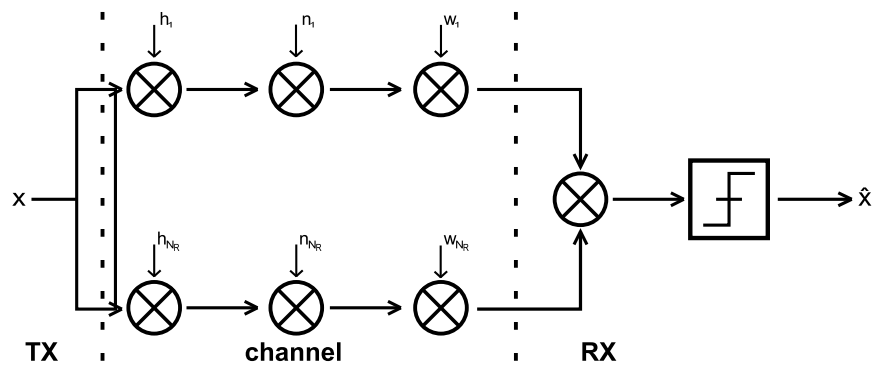


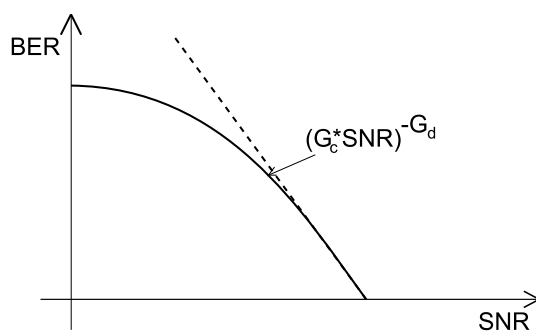
# 1 SIMO Systems

## Remarks

- In SIMO Systems only coding and diversity gains can be exploited (no multiplexing gains)
- To realize these gains diversity combining has to be performed
- Diversity combining schemes vary in complexity and performance
- There are many diversity combining schemes. Here we consider:
  - Maximal ratio combining (MRC)
  - Equal gain combining (EGC)
  - Selection combining (SC)
- Diversity combining problem



- how to choose combining weights  $w_n$ ?
- what performance (e.g. error rate, outage probability) is achieved?
- what diversity and coding/combining gain is achieved?



- $G_c$  : Coding gain
- $G_d$  : Diversity gain

## 1.1 Preliminaries

Consider an equivalent system:

$$y = hx + n;$$

$$\mathcal{E}\{|x|^2\} = \epsilon_s; \quad \mathcal{E}\{|n|^2\} = \sigma_n^2; \quad \mathcal{E}\{|h|^2\} = 1$$

- Instantaneous SNR:  $\gamma_t = \frac{\epsilon_s}{\sigma_n^2} \times |h|^2$
- Average SNR:  $\bar{\gamma}_t = \mathcal{E}\{\gamma_t\} = \frac{\epsilon_s}{\sigma_n^2}$

### Bit and Symbol Error Rate

- The Bit and Symbol Error Rate of many modulation schemes can be expressed for given  $\gamma_t$  as:

$$P_e(\gamma_t) = aQ\{\sqrt{b\gamma_t}\}$$

where:

- $Q(x) = \frac{1}{\sqrt{2\pi}} \times \int_x^\infty e^{-\frac{t^2}{2}} dt$
- $P_e(\gamma_t)$  may be exact result or approximation
- BPSK: exact with  $a = 1, b = 2$
- M-ary QAM: tight approximation with  $a = 4(1 - \frac{1}{\sqrt{M}}), b = \frac{3}{M-1}$

(*Einschub* : Gray – Code :  $BER = \frac{1}{\log_2 M} \times SER$ )

- Alternative representation of Q - function:

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}} d\theta$$

→ Integral limits are fixed and do not depend on integration variables!

- Average error probability

$$P_e = \mathcal{E}\{P_e(\gamma_t)\} = \int_0^\infty aQ(\sqrt{bx})p_{\gamma_t}(x) dx$$

- Integral may be difficult to solve analytically
- Integral has infinite support → numerical evaluation difficult
- Using alternative representation of Q-function we get:

$$P_e = \int_0^\infty \frac{a}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{bx}{2\sin^2\theta}} p_{\gamma_t}(x) d\theta dx$$

$$= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} \int_0^\infty p_{\gamma_t}(x) e^{-\frac{bx}{2\sin^2\theta}} dx d\theta = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\left(\frac{b}{2\sin^2\theta}\right) d\theta$$

where:

- $M_{\gamma_t}(s) = \int_0^\infty p_{\gamma_t}(x) e^{-sx} dx$  is the Laplace transform of  $p_{\gamma_t}$
- $M_{\gamma_t}(-s)$  is the so called Moment Generation Function (MGF) of  $p_{\gamma_t}$
- Here, we will also refer to  $M_{\gamma_t}(s)$  as MGF
- $M_{\gamma_t}(s)$  is sometimes easier to obtain than  $p_{\gamma_t}$
- The above integral can be easily evaluated numerically because of the finite integral limits

### Outage probability

- The outage probability is the probability that the channel cannot support a certain rate,  $R$ , i.e. (where  $\gamma_T$  is the threshold SNR):

$$C = \log_2(1 + \gamma_t) < R \quad \leftrightarrow \quad \gamma_t < 2^R - 1 \triangleq \gamma_T$$

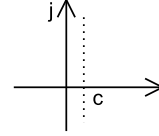
Thus, the outage probability is given by:

$$P_{out} = P_{\gamma_t < \gamma_T} = \int_0^{\gamma_T} p_{\gamma_t}(x) dx$$

- Using the inverse Laplace Transform

$$p_{\gamma_t}(x) = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) e^{sx} dx$$

where  $c > 0$  is a small constant that lies in the region of convergence of the integral, we obtain:



- 1.

$$P_{out} = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) \int_0^{\gamma_T} e^{sx} dx ds = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) e^{\gamma_T s} \frac{ds}{s}$$

(lower integral limit is 0 since  $p_{\gamma_t}(0) = 0$ )

- and 2.:

$$p_{\gamma_t}(x) = \int_0^x p_{\gamma_t}(t) dt = 0$$

$$\text{for } x = 0 \text{ note: } p_{\gamma_t}(x) \xleftrightarrow[\text{transform}]{\text{Laplace}} \frac{1}{s} M_{\gamma_t}(s)$$

### General combining scheme

$$y = \left( \sum_{n=1}^{N_R} h_n w_n \right) x + \sum_{n=1}^{N_R} w_n n_n$$

$$\gamma_t = \frac{\epsilon_s \left| \sum_{n=1}^{N_R} h_n w_n \right|^2}{\sigma_n^2 \sum_{n=1}^{N_R} |w_n|^2}$$

where  $w_n$  depends on the particular combining scheme.

## 1.2 MRC (Maximum Ratio Combining)

- what weight  $w_n$  maximize  $\gamma_t$ ?
  - Cauchy-Schwarz inequality

$$\left| \sum_{n=1}^{N_R} h_n w_n \right|^2 \leq \sum_{n=1}^{N_R} |h_n|^2 \cdot \sum_{n=1}^{N_R} |w_n|^2$$

where equality holds if and only if  $w_n = c \cdot h_n^*$  for some non-zero constant  $c$ .

- for  $w_n = h_n^*$ , we obtain

$$\gamma_t = \frac{\epsilon_s}{\sigma_n^2} \cdot \frac{\left( \sum_{n=1}^{N_R} |h_n|^2 \right)^2}{\sum_{n=1}^{N_R} |h_n|^2} = \frac{\epsilon_s}{\sigma_n^2} \sum_{n=1}^{N_R} |h_n|^2$$

- $w_n = h_n^* \forall n$  are the MRC combining weights.
- For performance analysis we assume independent identically distributed (IID) Rayleigh fading

$$\begin{aligned} \rightarrow \mathcal{E}\{|h_n|^2\} &= 1; \quad \bar{\gamma} = \frac{\epsilon_s}{\sigma_n^2}; \quad \gamma_n = \frac{\epsilon_s}{\sigma_n^2} |h_n|^2 \\ p_\gamma(x) &= \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0 \\ M_\gamma(s) &= \frac{1}{1 + s\bar{\gamma}} \end{aligned}$$

- Error rate

$$\gamma_t = \sum_{n=1}^{N_R} \gamma_n$$

$\rightarrow$  sum of IID random variables (r.v.s.)

$$M_{\gamma_t}(s) = \left( M_\gamma(s) \right)^{N_R} = \frac{1}{(1 + s\bar{\gamma})^{N_R}} = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{1}{\left(s + \frac{1}{\bar{\gamma}}\right)^{N_R}}$$

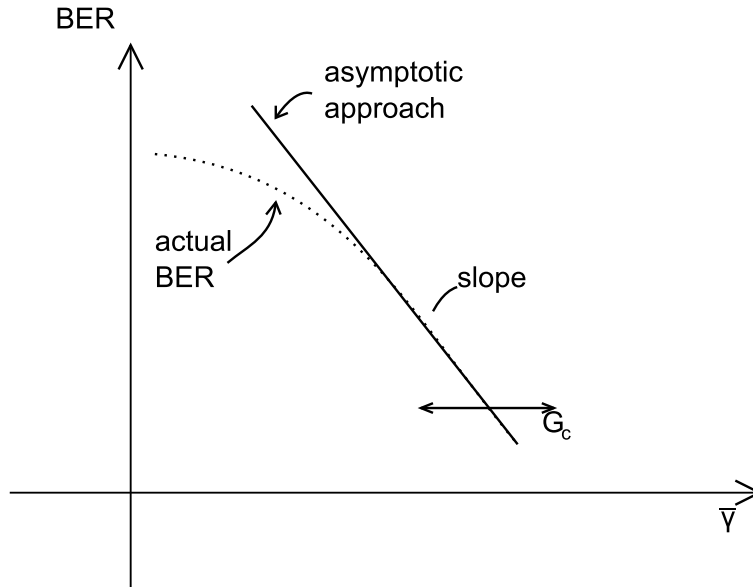
inverse Laplace-transform (from tables)

$$p_{\gamma_t}(x) = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{x^{N_R-1}}{(N_R-1)!} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0$$

- Direct approach

$$p_e = \int_0^\infty a \cdot Q(\sqrt{ax}) p_{\gamma_t}(x) dx = a \left( \frac{1-\mu}{2} \right)^{N_R} \cdot \sum_{n=0}^{N_R-1} \binom{N_R-1+n}{n} \left( \frac{1+\mu}{2} \right)^n$$

$$\text{where } \mu = \sqrt{\frac{b\bar{\gamma}}{2 + b\bar{\gamma}}}$$



- MGF approach

$$\begin{aligned}
 p_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t} \left( \frac{b}{2 \sin^2 \theta} \right) d\theta \\
 &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\bar{\gamma}^{N_R} \left( \frac{b}{\sin^2 \theta} + \frac{1}{\bar{\gamma}} \right)^{N_R}} d\theta \quad (\text{numerisch berechnen!})
 \end{aligned}$$

- high SNR:  $\bar{\gamma} \rightarrow \infty \iff \frac{1}{\bar{\gamma}} \rightarrow 0$

$$\begin{aligned}
 p_e &= \frac{a}{\pi} \cdot \frac{1}{\bar{\gamma}^{N_R}} \cdot \left( \frac{2}{b} \right)^{N_R} \int_0^{\frac{\pi}{2}} \sin^{2N_R} \theta d\theta \\
 (\text{from MGF approach: } \int_0^{\frac{\pi}{2}} \sin^{2N_R} \theta d\theta &= \frac{\pi}{2^{N_R+1}} \cdot \binom{2N_R}{N_R}) \\
 &= \frac{a}{2^{N_R+1} \cdot b^{N_R}} \binom{2N_R}{N_R} \frac{1}{\bar{\gamma}^{N_R}} \quad \text{as } \bar{\gamma} \rightarrow \infty \\
 &\stackrel{!}{=} \left( \frac{1}{G_c \bar{\gamma}} \right)
 \end{aligned}$$

where: Diversity gain:  $G_d = N_R$

$$\text{Combining/Coding gain: } G_c = 2b \left( \frac{a}{2} \binom{2N_R}{N_R} \right)^{-\frac{1}{N_R}}$$

- MRC exploits the maximal possible diversity
- Diversity gain is not affected by correlation as the branches are not fully correlated
- Diversity gain depends on fading distribution

## Outage probability

$$\begin{aligned} P_{out} &= \int_0^{\gamma_T} p_{\gamma_t}(x) dx = \frac{1}{\bar{\gamma}^{N_R}} \int_0^{\gamma_T} \frac{x^{N_R-1}}{(N_R-1)!} e^{-\frac{x}{\bar{\gamma}}} dx \\ &= 1 - e^{-\frac{\gamma_T}{\bar{\gamma}}} \cdot \sum_{n=1}^{N_R} \frac{\left(\frac{\gamma_T}{\bar{\gamma}}\right)^n}{(n-1)!} \end{aligned}$$

- Approximation (Taylor series):  $\bar{\gamma} \rightarrow \infty : -e^{-\frac{x}{\bar{\gamma}}} = 1 - \frac{x}{\bar{\gamma}} + O(\frac{1}{\bar{\gamma}})$  where a function  $f(x)$  is  $O(x)$  if  $\lim_{x \rightarrow \infty} \frac{f(x)}{x} = 0$ .

$$\Rightarrow P_{out} = \frac{1}{\bar{\gamma}^{N_R}} \int_0^{\gamma_T} \frac{x^{N_R-1}}{(N_R-1)!} \left(1 - \frac{x}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right)\right) dx$$

- Diversity and coding gain can also be defined for  $P_{out}$

## 1.3 EGC (Equal Gain Combining)

### Combining Weights

- For MRC, both, the amplitudes and phases of the channel gains  $h_n = |h_n|e^{j\varphi_n}$  have to be known (or estimated in practice)
- In EGC it is assumed that only the phases are known and weights  $w_n = e^{-j\varphi_n}$  are used.

$$\begin{aligned} \Rightarrow \gamma_t &= \frac{\mathcal{E}_s}{\sigma_n^2} \frac{\left| \sum_{n=1}^{N_R} |h_n| e^{j\varphi_n} e^{-j\varphi_n} \right|^2}{\sum_{n=1}^{N_R} |e^{-j\varphi_n}|^2} = \frac{\mathcal{E}_s}{\sigma_n^2} \frac{1}{N_R} \left( \sum_{n=1}^{N_R} |h_n| \right)^2 \\ &= \frac{1}{N_R} \left( \sum_{n=1}^{N_R} \sqrt{\gamma_n} \right)^2 ; \text{ with } \gamma_n = \frac{\mathcal{E}_s}{\sigma_n^2} |h_n|^2 \end{aligned}$$

### Performance Analysis

- i.i.d. case  
 $\Rightarrow \sqrt{\gamma_n}$  is Rayleigh distributed  
 $\Rightarrow$  Exact analysis is much more difficult than for MRC  $\Rightarrow$  see book by Simon & Alouini p.341
- Approximate result

$$P_e = \frac{a}{2} \left[ 1 - \sqrt{\frac{2b\bar{\gamma}}{5+2b\bar{\gamma}}} \sum_{n=0}^{N_R-1} \frac{\binom{2n}{n}}{4^n (1 + \frac{2}{5}b\bar{\gamma})^n} \right]$$

- high SNR

⇒ use high SNR analysis of Wang & Giannakis, 2003

⇒ at high SNR, only pdf of  $\gamma_n$  around 0 is relevant for performance

$$\stackrel{\text{Rayleigh}}{\Rightarrow} p_\gamma(x) = \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}} \stackrel{\text{Taylor Serie}}{=} \frac{1}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right) \text{ as } x \rightarrow 0$$

- need pdf  $\gamma_t$ : ( $\gamma_n$  bekannt,  $\rightarrow$  ges.: Wurzel, etc.)

(cumulative distribution function of  $\sqrt{\gamma}$  ( $\stackrel{\text{i.i.d.}}{=} \sqrt{\gamma_n}$ ) (cdf))

$$\begin{aligned} P_{\sqrt{\gamma}}(x) &= Pr\{\sqrt{\gamma} \leq x\} = Pr\{\gamma \leq x^2\} = P_\gamma(x^2) = \text{cdf of } \gamma \\ \rightarrow p_{\sqrt{\gamma}}(x) &= \frac{d}{dx} P_{\sqrt{\gamma}}(x) = 2x \cdot p_\gamma(x^2) = \frac{2x}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right) \end{aligned}$$

- Laplace Transformation to MGF

$$\rightarrow M_{\sqrt{\gamma}}(s) = \mathcal{L}\{p_{\sqrt{\gamma}}(x)\} = \frac{2}{\bar{\gamma}} \cdot \frac{1}{s^2} + O\left(\frac{1}{\bar{\gamma}}\right)$$

$$\sqrt{\gamma_t} = \sum_{n=1}^{N_R} \frac{\sqrt{\gamma_n}}{N_R}$$

$$\begin{aligned} M_{\sqrt{\gamma_t}}(s) &= \mathcal{E}\left\{\exp(-s\sqrt{\gamma_t})\right\} = \mathcal{E}\left\{\exp\left(-\frac{s}{\sqrt{N_R}} \cdot \sum_{n=1}^{N_R} \sqrt{\gamma_n}\right)\right\} = \left(\mathcal{E}\left\{\exp\left(-\frac{s}{\sqrt{N_R}} \cdot \sqrt{\gamma_n}\right)\right\}\right)^{N_R} \\ &= \left(M_{\sqrt{\gamma}}\left(\frac{s}{\sqrt{N_R}}\right)\right)^{N_R} = \left(\frac{2}{\bar{\gamma}} \cdot \frac{N_R}{s^2}\right)^{N_R} + O\left(\frac{1}{\bar{\gamma}^{N_R}}\right) \end{aligned}$$

- inverse Laplace Transform

$$\begin{aligned} p_{\sqrt{\gamma_t}}(x) &= \mathcal{L}^{-1}\left\{M_{\sqrt{\gamma_t}}(s)\right\} = \left(\frac{2N_R}{\bar{\gamma}}\right)^{N_R} \cdot \frac{x^{2N_R-1}}{(2N_R-1)!} + O\left(\frac{1}{\bar{\gamma}^{N_R}}\right) \\ P_{\gamma_t}(x) &= Pr\{\gamma_t \leq x\} = Pr\{\sqrt{\gamma_t} \leq \sqrt{x}\} = P_{\sqrt{\gamma_t}}(\sqrt{x}) \rightarrow \text{cdf of } \sqrt{\gamma_t} \\ p_{\gamma_t}(x) &= \frac{d}{dx} P_{\gamma_t}(x) = \frac{1}{2\sqrt{x}} \cdot p_{\gamma_t}(\sqrt{x}) = \frac{1}{2} \left(\frac{2N_R}{\bar{\gamma}}\right)^{N_R} \cdot \frac{x^{N_R-1}}{(2N_R-1)!} + O(\bar{\gamma}^{-N_R}) \\ \rightarrow M_{\gamma_t}(s) &= \mathcal{L}\{p_{\gamma_t}(x)\} = \frac{1}{2} \left(\frac{2N_R}{\bar{\gamma}}\right)^{N_R} \cdot \frac{(N_R-1)!}{(2N_R-1)!} \frac{1}{b^{N_R}} + O(\bar{\gamma}^{-N_R}) \end{aligned}$$

- Error Probability:

$$\begin{aligned}
P_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t} \left( \frac{b}{2 \sin^2(\theta)} \right) d\theta \\
&= \frac{a}{\pi} \frac{1}{2} \left( \frac{2N_R}{\bar{\gamma}} \right)^{N_R} \frac{(N_R - 1)!}{(2N_R - 1)!} \frac{2^{N_R}}{b^{N_R}} \underbrace{\int_0^{\frac{\pi}{2}} \sin^{2N_R}(\theta) d\theta}_{\frac{\pi}{2^{2N_R+1}} \binom{2N_R}{N_R} = \frac{\pi (2N_R)!}{2^{2N_R+1} (N_R!)^2}} + O\left(\frac{1}{\bar{\gamma}^{N_R}}\right) \\
&= \frac{aN_R^{N_R}}{2b^{N_R} N_R!} \frac{1}{\bar{\gamma}^{N_R}} + O\left(\frac{1}{\bar{\gamma}^{N_R}}\right) \stackrel{!}{=} \left(\frac{1}{G_c}\right)^{G_d} \\
&\implies \text{Diversity gain: } G_d = N_R \\
&\implies \text{Combining gain: } G_c = \frac{b}{N_R} \left(\frac{2N_R!}{a}\right)^{\frac{1}{N_R}}
\end{aligned}$$

vergleiche auch Blatt mit Kurven III und IV

A similar asymptotic analysis can be conducted for the outage probability.

## 1.4 SC (Selection Combining)

### Combining weights

- only the strongest branch is chosen
- strongest branch:  $\hat{n} = \underset{n}{\operatorname{argmax}} \gamma_n \longrightarrow \gamma_t = \gamma_{\hat{n}}$
- only on RF receiver chain required  $\rightarrow$  saves hardware complexity

### Performance analysis

- cdf of:  $\gamma_t$

$$\begin{aligned}
P_{\gamma_t}(x) &= \Pr\{\gamma_{\hat{n}} \leq x\} = \Pr\{\gamma_1 \leq x \cap \gamma_2 \leq x \cap \dots \cap \gamma_{N_R} \leq x\} \\
&\stackrel{(IID)}{=} \left(\Pr\{\gamma_n \leq x\}\right)^{N_R} = \left(P_{\gamma}(x)\right)^{N_R}
\end{aligned}$$

- pdf:

$$\begin{aligned}
p_{\gamma_t}(x) &= \frac{d}{dx} P_{\gamma_t}(x) = N_R (P_{\gamma}(x))^{N_R-1} \cdot p_{\gamma}(x) \\
\text{where: } p_{\gamma_t}(x) &= \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0 \\
P_{\gamma}(x) &= \int_0^x p_{\gamma}(x) dx = 1 - e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0 \\
\rightarrow p_{\gamma_t}(x) &= \frac{N_R}{\bar{\gamma}} \left(1 - e^{-\frac{x}{\bar{\gamma}}}\right)^{N_R-1} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0
\end{aligned}$$



### Error probability

- direct approach  $\rightarrow$  closed-form solution possible
- MGF approach
  - Binomial expansion

$$\begin{aligned} p_{\gamma_t}(x) &= \frac{N_R}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} 1^{N_R-1-n} \left(-e^{-\frac{x}{\bar{\gamma}}}\right)^n \\ &= \frac{N_R}{\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} \cdot (-1)^n e^{-\frac{x(n+1)}{\bar{\gamma}}}; \quad x \geq 0 \end{aligned}$$

- MGF

$$M_{\gamma_t}(s) = \frac{N_R}{\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} (-1)^n \frac{1}{s + \frac{n+1}{\bar{\gamma}}}$$

–

$$\begin{aligned} P_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\left(\frac{b}{2 \sin^2 \theta}\right) d\theta = \frac{aN_R}{\pi \bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} (-1)^n \int_0^{\frac{\pi}{2}} \frac{d\theta}{\frac{b}{2 \sin^2 \theta} + \frac{n+1}{\bar{\gamma}}} \\ &\quad \rightarrow \text{can be evaluated numerically} \end{aligned}$$