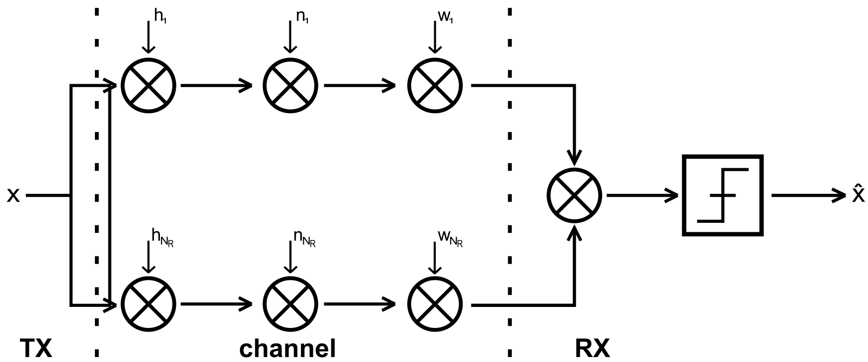
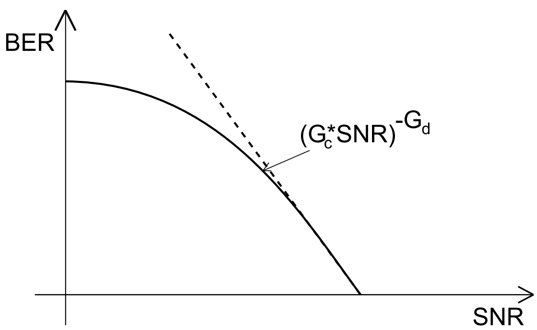
# 1 SIMO Systems

**Remarks**

- In SIMO Systems only coding and diversity gains can be exploited (no multiplexing gains)
- To realize these gains diversity combining has to be performed
- Diversity combining schemes vary in complexity and performance
- There are many diversity combining schemes. Here we consider:
  - Maximal ratio combining (MRC)
  - Equal gain combining (EGC)
  - Selection combining (SC)
- Diversity combining problem




  - how to choose combining weights $w_n$?
  - what performance (e.g. error rate, outage probability) is achieved?
  - what diversity and coding/combining gain is achieved?




- $G_c$ : Coding gain
- $G_d$ : Diversity gain

## 1.1 Preliminaries

Consider an equivalent system:

$$y = hx + n;$$
$$\mathcal{E}\{|x^2|\} = \epsilon_s; \qquad \mathcal{E}\{|n^2|\} = \sigma_n^2; \qquad \mathcal{E}\{|h|^2\} = 1$$

- Instantaneous SNR: $\gamma_t = \frac{\epsilon_s}{\sigma_n^2} \times |h|^2$
- Average SNR: $\bar{\gamma}_t = \mathcal{E}\{\gamma_t\} = \frac{\epsilon_s}{\sigma_n^2}$

**Bit and Symbol Error Rate**

- The Bit and Symbol Error Rate of many modulation schemes can be expressed for given $\gamma_t$ as:

$$P_e(\gamma_t) = aQ\big\{\sqrt{b\gamma_t}\big\}$$

where:

- $Q(x) = \frac{1}{\sqrt{2\pi}} \times \int_x^\infty e^{-\frac{t^2}{2}}\ dt$
- $P_e(\gamma_t)$ may be exact result or approximation
- BPSK: exact with $a = 1, b = 2$
- M-ary QAM: tight approximation with $a = 4\big(1 - \frac{1}{\sqrt{M}}\big), b = \frac{3}{M-1}$

$\big(Einschub : Gray - Code : BER = \frac{1}{\log_2 M} \times SER\big)$

- Alternative representation of Q - function:

$$Q(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2sin^2\theta}}\ d\theta$$

  $\rightarrow$ Integral limits are fixed and do not depend on integration variables!

- Average error probability

$$P_e = \mathcal{E}\big\{P_e(\gamma_t)\big\} = \int_0^\infty aQ\big(\sqrt{bx}\big)p_{\gamma_t}(x)\ dx$$

  - Integral may be difficult to solve analytically
  - Integral has infinite support $\rightarrow$ numerical evaluation difficult

- Using alternative representation of Q-function we get:

$$P_e = \int_0^\infty \frac{a}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{bx}{2sin^2\theta}} p_{\gamma_t}(x)\ d\theta\ dx$$
$$= \frac{a}{\pi}\int_0^{\frac{\pi}{2}}\int_0^\infty p_{\gamma_t}(x) e^{-\frac{b}{2sin^2\theta}}\ dx\ d\theta \qquad = \frac{a}{\pi}\int_0^{\frac{\pi}{2}} M_{\gamma_t}\big(\frac{b}{2sin^2\theta}\big)\ d\theta$$

  where:

- $M_{\gamma_t}(s) = \int_0^\infty p_{\gamma_t}(x) e^{-sx}\ dx$ is the Laplace transform of $p_{\gamma_t}$
- $M_{\gamma_t}(-s)$ is the so called Moment Generation Function (MGF) of $p_{\gamma_t}$
- Here, we will also refer to $M_{\gamma_t}(s)$ as MGF
- $M_{\gamma_t}(s)$ is sometimes easier to obtain than $p_{\gamma_t}$
- The above integral can be easily evaluated numerically because of the finite integral limits

**Outage probability**

- The outage probability is the probability that the channel cannot support a certain rate, R, i.e. (where $\gamma_T$ is the threshold SNR):

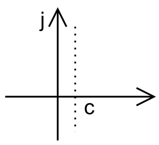
$$C = \log_2(1+\gamma_t) < R \quad \leftrightarrow \quad \gamma_t < 2^R - 1 \triangleq \gamma_T$$

  Thus, the outage probability is given by:

$$P_{out} = P_0 \gamma_t < \gamma - T \ = \int_0^{\gamma_T} p_{\gamma_t}(x)\ dx$$

- Using the inverse Laplace Transform

$$p_{\gamma_t}(x) = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) e^{sx}\ dx$$




  where $c > 0$ is a small constant that lies in the region of convergence of the integral, we obtain:

  - 1.

$$P_{out} = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) \int_0^{\gamma_T} e^{sx}\ dx\ ds = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) e^{\gamma_T s}\ \frac{ds}{s}$$

    $\big($lower integral limit is 0 since $p_{\gamma_t}(0) = 0\big)$

  - and 2.:

$$p_{\gamma_t}(x) = \int_0^x p_{\gamma_t}(t)\ dt = 0$$

    for $x = 0$ note: $p_{\gamma_t}(x) \xleftrightarrow[transform]{Laplace} \frac{1}{s} M_{\gamma_t}(s)$

**General combining scheme**

$$y = \Big(\sum_{n=1}^{N_R} h_n w_n\Big) x + \sum_{n=1}^{N_R} w_n n_n$$

$$\gamma_t = \frac{\epsilon_s \left|\sum_{n=1}^{N_R} h_n w_n\right|^2}{\sigma_n^2 \sum_{N=1}^{N_R} |w_n|^2}$$

where $w_n$ depends on the particular combining scheme.

## 1.2 MRC (Maximum Ratio Combining)

- what weight $w_n$ maximize $\gamma_t$?
  - Cauchy-Schwarz inequality

$$\Big|\sum_{n=1}^{N_R} h_n w_n\Big|^2 \leq \sum_{n=1}^{N_R} |h_n|^2 \cdot \sum_{n=1}^{N_R} |w_n|^2$$

    where equality holds if and only if $w_n = c \cdot h_n^*$ for some non-zero constant $c$.

  - for $w_n = h_n^*$, we obtain

$$\gamma_t = \frac{\epsilon_s}{\sigma_n^2} \cdot \frac{\left(\sum_{n=1}^{N_R} |h_n|^2\right)^2}{\sum_{n=1}^{N_R} |h_n|^2} = \frac{\epsilon_s}{\sigma_n^2} \sum_{n=1}^{N_R} |h_n|^2$$

  - $w_n = h_n^* \forall n$ are the MRC combining weights.

- For performance analysis we assume independent identically distributed (IID) Rayleigh fading

$$\begin{aligned}
\rightarrow \mathcal{E}\{|h_n|^2\} = 1; \quad \bar{\gamma} = \frac{\epsilon_s}{\sigma_n^2}; \quad \gamma_n = \frac{\epsilon_s}{\sigma_n^2}|h_n|^2 \\
p_\gamma(x) &= \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0 \\
M_\gamma(s) &= \frac{1}{1 + s\bar{\gamma}}
\end{aligned}$$

- Error rate

$$\gamma_t = \sum_{n=1}^{N_R} \gamma_n$$

  $\rightarrow$ sum of IID random variables (r.v.s.)

$$M_{\gamma_t}(s) = \Big(M_\gamma(s)\Big)^{N_R} = \frac{1}{(1 + s\bar{\gamma})^{N_R}} = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{1}{(s + \frac{1}{\bar{\gamma}})^{N_R}}$$

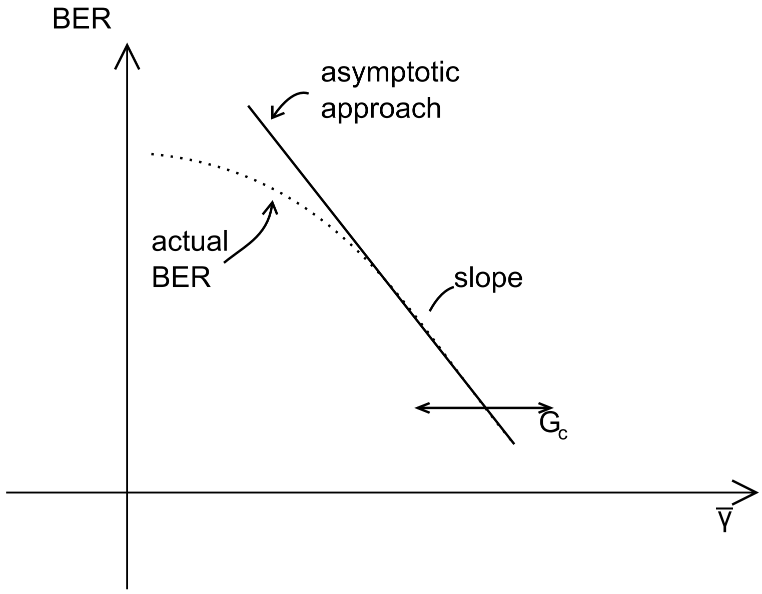
  inverse Laplace-transform (from tables)

$$p_{\gamma_t}(x) = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{x^{N_R - 1}}{(N_R - 1)!} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0$$

- Direct approach

$$p_e = \int_0^\infty a \cdot Q\big(\sqrt{ax}\big) p_{\gamma_t}(x)\ dx = a\Big(\frac{1-\mu}{2}\Big)^{N_R} \cdot \sum_{n=0}^{N_R - 1} \binom{N_R - 1 + n}{n} \Big(\frac{1+\mu}{2}\Big)^n$$

  where $\mu = \sqrt{\dfrac{b\bar{\gamma}}{2 + b\bar{\gamma}}}$




- MGF approach

$$
\begin{aligned}
p_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\Big(\frac{b}{2\sin^2\theta}\Big)\ d\theta \\
&= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\bar{\gamma}^{N_R}\big(\frac{b}{\sin^2\theta} + \frac{1}{\bar{\gamma}}\big)^{N_R}}\ d\theta \quad \text{(numerisch berechnen!)}
\end{aligned}
$$

- high SNR: $\quad \bar{\gamma} \to \infty \Longleftrightarrow \frac{1}{\bar{\gamma}} \to 0$

$$
p_e = \frac{a}{\pi} \cdot \frac{1}{\bar{\gamma}^{N_R}} \cdot \Big(\frac{2}{b}\Big)^{N_R} \int_0^{\frac{\pi}{2}} \sin^{2N_R}\theta\ d\theta
$$

$$
\begin{aligned}
\text{(from MGF approach: } &\int_0^{\frac{\pi}{2}} \sin^{2N_R}\theta\ d\theta = \frac{\pi}{2^{N_R+1}} \cdot \binom{2N_R}{N_R)} \\
&= \frac{a}{2^{N_R+1} \cdot b^{N_R}} \big(2N_R \quad N_R\big) \frac{1}{\bar{\gamma}^{N_R}} \quad \text{as } \bar{\gamma} \to \infty \\
&\stackrel{!}{=} \Big(\frac{1}{G_c \bar{\gamma}}\Big)
\end{aligned}
$$

where: Diversity gain: $G_d = N_R$

Combining/Coding gain: $G_c = 2b\Big(\frac{a}{2}\binom{2N_R}{N_R}\Big)^{-\frac{1}{N_R}}$

- MRC exploits the maximal possible diversity
- Diversity gain is not affected by correlation as the branches are not <u>fully</u> correlated
- Diversity gain depends on fading distribution

**Outage probability**

$$P_{out} = \int_0^{\gamma_T} p_{\gamma_t}(x)\ dx = \frac{1}{\bar{\gamma}^{N_R}} \int_0^{\gamma_T} \frac{x^{N_R-1}}{(N_R-1)!} e^{-\frac{x}{\bar{\gamma}}}\ dx$$
$$= 1 - e^{-\frac{\gamma_T}{\bar{\gamma}}} \cdot \sum_{n=1}^{N_R} \frac{\left(\frac{\gamma_T}{\bar{\gamma}}\right)^n}{(n-1)!}$$

- Approximation (Taylor series): $\bar{\gamma} \to \infty : -e^{-\frac{x}{\bar{\gamma}}} = 1 - \frac{x}{\bar{\gamma}} + O(\frac{1}{\bar{\gamma}})$ where a function $f(x)$ is $O(x)$ if $\lim\limits_{x\to\infty} \frac{f(x)}{x} = 0$.

$$\Rightarrow P_{out} = \frac{1}{\gamma^{N_R}} \int_0^{\gamma_T} \frac{x^{N_R-1}}{(N_R-1)!} \left(1 - \frac{x}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right)\right)$$

- Diversity and coding gain can also be defined for $P_{out}$

## 1.3 EGC (Equal Gain Combining)

**Combining Weights**

- For MRC, both, the amplitudes and phases of the channel gains $h_n = |h_n|e^{j\varphi_n}$ have to be known (or estimated in practice)
- In EGC it is assumed that only the phases are known and weights $w_n = e^{-j\varphi_n}$ are used.

$$\Rightarrow \gamma_t = \frac{\mathcal{E}_s}{\sigma_n^2} \frac{\left|\sum\limits_{n=1}^{N_R} |h_n| e^{j\varphi_n} e^{-j\varphi_n}\right|^2}{\sum\limits_{n=1}^{N_R} |e^{-j\varphi_n}|^2} = \frac{\mathcal{E}_s}{\sigma_n^2} \frac{1}{N_R} \left(\sum_{n=1}^{N_R} |h_n|\right)^2$$
$$= \frac{1}{N_R} \left(\sum_{n=1}^{N_R} \sqrt{\gamma_n}\right)^2 ; \text{ with } \gamma_n = \frac{\mathcal{E}_s}{\sigma_n^2} |h_n|^2$$

**Performance Analysis**

- IID case
  $\Rightarrow \sqrt{\gamma_n}$ is Rayleigh distributed
  $\Rightarrow$ Exact analysis is much more difficult than for MRC $\Rightarrow$ see book by Simon & Alouini p.341
- Approximate result

$$P_e = \frac{a}{2} \left[ 1 - \sqrt{\frac{2b\bar{\gamma}}{5 + 2b\bar{\gamma}}} \sum_{n=0}^{N_R-1} \frac{\binom{2n}{n}}{4^n (1 + \frac{2}{5} b\bar{\gamma})^n} \right]$$

- high SNR
  $\Rightarrow$ use high SNR analysis of Wang & Giannakis, 2003
  $\Rightarrow$ at high SNR, only pdf of $\gamma_n$ around 0 is relevant for performance

$$\Rightarrow \overset{\text{Rayleigh}}{p_\gamma(x)} = \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}} \overset{\text{Taylor Serie}}{=} \frac{1}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right) \text{ as } x \to 0$$

- need pdf $\gamma_t$: ($\gamma_n$ bekannt, $\to$ ges.: Wurzel, etc.)
  (cumulative distribution function of $\sqrt{\gamma}$ ($\overset{\text{i.i.d}}{=}$ $\sqrt{\gamma_n}$) (cdf))

$$P_{\sqrt{\gamma}}(x) = \Pr\{\sqrt{\gamma} \le x\} = \Pr\{\gamma \le x^2\} = P_\gamma(x^2) = \text{cdf of } \gamma$$
$$\to p_{\sqrt{\gamma}}(x) = \frac{d}{dx} P_{\sqrt{\gamma}}(x) = 2x \cdot p_\gamma(x^2) = \frac{2x}{\bar{\gamma}} + O\big(\frac{1}{\bar{\gamma}}\big)$$

- Laplace Transformation to MGF

$$\to M_{\sqrt{\gamma}}(s) = \mathcal{L}\{p_{\sqrt{\gamma}}(x)\} = \frac{2}{\bar{\gamma}} \cdot \frac{1}{s^2} + O\big(\frac{1}{\bar{\gamma}}\big)$$
$$\sqrt{\gamma_t} = \sum_{n=1}^{N_R} \frac{\sqrt{\gamma_n}}{N_R}$$
$$M_{\sqrt{\gamma_t}}(s) = \mathcal{E}\Big\{\exp(-s\sqrt{\gamma_t})\Big\} = \mathcal{E}\Big\{\exp(-\frac{s}{\sqrt{N_R}} \cdot \sum_{n=1}^{N_R} \sqrt{\gamma_n})\Big\} = \Big(\mathcal{E}\Big\{\exp(-\frac{s}{\sqrt{N_R}} \cdot \sqrt{\gamma_n}\Big\}\Big)^{N_R}$$
$$= \Big(M_{\sqrt{\gamma}}(\frac{s}{\sqrt{N_R}})\Big)^{N_R} = \Big(\frac{2}{\bar{\gamma}} \cdot \frac{N_R}{s^2}\Big)^{N_R} + O\Big(\frac{1}{\bar{\gamma}^{N_R}}\Big)$$

- inverse Laplace Transform

$$p_{\sqrt{\gamma_t}}(x) = \mathcal{L}^{-1}\Big\{M_{\sqrt{\gamma_t}}(s)\Big\} = \Big(\frac{2N_R}{\bar{\gamma}}\Big)^{N_R} \cdot \frac{x^{2N_R-1}}{(2N_R-1)!} + O\Big(\frac{1}{\bar{\gamma}^{N_R}}\Big)$$
$$P_{\gamma_t}(x) = \Pr\{\gamma_t \le x\} = \Pr\{\sqrt{\gamma_t} \le \sqrt{x}\} = P_{\sqrt{\gamma}_t}(\sqrt{x}) \to \text{cdf of } \sqrt{\gamma_t}$$
$$p_{\gamma_t}(x) = \frac{d}{dx} P_{\gamma_t}(x) = \frac{1}{2\sqrt{x}} \cdot p_{\gamma_t}(\sqrt{x}) = \frac{1}{2}\Big(\frac{2N_R}{\bar{\gamma}}\Big)^{N_R} \cdot \frac{x^{N_R-1}}{(2N_R-1)!} + O\big(\bar{\gamma}^{-N_R}\big)$$
$$\to M_{\gamma_t}(s) = \mathcal{L}\{p_{\gamma_t}(x)\} = \frac{1}{2}\Big(\frac{2N_R}{\bar{\gamma}}\Big)^{N_R} \cdot \frac{(N_R-1)!}{(2N_R-1)!\; b^{N_R}} + O\big(\bar{\gamma}^{-N_R}\big)$$

- Error Probability:

$$
\begin{aligned}
P_e &= \frac{a}{\pi} \int\limits_0^{\frac{\pi}{2}} M_{\gamma_t}\left(\frac{b}{2\sin^2(\theta)}\right) \mathrm{d}\theta \\
&= \frac{a}{\pi}\frac{1}{2}\left(\frac{2N_R}{\bar{\gamma}}\right)^{N_R} \frac{(N_R-1)!}{(2N_R-1)!}\frac{2^{N_R}}{b^{N_R}} \underbrace{\int\limits_0^{\frac{\pi}{2}} \sin^{2N_R}(\theta)\mathrm{d}\theta}_{\frac{\pi}{2^{2N_R+1}}\binom{2N_R}{N_R}=\frac{\pi(2N_R)!}{2^{2N_R+1}(N_R!)^2}} + O\left(\frac{1}{\bar{\gamma}^{N_R}}\right) \\
&= \frac{aN_R^{N_R}}{2b^{N_R}N_R!}\frac{1}{\bar{\gamma}^{N_R}} + O\left(\frac{1}{\bar{\gamma}^{N_R}}\right) \overset{!}{=} \left(\frac{1}{G_c}\right)^{G_d}
\end{aligned}
$$

$\Longrightarrow$ Diversity gain: $G_d = N_R$

$\Longrightarrow$ Combining gain: $G_c = \frac{b}{N_R}\left(\frac{2N_R!}{a}\right)^{\frac{1}{N_R}}$

vergleiche auch Blatt mit Kurven $\overline{\underline{\text{III}}}$ und $\overline{\underline{\text{IV}}}$

A similar asymptotic analysis can be conducted for the outage probability.

## 1.4 SC (Selection Combining)

### Combining weights

- only the strongest branch is chosen
- strongest branch: $\hat{n} = \underset{n}{\operatorname{argmax}} \gamma_n \longrightarrow \gamma_t = \gamma_{\hat{n}}$
- only on RF receiver chain required $\rightarrow$ saves hardware complexity

### Performance analysis

- cdf of: $\gamma_t$

$$
\begin{aligned}
P_{\gamma_t}(x) &= \Pr\{\gamma_{\hat{n}} \le x\} = \Pr\{\gamma_1 \le x \cap \gamma_2 \le x \cap \ldots \gamma_{N_R} \le x\} \\
&\overset{(IID)}{=} \Big(\Pr\{\gamma_n \le x\}\Big)^{N_R} = \Big(P_\gamma(x)\Big)^{N_R}
\end{aligned}
$$

- pdf:

$$
\begin{aligned}
p_{\gamma_t}(x) &= \frac{d}{dx}P_{\gamma_t}(x) = N_R\big(P_\gamma(x)\big)^{N_R-1} \cdot p_\gamma(x) \\
\text{where:} \qquad p_{\gamma_t}(x) &= \frac{1}{\bar{\gamma}}e^{-\frac{x}{\bar{\gamma}}}; \quad x \ge 0 \\
P_\gamma(x) &= \int_0^x p_\gamma(x)\ dx = 1 - e^{-\frac{x}{\bar{\gamma}}}; \quad x \ge 0 \\
\rightarrow p_{\gamma_t}(x) &= \frac{N_R}{\bar{\gamma}}\big(1 - e^{-\frac{x}{\bar{\gamma}}}\big)^{N_R-1} e^{-\frac{x}{\bar{\gamma}}}; \quad x \ge 0
\end{aligned}
$$

**Error probability**

- direct approach → closed-form solution possible
- MGF approach
  - Binomial expansion

$$
\begin{aligned}
p_{\gamma_t}(x) &= \frac{N_R}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} 1^{N_R-1-n} \left(-e^{-\frac{x}{\bar{\gamma}}}\right)^n \\
&= \frac{N_R}{\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} \cdot (-1)^n e^{-\frac{x(n+1)}{\bar{\gamma}}}; \quad x \geq 0
\end{aligned}
$$

  - MGF

$$
M_{\gamma_t}(s) = \frac{N_R}{\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} (-1)^n \frac{1}{s + \frac{n+1}{\bar{\gamma}}}
$$

  -

$$
P_e = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\Big(\frac{b}{2\sin^2\theta}\Big)\, d\theta = \frac{aN_R}{\pi\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} (-1)^n \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\frac{b}{2\sin^2\theta} + \frac{n+1}{\bar{\gamma}}}
$$

→ can be evaluated numerically

  - high SNR approach ⇒ $\bar{\gamma} \to \infty$

$$
\begin{aligned}
p_{\gamma_t} &= \frac{N_R}{\bar{\gamma}} \left[1 - \exp\left(-\frac{x}{\bar{\gamma}}\right)\right]^{N_R-1} \exp\left(-\frac{x}{\bar{\gamma}}\right) \\
&\overset{\bar{\gamma}\to\infty}{=} \frac{N_R}{\bar{\gamma}} \left[1 - \left(1 - \frac{x}{\bar{\gamma}} + O\left(\bar{\gamma}^{-1}\right)\right)\right]^{N_R-1} \left(1 - \frac{x}{\bar{\gamma}} + O\left(\bar{\gamma}^{-1}\right)\right) i \\
&= \frac{N_R}{\bar{\gamma}^{N_R}} x^{N_R-1} + o\left(\bar{\gamma}^{-N_R}\right)
\end{aligned}
$$

  - MGF:

$$
\begin{aligned}
M_{\gamma_t}(s) &= \frac{N_R}{\bar{\gamma}^{N_R}} \frac{(N_R-1)!}{s^{N_R}} + O\left(\bar{\gamma}^{-N_R}\right) \\
\Big[\to P_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\left(\frac{b}{2\sin^2(\theta)}\right) \mathrm{d}\theta\Big] \\
&= \frac{a(2N_R)!}{b^{N_R} 2^{N_R+1} N_R!} \frac{1}{\bar{\gamma}^{N_R}} + O(\bar{\gamma}^{-N_R})
\end{aligned}
$$

$\Longrightarrow$ Diversity gain: $G_d = N_R$

$\Longrightarrow$ Combining gain: $G_c = 2b \left(\frac{2N_R!}{a(2N_R)!}\right)^{\frac{1}{N_R}}$

– Outage Probability

$$P_{out} = \Pr\{\gamma_{\hat{n}} \le \gamma_T\} = P_{\gamma_{\hat{n}}}(\gamma_T) = \left[1 - \exp\left(-\frac{\gamma_T}{\bar{\gamma}}\right)\right]^{N_R}$$

$$\text{high SNR: } P_{out} = \left(\frac{\gamma_T}{\bar{\gamma}}\right)^{N_R} + O\left(\bar{\gamma}^{-N_R}\right)$$

## 1.5 Comparison

- Diversity Gain:
  MRC, EGC and SC all achieve the maximum possible diversity gain of $G_d = N_R$
- Combining Gain:
  The combining gains of MRC, EGC and SC are different
  – MRC/EGC:

$$\frac{G_C^{EGC}}{G_C^{MRC}} = \frac{\frac{1}{2b}\left(\frac{a}{2}\binom{2N_R}{N_R}\right)^{\frac{1}{N_R}}}{\frac{N_R}{b}\left(\frac{a}{2}\frac{1}{N_R!}\right)^{\frac{1}{N_R}}} = \frac{[(2N_R)!]^{\frac{1}{N_R}}}{2N_R(N_R)^{\frac{1}{N_R}}} \le 1$$

(independent of a or b which are modulation parameters, only depends on number of antennas)

$$N_R \gg 1: \qquad N_R! \approx \sqrt{2\pi} e^{-N_R} N_R^{N_R + \frac{1}{2}} \qquad (Stirling)$$

$$\left.\frac{G_C^{EGC}}{G_C^{MRC}}\right|_{N_R \gg 1} = \frac{\left(\sqrt{2\pi} e^{-2N_R}(2N_R)^{2N_R+\frac{1}{2}}\right)^{\frac{1}{N_R}}}{2N_R\left(\sqrt{2\pi} e^{-N_R} N_R^{N_R+\frac{1}{2}}\right)^{\frac{1}{N_R}}} = \frac{2 \cdot 2^{\frac{1}{2N_R}}}{2} \overset{N_R \to \infty}{\to} \frac{2}{e} \equiv -1.3\text{dB}$$

  – MRC/SC:

$$\frac{G_C^{SC}}{G_C^{MRC}} = \frac{2b\left(\frac{a}{2}\binom{2N_R}{N_R}\right)^{\frac{1}{N_R}}}{2b\left(\frac{a}{2}\frac{(2N_R)!}{N_R!}\right)^{\frac{1}{N_R}}} = \frac{1}{(N_R!)^{\frac{1}{N_R}}} \le 1$$

$$\left.\frac{G_C^{SC}}{G_C^{MRC}}\right|_{N_R \gg 1} = \frac{1}{\sqrt{2\pi}^{\frac{1}{N_R}} e^{-1} N_R^{1+\frac{1}{2N_R}}} N_R \overset{\to}{\to} \infty \frac{e}{N_R}$$

$\to$ loss increases with $N_R$

# 2 MISO Systems

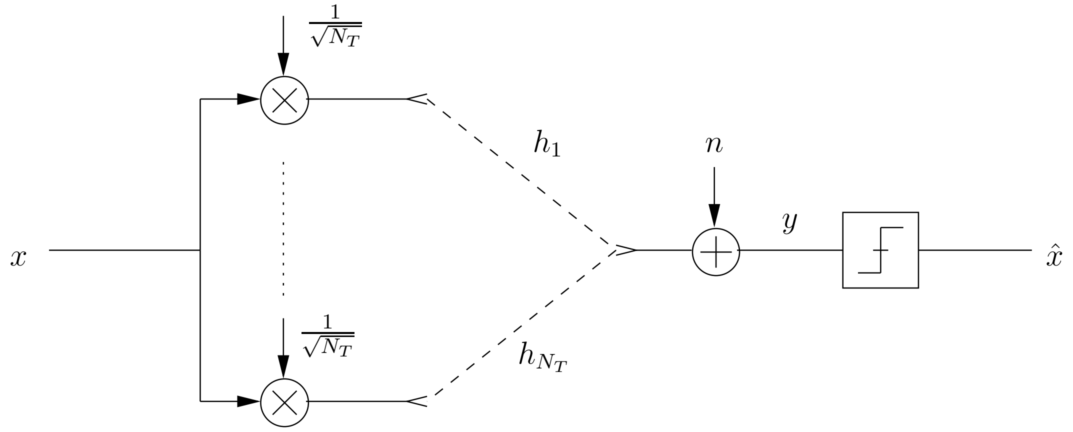
### Remarks

- Similar to SIMO systems, in MISO systems only coding and diversity gains can be obtained.

- To realize these gains, a careful transmitter design is necessary
- System design depends on whether or not channel state information (**CSI**) is available at transmitter

## 2.1 Naive Approach

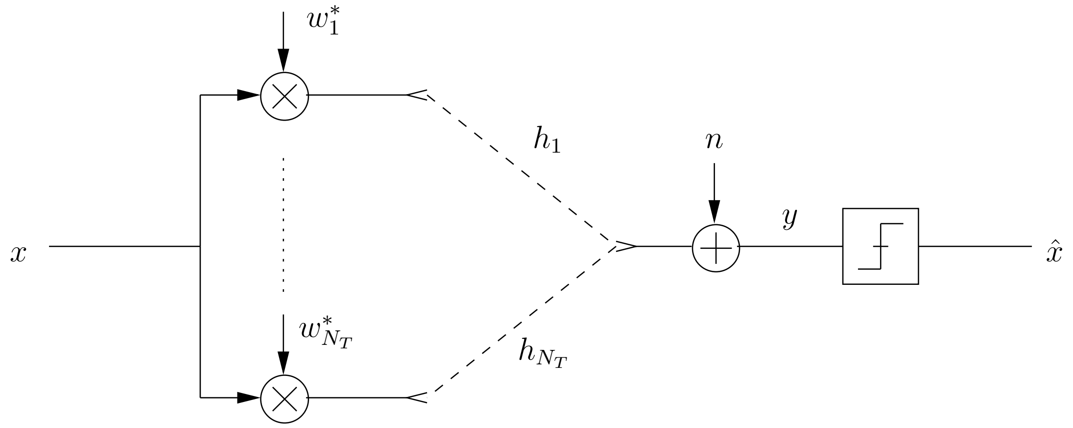
- Assume we simply send the same signal over all $N_T$ transmit antennas




- Transmit power: $\mathcal{E}\left\{\left|\frac{1}{\sqrt{N_T}}x\right|^2 +, \ldots, \left|\frac{1}{\sqrt{N_T}}x\right|^2\right\} = \mathcal{E}\left\{N_T \frac{1}{N_T}|x|^2\right\} = \mathcal{E}_s$
- Received signal: $y = \frac{1}{\sqrt{N_T}} \sum_{n=1}^{N_T} h_n \cdot x + n$
- Rayleigh fading: $h_n$ are zero mean complex gaussian random variables
  $\rightarrow h$ is also zero mean complex gaussian
- i.i.d.:
  - $\mathcal{E}\{|h_n|^2\} = 1 \; \forall n$
  - $\mathcal{E}\{|h|^2\} = \frac{1}{N_T}\mathcal{E}\left\{\left|\sum_{n=1}^{N_T} h_n\right|^2\right\} = \frac{1}{N_T}\mathcal{E}\left\{\sum_{n=1}^{N_T} |h_n|^2\right\} = 1$
  - statistical properties of h are independent of $N_T$
  - the multiple transmit antennas have no benefit at all
  - <u>more sophisticated transmitter designs necessary</u>

## 2.2 Full CSI Available at the Transmitter

- $h_n, n \in \{1, \ldots, N_T\}$ is known at the transmitter
- Perform "precoding" (beamforming) with coefficients $w_n$




- Transmit Power: Two constraints maybe considered
  - Average transmit power constraint

$$P_{av} = \mathcal{E}\left\{\sum_{n=1}^{N_T} |w_n^* x|^2\right\} = \sum_{n=1}^{N_T} |w_n|^2 \underbrace{\mathcal{E}\{|x|^2\}}_{\mathcal{E}_s} = \mathcal{E}_s \Rightarrow \sum_{n=1}^{N_T} |w_n|^2 = 1$$

  - Power constraint for each transmit antenna

$$\rightarrow |w_n| = \frac{1}{\sqrt{N_T}} \qquad \rightarrow P_{av} = \mathcal{E}_s$$

- Recveived signal: $y = \underbrace{\sum_{n=1}^{N_T} w_n^* h_n}_{h} x + n$ (equivalent SISO channel)

**Maximum Ratio Transmission (MRT)**

- we have only the average power constraint: $\sum_{n=1}^{N_T} |w_n|^2 = 1$
- SNR: $\gamma_t = \frac{\mathcal{E}_s |h|^2}{\sigma_n^2} = \frac{\mathcal{E}_s \left|\sum_{n=1}^{N_T} w_n^* \cdot h_n\right|^2}{\sigma_n^2}$
- Maximize SNR under constraint $\sum_{n=1}^{N_T} |w_n|^2 = 1$
- constraint optimization problem $\rightarrow$ Lagrange method

$$L = \frac{\mathcal{E}_s}{\sigma_n^2} \left|\sum_{n=1}^{N_T} w_n^* \cdot h_n\right|^2 + \lambda \left(\sum_{n=1}^{N_T} |w_n|^2 - 1\right); \quad \text{where: } \lambda = \text{Lagrange Multiplier}$$

⇒ Wirtinger Kalkül: treat $z$ and $z^*$ as independent variables for differentiation:

$$\frac{\partial z^*}{\partial z} = 0; \quad \frac{\partial |z|^2}{\partial z} = \frac{\partial z \cdot z^*}{\partial z} = z^*$$
$$\frac{\partial x^2}{\partial x} = 2x; \quad \frac{\partial (z^*)^2}{\partial z^*} = 2 \cdot z^*; \frac{\partial |z|^2}{\partial z} = z^*$$

$$\frac{\partial L}{\partial w_m^*} = \frac{\epsilon_s}{\sigma_n^2} \Big( \sum_{n=1}^{N_T} w_n^* \cdot h_n \Big)^* h_m + \lambda w_m$$

$$\rightarrow w_m = \underbrace{\frac{\epsilon_s}{\sigma_n^2 \cdot \lambda} \Big( \sum_{n=1}^{N_T} w_n^* h_n \Big)^*}_{\text{const., independent of } m := c} h_m$$

$$\rightarrow w_m = c \cdot h_m$$

$$\rightarrow \sum_{n=1}^{N_T} |w_n|^2 = 1 \rightarrow c^2 = \frac{1}{\sum\limits_{n=1}^{N_T} |h_n|^2}$$

$$\rightarrow w_n = \frac{h_n}{\sqrt{\sum\limits_{n=1}^{N_T} |h_n|^2}} \equiv \text{MRT gains}$$

$$\rightarrow \text{SNR} = \frac{\mathcal{E}_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} \frac{|h_n|^2}{\sqrt{\sum_{n=1}^{N_T} |h_m|^2}} \right|^2 = \frac{\epsilon_s}{\sigma_n^2} \sum_{n=1}^{N_T} |h_n|^2$$

⇒ same SNR as for maximum ration combining (MRC)

⇒ MRT with $N_T$ transmit antennas achieves the same performance as MRC with $N_T$ receive antennas

⇒ MRT/MRC can be extended to $N_T \times N_R$ MIMO systems

→ has the same performance as MRC with $N_T \cdot N_R$ receive antennas and one transmit antenna

**Equal Gain Transmission (EGT)**

- we employ gains: $w_n = \frac{1}{\sqrt{N_T}} \cdot \frac{h_n}{|h_m|} \rightarrow |w_n| = \frac{1}{\sqrt{N_T}}$

- SNR:

$$\begin{aligned}\gamma_t &= \frac{\mathcal{E}_s}{\sigma_n^2}\left|\sum_{n=1}^{N_T} w_n^* h_n\right|^2 \\ &= \frac{\mathcal{E}_s}{\sigma_n^2}\left|\sum_{n=1}^{N_T} \frac{1}{\sqrt{N_T}\cdot\frac{|h_n|^2}{|h_n|}}\right|^2 = \frac{1}{N_T}\cdot\frac{\mathcal{E}_s}{\sigma_n^2}\left|\sum_{n=1}^{N_T}|h_n|\right|^2 \\ \gamma_n &= \frac{\mathcal{E}_s}{\sigma_n^2}|h_n|^2 \\ &\text{same SNR as for EGC} \rightarrow \gamma_t = \frac{1}{N_T}\left|\sum_{n=1} N_T\sqrt{\gamma_n}\right|^2\end{aligned}$$

$\rightarrow$ EGC with $N_T$ transmit antennas achieves the same performance as EGC with $N_T$ receive antennas

**Transmit Antennas Selection**

- select antenna with maximum channel gain for transmission:

$$w_n = \begin{cases}\frac{h_n}{|h_n|}, & \text{if } n = \hat{n} \\ 0, & \text{otherwise}\end{cases} \text{ where } \hat{n} = \underset{n}{\operatorname{argmax}}|h_n|$$

- antenna selection with $N_T$ transmit antennas achieves the same performance as *Selection Combining* with $N_T$ receive antennas

## 2.3 No CSI at Transmitter - Space-Time-Coding

- $h_n, n \in \{1, \ldots, N_T\}$, is only known at the receiver
- "Space-time-coding" has to be employed to realize diversity gain
- $T \times N_T$ matrics $\mathbf{X}$ are transmitted in $T$ symbol intervals over $N_T$ antennas
- $\mathbf{X}$ is drawn from a matrix alphabet $\mathcal{X}$
- Example:

$$\mathbf{X} = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,N_T} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,N_T} \\ \vdots & \vdots & \ddots & \vdots \\ x_{T,1} & x_{T,2} & \cdots & x_{T,N_T}\end{pmatrix}$$

- We distinguish:
  - Space-time-block-codes (STBCs)
    $\rightarrow$ $\mathbf{X}$ is obtained by mapping $K$ scalar symbols $s_k,\ k = 1, \ldots, K$ from a scalar alphabet $\mathcal{A}$ to matrix $\mathbf{X}$

- Space-time-trellis-codes (STTCs)
  $\rightarrow$ $\mathbf{X}$ is obtained from scalar symbols $s_k$ through a trellis encoding process.
  [see: Tarokh, Seshadri, Calderbank: Space-time-codes for high datarate wireless communication: Performance criterions and coder construction; IEEE Trans. Inf. Theory 1998]
- here: We concentrate on space-time-block-codes (STBCs), but many results can be easily extended to space-time-trellis-codes

- STBCs:
  - $K$ $M$-ary scalar symbols (e.g. $M$-PSK symbols) are mapped to STBC matrices $\mathbf{X}$
    $\mathbf{S} = [s_1, \ldots, s_K] \rightarrow \mathbf{X}$
    $s_k \in \mathcal{A} \rightarrow x \in \mathcal{X}$ with $|\mathcal{X}| = M^K$
  - Example: "Alamouti"-Code

$$\mathbf{X} = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix}$$

    [Alamouti: A simple transmit diversity technique for wireless communication, IEEE JSAC 1998]

**Optimal Detection**

- Signal model:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_T \end{pmatrix} = \mathbf{X} \begin{pmatrix} h_1 \\ \vdots \\ h_{N_T} \end{pmatrix} + \begin{pmatrix} n_1 \\ \vdots \\ n_T \end{pmatrix}$$

$$\mathbf{y} = \mathbf{X} \cdot \mathbf{h} + \mathbf{n}$$

- Optimal detection - ML-detection
  - $\mathbf{h}$ is known at receiver
  - $\mathbf{n}$ is AWGN with $\mathcal{E}\{\mathbf{n} \cdot \mathbf{n}^{\mathbf{H}}\} = \sigma_{\mathbf{n}}^2 \cdots \mathbf{I}_{T\times T}$
  - $p(\mathbf{y}|\mathbf{x})$

$$= \frac{1}{\pi^T |\sigma_n^2 \mathbf{I}_{T\times T}|} \exp\left(-(\mathbf{y} - \mathbf{xh})^H (\sigma_n^2 \mathbf{I}_{T\times T})^{-1} (\mathbf{y} - \mathbf{xh})\right)$$
$$= \frac{1}{\pi^T \sigma_n^{2T}} \exp\left(-\frac{1}{\sigma_n^2} (\mathbf{y} - \mathbf{xh})^H (\mathbf{y} - \mathbf{xh})\right) = \frac{1}{\pi^T \sigma_n^{2T}} \exp\left(||\mathbf{y} - \mathbf{xh}||^2\right)$$

    $\rightarrow$ the optimal estimate $\hat{\mathbf{X}}$ or equivalently the optimal estimate $\hat{\mathbf{s}}$ can be obtained as

$$\hat{\mathbf{s}} = \underset{\mathbf{s}\in\mathcal{A}^K}{\operatorname{argmax}}\, p(\mathbf{y}|\mathbf{x}) = \underset{\mathbf{s}\in\mathcal{A}^K}{\operatorname{argmin}} ||\mathbf{y} - \mathbf{hx}||^2$$

  - Disadvantage: In general, metric $||\mathbf{y} - \mathbf{hx}||^2$ has to be calculated $M^K$ times
    $\rightarrow$ complexity increases exponentially with $K$