow \frac{\sqrt{2}}{\sqrt{|h_1|^2 + |h_2|^2}} \cdot \mathbf{F}$  is unitary matrix

$\rightarrow \frac{2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{y}} = \mathbf{s} + \frac{2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{n}}$

( $\frac{2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{n}}$  is AWGN vector with covariance matrix  $\frac{2\sigma_n^2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{I}_{T \times T}$ )

$\rightarrow$  ML decision:  $\hat{\mathbf{s}} = \underset{\mathbf{s}}{\text{argmin}} \left\| \frac{2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{y}} - \mathbf{s} \right\|^2$

$\rightarrow$  independent ML decoding

$$\begin{aligned} \hat{s}_1 &= \underset{s_1}{\text{argmin}} \left| s_1 - \frac{h_1^* y_1 + h_2 y_2^*}{\frac{1}{\sqrt{2}}(|h_1|^2 + |h_2|^2)} \right| \\ \hat{s}_2 &= \underset{s_2}{\text{argmin}} \left| s_2 - \frac{h_1^* y_1 - h_2 y_2^*}{\frac{1}{\sqrt{2}}(|h_1|^2 + |h_2|^2)} \right| \end{aligned}$$

- independent decoding property can be proved for all OSTBCs
- low complexity is at the expense of a rate-loss compared to other STBCs for  $N_T > 2$ 
  - Frequenzhopping
  - keine Kanalinformation aus vorher empfangenen Symbolen möglich  $\Rightarrow$  Kanal ändert sich ständig: nur Entscheidung, ob Rauschen oder Signal + Rauschen
- Performance Analysis of Alamouti Code
  - Decision-variables after combining

$$r_1 = \sqrt{2} \frac{h_1^* y_1 + h_2 y_2^*}{|h_1|^2 + |h_2|^2}$$

$$r_2 = \sqrt{2} \frac{h_1^* y_1 - h_2 y_2^*}{|h_1|^2 + |h_2|^2}$$

because of symmetry it suffices to consider  $r_1$

$$r_1 = \sqrt{2} \frac{h_1^* \left( \frac{1}{\sqrt{2}} s_1 h_1 + \frac{1}{\sqrt{2}} h_2 s_2 + n_1 \right) + h_2 \left( -\frac{1}{\sqrt{2}} h_2 s_1^* + \frac{1}{\sqrt{2}} h_1 s_2^* + n_2 \right)^*}{|h_1|^2 + |h_2|^2}$$

$$= \sqrt{2} \frac{\frac{1}{\sqrt{2}} (|h_1|^2 + |h_2|^2) s_1 + h_1^* n_1 + h_2 n_2^*}{|h_1|^2 + |h_2|^2}$$

$$= 1 \cdot s_1 + n_{eq}$$

where

$$n_{eq} = \sqrt{2} \frac{h_1^* n_1 + h_2 n_2^*}{|h_1|^2 + |h_2|^2}$$

$$\text{SNR} \rightarrow \gamma_t = \frac{E_s \cdot 1^2}{\sigma_{eq}^2} \quad \text{with} \quad \mathcal{E}\{|s_1|^2\} = \mathcal{E}_s$$

$$\sigma_{eq}^2 = 2 \frac{|h_1|^2 \sigma_n^2 + |h_2|^2 \sigma_{eq}^2}{(|h_1|^2 + |h_2|^2)^2} = \frac{2\sigma_n^2}{|h_1|^2 + |h_2|^2}$$

- $\gamma_t = \frac{1}{2} \frac{E_s}{\sigma_n^2} (|h_1|^2 + |h_2|^2)$
- $\text{SNR}_{\text{Alamouti}} = \frac{1}{2} \text{SNR}_{\text{MRC}} = \frac{1}{2} \text{SNR}_{\text{MRT}}$
- Alamouti code has diversity gain  $G_d = 2$
- Transmission with Alamouti STBC requires 3dB higher SNR to achieve same performance as MRT  $\rightarrow$  3dB loss in coding gain  $G_c$
- Lack of CSI knowledge at transmitter “costs” 3dB in power efficiency
- General:
  - OSTBCs achieve a diversity gain of  $G_d = N_T$  if only one receive antenna is available
  - if  $N_R$  receive antennas are available, MRC can be used at the receiver to yield a diversity gain of  $G_d = N_T N_R$
- Other STBCs:

- Quasi orthogonal STBCs
  - \* higher rate than OSTBCs
  - \* only subset of symbols have to be decoded jointly
  - \* Example:  $K = N_T = T = 4$

$$\mathbf{X} = \frac{1}{2} \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^* & s_1^* & -s_4^* & s_3^* \\ -s_3^* & -s_4^* & s_1^* & s_2^* \\ s_4 & -s_3 & -s_2 & s_1 \end{pmatrix}$$

- \* Anmerkung 1:  $\mathbf{X}$  ist ähnlich zu Alamouti Code
- \* Anmerkung 2:  $\mathbf{X}^H \mathbf{X}$ : viele Nicht-diagonal Elemente sind Null; die, die ungleich Null sind, zeigen, welche Symbole gemeinsam entschlüsselt werden müssen
- Golden Code for  $N_T = N_R = 2$ : achieves a rate of  $R_{STBC} = 2$  and full diversity of  $G_d = N_T, N_R = 4$
- Differential STBCs:  $\mathbf{X}_k = \mathbf{X}_{k-1} \cdot \mathbf{D}_k$ .  $\mathbf{X}_k$  is transmitted,  $\mathbf{D}_k$  is transmitted
- Linear dispersion codes: designed to achieve high mutual information
- noncoherent STBCs (On-Off-Keying)

#### Space Time Code Design Given:

- Code  $\mathcal{X} = \{\mathbf{X}_1, \dots, \mathbf{X}_{|\mathcal{X}|}\}$
- Channel: IID Rayleigh-fading:
  - $h_n \sim \mathcal{CN}(0, 1)$ ;  $n \in \{1, 2, \dots, N_T\}$
  - AWGN  $n \sim \mathcal{CN}(0, \sigma_n^2)$

Problem: How should we design codebook  $\mathcal{X}$ ?

- Need to derive error rate for general codebooks  $\mathcal{X}$ !
  - Codeword error rate

$$P_e = \frac{1}{|\mathcal{X}|} \sum_{i=1}^{|\mathcal{X}|} \Pr\{\mathbf{x}_i \neq \hat{\mathbf{x}}_i\}$$

where  $\hat{\mathbf{x}}_i$  is the detected codeword and we assume that all codewords are equally likely

Problem:  $\Pr\{\mathbf{x}_i \neq \hat{\mathbf{x}}_i\}$  is not tractable in general

- Use union bound to upper bound  $\Pr\{\mathbf{x}_i \neq \hat{\mathbf{x}}_i\}$  as upper sum over pairwise error probabilities (PEP)  $\Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\}$  where it is assumed that  $\mathbf{x}_i$  was transmitted and  $\mathbf{x}_i$  and  $\mathbf{x}_j$  are the only codewords in the codebook

$$P_e \leq \frac{1}{|\mathcal{X}|} \sum_{i=1}^{|\mathcal{X}|} \sum_{j=1}^{|\mathcal{X}|} \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} \text{ where } j \neq i$$

Calculation of PEPs

Recall:  $\hat{\mathbf{x}} = \underset{\mathbf{x} \in \mathcal{X}}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{x}\mathbf{h}\|^2$

Now,  $\mathbf{x}_i$  and  $\mathbf{x}_j$  are the only alternatives and an error is made if  $\|\mathbf{y} - \mathbf{x}_i\mathbf{h}\|^2 > \|\mathbf{y} - \mathbf{x}_j\mathbf{h}\|^2$  since  $\mathbf{x}_i$  was sent but  $\mathbf{x}_j$  was detected

$$\begin{aligned} \rightarrow \|\mathbf{x}_i\mathbf{h} + \mathbf{n} - \mathbf{x}_j\mathbf{h}\|^2 &> \|\mathbf{x}_i\mathbf{h} + \mathbf{n} - \mathbf{x}_j\mathbf{h}\|^2 \\ \|\mathbf{n}\| &> \|(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h} + \mathbf{n}\|^2 \\ \rightarrow \|\mathbf{n}\| &> \underbrace{\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h}}_{\Delta} + \mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{n} + \mathbf{n}^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h} + \|\mathbf{n}\|^2 \\ &\rightarrow \underbrace{-\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)^H\mathbf{n} - \mathbf{n}^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h}}_z > \Delta \end{aligned}$$

for given  $\mathbf{h}$ ,  $z$  is a gaussian random variable

$$\begin{aligned} \sigma_z^2 &= \mathcal{E}\{|z|^2\} = \mathcal{E}\{2\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j) \underbrace{\mathbf{n}\mathbf{n}^H}_{\sigma_n^2 \mathbf{I}}(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h} + 2\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)^H \underbrace{\mathbf{n}\mathbf{n}^T}_{=0}(\mathbf{x}_i - \mathbf{x}_j)^*\mathbf{h}^*\} \\ &= 2\sigma_n^2 \Delta + 0 \end{aligned}$$

$$\begin{aligned} \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} &= \int_{\Delta}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{z^2}{2\sigma_z^2}\right) dz, \quad t = \frac{z}{\sigma_z} \\ &= \frac{1}{\sqrt{2\pi}} \int_{\frac{\Delta}{\sigma_z}}^{\infty} e^{-\frac{t^2}{2}} dt = Q\left(\frac{\Delta}{\sigma_z}\right) = Q\left(\frac{\Delta}{\sqrt{2\sigma_n^2 \Delta}}\right) \\ &= Q\left(\sqrt{\frac{\Delta}{2\sigma_n^2}}\right) \end{aligned}$$

- $\Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} = \mathcal{E}\left\{Q\left(\sqrt{\frac{\Delta}{2\sigma_n^2}}\right)\right\}$   
 – to avoid cumbersome Q-function we use Chernoff bound:

$$\boxed{Q(x) \leq \frac{1}{2}e^{-\frac{x^2}{2}}}$$

$$\begin{aligned} \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} &\leq \frac{1}{2} \mathcal{E}_h \left\{ \exp\left(-\frac{\mathbf{h}^H \mathbf{Q} \mathbf{h}}{4\sigma_n^2}\right) \right\} \\ \text{where } \mathbf{Q} &= (\mathbf{x}_i - \mathbf{x}_j)^H(\mathbf{x}_i - \mathbf{x}_j) \end{aligned}$$

- Eigendecomposition:  $\mathbf{Q} = \mathbf{U}^H \mathbf{\Lambda} \mathbf{U}$  with  $\mathbf{\Lambda} = \operatorname{diag}\{\lambda_1, \dots, \lambda_r, 0, \dots, 0\}$   $r = \operatorname{rank}\{\mathbf{Q}\}$
- Elements  $\mathbf{h}$  are i.i.d. Gaussian

–  $\underline{\beta} = \mathbf{U}\mathbf{h}$  has also i.i.d. Gaussian random variables as elements since  $\mathbf{U}$  is unitary matrix

–  $\mathbf{h}^H \mathbf{Q} \mathbf{h} = \underbrace{\mathbf{h}^H \mathbf{U}^H}_{\underline{\beta}^*} \underbrace{\mathbf{\Lambda} \mathbf{U} \mathbf{h}}_{\underline{\beta}} = \sum_{i=1}^r \lambda_i |\beta_i|^2$  with  $\underline{\beta} = [\beta_1, \dots, \beta_{N_T}]$

$$\begin{aligned} \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} &= \frac{1}{2} \mathcal{E}_{\underline{\beta}} \left\{ \exp \left( -\frac{\sum_{i=1}^r \lambda_i |\beta_i|^2}{4\sigma_n^2} \right) \right\} \\ &= \frac{1}{2} \mathcal{E}_{\underline{\beta}} \left\{ \prod_{i=1}^r e^{-\frac{\lambda_i}{4\sigma_n^2} |\beta_i|^2} \right\} \\ &= \frac{1}{2} \prod_{i=1}^r \mathcal{E}_{\beta_i} \left\{ e^{-\frac{\lambda_i}{4\sigma_n^2} |\beta_i|^2} \right\} \\ &= \frac{1}{2} \prod_{i=1}^r \mathcal{E}_{|\beta_i|^2} \left\{ e^{-\frac{\lambda_i}{4\sigma_n^2} |\beta_i|^2} \right\} \triangleq \text{MGF of exponentially distributed variable } \alpha_i = |\beta_i|^2 \end{aligned}$$

$$\rightarrow P_{\alpha_i}(x) = e^{-x}, \quad x \geq 0$$

$$\begin{aligned} \rightarrow \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} &\leq \frac{1}{2} \prod_{i=1}^r \frac{1}{1 + \frac{\lambda_i}{4\sigma_n^2}} \\ &\leq \prod_{i=1}^r \frac{1}{\frac{\lambda_i}{4\sigma_n^2}} = 2^{2r-1} \frac{1}{\prod_{i=1}^r \lambda_i} \left( \underbrace{\frac{1}{\sigma_n^2}}_{\triangleq SNR} \right)^{-r} \end{aligned}$$

- upper bound on  $P_e$ :

$$\begin{aligned} \lambda_n(i, j) &= n\text{th eigenvalue of } (\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j) \\ r(i, j) &= \text{rank of } (\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j) \end{aligned}$$

$$\rightarrow P_e \leq \frac{1}{|\mathcal{X}|} \sum_{i=1}^{|\mathcal{X}|} \sum_{j=1}^{|\mathcal{X}|} 2^{2r(i,j)-1} \frac{1}{\prod_{n=1}^{r(i,j)} \lambda_n(i, j)} \left( \frac{1}{\sigma_n^2} \right)^{-r(i,j)}$$

- generally loose bound but offers significant insight for code design

Two criteria:

**Rank criterion:** The diversity gain of a ST code is given by

$$G_d = \min_{i,j} (r(i,j)) = \min_{i,j} \text{rank}((\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j))$$

→ Design code such that minimum rank of all possible matrices  $(\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j)$  is maximized

$$T \overset{N_T}{\updownarrow} \mathbf{X}_i \Rightarrow r(i,j) = N_T \quad \forall i \neq j$$

**Determinant criterion:** To maximize the coding gain among all codes with  $r(i,j) = N_T$ ,

$$\text{we need to maximize } \max_{i,j} \min_{n=1}^{N_T} \lambda_n(i,j) = \max_{i,j} \min_{i,j} |(\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j)| \quad \forall i \neq j$$

- Rank and determinant criterion can be used for the search for good space-time block codes and space-time trellis codes. These two criteria were first derived by Tarokh, et. al. 1998.
- diversity increases to  $N_T N_R$  if  $N_K$  receive antennas are available
- Example: see B  ro, Bauch, Hansmann: Improved codes for space-time trellis coded modulation. IEEE Comm. Letters, 2000.

### 2.3.4 Partial or Imperfect CSI at the Transmitter

- In practice, the CSI cannot be perfect. Channel estimation, quantization and noisy feedback channels introduce errors.
- If the system is optimized for perfect CSI (*e.g.* using MRT or EGT), the performance for imperfect CSI may be worse than for a system designed for no CSI (*e.g.* space-time coding)
- In this case, it is advantageous to use a hybrid approach and combine beamforming and space-time coding.

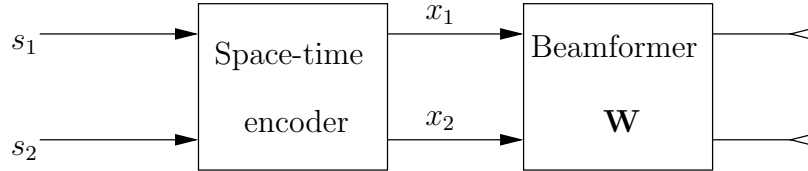


Figure 6: Block Diagramm of MISO with Beamforming

- $\mathbf{W}$  is the beamforming matrix which depends on the reliability of the CSI
- CSI is modeled as

$$\hat{h}_i = \rho h_i + \sqrt{1 - \rho^2} e_i$$

where:

- $\hat{h}_i$  is the CSI estimate
- $\rho$  is the correlation between  $\hat{h}_i$  and  $h_i$

- $e_i$  is the CSI error modeled as AWGN

extreme cases:

- $\rho = 0$  :  $\hat{h}_i$  independent of  $h_i \rightarrow$  no CSI ( $\mathbf{W} = \mathbf{I}$ )
- $\rho = 1$  :  $\hat{h}_i = h_i \rightarrow$  perfect CSI ( $\mathbf{W}$  performs MRT)
- $\mathbf{W}$  can be optimized under the assumptions for given  $\rho$  and  $\hat{h}_i$   
 $\rightarrow$  see for details: Jöngren, Skoglund and Ottersten: "Combining Beamforming and Orthogonal Space-time Block Coding", IEEE on IT, 2002.

## 2.4 MIMO Systems without CSI at the transmitter

- We consider  $N_T \times N_R$  MIMO system and assume that the channel matrix  $\mathbf{H}$  is not known at the transmitter  
 $\rightarrow$  no CSI at the transmitter (CSIT)
- signal model:

$$N_R \updownarrow \mathbf{y} = N_R \updownarrow \overset{N_T}{\mathbf{H}} \mathbf{x} \updownarrow N_T + \mathbf{n} \updownarrow N_R$$

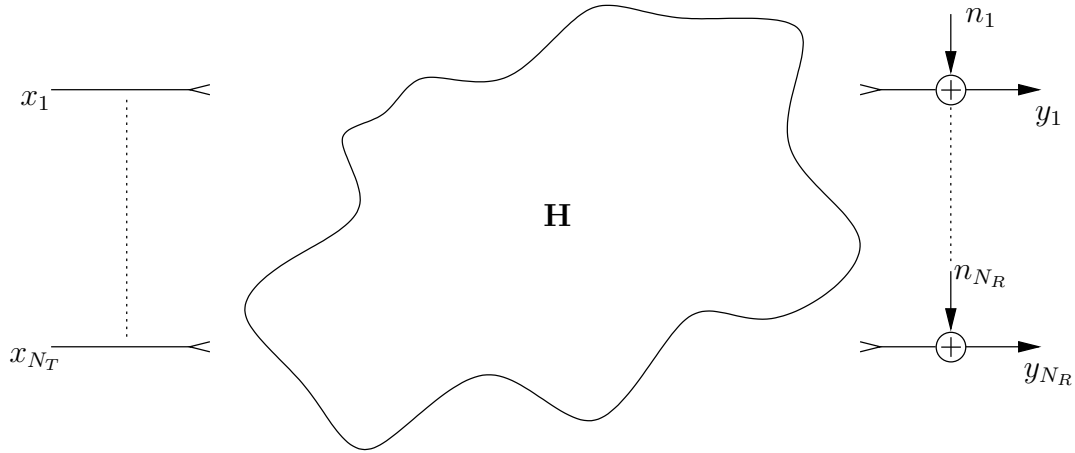


Figure 7: Block Diagramm of MISO without CSI

- $x_n$  are  $M$ -ary i.i.d. scalar symbols taken *e.g* from an  $M$ -PSK or  $M$ -QAM symbol alphabet  $\mathcal{A}$
- This scheme is often called "spatial multiplexing"
- We transmit  $N_T$  symbols per symbol interval  
 $\rightarrow$  rate  $R = \log_2(M) \cdot N_T$  for uncoded transmission
- Problem: How to detect  $\mathbf{x}$  at the receiver considering
  - performance and
  - complexity?

### 2.4.1 Optimum Detection

- Elements of  $\mathbf{n}$  are gaussian random variables with variance  $\sigma_n^2$
- $\mathbf{H}$  is known at the receiver

$$\begin{aligned} p(\mathbf{y}|\mathbf{x}) &= \frac{1}{\pi^{N_R} \sigma_n^2 \mathbf{I}_{N_R \times N_R}} \exp \left( -(\mathbf{y} - \mathbf{H}\mathbf{x})^H (\sigma_n^2 \mathbf{I}_{N_R \times N_R})^{-1} (\mathbf{y} - \mathbf{H}\mathbf{x}) \right) \\ &= \frac{1}{\pi^{N_R} \sigma_n^2} \exp \left( -\frac{1}{\sigma_n^2} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2 \right) \end{aligned}$$

- ML-Detection

$$\hat{x} = \underset{\mathbf{x} \in \mathcal{A}^{N_T}}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2 = \underset{\mathbf{x} \in \mathcal{A}^{N_T}}{\operatorname{argmax}} p(\mathbf{y}|\mathbf{x})$$

→  $M^{N_T}$  metric calculations → complexity is exponential in  $N_T$ !!

→ in general too complex in practice

- Performance
  - consider worst case pairwise error probability (PEP) to evaluate diversity gain
  - PEP →  $x_i$  is transmitted but  $x_j \neq x_i$  is detected  
this happens if  $\|\mathbf{y} - \mathbf{H}\mathbf{x}_i\|^2 > \|\mathbf{y} - \mathbf{H}\mathbf{x}_j\|^2$   
→  $\|\mathbf{n}\|^2 > \|\mathbf{H}(\mathbf{x}_i - \mathbf{x}_j) + \mathbf{n}\|^2$
  - the “worst case” is if  $\mathbf{x}_i$  &  $\mathbf{x}_j$  differ only in one element *i.e.*,

$$\mathbf{x}_i - \mathbf{x}_j = (x_{ni} - x_{nj}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{“1” in position } n$$

where  $\mathbf{x}_i = [x_{1i}, x_{2i}, \dots, x_{N_T i}]$

- $\|\mathbf{n}\|^2 > \left\| \underbrace{\mathbf{h}_n}_{\text{nth column of } \mathbf{H}} \underbrace{(x_{ni} - x_{nj})}_{\Delta x_n(i,j)} + \mathbf{n} \right\|^2$
- $\|\mathbf{n}\|^2 > \mathbf{h}_n^H \mathbf{n} \Delta x_n^*(i,j) + \mathbf{n}^H \mathbf{h}_n \Delta x_n(i,j) + \|\mathbf{n}\|^2 + \|\mathbf{h}_n\|^2 - |\Delta x_n(i,j)|^2$   

$$\|\mathbf{h}_n\|^2 |\Delta x_n(i,j)|^2 < \underbrace{-\mathbf{h}_n^H \mathbf{n} \Delta x_n(i,j) - \mathbf{n}^H \mathbf{h}_n \Delta x_n(i,j)}_{\text{Gaussian random variable with variance } \sigma_{eq}^2 = 2\sigma_n^2 |\Delta x_n(i,j)|^2 \|\mathbf{h}_n\|^2}$$
- $\Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j | \mathbf{H}\} = Q \left( \sqrt{\frac{\|\mathbf{h}_n\|^2 |\Delta x_n(i,j)|^2}{2\sigma_n^2}} \right)$



- $\Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} = \mathcal{E} \left\{ Q \left( \sqrt{\frac{\|\mathbf{h}_n\|^2 |\Delta x_n(i,j)|^2}{2\sigma_n^2}} \right) \right\}$   
 → use same approach as for space-time code design to get diversity order  
 or : SNR is

$$\gamma_t = \frac{\|\mathbf{h}_n\|^2 |\Delta x_n(i,j)|^2}{2\sigma_n^2} = \frac{|\Delta x_n(i,j)|^2}{2\sigma_n^2} (|h_{1n}|^2 + |h_{2n}|^2 + \dots + |h_{N_R n}|^2)$$

- same form as SNR of MRC with  $N_R$  receive antennas
- diversity gain of spatial multiplexing with ML-decoding is

$$G_d = N_R$$

- diversity of  $N_T$  transmit antennas is not exploited with spatial multiplexing
- to exploit this additional gain, coding across space is required (at the expense of rate)  
 (Hier gehören die detection performance kurven für BPSK hin)

## 2.4.2 Linear Receivers

- How can we avoid the complexity associated with the joint detection of the elements of  $\mathbf{x}$ ?
- Idea: Employ linear filter (matrix) to separate the elements of  $\mathbf{x}$
- Requires:  $N_T \leq N_R$
- We form

$$\mathbf{r} = N_T \overset{N_R}{\updownarrow} \mathbf{F} \mathbf{y} = [r_1, \dots, r_{N_T}]^T$$

where  $\mathbf{F}$  is the filter matrix and  $\mathbf{y}$  is the received vector

such that  $x_n$  can be obtained from

$$\hat{x}_n = \underset{x_n \in \mathcal{A}}{\operatorname{argmin}} |r_i - x_n|^2 \quad \text{where } \mathbf{F} \in \mathbb{C}^{N_T \times N_R}$$

- Two popular design criteria for  $\mathbf{F}$ 
  - Zero-forcing (ZF) criterion
  - minimum mean squared error (MMSE) criterion

## ZF Detection

$$\mathbf{r} = \mathbf{F}\mathbf{y} = \mathbf{F}(\mathbf{H}\mathbf{x} + \mathbf{n}) = \mathbf{F}\mathbf{H}\mathbf{x} + \mathbf{F}\mathbf{n}$$

ZF  $\leftrightarrow$  we require  $\mathbf{F}\mathbf{H} = \mathbf{I}_{N_T \times N_T}$

- noise covariance matrix

$$\Phi_{ee} = \mathcal{E}\{\mathbf{F}\mathbf{n}(\mathbf{F}\mathbf{n})^H\} = \sigma_n^2 \mathbf{F}\mathbf{F}^H$$

- $N_T = N_R \rightarrow \mathbf{F}\mathbf{H} = \mathbf{I}_{N_T \times N_T} \rightarrow \mathbf{F} = \mathbf{H}^{-1}$  assuming  $\mathbf{H}$  is invertible
- $\rightarrow N_T \leq N_R \rightarrow$  which one of the many  $\mathbf{F}$  that yield  $\mathbf{F}\mathbf{H} = \mathbf{I}_{N_T \times N_T}$ ?
- choose  $\mathbf{F}$  that leads to the smallest noise enhancement
- optimal  $\mathbf{F}$  is the solution to the following problem:

$$\min_{\mathbf{F}} \text{tr}\{\sigma_n^2 \mathbf{F}\mathbf{F}^H\}$$

$$\text{s.t } \mathbf{F}\mathbf{H} = \mathbf{I}_{N_T \times N_T}$$

$$\text{the constraint is equivalent to } \text{tr}\{(\mathbf{F}\mathbf{H} - \mathbf{I})(\mathbf{F}\mathbf{H} - \mathbf{I})^H\} = 0$$

Lagrangian:

$$\begin{aligned} L(\mathbf{F}) &= \text{tr}\{\sigma_n^2 \mathbf{F}\mathbf{F}^H\} + \lambda \text{tr}\{\mathbf{F}\mathbf{H}\mathbf{H}^H \mathbf{F} - \mathbf{F}\mathbf{H} - \mathbf{H}^H \mathbf{F}^H + \mathbf{I}\} \\ &= \sigma_n^2 \text{tr}\{\mathbf{F}\mathbf{F}^H\} + \lambda \text{tr}\{\mathbf{F}\mathbf{H}\mathbf{H}^H \mathbf{F}^H\} - \lambda \text{tr}\{\mathbf{F}\mathbf{H}\} - \lambda \text{tr}\{\mathbf{H}^H \mathbf{F}^H\} + \lambda N_T \end{aligned}$$

- use rules for complex matrix differentiation in Table IV in paper by Hjørungnes & Gesbert

$$\begin{aligned} \frac{\delta L(\mathbf{F})}{\delta \mathbf{F}^*} &= \sigma_n^2 \mathbf{F} + \lambda \mathbf{F}\mathbf{H}\mathbf{H}^H - \lambda \mathbf{H}^H = 0 \\ &\rightarrow \mathbf{F}(\sigma_n^2 \mathbf{I} + \lambda \mathbf{H}\mathbf{H}^H) = \lambda \mathbf{H}^H \\ &\rightarrow \mathbf{F} = \lambda \mathbf{H}^H (\sigma_n^2 \mathbf{I} + \lambda \mathbf{H}\mathbf{H}^H)^{-1} \end{aligned}$$

use matrix inversion lemma

$$(\mathbf{A} + \mathbf{U}\mathbf{B}\mathbf{V})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{B}^{-1} + \mathbf{V}\mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}\mathbf{A}^{-1}$$

$$\begin{aligned} \rightarrow \mathbf{F} &= \lambda \mathbf{H}^H \left[ \frac{1}{\sigma_n^2} \mathbf{I} - \frac{1}{\sigma_n^2} \mathbf{H} \left[ \frac{1}{\lambda} \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \right]^{-1} \mathbf{H}^H \frac{1}{\sigma_n^2} \right] \\ &= \frac{\lambda}{\sigma_n^2} \left[ \begin{array}{c} \mathbf{I} \\ \left( \left( \frac{1}{\lambda} \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \right) \left( \frac{1}{\lambda} \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \right)^{-1} - \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \left[ \frac{1}{\lambda} \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \right] \right) \mathbf{H}^H \end{array} \right] \mathbf{H}^H \\ &= \frac{\lambda}{\sigma_n^2} \left[ \frac{1}{\lambda} \mathbf{I} + \frac{1}{\lambda} \mathbf{H}^H \mathbf{H} - \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \right] \left( \frac{1}{\lambda} \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \right)^{-1} \mathbf{H}^H \\ &= \frac{1}{\sigma_n^2} \left( \frac{1}{\lambda} \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \right) \mathbf{H}^H \end{aligned}$$

- How to choose  $\lambda$

$$\begin{aligned} \mathbf{F}\mathbf{H} &= \frac{1}{\sigma_n^2} \left( \frac{1}{\lambda} \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \right)^{-1} \mathbf{H}^H \mathbf{H} = \mathbf{I} \\ \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} &= \frac{1}{\lambda} \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{H} \\ \Rightarrow \lambda &\rightarrow \infty \end{aligned}$$

$$\Rightarrow \boxed{\mathbf{F} = (\mathbf{H}^H \mathbf{H})^{-1}} \hat{=} \text{Moore-Penrose pseudoinverse}$$

noise covariance:

$$\Phi_{ee} = \sigma_n^2 \mathbf{F} \mathbf{F}^H = \sigma_n^2 (\mathbf{H}^H \mathbf{H})^{-1} \underbrace{\mathbf{H}^H \mathbf{H} (\mathbf{H} \mathbf{H})^{-1}}_{\mathbf{I}} = \sigma_n^2 (\mathbf{H}^H \mathbf{H})^{-1}$$

$\Phi_{ee}$  is not in general a diagonal matrix

- effective noise  $\mathbf{F} \mathbf{n}$  is spatially correlated
- "equalization of channel leads to coloring of noise"

- Interpretation:  
we have

$$\mathbf{F} \mathbf{H} \mathbf{x} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_{N_T} \end{bmatrix} \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \dots & \mathbf{h}_{N_T} \end{bmatrix} \mathbf{x} = \mathbf{x}$$

- $\mathbf{f}_i \mathbf{h}_i = 1 \quad \mathbf{f}_i \mathbf{h}_j = 0 \quad \forall i \neq j$
- $\mathbf{f}_i^T$  is orthogonal to  $[\mathbf{h}_1 \quad \dots \quad \mathbf{h}_{i-1} \quad \mathbf{h}_{i+1} \quad \dots \quad \mathbf{h}_{N_T}] \uparrow N_R$
- $\mathbf{f}_i^T$  is confined to an  $N_R - (N_T - 1)$  dimensional subspace of the  $N_R$  dimensional space spanned by  $\mathbf{H}$

- Diversity gain

- e.g. SISO model:  $r_i = \mathbf{f}_i \cdot \mathbf{h}_i \mathbf{x}_i + \mathbf{f}_i \cdot \mathbf{n}_i$

$$\rightarrow \text{SNR}_{\text{eq}} = \frac{\epsilon_s |\mathbf{f}_i \mathbf{h}_i|^2}{\sigma_n^2 \|\mathbf{f}_i\|^2} = \frac{\epsilon_s}{\sigma_n^2} \left| \tilde{\mathbf{f}}_i \cdot \mathbf{h}_i \right|, \quad \text{where } \mathbf{f}_i = \alpha \tilde{\mathbf{f}}_i \text{ with } \|\tilde{\mathbf{f}}_i\|^2 = 1$$

we can represent  $\mathbf{f}_i$  as:  $\tilde{\mathbf{f}}_i^T = \alpha \mathbf{M} \boldsymbol{\beta}$ ,

where:  $\mathbf{M} \in \mathbb{C}^{N_R \times (N_R - N_T + 1)}$  and  $\boldsymbol{\beta} \in \mathbb{C}^{(N_R - N_T + 1) \times 1} \hat{=} \text{basis of subspace}$

$$\rightarrow \tilde{\mathbf{f}}_i \mathbf{M}_i = \boldsymbol{\beta}^T \mathbf{M}_i^T \mathbf{M}_i,$$

where:  $\mathbf{M}^H \mathbf{M} = \tilde{\mathbf{M}} = \mathbf{I}$  and  $\tilde{\mathbf{M}}_i \rightarrow \mathcal{CN}(0, \sigma_n^2 \mathbf{I}_{(N_R - N_T + 1)})$

(since rows of  $\mathbf{M}^T$  are orthogonal)

$$\text{SNR}_{\text{eq}} = \frac{\epsilon_s}{\sigma_n^2} \alpha^2 \left| \sum_{j=1}^{N_R - N_T + 1} \beta_{ji} \tilde{\mathbf{h}}_j \right|^2; \tilde{\mathbf{h}}_i = (\tilde{h}_{1i}, \quad \tilde{h}_{2i}, \quad \dots, \quad \tilde{h}_{N_R - N_T + 1, i})^T$$

$$\boldsymbol{\beta}_i = (\beta_{1i}, \quad \dots, \quad \beta_{N_R - N_T + 1, i})^T$$

$\rightarrow \text{SNR}_{\text{eq}}$  includes only  $N_R - N_T + 1$  independent Gaussian RV

$\rightarrow$  diversity gain is limited to:  $G_d = N_R - N_T + 1$

Example:

$$N_T = N_R = 3$$

$$G_d^{ZF} = 1 \text{ but } G_d^{ML} = N_R = 3$$

→ huge performance loss because of linear ZF

### MMSE detection

- ZF criterion may be too strict and leads to noise enhancement
  - maybe it is better to allow some interferences between signals but reduce noise enhancement
  - What is the optimal trade-off between interference and noise?
  - MMSE criterion
- MMSE criterion
  - error signal:  $\mathbf{e} = \mathbf{F}\mathbf{y} - \mathbf{x}$
  - total error variance:  $\sigma_e^2 = \mathcal{E}\{\|\mathbf{e}\|^2\} = \mathcal{E}\{\text{tr}\{\mathbf{e}\mathbf{e}^H\}\} = \text{tr}\{\mathcal{E}\{\mathbf{e}\mathbf{e}^H\}\} = \text{tr}\{\Phi_{ee}\}$
  - $\Phi_{ee}$ : error covariance matrix
  - optimal filter:  $\mathbf{F}_{\text{opt}} = \underset{\mathbf{F}}{\text{argmin}} \text{tr}\{\Phi_{ee}\}$

#### • Deviation of $\mathbf{F}_{\text{opt}}$

$$- \Phi_{ee} = \mathcal{E}\{\mathbf{e}\mathbf{e}^H\} = \mathcal{E}\{(\mathbf{F}\mathbf{y} - \mathbf{x})(\mathbf{F}\mathbf{y} - \mathbf{x})^H\} = \mathbf{F} \cdot \Phi_{yy} \cdot \mathbf{F}^H - \mathbf{F} \cdot \Phi_{yx} - \Phi_{xx} \cdot \Phi_{xy} \cdot \mathbf{F}^H + \Phi_{xx}$$

with:

$$\Phi_{yy} = \mathcal{E}\{\mathbf{y}\mathbf{y}^H\} = \mathcal{E}\{(\mathbf{H}\mathbf{x} + \mathbf{n})(\mathbf{H}\mathbf{x} + \mathbf{n})^H\} = \epsilon_s \cdot \mathbf{H}\mathbf{H}^H + \sigma_n^2 \cdot \mathbf{I}_{N_R \times N_T}$$

$$\Phi_{yx} = \mathcal{E}\{\mathbf{y}\mathbf{x}^H\} = \mathcal{E}\{(\mathbf{H}\mathbf{x} + \mathbf{n}) \cdot \mathbf{x}^H\} = \epsilon_s \cdot \mathbf{H}^H = \Phi_{xy}^H$$

$$\Phi_{xx} = \epsilon_s \cdot \mathbf{I}_{N_T \times N_R}$$

$$- \mathbf{F}_{\text{opt}} \rightarrow \frac{d}{d\mathbf{F}^*} \left( \text{tr}\{\mathbf{F}\Phi_{yy}\mathbf{F}^H\} - \text{tr}\{\mathbf{F}\Phi_{yx}\} - \text{tr}\{\Phi_{xy}\mathbf{F}^H\} + \text{tr}\{\Phi_{xx}\} \right) \stackrel{!}{=} 0$$

with Table IV in paper by Hjørnager & Gesbert:

$$\Rightarrow \mathbf{F} \cdot \Phi_{yy} - \Phi_{xy} = 0$$

$$\Rightarrow \mathbf{F}_{\text{opt}} = \Phi_{xy} \cdot \Phi_{yy}^{-1} = \epsilon_s \mathbf{H}^H (\epsilon_s \mathbf{H}\mathbf{H}^H + \sigma_n^2 \mathbf{I})^{-1}$$

$$= (\text{Matrix inversion Lemma}) =$$

$$= \left( \mathbf{H}^H \mathbf{H} + \frac{\sigma_n^2}{\epsilon_s} \mathbf{I} \right)^{-1} \cdot \mathbf{H}^H$$

– Comparison:

$$\mathbf{F}_{\text{MMSE}} = \left( \mathbf{H}^H \mathbf{H} + \frac{\sigma_n^2}{\epsilon_s} \mathbf{I} \right)^{-1} \cdot \mathbf{H}^H \xrightarrow[\frac{\sigma_n^2}{\epsilon_s} \rightarrow 0]{} (\mathbf{H}^H \cdot \mathbf{H})^{-1} \mathbf{H}^H = \mathbf{F}_{ZF}$$

$$\xrightarrow[\frac{\sigma_n^2}{\epsilon_s} \rightarrow \infty]{\frac{\epsilon_s}{\sigma_n^2}} \frac{\epsilon_s}{\sigma_n^2} \cdot \mathbf{H}^H = \mathbf{F}_{MF} \hat{=} \text{matched filter}$$

⇒ For high SNR,  $\frac{\epsilon_s}{\sigma_n^2}$ , the MMSE filter approaches the ZF-Filter, for low SNR, it approaches the matched filter.

→ MMSE receiver yields the same diversity gain as the ZF receiver

$$G_d^{MMSE} = G_d^{ZF} = N_R - N_T + 1 \leq G_d^{ML} = N_R$$

– End-to-End Channel:  $\mathbf{K} = \mathbf{F}\mathbf{H} = (\mathbf{H}^H\mathbf{H} + \frac{\sigma_n^2}{\epsilon_s} \cdot \mathbf{I})^{-1} \cdot \mathbf{H}^H \cdot \mathbf{H} \neq$  diagonal matrix

⇒ crosstalk/interference between elements  $\mathbf{x}$  in received signal after filtering  $\mathbf{r}$ .

elements of  $\mathbf{K} : K_{l,n}$

– Covariance for  $\mathbf{F}_{\text{opt}}$

$$\begin{aligned}\Phi_{ee} &= \Phi_{xy} \cdot \overbrace{\Phi_{yy}^{-1} \cdot \Phi_{yy} \cdot \Phi_{yy}^{-1}}^{\Phi_{yy}^{-1}} \cdot \Phi_{xy}^H - \Phi_{xy} \cdot \Phi_{yy}^{-1} \cdot \Phi_{yx} - \Phi_{xy} \cdot \Phi_{yy}^{-1} \cdot \Phi_{xy}^H + \Phi_{xx} \\ &= \Phi_{xx} - \overbrace{\Phi_{xy} \cdot \Phi_{yy}^{-1} \cdot \Phi_{yx}}^{\mathbf{F}_{\text{opt}}} \\ &= \epsilon_s \mathbf{I} - \epsilon_s (\mathbf{H}^H \mathbf{H} + \frac{\sigma_n^2}{\epsilon_s} \cdot \mathbf{I})^{-1} \cdot \mathbf{H}^H \mathbf{H} = (\text{Matrix inversion Lemma}) \\ &= \sigma_n^2 (\mathbf{H}^H \mathbf{H} + \frac{\sigma_n^2}{\epsilon_s} \cdot \mathbf{I})^{-1} \\ \Phi_{ee} &= \epsilon_s (\mathbf{I} - \mathbf{K})\end{aligned}$$

→  $0 \leq K_{m,m} \leq 1$  since main diagonal elements of  $\Phi_{ee}$  are  $0 \leq [\Phi_{ee}]_{m,m} \leq \epsilon_s$

siehe auch Abbildung ??

## SNR (biased vs. unbiased)

### a) biased SNR

$$\text{SNR}_{\text{bias},m} = \frac{\epsilon_s}{[\Phi_{ee}]_{mm}} = \frac{\epsilon_s}{\epsilon_s(1 - K_{mm})} = \frac{1}{1 - K_{mm}}, \quad 1 \leq m \leq 4$$

Anmerkung:  $\mathbf{K} = \mathbf{F}_{\text{opt}} \cdot \mathbf{H} \rightarrow \text{SNR} = 1$  falls  $\mathbf{K} = \text{zeros}() \Rightarrow$  woher  $\text{SNR} = 1$  bei keiner Uebertragung?  $\Rightarrow$  nicht vorteilhaft: siehe: b) unbiased SNR but:

- $\text{SNR}_{\text{bias},m}$  does not represent the actual SNR since the main diagonal elements of  $\mathbf{K}$  are smaller than 1

- $\mathbf{r} = \mathbf{F}\mathbf{H}\mathbf{x} + \mathbf{F}\mathbf{n} = \mathbf{K}\mathbf{x} + \underbrace{\mathbf{F}\mathbf{n}}_{\tilde{\mathbf{n}}=[\tilde{n}_1, \dots, \tilde{n}_{N_T}]^T}$

- $r_m = \underbrace{K_{mm}}_{<1} x_m + \sum_{\substack{n=1 \\ n \neq m}}^{N_T} K_{mn} x_n + \tilde{n}_m$

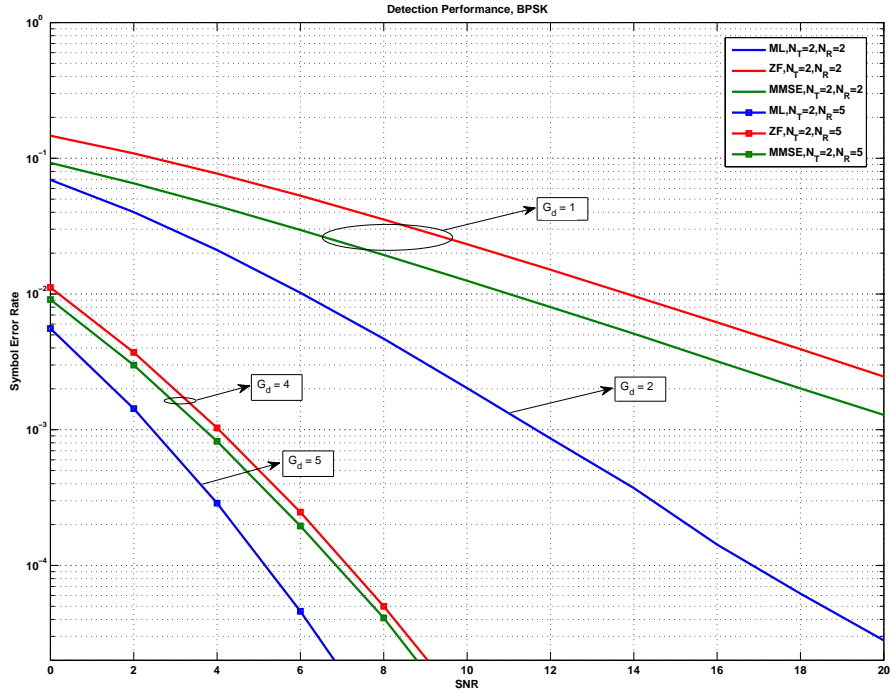


Figure 8: MMSE

### b) unbiased SNR

- remove bias:  $r'_m = \frac{r_m}{K_{mm}} = x_m + \underbrace{\frac{\tilde{e}_m}{K_{mm}}}_{e'_m}$

- SNR?

- scaling matrix:  $\mathbf{C} = \text{diag}\left\{\frac{1}{K_{11}}, \frac{1}{K_{22}}, \dots, \frac{1}{K_{N_T N_T}}\right\}$

- $\mathbf{r}' = \mathbf{C}\mathbf{r} \rightarrow \mathbf{e}' = [e'_1, \dots, e'_{N_T}]^T = \mathbf{r}' - \mathbf{x} = \mathbf{C}\mathbf{r} - \mathbf{x} = \mathbf{C}\mathbf{F}\mathbf{y} - \mathbf{x}$

- $\Phi_{e'e'} = \mathcal{E}\left\{(\mathbf{C}\mathbf{F}\mathbf{y} - \mathbf{x})(\mathbf{C}\mathbf{F}\mathbf{y} - \mathbf{x})^H\right\} = \mathbf{C}\mathbf{F}\Phi_{yy}\mathbf{F}^H\mathbf{C}^H - \mathbf{C}\mathbf{F}\Phi_{yx} - \Phi_{xy}\mathbf{F}^H\mathbf{C}^H + \Phi_{xx}$

- $\mathbf{F} = \mathbf{F}_{\text{opt}} \rightarrow \mathbf{F}_{\text{opt}}\Phi_{yy} = \Phi_{xy} = \epsilon_s \cdot \mathbf{H}^H$

$$\begin{aligned} \rightarrow \Phi_{e'e'} &= \epsilon_s \underbrace{\mathbf{C}\mathbf{H}^H\mathbf{F}^H}_{\mathbf{K}^H} \mathbf{C}^H - \epsilon_s \underbrace{\mathbf{C}\mathbf{F}\mathbf{H}}_{\mathbf{K}} - \epsilon_s \underbrace{\mathbf{H}^H\mathbf{F}^H}_{\mathbf{K}^H} \mathbf{C} + \epsilon_s \mathbf{I} \\ &= \epsilon_s [\mathbf{I} + (\mathbf{C} - \mathbf{I})\mathbf{K}^H\mathbf{C}^H - \mathbf{C}\mathbf{K}] \end{aligned}$$

Anmerkung: nur Hauptdiagonalelemente interessieren, da diese die Varianz darstellen

$$\begin{aligned} \rightarrow \text{maindiagonal elements of } \Phi_{e'e'} &= \text{variances of } e'_m = \epsilon_s \left(1 + \left(\frac{1}{K_{mm}} - 1\right)K_{mm}\frac{1}{K_{mm}} - \frac{1}{K_{mm}}K_{mm}\right) = \epsilon_s \frac{1-K_{mm}}{K_{mm}} \end{aligned}$$

→ vgl. Abbildung ??

$$\rightarrow SNR_{unbiased} = \frac{E_s}{[\Phi_{e'e'}]_{mm}} = \frac{E_s}{E_s \frac{1 - K_{mm}}{K_{mm}}} = \frac{K_{mm}}{1 - K_{mm}}, \quad 1 \leq m \leq N_T$$

→ the SNR after bias removal is by “1” smaller than the biased SNR → general result valid for any type of MMSE estimation

### 2.4.3 Decision - Feedback Equalization (Detection)

- Also known as:
  - BLAST (Bell Laboratories space-time system)
  - successive interference cancellation
- Problem of linear receiver: Noise enhancement because of linear filtering → nonlinear filtering processing necessary

#### Basic Idea

- Recall (linear filter):  $\mathbf{F}\mathbf{H} = \mathbf{I}$  for linear ZF receiver →  $i$ th row of  $\mathbf{F}$ ,  $\mathbf{f}_i$ , is orthogonal to the  $j$ th column of  $\mathbf{H}$ ,  $\mathbf{h}_j$ , if  $i \neq j$  (if  $i = j$ : inner product = 1)
- we can detect  $x_i$ , based on  $r_i = \mathbf{f}_i \mathbf{y}$
- Once we have detected  $x_i$ , we can subtract its contribution from  $\mathbf{y}$ :  $\mathbf{y}_1 = \mathbf{y} - \mathbf{h}_i \hat{x}_i$  ( $\hat{x}_i$  is detected symbol, we assume for now,  $\hat{x}_i = x_i$ )
- $\mathbf{y}_1$  can be expressed as  $\mathbf{y}_1 = \mathbf{H}_i \mathbf{x}_i + \mathbf{n}$  where

$$\mathbf{H}_i = [\mathbf{h}_1, \dots, \mathbf{h}_{i-1}, \mathbf{h}_{i+1}, \dots, \mathbf{h}_{N_T}]$$

$$\mathbf{x}_i = [x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{N_T}]$$

→ we have reduced the number of signal streams to  $N_T - 1$  ( $N_R$  bleibt gleich)

- apply now linear ZF filter for symbol to detected next, e.g.  $x_j$ , where  $j \in \{1, \dots, i - 1, i + 1, \dots, N_T\}$

→  $\mathbf{r}_j = \mathbf{f}_j \mathbf{y}_1$  where  $\mathbf{f}_j$  is the ZF filter for  $\mathbf{H}_1$

- subtract contribution of  $x_j$  from  $\mathbf{y}_1$ :  $\mathbf{y}_2 = \mathbf{y}_1 - \mathbf{h}_j x_j$
- subtract until last symbol is detected
- Blockdiagram see figure??
- Observations:
  - The order in which the  $x_1$  are detected can be freely chosen and effects the performance →  $N_T!$  possible orders → cannot explore all of them
  - Practical approach: Select in each step that  $x_i$  for which the noise variance enhancement is minimum, i.e. which has the smallest  $\mathcal{E}\{|\mathbf{f}_i \mathbf{n}|^2\} = \sigma_n^2 \|\mathbf{f}_i\|^2$
- Diversity order:

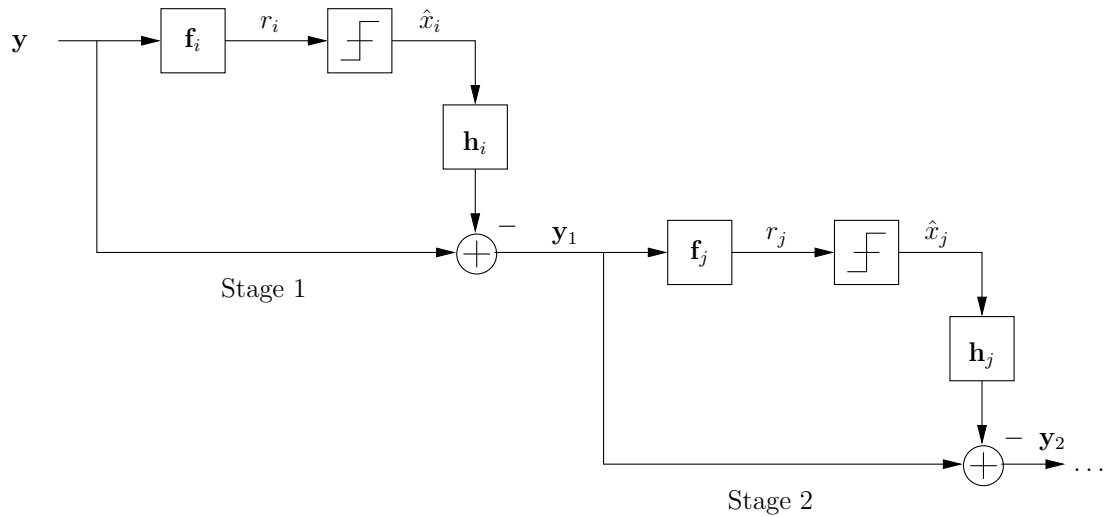


Figure 9: DFE Blockdiagram

- stage 1:  $G_d^1 = N_R - N_T + 1$
- stage 2:  $G_d^2 = G_d^1 + 1 = N_R - N_T + 2$
- $\vdots$
- stage  $N_T$ :  $G_d^{N_T} = N_R$
- overall:  $G_d = N_R - N_T + 1$
- (Anmerkung: der schlechteste Fall dominiert (Stage 1), weitere koennen nur schwer beeinflussen)

### ZF - DFE - Matrix Model

- Signal model:  $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} = \underbrace{\mathbf{H}\mathbf{P}}_{\tilde{\mathbf{H}}} \cdot \underbrace{\mathbf{P}^{-1}\mathbf{x}}_{\tilde{\mathbf{x}}} + \mathbf{n}$  with permutation matrix  $\mathbf{P}$
- $\mathbf{P}$  has one „1“ per column and row, all other elements are „0“
- can change the detection order to maximize performance
- note:  $\mathbf{P}^T = \mathbf{P}^{-1}$
- Example:

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} [r] \rightarrow \mathbf{P}^{-1} = \mathbf{P}^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\tilde{\mathbf{x}} = \mathbf{P}^T \cdot \mathbf{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}$$

- Blockdiagram see figure ??



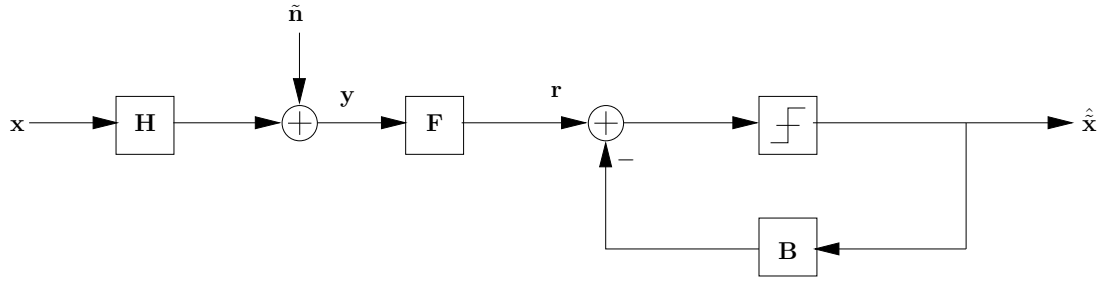


Figure 10: ZF-DFE Matrix Model

- DFE Filters:
  - $\mathbf{F}$  feedforward filter
  - $\mathbf{B}$  feedback filter
- Filter calculation
  - Cholesky factorization:  $\tilde{\mathbf{H}}^H \tilde{\mathbf{H}} = \mathbf{L}^H \mathbf{D} \mathbf{L}$  with diagonal matrix  $\mathbf{D}$  and lower triangular matrix  $\mathbf{L}$  (maindiagonal elements of  $\mathbf{L}$  are “1”)

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ l_{21} & 1 & \ddots & \ddots & \vdots \\ l_{31} & l_{32} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \cdots & 1 \end{bmatrix}$$

$$- \mathbf{F} = \mathbf{D}^{-1} \mathbf{L}^{-H} \tilde{\mathbf{H}}^H$$

$$\rightarrow \mathbf{r} = \mathbf{F} \mathbf{y} = \underbrace{\mathbf{D}^{-1} \mathbf{L}^{-H} \tilde{\mathbf{H}}^H \tilde{\mathbf{H}}}_{\mathbf{L}} \tilde{\mathbf{x}} + \underbrace{\mathbf{D}^{-1} \mathbf{L}^{-H} \tilde{\mathbf{H}}^H}_{\tilde{\mathbf{n}}} \mathbf{n}$$

$$- \mathbf{B} = \mathbf{L} - \mathbf{I} = \text{lower triangular matrix with maindiagonal elements "0"}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ l_{21} & 0 & \ddots & \ddots & \vdots \\ l_{31} & l_{32} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

- Interpretation:

– after feedforward filter

$$\mathbf{r} = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ l_{21} & 1 & \ddots & \ddots & \vdots \\ l_{31} & l_{32} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & \dots & \dots & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_{N_T} \end{bmatrix} + \tilde{\mathbf{n}}$$

• from feedback filter

$$\tilde{\mathbf{r}} = \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ l_{21} & 0 & \ddots & \ddots & \vdots \\ l_{31} & l_{32} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & \dots & \dots & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_{N_T} \end{bmatrix} \quad \hat{x}_n : \text{decision on } \tilde{x}_n$$

$$\rightarrow \mathbf{r} - \tilde{\mathbf{r}} = \begin{bmatrix} \tilde{x}_1 + 0 \\ l_{21}(\tilde{x}_1 - \hat{x}_1) + \tilde{x}_2 + 0 \\ l_{31}(\tilde{x}_1 - \hat{x}_2) + l_{32}(\tilde{x}_2 - \hat{x}_2) + \tilde{x}_3 + 0 \\ \vdots \end{bmatrix} + \tilde{\mathbf{n}}$$

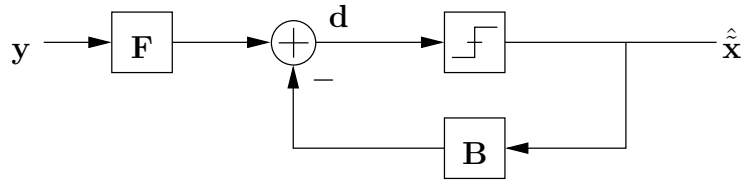


Figure 11: MMSE-DFE Blockdiagram

### MMSE-DFE

- $\mathbf{B} = \mathbf{L} - \mathbf{I}$  has to be lower triangular matrix with all-zero main diagonal elements because of causality.

- $\mathbf{d} = \mathbf{F}\mathbf{y} - \mathbf{B}\hat{\mathbf{x}} \rightarrow \mathbf{e} = \mathbf{d} - \hat{\mathbf{x}} = \mathbf{F}\mathbf{y} - \underbrace{(\mathbf{B} + \mathbf{I})}_{\mathbf{L}}\hat{\mathbf{x}}$

where we assume  $\hat{\hat{\mathbf{x}}} = \hat{\mathbf{x}}$  for filter optimization

- Optimal  $\mathbf{F}$  for given  $\mathbf{L}$

$$\sigma_e^2 = \text{tr} \left\{ \underbrace{\mathcal{E}\{\mathbf{e}\mathbf{e}^H\}}_{\Phi_{ee}} \right\} = \text{tr} \{ \mathbf{F}\Phi_{yy}\mathbf{F}^H \} - \text{tr} \{ \mathbf{F}\Phi_{y\hat{x}}\mathbf{L}^H \} - \text{tr} \{ \mathbf{L}\Phi_{\hat{x}y}\mathbf{F}^H \} + \text{tr} \{ \mathbf{L}\Phi_{\hat{x}\hat{x}}\mathbf{L}^H \}$$

$$\frac{\delta}{\delta \mathbf{F}^*} \sigma_e^2 = \mathbf{F} \Phi_{yy} - \mathbf{L} \Phi_{\tilde{x}y} = 0 \quad \rightarrow \mathbf{F} = \mathbf{L} \Phi_{\tilde{x}y} \Phi_{yy}^{-1}$$

where

$$\begin{aligned} \Phi_{yy} &= \mathcal{E}_s \tilde{\mathbf{H}} \tilde{\mathbf{H}}^H + \sigma_n^2 \mathbf{I} \\ \Phi_{\tilde{x}y} &= \mathcal{E}_s \tilde{\mathbf{H}}^H \end{aligned}$$

$$\rightarrow \mathbf{F} = \mathbf{L} \tilde{\mathbf{H}}^H (\tilde{\mathbf{H}} \tilde{\mathbf{H}}^H + \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I})^{-1} = \mathbf{L} \overbrace{(\tilde{\mathbf{H}}^H \tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I})^{-1} \tilde{\mathbf{H}}^H}^{\text{MMSE Linear Equalizer}}$$

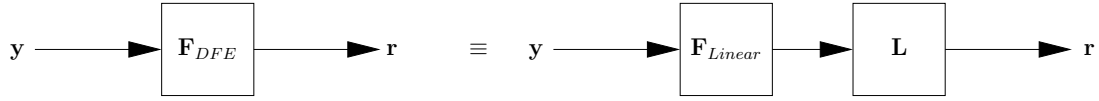


Figure 12: MMSE-DFE Equivalence

- Optimal  $\mathbf{L}$ :

$$\begin{aligned} \Phi_{ee} &= \mathbf{L} \Phi_{\tilde{x}y} \Phi_{yy}^{-1} \Phi_{yy} \Phi_{yy}^{-1} \Phi_{\tilde{x}y}^H \mathbf{L}^H - \mathbf{L} \Phi_{\tilde{x}y} \Phi_{yy}^{-1} \Phi_{y\tilde{x}} \mathbf{L}^H - \mathbf{L} \Phi_{\tilde{x}y} \Phi_{yy}^{-1} \Phi_{\tilde{x}y}^H \mathbf{H}^H + \mathbf{L} \Phi_{\tilde{x}\tilde{x}} \mathbf{L}^H \\ &= \mathbf{L} (\Phi_{\tilde{x}\tilde{x}} - \Phi_{\tilde{x}y} \Phi_{yy}^{-1} \Phi_{\tilde{x}y}^H) \mathbf{L}^H \end{aligned}$$

= MMSE covariance matrix for a linear MMSE filter

$$\rightarrow \Phi_{ee} = \sigma_n^2 \mathbf{L} (\tilde{\mathbf{H}}^H \tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I})^{-1} \mathbf{L}^H$$

$\rightarrow$  need lower triangular matrix  $\mathbf{L}$  which minimizes  $\text{tr}\{\Phi_{ee}\}$

$\rightarrow$  the optimum  $\mathbf{L}$  whitens  $\Phi_{ee} \rightarrow \Phi_{ee}$  becomes diagonal matrix i.e.  $\mathbf{L}$  exploits the correlation after linear MMSE filtering to reduce noise variance!

$\rightarrow \mathbf{L}$  is obtained via Cholesky factorization

$$\tilde{\mathbf{H}}^H \tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I} = \mathbf{L} \mathbf{D} \mathbf{L}^H$$

$$\rightarrow \Phi_{ee} = \sigma_n^2 \mathbf{L} (\mathbf{L} \mathbf{D} \mathbf{L}^H)^{-1} \mathbf{L} = \sigma_n^2 \mathbf{L} \mathbf{L}^{-1} \mathbf{D}^{-1} \mathbf{L}^{-H} \mathbf{L} = \sigma_n^2 \mathbf{D}^{-1} \hat{=} \text{diagonal matrix}$$

- Summary of MMSE calculation

$$- \mathbf{B} = \mathbf{L} - \mathbf{I} \text{ where } \mathbf{H}^H \mathbf{H} + \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I} = \mathbf{L} \mathbf{D} \mathbf{L}^H$$

$$- \mathbf{F} = \mathbf{L} \underbrace{(\mathbf{H}^H \mathbf{H} + \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I})^{-1} \tilde{\mathbf{H}}}_{\mathbf{F}_{Linear}}$$

-  $\mathbf{L}$  is a prediction error filter, which whitens the error signal  $\mathbf{e}$

– SNR:

$$\mathbf{D} = \text{diag}\{\xi_1, \dots, \xi_{N_T}\}$$

$$\mathbf{\Phi}_{ee} = \text{diag}\left\{\frac{\sigma_n^2}{\xi_1}, \dots, \frac{\sigma_n^2}{\xi_{N_T}}\right\}$$

– end-to-end channel

$$\begin{aligned}\mathbf{K} &= \mathbf{F}\tilde{\mathbf{H}} \\ &= \underbrace{\mathbf{L}(\tilde{\mathbf{H}}^H \tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I})^{-1} \mathbf{L}^H \mathbf{L}^{-H} \tilde{\mathbf{H}}^H \tilde{\mathbf{H}}}_{\frac{1}{\sigma_n^2} \mathbf{\Phi}_{ee}} \\ &= \frac{1}{\sigma_n^2} \mathbf{\Phi}_{ee} \mathbf{L}^{-H} (\tilde{\mathbf{H}}^H \tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I} - \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I}) \mathbf{L}^{-1} \mathbf{L} \\ &= \frac{1}{\sigma_n^2} [\underbrace{\mathbf{L}^{-H} (\tilde{\mathbf{H}}^H \tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{I}) \mathbf{L}^{-1} \mathbf{L}}_{\sigma_n^2 \mathbf{\Phi}_{ee}^{-1}} - \frac{\sigma_n^2}{\mathcal{E}_s} \mathbf{L}^{-H}] \\ &= \mathbf{L} - \frac{1}{\mathcal{E}_s} \mathbf{\Phi}_{ee} \mathbf{L}^{-H}\end{aligned}$$

→ main diagonal of  $\mathbf{K}$ :

$$K_{mm} = 1 - \frac{\sigma_n^2}{\mathcal{E}_s \xi_m} < 1$$

– biased SNR

$$SNR_{m,bias} = \frac{\mathcal{E}_S}{[\mathbf{\Phi}_{ee}]_{mm}} = \frac{\mathcal{E}_s}{\sigma_n^2} \xi_m = \frac{1}{1 - K_{mm}}, \quad 1 \leq m \leq N_T$$

– unbiased SNR

$$SNR_{m,unbiased} = SNR_{m,bias} - 1 = \frac{1}{1 - K_{mm}} - 1 = \frac{K_{mm}}{1 - K_{mm}}, \quad 1 \leq m \leq N_T$$

#### 2.4.4 Sphere Decoding

- Linear and DFE receivers cannot approach performance of ML - detector
- ML - detector:  $\hat{\mathbf{x}} = \underset{\mathbf{x} \in A^{N_T}}{\text{argmin}} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2$

→ high complexity for brute force search

#### Main Idea

- can we search ML metric in a “smarter” way, akin to the smart search of the viterbi algorithm for problems with a trellis structure
- we need to find a way to prune/dismiss non-ML sequences/vectores early on

→ this is accomplished by sphere decoding

**Step 1** Bring metric into a suitable form:

- real representation:  $\|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2 = \|\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\mathbf{x}}\|^2$
- where:

$$\begin{aligned}\tilde{\mathbf{y}} &= [\operatorname{Re}\{\mathbf{y}^T\} \quad \operatorname{Im}\{\mathbf{y}^T\}]^T \\ \tilde{\mathbf{x}} &= [\operatorname{Re}\{\mathbf{x}^T\} \quad \operatorname{Im}\{\mathbf{x}^T\}]^T \\ \tilde{\mathbf{H}} &= \begin{bmatrix} \operatorname{Re}\{\mathbf{H}\} & -\operatorname{Im}\{\mathbf{H}\} \\ \operatorname{Im}\{\mathbf{H}\} & \operatorname{Re}\{\mathbf{H}\} \end{bmatrix}^T\end{aligned}$$

→ QAM:  $\tilde{\mathbf{x}}$  defines points in an  $2N_T$  dimensional lattice

- QL decomposition:  $\tilde{\mathbf{H}} = \mathbf{Q}\mathbf{L}$  with
  - orthogonal matrix  $\mathbf{Q} \in \mathbb{R}^{2N_T \times 2N_T}$ ;  $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$
  - and lower triangular matrix  $\mathbf{L} \in \mathbb{R}^{2N_T \times 2N_T}$

$$\begin{aligned}\|\tilde{\mathbf{y}} - \mathbf{Q}\mathbf{L}\tilde{\mathbf{x}}\|^2 &= \|\mathbf{Q}\mathbf{Q}^T(\tilde{\mathbf{y}} - \mathbf{Q}\mathbf{L}\tilde{\mathbf{x}})\|^2 + \|(\mathbf{I} - \mathbf{Q}\mathbf{Q}^T)(\tilde{\mathbf{y}} - \mathbf{Q}\mathbf{L}\tilde{\mathbf{x}})\|^2 \\ &= \|\mathbf{Q}(\mathbf{Q}^T\tilde{\mathbf{y}} - \mathbf{L}\tilde{\mathbf{x}})\|^2 + \|(\mathbf{I} - \mathbf{Q}\mathbf{Q}^T)\tilde{\mathbf{y}} - \underbrace{(\mathbf{Q} - \mathbf{Q})\mathbf{L}\tilde{\mathbf{x}}}_{=0}\|^2 \\ &= \|\mathbf{Q}^T\tilde{\mathbf{y}} - \mathbf{L}\tilde{\mathbf{x}}\|^2 + \underbrace{\|(\mathbf{I} - \mathbf{Q}\mathbf{Q}^T)\tilde{\mathbf{y}}\|^2}_{\text{unabhängig von } \tilde{\mathbf{x}}}\end{aligned}$$

$$\Rightarrow \operatorname{argmin}_{\tilde{\mathbf{x}} \in A^{2N_T}} \|\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\mathbf{x}}\|^2 = \operatorname{argmin}_{\tilde{\mathbf{x}} \in A^{2N_T}} \underbrace{\|\mathbf{Q}^T\tilde{\mathbf{y}} - \mathbf{L}\tilde{\mathbf{x}}\|^2}_{\tilde{\mathbf{y}}}$$

where  $A^{2N_T}$  contains all  $M^{N_T}$  possible vectors  $\tilde{\mathbf{x}}$

- Observation:
  - with

$$\begin{aligned}\mathbf{L} &= \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ l_{2N_T-1} & \cdots & & l_{2N_T-2N_T} \end{bmatrix} \\ \bar{\mathbf{y}} &= [\bar{y}_1 \quad \cdots \quad \bar{y}_{2N_T}]^T ; \quad \tilde{\mathbf{x}} = [\tilde{x}_1 \quad \cdots \quad \tilde{x}_{2N_T}]\end{aligned}$$

- we have

$$\|\bar{\mathbf{y}} - \mathbf{L}\tilde{\mathbf{x}}\|^2 = (\bar{y} - l_{11}\tilde{x}_1)^2 + (\bar{y}_2 - l_{21}\tilde{x}_1 - l_{22}\tilde{x}_2)^2 + (\bar{y}_3 - l_{31}\tilde{x}_1 - l_{32}\tilde{x}_2 - l_{33}\tilde{x}_3)^2 + \dots \quad (1)$$

→ the  $n$ -th term in the above sum contains only  $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$

**Step 2** Sphere decoding algorithm**Define:**

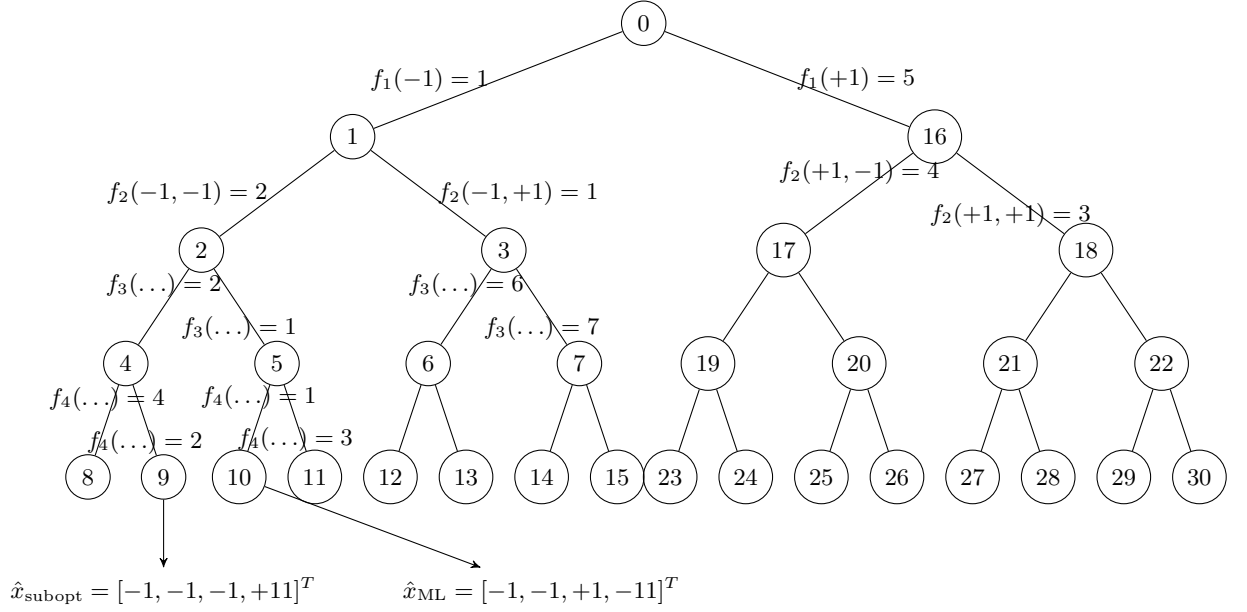
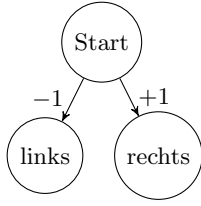
$$d(\tilde{\mathbf{x}}) = \sum_{n=1}^{2N_T} f_n(\tilde{\mathbf{x}}_n) ; \quad \text{with } f_n(\tilde{\mathbf{x}}_n) = \left( \tilde{y}_n - \sum_{i=1}^n l_{ni} \tilde{x}_i \right)^2 ; \quad \tilde{\mathbf{x}}_n = [\tilde{x}_1 \quad \dots \quad \tilde{x}_n]^T$$

$$d_n(\tilde{x}_n) = \sum_{m=1}^n f_n(\tilde{\mathbf{x}}_m)$$

**Main Idea:**

- Assume we know that  $d(\tilde{\mathbf{x}}) \leq R$  holds for some  $\tilde{\mathbf{x}}$ , where  $R$  is the so called “sphere radius”  $\rightarrow$  any  $\tilde{\mathbf{x}}$  with  $d(\tilde{\mathbf{x}}) \geq R$  cannot be the ML solution and can be discarded
- since  $d_{n+1}(\tilde{\mathbf{x}}'_n, \tilde{x}_{n+1}) \geq d_n(\tilde{\mathbf{x}}'_n)$ , we can easily discard  $\mathbf{x}'_n$  and all possible  $\tilde{\mathbf{x}} = [\tilde{\mathbf{x}}'_n \quad \tilde{x}_{n+1} \quad \tilde{x}_{n+2} \quad \dots \quad \tilde{x}_{2N_T}]^T$  if we find  $d_n(\tilde{\mathbf{x}}'_n) > R$   
 $\Rightarrow$  we can exclude many possible vectors  $\tilde{\mathbf{x}}$  without evaluating any metrics for them
- How to find a suitable  $R$ ?
  - Initial  $R$ :  $R = d(\tilde{\mathbf{x}}_{\text{subopt}})$  where  $\tilde{\mathbf{x}}_{\text{subopt}}$  was obtained with some suboptimum receiver
  - $R$  is updated as  $R = R_{\text{new}} = d(\tilde{\mathbf{x}}_{\text{cand}})$  where  $\tilde{\mathbf{x}}_{\text{cand}}$  is an  $\tilde{\mathbf{x}}$  which yields  $d(\tilde{\mathbf{x}}_{\text{cand}}) < R_{\text{old}} = R$
- Use tree structure to represent all possible  $\tilde{\mathbf{x}}$

**Example:** BPSK = 2,  $N_T = 2$



- Assume  $\tilde{\mathbf{x}}_{\text{subopt}} = [-1 \ -1 \ -1 \ 1]^T$   
 $\Rightarrow d(\tilde{\mathbf{x}}_{\text{subopt}}) = 1 + 2 + 2 + 2 = 7 = R$
- For  $\tilde{x}_1 = 1$  we find  $d_2(1, 1) = 5 + 3 = 8 > R$  and  $d(1, -1) = 5 + 4 > R$   
 $\Rightarrow$  we don't have to calculate metrics for remaining branches, i.e., for nodes 19, ..., 31
- For  $\tilde{x}_1 = -1$  and  $\tilde{x}_2 = 1$ , we find that  $d_3(-1, 1, -1) = 8 > R$  and  $d_3(-1, 1, 1) = 9 > R$   
 $\Rightarrow$  remaining branches emerging from nodes 6 and 7 can be discarded
- For the ML solution we find  $d(-1, -1, 1, -1) = 5 < R$
- Different strategies exist, regarding in which order the nodes in the tree (points in the lattice) are investigated
  - Pohst strategy
  - Schnorr - Enchner strategy
  - rich literature on lattice decoding algorithms
- Complexity of sphere decoding
  - worst-case complexity is still exponential in  $N_T$
  - (for practical case:) for sufficiently high SNR, the average complexity is only polynomial in  $N_T \Rightarrow$  efficient ML decoding

- sphere decoding has found application in many fields:
  - ML detection in MIMO and multiuser systems
  - precoding
  - source coding
  - multiple symbol differential detection