MIMO Skript - Wintersemester 2012/13
Kapitel 2 - 4

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2 Point - to - Point MIMO Systems

2.1 MIMO Channel Capacity

Inhalt: s. Skript in VL ausgeteilt

2.2 SIMO Systems

Remarks

- In SIMO Systems only coding and diversity gains can be exploited (no multiplexing gains)
- To realize these gains diversity combining has to be performed
- Diversity combining schemes vary in complexity and performance
- There are many diversity combining schemes. Here we consider:
  - Maximal ratio combining (MRC)
  - Equal gain combining (EGC)
  - Selection combining (SC)
- Diversity combining problem

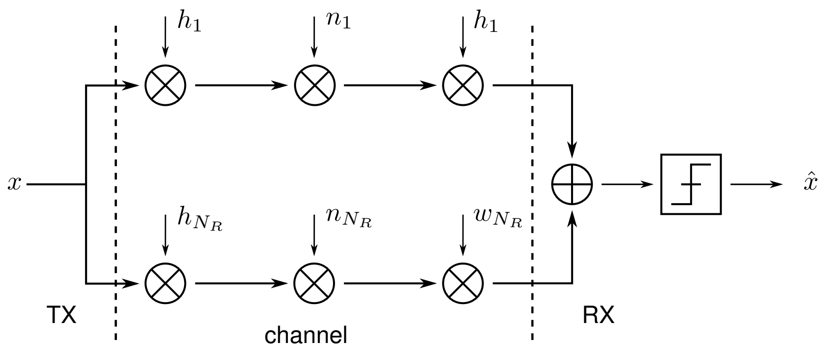


Abbildung 1: Block Diagramm for SIMO

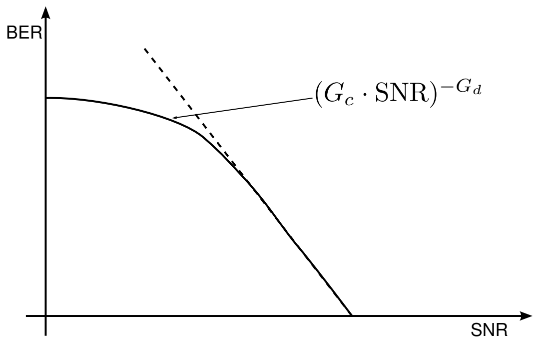
  - how to choose combining weights w_n?
  - what performance (e.g. error rate, outage probability) is achieved?
  - what diversity and coding/combining gain is achieved?

2.2.1 Preliminaries

Consider an equivalent system:

$$y = hx + n;$$

$$\mathcal{E}\{|x^2|\} = \mathcal{E}_s; \qquad \mathcal{E}\{|n^2|\} = \sigma_n^2; \qquad \mathcal{E}\{|h|^2\} = 1$$



- G_c : Coding gain
- G_d : Diversity gain

Abbildung 2: Exemplary BER for SIMO

- Instantaneous SNR: $\gamma_t = \frac{\mathcal{E}_s}{\sigma_n^2} \cdot |h|^2$
- Average SNR: $\bar{\gamma}_t = \mathcal{E}\{\gamma_t\} = \frac{\mathcal{E}_s}{\sigma_n^2}$

Bit and Symbol Error Rate

- The Bit and Symbol Error Rate of many modulation schemes can be expressed for given γ_t as:

$$P_e(\gamma_t) = aQ\big(\sqrt{b\gamma_t}\big)$$

where:

- $Q(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_x^\infty e^{-\frac{t^2}{2}}\ dt$
- $P_e(\gamma_t)$ may be exact result or approximation
- BPSK: exact with $a = 1, b = 2$
- M-ary QAM: tight approximation with $a = 4\big(1 - \frac{1}{\sqrt{M}}\big), b = \frac{3}{M-1}$

$\big(Einschub : Gray - Code : BER = \frac{1}{\log_2 M} \cdot SER\big)$

- Alternative representation of Q - function:

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}\ d\theta$$

  $\rightarrow$ Integral limits are fixed and do not depend on integration variables!

- Average error probability

$$P_e = \mathcal{E}\big\{P_e(\gamma_t)\big\} = \int_0^\infty aQ\big(\sqrt{bx}\big)p_{\gamma_t}(x)\ \mathrm{d}x$$

  - Integral may be difficult to solve analytically
  - Integral has infinite support $\rightarrow$ numerical evaluation difficult

- Using alternative representation of Q-function we get:

$$P_e = \int_0^\infty \frac{a}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{bx}{2\sin^2\theta}} p_{\gamma_t}(x)\ d\theta\ dx$$
$$= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} \int_0^\infty p_{\gamma_t}(x) e^{-\frac{b}{2\sin^2\theta}x}\ dx\ d\theta \qquad = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}(\frac{b}{2\sin^2\theta})\ d\theta$$

where:

  - $M_{\gamma_t}(s) = \int_0^\infty p_{\gamma_t}(x) e^{-sx}\ dx$ is the Laplace transform of p_{γ_t}
  - $M_{\gamma_t}(-s)$ is the so called Moment Generation Function (MGF) of p_{γ_t}
  - Here, we will also refer to $M_{\gamma_t}(s)$ as MGF
  - $M_{\gamma_t}(s)$ is sometimes easier to obtain than p_{γ_t}
  - The above integral can be easily evaluated numerically because of the finite integral limits

Outage probability

- The outage probability is the probability that the channel cannot support a certain rate, R, i.e. (where γ_T is the threshold SNR):

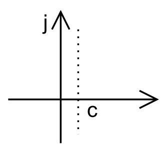
$$C = \log_2(1+\gamma_t) < R \quad \leftrightarrow \quad \gamma_t < 2^R - 1 \hat{=} \gamma_T$$

Thus, the outage probability is given by:

$$P_{out} = P_0\{\gamma_t < \gamma_T\} = \int_0^{\gamma_T} p_{\gamma_t}(x)\ dx$$

- Using the inverse Laplace Transform

$$p_{\gamma_t}(x) = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) e^{sx}\ dx$$



where $c > 0$ is a small constant that lies in the region of convergence of the integral, we obtain:

  - 1.

$$P_{out} = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) \int_0^{\gamma_T} e^{sx}\ dx\ ds = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) e^{\gamma_T s}\ \frac{ds}{s}$$

  (lower integral limit is 0 since $p_{\gamma_t}(0) = 0$)

  - and 2.:

$$p_{\gamma_t}(x) = \int_0^x p_{\gamma_t}(t)\ dt = 0$$

for $x = 0$ note: $p_{\gamma_t}(x) \xleftrightarrow[transform]{Laplace} \frac{1}{s} M_{\gamma_t}(s)$

General combining scheme

$$y = \Big(\sum_{n=1}^{N_R} h_n w_n\Big) x + \sum_{n=1}^{N_R} w_n n_n$$

$$\gamma_t = \frac{\mathcal{E}_s \Big| \sum_{n=1}^{N_R} h_n w_n \Big|^2}{\sigma_n^2 \sum_{n=1}^{N_R} |w_n|^2}$$

where w_n depends on the particular combining scheme.

2.2.2 MRC (Maximum Ratio Combining)

- what weight w_n maximize γ_t?
  - Cauchy-Schwarz inequality

    $$\Big| \sum_{n=1}^{N_R} h_n w_n \Big|^2 \leq \sum_{n=1}^{N_R} |h_n|^2 \cdot \sum_{n=1}^{N_R} |w_n|^2$$

    where equality holds if and only if $w_n = c \cdot h_n^*$ for some non-zero constant c.
  - for $w_n = h_n^*$, we obtain

    $$\gamma_t = \frac{\mathcal{E}_s}{\sigma_n^2} \cdot \frac{\Big(\sum_{n=1}^{N_R} |h_n|^2\Big)^2}{\sum_{n=1}^{N_R} |h_n|^2} = \frac{\mathcal{E}_s}{\sigma_n^2} \sum_{n=1}^{N_R} |h_n|^2$$

  - $w_n = h_n^* \, \forall \, n$ are the MRC combining weights.
- For performance analysis we assume <u>independent identically distributed (IID)</u> Rayleigh fading

  $$\rightarrow \mathcal{E}\{|h_n|^2\} = 1; \quad \bar{\gamma} = \frac{\mathcal{E}_s}{\sigma_n^2}; \quad \gamma_n = \frac{\mathcal{E}_s}{\sigma_n^2} |h_n|^2$$

  $$p_\gamma(x) = \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0$$

  $$M_\gamma(s) = \frac{1}{1 + s\bar{\gamma}}$$

- Error rate

  $$\gamma_t = \sum_{n=1}^{N_R} \gamma_n$$

  $\rightarrow$ sum of IID random variables (r.v.s.)

  $$M_{\gamma_t}(s) = \Big(M_\gamma(s)\Big)^{N_R} = \frac{1}{(1 + s\bar{\gamma})^{N_R}} = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{1}{(s + \frac{1}{\bar{\gamma}})^{N_R}}$$

  inverse Laplace-transform (from tables)

  $$p_{\gamma_t}(x) = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{x^{N_R - 1}}{(N_R - 1)!} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0$$

- Direct approach

$$P_e = \int_0^\infty a \cdot Q\big(\sqrt{ax}\big) p_{\gamma_t}(x)\ dx = a\Big(\frac{1-\mu}{2}\Big)^{N_R} \cdot \sum_{n=0}^{N_R-1} \binom{N_R - 1 + n}{n} \Big(\frac{1+\mu}{2}\Big)^n$$

where $\mu = \sqrt{\dfrac{b\bar{\gamma}}{2 + b\bar{\gamma}}}$

- MGF approach

$$\begin{aligned} P_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\Big(\frac{b}{2\sin^2\theta}\Big)\ d\theta \\ &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\bar{\gamma}^{N_R}\big(\frac{b}{2\sin^2\theta} + \frac{1}{\bar{\gamma}}\big)^{N_R}}\ d\theta \quad \text{(numerisch berechnen!)} \end{aligned}$$

- with high SNR: $\bar{\gamma} \to \infty \Longleftrightarrow \frac{1}{\bar{\gamma}} \to 0$ and with MGF approach this leads to Average Error Probability P_e:

$$\begin{aligned} P_e &= \frac{a}{\pi} \cdot \frac{1}{\bar{\gamma}^{N_R}} \cdot \Big(\frac{2}{b}\Big)^{N_R} \int_0^{\frac{\pi}{2}} \sin^{2N_R}\theta\ d\theta \\ &= \frac{a}{2^{N_R+1} \cdot b^{N_R}} \binom{2N_R}{N_R} \frac{1}{\bar{\gamma}^{N_R}} \quad \text{as } \bar{\gamma} \to \infty \\ &\overset{!}{=} \Big(\frac{1}{G_c\bar{\gamma}}\Big) \end{aligned}$$

where:

  - $\int_0^{\frac{\pi}{2}} \sin^{2N_R}\theta\ d\theta = \frac{\pi}{2^{N_R+1}} \cdot \binom{2N_R}{N_R}$
  - Diversity gain: $G_d = N_R$
  - Combining/Coding gain: $G_c = 2b\bigg(\frac{a}{2}\binom{2N_R}{N_R}\bigg)^{-\frac{1}{N_R}}$

- MRC exploits the maximal possible diversity
- Diversity gain is not affected by correlation as the branches are not fully correlated
- Diversity gain depends on fading distribution

Outage probability

$$\begin{aligned} P_{out} &= \int_0^{\gamma_T} p_{\gamma_t}(x)\ dx = \frac{1}{\bar{\gamma}^{N_R}} \int_0^{\gamma_T} \frac{x^{N_R-1}}{(N_R-1)!} e^{-\frac{x}{\bar{\gamma}}}\ dx \\ &= 1 - e^{-\frac{\gamma_T}{\bar{\gamma}}} \cdot \sum_{n=1}^{N_R} \frac{\big(\frac{\gamma_T}{\bar{\gamma}}\big)^n}{(n-1)!} \end{aligned}$$

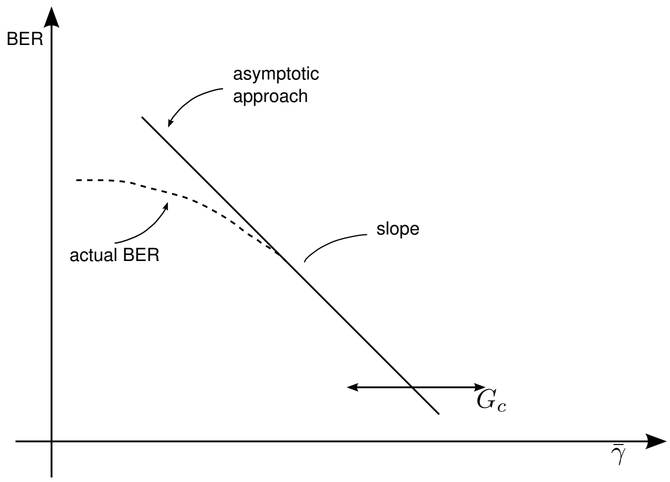


Abbildung 3: BER for average SNR $\bar{\gamma}_t$

- Approximation (Taylor series): $\bar{\gamma} \rightarrow \infty : -e^{-\frac{x}{\bar{\gamma}}} = 1 - \frac{x}{\bar{\gamma}} + O(\frac{1}{\bar{\gamma}})$ where a function $f(x)$ is $O(x)$ if $\lim\limits_{x\rightarrow\infty} \frac{f(x)}{x} = 0$.

$$\Rightarrow P_{out} = \frac{1}{\gamma^{N_R}} \int\limits_0^{\gamma_T} \frac{x^{N_R-1}}{(N_R-1)!} \left(1 - \frac{x}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right)\right)$$

- Diversity and coding gain can also be defined for P_{out}

2.2.3 EGC (Equal Gain Combining)

Combining Weights

- For MRC, both, the amplitudes and phases of the channel gains $h_n = |h_n|e^{j\varphi_n}$ have to be known (or estimated in practice)
- In EGC it is assumed that only the phases are known and weights $w_n = e^{-j\varphi_n}$ are used.

$$\Rightarrow \gamma_t = \frac{\mathcal{E}_s}{\sigma_n^2} \frac{\left|\sum\limits_{n=1}^{N_R} |h_n| e^{j\varphi_n} e^{-j\varphi_n}\right|^2}{\sum\limits_{n=1}^{N_R} |e^{-j\varphi_n}|^2} = \frac{\mathcal{E}_s}{\sigma_n^2} \frac{1}{N_R} \left(\sum_{n=1}^{N_R} |h_n|\right)^2$$

$$= \frac{1}{N_R} \left(\sum_{n=1}^{N_R} \sqrt{\gamma_n}\right)^2 ; \text{ with } \gamma_n = \frac{\mathcal{E}_s}{\sigma_n^2} |h_n|^2$$

Performance Analysis

- IID case
  $\Rightarrow \sqrt{\gamma_n}$ is Rayleigh distributed

$\Rightarrow$ Exact analysis is much more difficult than for MRC $\Rightarrow$ see book by Simon & Alouini p.341

- Approximate result

$$P_e = \frac{a}{2}\left[1 - \sqrt{\frac{2b\bar{\gamma}}{5+2b\bar{\gamma}}}\sum_{n=0}^{N_R-1}\frac{\begin{pmatrix}2n\\n\end{pmatrix}}{4^n(1+\frac{2}{5}b\bar{\gamma})^n}\right]$$

- high SNR
$\Rightarrow$ use high SNR analysis of Wang & Giannakis, 2003
$\Rightarrow$ at high SNR, only pdf of γ_n around 0 is relevant for performance

$$\Rightarrow \overset{\text{Rayleigh}}{p_\gamma(x)} = \frac{1}{\bar{\gamma}}e^{-\frac{x}{\bar{\gamma}}} \overset{\text{Taylor Serie}}{=} \frac{1}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right) \quad \text{as } x \to 0$$

- need pdf γ_t: (γ_n bekannt, $\to$ ges.: Wurzel, etc.)
(cumulative distribution function of $\sqrt{\gamma}$ (cdf))

$$P_{\sqrt{\gamma}}(x) = \Pr\{\sqrt{\gamma} \le x\} = \Pr\{\gamma \le x^2\} = P_\gamma(x^2) = \text{cdf of } \gamma$$
$$\to p_{\sqrt{\gamma}}(x) = \frac{d}{dx}P_{\sqrt{\gamma}}(x) = 2x \cdot p_\gamma(x^2) = \frac{2x}{\bar{\gamma}} + O\big(\frac{1}{\bar{\gamma}}\big)$$

- Laplace Transformation to MGF

$$\to M_{\sqrt{\gamma}}(s) = \mathcal{L}\{p_{\sqrt{\gamma}}(x)\} = \frac{2}{\bar{\gamma}}\cdot\frac{1}{s^2} + O\big(\frac{1}{\bar{\gamma}}\big)$$
$$\sqrt{\gamma_t} = \sum_{n=1}^{N_R}\frac{\sqrt{\gamma_n}}{N_R}$$
$$M_{\sqrt{\gamma_t}}(s) = \mathcal{E}\Big\{\exp(-s\sqrt{\gamma_t})\Big\} = \mathcal{E}\Big\{\exp(-\frac{s}{\sqrt{N_R}}\cdot\sum_{n=1}^{N_R}\sqrt{\gamma_n})\Big\} = \Big(\mathcal{E}\Big\{\exp(-\frac{s}{\sqrt{N_R}}\cdot\sqrt{\gamma_n}\Big\}\Big)^{N_R}$$
$$= \Big(M_{\sqrt{\gamma}}(\frac{s}{\sqrt{N_R}})\Big)^{N_R} = \Big(\frac{2}{\bar{\gamma}}\cdot\frac{N_R}{s^2}\Big)^{N_R} + O\Big(\frac{1}{\bar{\gamma}^{N_R}}\Big)$$

- inverse Laplace Transform

$$p_{\sqrt{\gamma_t}}(x) = \mathcal{L}^{-1}\Big\{M_{\sqrt{\gamma_t}}(s)\Big\} = \Big(\frac{2N_R}{\bar{\gamma}}\Big)^{N_R}\cdot\frac{x^{2N_R-1}}{(2N_R-1)!} + O\Big(\frac{1}{\bar{\gamma}^{N_R}}\Big)$$
$$P_{\gamma_t}(x) = \Pr\{\gamma_t \le x\} = \Pr\{\sqrt{\gamma_t} \le \sqrt{x}\} = P_{\sqrt{\gamma_t}}(\sqrt{x}) \to \text{cdf of } \sqrt{\gamma_t}$$
$$p_{\gamma_t}(x) = \frac{d}{dx}P_{\gamma_t}(x) = \frac{1}{2\sqrt{x}}\cdot p_{\gamma_t}(\sqrt{x}) = \frac{1}{2}\Big(\frac{2N_R}{\bar{\gamma}}\Big)^{N_R}\cdot\frac{x^{N_R-1}}{(2N_R-1)!} + O\big(\bar{\gamma}^{-N_R}\big)$$
$$\to M_{\gamma_t}(s) = \mathcal{L}\{p_{\gamma_t}(x)\} = \frac{1}{2}\Big(\frac{2N_R}{\bar{\gamma}}\Big)^{N_R}\cdot\frac{(N_R-1)!}{(2N_R-1)!\;b^{N_R}} + O\big(\bar{\gamma}^{-N_R}\big)$$

- Error Probability:

$$
\begin{aligned}
P_e &= \frac{a}{\pi} \int\limits_0^{\frac{\pi}{2}} M_{\gamma_t}\left(\frac{b}{2\sin^2(\theta)}\right) \mathrm{d}\theta \\
&= \frac{a}{\pi}\frac{1}{2}\left(\frac{2N_R}{\bar{\gamma}}\right)^{N_R} \frac{(N_R-1)!}{(2N_R-1)!}\frac{2^{N_R}}{b^{N_R}} \underbrace{\int\limits_0^{\frac{\pi}{2}} \sin^{2N_R}(\theta)\mathrm{d}\theta}_{\frac{\pi}{2^{2N_R+1}}\binom{2N_R}{N_R}=\frac{\pi(2N_R)!}{2^{2N_R+1}(N_R!)^2}} + O\left(\frac{1}{\bar{\gamma}^{N_R}}\right) \\
&= \frac{aN_R^{N_R}}{2b^{N_R}N_R!}\frac{1}{\bar{\gamma}^{N_R}} + O\left(\frac{1}{\bar{\gamma}^{N_R}}\right) \stackrel{!}{=} \left(\frac{1}{G_c\cdot\bar{\gamma}}\right)^{G_d}
\end{aligned}
$$

$\Longrightarrow$ Diversity gain: $G_d = N_R$

$\Longrightarrow$ Combining gain: $G_c = \frac{b}{N_R}\left(\frac{2N_R!}{a}\right)^{\frac{1}{N_R}}$

vergleiche auch Blatt mit Kurven III und IV

A similar asymptotic analysis can be conducted for the outage probability.

2.2.4 SC (Selection Combining)

Combining weights

- only the strongest branch is chosen
- strongest branch: $\hat{n} = \underset{n}{\operatorname{argmax}} \gamma_n \longrightarrow \gamma_t = \gamma_{\hat{n}}$
- only on RF receiver chain required $\rightarrow$ saves hardware complexity

Performance analysis

- cdf of: γ_t

$$
\begin{aligned}
P_{\gamma_t}(x) &= \Pr\{\gamma_{\hat{n}} \le x\} = \Pr\{\gamma_1 \le x \cap \gamma_2 \le x \cap \dots \gamma_{N_R} \le x\} \\
&\stackrel{(IID)}{=} \Big(\Pr\{\gamma_n \le x\}\Big)^{N_R} = \Big(P_\gamma(x)\Big)^{N_R}
\end{aligned}
$$

- pdf:

$$
\begin{aligned}
p_{\gamma_t}(x) &= \frac{d}{dx}P_{\gamma_t}(x) = N_R\big(P_\gamma(x)\big)^{N_R-1}\cdot p_\gamma(x) \\
\text{where:} \quad p_\gamma(x) &= \frac{1}{\bar{\gamma}}e^{-\frac{x}{\bar{\gamma}}}; \quad x \ge 0 \\
P_\gamma(x) &= \int_0^x p_\gamma(x)\; dx = 1 - e^{-\frac{x}{\bar{\gamma}}}; \quad x \ge 0 \\
\rightarrow p_{\gamma_t}(x) &= \frac{N_R}{\bar{\gamma}}\big(1 - e^{-\frac{x}{\bar{\gamma}}}\big)^{N_R-1} e^{-\frac{x}{\bar{\gamma}}}; \quad x \ge 0
\end{aligned}
$$

Error probability

- direct approach $\rightarrow$ closed-form solution possible
- MGF approach (with Binomial expansion):

$$\begin{aligned} p_{\gamma_t}(x) &= \frac{N_R}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} 1^{N_R-1-n} \left(-e^{-\frac{x}{\bar{\gamma}}}\right)^n \\ &= \frac{N_R}{\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} \cdot (-1)^n e^{-\frac{x(n+1)}{\bar{\gamma}}}; \quad x \geq 0 \end{aligned}$$

$$M_{\gamma_t}(s) = \frac{N_R}{\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} (-1)^n \frac{1}{s + \frac{n+1}{\bar{\gamma}}}$$

$$P_e = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\Big(\frac{b}{2\sin^2\theta}\Big)\, d\theta = \frac{aN_R}{\pi\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} (-1)^n \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\frac{b}{2\sin^2\theta} + \frac{n+1}{\bar{\gamma}}}$$

$\rightarrow$ can be evaluated numerically

- high SNR approach $\Rightarrow$ $\bar{\gamma} \rightarrow \infty$

$$\begin{aligned} p_{\gamma_t} &= \frac{N_R}{\bar{\gamma}} \left[1 - \exp\left(-\frac{x}{\bar{\gamma}}\right)\right]^{N_R-1} \exp\left(-\frac{x}{\bar{\gamma}}\right) \\ &\overset{\bar{\gamma}\rightarrow\infty}{=} \frac{N_R}{\bar{\gamma}} \left[1 - \left(1 - \frac{x}{\bar{\gamma}} + O\left(\bar{\gamma}^{-1}\right)\right)\right]^{N_R-1} \left(1 - \frac{x}{\bar{\gamma}} + O\left(\bar{\gamma}^{-1}\right)\right) \\ &= \frac{N_R}{\bar{\gamma}^{N_R}} x^{N_R-1} + o\left(\bar{\gamma}^{-N_R}\right) \end{aligned}$$

$$\begin{aligned} M_{\gamma_t}(s) &= \frac{N_R}{\bar{\gamma}^{N_R}} \frac{(N_R-1)!}{s^{N_R}} + O\left(\bar{\gamma}^{-N_R}\right) \\ \Big[\rightarrow P_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\left(\frac{b}{2\sin^2(\theta)}\right) \mathrm{d}\theta\Big] \\ &= \frac{a(2N_R)!}{b^{N_R} 2^{N_R+1} N_R!} \frac{1}{\bar{\gamma}^{N_R}} + O(\bar{\gamma}^{-N_R}) \end{aligned}$$

$\Longrightarrow$ Diversity gain: $G_d = N_R$

$\Longrightarrow$ Combining gain: $G_c = 2b\left(\frac{2N_R!}{a(2N_R)!}\right)^{\frac{1}{N_R}}$

Outage Probability

$$P_{out} = \Pr\{\gamma_{\hat{n}} \leq \gamma_T\} = P_{\gamma_{\hat{n}}}(\gamma_T) = \left[1 - \exp\left(-\frac{\gamma_T}{\bar{\gamma}}\right)\right]^{N_R}$$

high SNR: $P_{out} = \left(\frac{\gamma_T}{\bar{\gamma}}\right)^{N_R} + O\left(\bar{\gamma}^{-N_R}\right)$

2.2.5 Comparison

- Diversity Gain:
  MRC, EGC and SC all achieve the maximum possible diversity gain of $G_d = N_R$
- Combining Gain:
  The combining gains of MRC, EGC and SC are different
  - MRC/EGC:

$$\frac{G_C^{EGC}}{G_C^{MRC}} = \frac{\frac{1}{2b}\left(\frac{a}{2}\binom{2N_R}{N_R}\right)^{\frac{1}{N_R}}}{\frac{N_R}{b}\left(\frac{a}{2}\frac{1}{N_R!}\right)^{\frac{1}{N_R}}} = \frac{[(2N_R)!]^{\frac{1}{N_R}}}{2N_R(N_R)^{\frac{1}{N_R}}} \leq 1$$

    (independent of a or b which are modulation parameters, only depends on number of antennas)

$$N_R \gg 1: \qquad N_R! \approx \sqrt{2\pi}e^{-N_R}N_R^{N_R+\frac{1}{2}} \qquad (Stirling)$$

$$\left.\frac{G_c^{EGC}}{G_c^{MRC}}\right|_{N_R\gg 1} = \frac{\left(\sqrt{2\pi}e^{-2N_R}(2N_R)^{2N_R+\frac{1}{2}}\right)^{\frac{1}{N_R}}}{2N_R\left(\sqrt{2\pi}e^{-N_R}N_R^{N_R+\frac{1}{2}}\right)^{\frac{1}{N_R}}} = \frac{2\cdot 2^{\frac{1}{2N_R}}}{2} \overset{N_R\to\infty}{\rightarrow} \frac{2}{e} \equiv -1.3\text{dB}$$

  - MRC/SC:

$$\frac{G_c^{SC}}{G_c^{MRC}} = \frac{2b\left(\frac{a}{2}\binom{2N_R}{N_R}\right)^{\frac{1}{N_R}}}{2b\left(\frac{a}{2}\frac{(2N_R)!}{N_R!}\right)^{\frac{1}{N_R}}} = \frac{1}{(N_R!)^{\frac{1}{N_R}}} \leq 1$$

$$\left.\frac{G_c^{SC}}{G_c^{MRC}}\right|_{N_R\gg 1} = \frac{1}{\sqrt{2\pi}^{\frac{1}{N_R}}e^{-1}N_R^{1+\frac{1}{2N_R}}} N_R \overset{\rightarrow}{\rightarrow} \infty \frac{e}{N_R}$$

    $\rightarrow$ loss increases with N_R

- Ergebnis:
  - unterschiedliche Kurven in Diagramm „Combiner Performance Comparison, BPSK“ ergeben sich durch G_c
  - alle Kurven haben die gleiche Steigung $\Rightarrow$ G_d ist überall gleich
  - nur Abstände sind unterschiedlich $\Rightarrow$ G_c wird in MRC besser „genutzt“ als in EGC und SC

2.3 MISO Systems

Remarks

- Similar to SIMO systems, in MISO systems only coding and diversity gains can be obtained.
- To realize these gains, a careful transmitter design is necessary
- System design depends on whether or not channel state information (**CSI**) is available at transmitter

2.3.1 Naive Approach

- Assume we simply send the same signal over all N_T transmit antennas

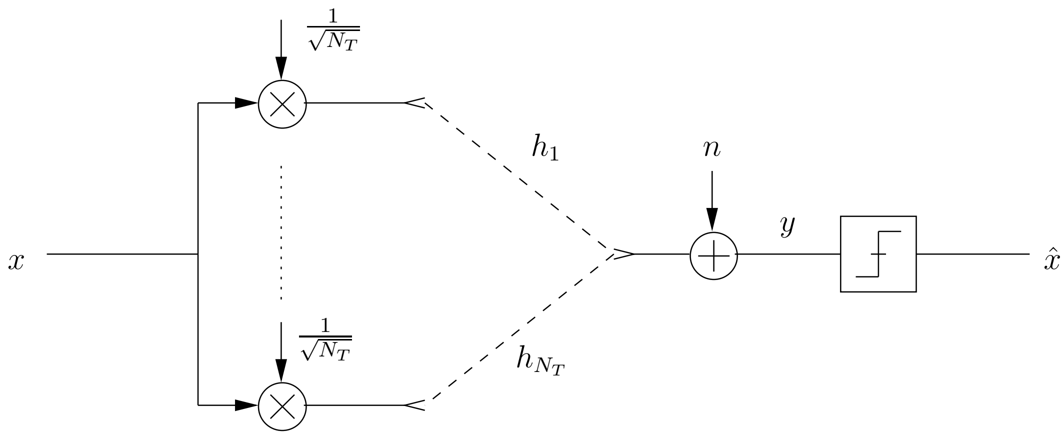


Abbildung 4: Block Diagramm of naive MISO

- Transmit power: $\mathcal{E}\left\{\left|\frac{1}{\sqrt{N_T}}x\right|^2 +, \ldots, \left|\frac{1}{\sqrt{N_T}}x\right|^2\right\} = \mathcal{E}\left\{N_T \frac{1}{N_T}|x|^2\right\} = \mathcal{E}_s$
- Received signal: $y = \frac{1}{\sqrt{N_T}} \sum\limits_{n=1}^{N_T} h_n \cdot x + n$
- Rayleigh fading: h_n are zero mean complex gaussian random variables
  $\rightarrow h$ is also zero mean complex gaussian
- i.i.d.:
  - $\mathcal{E}\{|h_n|^2\} = 1 \; \forall n$
  - $\mathcal{E}\{|h|^2\} = \frac{1}{N_T}\mathcal{E}\left\{\left|\sum\limits_{n=1}^{N_T} h_n\right|^2\right\} = \frac{1}{N_T}\mathcal{E}\left\{\sum\limits_{n=1}^{N_T} |h_n|^2\right\} = 1$
  - statistical properties of h are independent of N_T
  - the multiple transmit antennas have no benefit at all
  - more sophisticated transmitter designs necessary

2.3.2 Full CSI Available at the Transmitter

- $h_n, n \in \{1, \ldots, N_T\}$ is known at the transmitter
- Perform "precoding" (beamforming) with coefficients w_n

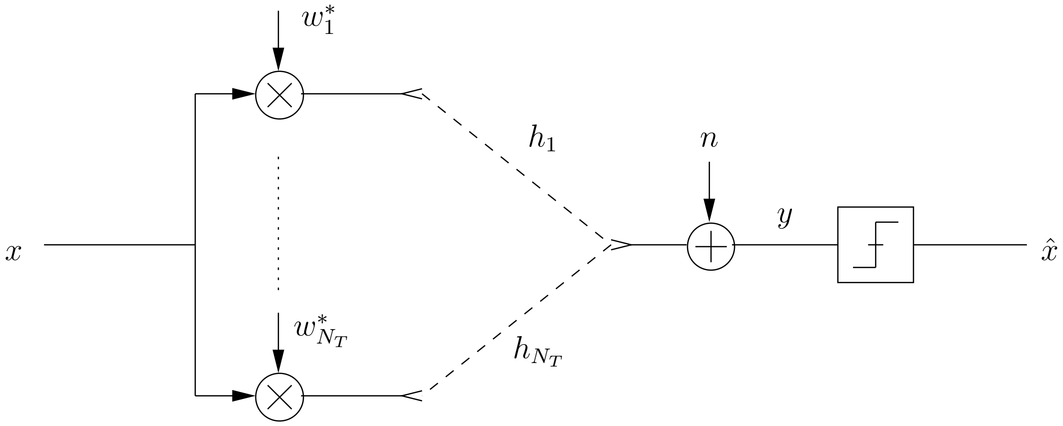


Abbildung 5: Block Diagramm of MISO with CSI

- Transmit Power: Two constraints maybe considered
  - Average transmit power constraint

$$P_{av} = \mathcal{E}\left\{\sum_{n=1}^{N_T} |w_n^* x|^2\right\} = \sum_{n=1}^{N_T} |w_n|^2 \underbrace{\mathcal{E}\{|x|^2\}}_{\mathcal{E}_s} = \mathcal{E}_s \Rightarrow \sum_{n=1}^{N_T} |w_n|^2 = 1$$

  - Power constraint for each transmit antenna

$$\rightarrow |w_n| = \frac{1}{\sqrt{N_T}} \qquad \rightarrow P_{av} = \mathcal{E}_s$$

- Recveived signal: $y = \underbrace{\sum_{n=1}^{N_T} w_n^* h_n}_{h} x + n$ (equivalent SISO channel)

Maximum Ratio Transmission (MRT)

- we have only the average power constraint: $\sum_{n=1}^{N_T} |w_n|^2 = 1$
- SNR: $\gamma_t = \frac{\mathcal{E}_s |h|^2}{\sigma_n^2} = \frac{\mathcal{E}_s \left|\sum_{n=1}^{N_T} w_n^* \cdot h_n\right|^2}{\sigma_n^2}$
- Maximize SNR under constraint $\sum_{n=1}^{N_T} |w_n|^2 = 1$
- constraint optimization problem $\rightarrow$ Lagrange method

$$L = \frac{\mathcal{E}_s}{\sigma_n^2}\left|\sum_{n=1}^{N_T} w_n^* \cdot h_n\right|^2 + \lambda\left(\sum_{n=1}^{N_T} |w_n|^2 - 1\right); \quad \text{where: } \lambda = \text{Lagrange Multiplier}$$

$\Rightarrow$ Wirtinger Kalkül: treat z and z^* as independent variables for differentiation:

$$\frac{\partial z^*}{\partial z} = 0; \quad \frac{\partial |z|^2}{\partial z} = \frac{\partial z \cdot z^*}{\partial z} = z^*$$
$$\frac{\partial x^2}{\partial x} = 2x; \quad \frac{\partial (z^*)^2}{\partial z^*} = 2 \cdot z^*; \frac{\partial |z|^2}{\partial z} = z^*$$

$$\frac{\partial L}{\partial w_m^*} = \frac{\mathcal{E}_s}{\sigma_n^2} \left(\sum_{n=1}^{N_T} w_n^* \cdot h_n \right)^* h_m + \lambda w_m$$

$\rightarrow$ $w_m = \underbrace{\frac{\mathcal{E}_s}{\sigma_n^2 \cdot \lambda} \left(\sum_{n=1}^{N_T} w_n^* h_n \right)^*}_{\text{const., independent of } m := c} h_m$

$\rightarrow$ $w_m = c \cdot h_m$

$\rightarrow$ $\sum_{n=1}^{N_T} |w_n|^2 = 1 \rightarrow c^2 = \frac{1}{\sum\limits_{n=1}^{N_T} |h_n|^2}$

$\rightarrow$ $w_n = \frac{h_n}{\sqrt{\sum_{n=1}^{N_T} |h_n|^2}} \equiv$ MRT gains

$\rightarrow$ SNR $= \frac{\mathcal{E}_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} \frac{|h_n|^2}{\sqrt{\sum_{m=1}^{N_T} |h_m|^2}} \right|^2 = \frac{\mathcal{E}_s}{\sigma_n^2} \sum_{n=1}^{N_T} |h_n|^2$

$\Rightarrow$ same SNR as for maximum ration combining (MRC)

$\Rightarrow$ MRT with N_T transmit antennas achieves the same performance as MRC with N_T receive antennas

$\Rightarrow$ MRT/MRC can be extended to $N_T \times N_R$ MIMO systems

- $\rightarrow$ has the same performance as MRC with $N_T \cdot N_R$ receive antennas and one transmit antenna

Equal Gain Transmission (EGT)

- we employ gains: $w_n = \frac{1}{\sqrt{N_T}} \cdot \frac{h_n}{|h_n|} \rightarrow |w_n| = \frac{1}{\sqrt{N_T}}$
- SNR:

$$\begin{aligned} \gamma_t &= \frac{\mathcal{E}_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} w_n^* h_n \right|^2 \\ &= \frac{\mathcal{E}_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} \frac{1}{\sqrt{N_T}} \cdot \frac{|h_n|^2}{|h_n|} \right|^2 = \frac{1}{N_T} \cdot \frac{\mathcal{E}_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} |h_n| \right|^2 \\ \gamma_n &= \frac{\mathcal{E}_s}{\sigma_n^2} |h_n|^2 \end{aligned}$$

$$\text{same SNR as for EGC} \rightarrow \gamma_t = \frac{1}{N_T} \left| \sum_{n=1} N_T \sqrt{\gamma_n} \right|^2$$

$\rightarrow$ EGC with N_T transmit antennas achieves the same performance as EGC with N_T receive antennas

Transmit Antennas Selection

- select antenna with maximum channel gain for transmission:

$$w_n = \begin{cases} \frac{h_n}{|h_n|}, & \text{if } n = \hat{n} \\ 0, & \text{otherwise} \end{cases} \text{ where } \hat{n} = \underset{n}{\operatorname{argmax}} |h_n|$$

- antenna selection with N_T transmit antennas achieves the same performance as *Selection Combining* with N_T receive antennas

2.3.3 No CSI at Transmitter - Space - Time - Coding

- $h_n, n \in \{1, \dots, N_T\}$, is only known at the receiver
- "Space-time-coding" has to be employed to realize diversity gain
- $T \times N_T$ matrics $\mathbf{X}$ are transmitted in T symbol intervals over N_T antennas
- $\mathbf{X}$ is drawn from a matrix alphabet $\mathcal{X}$
- Example:

$$\mathbf{X} = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,N_T} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,N_T} \\ \vdots & \vdots & \ddots & \vdots \\ x_{T,1} & x_{T,2} & \cdots & x_{T,N_T} \end{pmatrix}$$

- We distinguish:
  - Space-time-block-codes (STBCs)
    $\rightarrow$ $\mathbf{X}$ is obtained by mapping K scalar symbols $s_k,\ k = 1, \dots, K$ from a scalar alphabet $\mathcal{A}$ to matrix $\mathbf{X}$
  - Space-time-trellis-codes (STTCs)
    $\rightarrow$ $\mathbf{X}$ is obtained from scalar symbols s_k through a trellis encoding process.
    [see: Tarokh, Seshadri, Calderbank: Space-time-codes for high datarate wireless communication: Performance criterions and coder construction; IEEE Trans. Inf. Theory 1998]
  - here: We concentrate on space-time-block-codes (STBCs), but many results can be easily extended to space-time-trellis-codes
- STBCs:
  - K M-ary scalar symbols (e.g. M-PSK symbols) are mapped to STBC matrices $\mathbf{X}$
    $\mathbf{S} = [s_1, \dots, s_K] \rightarrow \mathbf{X}$
    $s_k \in \mathcal{A} \rightarrow x \in \mathcal{X}$ with $|\mathcal{X}| = M^K$
  - Example: "Alamouti"-Code

$$\mathbf{X} = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix}$$

[Alamouti: A simple transmit diversity technique for wireless communication, IEEE JSAC 1998]

Optimal Detection

- Signal model:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_T \end{pmatrix} = \mathbf{X} \begin{pmatrix} h_1 \\ \vdots \\ h_{N_T} \end{pmatrix} + \begin{pmatrix} n_1 \\ \vdots \\ n_T \end{pmatrix}$$

$$\mathbf{y} = \mathbf{X} \cdot \mathbf{h} + \mathbf{n}$$

- Optimal detection - ML-detection
  - $\mathbf{h}$ is known at receiver
  - $\mathbf{n}$ is AWGN with $\mathcal{E}\{\mathbf{n} \cdot \mathbf{n^H}\} = \sigma_{\mathbf{n}}^2 \cdots \mathbf{I}_{T\times T}$

$$\begin{aligned} p(\mathbf{y}|\mathbf{X}) &= \frac{1}{\pi^T |\sigma_n^2 \mathbf{I}_{T\times T}|} \exp\left(-(\mathbf{y}-\mathbf{Xh})^H (\sigma_n^2 \mathbf{I}_{T\times T})^{-1} (\mathbf{y}-\mathbf{Xh})\right) \\ &= \frac{1}{\pi^T \sigma_n^{2T}} \exp\left(-\frac{1}{\sigma_n^2} (\mathbf{y}-\mathbf{Xh})^H (\mathbf{y}-\mathbf{Xh})\right) = \frac{1}{\pi^T \sigma_n^{2T}} \exp\left(||\mathbf{y}-\mathbf{Xh}||^2\right) \end{aligned}$$

  $\rightarrow$ the optimal estimate $\hat{\mathbf{X}}$ or equivalently the optimal estimate $\hat{\mathbf{s}}$ can be obtained as

$$\hat{\mathbf{s}} = \underset{\mathbf{s}\in\mathcal{A}^K}{\operatorname{argmax}}\, p(\mathbf{y}|\mathbf{X}) = \underset{\mathbf{s}\in\mathcal{A}^K}{\operatorname{argmin}} ||\mathbf{y}-\mathbf{Xh}||^2$$

  - Disadvantage: In general, metric $||\mathbf{y}-\mathbf{Xh}||^2$ has to be calculated M^K times
    $\rightarrow$ complexity increases exponentially with K

Types of STBCs

- Orthogonal STBCs (OSTBCs)
  - OSTBCs are a special class of STBCs which allow independent detection of each s_k $\rightarrow$ only $K \cdot M$ metrics have to be evaluated
  - Rate STBCs: $R_{STBC} = \frac{K}{T}$
  - Examples:
    * Alamouti Code ($K = 2, T = 2$) $\rightarrow$ $R_{STBC} = 1$

$$\mathbf{X} = \frac{1}{\sqrt{2}} \underbrace{\begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix}}_{\longleftrightarrow N_T} \updownarrow T$$

      $\rightarrow$ only "full rate OSTBC for complex s_k

    * $N_T = 3, K = 3, T = 4$

$$\mathbf{X} = \frac{1}{\sqrt{3}} \begin{pmatrix} s_1 & s_2 & s_3 \\ -s_2^* & s_1^* & 0 \\ s_3^* & 0 & -s_3^* \\ 0 & -s_3^* & s_2^* \end{pmatrix} \rightarrow R_{STBC} = \frac{K}{T} = \frac{3}{4}$$

- Orthogonality: $\mathbf{X}^H\mathbf{X} = const \cdot \mathbf{I}_{N_T \times N_T}$
- Independent detection of s_1 & s_2 for Alamouti Code

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$
$$\rightarrow \underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{\tilde{y}} = \frac{1}{\sqrt{2}} \underbrace{\begin{pmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{pmatrix}}_{\tilde{F}} \underbrace{\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}}_{s} + \underbrace{\begin{pmatrix} n_1 \\ n_2^* \end{pmatrix}}_{\tilde{n}}$$

$\Big($ Anmerkung: $\underline{\text{nur}}$ $\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$ gewünscht, nicht: $s_1^*, s_2^*\Big)$

$$\mathbf{F}^H\mathbf{F} = \frac{1}{2}\begin{pmatrix} h_1^* & h_2 \\ h_2^* & -h_1 \end{pmatrix}\begin{pmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{pmatrix} = \frac{1}{2}\begin{pmatrix} |h_1|^2+|h_2|^2 & 0 \\ 0 & |h_1|^2+|h_2|^2 \end{pmatrix}$$

$\rightarrow$ $\frac{\sqrt{2}}{\sqrt{|h_1|^2+|h_2|^2}} \cdot \mathbf{F}$ is unitary matrix

$\rightarrow$ $\frac{2}{|h_1|^2+|h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{y}} = \mathbf{s} + \frac{2}{|h_1|^2+|h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{n}}$
$\big(\frac{2}{|h_1|^2+|h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{n}}$ is AWGN vector with covariance matrix $\frac{2\sigma_n^2}{|h_1|^2+|h_2|^2} \cdot \mathbf{I}_{T\times T}\big)$

$\rightarrow$ ML decision: $\hat{\mathbf{s}} = \underset{\mathbf{s}}{\text{argmin}}\big|\big|\frac{2}{|h_1|^2+|h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{y}} - \mathbf{s}\big|\big|^2$

$\rightarrow$ independent ML decoding

$$\hat{s}_1 = \underset{s_1}{\text{argmin}}\left|s_1 - \frac{h_1^* y_1 + h_2 y_2^*}{\frac{1}{\sqrt{2}}(|h_1|^2+|h_2|^2)}\right|$$
$$\hat{s}_2 = \underset{s_2}{\text{argmin}}\left|s_2 - \frac{h_1^* y_1 - h_2 y_2^*}{\frac{1}{\sqrt{2}}(|h_1|^2+|h_2|^2)}\right|$$

- independent decoding property can be proved for all OSTBCs
- low complexity is at the expense of a rate-loss compared to other STBCs for $N_T > 2$
  - $\rightarrow$ Frequenzhopping
  - $\rightarrow$ keine Kanalinformation aus vorher empfangenen Symbolen möglich $\Rightarrow$ Kanal ändert sich ständig: nur Entscheidung, ob Rauschen oder Signal + Rauschen

- Performance Analysis of Alamouti Code
  - Decision-variables after combining

$$r_1 = \sqrt{2}\frac{h_1^* y_1 + h_2 y_2^*}{|h_1|^2+|h_2|^2}$$
$$r_2 = \sqrt{2}\frac{h_1^* y_1 - h_2 y_2^*}{|h_1|^2+|h_2|^2}$$

because of symmetry it suffices to consider r_1

$$
\begin{aligned}
r_1 &= \sqrt{2}\frac{h_1^*\left(\frac{1}{\sqrt{2}}s_1h_1 + \frac{1}{\sqrt{2}}h_2s_2 + n_1\right) + h_2\left(-\frac{1}{\sqrt{2}}h_2s_1^* + \frac{1}{\sqrt{2}}h_1s_2^* + n_2\right)^*}{|h_1|^2 + |h_2|^2} \\
&= \sqrt{2}\frac{\frac{1}{\sqrt{2}}\left(|h_1|^2 + |h_2|^2\right)s_1 + h_1^*n_1 + h_2n_2^*}{|h_1|^2 + |h_2|^2} \\
&= 1 \cdot s_1 + n_{eq}
\end{aligned}
$$

where

$$
\begin{aligned}
n_{eq} &= \sqrt{2}\,\frac{h_1^*n_1 + h_2n_2^*}{|h_1|^2 + |h_2|^2} \\
\text{SNR} \rightarrow \gamma_t &= \frac{\mathcal{E}_s \cdot 1^2}{\sigma_{eq}^2} \quad \text{with} \quad \mathcal{E}\left\{|s_1|^2\right\} = \mathcal{E}_s \\
\sigma_{eq}^2 &= 2\frac{|h_1|^2\sigma_n^2 + |h_2|^2\sigma_n^2}{\left(|h_1|^2 + |h_2|^2\right)^2} = \frac{2\sigma_n^2}{|h_1|^2 + |h_2|^2}
\end{aligned}
$$

→ $\gamma_t = \frac{1}{2}\frac{\mathcal{E}_s}{\sigma_n^2}\left(|h_1|^2 + |h_2|^2\right)$

→ $\text{SNR}_{\text{Alamouti}} = \frac{1}{2}\text{SNR}_{\text{MRC}} = \frac{1}{2}\text{SNR}_{\text{MRT}}$

→ Alamouti code has diversity gain $G_d = 2$

→ Transmission with Alamouti STBC requires 3dB higher SNR to achieve same performance as MRT → 3dB loss in coding gain G_c

→ Lack of CSI knowledge at transmitter “costs” 3dB in power efficiency

→ General:

- OSTBCs achieve a diversity gain of $G_d = N_T$ if only one receive antenna is available
- if N_R receive antennas are available, MRC can be used at the receiver to yield a diversity gain of $\underline{G_d = N_T N_R}$

- Other STBCs:
  - Quasi orthogonal STBCs
    - higher rate than OSTBCs
    - only subset of symbols have to be decoded jointly
    - Example: $K = N_T = T = 4$

$$
\mathbf{X} = \frac{1}{2}\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^* & s_1^* & -s_4^* & s_3^* \\ -s_3^* & -s_4^* & s_1^* & s_2^* \\ s_4 & -s_3 & -s_2 & s_1 \end{pmatrix}
$$

    - Anmerkung 1: $\mathbf{X}$ ist ähnlich zu Alamouti Code
    - Anmerkung 2: $\mathbf{X}^H\mathbf{X}$: viele Nicht-diagonal Elemente sind Null; die, die ungleich Null sind, zeigen, welche Symbole gemeinsam entschlüsselt werden müssen

– Golden Code for $N_T = N_R = 2$: achieves a rate of $R_{STBC} = 2$ and full diversity of $G_d = N_T, N_R = 4$

– Differential STBCs: $\mathbf{X}_k = \mathbf{X}_{k-1} \cdot \mathbf{D}_k$. $\mathbf{X}_k$ is transmitted, $\mathbf{D}_k$ is transmitted

– Linear dispersion codes: designed to achieve high mutual information

– noncoherent STBCs (On-Off-Keying)

Space Time Code Design Given:

- Code $\mathscr{X} = \left\{\mathbf{X}_1, \ldots, \mathbf{X}_{|\mathscr{X}|}\right\}$
- Channel: IID Rayleigh-fading:
  - $h_n \sim \mathcal{CN}(0,1); \quad n \in \{1, 2, \ldots, N_T\}$
  - AWGN $n \sim \mathcal{CN}(0, \sigma_n^2)$

Problem: How should we design codebook $\mathscr{X}$?

- Need to derive error rate for general codebooks $\mathscr{X}$!
  - Codeword error rate

$$P_e = \frac{1}{|\mathscr{X}|} \sum_{i=1}^{|\mathscr{X}|} \Pr\{\mathbf{x}_i \neq \hat{\mathbf{x}}_i\}$$

  where $\hat{\mathbf{x}}_i$ is the detected codeword and we assume that all codewords are equally likely

  Problem: $\Pr\{\mathbf{x}_i \neq \hat{\mathbf{x}}_i\}$ is not tractable in general

- Use union bound to upper bound $\Pr\{\mathbf{x}_i \neq \hat{\mathbf{x}}_i\}$ as upper sum over pairwise error probabilities(PEP) $\Pr\{\mathbf{x}_i \to \mathbf{x}_j\}$ where it is assumed that $\mathbf{x}_i$ was transmitted and $\mathbf{x}_i$ and $\mathbf{x}_j$ are the only codewords in the codebook

$$\boxed{P_e \leq \frac{1}{|\mathscr{X}|} \sum_{i=1}^{|\mathscr{X}|} \sum_{j=1}^{|\mathscr{X}|} \Pr\{\mathbf{x}_i \to \mathbf{x}_j\} \text{ where } j \neq i}$$

  Calculation of PEPs
  Recall: $\hat{\mathbf{x}} = \underset{\mathbf{x} \in \mathscr{X}}{\operatorname{argmin}} ||\mathbf{y} - \mathbf{x}\mathbf{h}||^2$
  Now, $\mathbf{x}_i$ and $\mathbf{x}_j$ are the only alternatives and an error is made if $||\mathbf{y} - \mathbf{x}_i\mathbf{h}||^2 > ||\mathbf{y} - \mathbf{x}_j\mathbf{h}||^2$ since $\mathbf{x}_i$ was sent but $\mathbf{x}_j$ was detected

$$\rightarrow ||\mathbf{x}_i\mathbf{h} + \mathbf{n} - \mathbf{x}_i\mathbf{h}||^2 > ||\mathbf{x}_i\mathbf{h} + \mathbf{n} - \mathbf{x}_j\mathbf{h}||^2$$

$$||\mathbf{n}||^2 > ||(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h} + \mathbf{n}||^2$$

$$\rightarrow ||\mathbf{n}||^2 > \underbrace{\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h}}_{\Delta} + \mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{n} + \mathbf{n}^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h} + ||\mathbf{n}||^2$$

$$\rightarrow \underbrace{-\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)^H\mathbf{n} - \mathbf{n}^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h}}_{z} > \Delta$$

for given $\mathbf{h}$, z is a gaussian random variable

$$\begin{aligned}\sigma_z^2 &= \mathcal{E}\{|z|^2\} = \mathcal{E}\{2\mathbf{h}^H(\mathbf{x}_i-\mathbf{x}_j)\overbrace{\mathbf{n}\mathbf{n}^H}^{\sigma_n^2\mathbf{I}}(\mathbf{x}_i-\mathbf{x}_j)\mathbf{h} + 2\mathbf{h}^H(\mathbf{x}_i-\mathbf{x}_j)^H\overbrace{\mathbf{n}\mathbf{n}^T}^{=0}(\mathbf{x}_i-\mathbf{x}_j)^*\mathbf{h}^*\}\\ &= 2\sigma_n^2\Delta + 0\end{aligned}$$

$$\begin{aligned}\Pr\{\mathbf{x}_i \to \mathbf{x}_j\} &= \int_{\Delta}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_z}\exp\left(-\frac{z^2}{2\sigma_z^2}\right)\mathrm{d}z,\ t=\frac{z}{\sigma_z}\\ &= \frac{1}{\sqrt{2\pi}}\int_{\frac{\Delta}{\sigma_z}}^{\infty} e^{-\frac{t^2}{2}}\mathrm{d}t = Q\left(\frac{\Delta}{\sigma_z}\right) = Q\left(\frac{\Delta}{\sqrt{2\sigma_n^2\Delta}}\right)\\ &= Q\left(\sqrt{\frac{\Delta}{2\sigma_n^2}}\right)\end{aligned}$$

- $\Pr\{\mathbf{x}_i \to \mathbf{x}_j\} = \mathcal{E}\left\{Q\left(\sqrt{\frac{\Delta}{2\sigma_n^2}}\right)\right\}$
  - to avoid cumbersome Q-function we use Chernoff bound:

$$\boxed{Q(x) \le \frac{1}{2}e^{-\frac{x^2}{2}}}$$

$$\Pr\{\mathbf{x}_i \to \mathbf{x}_j\} \le \frac{1}{2}\mathcal{E}_h\left\{\exp\left(-\frac{\mathbf{h}^H\mathbf{Q}\mathbf{h}}{4\sigma_n^2}\right)\right\}$$
$$\text{where } \mathbf{Q} = (\mathbf{x}_i-\mathbf{x}_j)^H(\mathbf{x}_i-\mathbf{x}_j)$$

- Eigendecomposition: $\mathbf{Q} = \mathbf{U}^H\mathbf{\Lambda}\mathbf{U}$ with $\mathbf{\Lambda} = \mathrm{diag}\{\lambda_1,\ldots,\lambda_r,0,\ldots,0\}$ $\quad r = \mathrm{rank}\{\mathbf{Q}\}$

- Elements $\mathbf{h}$ are i.i.d. Gaussian
  - $\underline{\beta} = \mathbf{U}\mathbf{h}$ has also i.i.d. Gaussian random variables as elements since $\mathbf{U}$ is unitary matrix
  - $\mathbf{h}^H\mathbf{Q}\mathbf{h} = \underbrace{\mathbf{h}^H\mathbf{U}^H}_{\underline{\beta}^*}\mathbf{\Lambda}\underbrace{\mathbf{U}\mathbf{h}}_{\underline{\beta}} = \sum_{i=1}^{r}\lambda_i|\beta_i|^2$ with $\underline{\beta} = [\beta_1,\ldots,\beta_{N_T}]$

$$
\begin{aligned}
\Pr\{\mathbf{x}_i \to \mathbf{x}_j\} &= \frac{1}{2}\mathcal{E}_{\underline{\beta}}\left\{\exp\left(-\frac{\sum\limits_{i=1}^{r}\lambda_i|\beta_i|^2}{4\sigma_n^2}\right)\right\} \\
&= \frac{1}{2}\mathcal{E}_{\underline{\beta}}\left\{\prod_{i=1}^{r} e^{-\frac{\lambda_i}{4\sigma_n^2}|\beta_i|^2}\right\} \\
&= \frac{1}{2}\prod_{i=1}^{r}\mathcal{E}_{\beta_i}\left\{e^{-\frac{\lambda_i}{4\sigma_n^2}|\beta_i|^2}\right\} \\
&= \frac{1}{2}\prod_{i=1}^{r}\mathcal{E}_{|\beta_i|^2}\left\{e^{-\frac{\lambda_i}{4\sigma_n^2}|\beta_i|^2}\right\} \mathrel{\widehat{=}} \text{MGF of exponentially distributed variable } \alpha_i = |\beta_i|^2
\end{aligned}
$$

$\to P_{\alpha_i}(x) = e^{-x},\ x \geq 0$

$$
\begin{aligned}
\to \Pr\{\mathbf{x}_i \to \mathbf{x}_j\} &\leq \frac{1}{2}\prod_{i=1}^{r}\frac{1}{1+\frac{\lambda_i}{4\sigma_n^2}} \\
&\leq \prod_{i=1}^{r}\frac{1}{\frac{\lambda_i}{4\sigma_n^2}} = 2^{2r-1}\frac{1}{\prod\limits_{i=1}^{r}\lambda_i}\Bigg(\underbrace{\frac{1}{\sigma_n^2}}_{\widehat{=}SNR}\Bigg)^{-r}
\end{aligned}
$$

- upper bound on P_e:

$$
\begin{aligned}
\lambda_n(i,j) &= n\text{th eigenvalue of } (\mathbf{x}_i - \mathbf{x}_j)^H(\mathbf{x}_i - \mathbf{x}_j) \\
r(i,j) &= \text{ rank of } (\mathbf{x}_i - \mathbf{x}_j)^H(\mathbf{x}_i - \mathbf{x}_j)
\end{aligned}
$$

$$
\to P_e \leq \frac{1}{|\mathscr{X}|}\sum_{i=1}^{|\mathscr{X}|}\sum_{j=1}^{|\mathscr{X}|} 2^{2r(i,j)-1}\frac{1}{\prod\limits_{n=1}^{r(i,j)}\lambda_n(i,j)}\left(\frac{1}{\sigma_n^2}\right)^{-r(i,j)}
$$

- generally loose bound but offers significant insight for code design

<u>Two criteria:</u>

<u>Rank criterion:</u> The diversity gain of a ST code is given by

$G_d = \min\limits_{i,j}(r(i,j)) = \min\limits_{i,j} \text{rank}\left((\mathbf{x}_i - \mathbf{x}_j)^H(\mathbf{x}_i - \mathbf{x}_j)\right)$

$\to$ Design code such that minimum rank of all possible matrices $(\mathbf{x}_i - \mathbf{x}_j)^H(\mathbf{x}_i - \mathbf{x}_j)$ is maximized

$$
T \updownarrow \overset{\overset{N_T}{\longleftrightarrow}}{\mathbf{X}_i} \Rightarrow r(i,j) = N_T \qquad \forall i \neq j
$$

Determinant criterion: To maximize the coding gain among all codes with $r(i,j) = N_T$,

we need to maximize $\max \min_{i,j} \prod_{n=1}^{N_T} \lambda_n(i,j) = \max \min_{i,j} \left| (\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j) \right| \qquad \forall i \neq j$

- Rank and determinant criterion can be used for the search for good space-time block codes and space-time trellis codes. These two criteria were first derived by Tarokh, et. al. 1998.
- diversity increases to $N_T N_R$ if N_K receive antennas are available
- Example: see Bäro, Bauch, Hansmann: Improved codes for space-time trellis coded modulation. IEEE Comm. Letters, 2000.

2.3.4 Partial or Imperfect CSI at the Transmitter

- In practice, the CSI cannot be perfect. Channel estimation, quantization and noisy feedback channels introduce errors.
- If the system is optimized for perfect CSI (*e.g.* using MRT or EGT), the performance for imperfect CSI may be worse than for a system designed for no CSI(*e.g.* space-time coding)
- In this case, it is advantageous to use a hybrid approach and combine beamforming and space-time coding.

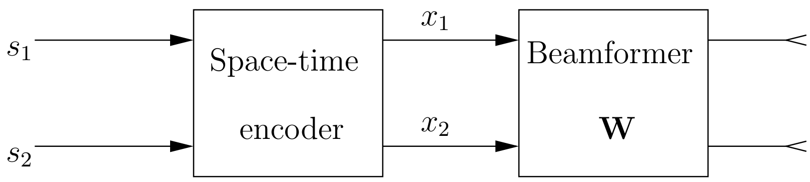


Abbildung 6: Block Diagramm of MISO with Beamforming

- $\mathbf{W}$ is the beamforming matrix which depends on the reliability of the CSI
- CSI is modeled as

$$\hat{h}_i = \rho h_i + \sqrt{1-\rho^2} e_i$$

  where:

  - $\hat{h}_i$ is the CSI estimate
  - ρ is the correlation between $\hat{h}_i$ and h_i
  - e_i is the CSI error modeled as AWGN

  extreme cases:

  - $\rho = 0$: $\hat{h}_i$ independent of h_i $\rightarrow$ no CSI ($\mathbf{W} = \mathbf{I}$)
  - $\rho = 1$: $\hat{h}_i = h_i$ $\rightarrow$ perfect CSI ($\mathbf{W}$ performs MRT)

- $\mathbf{W}$ can be optimized under the assumptions for given ρ and $\hat{h}_i$
  $\rightarrow$ see for details: Jöngren, Skorglund and Ottersten: "Combining Beamforming and Orthogonal Space-time Block Coding", IEEE on IT, 2002.

2.4 MIMO Systems without CSI at the transmitter

- We consider $N_T \times N_R$ MIMO system and assume that the channel matrix $\mathbf{H}$ is not known at the transmitter
  $\rightarrow$ no CSI at the transmitter (CSIT)
- signal model:

$$N_R \updownarrow \mathbf{y} = N_R \updownarrow \overset{\overset{N_T}{\longleftrightarrow}}{\mathbf{H}}\, \mathbf{x} \updownarrow N_T + \mathbf{n} \updownarrow N_R$$

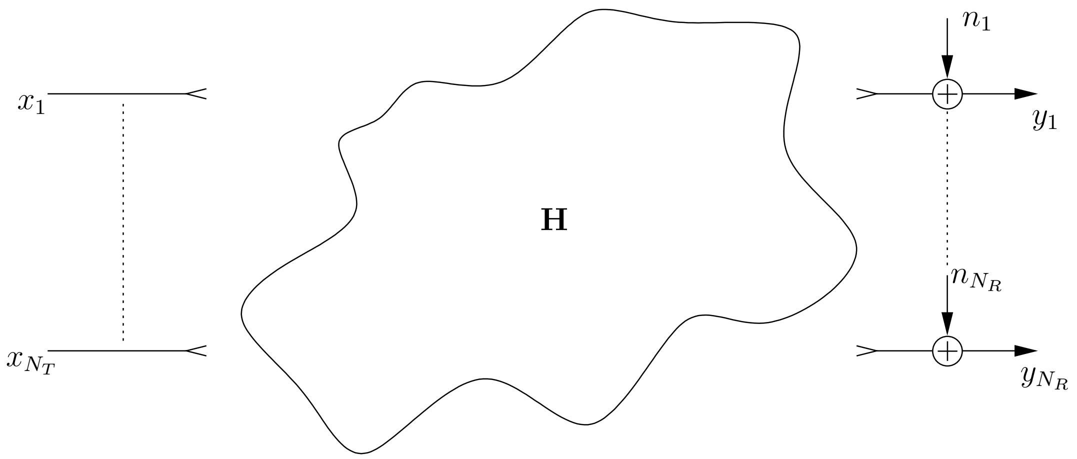


Abbildung 7: Block Diagramm of MISO without CSI

- x_n are M-ary i.i.d. scalar symbols taken *e.g* from an M-PSK or M-QAM symbol alphabet $\mathscr{A}$
- This scheme is often called "spatial multiplexing"
- We transmit N_T symbols per symbol interval
  $\rightarrow$ rate $R = \log_2(M) \cdot N_T$ for uncoded transmission
- Problem: How to detect $\mathbf{x}$ at the receiver considering
  - performance and
  - complexity?

2.4.1 Optimum Detection

- Elements of $\mathbf{n}$ are gaussian random variables with variance σ_n^2
- $\mathbf{H}$ is known at the receiver

$$\begin{aligned} p(\mathbf{y}|\mathbf{x}) &= \frac{1}{\pi^{N_R} \sigma_n^2 \mathbf{I}_{N_R \times N_R}} \exp\left(-(\mathbf{y} - \mathbf{Hx})^H (\sigma_n^2 \mathbf{I}_{N_R \times N_R})^{-1} (\mathbf{y} - \mathbf{Hx})\right) \\ &= \frac{1}{\pi^{N_R} \sigma_n^{2N_R}} \exp\left(-\frac{1}{\sigma_n^2} ||\mathbf{y} - \mathbf{xH}||^2\right) \end{aligned}$$

- ML-Detection

$$\hat{x} = \underset{\mathbf{x}\in\mathscr{A}^{N_T}}{\text{argmin}}||\mathbf{y}-\mathbf{xH}||^2 = \underset{\mathbf{x}\in\mathscr{A}^{N_T}}{\text{argmax}} \quad p(\mathbf{y}|\mathbf{x})$$

$\rightarrow$ M^{N_T} metric calculations $\rightarrow$ complexity is exponential in N_T!!

$\rightarrow$ in general too complex in practice

- Performance
  - consider worst case pairwise error probability (PEP) to evaluate diversity gain
  - PEP $\rightarrow x_i$ is transmitted but $x_j \neq x_i$ is detected
    this happens if $||\mathbf{y}-\mathbf{H}\mathbf{x}_i||^2 > ||\mathbf{y}-\mathbf{H}\mathbf{x}_j||^2$
    $\rightarrow$ $||\mathbf{n}||^2 > ||\mathbf{H}(\mathbf{x}_i-\mathbf{x}_j)+\mathbf{n}||^2$
  - the "worst case" is if $\mathbf{x}_i$ & $\mathbf{x}_j$ differ only in one element *i.e.*,

$$\mathbf{x}_i - \mathbf{x}_j = (x_{ni}-x_{nj}) = \begin{bmatrix} 0\\0\\0\\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{“1” in position } n$$

  where $\mathbf{x}_i = [x_{1i}, x_{2i}, \ldots, x_{N_T i}]$
  - $||\mathbf{n}||^2 > ||\underbrace{\mathbf{h}_n}_{n\text{th column of } \mathbf{H}} \underbrace{(x_{ni}-x_{nj})}_{\Delta x_n(i,j)} + \mathbf{n}||^2$
  - $||\mathbf{n}||^2 > \mathbf{h}_n^H \mathbf{n} \Delta x_n^*(i,j) + \mathbf{n}^H \mathbf{h}_n \Delta x_n(i,j) + ||\mathbf{n}||^2 + ||\mathbf{h}_n||^2 - |\Delta x_n(i,j)|^2$

$$||\mathbf{h}_n||^2|\Delta x_n(i,j)|^2 < \underbrace{-\mathbf{h}_n^H\mathbf{n}\Delta x_n(i,j) - \mathbf{n}^H\mathbf{h}_n\Delta x_n(i,j)}_{\text{Gaussian random variable with variance } \sigma_{eq}^2 = 2\sigma_n^2|\Delta x_n(i,j)|^2||\mathbf{h}_n||^2}$$

  - $\Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j | \mathbf{H}\} = Q\left(\sqrt{\frac{||\mathbf{h}_n||^2|\Delta x_n(i,j)|^2}{2\sigma_n^2}}\right)$
  - $\Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} = \mathcal{E}\left\{Q\left(\sqrt{\frac{||\mathbf{h}_n||^2|\Delta x_n(i,j)|^2}{2\sigma_n^2}}\right)\right\}$
    $\rightarrow$ use same approach as for space-time code design to get diversity order
    or : SNR is

$$\gamma_t = \frac{||\mathbf{h}_n||^2|\Delta x_n(i,j)|^2}{2\sigma_n^2} = \frac{|\Delta x_n(i,j)|^2}{2\sigma_n^2}(|h_{1n}|^2 + |h_{2n}^2 + \cdots + |h_{N_R n}|^2)$$

  - same form as SNR of MRC with N_R receive antennas
  - diversity gain of spatial multiplexing wit ML-decoding is

$$G_d = N_R$$

– diversity of N_T transmit antennas is not exploited with spatial multiplexing

– to exploit this additional gain, coding across space is required (at the expense of rate)
(Hier gehören die detection performance kurven für BPSK hin)

2.4.2 Linear Receivers

- How can we avoid the complexity associated wit the joint detection of the elements of $\mathbf{x}$?
- Idea: Employ linear filter (matrix) to seperate the elements of $\mathbf{x}$
- Requires: $N_T \leq N_R$
- We form

$$\mathbf{r} = N_T \updownarrow \overset{\overset{N_R}{\longleftrightarrow}}{\mathbf{F}}\ \mathbf{y} = [r_1, \ldots, r_{N_T}]^T$$

where $\mathbf{F}$ is the filter matrix and $\mathbf{y}$ is the received vector

such that x_n can be obtained from

$$\hat{x}_n = \underset{x_n \in \mathscr{A}}{\operatorname{argmin}} |r_i - x_n|^2 \qquad \text{where } \mathbf{F} \in \mathbb{C}^{N_T \times N_R}$$

- Two popular design criteria for $\mathbf{F}$
  – Zero-forcing (ZF) criterion
  – minimum mean squared error (MMSE) crtiterion

ZF Detection

$$\mathbf{r} = \mathbf{F}\mathbf{y} = \mathbf{F}(\mathbf{H}\mathbf{x} + \mathbf{n}) = \mathbf{F}\mathbf{H}\mathbf{x} + \mathbf{F}\mathbf{n}$$

ZF $\leftrightarrow$ we require $\mathbf{F}\mathbf{H} = \mathbf{I}_{N_T \times N_T}$

- noise covariance matrix

$$\Phi_{ee} = \mathcal{E}\{\mathbf{F}\mathbf{n}(\mathbf{F}\mathbf{n})^H\} = \sigma_n^2 \mathbf{F}\mathbf{F}^H$$

- $N_T = N_R \ \rightarrow \mathbf{F}\mathbf{H} = \mathbf{I}_{N_T \times N_T} \ \rightarrow \mathbf{F} = \mathbf{H}^{-1}$ assuming $\mathbf{H}$ is invertible
- $\rightarrow N_T \leq N_R \rightarrow$ which one of the many $\mathbf{F}$ that yield $\mathbf{F}\mathbf{H} = \mathbf{I}_{N_T \times N_T}$?
- choose $\mathbf{F}$ that leads to the smallest noise enhancement
- optimal $\mathbf{F}$ is the solution to the following problem:

$$\min_{\mathbf{F}} \operatorname{tr}\{\sigma_n^2 \mathbf{F}\mathbf{F}^H\}$$
$$\text{s.t } \mathbf{F}\mathbf{H} = \mathbf{I}_{N_T \times N_T}$$

the constraint is equivalent to $\operatorname{tr}\{(\mathbf{F}\mathbf{H} - \mathbf{I})(\mathbf{F}\mathbf{H} - \mathbf{I})^H\} = 0$

Lagrangian:

$$\begin{aligned}L(\mathbf{F}) &= \mathrm{tr}\{\sigma_n^2 \mathbf{F}\mathbf{F}^H\} + \lambda \mathrm{tr}\{\mathbf{F}\mathbf{H}\mathbf{H}^H\mathbf{F} - \mathbf{F}\mathbf{H} - \mathbf{H}^H\mathbf{F}^H + \mathbf{I}\} \\ &= \sigma_n^2 \mathrm{tr}\{\mathbf{F}\mathbf{F}^H\} + \lambda \mathrm{tr}\{\mathbf{F}\mathbf{H}\mathbf{H}^H\mathbf{F}^H\} - \lambda \mathrm{tr}\{\mathbf{F}\mathbf{H}\} - \lambda \mathrm{tr}\{\mathbf{H}^H\mathbf{F}^H\} + \lambda N_T\end{aligned}$$

- use rules for complex matrix differentiation in Table IV in paper by Hjörunges & Gesbert

$$\begin{aligned}\frac{\delta L(\mathbf{F})}{\delta \mathbf{F}^*} &= \sigma_n^2 \mathbf{F} + \lambda \mathbf{F}\mathbf{H}\mathbf{H}^H - \lambda \mathbf{H}^H = 0 \\ &\rightarrow \mathbf{F}(\sigma_n^2 \mathbf{I} + \lambda \mathbf{H}\mathbf{H}^H) = \lambda \mathbf{H}^H \\ &\rightarrow \mathbf{F} = \lambda \mathbf{H}^H (\sigma_n^2 \mathbf{I} + \lambda \mathbf{H}\mathbf{H}^H)^{-1}\end{aligned}$$

use matrix inversion lemma

$$(\mathbf{A} + \mathbf{U}\mathbf{B}\mathbf{V})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{B}^{-1} + \mathbf{V}\mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}\mathbf{A}^{-1}$$

$$\begin{aligned}\rightarrow \mathbf{F} &= \lambda \mathbf{H}^H \left[\frac{1}{\sigma_n^2}\mathbf{I} - \frac{1}{\sigma_n^2}\mathbf{H}\left[\frac{1}{\lambda}\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H}\right]^{-1}\mathbf{H}^H \frac{1}{\sigma_n^2}\right] \\ &= \frac{\lambda}{\sigma_n^2}\left[\overbrace{\left(\tfrac{1}{\lambda}\mathbf{I} + \tfrac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H}\right)\left(\tfrac{1}{\lambda}\mathbf{I} + \tfrac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H}\right)^{-1}}^{\mathbf{I}} - \frac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H}\left[\frac{1}{\lambda}\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H}\right]\right]\mathbf{H}^H \\ &= \frac{\lambda}{\sigma_n^2}\left[\frac{1}{\lambda}\mathbf{I} + \frac{1}{\lambda}\mathbf{H}^H\mathbf{H} - \frac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H}\right]\left(\frac{1}{\lambda}\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H}\right)^{-1}\mathbf{H}^H \\ &= \frac{1}{\sigma_n^2}\left(\frac{1}{\lambda}\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H}\right)\mathbf{H}^H\end{aligned}$$

- How to choose λ

$$\begin{aligned}\mathbf{F}\mathbf{H} &= \frac{1}{\sigma_n^2}\left(\frac{1}{\lambda}\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H}\right)^{-1}\mathbf{H}^H\mathbf{H} = \mathbf{I} \\ \frac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H} &= \frac{1}{\lambda}\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}^H\mathbf{H} \\ &\Rightarrow \lambda \rightarrow \infty\end{aligned}$$

$$\Rightarrow \boxed{\mathbf{F} = (\mathbf{H}^H\mathbf{H})^{-1}} \mathrel{\widehat{=}} \text{Moore-Penrose pseudoinverse}$$

noise covariance:

$$\Phi_{ee} = \sigma_n^2 \mathbf{F}\mathbf{F}^H = \sigma_n^2(\mathbf{H}^H\mathbf{H})^{-1}\underbrace{\mathbf{H}^H\mathbf{H}(\mathbf{H}\mathbf{H})^{-1}}_{\mathbf{I}} = \sigma_n^2(\mathbf{H}^H\mathbf{H})^{-1}$$

Φ_{ee} is not in general a diagonal matrix

  - effective noise $\mathbf{Fn}$ is spatially correlated

– "equalization of channel leads to coloring of noise

- Interpretation:
we have

$$\mathbf{F}\mathbf{H}\mathbf{x} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_{N_T} \end{bmatrix} \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \dots & \mathbf{h}_{N_T} \end{bmatrix} \mathbf{x} = \mathbf{x}$$

  – $\mathbf{f}_i \mathbf{h}_i = 1 \quad \mathbf{f}_i \mathbf{h}_j = 0 \quad \forall i \neq j$
  – $\mathbf{f}_i^T$ is orthogonal to $\begin{bmatrix} \mathbf{h}_1 & \dots \mathbf{h}_{i-1} & \mathbf{h}_{i+1} & \dots & \mathbf{h}_{N_T} \end{bmatrix} \updownarrow N_R$
  – $\mathbf{f}_i^T$ is confined to an $N_R - (N_T - 1)$ dimensional subspace of the N_R dimensional space spanned by $\mathbf{H}$

- Diversity gain
  – e.g. SISO model: $r_i = \mathbf{f}_i \cdot \mathbf{h}_i \mathbf{x}_i + \mathbf{f}_i \cdot \mathbf{n}_i$

    $\rightarrow$ $\mathrm{SNR_{eq}}$= $\frac{\mathcal{E}_s |\mathbf{f}_i \mathbf{h}_i|^2}{\sigma_n^2 \|\mathbf{f}_i\|^2} = \frac{\mathcal{E}_s}{\sigma_n^2} \left| \tilde{\mathbf{f}}_i \cdot \mathbf{h}_i \right|$, where $\mathbf{f}_i = \alpha \mathbf{f}_i$ with $\|\tilde{\mathbf{f}}_i\|^2 = 1$

    we can represent $\mathbf{f}_i$ as: $\tilde{\mathbf{f}}_i^T = \alpha \mathbf{M} \boldsymbol{\beta}$,

    where: $\mathbf{M} \in \mathbb{C}^{N_R \times (N_R - N_T + 1)}$ and $\boldsymbol{\beta} \in \mathbb{C}^{(N_R - N_T + 1) \times 1}$ $\quad \widehat{=}$ basis of subspace

    $\rightarrow$ $\tilde{\mathbf{f}}_i \mathbf{M}_i = \boldsymbol{\beta}^T \mathbf{M}_i^T \mathbf{M}_i$,

    where: $\mathbf{M}^H \mathbf{M} = \tilde{\mathbf{M}}_i = \mathbf{I}$ and $\tilde{\mathbf{M}}_i \rightarrow \mathcal{CN}(0, \sigma_n^2 \mathbf{I}_{(N_R - N_T + 1)}$

    (since rows of $\mathbf{M}^T$ are orthogonal)

$$\mathrm{SNR_{eq}} = \frac{\mathcal{E}_s}{\sigma_n^2} \alpha^2 \left| \sum_{j=1}^{N_R - N_T + 1} \boldsymbol{\beta}_{ji} \tilde{\mathbf{h}}_j i \right|^2 ; \tilde{\mathbf{h}}_i = \begin{pmatrix} \tilde{h}_{1i}, & \tilde{h}_{2i}, & \dots, & \tilde{h}_{N_R - N_T + 1} \end{pmatrix}^T$$

$$\boldsymbol{\beta}_i = \begin{pmatrix} \beta_{1i}, & \dots, & \beta_{N_R - N_T + 1, i} \end{pmatrix}^T$$

    $\rightarrow$ $\mathrm{SNR_{eq}}$ includes only $N_R - N_T + 1$ independent Gaussian RV

    $\rightarrow$ diversity gain is limited to: $\underline{G_d = N_R - N_T + 1}$

  Example:

$$N_T = N_R = 3$$
$$G_d^{ZF} = 1 \text{ but } G_d^{ML} = N_R = 3$$

  $\rightarrow$ huge performance loss because of linear ZF

MMSE detection

- ZF criterion may be too strict and leads to noise enhancement

  $\rightarrow$ maybe it is better to allow some interferences between signals but reduce noise enhancement

→ What is the optimal trade-off between interference and noise?

→ MMSE criterion

- MMSE criterion
  - error signal: $\mathbf{e} = \mathbf{F}\mathbf{y} - \mathbf{x}$
  - total error variance: $\sigma_e^2 = \mathcal{E}\big\{\|\mathbf{e}\|^2\big\} = \mathcal{E}\big\{\mathrm{tr}\{\mathbf{e}\mathbf{e}^H\}\big\} = \mathrm{tr}\big\{\mathcal{E}\{\mathbf{e}\mathbf{e}^H\}\big\} = \mathrm{tr}\{\Phi_{ee}\}$
  - Φ_{ee}: error covariance matrix
  - optimal filter: $\mathbf{F}_{\text{opt}} = \underset{\mathbf{F}}{\operatorname{argmin}}\,\mathrm{tr}\{\Phi_{ee}\}$
- Deviation of $\mathbf{F}_{\text{opt}}$
  - $\Phi_{ee} = \mathcal{E}\{\mathbf{e}\mathbf{e}^H\} = \mathcal{E}\{(\mathbf{F}\mathbf{y}-\mathbf{x})(\mathbf{F}\mathbf{y}-\mathbf{x})^H\} = \mathbf{F}\cdot\mathbf{\Phi}_{yy}\cdot\mathbf{F}^H - \mathbf{F}\cdot\mathbf{\Phi}_{yx} - \mathbf{\Phi}_{xx}\cdot\mathbf{\Phi}_{xy}\cdot\mathbf{F}^H + \mathbf{\Phi}_{xx}$

    with:

$$\begin{aligned}
\mathbf{\Phi}_{yy} &= \mathcal{E}\{\mathbf{y}\mathbf{y}^H\} = \mathcal{E}\{(\mathbf{H}\mathbf{x}+\mathbf{n})(\mathbf{H}\mathbf{x}+\mathbf{n})^H\} = \mathcal{E}_s\cdot\mathbf{H}\mathbf{H}^H + \sigma_n^2\cdot\mathbf{I}_{N_R\times N_T}\\
\mathbf{\Phi}_{yx} &= \mathcal{E}\{\mathbf{y}\mathbf{x}^H\} = \mathcal{E}\{(\mathbf{H}\mathbf{x}+\mathbf{n})\cdot\mathbf{x}^H\} = \mathcal{E}_s\cdot\mathbf{H}^H = \mathbf{\Phi}_{xy}^H\\
\mathbf{\Phi}_{xx} &= \mathcal{E}_s\cdot\mathbf{I}_{N_T\times N_R}
\end{aligned}$$

  - $\mathbf{F}_{\text{opt}} \quad \rightarrow \quad \frac{d}{d\mathbf{F}^*}\Big(\mathrm{tr}\{\mathbf{F}\mathbf{\Phi}_{yy}\mathbf{F}^H\} - \mathrm{tr}\{\mathbf{F}\mathbf{\Phi}_{yx}\} - \mathrm{tr}\{\mathbf{\Phi}_{xy}\mathbf{F}^H\} + \mathrm{tr}\{\mathbf{\Phi}_{xx}\}\Big) \overset{!}{=} 0$

    with Table IV in paper by Hjoranges & Gesbert:

    ⇒ $\mathbf{F}\cdot\mathbf{\Phi}_{yy} - \mathbf{\Phi}_{xy} = 0$

    ⇒ $\mathbf{F}_{\text{opt}} = \mathbf{\Phi}_{xy}\cdot\mathbf{\Phi}_{yy}^{-1} = \mathcal{E}_s\mathbf{H}^H\big(\mathcal{E}_s\mathbf{H}\mathbf{H}^H + \sigma_n^2\mathbf{I}\big)^{-1}$

    $=$ (Matrix inversion Lemma) $=$

    $= \big(\mathbf{H}^H\mathbf{H} + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I}\big)^{-1}\cdot\mathbf{H}^H$
  - Comparison:

$$\begin{aligned}
\mathbf{F}_{\text{MMSE}} = \big(\mathbf{H}^H\mathbf{H} + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I}\big)^{-1}\cdot\mathbf{H}^H &\xrightarrow[\frac{\sigma_n^2}{\mathcal{E}_s}\to 0]{} \big(\mathbf{H}^H\cdot\mathbf{H}\big)^{-1}\mathbf{H}^H = \mathbf{F}_{ZF}\\
&\xrightarrow[\frac{\sigma_n^2}{\mathcal{E}_s}\to\infty]{} \frac{\mathcal{E}_s}{\sigma_n^2}\cdot\mathbf{H}^H = \mathbf{F}_{MF} \quad \widehat{=}\text{matched filter}
\end{aligned}$$

⇒ For high SNR, $\frac{\mathcal{E}_s}{\sigma_n^2}$, the MMSE filter approaches the ZF-Filter, for low SNR, it approaches the matched filter.

→ MMSE receiver yields the same diversity gain as the ZF receiver

$G_d^{MMSE} = G_d^{ZF} = N_R - N_T + 1 \leq G_d^{ML} = N_R$

  - End-to-End Channel: $\mathbf{K} = \mathbf{F}\mathbf{H} = \big(\mathbf{H}^H\mathbf{H} + \frac{\sigma_n^2}{\mathcal{E}_s}\cdot\mathbf{I}\big)^{-1}\cdot\mathbf{H}^H\cdot\mathbf{H} \neq$ diagonal matrix

    ⇒ crosstalk/interference between elements $\mathbf{x}$ in received signal after filtering $\mathbf{r}$.

    elements of $\mathbf{K}$: $K_{l,n}$

– Covariance for $\mathbf{F}_{\text{opt}}$

$$
\begin{aligned}
\mathbf{\Phi}_{ee} &= \mathbf{\Phi}_{xy} \cdot \overbrace{\mathbf{\Phi}_{yy}^{-1} \cdot \mathbf{\Phi}_{yy} \cdot \mathbf{\Phi}_{yy}^{-1}}^{\mathbf{\Phi}_{yy}^{-1}} \cdot \mathbf{\Phi}_{xy}^{H} - \mathbf{\Phi}_{xy} \cdot \mathbf{\Phi}_{yy}^{-1} \cdot \mathbf{\Phi}_{yx} - \mathbf{\Phi}_{xy} \cdot \mathbf{\Phi}_{yy}^{-1} \cdot \mathbf{\Phi}_{xy}^{H} + \mathbf{\Phi}_{xx} \\
&= \mathbf{\Phi}_{xx} - \overbrace{\mathbf{\Phi}_{xy} \cdot \mathbf{\Phi}_{yy}^{-1}}^{\mathbf{F}_{\text{opt}}} \cdot \mathbf{\Phi}_{yx} \\
&= \mathcal{E}_s \mathbf{I} - \mathcal{E}_s \big(\mathbf{H}^H \mathbf{H} + \frac{\sigma_n^2}{\mathcal{E}_s} \cdot \mathbf{I}\big)^{-1} \cdot \mathbf{H}^H \mathbf{H} = \text{(Matrix inversion Lemma)} \\
&= \sigma_n^2 \big(\mathbf{H}^H \mathbf{H} + \frac{\sigma_n^2}{\mathcal{E}_s} \cdot \mathbf{I}\big)^{-1} \\
\mathbf{\Phi}_{ee} &= \mathcal{E}_s (\mathbf{I} - \mathbf{K})
\end{aligned}
$$

$\rightarrow$ $0 \leq K_{m,m} \leq 1$ since main diagonal elements of $\mathbf{\Phi}_{ee}$ are $0 \leq [\mathbf{\Phi}_{ee}]_{m,m} \leq \mathcal{E}_s$ siehe auch Abbildung **??**

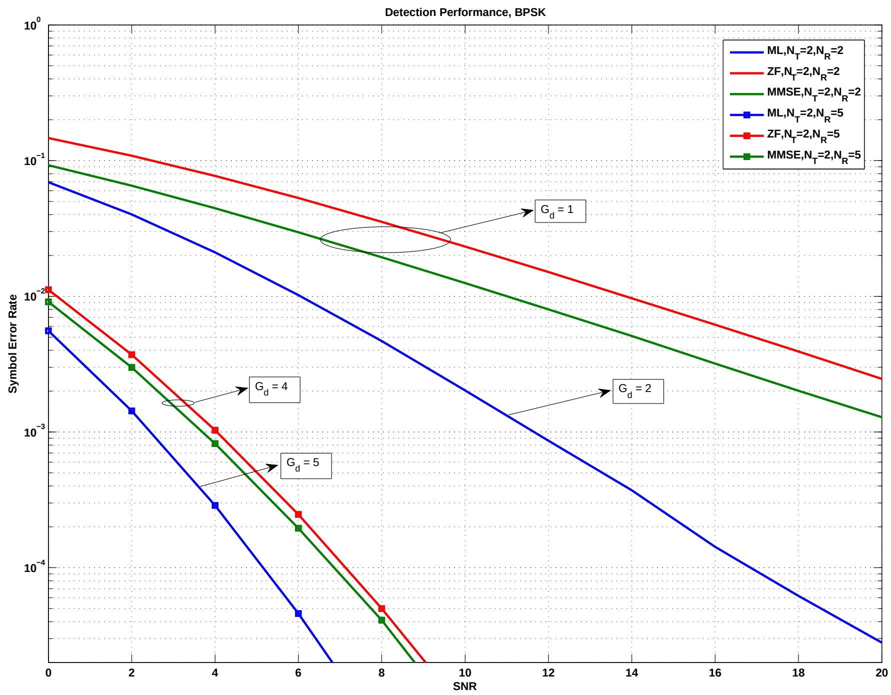


Abbildung 8: MMSE

SNR (biased vs. unbiased)

a) biased SNR

$$\mathrm{SNR}_{\mathrm{bias,m}} = \frac{\mathcal{E}_s}{[\mathbf{\Phi}_{ee}]_{mm}} = \frac{\mathcal{E}_s}{\mathcal{E}_s(1-K_{mm})} = \frac{1}{1-K_{mm}}, \quad 1 \leq m \leq 4$$

Anmerkung: $\mathbf{K} = \mathbf{F}_{\mathrm{opt}} \cdot \mathbf{H} \rightarrow \mathrm{SNR} = 1$ *falls* $\mathbf{K} = \mathrm{zeros}() \Rightarrow$ woher SNR $= 1$ *bei keiner Uebertragung?* $\Rightarrow$ *nicht vorteilhaft: siehe: b) unbiased SNR* but:

- $\mathrm{SNR}_{\mathrm{bias,m}}$ does not represent the actual SNR since the main diagonal elements of $\mathbf{K}$ are smaller than 1
- $\mathbf{r} = \mathbf{FHx} + \mathbf{Fn} = \mathbf{Kx} + \underbrace{\mathbf{Fn}}_{\tilde{\mathbf{n}}=[\tilde{n}_1,\ldots,\tilde{n}_{N_T}]^T}$
- $r_m = \underbrace{K_{mm}}_{<1} x_m + \sum_{\substack{n=1\\n\neq m}}^{N_T} K_{mn}x_n + \tilde{n}_m$

b) unbiased SNR

- remove bias: $r'_m = \dfrac{r_m}{K_{mm}} = x_m + \underbrace{\dfrac{\tilde{e}_m}{K_{mm}}}_{e'_m}$
- SNR?
- scaling matrix: $\mathbf{C} = \mathrm{diag}\{\frac{1}{K_{11}}, \frac{1}{K_{22}}, \ldots, \frac{1}{K_{N_T N_T}}\}$
- $\mathbf{r}' = \mathbf{Cr} \rightarrow \mathbf{e}' = [e'_1, \ldots, e'_{N_T}]^T = \mathbf{r}' - \mathbf{x} = \mathbf{Cr} - \mathbf{x} = \mathbf{CFy} - \mathbf{x}$
- $\mathbf{\Phi}_{e'e'} = \mathcal{E}\Big\{\big(\mathbf{CFy}-\mathbf{x}\big)\big(\mathbf{CFy}-\mathbf{x}\big)^H\Big\} = \mathbf{CF\Phi}_{yy}\mathbf{F}^H\mathbf{C}^H - \mathbf{CF\Phi}_{yx} - \mathbf{\Phi}_{xy}\mathbf{F}^H\mathbf{C}^H + \mathbf{\Phi}_{xx}$
- $\mathbf{F} = \mathbf{F}_{\mathrm{opt}} \rightarrow \mathbf{F}_{\mathrm{opt}}\mathbf{\Phi}_{yy} = \mathbf{\Phi}_{xy} = \mathcal{E}_s \cdot \mathbf{H}^H$

$\rightarrow$ $\mathbf{\Phi}_{e'e'} = \mathcal{E}_s\mathbf{C}\underbrace{\mathbf{H}^H\mathbf{F}^H}_{\mathbf{K}^H}\mathbf{C}^H - \mathcal{E}_s\mathbf{C}\underbrace{\mathbf{FH}}_{\mathbf{K}} - \mathcal{E}_s\underbrace{\mathbf{H}^H\mathbf{F}^H}_{\mathbf{K}^H}\mathbf{C} + \mathcal{E}_s\mathbf{I}$

$= \mathcal{E}_s\big[\mathbf{I} + (\mathbf{C}-\mathbf{I})\mathbf{K}^H\mathbf{C}^H - \mathbf{CK}\big]$

Anmerkung: nur Hauptdiagonalelemente interessieren, da diese die Varianz darstellen

$\rightarrow$ maindiagonal elements of $\mathbf{\Phi}_{e'e'}$ = variances of e'_m $= \mathcal{E}_s\Big(1 + \big(\frac{1}{K_{mm}} - 1\big)K_{mm}\frac{1}{K_{mm}} - \frac{1}{K_{mm}}K_{mm}\Big) = \frac{1-K_{mm}}{K_{mm}}$

$\rightarrow$ vgl. Abbildung **??**

$\rightarrow$ $SNR_{unbiased} = \dfrac{\mathcal{E}_s}{[\mathbf{\Phi}_{e'e'}]_{mm}} = \dfrac{\mathcal{E}_s}{\mathcal{E}_s\dfrac{1-K_{mm}}{K_{mm}}} = \dfrac{K_{mm}}{1-K_{mm}}, \quad 1 \leq m \leq N_T$

$\rightarrow$ the SNR after bias removal is by "1" smaller than the biased SNR $\rightarrow$ general result valid for any type of MMSE estimation

2.4.3 Decision - Feadback Equalization (Detection)

- Also known as:
  - BLAST (Bell Laboratories space-time system)
  - successive interference cancellation
- Problem of linear receiver: Noise enhancementbecause of linear filtering $\rightarrow$ nonlinear filtering processing necessary

Basic Idea

- Recall (linear filter): $\mathbf{FH} = \mathbf{I}$ for linear ZF receiver $\rightarrow$ ith row of $\mathbf{F}, \mathbf{f}_i$, is orthogonal to the jth column of $\mathbf{H}, \mathbf{h}_j$, if $i \neq j$ (if $i = j$: inner product $= 1$)
- we can detect x_i, based on $r_i = \mathbf{f}_i \mathbf{y}$
- Once we have detected x_i, we can substract its contribution from $\mathbf{y}$: $\mathbf{y}_1 = \mathbf{y} - \mathbf{h}_i \hat{x}_i$ ($\hat{x}_i$ is detected symbol, we assume for now, $\hat{x}_i = x_i$)
- $\mathbf{y}_1$ can be expressed as $\mathbf{y}_1 = \mathbf{H}_i \mathbf{x}_i + \mathbf{n}$ where

$$\mathbf{H}_i = \left[\mathbf{h}_1, \dots, \mathbf{h}_{i-1}, \mathbf{h}_{i+1}, \dots, \mathbf{h}_{N_T}\right]$$
$$\mathbf{x}_i = \left[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{N_T}\right]$$

  $\rightarrow$ we have reduced the number of signal streams to $N_T - 1$ (N_R bleibt gleich)

- apply now linear ZF filter for symbol to detected next, e.g. x_j, where $j \in \left\{1, \dots, i-1, i+1, \dots, N_T\right\}$

$\rightarrow$ $\mathbf{r}_j = \mathbf{f}_j \mathbf{y}_1$ where $\mathbf{f}_j$ is the ZF filter for $\mathbf{H}_1$

- substract contribution of x_j from $\mathbf{y}_1$: $\mathbf{y}_2 = \mathbf{y}_1 - \mathbf{h}_j x_j$
- substract until last symbol is detected
- Blockdiagram see figure**??**
- Observations:
  - The order in which the x_1 are detected can be freely chosen and effects the performance $\rightarrow$ $N_T!$ possible orders $\rightarrow$ cannot explore all of them
  - Practical approach: Select in each step that x_i for which the noise variance enhancement is minimum, i.e. which has the smallest $\mathcal{E}\left\{|\mathbf{f}_i \mathbf{n}|^2\right\} = \sigma_n^2 |||\mathbf{f}_i|||^2$
- Diversity order:
  - stage 1: $G_d^1 = N_R - N_T + 1$
  - stage 2: $G_d^2 = G_d^1 + 1 = N_R - N_T + 2$

    $\vdots$
  - stage N_T: $G_d^{N_T} = N_R$
  - overall: $\underline{G_d = N_R - N_T + 1}$

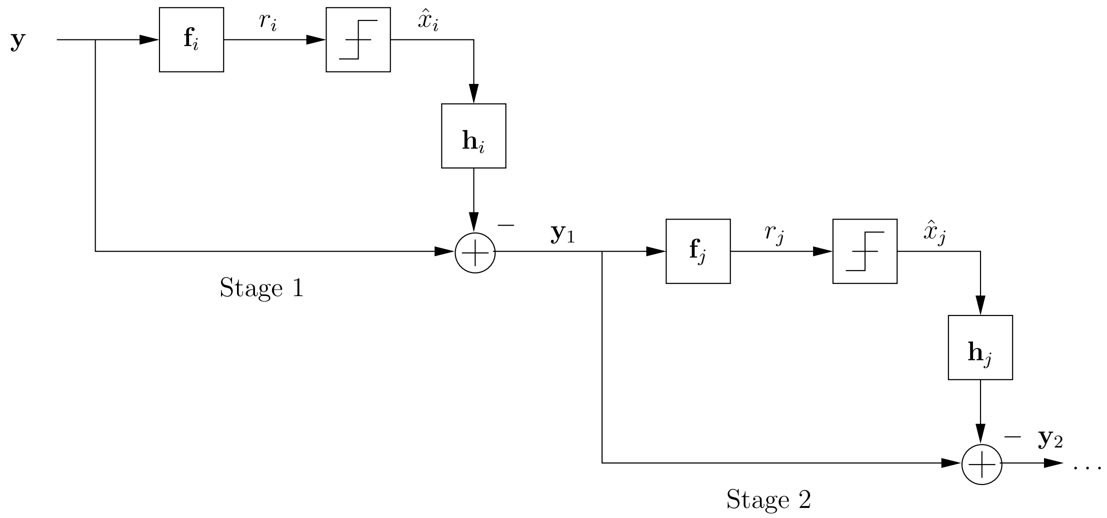


Abbildung 9: DFE Blockdiagram

  - *(Anmerkung: der schlechteste Fall dominiert (Stage 1), weitere koennen nur schwer beeinflussen)*

ZF - DFE - Matrix Model

- Signal model: $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} = \underbrace{\mathbf{H}\mathbf{P}}_{\tilde{\mathbf{H}}} \cdot \underbrace{\mathbf{P}^{-1}\mathbf{x}}_{\tilde{\mathbf{x}}} + \mathbf{n}$ with permutation matrix $\mathbf{P}$
- $\mathbf{P}$ has one „1“ per column and row, all other elements are „0“

→ can change the detection order to maximize performance

- note: $\mathbf{P}^T = \mathbf{P}^{-1}$
- Example:

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} [r] \rightarrow \mathbf{P}^{-1} = \mathbf{P}^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\tilde{\mathbf{x}} = \mathbf{P}^T \cdot \mathbf{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}$$

- Blockdiagram see figure **??**
- DFE Filters:
  - $\mathbf{F}$ feedworward filter
  - $\mathbf{B}$ feedback filter
- Filter calculation

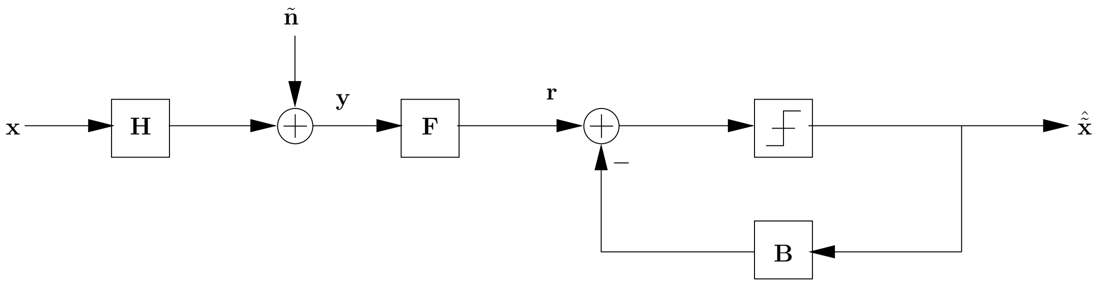


Abbildung 10: ZF-DFE Matrix Model

- Cholesky factorization: $\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} = \mathbf{L}^H\mathbf{D}\mathbf{L}$ with diagonal matrix $\mathbf{D}$ and lower triangular matrix $\mathbf{L}$ (maindiagonal elements of $\mathbf{L}$ are "1")

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ l_{21} & 1 & \ddots & \ddots & \vdots \\ l_{31} & l_{32} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \cdots & 1 \end{bmatrix}$$

- $\mathbf{F} = \mathbf{D}^{-1}\mathbf{L}^{-H}\tilde{\mathbf{H}}^H$

$$\rightarrow \mathbf{r} = \mathbf{F}\mathbf{y} = \underbrace{\mathbf{D}^{-1}\mathbf{L}^{-H}\underbrace{\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}}_{\mathbf{L}^H\mathbf{D}\mathbf{L}}}_{\mathbf{L}}\tilde{\mathbf{x}} + \underbrace{\mathbf{D}^{-1}\mathbf{L}^{-H}\tilde{\mathbf{H}}^H\mathbf{n}}_{\tilde{\mathbf{n}}}$$

- $\mathbf{B} = \mathbf{L} - \mathbf{I} =$ lower triangular matrix with maindiagonal elements "0"

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ l_{21} & 0 & \ddots & \ddots & \vdots \\ l_{31} & l_{32} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

- Interpretation:
  - after feedforward filter

$$\mathbf{r} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ l_{21} & 1 & \ddots & \ddots & \vdots \\ l_{31} & l_{32} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \cdots & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_{N_T} \end{bmatrix} + \tilde{\mathbf{n}}$$

- from feedback filter

$$\tilde{\mathbf{r}} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ l_{21} & 0 & \ddots & \ddots & \vdots \\ l_{31} & l_{32} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} \hat{\tilde{x}}_1 \\ \hat{\tilde{x}}_2 \\ \vdots \\ \hat{\tilde{x}}_{N_T} \end{bmatrix} \quad \hat{\tilde{x}}_n : \text{decision on } \tilde{x}_n$$

$$\rightarrow \mathbf{r} - \tilde{\mathbf{r}} = \begin{bmatrix} \tilde{x}_1 + 0 \\ l_{21}(\tilde{x}_1 - \hat{\tilde{x}}_1) + \tilde{x}_2 + 0 \\ l_{31}(\tilde{x}_1 - \hat{\tilde{x}}_2) + l_{32}(\tilde{x}_2 - \hat{\tilde{x}}_2) + \tilde{x}_3 + 0 \\ \vdots \end{bmatrix} + \tilde{\mathbf{n}}$$

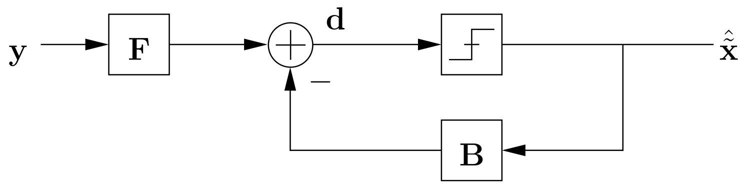


Abbildung 11: MMSE-DFE Blockdiagram

MMSE-DFE

- $\mathbf{B} = \mathbf{L} - \mathbf{I}$ has to be lower triangular matrx with all-zero main diagonal elements because of causality.
- $\mathbf{d} = \mathbf{F}\mathbf{y} - \mathbf{B}\hat{\tilde{\mathbf{x}}} \quad \rightarrow \mathbf{e} = \mathbf{d} - \hat{\mathbf{x}} = \mathbf{F}\mathbf{y} - \underbrace{(\mathbf{B} + \mathbf{I})}_{\mathbf{L}}\tilde{\mathbf{x}}$

  where we assume $\hat{\tilde{\mathbf{x}}} = \tilde{\mathbf{x}}$ for filter optimization
- Optimal $\mathbf{F}$ for given $\mathbf{L}$

$$\sigma_e^2 = \operatorname{tr}\left\{\underbrace{\mathcal{E}\{\mathbf{e}\mathbf{e}^H\}}_{\mathbf{\Phi}_{ee}}\right\} = \operatorname{tr}\left\{\mathbf{F}\mathbf{\Phi}_{yy}\mathbf{F}^H\right\} - \operatorname{tr}\left\{\mathbf{F}\mathbf{\Phi}_{y\tilde{x}}\mathbf{L}^H\right\} - \operatorname{tr}\left\{\mathbf{L}\mathbf{\Phi}_{\tilde{x}y}\mathbf{F}^H\right\} + \operatorname{tr}\left\{\mathbf{L}\mathbf{\Phi}_{\tilde{x}\tilde{x}}\mathbf{L}^H\right\}$$

$$\frac{\delta}{\delta \mathbf{F}^*}\sigma_e^2 = \mathbf{F}\mathbf{\Phi}_{yy} - \mathbf{L}\mathbf{\Phi}_{\tilde{x}y} = 0 \quad \rightarrow \mathbf{F} = \mathbf{L}\mathbf{\Phi}_{\tilde{x}y}\mathbf{\Phi}_{yy}^{-1}$$

where

$$\mathbf{\Phi}_{yy} = \mathcal{E}_s\tilde{\mathbf{H}}\tilde{\mathbf{H}}^H + \sigma_n^2\mathbf{I}$$
$$\mathbf{\Phi}_{\tilde{x}y} = \mathcal{E}_s\tilde{\mathbf{H}}^H$$

$$\rightarrow \mathbf{F} = \mathbf{L}\tilde{\mathbf{H}}^H(\tilde{\mathbf{H}}\tilde{\mathbf{H}}^h + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I})^{-1} = \mathbf{L}\,\overbrace{(\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I})^{-1}\tilde{\mathbf{H}}^H}^{\text{MMSE Linear Equalizer}}$$

$$\mathbf{y} \longrightarrow \boxed{\mathbf{F}_{DFE}} \longrightarrow \mathbf{r} \quad \equiv \quad \mathbf{y} \longrightarrow \boxed{\mathbf{F}_{Linear}} \longrightarrow \boxed{\mathbf{L}} \longrightarrow \mathbf{r}$$

Abbildung 12: MMSE-DFE Equivalence

- Optimal $\mathbf{L}$:

$$\begin{aligned}\mathbf{\Phi}_{ee} &= \mathbf{L}\mathbf{\Phi}_{\tilde{x}y}\mathbf{\Phi}_{yy}^{-1}\mathbf{\Phi}_{yy}\mathbf{\Phi}_{yy}^{-1}\mathbf{\Phi}_{\tilde{x}y}^{H}\mathbf{L}^{H} - \mathbf{L}\mathbf{\Phi}_{\tilde{x}y}\mathbf{\Phi}_{yy}^{-1}\mathbf{\Phi}_{y\tilde{x}}\mathbf{L}^{H} - \mathbf{L}\mathbf{\Phi}_{\tilde{x}y}\mathbf{\Phi}_{yy}^{-1}\mathbf{\Phi}_{\tilde{x}y}^{H}\mathbf{H}^{H} + \mathbf{L}\mathbf{\Phi}_{\tilde{x}\tilde{x}}\mathbf{L}^{H} \\ &= \mathbf{L}(\mathbf{\Phi}_{\tilde{x}\tilde{x}} - \mathbf{\Phi}_{\tilde{x}y}\mathbf{\Phi}_{yy}^{-1}\mathbf{\Phi}_{\tilde{x}y}^{H})\mathbf{L}^{H}\end{aligned}$$

= MMSE covariance matrix for a linear MMSE filter

→ $\mathbf{\Phi}_{ee} = \sigma_n^2\mathbf{L}(\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I})^{-1}\mathbf{L}^H$

→ need lower triangular matrix $\mathbf{L}$ which minimizes $\mathrm{tr}\{\mathbf{\Phi}_{ee}\}$

→ the optimum $\mathbf{L}$ whitenes $\mathbf{\Phi}_{ee}$ → $\mathbf{\Phi}_{ee}$ becomes diagonal matrix i.e. $\mathbf{L}$ exploits the correlation after linear MMSE filtering to reduce noise variance!

→ $\mathbf{L}$ is obtained via Cholesky factorization

$$\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I} = \mathbf{L}\mathbf{D}\mathbf{L}^H$$

→ $\mathbf{\Phi}_{ee} = \sigma_n^2\mathbf{L}(\mathbf{L}\mathbf{D}\mathbf{L}^H)^{-1}\mathbf{L} = \sigma_n^2\mathbf{L}\mathbf{L}^{-1}\mathbf{D}^{-1}\mathbf{L}^{-H}\mathbf{L} = \sigma_n^2\mathbf{D}^{-1} \widehat{=}$ diagonal matrix

- Summary of MMSE calculation
  - $\mathbf{B} = \mathbf{L} - \mathbf{I}$ where $\mathbf{H}^H\mathbf{H} + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I} = \mathbf{L}\mathbf{D}\mathbf{L}^H$
  - $\mathbf{F} = \mathbf{L}\underbrace{(\mathbf{H}^H\mathbf{H} + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I})^{-1}\tilde{\mathbf{H}}}_{\mathbf{F}_{Linear}}$
  - $\mathbf{L}$ is a prediction error filter, which whitenes the error signal $\mathbf{e}$
  - SNR:

$$\begin{aligned}\mathbf{D} &= \mathrm{diag}\{\xi_1, \ldots, \xi_{N_T}\} \\ \mathbf{\Phi}_{ee} &= \mathrm{diag}\{\frac{\sigma_n^2}{\xi_1}, \ldots, \frac{\sigma_n^2}{\xi_{N_T}}\}\end{aligned}$$

– end-to-end channel

$$\begin{aligned}
\mathbf{K} &= \mathbf{F}\tilde{\mathbf{H}} \\
&= \underbrace{\mathbf{L}(\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I})^{-1}\mathbf{L}^H}_{\frac{1}{\sigma_n^2}\mathbf{\Phi}_{ee}}\,\mathbf{L}^{-H}\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} \\
&= \frac{1}{\sigma_n^2}\mathbf{\Phi}_{ee}\mathbf{L}^{-H}(\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I} - \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I})\mathbf{L}^{-1}\mathbf{L} \\
&= \frac{1}{\sigma_n^2}[\underbrace{\mathbf{L}^{-H}(\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} + \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{I})\mathbf{L}^{-1}}_{\sigma_n^2\mathbf{\Phi}_{ee}^{-1}}\,\mathbf{L} - \frac{\sigma_n^2}{\mathcal{E}_s}\mathbf{L}^{-H}] \\
&= \mathbf{L} - \frac{1}{\mathcal{E}_s}\mathbf{\Phi}_{ee}\mathbf{L}^{-H}
\end{aligned}$$

→ main diagonal of $\mathbf{K}$:

$$K_{mm} = 1 - \frac{\sigma_n^2}{\mathcal{E}_s\xi_m} < 1$$

– biased SNR

$$SNR_{m,bias} = \frac{\mathcal{E}_S}{[\mathbf{\Phi}_{ee}]_{mm}} = \frac{\mathcal{E}_s}{\sigma_n^2}\xi_m = \frac{1}{1-K_{mm}}, \quad 1 \leq m \leq N_T$$

– unbiased SNR

$$SNR_{m,unbiased} = SNR_{m,bias} - 1 = \frac{1}{1-K_{mm}} - 1 = \frac{K_{mm}}{1-K_{mm}}, \quad 1 \leq m \leq N_T$$

2.4.4 Sphere Decoding

- Linear and DFE receivers cannot approach performance of ML - detector
- ML - detector: $\hat{\mathbf{x}} = \underset{\mathbf{x}\in A^{N_T}}{\operatorname{argmin}} ||\mathbf{y} - \mathbf{H}\mathbf{x}||^2$

→ high complexity for brute force search

Main Idea

- can we search ML metric in a "smarter" way, akin to the smart search of the viterbi algorithm for problems with a trellis structure
- we need to find a way to prune/dismiss non - ML sequences/vectores early on

→ this is accomplished by sphere decoding

Step 1 Bring metric into a suitable form:

- real represantation: $||\mathbf{y} - \mathbf{H}\mathbf{x}||^2 = ||\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\mathbf{x}}||^2$
- where:

$$\tilde{\mathbf{y}} = \begin{bmatrix}\mathrm{Re}\{\mathbf{y}^T\} & \mathrm{Im}\{\mathbf{y}^T\}\end{bmatrix}^T$$
$$\tilde{\mathbf{x}} = \begin{bmatrix}\mathrm{Re}\{\mathbf{x}^T\} & \mathrm{Im}\{\mathbf{x}^T\}\end{bmatrix}^T$$
$$\tilde{\mathbf{H}} = \begin{bmatrix}\mathrm{Re}\{\mathbf{H}\} & -\mathrm{Im}\{\mathbf{H}\} \\ \mathrm{Im}\{\mathbf{H}\} & \mathrm{Re}\{\mathbf{H}\}\end{bmatrix}^T$$

$\rightarrow$ QAM: $\tilde{\mathbf{x}}$ defines points in an $2N_T$ dimensional lattice

- QL decomposition: $\tilde{\mathbf{H}} = \mathbf{QL}$ with
  - orthogonal matrix $\mathbf{Q} \in \mathbb{R}^{2N_T \times 2N_T}$; $\quad \mathbf{Q}\mathbf{Q}^T = \mathbf{I}$
  - and lower triangular matrix $\mathbf{L} \in \mathbb{R}^{2N_T \times 2N_T}$

$$\begin{aligned}
||\tilde{\mathbf{y}} - \mathbf{QL}\tilde{\mathbf{x}}||^2 &= ||\mathbf{Q}\mathbf{Q}^T(\tilde{\mathbf{y}} - \mathbf{QL}\tilde{\mathbf{x}})||^2 + ||(\mathbf{I} - \mathbf{Q}\mathbf{Q}^T)(\tilde{\mathbf{y}} - \mathbf{QL}\tilde{\mathbf{x}})||^2 \\
&= ||\mathbf{Q}(\mathbf{Q}^T\tilde{\mathbf{y}} - \mathbf{L}\tilde{\mathbf{x}})||^2 + ||(\mathbf{I} - \mathbf{Q}\mathbf{Q}^T)\tilde{\mathbf{y}} - \underbrace{(\mathbf{Q} - \mathbf{Q})}_{=0}\mathbf{L}\tilde{\mathbf{x}}||^2 \\
&= ||\mathbf{Q}^T\tilde{\mathbf{y}} - \mathbf{L}\tilde{\mathbf{x}}||^2 + \underbrace{||(\mathbf{I} - \mathbf{Q}\mathbf{Q}^T)\tilde{\mathbf{y}}||^2}_{\text{unabhängig von } \tilde{\mathbf{x}}}
\end{aligned}$$

$\Rightarrow$ $\underset{\tilde{\mathbf{x}} \in A^{2N_T}}{\mathrm{argmin}}||\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\mathbf{x}}||^2 = \underset{\tilde{\mathbf{x}} \in A^{2N_T}}{\mathrm{argmin}}||\underbrace{\mathbf{Q}^T\tilde{\mathbf{y}}}_{\bar{\mathbf{y}}} - \mathbf{L}\tilde{\mathbf{x}}||^2$
where A^{2N_T} contains all M^{N_T} possible vectors $\tilde{\mathbf{x}}$

- Observation:
  - with

$$\mathbf{L} = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ l_{2N_T\,1} & \cdots & & l_{2N_T\,2N_T} \end{bmatrix}$$
$$\bar{\mathbf{y}} = \begin{bmatrix}\bar{y}_1 & \cdots & \bar{y}_{2N_T}\end{bmatrix}^T; \quad \tilde{\mathbf{x}} = \begin{bmatrix}\tilde{x}_1 & \cdots & \tilde{x}_{2N_T}\end{bmatrix}$$

  - we have

$$||\bar{\mathbf{y}} - \mathbf{L}\tilde{\mathbf{x}}||^2 = (\bar{y} - l_{11}\tilde{x}_1)^2 + (\bar{y}_2 - l_{21}\tilde{x}_1 - l_{22}\tilde{x}_2)^2 + (\bar{y}_3 - l_{31}\tilde{x}_1 - l_{32}\tilde{x}_2 - l_{33}\tilde{x}_3)^2 + \ldots \tag{1}$$

$\rightarrow$ the n-th term in the above sum contains only $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$

Step 2 Sphere decoding algorithm

Define:

$$d(\tilde{\mathbf{x}}) = \sum_{n=1}^{2N_T} f_n(\tilde{\mathbf{x}}_n) \; ; \quad \text{with } f_n(\tilde{\mathbf{x}}_n) = \big(\bar{y}_n - \sum_{i=1}^{n} l_{ni}\tilde{x}_i\big)^2 \; ; \quad \tilde{\mathbf{x}}_n = \begin{bmatrix}\tilde{x}_1 & \dots & \tilde{x}_n\end{bmatrix}^T$$

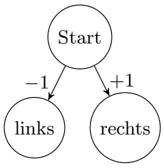
$$d_n(\tilde{x}_n) = \sum_{m=1}^{n} f_n(\tilde{\mathbf{x}}_m)$$

Main Idea:

- Assume we know that $d(\tilde{\mathbf{x}}) \leq R$ holds for some $\tilde{\mathbf{x}}$, where R is the so called "sphere radius" $\rightarrow$ any $\tilde{\mathbf{x}}$ with $d(\tilde{\mathbf{x}}) \geq R$ cannot be the ML solution and can be discarded
- since $d_{n+1}(\tilde{\mathbf{x}}'_n, \tilde{x}_{n+1}) \geq d_n(\tilde{\mathbf{x}}'_n)$, we can easily discard $\mathbf{x}'_n$ and all possible $\tilde{\mathbf{x}} = \begin{bmatrix}\tilde{\mathbf{x}}'_n & \tilde{x}_{n+1} & \tilde{x}_{n+2} & \dots & \tilde{x}_{2N_T}\end{bmatrix}^T$ if we find $d_n(\tilde{\mathbf{x}}'_n) > R$
  $\Rightarrow$ we can exclude many possible vectors $\tilde{\mathbf{x}}$ without evaluating any metrics for them
- How to find a suitable R?
  - Initial R: $R = d(\tilde{\mathbf{x}}_{\text{subopt}})$ where $\tilde{\mathbf{x}}_{\text{subopt}}$ was obtained with some suboptimum receiver
  - R is updated as $R = R_{\text{new}} = d(\tilde{\mathbf{x}}_{\text{cand}})$ where $\tilde{\mathbf{x}}_{\text{cand}}$ is an $\tilde{\mathbf{x}}$ which yields $d(\tilde{\mathbf{x}}_{\text{cand}}) < R_{\text{old}} = R$
- Use tree structure to represent all possible $\tilde{\mathbf{x}}$

Example:

- Siehe dazu Abbildung 13 auf S. 39
- Assume $\tilde{\mathbf{x}}_{\text{subopt}} = \begin{bmatrix}-1 & -1 & -1 & 1\end{bmatrix}^T$
  $\Rightarrow d(\tilde{x}_{\text{subopt}} = 1+2+2+2 = 7 = R$
- For $\tilde{x}_1 = 1$ we find $d_2(1,1) = 5+3 = 8 > R$ and $d(1,-1) = 5+4 > R$
  $\Rightarrow$ we don't have to calculate metrics for remaining branches, i.e., for nodes $19, \dots, 31$
- For $\tilde{x}_1 = -1$ and $\tilde{x}_2 = 1$, we find that $d_3(-1,1,-1) = 8 > R$ and $d_3(-1,1,1) = 9 > R$
  $\Rightarrow$ remaining branches emerging from nodes 6 and 7 can be discarded
- For the ML solution we find $d(-1,-1,1,-1) = 5 < R$
- Different strategies exist, regarding in which order the nodes in the tree (points in the lattice) are investigated
  - Pohst strategy
  - Schnoir - Enchner strategy
  - rich literature on lattice decoding algorithms
- Complexity of sphere decoding
  - worst-case complexity is still exponential in N_T
  - (for practical case:) for sufficiently high SNR, the average complexity is only polynomial in $N_T \Rightarrow$ efficient ML decoding



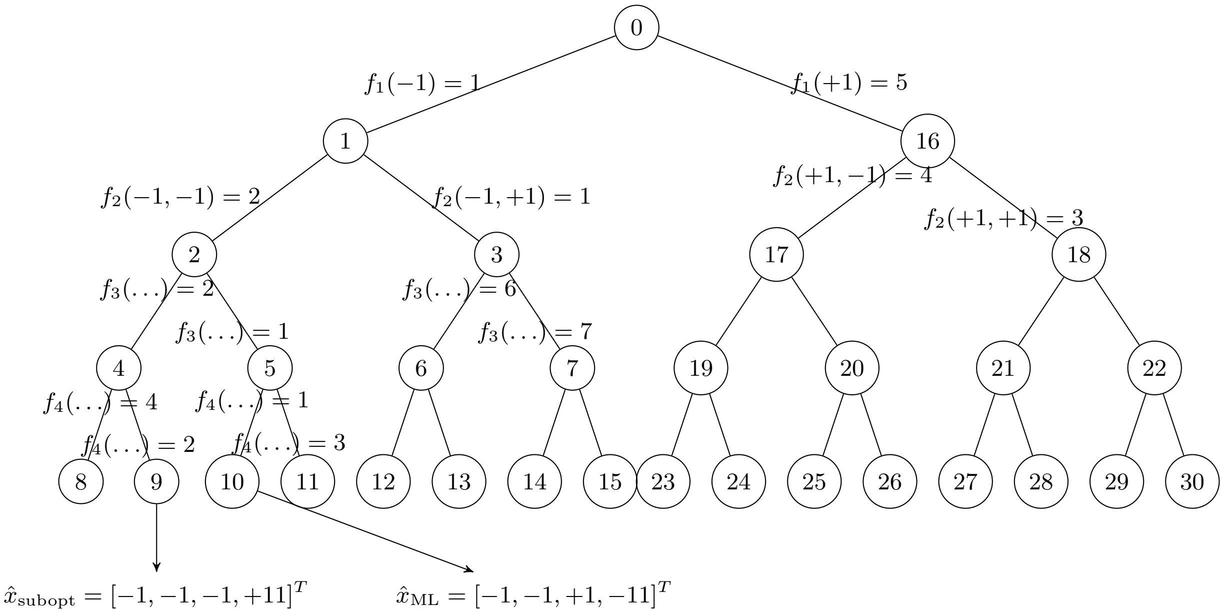


Abbildung 13: BPSK = 2, $N_T = 2$

- sphere decoding has found application in many fields:
  - ML detection in MIMO and multiuser systems
  - precoding
  - source coding
  - multiple symbol differential detection

3 Multiuser MIMO

- We distinguish two cases:
  - multipoint - to - point transmission
  - point - to - multipoint transmission
- Multipoint - to - point transmission
  - typical uplink scenario in cellular systems
  - information theoretical channel model: Multiple Access Channel (MAC)

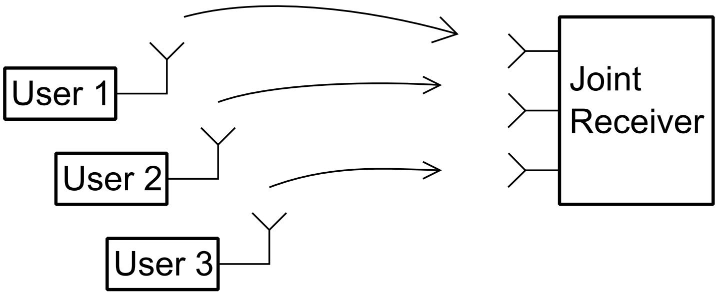


Abbildung 14: Multiple Access Channel

- Point - to -multipoint transmission
  - typical downlink scenarion in cellular systems
  - information theoretical channel model: Broadcast Channel (BC)

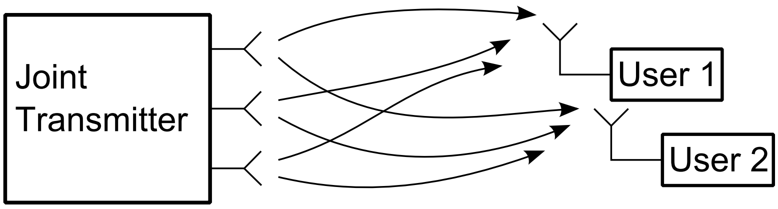


Abbildung 15: Broadcast Channel

- Advantage of multiuser MIMO compared to point - to - point MIMO
  - multiplexing gain can be exploited even if users have only single antenna
  - users are spatially distributed in cell $\rightarrow$ channels to different users are independent

3.1 Multiple Access Channel (MAC)

We consider two aspects:

- Detector structures

- Rate region

3.1.1 Detector structures

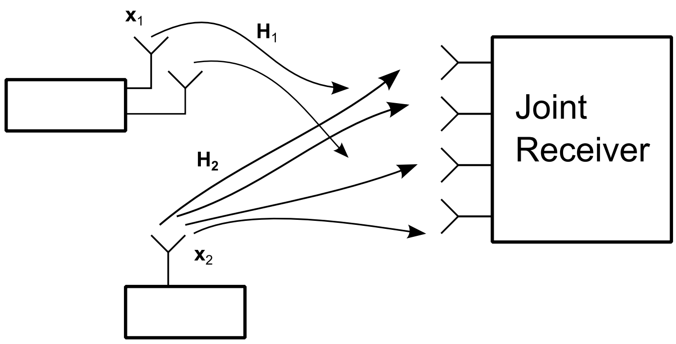


Abbildung 16: Block Diagramm of Channel Model

Channel model: $\rightarrow$ general MAC: $\mathbf{y} = \sum\limits_{k=1}^{K} \mathbf{H}_k \mathbf{x}_k + \mathbf{n}$

with:

- K users
- user k has $N_{T,k}$ transmit antennas
- N_R receive antennas
- $\mathbf{H}_k \in \mathbb{C}^{N_R \times N_{T,k}}$

$$\mathbf{y} = \underbrace{\begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 & \dots & \mathbf{H}_k \end{bmatrix}}_{\mathbf{H}} \cdot \underbrace{\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_k \end{bmatrix}}_{\mathbf{x}} + \mathbf{n}$$

Observation:

- same equivalent channel model as for a point-to-point MIMO system transmitting $N_T = \sum_{k=1}^{K} N_{T,k}$ independent signal streams *(Anmerkung: kein Unterschied für Empfänger, ob Signale von einem Nutzer oder von mehreren)*
- the receiver (e.g. base station) can use detection schemes as for point-to-point MIMO systems
  - linear receiver
  - DFG
  - sphere decoder

Typical problems in uplink multiuser MIMO For given receiver structure:

- calculate SNR_k for all users k based on the expressions developed in Chapter 2.4
- optimize transmit power of users, $E_k = \mathcal{E}\{||x_k||^2\}$ for maximization of the sumrate or maximization of the minimum SNR_k *(Anmerkung: Maximierung der sumrate kann durch Maximierung des SNR des Users mit bestem Kanal erfolgen, aber: unfair anderen Usern gegenüber $\Rightarrow$ starving)*

3.1.2 Rate region

For point - to - point links, we can decode error free, if the rate, R, meets

a) SISO $R < \log_2\left(1 + \frac{\mathcal{E}_s}{\sigma_n^2}\right)$

b) MIMO $R < \log_2 \underbrace{\left|\mathbf{I} + \frac{\mathcal{E}_s}{N_T \sigma_n^2}\mathbf{H}\mathbf{H}^H\right|}_{\det}$

Questions: What happens if there are multiple users?

Rate Region for Single Antenna Users and Receivers

- Gaussian channel
- $N_R = N_{T,k} = 1 \forall \quad k$
- received signal:

$$y = \sum_{k=1}^{K} x_k + n$$
$$*\mathcal{E}_k = \mathcal{E}\{||x_k||^2\}$$
$$*\sigma_n^2 = \{||n||^2\}$$

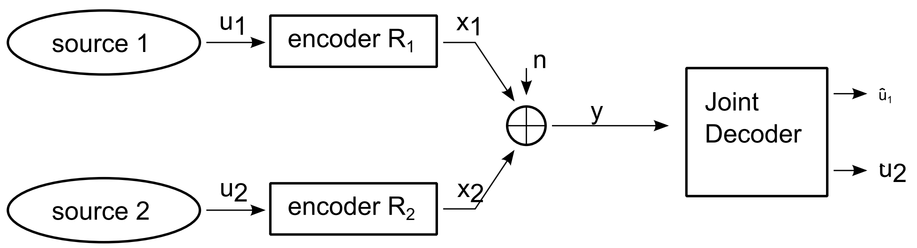


Abbildung 17: Block Diagramm

Example: 2 Users

- How should we choose R_1 and R_2 to ensure error free decoding of <u>both</u> signal streams?
- It is no longer sufficient to maximizie a single rate. Instead we have to consider rate pairs (R_1, R_2)

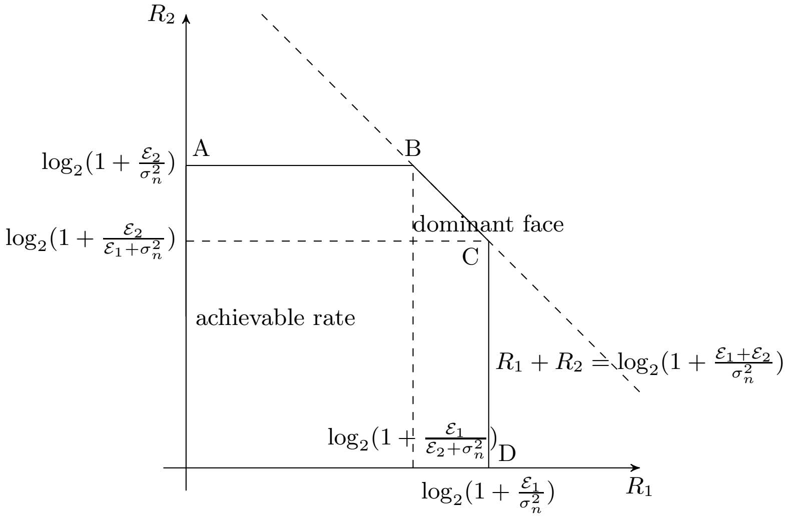
- All possible rate points, that allows error free decoding, define the rate region $\underline{C}$
- Possible desing goals of the system:
  - maximized sumrate $R_{\text{sum}} = \max\limits_{(R_1,R_2)\in\underline{C}} R_1 + R_2$
  - maximize minimum user rate: $R_{\text{max-min}} = \max\limits_{(R_1,R_2)\in\underline{C}} \min\limits_{i\in\{1,2\}} R_i$
- Rate Region of two user Gaussian MAC *Anmerkung: Einschränkungen*

$$R_1 < \log_2\left(1 + \frac{\mathcal{E}_1}{\sigma_n^2}\right) \tag{2}$$

$$R_2 < \log_2\left(1 + \frac{\mathcal{E}_2}{\sigma_n^2}\right) \tag{3}$$

$$R_1 + R_2 < \log_2\left(1 + \frac{\mathcal{E}_1 + \mathcal{E}_2}{\sigma_n^2}\right) \tag{4}$$

- Interpretation:
  - (2) and (3) (= single-to user constraint) are the "single-user bounds, i.e., the maximum rates of user 1 and 2, if the other user was not there
  - (4) can be interpreted as the maximum rate if streams of users 1 and 2 were jointly encoded. The separate encoding in the MAC cannot yield a better performance
  - Graphical represantation:



- Observations:
  - A-B is defined by (3)
  - C-D is defined by (2)
  - B-C is defined by (4)
  - A-B suggests that even if user 2 transmits with the same max. rate as in the single user case, user 1 can transmit with non-zero rate! → Multiuser communication enables "free rate gains!

- Which point on A - B - C,- D we choose, depends on the design criterion

- How do we achieve points on A - B - C,- D?
  - Both user use Gaussian codebooks
  - B:
    * signal of user 1, x_1, is decoded first and x_2 is treated as noise:

$$y = x_1 + \underbrace{x_2 + n}_{\text{treat as noise}}$$
$$\rightarrow R_1 < \log_2\big(1 + \frac{\mathcal{E}_1}{\mathcal{E}_2 + \sigma_n^2}\big)$$

    * once x_1 is known, we form

$$y - x_1 = x_2 + n$$
$$\rightarrow R_2 < \log_2\big(1 + \frac{\mathcal{E}_s}{\sigma_n^2}\big)$$

    * this approach is referred to as successive interference cancellation (SIC) and is a direct result of the chain rule in information theory: $I(X_1, X_2, Y) = I(X_1, Y) + I(X_2; Y|X_1)$
  - C: same as B, but X_1 and X_2 change rules
  - Points on A - B, C - D can be achieved by decreasing the rate of users 1 and 2 respectively (not desirable)
  - Points on B - C (dominant face): Achievable by "time-sharing, i.e., $\theta \cdot 100\%$ of the time we decode user 1 first and $(1-\theta)100\%$ of the time we decode user 2 first, $0 \leq \theta \leq 1$

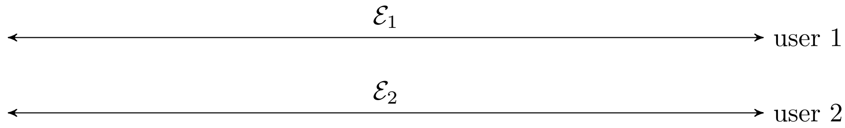
$$\begin{aligned}
R_1 &< \theta \log_2\big(1 + \frac{\mathcal{E}_1}{\mathcal{E}_2 + \sigma_n^2}\big) + \big(1-\theta\big)\log_2\big(1 + \frac{\mathcal{E}_1}{\sigma_n^2}\big) \\
R_2 &< \theta \log_2\big(1 + \frac{\mathcal{E}_2}{\sigma_n^2}\big) + \big(1-\theta\big)\log_2\big(1 + \frac{\mathcal{E}_2}{\mathcal{E}_1 + \sigma_n^2}\big) \\
\rightarrow R_1 + R_2 &< \theta\Big(\log_2\big(1 + \frac{\mathcal{E}_1}{\mathcal{E}_2 + \sigma_n^2}\big) + \log_2\big(1 + \frac{\mathcal{E}_2}{\sigma_n^2}\big)\Big) + \\
&+ \big(1-\theta\big)\Big(\log_2\big(1 + \frac{\mathcal{E}_1}{\sigma_n^2}\big) + \log_2\big(1 + \frac{\mathcal{E}_2}{\mathcal{E}_1 + \sigma_n^2}\big)\Big) = \\
&= \theta \log_2\Big(\frac{\mathcal{E}_1 + \mathcal{E}_2 + \sigma_n^2}{\mathcal{E}_2 + \sigma_n^2} \cdot \frac{\mathcal{E}_2 + \sigma_n^2}{\sigma_n^2}\Big) \cdot \big(1-\theta\big)\log_2\Big(\frac{\mathcal{E}_1 + \sigma_n^2}{\sigma_n^2} \cdot \frac{\mathcal{E}_1 + \mathcal{E}_2 + \sigma_n^2}{\mathcal{E}_1 + \sigma_n^2}\Big) = \\
&= \log_2\Big(1 + \frac{\mathcal{E}_1 + \mathcal{E}_2}{\sigma_n^2}\Big)
\end{aligned}$$

- Comparison with orthogonal transmission
  - User 1 transmits for $\theta \cdot 100\%$ of the time and user 2 transmits for $(1-\theta) \cdot 100\%$ of the time, $0 \leq \theta \leq 1$

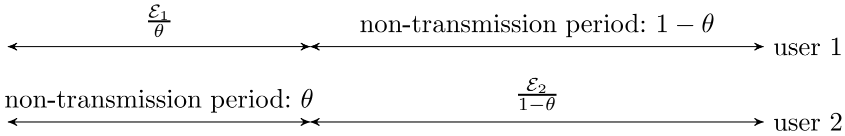
- to keep average transmit power independent of θ, the users transmit with powers $\frac{\mathcal{E}_1}{\theta}$ and $\frac{\mathcal{E}_2}{1-\theta}$
- Rates:

$$
\begin{aligned}
R_1 &< \theta \log_2\Big(1 + \frac{\mathcal{E}_1}{\theta\sigma_n^2}\Big) \\
R_2 &< \big(1-\theta\big)\log_2\Big(1 + \frac{\mathcal{E}_2}{(1-\theta)\sigma_n^2}\Big)
\end{aligned}
$$

multiuser:



orthogonal:



- sumrate:

$$
R_1 + R_2 < \theta \log_2\Big(1 + \frac{\mathcal{E}_1}{\theta\sigma_n^2}\Big) + \big(1-\theta\big)\log_2\Big(1 + \frac{\mathcal{E}_2}{(1-\theta)\sigma_n^2}\Big) = R_{\mathrm{sum}}
$$

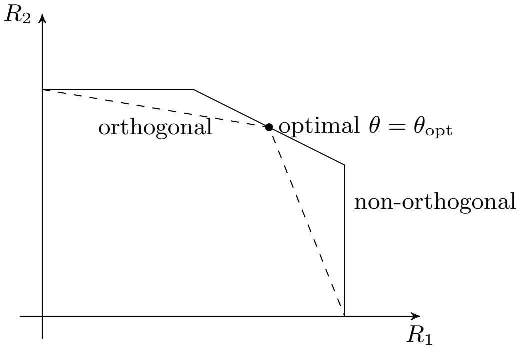
- Which θ maximizes sumrate?

$$
\frac{\delta R_{\mathrm{sum}}}{\delta\theta} \overset{!}{=} 0 \text{ leads to } \theta_{\mathrm{opt}} = \frac{\mathcal{E}_1}{\mathcal{E}_1 + \mathcal{E}_2}
$$

- Maximum sumrate

$$
\begin{aligned}
R_{\mathrm{sum}} &= \frac{\mathcal{E}_1}{\mathcal{E}_1 + \mathcal{E}_2}\log_2\Big(1 + \frac{\mathcal{E}_1 + \mathcal{E}_2}{\sigma_n^2}\Big) + \frac{\mathcal{E}_2}{\mathcal{E}_1 + \mathcal{E}_2}\log_2\Big(1 + \frac{\mathcal{E}_1 + \mathcal{E}_2}{\sigma_n^2}\Big) = \\
&= \log_2\Big(1 + \frac{\mathcal{E}_1 + \mathcal{E}_2}{\sigma_n^2}\Big) \\
&\rightarrow \text{ same value as for general non-orthogonal transmission!}
\end{aligned}
$$

– But: In general, orthogonal transmission is suboptimal!



- 3 users case:

$$R_1 < \log_2\Big(1 + \frac{\mathcal{E}_1}{\sigma_n^2}\Big)$$
$$R_2 < \log_2\Big(1 + \frac{\mathcal{E}_2}{\sigma_n^2}\Big)$$
$$R_3 < \log_2\Big(1 + \frac{\mathcal{E}_3}{\sigma_n^2}\Big)$$
$$R_i + R_j < \log_2\Big(1 + \frac{\mathcal{E}_i + \mathcal{E}_j}{\sigma_n^2}\Big), \quad i \neq j$$
$$R_1 + R_2 + R_3 < \log_2\Big(1 + \frac{\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3}{\sigma_n^2}\Big)$$

$\rightarrow$ rate region $\mathcal{C}$ has $3! = 6$ corner points

- general case of K users
  - define all non-empty subsets of $\mathbf{K} = \big\{1, \dots, K\big\}$ as $\mathbf{S} \in \mathbf{K}$, e.g. $K = 2$: $\mathbf{K} = \big\{1, 2\big\}, \mathbf{S} = \big\{\{1\}, \{2\}, \{1, 2\}\big\}$
- rate region $\mathcal{C}$ is defined by

$$\sum_{k \in \mathbf{S}} R_k < \log_2\Big(1 + \frac{\sum_{k \in \mathbf{S}} \mathcal{E}_k}{\sigma_n^2}\Big) \quad \forall \mathbf{S}$$

$\rightarrow$ $\mathcal{C}$ has $K!$ corner points which can all be achieved by successive interference cancellation (SIC)

Rate region for MIMO Users and Receivers

- Channel Model: $\mathbf{y} = \sum_{k=1}^{K} \mathbf{H}_k \mathbf{x}_k + \mathbf{n}$, with:
  - User k has $N_{T,k}$ transmit antennas
  - N_R receive antennas
  - $\mathbf{n}$: AWGN vector $\mathcal{N}(\mathbf{0}, \sigma_n^2 \mathbf{I})$

- 2 Users case:

$$\mathbf{y} = \mathbf{H}_1\mathbf{x}_1 + \mathbf{H}_2\mathbf{x}_2 + \mathbf{n} \tag{5}$$

  - Covariance matrix of the TX signal of user k: $\mathbf{Q}_k = \mathcal{E}\{\mathbf{x}_k\mathbf{x}_k^H\}$
  - transmit power: $\mathcal{E}_k = \text{tr}\{\mathbf{Q}_k\}$
- rate region for 2 user case and given $\mathbf{Q}_k$
  - $\mathbf{Q}_k$ given, for example
    a) $\mathbf{Q}_k$ optimal for single user case $\rightarrow \mathbf{Q}_k = \mathbf{U}_k\mathbf{\Lambda}_k\mathbf{U}_k^H$, where:
      · $\mathbf{U}_k$ is an unitary matrix
      · obtained from $\mathbf{H}_k = \mathbf{U}_k\mathbf{\Sigma}_k\mathbf{V}_k^H$
      · $\mathbf{\Lambda}_k = \text{diag}\{\mathcal{E}_{k,1}, \mathcal{E}_{k,2}, \ldots, \mathcal{E}_{k,N_T}\}$ with $\mathcal{E}_{l,i}$ obtained from waterfilling and $\sum_{i=1}^{N_{Tk}} \mathcal{E}_{k,i} = \mathcal{E}_k$

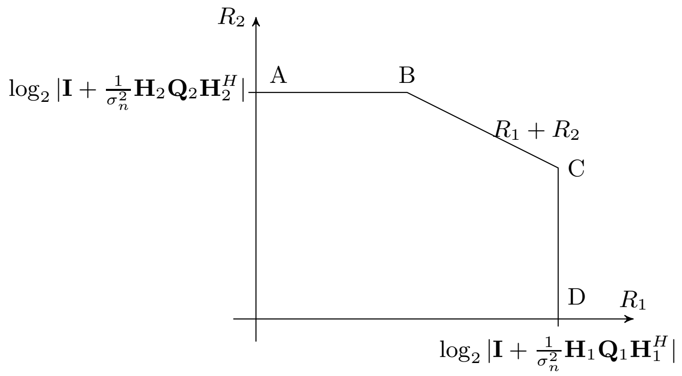
    b) $\mathbf{Q}_k = \frac{\mathcal{E}_k}{N_{T,k}}\mathbf{I}_{N_{T,k}}$ if $\mathbf{H}_k$ is not known at transmitter
  - for given $\mathbf{Q}_1$ and $\mathbf{Q}_2$ we can obtain the rate region as direct extension of the SISO case

$$R_1 < \log_2\left|\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}_1\mathbf{Q}_1\mathbf{H}_1^H\right| \tag{6}$$

$$R_2 < \log_2\left|\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}_2\mathbf{Q}_2\mathbf{H}_2^H\right| \tag{7}$$

$$R_1 + R_2 < \log_2\left|\mathbf{I} + \frac{1}{\sigma_n^2}\sum_{i=1}^{2}\mathbf{H}_i\mathbf{Q}_i\mathbf{H}_i^H\right| \tag{8}$$

    * equation 6 and equation 7 are the single user bounds,
    * equation 8 is the bound for the joint encoding of both users
  - graphical represantation



  - Points on A - B - C - D can be achieved in a similar manner as for SISO case
  - e.g. bound C can be achieved by SIC

– At B we have

$$R_2 = \log_2\left|\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}_2\mathbf{Q}_2\mathbf{H}_2^H\right|$$

$$R_1 = \log_2\left|\mathbf{I} + \frac{1}{\sigma_n^2}\sum_{i=1}^{2}\mathbf{H}_i\mathbf{Q}_i\mathbf{H}_i^H\right| - R_2$$

$\to$ user 1 transmits with rate

$$R_1 = \log_2\left|\mathbf{I} + \frac{1}{\sigma_n^2}(\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}_2\mathbf{Q}_2\mathbf{H}_2^H)^{-1}\mathbf{H}_1\mathbf{Q}_1\mathbf{H}_1^H\right|$$

– How to achieve rates at B? $\to$ Treat $\mathbf{H}_2\mathbf{x}_2+\mathbf{n}$ in equation 5 as noise with covariance matrix $\mathbf{Q}_N = \mathbf{H}_2\mathbf{Q}_2\mathbf{H}_2^H + \sigma_n^2\mathbf{I}$

$\to$ equivalent channel matrix with white noise: $\mathbf{r} = \mathbf{Q}_N^{-\frac{1}{2}}\mathbf{y} = \mathbf{Q}_N^{-\frac{1}{2}}\mathbf{H}_1\mathbf{x}_1 + \tilde{\mathbf{n}}$ where $\tilde{\mathbf{n}}$ is white noise with covariance $\mathbf{I}$.
Anmerkung: Rauschen war vorher farbig, muss "geweißt" werden.

$$\to R_1 = \log_2\left|\mathbf{I} + \underbrace{\mathbf{Q}_N^{-\frac{1}{2}}\mathbf{H}_1}_{\mathbf{H}}\mathbf{Q}_1\underbrace{\mathbf{H}_1^H\mathbf{Q}_N^{-\frac{1}{2}}}_{\mathbf{H_{eq}^H}}\right| = \log_2\left(\left|\mathbf{Q}_N^{\frac{1}{2}} + \mathbf{Q}_N^{-\frac{1}{2}}\mathbf{H}_1\mathbf{Q}_1\mathbf{H}_1^H\right| \cdot \left|\mathbf{Q}_N^{-\frac{1}{2}}\right|\right)$$

$$= \log_2\left|\mathbf{I} + \mathbf{Q}_N^{-1}\mathbf{H}_1\mathbf{Q}_1\mathbf{H}_1^H\right| =$$

$$= \log_2\left|\mathbf{I} + \frac{1}{\sigma_n^2}\left(\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}_2\mathbf{Q}_2\mathbf{H}_2^H\right)^{-1}\mathbf{H}_1\mathbf{Q}_1\mathbf{H}_1^H\right|$$

$\to$ we can achieve R_1 in B by treating user 2 as noise

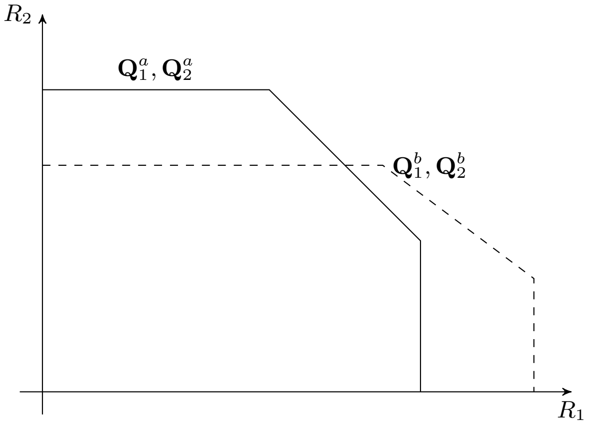
$\to$ once user 1 is detected, we can subtract its contribution from the received signal and detect user 2

$\Rightarrow$ user 2 can transmit with maximum single user rate

$\to$ bound C can be achieved by SIC similar to SISO case

– points on B - C are achieved through time sharing

- extension to K user case $\to$ analogous to SISO case
- Note: Different choices for Q_k will lead to different rate regions



- Example: $K = 2$

→ $\mathbf{Q}_1$ and $\mathbf{Q}_2$ can be optimized to achieve desired trade-off between performance of users 1 and 2

3.2 Broadcast Channel

We consider:

- uplink - downlink duality
- rate region

3.2.1 Multiplexing Gain - Degrees of freedom

Downlink scenarios:

- N_R antennas at transmitter, single antennas at the users
- user k receives: $\mathbf{y}_k = \mathbf{h}_k^H \mathbf{x} + \mathbf{n}_k$, with:
  - N_R dimensional channel vector of user k: $\mathbf{h}_k^H$
  - n_k: AWGN at user k
  - $\mathbf{x}$ transmit vector

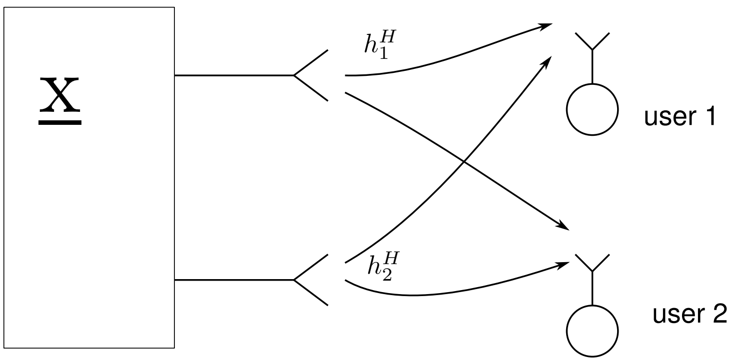


Abbildung 18: Scenario for Broadcast Channel

- How many independent signal streams can we transmit?
  - Consider transmit signal: $\mathbf{x} = \sum_{k=1}^{K} \mathbf{h}_k \mathbf{x}_k$, with TX transmit vector $\mathbf{n}_k$ and symbol x_k is intended for user k
- received signal of user k: $y_k = \sum_{k=1}^{K} (\mathbf{h}_k^H \mathbf{n}_i) \mathbf{x}_i + n_k$
- if all $\mathbf{h}_k$ were orthogonal and we chose $\mathbf{n}_k = \mathbf{h}_k$, the received signal would be: $y_k = ||h_k||^2 \mathbf{x}_k + n_k$
- if $N_R \geq K$, we can transmit simultaneously and interference free to all K users ⇒ multiplexing gain = min$\{K, N_k\}$

- In practice, the $\mathbf{h}_k$ will not be orthogonal

  $\rightarrow$ choose $\mathbf{n}_k$ such that it lies in the null space of $\begin{bmatrix}\mathbf{h}_1 & \dots & \mathbf{h}_{k-1} & \mathbf{h}_{k+1} & \dots & \mathbf{h}_K\end{bmatrix}$

  $\rightarrow$ always possible if $\mathbf{h}_1, \dots, \mathbf{h}_K$ are linearly independent

  $\rightarrow$ multiplexing gain (= degrees of freedom) is equal to $\min\{K, N_R\}$

- we can transmit interference free to $K \leq N_R$ users
  Anmerkung: Falls TX viele Antennen, aber RX nur eine hat $\Rightarrow$ begrenzter Nutzen: SNR Verbesserung, kein Multiplexing Gain; falls TX viele Antennen und viele RX vorhanden sind $\Rightarrow$ RX erscheinen als Antennenarray $\rightarrow$ hoher Multiplexing Gain

3.2.2 Uplink - Downlink Duality

- How should we choose signature vectors to achieve a certain SNR at users?
- Difficult problem since optimal (in the SINR sense) $\mathbf{u}_k$ are not orthogonal $\rightarrow$ signature of user k, $\mathbf{u}_k$, influences SINR at all other users!
- On the other hand, the uplink problem was much easier to solve, since the receive filter of user k, $\mathbf{f}_k$, was not affected by receive filters of other users! (vgl. Point - to - Point $\rightarrow$ detection problem

$\Rightarrow$ We establish a duality between the uplink and downlink, that allows us to solve the more challenging downlink problem by solving an equivalent uplink problem.

Downlink:

- transmit signal

$$\mathbf{x}_{dl} = \sum_{k=1}^{K} \mathbf{u}_k x_{dl,k}$$

- received signal at user k

$$y_{dl,k} = \mathbf{h}_k^H \mathbf{u}_k x_{dl,k} + \sum_{j \neq k} \mathbf{h}_k^H \mathbf{u}_j x_{dl,j} + u_{dl,k}$$

- SINR of user k

$$\mathrm{SINR}_k^{dl} = \frac{\mathcal{E}_{dl,k} |\mathbf{u}_k^H \mathbf{h}_k|^2}{\sigma_n^2 + \sum_{j \neq k} \mathcal{E}_{dl,j} |\mathbf{h}_j^H \mathbf{h}_k|^2}, \quad 1 \leq k \leq K$$

- where: $\mathcal{E}_{dl,k} = \mathcal{E}\{|x_{dl,k}|^2\}$; $\quad \sigma_n^2 = \mathcal{E}\{|n_{dl,k}|^2\}$
- Using:

$$a_k = \frac{\mathrm{SINR}_k^{dl}}{(1 + \mathrm{SINR}_k^{dl}) |\mathbf{h}_k^H \mathbf{u} K|^2}$$

we can rewrite the the SINR expressions as:

$$\boxed{\big(\mathbf{I}_K - \mathrm{diag}\{a_1, \ldots, a_k\}\mathbf{A}\big)\mathbf{p}_{dl} = \sigma_n^2 \mathbf{a}}$$

where:

$$\mathbf{a} = \big[a_1, \ldots, a_k\big]^T$$
$$\mathbf{A} = \begin{bmatrix} |u_1^H h_1|^2 & |u_2^H h_1|^2 & \ldots & |u_K^H h_1|^2 \\ \vdots & & & \\ |u_1^H h_K|^2 & & \ldots & |u_K^H h_K|^2 \end{bmatrix}$$
$$\mathbf{p}_{dl} = \big[\mathcal{E}_{dl,1}, \ldots, \mathcal{E}_{dl,K}\big]^T$$

→ We can easily calculate transmit powers $\mathcal{E}_{dl,k}$ required to achieve desired $\mathrm{SINR}_k^{dl}, 1 \leq k_1 \leq K$, for given signature (precoding) vectores $u_k, 1 \leq k \leq K$

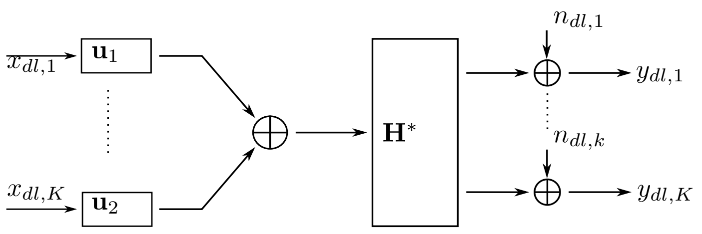


Abbildung 19: Block Diagramm of MIMO in Downlink

Uplink: Use downlink signatures, $\mathbf{h}_k$, as receive filters, $\mathbf{f}_k$

- block diagramm:

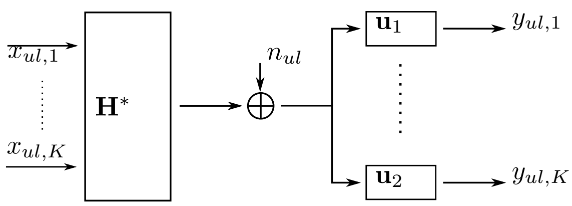


Abbildung 20: Block Diagramm of MIMO in Uplink

- Signal model:

$$y_{ul,k} = \mathbf{u}_k^H \big(\mathbf{H}\mathbf{x}_{ul} + \mathbf{u}_{ul}\big)$$

with

$$\mathbf{H} = \begin{bmatrix}\mathbf{h}_1 & \ldots & \mathbf{h}_K\end{bmatrix}, \quad \mathbf{x}_{ul} = \begin{bmatrix}x_{ul,1} & \ldots & x_{ul,K}\end{bmatrix}^T, \quad \mathbf{u}_{ul} = \begin{bmatrix}u_{ul,1} & \ldots & u_{ul,K}\end{bmatrix}^T$$

$$\rightarrow y_{ul,k} = \mathbf{u}_k^H \mathbf{h}_k x_{ul,k} + \sum_{j \neq k} \mathbf{u}_k \mathbf{h}_j x_{ul,j} + \mathbf{n}_k^H \mathbf{u}_{ul} \tag{9}$$

$$\rightarrow \mathrm{SINR}_k^{ul} = \frac{\mathcal{E}_{ul,k} |\mathbf{u}^H \mathbf{h}_k|^2}{\sigma_n^2 + \sum\limits_{j \neq k} \mathcal{E}_{ul,j} |\mathbf{u}_k^H \mathbf{h}_j|^2} \tag{10}$$

where we used $\mathbf{u}_k^H \mathbf{u}_k = 1$ and $\mathcal{E}_{ul,K} = \mathcal{E}\{|x_{ul,K}|^2\}$

- we define: $b_k = \frac{\mathrm{SINR}_k^{ul}}{(1+\mathrm{SINR}_k^{ul})|\mathbf{u}_k^H \mathbf{h}_k|^2}$
- we can rewrite SINR epression (10) as :

$$\sigma_n^2 + \sum_{j \neq k} \mathcal{E}_{ul,j} \left|\mathbf{u}_k^H \mathbf{h}_j\right|^2 = \frac{1}{\mathrm{SINR}_k^{ul}} \mathcal{E}_{ul,k} \left|\mathbf{u}_k^H \mathbf{h}_k\right|^2$$

$$\underbrace{\big(1 + \frac{1}{\mathrm{SINR}_k^{ul}}\big) \left|\mathbf{u}_k^H \mathbf{h}_k\right|^2}_{\frac{1}{b_k}} \mathcal{E}_{ul,k} - \sum_{j=1}^{K} \mathcal{E}_{ul,k} \left|\mathbf{u}_k^H \mathbf{h}_k\right|^2 = \sigma_n^2$$

$$\rightarrow \mathcal{E}_{ul,k} - b_k \sum_{j=1}^{K} \mathcal{E}_{ul,j} \left|\mathbf{u}_k^H \mathbf{h}_j\right|^2 = b_k \sigma_n^2$$

- matrix notation:

$$\big[\mathbf{I}_K - \mathrm{diag}\{b_1, \ldots, b_K\} \mathbf{A}^T \mathbf{p}_{ul} = \sigma_n^2 \cdot \mathbf{b}\big]$$

- where: $\mathbf{A}$ was defined for downlink case

$$\mathbf{p}_{ul} = \begin{bmatrix} \mathcal{E}_{ul,1} & \ldots & \mathcal{E}_{ul,K} \end{bmatrix}^T$$
$$\mathbf{b} = \begin{bmatrix} b_1 & \ldots & b_K \end{bmatrix}^T$$

- we can calculate power allocation vector $\mathbf{p}_{ul}$ for given SINR_1^{ul} and $\mathbf{u}_k$, $\quad 1 \leq k \leq K$

Comparison: Assume, we want to achieve same SINR in uplink and downlink

$\rightarrow$ $\mathrm{SINR}_k^{ul} = \mathrm{SINR}_k^{dl} \quad \forall \quad k$ or equivalently $a_k = b_k, \quad \forall \quad k$!

What sum power do we need in both uses?

$$\mathbf{p}_{dl} = \sigma_n^2 \big(\mathbf{I} - \mathrm{diag}\{a_1, \ldots, a_K\} \mathbf{A}\big)^{-1} \mathbf{a} =$$
$$= \sigma_n^2 \big(\mathbf{D}_a - \mathbf{A}\big)^{-1} \cdot \mathbf{1}$$

where: $\mathbf{D}_a = \mathrm{diag}\{\frac{1}{a_1}, \ldots, \frac{1}{a_K}\}$ and $\mathbf{1} = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \end{bmatrix}^T$

$$\mathbf{p}_{ul} = \sigma_n^2 (\mathbf{D}_b - \mathbf{A}^T)^{-1} \cdot \mathbf{1}$$

where: $\mathbf{D}_b = \text{diag}\{\frac{1}{b_1}, \dots, \frac{1}{b_K}\}$

$$\begin{aligned}\sum_{k=1}^{K} \mathcal{E}_{dl,k} &= \mathbf{1}^T \mathbf{p}_{dl} = \sigma_n^2 \mathbf{1}^T \big(\mathbf{D}_a - \mathbf{A}\big)^{-1} \cdot \mathbf{1} \\ &= \sigma_n^2 \mathbf{1}^T \big(\mathbf{D}_b - \mathbf{A}\big)^{-1} \mathbf{1} \\ &= \sigma_n^2 \big[\mathbf{1}^T \big(\mathbf{D}_b - \mathbf{A}\big)^{-1} \mathbf{1}\big]^T \\ &= \sigma_n^2 \mathbf{1}^T \big[\big(\mathbf{D}_b - \mathbf{A}\big)^{-1}\big]^T \mathbf{1} \\ &= \sigma_n^2 \mathbf{1}^T \big(\underbrace{\mathbf{D}_b^T}_{\mathbf{D}_b} - \mathbf{A}^T\big)^{-1}\end{aligned}$$

$$cdot\mathbf{1} = \sum_{k=1}^{K} \mathcal{E}_{ul,k}$$

Conclusions:

- We can achieve any desired $\text{SINR}_k^{dl}, \forall\, k$, in the downlink by using filters optimized for uplink transmission as signature (precoding) vectors and the same sum power as in the uplink
- suitable filters may be MMSE or ZF filters $\mathbf{u}_k = \mathbf{f}_k, \quad \forall\, k$
- Note that, in general, $\mathcal{E}_{ul,k} \neq \mathcal{E}_{dl,k}$, only the sum powers are equal!

→ For linear precoding, we can solve the more challenging problem via solving an equivalent uplink problem!

Extension: This concept can be extended to nonlinear receivers as well. In this case the DFE receiver in the uplink is dual to a nonlinear Tamlinson-Harashima (TH) precoder in the downlink.

3.2.3 Rate Region (only SISO)

- two user case: $y_k = h_k x + n_k\,, \quad k \in \{1, 2\}$
- Capacity can be achieved with so-called *Costa Cor (dirty paper)* precoding
- In this scheme, the signal of ine user is processed such that it does not impair the receiver of the other user, while the signal of the other user is treated as Gaussian noise at the receiver of the first user.

→ Thus, the achievable rates would be

$$\begin{aligned} R_1 &< \log_2\big(1 + \frac{|h_1|^2 \mathcal{E}_1}{\sigma_n^2}\big) \\ R_2 &< \log_2\big(1 + \frac{|h_2|^2 \mathcal{E}_2}{\sigma_n^2 + |h_1|^2 \mathcal{E}_1}\big)\end{aligned}$$

- $\mathcal{E}_1$ and $\mathcal{E}_2$ can be optimized as they both occure at the same transmitter

4 Distributed MIMO

- This research topic emerged ca. 10 years ago and is still a very active area of research
- Simple relaying schemes have been included in recent standards such as IEEE 802.16 (WiMAX) and LTE - Advanced
- Advantages: relay - assisted communications:
  - Relays can help to reduce the effective overall pathloss
  - Relays can also combat small - scale fading effects
  - Relays can help to realize MIMO gains with single - antenna nodes
- Challenges in relays - assisted communication:
  - Network architectures are becoming more complex
  - Synchronization across different nodes may be necessary *(Anm.: untersch. Trägerfrequenzen der Relays → Offset, Fehler, etc.)*
  - Exchange of channel state information (CSI) across nodes may be required

4.1 Half - Duplex One - Way Relaying

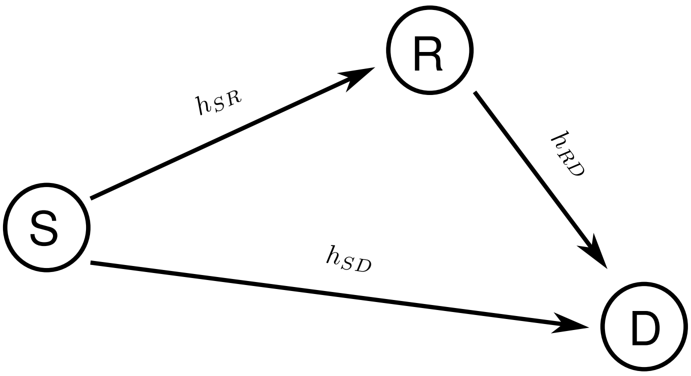


Abbildung 21: Basic Relay Network

Basic Relay Network

- Relay R assists source S in communication with destination D
- Two basic nodes of transmission (at the relay):

Full - Duplex relaying: R can receive and transmit at the same time and in the same frequency band *(Anm.: effizient, da restliche Zeit und restliche Frequenzband von anderen genutzt werden kann)*

→ Since the TX signal power is orders of magnitude larger than the RX power, there is self - interference (at the relay)

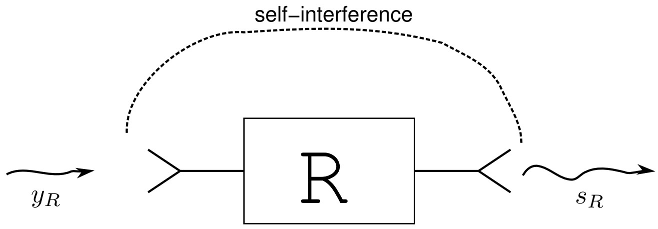


Abbildung 22: Relay with self-interference

→ Full - duplex relays are difficult to implement. The design of full - duplex relays is an active area of research.

→ Majority of existing literature assumes half - duplex relaying.

Half - duplex relaying: R transmits and receives in different time slots and/ or different frequency bands. Typically, a two - phase protocoll is used:

Phase 1: S transmits, R and D receive

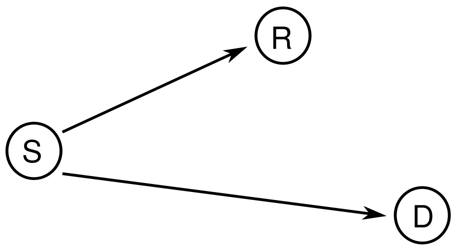


Abbildung 23: Half-duplex Relaying: Phase 1

Phase 2: R transmits, D receives, S may or may not transmit

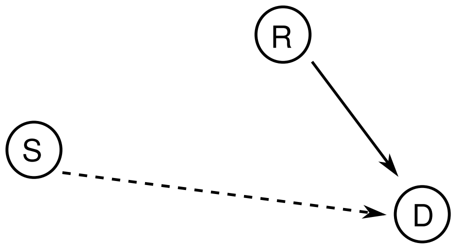


Abbildung 24: Half-duplex Relaying: Phase 2

There are different relaying strategies that differ in the processing applied at the relay. The most popular are:

- Decode - and - Forward
- Amplify - and - Forward

- (Compress - and - Forward)

4.1.1 Decode - and - Forward (DF) Relaying

In DF relaying, the relay detects and decodes the signal received from the source before encoding it and forwarding it to the destination.

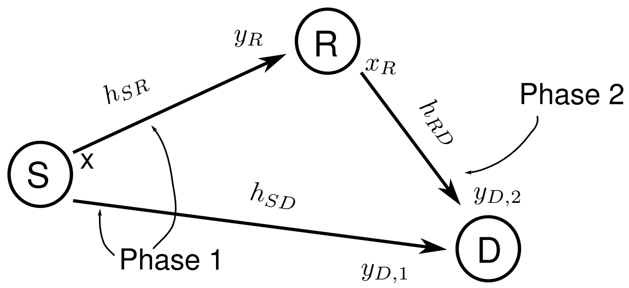


Abbildung 25: Block diagramm Decode - and - forward Relaying

Phase 1:

- R receives: $y_R = h_{SR}x + n_R$
- D receives: $y_{D1} = h_{SD}x + n_{D1}$
- with:
  - transmit signal $x, \mathcal{E}_s = \mathcal{E}\{|x|^2\}$
  - AWGN n_R and n_{D1}, $\sigma_n^2 = \mathcal{E}\{|n_R|^2\} = \mathcal{E}\{|n_{D1}|^2\}$

Phase 2:

- R decodes and forwards x_R (estimate of x)
- D receives: $y_{D2} = h_{RD}x_R + n_{D2}$
  - x_R is estimate of x after decoding at R
  - $\sigma_n^2 = \mathcal{E}\{|n_{D2}|^2\}\,; \quad \mathcal{E}_R = \mathcal{E}\{|x_R|^2\}$
  - weassume: S is silent in Phase 2

- The capacity at the three node relay channel is not known!
- We provide an achievable rate under the following simplifying assumption: The direct source - relay link is not used/ exploited.

- Achievable rate without S - D link:

$$\boxed{C_{DF} = \frac{1}{2}\min\left\{\log_2\left(1 + \frac{\mathcal{E}_S|h_{SR}|^2}{\sigma_n^2}\right),\ \log_2\left(1 + \frac{\mathcal{E}_R|h_{RD}|^2}{\sigma_n^2}\right)\right\}}$$

  - factor $\frac{1}{2}$ is due to the fact that we use two time slots to transmit one packet
  - $\min\{\ldots\}$ means we are limited by the weaker link (bottle - neck)
  - If power allocation is possible, the total power $\mathcal{E} = \mathcal{E}_S + \mathcal{E}_R$ should be divided between S and R to guarantee:

$$\begin{aligned}
\frac{\mathcal{E}_S|h_{SR}|^2}{\sigma_n^2} &= \frac{\mathcal{E}_R|h_{RD}|^2}{\sigma_n^2}, \\
\mathcal{E}_R &= \frac{|h_{SR}|^2}{|h_{SR}|^2 + |h_{RD}|^2} \cdot \mathcal{E}, \\
\mathcal{E}_S &= \frac{|h_{RD}|^2}{|h_{SR}|^2 + |h_{RD}|^2} \cdot \mathcal{E}
\end{aligned}$$

- Outage - probability in fading:
  - We transmit with fixed rate R
  - An outage occurs, if:

$$\begin{aligned}
&\frac{1}{2}\log_2\Big(1 + \underbrace{\frac{\mathcal{E}_S|h_{SR}|^2}{\sigma_n^2}}_{=\gamma_{SR}}\Big) < R \quad \text{or} \\
&\frac{1}{2}\log_2\Big(1 + \underbrace{\frac{\mathcal{E}_R|h_{RD}|^2}{\sigma_n^2}}_{=\gamma_{RD}}\Big) < R
\end{aligned}$$

$$\boxed{\gamma_{SR} < \underbrace{2^{2R} - 1}_{\gamma_T} \quad \text{or} \quad \gamma_{RD} < 2^{2R} - 1}$$

$$\begin{aligned}
P_{\text{out}} &= \Pr\big\{\gamma_{SR} < \gamma_T \vee \gamma_{RD} < \gamma_T\big\} = \Pr\big\{\underbrace{\min\{\gamma_{SR}, \gamma_{RD}\}}_{=\gamma_{eq}} < \gamma_T\big\} \\
&= 1 - \Pr\big\{\gamma_{SR} > \gamma_T \wedge \gamma_{RD} > \gamma_T\big\} = 1 - \Pr\big\{\gamma_{SR} > \gamma_T\big\}\Pr\big\{\gamma_{RD} > \gamma_T\big\} = \\
&= 1 - \big(1 - F_{\gamma_{SR}}(\gamma_T)\big)\big(1 - F_{\gamma_{RD}}(\gamma_T)\big) = \\
&= \underline{F_{\gamma_{SR}}(\gamma_T) + F_{\gamma_{RD}}(\gamma_T) - F_{\gamma_{SR}}(\gamma_T) \cdot F_{\gamma_{RD}}(\gamma_T)}
\end{aligned}$$

with CDFs: $F_{\gamma_{SR}}(\cdot)$ and $F_{\gamma_{RD}}(\cdot)$

- Rayleigh Fading:

$$\rightarrow F_{\gamma_{SR}}(\gamma) = 1 - \exp\left(\frac{-\gamma}{\bar{\gamma}_{SR}}\right); \quad \bar{\gamma}_{SR} = \mathcal{E}\{\gamma_{SR}\}$$

$$F_{\gamma_{RD}}(\gamma) = 1 - \exp\left(\frac{-\gamma}{\bar{\gamma}_{RD}}\right); \quad \bar{\gamma}_{RD} = \mathcal{E}\{\gamma_{RD}\}$$

$$\rightarrow P_{\text{out}} = 1 - \exp\left(\frac{-\gamma_T}{\bar{\gamma}_{SR}}\right) + 1 - \exp\left(\frac{-\gamma_T}{\bar{\gamma}_{RD}}\right) - \Big(1 - \exp\left(\frac{-\gamma_T}{\bar{\gamma}_{SR}}\right)\Big)\Big(1 - \exp\left(\frac{-\gamma_T}{\bar{\gamma}_{RD}}\right)\Big) =$$

$$= 1 - \exp\Big(-\frac{\bar{\gamma}_{SR} + \bar{\gamma}_{RD}}{\bar{\gamma}_{SR}\bar{\gamma}_{RD}} \cdot \gamma_T\Big)$$

$\rightarrow$ equivalent SNR $\gamma_{eq} = \min\big\{\gamma_{SR}, \gamma_{RD}\big\}$ is also exponentially distributed with $\bar{\gamma}_{eq} = \frac{\bar{\gamma}_{SR}\bar{\gamma}_{RD}}{\bar{\gamma}_{SR}+\bar{\gamma}_{RD}}$

- Diversity gain: Assume $\bar{\gamma}_{SR} = \alpha\bar{\gamma}$

$$\rightarrow P_{out} \xrightarrow{\bar{\gamma}\rightarrow\inf} 1 - \Big(1 - \frac{\alpha+\beta}{\alpha\beta} \cdot \frac{\gamma_T}{\bar{\gamma}}\Big) + \mathcal{O}\big(\bar{\gamma}^{-1}\big) =$$

$$= \frac{\alpha+\beta}{\alpha\beta} \cdot \frac{\gamma_T}{\bar{\gamma}} + \mathcal{O}\big(\bar{\gamma}^{-1}\big) =$$

$$\rightarrow \boxed{G_d = 1}$$

- Bit error rate (BER) of BPSK (uncoded)
  - $\text{BER}\big(\gamma_{SR}, \gamma_{RD}\big) = \Big(1 - \text{BER}_{SR}\big(\gamma_{SR}\big)\Big)\text{BER}_{RD}\big(\gamma_{RD}\big) + \Big(1 - \text{BER}_{RD}\big(\gamma_{RD}\big)\Big)\text{BER}_{SR}\big(\gamma_{SR}\big)$
    * with BER of the S - R link, $\text{BER}_{SR}(\gamma_{SR})$ and BER of the R - D link $\text{BER}_{RD}(\gamma_{RD})$
    * for sufficiently high SNR $\rightsquigarrow \text{BER}_{SR}(\gamma_{SR}), \text{BER}_{RD}(\gamma_{RD}) \ll \text{BER}_{SR}(\gamma_{SR}) + \text{BER}_{RD}(\gamma_{RD})$

  $\rightarrow$ $\underline{\text{BER}(\gamma_{SR}, \gamma_{RD}) \approx \text{BER}_{SR}(\gamma_{SR}) + \text{BER}_{RD}(\gamma_{RD})}$

    * to average BER (Rayleigh Fading):

$$\text{BER} = \mathcal{E}_{\gamma_{SR},\gamma_{RD}}\Big\{\text{BER}\big(\gamma_{SR}, \gamma_{RD}\big)\Big\} = \frac{1}{2}\Big(1 - \sqrt{\frac{1}{1+\frac{1}{\bar{\gamma}_{SR}}}} + \frac{1}{2}\Big(1 - \sqrt{\frac{1}{1+\frac{1}{\bar{\gamma}_{RD}}}}\Big)$$

    * high SNR:

$$\text{BER} \approx \frac{1}{2}\big(1 - 1 + \frac{1}{2}\frac{1}{\bar{\gamma}_{SR}}\big) + \frac{1}{2}\big(1 - 1 + \frac{1}{2}\frac{1}{\bar{\gamma}_{RD}}\big) =$$

$$= \frac{1}{4}\big(\frac{1}{\bar{\gamma}_{SR}} + \frac{1}{\bar{\gamma}_{RD}}\big)$$

  $\rightsquigarrow$ also indicates diversity gain $G_d = 1$

4.1.2 Amplify - and - Forward (AF) Relaying

- Relay does not decode signal received from source but only amplifies it before forwarding it to the destination

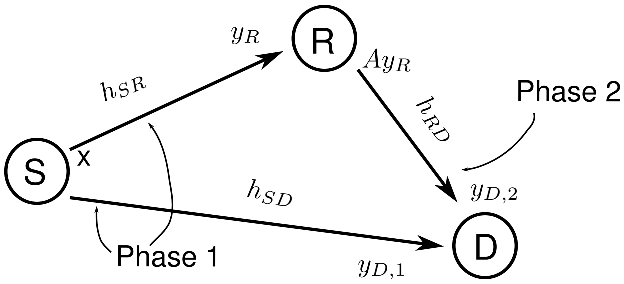


Abbildung 26: Block diagramm AF - Relaying

- Amplification gain A may be constant on channel dependent and ensures a certain (average) transmit power

  Phase 1:

  - R receives: $y_D = h_{SR}x + n_R$
  - D receives: $y_{D,1} = h_{SD}x + n_{D,1}$

  Phase 2:

  - R transmits: $s_R = Ay_R = A(h_{SR}x + n_R)$
  - D receives: $y_{D,2} = h_{RD}Ay_R + n_{D,2}$

- We can use MRC to combine $y_{D,1}$ and $y_{D,2}$ at D: $y_{D,2} = Ah_{RD}h_{SR}x + h_{RD}An_R + n_{D,2}$, where: $h_{RD}An_R + n_{D,2}$ is effective noise n_{eff} with variance $\sigma^2_{n_{\text{eff}}} = \sigma^2_n\big(|h_{RD}|^2A^2 + 1\big)$

→ make noise variances of both branches equal

$$\bar{y}_{D,2} = \frac{1}{\sqrt{|h_{RD}|^2A^2 + 1}} \cdot y_{D,2} = \frac{Ah_{RD}h_{SR}}{\sqrt{A^2|h_{RD}|^2 + 1}} \cdot x + \tilde{n}_{\text{eff}}$$

$$\text{MRC:} \quad r = h^*_{SD}y_{D,1} + \frac{Ah^*_{RD}h^*_{SR}}{\sqrt{A^2|h_{RD}|^2 + 1}} \cdot \bar{y}_{D,2} = \underbrace{h^*_{SD}y_{D,1} + \frac{Ah^*_{RD}h^*_{SR}}{A^2|h_{RD}|^2 + 1} \cdot y_{D,2}}_{=\text{decision variable!}}$$

- Choice of A: The goal is to ensure en (average) transmit power of $\mathcal{E}_R$

a) Variable gain relaying: In this case we introduce an instantaneous power constraint. *Anm.: A muss abhängig von h_{SR} sein, um es kompensieren zu können.*

$$\begin{aligned}\mathcal{E}_{x,n}\big\{|S_R|^2\big\} &= \mathcal{E}_{x,n}\big\{A^2\big(|h_{SR}|^2|x|^2 + |n_R|^2\big)\big\} = \\ &= A^2\big(|h_{SR}|^2\mathcal{E}_S + \sigma^2_n\big) \overset{!}{=} \mathcal{E}_R \\ \rightarrow A^2 &= \frac{\mathcal{E}_R}{|h_{SR}|^2\mathcal{E}_S + \sigma^2_n}\end{aligned}$$

- A is channel dependent
- Instantaneous transmit power is <u>not</u> channel dependent

b) Fixed gain relaying: In this case, we introduce an average (with respect to the channel) power constraint

$$\begin{aligned}\mathcal{E}\{|S_R|^2\} &= \mathcal{E}\{A^2(|h_{SR}|^2|x|^2 + |n_R|^2)\} = \\ &= A^2(\underbrace{\mathcal{E}\{|h_{SR}|^2\}}_{\sigma_{SR}^2}\mathcal{E}_S + \sigma_n^2) \overset{!}{=} \mathcal{E}_R\end{aligned}$$

$$\rightarrow A^2 = \frac{\mathcal{E}_R}{\mathcal{E}_S\sigma_{SR}^2 + \sigma_n^2}$$

- A is not channel dependent
- Instantaneous power of S_R depends on channel and may actually vary widely

Equivalent SNR for variable gain AF relaying (inly relayed link)

$$y_{D,2} = Ah_{RD}h_{SR}x + h_{RD}An_R + n_{D,2}$$

$$\begin{aligned}\text{SNR:}\quad \gamma_{eq}^{AF} &= \frac{A^2|h_{SR}|^2|h_{RD}|^2\mathcal{E}_S}{A^2|h_{RD}|^2\sigma_n^2 + \sigma_n^2} = \frac{\frac{\mathcal{E}_S}{\sigma_n^2}|h_{SR}|^2|h_{RD}|^2}{|h_{RD}|^2 + \frac{1}{\mathcal{E}_R}(|h_{SR}|^2\mathcal{E}_S + \sigma_n^2)} = \\ &= \frac{\frac{\mathcal{E}_S}{\sigma_n^2}|h_{SR}|^2 \cdot \frac{\mathcal{E}_R}{\sigma_n^2}|h_{RD}|^2}{\frac{\mathcal{E}_R}{\sigma_n^2}|h_{RD}|^2 + \frac{\mathcal{E}_S}{\sigma_n^2}|h_{SR}|^2 + 1} = \\ &= \frac{\gamma_{SR}\gamma_{RD}}{\gamma_{SR} + \gamma_{RD} + 1}\end{aligned}$$

high SNR: γ_{SR}, $\gamma_{RD} \gg 1$

$$\boxed{\gamma_{eq}^{AF} = \frac{\gamma_{SR}\gamma_{RD}}{\gamma_{SR} + \gamma_{RD}}} \tag{11}$$

Anm.: Vgl. Formel 11 mit Berechnung zweier paralleler Widerstände. Comparison with equivalent SNF of DF:

$$\boxed{\gamma_{eq}^{DF} = \min\{\gamma_{SR}, \gamma_{RD}\}} \tag{12}$$

<u>3 cases:</u>

$$\text{a)}\quad \gamma_{SR} = \gamma_{RD} = \gamma \quad \rightarrow \quad \gamma_{eq}^{AF} = \frac{1}{2}\gamma = \frac{1}{2}\gamma_{eq}^{DF} \tag{13}$$

$$\text{b)}\quad \gamma_{SR} \gg \gamma_{RD} \quad \rightarrow \quad \gamma_{eq}^{AF} = \gamma_{RD} = \gamma_{eq}^{DF} \tag{14}$$

$$\text{c)}\quad \gamma_{SR} \ll \gamma_{RD} \quad \rightarrow \quad \gamma_{eq}^{AF} = \gamma_{SR} = \gamma_{eq}^{DF} \tag{15}$$

Anm.: Fälle 14 und 15 sind am wahrscheinlichsten. Decision errors mostly occur if one of the two link SNRs is much smaller than the other. The probability, that both SNRs are small at the same time is much smaller, than the probability, that just one link SNR is small.

→ $\gamma_{eq}^{AF} = \gamma_{eq}^{DF}$ holds most of the time

→ AF relaying with variable gain has the same performance as DF relaying in high SNR, vgl Plot vom 07.02.13

4.1.3 Buffer - aided DF Relaying

- For conventional relaying, the performance is always limited by the weaker (bottleneck) link since the relay has to immediately retransmit

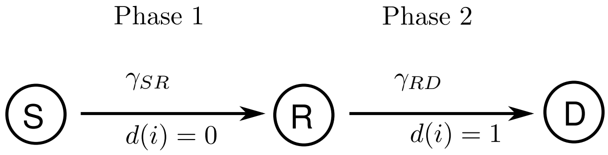


Abbildung 27: Buffer - aided DF Relaying

- In practice, the nodes in the network have buffers. Thus, we can use the stronger link and wait until the channel conditions of the weaker link have sufficiently improved.
- To avoid buffer over- or underflow at the relay, we demand that the average rate of the source relay channel (S - R) is equal to the average rate of the R - D channel
- We introduce a binary selection variable $d(i)$ for time slot $i = \{1, 2, \ldots\}$, where:

$$\begin{aligned} d(i) = 0 \quad &\Rightarrow \quad \text{S transmits, R receives} \\ d(i) = 1 \quad &\Rightarrow \quad \text{R transmits, D receives} \end{aligned}$$

- Note: For conventional relaying we have $d(1) = 0$, $d(2) = 1$, $d(3) = 0$, $d(4) = 1, \ldots$
- The average rate in the S - R link is:

$$R_{SR} = \frac{1}{N} \sum_{i=1}^{N} \big(1 - d(i)\big) \log_2\big(1 + \gamma_{SR}(i)\big)$$

  and that of the R - D link is:

$$R_{RD} = \frac{1}{N} \sum_{i=1}^{N} d(i) \log_2\big(1 + \gamma_{RD}(i)\big)$$

  where N denotes the total number of time slots.
- $\gamma_{SR}(i)$ and $\gamma_{RD}(i)$ change from one time slot to the next following e.g. a Rayleigh distribution
- At the relay, we have the constraint $R_{SR} = R_{RD}$ to avoid buffer over - / underflow
- To maximize the achievable throughput, we formulate an optimization problem:

$$\begin{aligned} &\max_{d(i)\forall i} R_{RD} \\ \text{subject to:}\ &\text{C1:} \quad R_{RD} = R_{SR} \\ &\text{C2:} \quad d(i) \in \{0, 1\} \end{aligned}$$

- For finite N, this problem is very difficult to solve.
- For infinte N, a simple solution exists.

→ Solution can be found by Langrange method

- Solution (for $N \to \inf$): The optimal $d(i)$ is given by:

$$d(i) = \begin{cases} 1 & \text{if} \quad \log_2\big(1+\gamma_{RD}(i)\big) \geq \rho \log_2\big(1+\gamma_{SR}(i)\big) \\ 0 & \text{otherwise} \end{cases}$$

  where ρ is a constant, that only depends on the statistics of $\gamma_{SR}(i)$ and $\gamma_{RD}(i)$ and can be obtained from (numerical search needed):

$$\mathcal{E}_{\gamma_{SR}(i)}\big\{\big(1-d(i)\big)\log_2\big(1+\gamma_{SR}(i)\big)\big\} = \mathcal{E}_{\gamma_{RD}(i)}\big\{d(i)\cdot\log_2\big(1+\gamma_{RD}(i)\big)\big\}$$

- Since always: the „best“ of two links is selected, this scheme can achieve a diversity gain of $G_d = 2$ (Rayleigh Fading)