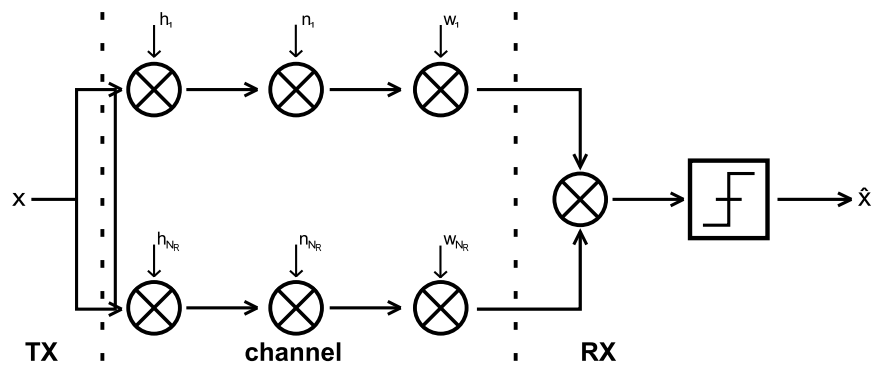


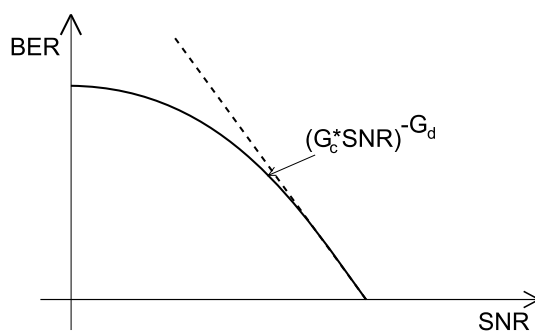
1 SIMO Systems

Remarks

- In SIMO Systems only coding and diversity gains can be exploited (no multiplexing gains)
- To realize these gains diversity combining has to be performed
- Diversity combining schemes vary in complexity and performance
- There are many diversity combining schemes. Here we consider:
 - Maximal ratio combining (MRC)
 - Equal gain combining (EGC)
 - Selection combining (SC)
- Diversity combining problem



- how to choose combining weights w_n ?
- what performance (e.g. error rate, outage probability) is achieved?
- what diversity and coding/combining gain is achieved?



- G_c : Coding gain
- G_d : Diversity gain

1.1 Preliminaries

Consider an equivalent system:

$$y = hx + n; \quad \mathcal{E}\{|x^2|\} = \mathcal{E}_s; \quad \mathcal{E}\{|n^2|\} = \sigma_n^2; \quad \mathcal{E}\{|h|^2\} = 1$$

- Instantaneous SNR: $\gamma_t = \frac{\mathcal{E}_s}{\sigma_n^2} \times |h|^2$
- Average SNR: $\bar{\gamma}_t = \mathcal{E}\{\gamma_t\} = \frac{\mathcal{E}_s}{\sigma_n^2}$

Bit and Symbol Error Rate

- The Bit and Symbol Error Rate of many modulation schemes can be expressed for given γ_t as:

$$P_e(\gamma_t) = aQ\{\sqrt{b\gamma_t}\}$$

where:

- $Q(x) = \frac{1}{\sqrt{2\pi}} \times \int_x^\infty e^{-\frac{t^2}{2}} dt$
- $P_e(\gamma_t)$ may be exact result or approximation
- BPSK: exact with $a = 1, b = 2$
- M-ary QAM: tight approximation with $a = 4(1 - \frac{1}{\sqrt{M}}), b = \frac{3}{M-1}$

(*Einschub : Gray - Code : BER = $\frac{1}{\log_2 M} \times SER$*)

- Alternative representation of Q - function:

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}} d\theta$$

→ Integral limits are fixed and do not depend on integration variables!

- Average error probability

$$P_e = \mathcal{E}\{P_e(\gamma_t)\} = \int_0^\infty aQ(\sqrt{bx})p_{\gamma_t}(x) dx$$

- Integral may be difficult to solve analytically
- Integral has infinite support → numerical evaluation difficult
- Using alternative representation of Q-function we get:

$$\begin{aligned} P_e &= \int_0^\infty \frac{a}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{bx}{2\sin^2\theta}} p_{\gamma_t}(x) d\theta dx \\ &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} \int_0^\infty p_{\gamma_t}(x) e^{-\frac{bx}{2\sin^2\theta}} dx d\theta = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\left(\frac{b}{2\sin^2\theta}\right) d\theta \end{aligned}$$

where:

- $M_{\gamma_t}(s) = \int_0^\infty p_{\gamma_t}(x) e^{-sx} dx$ is the Laplace transform of p_{γ_t}
- $M_{\gamma_t}(-s)$ is the so called Moment Generation Function (MGF) of p_{γ_t}
- Here, we will also refer to $M_{\gamma_t}(s)$ as MGF
- $M_{\gamma_t}(s)$ is sometimes easier to obtain than p_{γ_t}
- The above integral can be easily evaluated numerically because of the finite integral limits

Outage probability

- The outage probability is the probability that the channel cannot support a certain rate, R , i.e. (where γ_T is the threshold SNR):

$$C = \log_2(1 + \gamma_t) < R \quad \leftrightarrow \quad \gamma_t < 2^R - 1 \triangleq \gamma_T$$

Thus, the outage probability is given by:

$$P_{out} = P_{\gamma_t < \gamma_T} = \int_0^{\gamma_T} p_{\gamma_t}(x) dx$$

- Using the inverse Laplace Transform

$$p_{\gamma_t}(x) = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) e^{sx} dx$$

where $c > 0$ is a small constant that lies in the region of convergence of the integral, we obtain:



- 1.

$$P_{out} = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) \int_0^{\gamma_T} e^{sx} dx ds = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) e^{\gamma_T s} \frac{ds}{s}$$

(lower integral limit is 0 since $p_{\gamma_t}(0) = 0$)

- and 2.:

$$p_{\gamma_t}(x) = \int_0^x p_{\gamma_t}(t) dt = 0$$

$$\text{for } x = 0 \text{ note: } p_{\gamma_t}(x) \xleftrightarrow[\text{transform}]{\text{Laplace}} \frac{1}{s} M_{\gamma_t}(s)$$

General combining scheme

$$y = \left(\sum_{n=1}^{N_R} h_n w_n \right) x + \sum_{n=1}^{N_R} w_n n_n$$

$$\gamma_t = \frac{\epsilon_s \left| \sum_{n=1}^{N_R} h_n w_n \right|^2}{\sigma_n^2 \sum_{n=1}^{N_R} |w_n|^2}$$

where w_n depends on the particular combining scheme.

1.2 MRC (Maximum Ratio Combining)

- what weight w_n maximize γ_t ?
 - Cauchy-Schwarz inequality

$$\left| \sum_{n=1}^{N_R} h_n w_n \right|^2 \leq \sum_{n=1}^{N_R} |h_n|^2 \cdot \sum_{n=1}^{N_R} |w_n|^2$$

where equality holds if and only if $w_n = c \cdot h_n^*$ for some non-zero constant c .

- for $w_n = h_n^*$, we obtain

$$\gamma_t = \frac{\epsilon_s}{\sigma_n^2} \cdot \frac{\left(\sum_{n=1}^{N_R} |h_n|^2 \right)^2}{\sum_{n=1}^{N_R} |h_n|^2} = \frac{\epsilon_s}{\sigma_n^2} \sum_{n=1}^{N_R} |h_n|^2$$

- $w_n = h_n^* \forall n$ are the MRC combining weights.
- For performance analysis we assume independent identically distributed (IID) Rayleigh fading

$$\begin{aligned} \rightarrow \mathcal{E}\{|h_n|^2\} &= 1; \quad \bar{\gamma} = \frac{\epsilon_s}{\sigma_n^2}; \quad \gamma_n = \frac{\epsilon_s}{\sigma_n^2} |h_n|^2 \\ p_\gamma(x) &= \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0 \\ M_\gamma(s) &= \frac{1}{1 + s\bar{\gamma}} \end{aligned}$$

- Error rate

$$\gamma_t = \sum_{n=1}^{N_R} \gamma_n$$

\rightarrow sum of IID random variables (r.v.s.)

$$M_{\gamma_t}(s) = \left(M_\gamma(s) \right)^{N_R} = \frac{1}{(1 + s\bar{\gamma})^{N_R}} = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{1}{\left(s + \frac{1}{\bar{\gamma}}\right)^{N_R}}$$

inverse Laplace-transform (from tables)

$$p_{\gamma_t}(x) = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{x^{N_R-1}}{(N_R-1)!} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0$$

- Direct approach

$$p_e = \int_0^\infty a \cdot Q(\sqrt{ax}) p_{\gamma_t}(x) dx = a \left(\frac{1-\mu}{2} \right)^{N_R} \cdot \sum_{n=0}^{N_R-1} \binom{N_R-1+n}{n} \left(\frac{1+\mu}{2} \right)^n$$

$$\text{where } \mu = \sqrt{\frac{b\bar{\gamma}}{2 + b\bar{\gamma}}}$$



- MGF approach

$$\begin{aligned}
 p_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t} \left(\frac{b}{2 \sin^2 \theta} \right) d\theta \\
 &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\bar{\gamma}^{N_R} \left(\frac{b}{\sin^2 \theta} + \frac{1}{\bar{\gamma}} \right)^{N_R}} d\theta \quad (\text{numerisch berechnen!})
 \end{aligned}$$

- high SNR: $\bar{\gamma} \rightarrow \infty \iff \frac{1}{\bar{\gamma}} \rightarrow 0$

$$\begin{aligned}
 p_e &= \frac{a}{\pi} \cdot \frac{1}{\bar{\gamma}^{N_R}} \cdot \left(\frac{2}{b} \right)^{N_R} \int_0^{\frac{\pi}{2}} \sin^{2N_R} \theta d\theta \\
 (\text{from MGF approach: } \int_0^{\frac{\pi}{2}} \sin^{2N_R} \theta d\theta &= \frac{\pi}{2^{N_R+1}} \cdot \binom{2N_R}{N_R}) \\
 &= \frac{a}{2^{N_R+1} \cdot b^{N_R}} \binom{2N_R}{N_R} \frac{1}{\bar{\gamma}^{N_R}} \quad \text{as } \bar{\gamma} \rightarrow \infty \\
 &\stackrel{!}{=} \left(\frac{1}{G_c \bar{\gamma}} \right)
 \end{aligned}$$

where: Diversity gain: $G_d = N_R$

$$\text{Combining/Coding gain: } G_c = 2b \left(\frac{a}{2} \binom{2N_R}{N_R} \right)^{-\frac{1}{N_R}}$$

- MRC exploits the maximal possible diversity
- Diversity gain is not affected by correlation as the branches are not fully correlated
- Diversity gain depends on fading distribution

Outage probability

$$\begin{aligned} P_{out} &= \int_0^{\gamma_T} p_{\gamma_t}(x) dx = \frac{1}{\bar{\gamma}^{N_R}} \int_0^{\gamma_T} \frac{x^{N_R-1}}{(N_R-1)!} e^{-\frac{x}{\bar{\gamma}}} dx \\ &= 1 - e^{-\frac{\gamma_T}{\bar{\gamma}}} \cdot \sum_{n=1}^{N_R} \frac{\left(\frac{\gamma_T}{\bar{\gamma}}\right)^n}{(n-1)!} \end{aligned}$$

- Approximation (Taylor series): $\bar{\gamma} \rightarrow \infty : -e^{-\frac{x}{\bar{\gamma}}} = 1 - \frac{x}{\bar{\gamma}} + O(\frac{1}{\bar{\gamma}})$ where a function $f(x)$ is $O(x)$ if $\lim_{x \rightarrow \infty} \frac{f(x)}{x} = 0$.

$$\Rightarrow P_{out} = \frac{1}{\gamma^{N_R}} \int_0^{\gamma_T} \frac{x^{N_R-1}}{(N_R-1)!} \left(1 - \frac{x}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right)\right) dx$$

- Diversity and coding gain can also be defined for P_{out}

1.3 EGC (Equal Gain Combining)

Combining Weights

- For MRC, both, the amplitudes and phases of the channel gains $h_n = |h_n|e^{j\varphi_n}$ have to be known (or estimated in practice)
- In EGC it is assumed that only the phases are known and weights $w_n = e^{-j\varphi_n}$ are used.

$$\begin{aligned} \Rightarrow \gamma_t &= \frac{\mathcal{E}_s}{\sigma_n^2} \frac{\left| \sum_{n=1}^{N_R} |h_n| e^{j\varphi_n} e^{-j\varphi_n} \right|^2}{\sum_{n=1}^{N_R} |e^{-j\varphi_n}|^2} = \frac{\mathcal{E}_s}{\sigma_n^2} \frac{1}{N_R} \left(\sum_{n=1}^{N_R} |h_n| \right)^2 \\ &= \frac{1}{N_R} \left(\sum_{n=1}^{N_R} \sqrt{\gamma_n} \right)^2 ; \text{ with } \gamma_n = \frac{\mathcal{E}_s}{\sigma_n^2} |h_n|^2 \end{aligned}$$

Performance Analysis

- IID case
 $\Rightarrow \sqrt{\gamma_n}$ is Rayleigh distributed
 \Rightarrow Exact analysis is much more difficult than for MRC \Rightarrow see book by Simon & Alouini p.341
- Approximate result

$$P_e = \frac{a}{2} \left[1 - \sqrt{\frac{2b\bar{\gamma}}{5+2b\bar{\gamma}}} \sum_{n=0}^{N_R-1} \frac{\binom{2n}{n}}{4^n (1 + \frac{2}{5}b\bar{\gamma})^n} \right]$$

- high SNR

⇒ use high SNR analysis of Wang & Giannakis, 2003

⇒ at high SNR, only pdf of γ_n around 0 is relevant for performance

$$\stackrel{\text{Rayleigh}}{\Rightarrow} p_\gamma(x) = \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}} \stackrel{\text{Taylor Serie}}{=} \frac{1}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right) \text{ as } x \rightarrow 0$$

- need pdf γ_t : (γ_n bekannt, → ges.: Wurzel, etc.)

(cumulative distribution function of $\sqrt{\gamma}$ ($\stackrel{\text{i.i.d.}}{=} \sqrt{\gamma_n}$) (cdf))

$$\begin{aligned} P_{\sqrt{\gamma}}(x) &= \Pr\{\sqrt{\gamma} \leq x\} = \Pr\{\gamma \leq x^2\} = P_\gamma(x^2) = \text{cdf of } \gamma \\ \rightarrow p_{\sqrt{\gamma}}(x) &= \frac{d}{dx} P_{\sqrt{\gamma}}(x) = 2x \cdot p_\gamma(x^2) = \frac{2x}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right) \end{aligned}$$

- Laplace Transformation to MGF

$$\rightarrow M_{\sqrt{\gamma}}(s) = \mathcal{L}\{p_{\sqrt{\gamma}}(x)\} = \frac{2}{\bar{\gamma}} \cdot \frac{1}{s^2} + O\left(\frac{1}{\bar{\gamma}}\right)$$

$$\sqrt{\gamma_t} = \sum_{n=1}^{N_R} \frac{\sqrt{\gamma_n}}{N_R}$$

$$\begin{aligned} M_{\sqrt{\gamma_t}}(s) &= \mathcal{E}\left\{\exp(-s\sqrt{\gamma_t})\right\} = \mathcal{E}\left\{\exp\left(-\frac{s}{\sqrt{N_R}} \cdot \sum_{n=1}^{N_R} \sqrt{\gamma_n}\right)\right\} = \left(\mathcal{E}\left\{\exp\left(-\frac{s}{\sqrt{N_R}} \cdot \sqrt{\gamma_n}\right)\right\}\right)^{N_R} \\ &= \left(M_{\sqrt{\gamma}}\left(\frac{s}{\sqrt{N_R}}\right)\right)^{N_R} = \left(\frac{2}{\bar{\gamma}} \cdot \frac{N_R}{s^2}\right)^{N_R} + O\left(\frac{1}{\bar{\gamma}^{N_R}}\right) \end{aligned}$$

- inverse Laplace Transform

$$\begin{aligned} p_{\sqrt{\gamma_t}}(x) &= \mathcal{L}^{-1}\left\{M_{\sqrt{\gamma_t}}(s)\right\} = \left(\frac{2N_R}{\bar{\gamma}}\right)^{N_R} \cdot \frac{x^{2N_R-1}}{(2N_R-1)!} + O\left(\frac{1}{\bar{\gamma}^{N_R}}\right) \\ P_{\gamma_t}(x) &= \Pr\{\gamma_t \leq x\} = \Pr\{\sqrt{\gamma_t} \leq \sqrt{x}\} = P_{\sqrt{\gamma_t}}(\sqrt{x}) \rightarrow \text{cdf of } \sqrt{\gamma_t} \\ p_{\gamma_t}(x) &= \frac{d}{dx} P_{\gamma_t}(x) = \frac{1}{2\sqrt{x}} \cdot p_{\gamma_t}(\sqrt{x}) = \frac{1}{2} \left(\frac{2N_R}{\bar{\gamma}}\right)^{N_R} \cdot \frac{x^{N_R-1}}{(2N_R-1)!} + O(\bar{\gamma}^{-N_R}) \\ \rightarrow M_{\gamma_t}(s) &= \mathcal{L}\{p_{\gamma_t}(x)\} = \frac{1}{2} \left(\frac{2N_R}{\bar{\gamma}}\right)^{N_R} \cdot \frac{(N_R-1)!}{(2N_R-1)!} \frac{1}{s^{N_R}} + O(\bar{\gamma}^{-N_R}) \end{aligned}$$

- Error Probability:

$$\begin{aligned}
P_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t} \left(\frac{b}{2 \sin^2(\theta)} \right) d\theta \\
&= \frac{a}{\pi} \frac{1}{2} \left(\frac{2N_R}{\bar{\gamma}} \right)^{N_R} \frac{(N_R - 1)!}{(2N_R - 1)!} \frac{2^{N_R}}{b^{N_R}} \underbrace{\int_0^{\frac{\pi}{2}} \sin^{2N_R}(\theta) d\theta}_{\frac{\pi}{2^{2N_R+1}} \binom{2N_R}{N_R} = \frac{\pi (2N_R)!}{2^{2N_R+1} (N_R!)^2}} + O \left(\frac{1}{\bar{\gamma}^{N_R}} \right) \\
&= \frac{aN_R^{N_R}}{2b^{N_R} N_R!} \frac{1}{\bar{\gamma}^{N_R}} + O \left(\frac{1}{\bar{\gamma}^{N_R}} \right) \stackrel{!}{=} \left(\frac{1}{G_c} \right)^{G_d} \\
&\implies \text{Diversity gain: } G_d = N_R \\
&\implies \text{Combining gain: } G_c = \frac{b}{N_R} \left(\frac{2N_R!}{a} \right)^{\frac{1}{N_R}}
\end{aligned}$$

vergleiche auch Blatt mit Kurven III und IV

A similar asymptotic analysis can be conducted for the outage probability.

1.4 SC (Selection Combining)

Combining weights

- only the strongest branch is chosen
- strongest branch: $\hat{n} = \underset{n}{\operatorname{argmax}} \gamma_n \longrightarrow \gamma_t = \gamma_{\hat{n}}$
- only on RF receiver chain required \rightarrow saves hardware complexity

Performance analysis

- cdf of: γ_t

$$\begin{aligned}
P_{\gamma_t}(x) &= \Pr\{\gamma_{\hat{n}} \leq x\} = \Pr\{\gamma_1 \leq x \cap \gamma_2 \leq x \cap \dots \cap \gamma_{N_R} \leq x\} \\
&\stackrel{(IID)}{=} \left(\Pr\{\gamma_n \leq x\} \right)^{N_R} = \left(P_{\gamma}(x) \right)^{N_R}
\end{aligned}$$

- pdf:

$$\begin{aligned}
p_{\gamma_t}(x) &= \frac{d}{dx} P_{\gamma_t}(x) = N_R (P_{\gamma}(x))^{N_R-1} \cdot p_{\gamma}(x) \\
\text{where: } p_{\gamma_t}(x) &= \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0 \\
P_{\gamma}(x) &= \int_0^x p_{\gamma}(x) dx = 1 - e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0 \\
\rightarrow p_{\gamma_t}(x) &= \frac{N_R}{\bar{\gamma}} (1 - e^{-\frac{x}{\bar{\gamma}}})^{N_R-1} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0
\end{aligned}$$

Error probability

- direct approach \rightarrow closed-form solution possible
- MGF approach
 - Binomial expansion

$$\begin{aligned} p_{\gamma_t}(x) &= \frac{N_R}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} 1^{N_R-1-n} \left(-e^{-\frac{x}{\bar{\gamma}}}\right)^n \\ &= \frac{N_R}{\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} \cdot (-1)^n e^{-\frac{x(n+1)}{\bar{\gamma}}}; \quad x \geq 0 \end{aligned}$$

- MGF

$$M_{\gamma_t}(s) = \frac{N_R}{\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} (-1)^n \frac{1}{s + \frac{n+1}{\bar{\gamma}}}$$

–

$$\begin{aligned} P_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\left(\frac{b}{2 \sin^2 \theta}\right) d\theta = \frac{aN_R}{\pi \bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} (-1)^n \int_0^{\frac{\pi}{2}} \frac{d\theta}{\frac{b}{2 \sin^2 \theta} + \frac{n+1}{\bar{\gamma}}} \\ &\rightarrow \text{can be evaluated numerically} \end{aligned}$$

- high SNR approach $\Rightarrow \bar{\gamma} \rightarrow \infty$

$$\begin{aligned} p_{\gamma_t} &= \frac{N_R}{\bar{\gamma}} \left[1 - \exp\left(-\frac{x}{\bar{\gamma}}\right)\right]^{N_R-1} \exp\left(-\frac{x}{\bar{\gamma}}\right) \\ &\stackrel{\bar{\gamma} \rightarrow \infty}{\approx} \frac{N_R}{\bar{\gamma}} \left[1 - \left(1 - \frac{x}{\bar{\gamma}} + O(\bar{\gamma}^{-1})\right)\right]^{N_R-1} \left(1 - \frac{x}{\bar{\gamma}} + O(\bar{\gamma}^{-1})\right) \\ &= \frac{N_R}{\bar{\gamma}^{N_R}} x^{N_R-1} + o(\bar{\gamma}^{-N_R}) \end{aligned}$$

- MGF:

$$\begin{aligned} M_{\gamma_t}(s) &= \frac{N_R}{\bar{\gamma}^{N_R}} \frac{(N_R-1)!}{s^{N_R}} + O(\bar{\gamma}^{-N_R}) \\ \left[\rightarrow P_e = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\left(\frac{b}{2 \sin^2(\theta)}\right) d\theta\right] \\ &= \frac{a(2N_R)!}{b^{N_R} 2^{N_R+1} N_R!} \frac{1}{\bar{\gamma}^{N_R}} + O(\bar{\gamma}^{-N_R}) \end{aligned}$$

\Rightarrow Diversity gain: $G_d = N_R$

\Rightarrow Combining gain: $G_c = 2b \left(\frac{2N_R!}{a(2N_R)!}\right)^{\frac{1}{N_R}}$

– Outage Probability

$$P_{out} = \Pr\{\gamma_{\hat{n}} \leq \gamma_T\} = P_{\gamma_{\hat{n}}}(\gamma_T) = \left[1 - \exp\left(-\frac{\gamma_T}{\bar{\gamma}}\right)\right]^{N_R}$$

$$\text{high SNR: } P_{out} = \left(\frac{\gamma_T}{\bar{\gamma}}\right)^{N_R} + O(\bar{\gamma}^{-N_R})$$

1.5 Comparison

- Diversity Gain:
MRC, EGC and SC all achieve the maximum possible diversity gain of $G_d = N_R$
- Combining Gain:
The combining gains of MRC, EGC and SC are different
 - MRC/EGC:

$$\frac{G_C^{EGC}}{G_C^{MRC}} = \frac{\frac{1}{2b} \left(\frac{a}{2} \binom{2N_R}{N_R}\right)^{\frac{1}{N_R}}}{\frac{N_R}{b} \left(\frac{a}{2} \frac{1}{N_R!}\right)^{\frac{1}{N_R}}} = \frac{[(2N_R)!]^{\frac{1}{N_R}}}{2N_R(N_R)^{\frac{1}{N_R}}} \leq 1$$

(independent of a or b which are modulation parameters, only depends on number of antennas)

$$N_R \gg 1 : \quad N_R! \approx \sqrt{2\pi} e^{-N_R} N_R^{N_R + \frac{1}{2}} \quad (\text{Stirling})$$

$$\left. \frac{G_C^{EGC}}{G_C^{MRC}} \right|_{N_R \gg 1} = \frac{\left(\sqrt{2\pi} e^{-2N_R} (2N_R)^{2N_R + \frac{1}{2}}\right)^{\frac{1}{N_R}}}{2N_R \left(\sqrt{2\pi} e^{-N_R} N_R^{N_R + \frac{1}{2}}\right)^{\frac{1}{N_R}}} = \frac{2 \cdot 2^{\frac{1}{2N_R}}}{2} \xrightarrow{N_R \rightarrow \infty} \frac{2}{e} \equiv -1.3\text{dB}$$

– MRC/SC:

$$\frac{G_C^{SC}}{G_C^{MRC}} = \frac{2b \left(\frac{a}{2} \binom{2N_R}{N_R}\right)^{\frac{1}{N_R}}}{2b \left(\frac{a}{2} \frac{(2N_R)!}{N_R!}\right)^{\frac{1}{N_R}}} = \frac{1}{(N_R!)^{\frac{1}{N_R}}} \leq 1$$

$$\left. \frac{G_C^{SC}}{G_C^{MRC}} \right|_{N_R \gg 1} = \frac{1}{\sqrt{2\pi}^{\frac{1}{N_R}} e^{-1} N_R^{1 + \frac{1}{2N_R}}} N_R \xrightarrow{} \infty \frac{e}{N_R}$$

→ loss increases with N_R

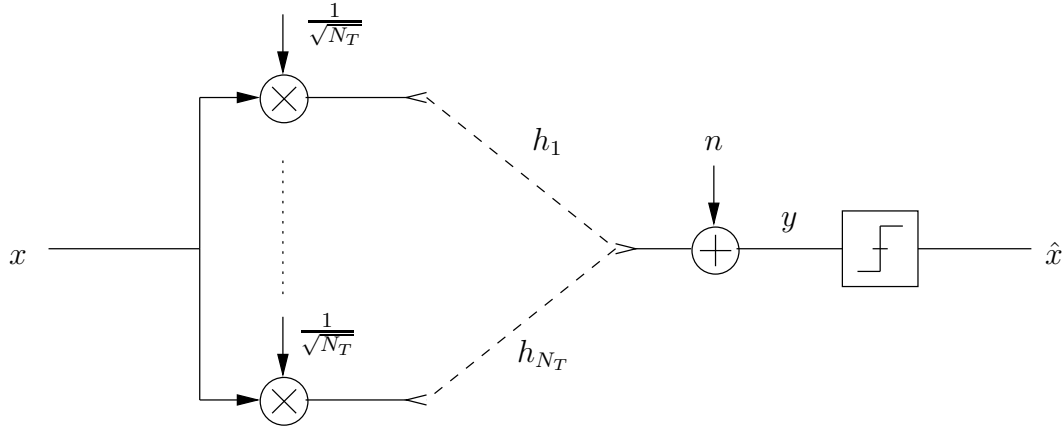
2 MISO Systems

Remarks

- Similar to SIMO systems, in MISO systems only coding and diversity gains can be obtained.
- To realize these gains, a careful transmitter design is necessary
- System design depends on whether or not channel state information (**CSI**) is available at transmitter

2.1 Naive Approach

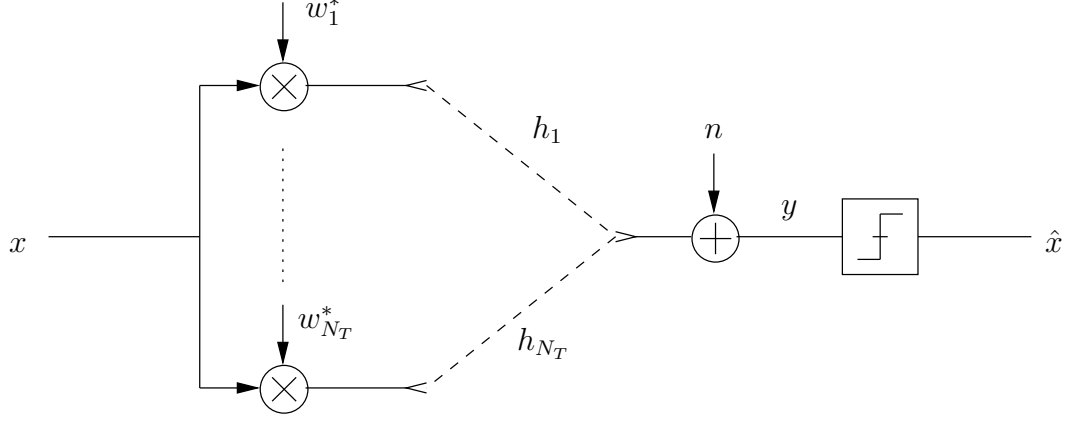
- Assume we simply send the same signal over all N_T transmit antennas



- Transmit power: $\mathcal{E} \left\{ \left| \frac{1}{\sqrt{N_T}} x \right|^2 + \dots + \left| \frac{1}{\sqrt{N_T}} x \right|^2 \right\} = \mathcal{E} \left\{ N_T \frac{1}{N_T} |x|^2 \right\} = \mathcal{E}_s$
- Received signal: $y = \frac{1}{\sqrt{N_T}} \sum_{n=1}^{N_T} h_n \cdot x + n$
- Rayleigh fading: h_n are zero mean complex gaussian random variables
 $\rightarrow h$ is also zero mean complex gaussian
- i.i.d.:
 - $\mathcal{E}\{|h_n|^2\} = 1 \ \forall n$
 - $\mathcal{E}\{|h|^2\} = \frac{1}{N_T} \mathcal{E} \left\{ \left| \sum_{n=1}^{N_T} h_n \right|^2 \right\} = \frac{1}{N_T} \mathcal{E} \left\{ \sum_{n=1}^{N_T} |h_n|^2 \right\} = 1$
 - statistical properties of h are independent of N_T
 - the multiple transmit antennas have no benefit at all
 - more sophisticated transmitter designs necessary

2.2 Full CSI Available at the Transmitter

- $h_n, n \in \{1, \dots, N_T\}$ is known at the transmitter
- Perform “precoding” (beamforming) with coefficients w_n



- Transmit Power: Two constraints maybe considered
 - Average transmit power constraint

$$P_{av} = \mathcal{E} \left\{ \sum_{n=1}^{N_T} |w_n^* x|^2 \right\} = \sum_{n=1}^{N_T} |w_n|^2 \underbrace{\mathcal{E}\{|x|^2\}}_{\mathcal{E}_s} = \mathcal{E}_s \Rightarrow \sum_{n=1}^{N_T} |w_n|^2 = 1$$

- Power constraint for each transmit antenna

$$\rightarrow |w_n| = \frac{1}{\sqrt{N_T}} \quad \rightarrow P_{av} = \mathcal{E}_s$$

- Received signal: $y = \underbrace{\sum_{n=1}^{N_T} w_n^* h_n}_{h} x + n$ (equivalent SISO channel)

Maximum Ratio Transmission (MRT)

- we have only the average power constraint: $\sum_{n=1}^{N_T} |w_n|^2 = 1$

$$\bullet \text{ SNR: } \gamma_t = \frac{\mathcal{E}_s |h|^2}{\sigma_n^2} = \frac{\mathcal{E}_s \left| \sum_{n=1}^{N_T} w_n^* \cdot h_n \right|^2}{\sigma_n^2}$$

- Maximize SNR under constraint $\sum_{n=1}^{N_T} |w_n|^2 = 1$

- constraint optimization problem \rightarrow Lagrange method

$$L = \frac{\epsilon_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} w_n^* \cdot h_n \right|^2 + \lambda \left(\sum_{n=1}^{N_T} |w_n|^2 - 1 \right); \quad \text{where: } \lambda = \text{Lagrange Multiplier}$$

\Rightarrow Wirtinger Kalkül: treat z and z^* as independent variables for differentiation:

$$\begin{aligned} \frac{\partial z^*}{\partial z} &= 0; & \frac{\partial |z|^2}{\partial z} &= \frac{\partial z \cdot z^*}{\partial z} = z^* \\ \frac{\partial x^2}{\partial x} &= 2x; & \frac{\partial (z^*)^2}{\partial z^*} &= 2 \cdot z^*; & \frac{\partial |z|^2}{\partial z^*} &= z^* \end{aligned}$$

$$\frac{\partial L}{\partial w_m^*} = \frac{\epsilon_s}{\sigma_n^2} \left(\sum_{n=1}^{N_T} w_n^* \cdot h_n \right)^* h_m + \lambda w_m$$

$$\rightarrow w_m = \frac{\frac{\epsilon_s}{\sigma_n^2} \cdot \lambda}{\text{const., independent of } m} \left(\sum_{n=1}^{N_T} w_n^* h_n \right)^* h_m$$

$$\rightarrow w_m = c \cdot h_m$$

$$\rightarrow \sum_{n=1}^{N_T} |w_n|^2 = 1 \rightarrow c^2 = \frac{1}{\sum_{n=1}^{N_T} |h_n|^2}$$

$$\rightarrow w_n = \frac{h_n}{\sqrt{\sum_{n=1}^{N_T} |h_n|^2}} \equiv \text{MRT gains}$$

$$\rightarrow \text{SNR} = \frac{\epsilon_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} \frac{|h_n|^2}{\sqrt{\sum_{n=1}^{N_T} |h_n|^2}} \right|^2 = \frac{\epsilon_s}{\sigma_n^2} \sum_{n=1}^{N_T} |h_n|^2$$

\Rightarrow same SNR as for maximum ratio combining (MRC)

\Rightarrow MRT with N_T transmit antennas achieves the same performance as MRC with N_T receive antennas

\Rightarrow MRT/MRC can be extended to $N_T \times N_R$ MIMO systems

\rightarrow has the same performance as MRC with $N_T \cdot N_R$ receive antennas and one transmit antenna

Equal Gain Transmission (EGT)

- we employ gains: $w_n = \frac{1}{\sqrt{N_T}} \cdot \frac{h_n}{|h_m|} \rightarrow |w_n| = \frac{1}{\sqrt{N_T}}$

- SNR:

$$\begin{aligned}
\gamma_t &= \frac{\mathcal{E}_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} w_n^* h_n \right|^2 \\
&= \frac{\mathcal{E}_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} \frac{1}{\sqrt{N_T} \cdot \frac{|h_n|^2}{|h_n|}} \right|^2 = \frac{1}{N_T} \cdot \frac{\mathcal{E}_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} |h_n| \right|^2 \\
\gamma_n &= \frac{\mathcal{E}_s}{\sigma_n^2} |h_n|^2 \\
\text{same SNR as for EGC} &\rightarrow \gamma_t = \frac{1}{N_T} \left| \sum_{n=1}^{N_T} \sqrt{\gamma_n} \right|^2
\end{aligned}$$

→ EGC with N_T transmit antennas achieves the same performance as EGC with N_T receive antennas

Transmit Antennas Selection

- select antenna with maximum channel gain for transmission:

$$w_n = \begin{cases} \frac{h_n}{|h_n|}, & \text{if } n = \hat{n} \\ 0, & \text{otherwise} \end{cases} \text{ where } \hat{n} = \underset{n}{\operatorname{argmax}} |h_n|$$

- antenna selection with N_T transmit antennas achieves the same performance as *Selection Combining* with N_T receive antennas

2.3 No CSI at Transmitter - Space-Time-Coding

- $h_n, n \in \{1, \dots, N_T\}$, is only known at the receiver
- “Space-time-coding” has to be employed to realize diversity gain
- $T \times N_T$ matrices \mathbf{X} are transmitted in T symbol intervals over N_T antennas
- \mathbf{X} is drawn from a matrix alphabet \mathcal{X}
- Example:

$$\mathbf{X} = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,N_T} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,N_T} \\ \vdots & \vdots & \ddots & \vdots \\ x_{T,1} & x_{T,2} & \cdots & x_{T,N_T} \end{pmatrix}$$

- We distinguish:
 - Space-time-block-codes (STBCs)
 - \mathbf{X} is obtained by mapping K scalar symbols s_k , $k = 1, \dots, K$ from a scalar alphabet \mathcal{A} to matrix \mathbf{X}

- Space-time-trellis-codes (STTCs)
 - \mathbf{X} is obtained from scalar symbols s_k through a trellis encoding process.
 - [see: Tarokh, Seshadri, Calderbank: Space-time-codes for high datarate wireless communication: Performance criteria and coder construction; IEEE Trans. Inf. Theory 1998]
- here: We concentrate on space-time-block-codes (STBCs), but many results can be easily extended to space-time-trellis-codes
- STBCs:
 - K M -ary scalar symbols (e.g. M -PSK symbols) are mapped to STBC matrices \mathbf{X}
 $\mathbf{S} = [s_1, \dots, s_K] \rightarrow \mathbf{X}$
 $s_k \in \mathcal{A} \rightarrow x \in \mathcal{X}$ with $|\mathcal{X}| = M^K$
 - Example: “Alamouti”-Code

$$\mathbf{X} = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix}$$

[Alamouti: A simple transmit diversity technique for wireless communication, IEEE JSAC 1998]

Optimal Detection

- Signal model:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_T \end{pmatrix} = \mathbf{X} \begin{pmatrix} h_1 \\ \vdots \\ h_{N_T} \end{pmatrix} + \begin{pmatrix} n_1 \\ \vdots \\ n_T \end{pmatrix}$$

$$\mathbf{y} = \mathbf{X} \cdot \mathbf{h} + \mathbf{n}$$

- Optimal detection - ML-detection
 - \mathbf{h} is known at receiver
 - \mathbf{n} is AWGN with $\mathcal{E}\{\mathbf{n} \cdot \mathbf{n}^H\} = \sigma_n^2 \cdots \mathbf{I}_{T \times T}$
 - $p(\mathbf{y}|\mathbf{x})$

$$\begin{aligned} &= \frac{1}{\pi^T |\sigma_n^2 \mathbf{I}_{T \times T}|} \exp \left(-(\mathbf{y} - \mathbf{xh})^H (\sigma_n^2 \mathbf{I}_{T \times T})^{-1} (\mathbf{y} - \mathbf{xh}) \right) \\ &= \frac{1}{\pi^T \sigma_n^{2T}} \exp \left(-\frac{1}{\sigma_n^2} (\mathbf{y} - \mathbf{xh})^H (\mathbf{y} - \mathbf{xh}) \right) = \frac{1}{\pi^T \sigma_n^{2T}} \exp (||\mathbf{y} - \mathbf{xh}||^2) \end{aligned}$$

→ the optimal estimate $\hat{\mathbf{X}}$ or equivalently the optimal estimate $\hat{\mathbf{s}}$ can be obtained as

$$\hat{\mathbf{s}} = \underset{\mathbf{s} \in \mathcal{A}^K}{\operatorname{argmax}} p(\mathbf{y}|\mathbf{x}) = \underset{\mathbf{s} \in \mathcal{A}^K}{\operatorname{argmin}} ||\mathbf{y} - \mathbf{hx}||^2$$

- Disadvantage: In general, metric $||\mathbf{y} - \mathbf{hx}||^2$ has to be calculated M^K times
 - complexity increases exponentially with K

Types of STBCs

- Orthogonal STBCs (OSTBCs)

- OSTBCs are a special class of STBCs which allow independent detection of each $s_k \rightarrow$ only $K \cdot M$ metrics have to be evaluated

- Rate STBCs: $R_{STBC} = \frac{K}{T}$

- Examples:

- * Alamouti Code ($K = 2, T = 2$) $\rightarrow R_{STBC} = 1$

$$\mathbf{X} = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix} \begin{matrix} \downarrow T \\ \longleftrightarrow N_T \end{matrix}$$

\rightarrow only “full rate” OSTBC for complex s_k

- * $N_T = 3, K = 3, T = 4$

$$\mathbf{X} = \frac{1}{\sqrt{3}} \begin{pmatrix} s_1 & s_2 & s_3 \\ -s_2^* & s_1^* & 0 \\ s_3^* & 0 & -s_3^* \\ 0 & -s_3^* & s_2^* \end{pmatrix} \rightarrow R_{STBC} = \frac{K}{T} = \frac{3}{4}$$

- Orthogonality: $\mathbf{X}^H \mathbf{X} = \text{const} \cdot \mathbf{I}_{N_T \times N_T}$

- Independent detection of s_1 & s_2 for Alamouti Code

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \\ \rightarrow \underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{\tilde{\mathbf{y}}} = \frac{1}{\sqrt{2}} \underbrace{\begin{pmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{pmatrix}}_{\mathbf{F}} \underbrace{\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}}_{\mathbf{s}} + \underbrace{\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}}_{\tilde{\mathbf{n}}}$$

(Anmerkung: nur $\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$ gewünscht, nicht: s_1^*, s_2^*)

$$\mathbf{F}^H \mathbf{F} = \frac{1}{2} \begin{pmatrix} h_1^* & h_2 \\ h_2^* & -h_1 \end{pmatrix} \begin{pmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{pmatrix} = \frac{1}{2} \begin{pmatrix} |h_1|^2 + |h_2|^2 & 0 \\ 0 & |h_1|^2 + |h_2|^2 \end{pmatrix}$$

$\rightarrow \frac{\sqrt{2}}{\sqrt{|h_1|^2 + |h_2|^2}} \cdot \mathbf{F}$ is unitary matrix

$\rightarrow \frac{2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{y}} = \mathbf{s} + \frac{2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{n}}$

$\left(\frac{2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{n}} \right)$ is AWGN vector with covariance matrix $\frac{2\sigma_n^2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{I}_{T \times T}$

\rightarrow ML decision: $\hat{\mathbf{s}} = \underset{\mathbf{s}}{\text{argmin}} \left\| \frac{2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{y}} - \mathbf{s} \right\|^2$

\rightarrow independent ML decoding

$$\hat{s}_1 = \underset{s_1}{\text{argmin}} \left| s_1 - \frac{h_1^* y_1 + h_2 y_2^*}{\frac{1}{\sqrt{2}}(|h_1|^2 + |h_2|^2)} \right| \\ \hat{s}_2 = \underset{s_2}{\text{argmin}} \left| s_2 - \frac{h_1^* y_1 - h_2 y_2^*}{\frac{1}{\sqrt{2}}(|h_1|^2 + |h_2|^2)} \right|$$

- independent decoding property can be proved for all OSTBCs
- low complexity is at the expense of a rate-loss compared to other STBCs for $N_T > 2$
 - Frequenzhopping
 - keine Kanalinformation aus vorher empfangenen Symbolen möglich \Rightarrow Kanal ändert sich ständig; nur Entscheidung, ob Rauschen oder Signal + Rauschen
- Performance Analysis of Alamouti Code
 - Decision-variables after combining

$$r_1 = \sqrt{2} \frac{h_1^* y_1 + h_2 y_2^*}{|h_1|^2 + |h_2|^2}$$

$$r_2 = \sqrt{2} \frac{h_1^* y_1 - h_2 y_2^*}{|h_1|^2 + |h_2|^2}$$

because of symmetry it suffices to consider r_1

$$r_1 = \sqrt{2} \frac{h_1^* \left(\frac{1}{\sqrt{2}} s_1 h_1 + \frac{1}{\sqrt{2}} h_2 s_2 + n_1 \right) + h_2 \left(-\frac{1}{\sqrt{2}} h_2 s_1^* + \frac{1}{\sqrt{2}} h_1 s_2^* + n_2 \right)^*}{|h_1|^2 + |h_2|^2}$$

$$= \sqrt{2} \frac{\frac{1}{\sqrt{2}} (|h_1|^2 + |h_2|^2) s_1 + h_1^* n_1 + h_2 n_2^*}{|h_1|^2 + |h_2|^2}$$

$$= 1 \cdot s_1 + n_{eq}$$

where

$$n_{eq} = \sqrt{2} \frac{h_1^* n_1 + h_2 n_2^*}{|h_1|^2 + |h_2|^2}$$

$$\text{SNR} \rightarrow \gamma_t = \frac{\mathcal{E}_s \cdot 1^2}{\sigma_{eq}^2} \quad \text{with} \quad \mathcal{E}\{|s_1|^2\} = \mathcal{E}_s$$

$$\sigma_{eq}^2 = 2 \frac{|h_1|^2 \sigma_n^2 + |h_2|^2 \sigma_{eq}^2}{(|h_1|^2 + |h_2|^2)^2} = \frac{2\sigma_n^2}{|h_1|^2 + |h_2|^2}$$

- $\gamma_t = \frac{1}{2} \frac{\mathcal{E}_s}{\sigma_n^2} (|h_1|^2 + |h_2|^2)$
- $\text{SNR}_{\text{Alamouti}} = \frac{1}{2} \text{SNR}_{\text{MRC}} = \frac{1}{2} \text{SNR}_{\text{MRT}}$
- Alamouti code has diversity gain $G_d = 2$
- Transmission with Alamouti STBC requires 3dB higher SNR to achieve same performance as MRT \rightarrow 3dB loss in coding gain G_c
- Lack of CSI knowledge at transmitter “costs” 3dB in power efficiency
- General:
 - OSTBCs achieve a diversity gain of $G_d = N_T$ if only one receive antenna is available
 - if N_R receive antennas are available, MRC can be used at the receiver to yield a diversity gain of $G_d = N_T N_R$
- Other STBCs:

- Quasi orthogonal STBCs
 - * higher rate than OSTBCs
 - * only subset of symbols have to be decoded jointly
 - * Example: $K = N_T = T = 4$

$$\mathbf{X} = \frac{1}{2} \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^* & s_1^* & -s_4^* & s_3^* \\ -s_3^* & -s_4^* & s_1^* & s_2^* \\ s_4 & -s_3 & -s_2 & s_1 \end{pmatrix}$$

- * Anmerkung 1: \mathbf{X} ist ähnlich zu Alamouti Code
- * Anmerkung 2: $\mathbf{X}^H \mathbf{X}$: viele Nicht-diagonal Elemente sind Null; die, die ungleich Null sind, zeigen, welche Symbole gemeinsam entschlüsselt werden müssen
- Golden Code for $N_T = N_R = 2$: achieves a rate of $R_{STBC} = 2$ and full diversity of $G_d = N_T, N_R = 4$
- Differential STBCs: $\mathbf{X}_k = \mathbf{X}_{k-1} \cdot \mathbf{D}_k$. \mathbf{X}_k is transmitted, \mathbf{D}_k is transmitted
- Linear dispersion codes: designed to achieve high mutual information
- noncoherent STBCs (On-Off-Keying)

Space Time Code Design

Given:

- Code $\mathcal{H} = \{\mathbf{X}_1, \dots, \mathbf{X}_{|\mathcal{H}|}\}$
- Channel: IID Rayleigh-fading:
 - $h_n \sim \mathcal{CN}(0, 1)$; $n \in \{1, 2, \dots, N_T\}$
 - AWGN $n \sim \mathcal{CN}(0, \sigma_n^2)$

Problem: How should we design codebook \mathcal{H} ?

- Need to derive error rate for general codebooks \mathcal{H} !
 - Codeword error rate

$$P_e = \frac{1}{|\mathcal{H}|} \sum_{i=1}^{|\mathcal{H}|} \Pr\{\mathbf{x}_i \neq \hat{\mathbf{x}}_i\}$$

where $\hat{\mathbf{x}}_i$ is the detected codeword and we assume that all codewords are equally likely

Problem: $\Pr\{\mathbf{x}_i \neq \hat{\mathbf{x}}_i\}$ is not tractable in general

- Use union bound to upper bound $\Pr\{\mathbf{x}_i \neq \hat{\mathbf{x}}_i\}$ as upper sum over pairwise error probabilities(PEP) $\Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\}$ where it is assumed that \mathbf{x}_i was transmitted and \mathbf{x}_i and \mathbf{x}_j are the only codewords in the codebook

$$P_e \leq \frac{1}{|\mathcal{H}|} \sum_{i=1}^{|\mathcal{H}|} \sum_{j=1}^{|\mathcal{H}|} \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} \text{ where } j \neq i$$

Calculation of PEPs

Recall: $\hat{\mathbf{x}} = \underset{\mathbf{x} \in \mathcal{H}}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{x}\mathbf{h}\|^2$

Now, \mathbf{x}_i and \mathbf{x}_j are the only alternatives and an error is made if $\|\mathbf{y} - \mathbf{x}_i\mathbf{h}\|^2 > \|\mathbf{y} - \mathbf{x}_j\mathbf{h}\|^2$ since \mathbf{x}_i was sent but \mathbf{x}_j was detected

$$\begin{aligned} \rightarrow \|\mathbf{x}_i\mathbf{h} + \mathbf{n} - \mathbf{x}_j\mathbf{h}\|^2 &> \|\mathbf{x}_i\mathbf{h} + \mathbf{n} - \mathbf{x}_j\mathbf{h}\|^2 \\ \|\mathbf{n}\| &> \|(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h} + \mathbf{n}\|^2 \\ \rightarrow \|\mathbf{n}\| &> \underbrace{\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h}}_{\Delta} + \mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{n} + \mathbf{n}^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h} + \|\mathbf{n}\|^2 \\ &\rightarrow \underbrace{-\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)^H\mathbf{n} - \mathbf{n}^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h}}_z > \Delta \end{aligned}$$

for given \mathbf{h} , z is a gaussian random variable

$$\begin{aligned} \sigma_z^2 &= \mathcal{E}\{|z|^2\} = \mathcal{E}\{2\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j) \underbrace{\mathbf{n}\mathbf{n}^H}_{\sigma_n^2 \mathbf{I}}(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h} + 2\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)^H \underbrace{\mathbf{n}\mathbf{n}^T}_{=0}(\mathbf{x}_i - \mathbf{x}_j)^*\mathbf{h}^*\} \\ &= 2\sigma_n^2 \Delta + 0 \end{aligned}$$

$$\begin{aligned} \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} &= \int_{\Delta}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{z^2}{2\sigma_z^2}\right) dz, \quad t = \frac{z}{\sigma_z} \\ &= \frac{1}{\sqrt{2\pi}} \int_{\frac{\Delta}{\sigma_z}}^{\infty} e^{-\frac{t^2}{2}} dt = Q\left(\frac{\Delta}{\sigma_z}\right) = Q\left(\frac{\Delta}{\sqrt{2\sigma_n^2 \Delta}}\right) \\ &= Q\left(\sqrt{\frac{\Delta}{2\sigma_n^2}}\right) \end{aligned}$$

- $\Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} = \mathcal{E}\left\{Q\left(\sqrt{\frac{\Delta}{2\sigma_n^2}}\right)\right\}$

– to avoid cumbersome Q-function we use Chernoff bound:

$$Q(x) \leq \frac{1}{2} e^{-\frac{x^2}{2}}$$

$$\begin{aligned} \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} &\leq \frac{1}{2} \mathcal{E}_h \left\{ \exp\left(-\frac{\mathbf{h}^H \mathbf{Q} \mathbf{h}}{4\sigma_n^2}\right) \right\} \\ \text{where } \mathbf{Q} &= (\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j) \end{aligned}$$

- Eigendecomposition: $\mathbf{Q} = \mathbf{U}^H \mathbf{\Lambda} \mathbf{U}$ with $\mathbf{\Lambda} = \text{diag}\{\lambda_1, \dots, \lambda_r, 0, \dots, 0\}$ $r = \text{rank}\{\mathbf{Q}\}$

- Elements \mathbf{h} are i.i.d. Gaussian

– $\underline{\beta} = \mathbf{U}\mathbf{h}$ has also i.i.d. Gaussian random variables as elements since \mathbf{U} is unitary matrix

$$- \mathbf{h}^H \mathbf{Q} \mathbf{h} = \underbrace{\mathbf{h}^H \mathbf{U}^H}_{\underline{\beta}^*} \mathbf{\Lambda} \underbrace{\mathbf{U} \mathbf{h}}_{\underline{\beta}} = \sum_{i=1}^r \lambda_i |\beta_i|^2 \text{ with } \underline{\beta} = [\beta_1, \dots, \beta_{N_T}]$$

$$\begin{aligned} \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} &= \frac{1}{2} \mathcal{E}_{\underline{\beta}} \left\{ \exp \left(-\frac{\sum_{i=1}^r \lambda_i |\beta_i|^2}{4\sigma_n^2} \right) \right\} \\ &= \frac{1}{2} \mathcal{E}_{\underline{\beta}} \left\{ \prod_{i=1}^r e^{-\frac{\lambda_i}{4\sigma_n^2} |\beta_i|^2} \right\} \\ &= \frac{1}{2} \prod_{i=1}^r \mathcal{E}_{\beta_i} \left\{ e^{-\frac{\lambda_i}{4\sigma_n^2} |\beta_i|^2} \right\} \\ &= \frac{1}{2} \prod_{i=1}^r \mathcal{E}_{|\beta_i|^2} \left\{ e^{-\frac{\lambda_i}{4\sigma_n^2} |\beta_i|^2} \right\} \triangleq \text{MGF of exponentially distributed variable } \alpha_i = |\beta_i|^2 \end{aligned}$$

$$\rightarrow P_{\alpha_i}(x) = e^{-x}, \quad x \geq 0$$

$$\begin{aligned} \rightarrow \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} &\leq \frac{1}{2} \prod_{i=1}^r \frac{1}{1 + \frac{\lambda_i}{4\sigma_n^2}} \\ &\leq \prod_{i=1}^r \frac{1}{\frac{\lambda_i}{4\sigma_n^2}} = 2^{2r-1} \frac{1}{\prod_{i=1}^r \lambda_i} \left(\frac{1}{\underbrace{\sigma_n^2}_{\triangleq SNR}} \right)^{-r} \end{aligned}$$

- upper bound on P_e :

$$\lambda_n(i, j) = n\text{th eigenvalue of } (\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j)$$

$$r(i, j) = \text{rank of } (\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j)$$

$$\rightarrow P_e \leq \frac{1}{|\mathcal{H}|} \sum_{i=1}^{|\mathcal{H}|} \sum_{j=1}^{|\mathcal{H}|} 2^{2r(i,j)-1} \frac{1}{\prod_{n=1}^{r(i,j)} \lambda_n(i, j)} \left(\frac{1}{\sigma_n^2} \right)^{-r(i,j)}$$

- generally loose bound but offers significant insight for code design