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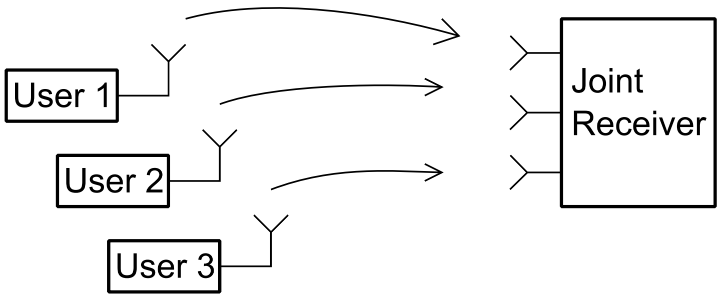
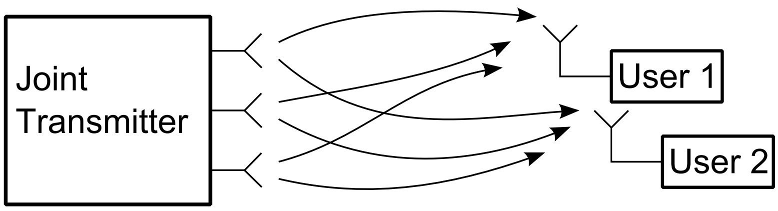
3 Multiuser MIMO

- We distinguish two cases:
  - multipoint - to - point transmission
  - point - to - multipoint transmission
- Multipoint - to - point transmission
  - typical uplink scenario in cellular systems
  - information theoretical channel model: Multiple Access Channel (MAC)



- Point - to -multipoint transmission
  - typical downlink scenarion in cellular systems
  - information theoretical channel model: Broadcast Channel (BC)



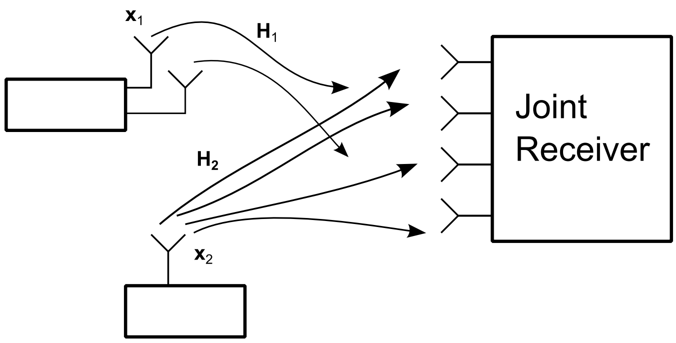
- Advantage of multiuser MIMO compared to point - to - point MIMO
  - multiplexing gain can be exploited even if users have only single antenna
  - users are spatially distributed in cell → channels to different users are independent

3.1 Multiple Access Channel (MAC)

We consider two aspects:

- Detector structures
- Rate region

3.1.1 Detector structures



Channel model: $\rightarrow$ general MAC: $\mathbf{y} = \sum\limits_{k=1}^{K} \mathbf{H}_k \mathbf{x}_k + \mathbf{n}$

with:

- K users
- user k has $N_{T,k}$ transmit antennas
- N_R receive antennas
- $\mathbf{H}_k \in \mathbb{C}^{N_R \times N_{T,k}}$

$$\mathbf{y} = \underbrace{\begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 & \dots & \mathbf{H}_k \end{bmatrix}}_{\mathbf{H}} \cdot \underbrace{\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_k \end{bmatrix}}_{\mathbf{x}} + \mathbf{n}$$

Observation:

- same equivalent channel model as for a point-to-point MIMO system transmitting $N_T = \sum_{k=1}^{K} N_{T,k}$ independent signal streams *(Anmerkung: kein Unterschied für Empfänger, ob Signale von einem Nutzer oder von mehreren)*
- the receiver (e.g. base station) can use detection schemes as for point-to-point MIMO systems
  - linear receiver
  - DFG
  - sphere decoder

Typical problems in uplink multiuser MIMO For given receiver structure:

- calculate SNR_k for all users k based on the expressions developed in Chapter 2.4
- optimize transmit power of users, $E_k = \mathcal{E}\{||x_k||^2\}$ for maximization of the sumrate or maximization of the minimum SNR_k *(Anmerkung: Maximierung der sumrate kann durch Maximierung des SNR des Users mit bestem Kanal erfolgen, aber: unfair anderen Usern gegenüber ⇒ starving)*

3.1.2 Rate region

For point-to-point links, we can decode error free, if the rate, R, meets

a) SISO $R < \log_2\left(1 + \frac{\mathcal{E}_s}{\sigma_n^2}\right)$

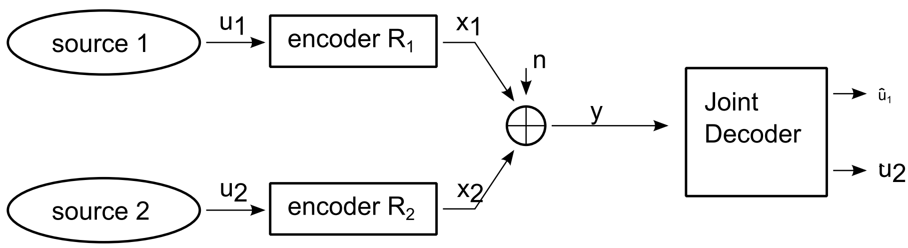
b) MIMO $R < \log_2 \underbrace{\left|\mathbf{I} + \frac{\mathcal{E}_s}{N_T \sigma_n^2}\mathbf{H}\mathbf{H}^H\right|}_{\det}$

Questions: What happens if there are multiple users?

Rate Region for Single Antenna Users and Receivers

- Gaussian channel
- $N_R = N_{T,k} = 1 \forall \quad k$
- received signal:

$$
\begin{aligned}
y &= \sum_{k=1}^{K} x_k + n \\
*\mathcal{E}_k &= \mathcal{E}\{||x_k||^2\} \\
*\sigma_n^2 &= \{||n||^2\}
\end{aligned}
$$



Example: 2 Users

- How should we choose R_1 and R_2 to ensure error free decoding of <u>both</u> signal streams?
- It is no longer sufficient to maximizie a single rate. Instead we have to consider rate pairs (R_1, R_2)
- All possible rate points, that allows error free decoding, define the rate region $\underline{C}$

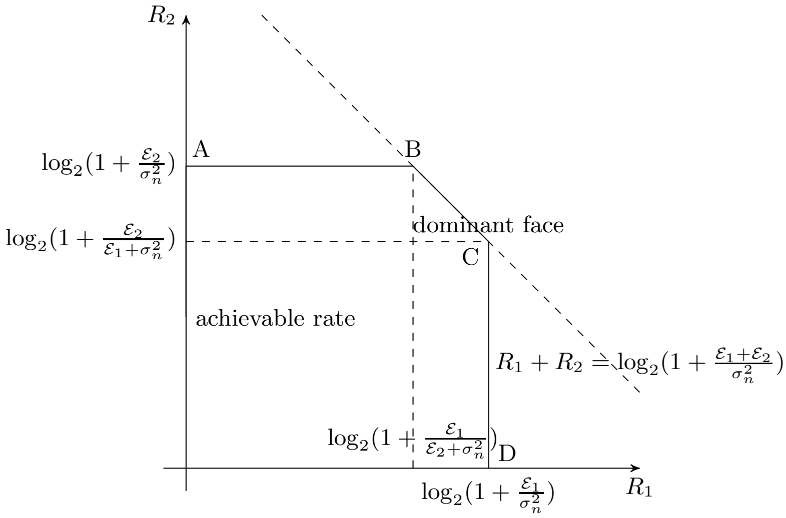
- Possible desing goals of the system:
  - maximized sumrate $R_{\text{sum}} = \max\limits_{(R_1,R_2)\in \underline{C}} R_1 + R_2$
  - maximize minimum user rate: $R_{\text{max-min}} = \max\limits_{(R_1,R_2)\in \underline{C}} \min\limits_{i\in\{1,2\}} R_i$
- Rate Region of two user Gaussian MAC *Anmerkung: Einschränkungen*

$$R_1 < \log_2\big(1 + \frac{\mathcal{E}_1}{\sigma_n^2}\big) \tag{1}$$

$$R_2 < \log_2\big(1 + \frac{\mathcal{E}_2}{\sigma_n^2}\big) \tag{2}$$

$$R_1 + R_2 < \log_2\big(1 + \frac{\mathcal{E}_1 + \mathcal{E}_2}{\sigma_n^2}\big) \tag{3}$$

- Interpretation:
  - (1) and (2) (= single-to user constraint) are the “single-user bounds, i.e., the maximum rates of user 1 and 2, if the other user was not there
  - (3) can be interpreted as the maximum rate if streams of users 1 and 2 were jointly encoded. The separate encoding in the MAC cannot yield a better performance
  - Graphical represantation:



- Observations:
  - A - B is defined by (2)
  - C - D is defined by (1)
  - B - C is defined by (3)
  - A - B suggests that even if user 2 transmits with the same max. rate as in the single user case, user 1 can transmit with non-zero rate! → Multiuser communication enables “free rate gains!
  - Which point on A - B - C,- D we choose, depends on the design criterion

- How do we achieve points on A - B - C,- D?
  - Both user use Gaussian codebooks
  - B:
    * signal of user 1, x_1, is decoded first and x_2 is treated as noise:

$$y = x_1 + \underbrace{x_2 + n}_{\text{treat as noise}}$$
$$\rightarrow R_1 < \log_2\big(1 + \frac{\mathcal{E}_1}{\mathcal{E}_2 + \sigma_n^2}\big)$$

    * once x_1 is known, we form

$$y - x_1 = x_2 + n$$
$$\rightarrow R_2 < \log_2\big(1 + \frac{\mathcal{E}_s}{\sigma_n^2}\big)$$

    * this approach is referred to as successive interference cancellation (SIC) and is a direct result of the chain rule in information theory: $I(X_1, X_2, Y) = I(X_1, Y) + I(X_2; Y|X_1)$
  - C: same as B, but X_1 and X_2 change rules
  - Points on A - B, C - D can be achieved by decreasing the rate of users 1 and 2 respectively (not desirable)
  - Points on B - C (dominant face): Achievable by "time-sharing, i.e., $\theta \cdot 100\%$ of the time we decode user 1 first and $(1 - \theta)100\%$ of the time we decode user 2 first, $0 \leq \theta \leq 1$

$$R_1 < \theta \log_2\big(1 + \frac{\mathcal{E}_1}{\mathcal{E}_2 + \sigma_n^2}\big) + \big(1 - \theta\big) \log_2\big(1 + \frac{\mathcal{E}_1}{\sigma_n^2}\big)$$
$$R_2 < \theta \log_2\big(1 + \frac{\mathcal{E}_2}{\sigma_n^2}\big) + \big(1 - \theta\big) \log_2\big(1 + \frac{\mathcal{E}_2}{\mathcal{E}_1 + \sigma_n^2}\big)$$
$$\begin{aligned}\rightarrow R_1 + R_2 < \theta\Big(\log_2\big(1 + \frac{\mathcal{E}_1}{\mathcal{E}_2 + \sigma_n^2}\big) + \log_2\big(1 + \frac{\mathcal{E}_2}{\sigma_n^2}\big)\Big) + \\ + \big(1 - \theta\big)\Big(\log_2\big(1 + \frac{\mathcal{E}_1}{\sigma_n^2}\big) + \log_2\big(1 + \frac{\mathcal{E}_2}{\mathcal{E}_1 + \sigma_n^2}\big)\Big) = \\ = \theta \log_2\Big(\frac{\mathcal{E}_1 + \mathcal{E}_2 + \sigma_n^2}{\mathcal{E}_2 + \sigma_n^2} \cdot \frac{\mathcal{E}_2 + \sigma_n^2}{\sigma_n^2}\Big) \cdot \big(1 - \theta\big) \log_2\Big(\frac{\mathcal{E}_1 + \sigma_n^2}{\sigma_n^2} \cdot \frac{\mathcal{E}_1 + \mathcal{E}_2 + \sigma_n^2}{\mathcal{E}_1 + \sigma_n^2} = \\ = \log_2\Big(1 + \frac{\mathcal{E}_1 + \mathcal{E}_2}{\sigma_n^2}\Big)\end{aligned}$$

- Comparison with orthogonal transmission
  - User 1 transmits for $\theta \cdot 100\%$ of the time and user 2 transmits for $(1 - \theta) \cdot 100\%$ of the time, $0 \leq \theta \leq 1$
  - to keep average transmit power independent of θ, the users transmit with powers $\frac{\mathcal{E}_1}{\theta}$ and $\frac{\mathcal{E}_2}{1-\theta}$

– Rates:

$$R_1 < \theta \log_2\Big(1 + \frac{\mathcal{E}_1}{\theta\sigma_n^2}\Big)$$
$$R_2 < \big(1-\theta\big) \log_2\Big(1 + \frac{\mathcal{E}_2}{(1-\theta)\sigma_n^2}\Big)$$

multiuser:

$$\xleftrightarrow{\qquad\qquad\qquad\mathcal{E}_1\qquad\qquad\qquad} \text{ user 1}$$

$$\xleftrightarrow{\qquad\qquad\qquad\mathcal{E}_2\qquad\qquad\qquad} \text{ user 2}$$

orthogonal:

$$\xleftrightarrow{\quad\frac{\mathcal{E}_1}{\theta}\quad}\xleftrightarrow{\text{non-transmission period: } 1-\theta} \text{ user 1}$$

$$\xleftrightarrow{\text{non-transmission period: } \theta}\xleftrightarrow{\quad\frac{\mathcal{E}_2}{1-\theta}\quad} \text{ user 2}$$

– sumrate:

$$R_1 + R_2 < \theta \log_2\Big(1 + \frac{\mathcal{E}_1}{\theta\sigma_n^2}\Big) + \big(1-\theta\big) \log_2\Big(1 + \frac{\mathcal{E}_2}{(1-\theta)\sigma_n^2}\Big) = R_{\text{sum}}$$

– Which θ maximizes sumrate?

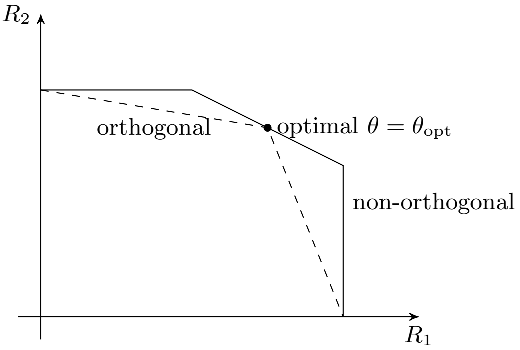
$$\frac{\delta R_{\text{sum}}}{\delta\theta} \overset{!}{=} 0 \text{ leads to } \theta_{\text{opt}} = \frac{\mathcal{E}_1}{\mathcal{E}_1 + \mathcal{E}_2}$$

– Maximum sumrate

$$R_{\text{sum}} = \frac{\mathcal{E}_1}{\mathcal{E}_1 + \mathcal{E}_2} \log_2\Big(1 + \frac{\mathcal{E}_1 + \mathcal{E}_2}{\sigma_n^2}\Big) + \frac{\mathcal{E}_2}{\mathcal{E}_1 + \mathcal{E}_2} \log_2\Big(1 + \frac{\mathcal{E}_1 + \mathcal{E}_2}{\sigma_n^2}\Big) =$$
$$= \log_2\Big(1 + \frac{\mathcal{E}_1 + \mathcal{E}_2}{\sigma_n^2}\Big)$$

$\rightarrow$ same value as for general non-orthogonal transmission!

– <u>But:</u> In general, orthogonal transmission is suboptimal!



- 3 users case:

$$
\begin{aligned}
R_1 &< \log_2\Big(1 + \frac{\mathcal{E}_1}{\sigma_n^2}\Big) \\
R_2 &< \log_2\Big(1 + \frac{\mathcal{E}_2}{\sigma_n^2}\Big) \\
R_3 &< \log_2\Big(1 + \frac{\mathcal{E}_3}{\sigma_n^2}\Big) \\
R_i + R_j &< \log_2\Big(1 + \frac{\mathcal{E}_i + \mathcal{E}_j}{\sigma_n^2}\Big), \quad i \neq j \\
R_1 + R_2 + R_3 &< \log_2\Big(1 + \frac{\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3}{\sigma_n^2}\Big) \\
&\rightarrow \text{rate region } \mathcal{C} \text{ has } 3! = 6 \text{ corner points}
\end{aligned}
$$

- general case of K users
  - define all non-empty subsets of $\mathbf{K} = \big\{1, \ldots, K\big\}$ as $\mathbf{S} \in \mathbf{K}$, e.g. $K = 2$: $\mathbf{K} = \big\{1, 2\big\}, \mathbf{S} = \big\{\{1\}, \{2\}, \{1, 2\}\big\}$
- rate region $\mathcal{C}$ is defined by

$$
\sum_{k \in \mathbf{S}} R_k < \log_2\Big(1 + \frac{\sum_{k \in \mathbf{S}} \mathcal{E}_k}{\sigma_n^2}\Big) \quad \forall \mathbf{S}
$$

$\rightarrow$ $\mathcal{C}$ has $K!$ corner points which can all be achieved by successive interference cancellation (SIC)

Rate region for MIMO Users and Receivers

- Channel Model: $\mathbf{y} = \sum_{k=1}^{K} \mathbf{H}_k \mathbf{x}_k + \mathbf{n}$, with:
  - User k has $N_{T,k}$ transmit antennas
  - N_R receive antennas
  - $\mathbf{n}$: AWGN vector $\mathcal{N}(\mathbf{0}, \sigma_n^2 \mathbf{I})$

- 2 Users case:

$$\mathbf{y} = \mathbf{H}_1\mathbf{x}_1 + \mathbf{H}_2\mathbf{x}_2 + \mathbf{n} \tag{4}$$

  - Covariance matrix of the TX signal of user k: $\mathbf{Q}_k = \mathcal{E}\{\mathbf{x}_k\mathbf{x}_k^H\}$
  - transmit power: $\mathcal{E}_k = \mathrm{tr}\{\mathbf{Q}_k\}$

- rate region for 2 user case and given $\mathbf{Q}_k$
  - $\mathbf{Q}_k$ given, for example
    a) $\mathbf{Q}_k$ optimal for single user case $\rightarrow \mathbf{Q}_k = \mathbf{U}_k\mathbf{\Lambda}_k\mathbf{U}_k^H$, where:
      · $\mathbf{U}_k$ is an unitary matrix
      · obtained from $\mathbf{H}_k = \mathbf{U}_k\mathbf{\Sigma}_k\mathbf{V}_k^H$
      · $\mathbf{\Lambda}_k = \mathrm{diag}\{\mathcal{E}_{k,1}, \mathcal{E}_{k,2}, \ldots, \mathcal{E}_{k,N_T}\}$ with $\mathcal{E}_{l,i}$ obtained from waterfilling and $\sum_{i=1}^{N_{Tk}} \mathcal{E}_{k,i} = \mathcal{E}_k$

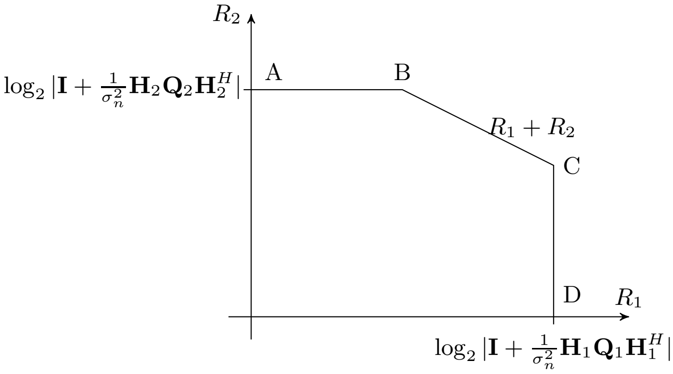
    b) $\mathbf{Q}_k = \frac{\mathcal{E}_k}{N_{T,k}}\mathbf{I}_{N_{T,k}}$ if $\mathbf{H}_k$ is not known at transmitter
  - for given $\mathbf{Q}_1$ and $\mathbf{Q}_2$ we can obtain the rate region as direct extension of the SISO case

$$R_1 < \log_2\left|\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}_1\mathbf{Q}_1\mathbf{H}_1^H\right| \tag{5}$$

$$R_2 < \log_2\left|\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}_2\mathbf{Q}_2\mathbf{H}_2^H\right| \tag{6}$$

$$R_1 + R_2 < \log_2\left|\mathbf{I} + \frac{1}{\sigma_n^2}\sum_{i=1}^{2}\mathbf{H}_i\mathbf{Q}_i\mathbf{H}_i^H\right| \tag{7}$$

    * equation 5 and equation 6 are the single user bounds,
    * equation 7 is the bound for the joint encoding of both users
  - graphical represantation



  - Points on A - B - C - D can be achieved in a similar manner as for SISO case
  - e.g. bound C can be achieved by SIC

- At B we have

$$R_2 = \log_2 \left| \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}_2 \mathbf{Q}_2 \mathbf{H}_2^H \right|$$

$$R_1 = \log_2 \left| \mathbf{I} + \frac{1}{\sigma_n^2} \sum_{i=1}^{2} \mathbf{H}_i \mathbf{Q}_i \mathbf{H}_i^H \right| - R_2$$

$\rightarrow$ user 1 transmits with rate

$$R_1 = \log_2 \left| \mathbf{I} + \frac{1}{\sigma_n^2} (\mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}_2 \mathbf{Q}_2 \mathbf{H}_2^H)^{-1} \mathbf{H}_1 \mathbf{Q}_1 \mathbf{H}_1^H \right|$$

- How to achieve rates at B? $\rightarrow$ Treat $\mathbf{H}_2\mathbf{x}_2 + \mathbf{n}$ in equation 9 as noise with covariance matrix $\mathbf{Q}_N = \mathbf{H}_2 \mathbf{Q}_2 \mathbf{H}_2^H + \sigma_n^2 \mathbf{I}$

$\rightarrow$ equivalent channel matrix with white noise: $\mathbf{r} = \mathbf{Q}_N^{-\frac{1}{2}} \mathbf{y} = \mathbf{Q}_N^{-\frac{1}{2}} \mathbf{H}_1 \mathbf{x}_1 + \tilde{\mathbf{n}}$ where $\tilde{\mathbf{n}}$ is white noise with covariance $\mathbf{I}$.
Anmerkung: Rauschen war vorher farbig, muss "geweißt" werden.

$$\rightarrow R_1 = \log_2 \left| \mathbf{I} + \underbrace{\mathbf{Q}_N^{-\frac{1}{2}} \mathbf{H}_1}_{\mathbf{H}} \mathbf{Q}_1 \underbrace{\mathbf{H}_1^H \mathbf{Q}_N^{-\frac{1}{2}}}_{\mathbf{H}_{\mathbf{eq}}^{\mathbf{H}}} \right| = \log_2 \left(\left| \mathbf{Q}_N^{\frac{1}{2}} + \mathbf{Q}_N^{-\frac{1}{2}} \mathbf{H}_1 \mathbf{Q}_1 \mathbf{H}_1^H \right| \cdot \left| \mathbf{Q}_N^{-\frac{1}{2}} \right| \right)$$

$$= \log_2 \left| \mathbf{I} + \mathbf{Q}_N^{-1} \mathbf{H}_1 \mathbf{Q}_1 \mathbf{H}_1^H \right| =$$

$$= \log_2 \left| \mathbf{I} + \frac{1}{\sigma_n^2} \left(\mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}_2 \mathbf{Q}_2 \mathbf{H}_2^H \right)^{-1} \mathbf{H}_1 \mathbf{Q}_1 \mathbf{H}_1^H \right|$$

$\rightarrow$ we can achieve R_1 in B by treating user 2 as noise

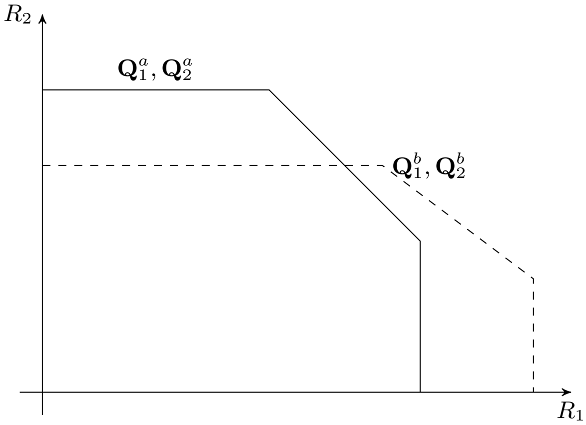
$\rightarrow$ once user 1 is detected, we can subtract its contribution from the received signal and detect user 2

$\Rightarrow$ user 2 can transmit with maximum single user rate

$\rightarrow$ bound C can be achieved by SIC similar to SISO case

- points on B - C are achieved through time sharing

- extension to K user case $\rightarrow$ analogous to SISO case
- Note: Different choices for Q_k will lead to different rate regions



- Example: $K = 2$

$\rightarrow$ $\mathbf{Q}_1$ and $\mathbf{Q}_2$ can be optimized to achieve desired trade-off between performance of users 1 and 2

3.2 Broadcast Channel

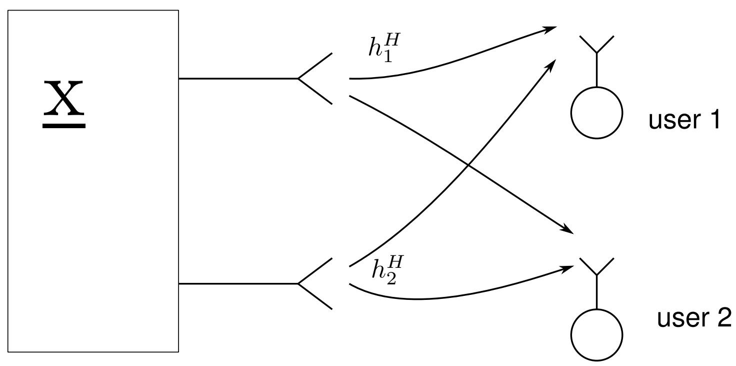
We consider:

- uplink - downlink duality
- rate region

3.2.1 Multiplexing Gain - Degrees of freedom

Downlink scenarios:

- N_R antennas at transmitter, single antennas at the users
- user k receives: $\mathbf{y}_k = \mathbf{h}_k^H \mathbf{x} + \mathbf{n}_k$, with:
  - N_R dimensional channel vector of user k: $\mathbf{h}_k^H$
  - n_k: AWGN at user k
  - $\mathbf{x}$ transmit vector



- How many independent signal streams can we transmit?
  - Consider transmit signal: $\mathbf{x} = \sum_{k=1}^{K} \mathbf{h}_k \mathbf{x}_k$, with TX transmit vector $\mathbf{n}_k$ and symbol x_k is intended for user k
- received signal of user k: $y_k = \sum_{k=1}^{K} (\mathbf{h}_k^H \mathbf{n}_i)\mathbf{x}_i + n_k$
- if all $\mathbf{h}_k$ were orthogonal and we chose $\mathbf{n}_k = \mathbf{h}_k$, the received signal would be: $y_k = ||h_k||^2 \mathbf{x}_k + n_k$
- if $N_R \geq K$, we can transmit simultaneously and interference free to all K users $\Rightarrow$ multiplexing gain $= \min\{K, N_k\}$
- In practice, the $\mathbf{h}_k$ will not be orthogonal

→ choose $\mathbf{n}_k$ such that it lies in the null space of $\begin{bmatrix}\mathbf{h}_1 & \dots & \mathbf{h}_{k-1} & \mathbf{h}_{k+1} & \dots & \mathbf{h}_K\end{bmatrix}$

→ always possible if $\mathbf{h}_1, \dots, \mathbf{h}_K$ are linearly independent

→ multiplexing gain (= degrees of freedom) is equal to $\min\{K, N_R\}$

- we can transmit interference free to $K \leq N_R$ users
  Anmerkung: Falls TX viele Antennen, aber RX nur eine hat ⇒ begrenzter Nutzen: SNR Verbesserung, kein Multiplexing Gain; falls TX viele Antennen und viele RX vorhanden sind ⇒ RX erscheinen als Antennenarray → hoher Multiplexing Gain

3.2.2 Uplink - Downlink Duality

- How should we choose signature vectors to achieve a certain SNR at users?
- Difficult problem since optimal (in the SINR sense) $\mathbf{u}_k$ are not orthogonal → signature of user k, $\mathbf{u}_k$, influences SINR at all other users!
- On the other hand, the uplink problem was much easier to solve, since the receive filter of user k, $\mathbf{f}_k$, was not affected by receive filters of other users! (vgl. Point - to - Point → detection problem

⇒ We establish a duality between the uplink and downlink, that allows us to solve the more challenging downlink problem by solving an equivalent uplink problem.

Downlink:

- transmit signal

$$\mathbf{x}_{dl} = \sum_{k=1}^{K} \mathbf{u}_k x_{dl,k}$$

- received signal at user k

$$y_{dl,k} = \mathbf{h}_k^H \mathbf{u}_k x_{dl,k} + \sum_{j \neq k} \mathbf{h}_k^H \mathbf{u}_j x_{dl,j} + u_{dl,k}$$

- SINR of user k

$$\mathrm{SINR}_k^{dl} = \frac{\mathcal{E}_{dl,k} |\mathbf{u}_k^H \mathbf{h}_k|^2}{\sigma_n^2 + \sum\limits_{j \neq k} \mathcal{E}_{dl,j} |\mathbf{h}_j^H \mathbf{h}_k|^2}, \quad 1 \leq k \leq K$$

- where: $\mathcal{E}_{dl,k} = \mathcal{E}\big\{|x_{dl,k}|^2\big\}$; $\quad \sigma_n^2 = \mathcal{E}\big\{|n_{dl,k}|^2\big\}$
- Using:

$$a_k = \frac{\mathrm{SINR}_k^{dl}}{(1 + \mathrm{SINR}_k^{dl}) |\mathbf{h}_k^H \mathbf{u} K|^2}$$

  we can rewrite the the SINR expressions as:

$$\boxed{\big(\mathbf{I}_K - \mathrm{diag}\{a_1, \dots, a_k\} \mathbf{A}\big) \mathbf{p}_{dl} = \sigma_n^2 \mathbf{a}}$$

where:

$$\mathbf{a} = \left[a_1, \dots, a_k\right]^T$$
$$\mathbf{A} = \begin{bmatrix} |u_1^H h_1|^2 & |u_2^H h_1|^2 & \dots & |u_K^H h_1|^2 \\ \vdots & & & \\ |u_1^H h_K|^2 & \dots & & |u_K^H h_K|^2 \end{bmatrix}$$
$$\mathbf{p}_{dl} = \left[\mathcal{E}_{dl,1}, \dots, \mathcal{E}_{dl,K}\right]^T$$

$\rightarrow$ We can easily calculate transmit powers $\mathcal{E}_{dl,k}$ required to achieve desired $\text{SINR}_k^{dl}, 1 \leq k_1 \leq K$, for given signature (precoding) vectores $u_k, 1 \leq k \leq K$

- Block diagramm: **Hier Bild einfügen**

Uplink: Use downlink signatures, $\mathbf{h}_k$, as receive filters, $\mathbf{f}_k$

- block diagramm: **Hier Bild einfügen**
- Signal model:

$$y_{ul,k} = \mathbf{u}_k^H \left(\mathbf{H}\mathbf{x}_{ul} + \mathbf{u}_{ul}\right)$$

with

$$\mathbf{H} = \begin{bmatrix}\mathbf{h}_1 & \dots & \mathbf{h}_K\end{bmatrix}, \quad \mathbf{x}_{ul} = \begin{bmatrix}x_{ul,1} & \dots & x_{ul,K}\end{bmatrix}^T, \quad \mathbf{u}_{ul} = \begin{bmatrix}u_{ul,1} & \dots & u_{ul,K}\end{bmatrix}^T$$

$$\rightarrow y_{ul,k} = \mathbf{u}_k^H \mathbf{h}_k x_{ul,k} + \sum_{j \neq k} \mathbf{u}_k \mathbf{h}_j x_{ul,j} + \mathbf{n}_k^H \mathbf{u}_{ul} \tag{8}$$

$$\rightarrow \text{SINR}_k^{ul} = \frac{\mathcal{E}_{ul,k}|\mathbf{u}^H \mathbf{h}_k|^2}{\sigma_n^2 + \sum\limits_{j \neq k} \mathcal{E}_{ul,j}|\mathbf{u}_k^H \mathbf{h}_j|^2} \tag{9}$$

where we used $\mathbf{u}_k^H \mathbf{u}_k = 1$ and $\mathcal{E}_{ul,K} = \mathcal{E}\left\{|x_{ul,K}|^2\right\}$

- we define: $b_k = \frac{\text{SINR}_k^{ul}}{(1+\text{SINR}_k^{ul})|\mathbf{u}_k^H \mathbf{h}_k|^2}$
- we can rewrite SINR epression (9) as :

$$\sigma_n^2 + \sum_{j \neq k} \mathcal{E}_{ul,j} \left|\mathbf{u}_k^H \mathbf{h}_j\right|^2 = \frac{1}{\text{SINR}_k^{ul}} \mathcal{E}_{ul,k} \left|\mathbf{u}_k^H \mathbf{h}_k\right|^2$$

$$\underbrace{\left(1 + \frac{1}{\text{SINR}_k^{ul}}\right) \left|\mathbf{u}_k^H \mathbf{h}_k\right|^2}_{\frac{1}{b_k}} \mathcal{E}_{ul,k} - \sum_{j=1}^{K} \mathcal{E}_{ul,k} \left|\mathbf{u}_k^H \mathbf{h}_k\right|^2 = \sigma_n^2$$

$$\rightarrow \mathcal{E}_{ul,k} - b_k \sum_{j=1}^{K} \mathcal{E}_{ul,j} \left|\mathbf{u}_k^H \mathbf{h}_j\right|^2 = b_k \sigma_n^2$$

- matrix notation:

$$\big[\mathbf{I}_K - \mathrm{diag}\big\{b_1,\ldots,b_K\big\}\mathbf{A}^T\mathbf{p}_{ul} = \sigma_n^2 \cdot \mathbf{b}\big]$$

- where: $\mathbf{A}$ was defined for downlink case

$$\mathbf{p}_{ul} = \begin{bmatrix}\mathcal{E}_{ul,1} & \ldots & \mathcal{E}_{ul,K}\end{bmatrix}^T$$
$$\mathbf{b} = \begin{bmatrix}b_1 & \ldots & b_K\end{bmatrix}^T$$

- we can calculate power allocation vector $\mathbf{p}_{ul}$ for given SINR_1^{ul} and $\mathbf{u}_k, \quad 1 \le k \le K$

Comparison: Assume, we want to achieve same SINR in uplink and downlink

→ $\mathrm{SINR}_k^{ul} = \mathrm{SINR}_k^{dl} \quad \forall \quad k$ or equivalently $a_k = b_k, \quad \forall \quad k$!

What sum power do we need in both uses?

$$\begin{aligned}\mathbf{p}_{dl} &= \sigma_n^2\big(\mathbf{I} - \mathrm{diag}\{a_1,\ldots,a_K\}\mathbf{A}\big)^{-1}\mathbf{a} = \\ &= \sigma_n^2\big(\mathbf{D}_a - \mathbf{A}\big)^{-1} \cdot \mathbf{1}\end{aligned}$$

where: $\mathbf{D}_a = \mathrm{diag}\{\frac{1}{a_1},\ldots,\frac{1}{a_K}\}$ and $\mathbf{1} = \begin{bmatrix}1 & 1 & 1 & \ldots & 1\end{bmatrix}^T$

$$\mathbf{p}_{ul} = \sigma_n^2(\mathbf{D}_b - \mathbf{A}^T)^{-1} \cdot \mathbf{1}$$

where: $\mathbf{D}_b = \mathrm{diag}\{\frac{1}{b_1},\ldots,\frac{1}{b_K}\}$

$$\begin{aligned}\sum_{k=1}^{K}\mathcal{E}_{dl,k} = \mathbf{1}^T\mathbf{p}_{dl} &= \sigma_n^2\mathbf{1}^T\big(\mathbf{D}_a - \mathbf{A}\big)^{-1} \cdot \mathbf{1} \\ &= \sigma_n^2\mathbf{1}^T\big(\mathbf{D}_b - \mathbf{A}\big)^{-1}\mathbf{1} \\ &= \sigma_n^2\big[\mathbf{1}^T\big(\mathbf{D}_b - \mathbf{A}\big)^{-1}\mathbf{1}\big]^T \\ &= \sigma_n^2\mathbf{1}^T\big[\big(\mathbf{D}_b - \mathbf{A}\big)^{-1}\big]^T\mathbf{1} \\ &= \sigma_n^2\mathbf{1}^T\big(\underbrace{\mathbf{D}_b^T}_{\mathbf{D}_b} - \mathbf{A}^T\big)^{-1}\end{aligned}$$

$$cdot\mathbf{1} = \sum_{k=1}^{K}\mathcal{E}_{ul,k}$$

Conclusions:

- We can achieve any desired $\mathrm{SINR}_k^{dl}, \forall\, k$, in the downlink by using filters optimized for uplink transmission as signature (precoding) vectors and the same sum power as in the uplink
- suitable filters may be MMSE or ZF filters $\mathbf{u}_k = \mathbf{f}_k, \quad \forall\, k$
- Note that, in general, $\mathcal{E}_{ul,k} \neq \mathcal{E}_{dl,k}$, only the sum powers are equal!

→ For linear precoding, we can solve the more challenging problem via solving an equivalent uplink problem!

Extension: This concept can be extended to nonlinear receivers as well. In this case the DFE receiver in the uplink is dual to a nonlinear Tamlinson - Harashima (TH) precoder in the downlink.