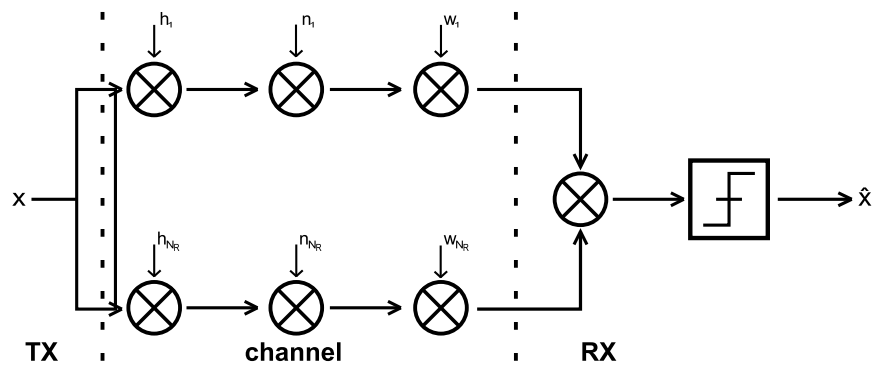


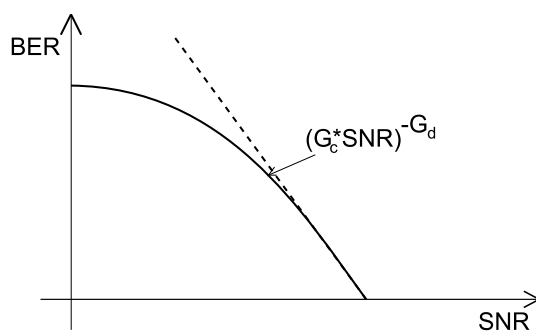
# 1 SIMO Systems

## Remarks

- In SIMO Systems only coding and diversity gains can be exploited (no multiplexing gains)
- To realize these gains diversity combining has to be performed
- Diversity combining schemes vary in complexity and performance
- There are many diversity combining schemes. Here we consider:
  - Maximal ratio combining (MRC)
  - Equal gain combining (EGC)
  - Selection combining (SC)
- Diversity combining problem



- how to choose combining weights  $w_n$ ?
- what performance (e.g. error rate, outage probability) is achieved?
- what diversity and coding/combining gain is achieved?



- $G_c$  : Coding gain
- $G_d$  : Diversity gain

## 1.1 Preliminaries

Consider an equivalent system:

$$y = hx + n; \quad \mathcal{E}\{|x|^2\} = E_s; \quad \mathcal{E}\{|n|^2\} = \sigma_n^2; \quad \mathcal{E}\{|h|^2\} = 1$$

- Instantaneous SNR:  $\gamma_t = \frac{E_s}{\sigma_n^2} \times |h|^2$
- Average SNR:  $\bar{\gamma}_t = \mathcal{E}\{\gamma_t\} = \frac{E_s}{\sigma_n^2}$

### Bit and Symbol Error Rate

- The Bit and Symbol Error Rate of many modulation schemes can be expressed for given  $\gamma_t$  as:

$$P_e(\gamma_t) = aQ\{\sqrt{b\gamma_t}\}$$

where:

- $Q(x) = \frac{1}{\sqrt{2\pi}} \times \int_x^\infty e^{-\frac{t^2}{2}} dt$
- $P_e(\gamma_t)$  may be exact result or approximation
- BPSK: exact with  $a = 1, b = 2$
- M-ary QAM: tight approximation with  $a = 4(1 - \frac{1}{\sqrt{M}}), b = \frac{3}{M-1}$

(*Einschub : Gray – Code : BER =  $\frac{1}{\log_2 M} \times SER$* )

- Alternative representation of Q - function:

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}} d\theta$$

→ Integral limits are fixed and do not depend on integration variables!

- Average error probability

$$P_e = \mathcal{E}\{P_e(\gamma_t)\} = \int_0^\infty aQ(\sqrt{bx})p_{\gamma_t}(x) dx$$

- Integral may be difficult to solve analytically
- Integral has infinite support → numerical evaluation difficult
- Using alternative representation of Q-function we get:

$$\begin{aligned} P_e &= \int_0^\infty \frac{a}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{bx}{2\sin^2\theta}} p_{\gamma_t}(x) d\theta dx \\ &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} \int_0^\infty p_{\gamma_t}(x) e^{-\frac{bx}{2\sin^2\theta}} dx d\theta = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\left(\frac{b}{2\sin^2\theta}\right) d\theta \end{aligned}$$

where:

- $M_{\gamma_t}(s) = \int_0^\infty p_{\gamma_t}(x) e^{-sx} dx$  is the Laplace transform of  $p_{\gamma_t}$
- $M_{\gamma_t}(-s)$  is the so called Moment Generation Function (MGF) of  $p_{\gamma_t}$
- Here, we will also refer to  $M_{\gamma_t}(s)$  as MGF
- $M_{\gamma_t}(s)$  is sometimes easier to obtain than  $p_{\gamma_t}$
- The above integral can be easily evaluated numerically because of the finite integral limits

### Outage probability

- The outage probability is the probability that the channel cannot support a certain rate,  $R$ , i.e. (where  $\gamma_T$  is the threshold SNR):

$$C = \log_2(1 + \gamma_t) < R \quad \leftrightarrow \quad \gamma_t < 2^R - 1 \triangleq \gamma_T$$

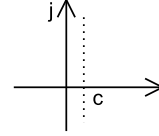
Thus, the outage probability is given by:

$$P_{out} = P_{\gamma_t < \gamma_T} = \int_0^{\gamma_T} p_{\gamma_t}(x) dx$$

- Using the inverse Laplace Transform

$$p_{\gamma_t}(x) = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) e^{sx} dx$$

where  $c > 0$  is a small constant that lies in the region of convergence of the integral, we obtain:



- 1.

$$P_{out} = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) \int_0^{\gamma_T} e^{sx} dx ds = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} M_{\gamma_t}(s) e^{\gamma_T s} \frac{ds}{s}$$

(lower integral limit is 0 since  $p_{\gamma_t}(0) = 0$ )

- and 2.:

$$p_{\gamma_t}(x) = \int_0^x p_{\gamma_t}(t) dt = 0$$

$$\text{for } x = 0 \text{ note: } p_{\gamma_t}(x) \xleftrightarrow[\text{transform}]{\text{Laplace}} \frac{1}{s} M_{\gamma_t}(s)$$

### General combining scheme

$$y = \left( \sum_{n=1}^{N_R} h_n w_n \right) x + \sum_{n=1}^{N_R} w_n n_n$$

$$\gamma_t = \frac{\epsilon_s \left| \sum_{n=1}^{N_R} h_n w_n \right|^2}{\sigma_n^2 \sum_{n=1}^{N_R} |w_n|^2}$$

where  $w_n$  depends on the particular combining scheme.

## 1.2 MRC (Maximum Ratio Combining)

- what weight  $w_n$  maximize  $\gamma_t$ ?
  - Cauchy-Schwarz inequality

$$\left| \sum_{n=1}^{N_R} h_n w_n \right|^2 \leq \sum_{n=1}^{N_R} |h_n|^2 \cdot \sum_{n=1}^{N_R} |w_n|^2$$

where equality holds if and only if  $w_n = c \cdot h_n^*$  for some non-zero constant  $c$ .

- for  $w_n = h_n^*$ , we obtain

$$\gamma_t = \frac{\epsilon_s}{\sigma_n^2} \cdot \frac{\left( \sum_{n=1}^{N_R} |h_n|^2 \right)^2}{\sum_{n=1}^{N_R} |h_n|^2} = \frac{\epsilon_s}{\sigma_n^2} \sum_{n=1}^{N_R} |h_n|^2$$

- $w_n = h_n^* \forall n$  are the MRC combining weights.
- For performance analysis we assume independent identically distributed (IID) Rayleigh fading

$$\begin{aligned} \rightarrow \mathcal{E}\{|h_n|^2\} &= 1; \quad \bar{\gamma} = \frac{\epsilon_s}{\sigma_n^2}; \quad \gamma_n = \frac{\epsilon_s}{\sigma_n^2} |h_n|^2 \\ p_\gamma(x) &= \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0 \\ M_\gamma(s) &= \frac{1}{1 + s\bar{\gamma}} \end{aligned}$$

- Error rate

$$\gamma_t = \sum_{n=1}^{N_R} \gamma_n$$

$\rightarrow$  sum of IID random variables (r.v.s.)

$$M_{\gamma_t}(s) = \left( M_\gamma(s) \right)^{N_R} = \frac{1}{(1 + s\bar{\gamma})^{N_R}} = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{1}{\left(s + \frac{1}{\bar{\gamma}}\right)^{N_R}}$$

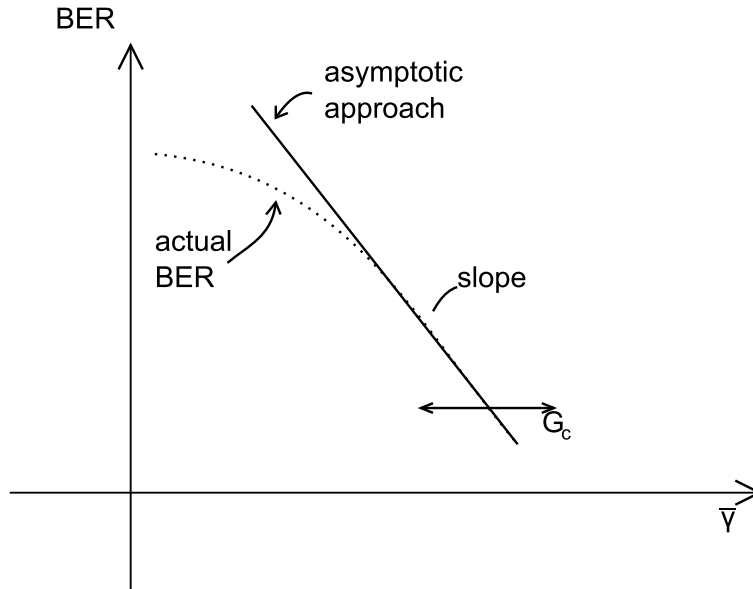
inverse Laplace-transform (from tables)

$$p_{\gamma_t}(x) = \frac{1}{\bar{\gamma}^{N_R}} \cdot \frac{x^{N_R-1}}{(N_R-1)!} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0$$

- Direct approach

$$p_e = \int_0^\infty a \cdot Q(\sqrt{ax}) p_{\gamma_t}(x) dx = a \left( \frac{1-\mu}{2} \right)^{N_R} \cdot \sum_{n=0}^{N_R-1} \binom{N_R-1+n}{n} \left( \frac{1+\mu}{2} \right)^n$$

$$\text{where } \mu = \sqrt{\frac{b\bar{\gamma}}{2 + b\bar{\gamma}}}$$



- MGF approach

$$\begin{aligned}
 p_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t} \left( \frac{b}{2 \sin^2 \theta} \right) d\theta \\
 &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\bar{\gamma}^{N_R} \left( \frac{b}{\sin^2 \theta} + \frac{1}{\bar{\gamma}} \right)^{N_R}} d\theta \quad (\text{numerisch berechnen!})
 \end{aligned}$$

- high SNR:  $\bar{\gamma} \rightarrow \infty \iff \frac{1}{\bar{\gamma}} \rightarrow 0$

$$\begin{aligned}
 p_e &= \frac{a}{\pi} \cdot \frac{1}{\bar{\gamma}^{N_R}} \cdot \left( \frac{2}{b} \right)^{N_R} \int_0^{\frac{\pi}{2}} \sin^{2N_R} \theta d\theta \\
 (\text{from MGF approach: } \int_0^{\frac{\pi}{2}} \sin^{2N_R} \theta d\theta &= \frac{\pi}{2^{N_R+1}} \cdot \binom{2N_R}{N_R}) \\
 &= \frac{a}{2^{N_R+1} \cdot b^{N_R}} \binom{2N_R}{N_R} \frac{1}{\bar{\gamma}^{N_R}} \quad \text{as } \bar{\gamma} \rightarrow \infty \\
 &\stackrel{!}{=} \left( \frac{1}{G_c \bar{\gamma}} \right)
 \end{aligned}$$

where: Diversity gain:  $G_d = N_R$

$$\text{Combining/Coding gain: } G_c = 2b \left( \frac{a}{2} \binom{2N_R}{N_R} \right)^{-\frac{1}{N_R}}$$

- MRC exploits the maximal possible diversity
- Diversity gain is not affected by correlation as the branches are not fully correlated
- Diversity gain depends on fading distribution

## Outage probability

$$\begin{aligned} P_{out} &= \int_0^{\gamma_T} p_{\gamma_t}(x) dx = \frac{1}{\bar{\gamma}^{N_R}} \int_0^{\gamma_T} \frac{x^{N_R-1}}{(N_R-1)!} e^{-\frac{x}{\bar{\gamma}}} dx \\ &= 1 - e^{-\frac{\gamma_T}{\bar{\gamma}}} \cdot \sum_{n=1}^{N_R} \frac{\left(\frac{\gamma_T}{\bar{\gamma}}\right)^n}{(n-1)!} \end{aligned}$$

- Approximation (Taylor series):  $\bar{\gamma} \rightarrow \infty : -e^{-\frac{x}{\bar{\gamma}}} = 1 - \frac{x}{\bar{\gamma}} + O(\frac{1}{\bar{\gamma}})$  where a function  $f(x)$  is  $O(x)$  if  $\lim_{x \rightarrow \infty} \frac{f(x)}{x} = 0$ .

$$\Rightarrow P_{out} = \frac{1}{\bar{\gamma}^{N_R}} \int_0^{\gamma_T} \frac{x^{N_R-1}}{(N_R-1)!} \left(1 - \frac{x}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right)\right) dx$$

- Diversity and coding gain can also be defined for  $P_{out}$

## 1.3 EGC (Equal Gain Combining)

### Combining Weights

- For MRC, both, the amplitudes and phases of the channel gains  $h_n = |h_n|e^{j\varphi_n}$  have to be known (or estimated in practice)
- In EGC it is assumed that only the phases are known and weights  $w_n = e^{-j\varphi_n}$  are used.

$$\begin{aligned} \Rightarrow \gamma_t &= \frac{E_s}{\sigma_n^2} \frac{\left| \sum_{n=1}^{N_R} |h_n| e^{j\varphi_n} e^{-j\varphi_n} \right|^2}{\sum_{n=1}^{N_R} |e^{-j\varphi_n}|^2} = \frac{E_s}{\sigma_n^2} \frac{1}{N_R} \left( \sum_{n=1}^{N_R} |h_n| \right)^2 \\ &= \frac{1}{N_R} \left( \sum_{n=1}^{N_R} \sqrt{\gamma_n} \right)^2 ; \text{ with } \gamma_n = \frac{E_s}{\sigma_n^2} |h_n|^2 \end{aligned}$$

### Performance Analysis

- IID case  
 $\Rightarrow \sqrt{\gamma_n}$  is Rayleigh distributed  
 $\Rightarrow$  Exact analysis is much more difficult than for MRC  $\Rightarrow$  see book by Simon & Alouini p.341
- Approximate result

$$P_e = \frac{a}{2} \left[ 1 - \sqrt{\frac{2b\bar{\gamma}}{5+2b\bar{\gamma}}} \sum_{n=0}^{N_R-1} \frac{\binom{2n}{n}}{4^n (1 + \frac{2}{5}b\bar{\gamma})^n} \right]$$

- high SNR

⇒ use high SNR analysis of Wang & Giannakis, 2003

⇒ at high SNR, only pdf of  $\gamma_n$  around 0 is relevant for performance

$$\stackrel{\text{Rayleigh}}{\Rightarrow} p_\gamma(x) = \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}} \stackrel{\text{Taylor Serie}}{=} \frac{1}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right) \text{ as } x \rightarrow 0$$

- need pdf  $\gamma_t$ : ( $\gamma_n$  bekannt,  $\rightarrow$  ges.: Wurzel, etc.)

(cumulative distribution function of  $\sqrt{\gamma}$  ( $\stackrel{\text{i.i.d.}}{=} \sqrt{\gamma_n}$ ) (cdf))

$$\begin{aligned} P_{\sqrt{\gamma}}(x) &= \Pr\{\sqrt{\gamma} \leq x\} = \Pr\{\gamma \leq x^2\} = P_\gamma(x^2) = \text{cdf of } \gamma \\ \rightarrow p_{\sqrt{\gamma}}(x) &= \frac{d}{dx} P_{\sqrt{\gamma}}(x) = 2x \cdot p_\gamma(x^2) = \frac{2x}{\bar{\gamma}} + O\left(\frac{1}{\bar{\gamma}}\right) \end{aligned}$$

- Laplace Transformation to MGF

$$\rightarrow M_{\sqrt{\gamma}}(s) = \mathcal{L}\{p_{\sqrt{\gamma}}(x)\} = \frac{2}{\bar{\gamma}} \cdot \frac{1}{s^2} + O\left(\frac{1}{\bar{\gamma}}\right)$$

$$\sqrt{\gamma_t} = \sum_{n=1}^{N_R} \frac{\sqrt{\gamma_n}}{N_R}$$

$$\begin{aligned} M_{\sqrt{\gamma_t}}(s) &= \mathcal{E}\left\{\exp(-s\sqrt{\gamma_t})\right\} = \mathcal{E}\left\{\exp\left(-\frac{s}{\sqrt{N_R}} \cdot \sum_{n=1}^{N_R} \sqrt{\gamma_n}\right)\right\} = \left(\mathcal{E}\left\{\exp\left(-\frac{s}{\sqrt{N_R}} \cdot \sqrt{\gamma_n}\right)\right\}\right)^{N_R} \\ &= \left(M_{\sqrt{\gamma}}\left(\frac{s}{\sqrt{N_R}}\right)\right)^{N_R} = \left(\frac{2}{\bar{\gamma}} \cdot \frac{N_R}{s^2}\right)^{N_R} + O\left(\frac{1}{\bar{\gamma}^{N_R}}\right) \end{aligned}$$

- inverse Laplace Transform

$$\begin{aligned} p_{\sqrt{\gamma_t}}(x) &= \mathcal{L}^{-1}\left\{M_{\sqrt{\gamma_t}}(s)\right\} = \left(\frac{2N_R}{\bar{\gamma}}\right)^{N_R} \cdot \frac{x^{2N_R-1}}{(2N_R-1)!} + O\left(\frac{1}{\bar{\gamma}^{N_R}}\right) \\ P_{\gamma_t}(x) &= \Pr\{\gamma_t \leq x\} = \Pr\{\sqrt{\gamma_t} \leq \sqrt{x}\} = P_{\sqrt{\gamma_t}}(\sqrt{x}) \rightarrow \text{cdf of } \sqrt{\gamma_t} \\ p_{\gamma_t}(x) &= \frac{d}{dx} P_{\gamma_t}(x) = \frac{1}{2\sqrt{x}} \cdot p_{\gamma_t}(\sqrt{x}) = \frac{1}{2} \left(\frac{2N_R}{\bar{\gamma}}\right)^{N_R} \cdot \frac{x^{N_R-1}}{(2N_R-1)!} + O(\bar{\gamma}^{-N_R}) \\ \rightarrow M_{\gamma_t}(s) &= \mathcal{L}\{p_{\gamma_t}(x)\} = \frac{1}{2} \left(\frac{2N_R}{\bar{\gamma}}\right)^{N_R} \cdot \frac{(N_R-1)!}{(2N_R-1)!} \frac{1}{b^{N_R}} + O(\bar{\gamma}^{-N_R}) \end{aligned}$$

- Error Probability:

$$\begin{aligned}
P_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t} \left( \frac{b}{2 \sin^2(\theta)} \right) d\theta \\
&= \frac{a}{\pi} \frac{1}{2} \left( \frac{2N_R}{\bar{\gamma}} \right)^{N_R} \frac{(N_R - 1)!}{(2N_R - 1)!} \frac{2^{N_R}}{b^{N_R}} \underbrace{\int_0^{\frac{\pi}{2}} \sin^{2N_R}(\theta) d\theta}_{\frac{\pi}{2^{2N_R+1}} \binom{2N_R}{N_R} = \frac{\pi (2N_R)!}{2^{2N_R+1} (N_R!)^2}} + O \left( \frac{1}{\bar{\gamma}^{N_R}} \right) \\
&= \frac{aN_R^{N_R}}{2b^{N_R} N_R!} \frac{1}{\bar{\gamma}^{N_R}} + O \left( \frac{1}{\bar{\gamma}^{N_R}} \right) \stackrel{!}{=} \left( \frac{1}{G_c} \right)^{G_d} \\
&\implies \text{Diversity gain: } G_d = N_R \\
&\implies \text{Combining gain: } G_c = \frac{b}{N_R} \left( \frac{2N_R!}{a} \right)^{\frac{1}{N_R}}
\end{aligned}$$

vergleiche auch Blatt mit Kurven III und IV

A similar asymptotic analysis can be conducted for the outage probability.

## 1.4 SC (Selection Combining)

### Combining weights

- only the strongest branch is chosen
- strongest branch:  $\hat{n} = \underset{n}{\operatorname{argmax}} \gamma_n \longrightarrow \gamma_t = \gamma_{\hat{n}}$
- only on RF receiver chain required  $\rightarrow$  saves hardware complexity

### Performance analysis

- cdf of:  $\gamma_t$

$$\begin{aligned}
P_{\gamma_t}(x) &= \Pr\{\gamma_{\hat{n}} \leq x\} = \Pr\{\gamma_1 \leq x \cap \gamma_2 \leq x \cap \dots \cap \gamma_{N_R} \leq x\} \\
&\stackrel{(IID)}{=} \left( \Pr\{\gamma_n \leq x\} \right)^{N_R} = \left( P_{\gamma}(x) \right)^{N_R}
\end{aligned}$$

- pdf:

$$\begin{aligned}
p_{\gamma_t}(x) &= \frac{d}{dx} P_{\gamma_t}(x) = N_R (P_{\gamma}(x))^{N_R-1} \cdot p_{\gamma}(x) \\
\text{where: } p_{\gamma_t}(x) &= \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0 \\
P_{\gamma}(x) &= \int_0^x p_{\gamma}(x) dx = 1 - e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0 \\
\rightarrow p_{\gamma_t}(x) &= \frac{N_R}{\bar{\gamma}} \left( 1 - e^{-\frac{x}{\bar{\gamma}}} \right)^{N_R-1} e^{-\frac{x}{\bar{\gamma}}}; \quad x \geq 0
\end{aligned}$$



### Error probability

- direct approach  $\rightarrow$  closed-form solution possible
- MGF approach
  - Binomial expansion

$$\begin{aligned} p_{\gamma_t}(x) &= \frac{N_R}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} 1^{N_R-1-n} \left(-e^{-\frac{x}{\bar{\gamma}}}\right)^n \\ &= \frac{N_R}{\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} \cdot (-1)^n e^{-\frac{x(n+1)}{\bar{\gamma}}}; \quad x \geq 0 \end{aligned}$$

- MGF

$$M_{\gamma_t}(s) = \frac{N_R}{\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} (-1)^n \frac{1}{s + \frac{n+1}{\bar{\gamma}}}$$

–

$$\begin{aligned} P_e &= \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\left(\frac{b}{2\sin^2\theta}\right) d\theta = \frac{aN_R}{\pi\bar{\gamma}} \sum_{n=0}^{N_R-1} \binom{N_R-1}{n} (-1)^n \int_0^{\frac{\pi}{2}} \frac{d\theta}{\frac{b}{2\sin^2\theta} + \frac{n+1}{\bar{\gamma}}} \\ &\rightarrow \text{can be evaluated numerically} \end{aligned}$$

- high SNR approach  $\Rightarrow \bar{\gamma} \rightarrow \infty$

$$\begin{aligned} p_{\gamma_t} &= \frac{N_R}{\bar{\gamma}} \left[1 - \exp\left(-\frac{x}{\bar{\gamma}}\right)\right]^{N_R-1} \exp\left(-\frac{x}{\bar{\gamma}}\right) \\ &\stackrel{\bar{\gamma} \rightarrow \infty}{\approx} \frac{N_R}{\bar{\gamma}} \left[1 - \left(1 - \frac{x}{\bar{\gamma}} + O(\bar{\gamma}^{-1})\right)\right]^{N_R-1} \left(1 - \frac{x}{\bar{\gamma}} + O(\bar{\gamma}^{-1})\right) \\ &= \frac{N_R}{\bar{\gamma}^{N_R}} x^{N_R-1} + o(\bar{\gamma}^{-N_R}) \end{aligned}$$

- MGF:

$$\begin{aligned} M_{\gamma_t}(s) &= \frac{N_R}{\bar{\gamma}^{N_R}} \frac{(N_R-1)!}{s^{N_R}} + O(\bar{\gamma}^{-N_R}) \\ \left[\rightarrow P_e = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_t}\left(\frac{b}{2\sin^2(\theta)}\right) d\theta\right] \\ &= \frac{a(2N_R)!}{b^{N_R} 2^{N_R+1} N_R!} \frac{1}{\bar{\gamma}^{N_R}} + O(\bar{\gamma}^{-N_R}) \end{aligned}$$

$\Rightarrow$  Diversity gain:  $G_d = N_R$

$\Rightarrow$  Combining gain:  $G_c = 2b \left(\frac{2N_R!}{a(2N_R)!}\right)^{\frac{1}{N_R}}$

– Outage Probability

$$P_{out} = \Pr\{\gamma_{\hat{n}} \leq \gamma_T\} = P_{\gamma_{\hat{n}}}(\gamma_T) = \left[1 - \exp\left(-\frac{\gamma_T}{\bar{\gamma}}\right)\right]^{N_R}$$

$$\text{high SNR: } P_{out} = \left(\frac{\gamma_T}{\bar{\gamma}}\right)^{N_R} + O(\bar{\gamma}^{-N_R})$$

## 1.5 Comparison

- Diversity Gain:  
MRC, EGC and SC all achieve the maximum possible diversity gain of  $G_d = N_R$
- Combining Gain:  
The combining gains of MRC, EGC and SC are different
  - MRC/EGC:

$$\frac{G_C^{EGC}}{G_C^{MRC}} = \frac{\frac{1}{2b} \left(\frac{a}{2} \binom{2N_R}{N_R}\right)^{\frac{1}{N_R}}}{\frac{N_R}{b} \left(\frac{a}{2} \frac{1}{N_R!}\right)^{\frac{1}{N_R}}} = \frac{[(2N_R)!]^{\frac{1}{N_R}}}{2N_R(N_R)^{\frac{1}{N_R}}} \leq 1$$

(independent of a or b which are modulation parameters, only depends on number of antennas)

$$N_R \gg 1 : \quad N_R! \approx \sqrt{2\pi} e^{-N_R} N_R^{N_R + \frac{1}{2}} \quad (\text{Stirling})$$

$$\left. \frac{G_C^{EGC}}{G_C^{MRC}} \right|_{N_R \gg 1} = \frac{\left(\sqrt{2\pi} e^{-2N_R} (2N_R)^{2N_R + \frac{1}{2}}\right)^{\frac{1}{N_R}}}{2N_R \left(\sqrt{2\pi} e^{-N_R} N_R^{N_R + \frac{1}{2}}\right)^{\frac{1}{N_R}}} = \frac{2 \cdot 2^{\frac{1}{2N_R}}}{2} \xrightarrow{N_R \rightarrow \infty} \frac{2}{e} \equiv -1.3\text{dB}$$

– MRC/SC:

$$\frac{G_C^{SC}}{G_C^{MRC}} = \frac{2b \left(\frac{a}{2} \binom{2N_R}{N_R}\right)^{\frac{1}{N_R}}}{2b \left(\frac{a}{2} \frac{(2N_R)!}{N_R!}\right)^{\frac{1}{N_R}}} = \frac{1}{(N_R!)^{\frac{1}{N_R}}} \leq 1$$

$$\left. \frac{G_C^{SC}}{G_C^{MRC}} \right|_{N_R \gg 1} = \frac{1}{\sqrt{2\pi}^{\frac{1}{N_R}} e^{-1} N_R^{1 + \frac{1}{2N_R}}} N_R \xrightarrow{} \infty \frac{e}{N_R}$$

→ loss increases with  $N_R$

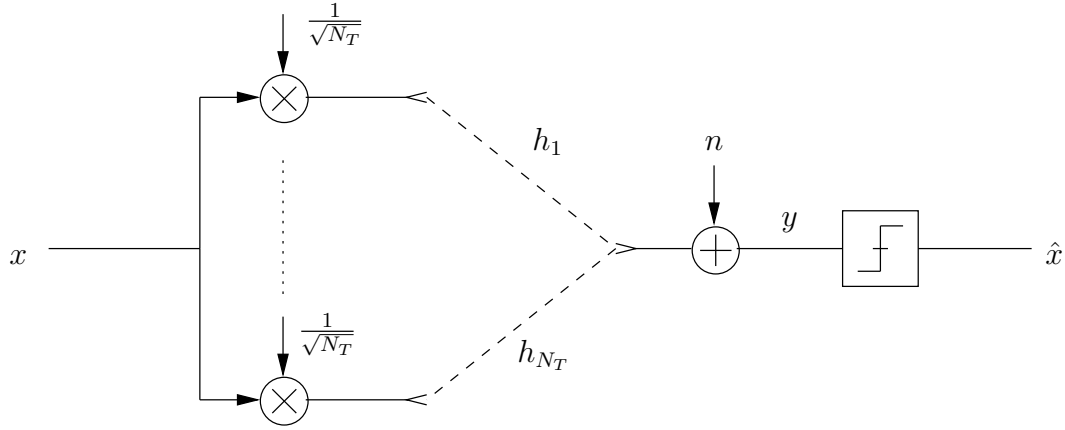
## 2 MISO Systems

### Remarks

- Similar to SIMO systems, in MISO systems only coding and diversity gains can be obtained.
- To realize these gains, a careful transmitter design is necessary
- System design depends on whether or not channel state information (**CSI**) is available at transmitter

### 2.1 Naive Approach

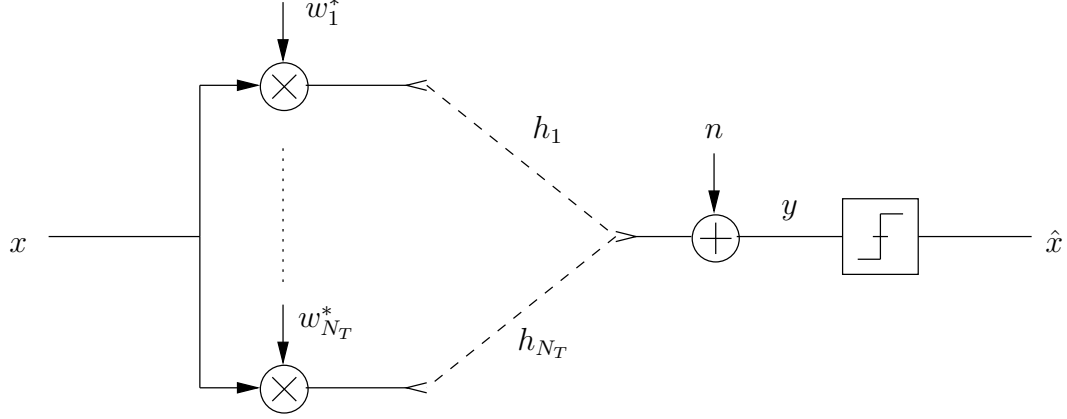
- Assume we simply send the same signal over all  $N_T$  transmit antennas



- Transmit power:  $\mathcal{E} \left\{ \left| \frac{1}{\sqrt{N_T}} x \right|^2 + \dots + \left| \frac{1}{\sqrt{N_T}} x \right|^2 \right\} = \mathcal{E} \left\{ N_T \frac{1}{N_T} |x|^2 \right\} = E_s$
- Received signal:  $y = \frac{1}{\sqrt{N_T}} \sum_{n=1}^{N_T} h_n \cdot x + n$
- Rayleigh fading:  $h_n$  are zero mean complex gaussian random variables  
 $\rightarrow h$  is also zero mean complex gaussian
- i.i.d.:
  - $\mathcal{E}\{|h_n|^2\} = 1 \ \forall n$
  - $\mathcal{E}\{|h|^2\} = \frac{1}{N_T} \mathcal{E} \left\{ \left| \sum_{n=1}^{N_T} h_n \right|^2 \right\} = \frac{1}{N_T} \mathcal{E} \left\{ \sum_{n=1}^{N_T} |h_n|^2 \right\} = 1$
  - statistical properties of  $h$  are independent of  $N_T$
  - the multiple transmit antennas have no benefit at all
  - more sophisticated transmitter designs necessary

## 2.2 Full CSI Available at the Transmitter

- $h_n, n \in \{1, \dots, N_T\}$  is known at the transmitter
- Perform “precoding” (beamforming) with coefficients  $w_n$



- Transmit Power: Two constraints maybe considered
  - Average transmit power constraint

$$P_{av} = \mathcal{E} \left\{ \sum_{n=1}^{N_T} |w_n^* x|^2 \right\} = \sum_{n=1}^{N_T} |w_n|^2 \underbrace{\mathcal{E}\{|x|^2\}}_{E_s} = \mathcal{E}_s \Rightarrow \sum_{n=1}^{N_T} |w_n|^2 = 1$$

- Power constraint for each transmit antenna

$$\rightarrow |w_n| = \frac{1}{\sqrt{N_T}} \quad \rightarrow P_{av} = E_s$$

- Received signal:  $y = \underbrace{\sum_{n=1}^{N_T} w_n^* h_n}_{h} x + n$  (equivalent SISO channel)

### Maximum Ratio Transmission (MRT)

- we have only the average power constraint:  $\sum_{n=1}^{N_T} |w_n|^2 = 1$

$$\bullet \text{ SNR: } \gamma_t = \frac{E_s |h|^2}{\sigma_n^2} = \frac{\mathcal{E}_s \left| \sum_{n=1}^{N_T} w_n^* \cdot h_n \right|^2}{\sigma_n^2}$$

- Maximize SNR under constraint  $\sum_{n=1}^{N_T} |w_n|^2 = 1$

- constraint optimization problem  $\rightarrow$  Lagrange method

$$L = \frac{E_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} w_n^* \cdot h_n \right|^2 + \lambda \left( \sum_{n=1}^{N_T} |w_n|^2 - 1 \right); \quad \text{where: } \lambda = \text{Lagrange Multiplier}$$

$\Rightarrow$  Wirtinger Kalkül: treat  $z$  and  $z^*$  as independent variables for differentiation:

$$\begin{aligned} \frac{\partial z^*}{\partial z} &= 0; & \frac{\partial |z|^2}{\partial z} &= \frac{\partial z \cdot z^*}{\partial z} = z^* \\ \frac{\partial x^2}{\partial x} &= 2x; & \frac{\partial (z^*)^2}{\partial z^*} &= 2 \cdot z^*; & \frac{\partial |z|^2}{\partial z^*} &= z^* \end{aligned}$$

$$\frac{\partial L}{\partial w_m^*} = \frac{\epsilon_s}{\sigma_n^2} \left( \sum_{n=1}^{N_T} w_n^* \cdot h_n \right)^* h_m + \lambda w_m$$

$$\rightarrow w_m = \frac{\epsilon_s}{\sigma_n^2 \cdot \lambda} \left( \sum_{n=1}^{N_T} w_n^* h_n \right)^* h_m$$

const., independent of  $m := c$

$$\rightarrow w_m = c \cdot h_m$$

$$\rightarrow \sum_{n=1}^{N_T} |w_n|^2 = 1 \rightarrow c^2 = \frac{1}{\sum_{n=1}^{N_T} |h_n|^2}$$

$$\rightarrow w_n = \frac{h_n}{\sqrt{\sum_{n=1}^{N_T} |h_n|^2}} \equiv \text{MRT gains}$$

$$\rightarrow \text{SNR} = \frac{E_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} \frac{|h_n|^2}{\sqrt{\sum_{n=1}^{N_T} |h_n|^2}} \right|^2 = \frac{\epsilon_s}{\sigma_n^2} \sum_{n=1}^{N_T} |h_n|^2$$

$\Rightarrow$  same SNR as for maximum ratio combining (MRC)

$\Rightarrow$  MRT with  $N_T$  transmit antennas achieves the same performance as MRC with  $N_T$  receive antennas

$\Rightarrow$  MRT/MRC can be extended to  $N_T \times N_R$  MIMO systems

$\rightarrow$  has the same performance as MRC with  $N_T \cdot N_R$  receive antennas and one transmit antenna

### Equal Gain Transmission (EGT)

- we employ gains:  $w_n = \frac{1}{\sqrt{N_T}} \cdot \frac{h_n}{|h_m|} \rightarrow |w_n| = \frac{1}{\sqrt{N_T}}$

- SNR:

$$\begin{aligned}
\gamma_t &= \frac{E_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} w_n^* h_n \right|^2 \\
&= \frac{E_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} \frac{1}{\sqrt{N_T} \cdot \frac{|h_n|^2}{|h_n|}} \right|^2 = \frac{1}{N_T} \cdot \frac{E_s}{\sigma_n^2} \left| \sum_{n=1}^{N_T} |h_n| \right|^2 \\
\gamma_n &= \frac{E_s}{\sigma_n^2} |h_n|^2 \\
\text{same SNR as for EGC} &\rightarrow \gamma_t = \frac{1}{N_T} \left| \sum_{n=1}^{N_T} \sqrt{\gamma_n} \right|^2
\end{aligned}$$

→ EGC with  $N_T$  transmit antennas achieves the same performance as EGC with  $N_T$  receive antennas

### Transmit Antennas Selection

- select antenna with maximum channel gain for transmission:

$$w_n = \begin{cases} \frac{h_n}{|h_n|}, & \text{if } n = \hat{n} \\ 0, & \text{otherwise} \end{cases} \quad \text{where } \hat{n} = \underset{n}{\operatorname{argmax}} |h_n|$$

- antenna selection with  $N_T$  transmit antennas achieves the same performance as *Selection Combining* with  $N_T$  receive antennas

## 2.3 No CSI at Transmitter - Space-Time-Coding

- $h_n, n \in \{1, \dots, N_T\}$ , is only known at the receiver
- “Space-time-coding” has to be employed to realize diversity gain
- $T \times N_T$  matrices  $\mathbf{X}$  are transmitted in  $T$  symbol intervals over  $N_T$  antennas
- $\mathbf{X}$  is drawn from a matrix alphabet  $\mathcal{X}$
- Example:

$$\mathbf{X} = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,N_T} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,N_T} \\ \vdots & \vdots & \ddots & \vdots \\ x_{T,1} & x_{T,2} & \cdots & x_{T,N_T} \end{pmatrix}$$

- We distinguish:
  - Space-time-block-codes (STBCs)
    - $\mathbf{X}$  is obtained by mapping  $K$  scalar symbols  $s_k, k = 1, \dots, K$  from a scalar alphabet  $\mathcal{A}$  to matrix  $\mathbf{X}$

- Space-time-trellis-codes (STTCs)
  - $\mathbf{X}$  is obtained from scalar symbols  $s_k$  through a trellis encoding process.
  - [see: Tarokh, Seshadri, Calderbank: Space-time-codes for high datarate wireless communication: Performance criteria and coder construction; IEEE Trans. Inf. Theory 1998]
- here: We concentrate on space-time-block-codes (STBCs), but many results can be easily extended to space-time-trellis-codes
- STBCs:
  - $K$   $M$ -ary scalar symbols (e.g.  $M$ -PSK symbols) are mapped to STBC matrices  $\mathbf{X}$ 
 $\mathbf{S} = [s_1, \dots, s_K] \rightarrow \mathbf{X}$ 
 $s_k \in \mathcal{A} \rightarrow x \in \mathcal{X}$  with  $|\mathcal{X}| = M^K$
  - Example: “Alamouti”-Code

$$\mathbf{X} = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix}$$

[Alamouti: A simple transmit diversity technique for wireless communication, IEEE JSAC 1998]

## Optimal Detection

- Signal model:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_T \end{pmatrix} = \mathbf{X} \begin{pmatrix} h_1 \\ \vdots \\ h_{N_T} \end{pmatrix} + \begin{pmatrix} n_1 \\ \vdots \\ n_T \end{pmatrix}$$

$$\mathbf{y} = \mathbf{X} \cdot \mathbf{h} + \mathbf{n}$$

- Optimal detection - ML-detection

- $\mathbf{h}$  is known at receiver
- $\mathbf{n}$  is AWGN with  $\mathcal{E}\{\mathbf{n} \cdot \mathbf{n}^H\} = \sigma_n^2 \cdots \mathbf{I}_{T \times T}$
- $p(\mathbf{y}|\mathbf{x})$

$$\begin{aligned} &= \frac{1}{\pi^T |\sigma_n^2 \mathbf{I}_{T \times T}|} \exp \left( -(\mathbf{y} - \mathbf{xh})^H (\sigma_n^2 \mathbf{I}_{T \times T})^{-1} (\mathbf{y} - \mathbf{xh}) \right) \\ &= \frac{1}{\pi^T \sigma_n^{2T}} \exp \left( -\frac{1}{\sigma_n^2} (\mathbf{y} - \mathbf{xh})^H (\mathbf{y} - \mathbf{xh}) \right) = \frac{1}{\pi^T \sigma_n^{2T}} \exp (||\mathbf{y} - \mathbf{xh}||^2) \end{aligned}$$

→ the optimal estimate  $\hat{\mathbf{X}}$  or equivalently the optimal estimate  $\hat{\mathbf{s}}$  can be obtained as

$$\hat{\mathbf{s}} = \underset{\mathbf{s} \in \mathcal{A}^K}{\operatorname{argmax}} p(\mathbf{y}|\mathbf{x}) = \underset{\mathbf{s} \in \mathcal{A}^K}{\operatorname{argmin}} ||\mathbf{y} - \mathbf{hx}||^2$$

- Disadvantage: In general, metric  $||\mathbf{y} - \mathbf{hx}||^2$  has to be calculated  $M^K$  times
  - complexity increases exponentially with  $K$

## Types of STBCs

- Orthogonal STBCs (OSTBCs)

- OSTBCs are a special class of STBCs which allow independent detection of each  $s_k \rightarrow$  only  $K \cdot M$  metrics have to be evaluated

- Rate STBCs:  $R_{STBC} = \frac{K}{T}$

- Examples:

- \* Alamouti Code ( $K = 2, T = 2$ )  $\rightarrow R_{STBC} = 1$

$$\mathbf{X} = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix} \begin{matrix} \updownarrow T \\ \longleftrightarrow N_T \end{matrix}$$

$\rightarrow$  only “full rate” OSTBC for complex  $s_k$

- \*  $N_T = 3, K = 3, T = 4$

$$\mathbf{X} = \frac{1}{\sqrt{3}} \begin{pmatrix} s_1 & s_2 & s_3 \\ -s_2^* & s_1^* & 0 \\ s_3^* & 0 & -s_3^* \\ 0 & -s_3^* & s_2^* \end{pmatrix} \rightarrow R_{STBC} = \frac{K}{T} = \frac{3}{4}$$

- Orthogonality:  $\mathbf{X}^H \mathbf{X} = \text{const} \cdot \mathbf{I}_{N_T \times N_T}$

- Independent detection of  $s_1$  &  $s_2$  for Alamouti Code

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \\ \rightarrow \underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{\tilde{\mathbf{y}}} = \frac{1}{\sqrt{2}} \underbrace{\begin{pmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{pmatrix}}_{\tilde{\mathbf{F}}} \underbrace{\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}}_{\mathbf{s}} + \underbrace{\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}}_{\tilde{\mathbf{n}}}$$

(Anmerkung: nur  $\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$  gewünscht, nicht:  $s_1^*, s_2^*$ )

$$\mathbf{F}^H \mathbf{F} = \frac{1}{2} \begin{pmatrix} h_1^* & h_2 \\ h_2^* & -h_1^* \end{pmatrix} \begin{pmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{pmatrix} = \frac{1}{2} \begin{pmatrix} |h_1|^2 + |h_2|^2 & 0 \\ 0 & |h_1|^2 + |h_2|^2 \end{pmatrix}$$

$\rightarrow \frac{\sqrt{2}}{\sqrt{|h_1|^2 + |h_2|^2}} \cdot \mathbf{F}$  is unitary matrix

$\rightarrow \frac{2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{y}} = \mathbf{s} + \frac{2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{n}}$

( $\frac{2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{n}}$  is AWGN vector with covariance matrix  $\frac{2\sigma_n^2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{I}_{T \times T}$ )

$\rightarrow$  ML decision:  $\hat{\mathbf{s}} = \underset{\mathbf{s}}{\text{argmin}} \left\| \frac{2}{|h_1|^2 + |h_2|^2} \cdot \mathbf{F}^H \cdot \tilde{\mathbf{y}} - \mathbf{s} \right\|^2$

$\rightarrow$  independent ML decoding

$$\hat{s}_1 = \underset{s_1}{\text{argmin}} \left| s_1 - \frac{h_1^* y_1 + h_2 y_2^*}{\frac{1}{\sqrt{2}}(|h_1|^2 + |h_2|^2)} \right| \\ \hat{s}_2 = \underset{s_2}{\text{argmin}} \left| s_2 - \frac{h_1^* y_1 - h_2 y_2^*}{\frac{1}{\sqrt{2}}(|h_1|^2 + |h_2|^2)} \right|$$



- independent decoding property can be proved for all OSTBCs
- low complexity is at the expense of a rate-loss compared to other STBCs for  $N_T > 2$ 
  - Frequenzhopping
  - keine Kanalinformation aus vorher empfangenen Symbolen möglich  $\Rightarrow$  Kanal ändert sich ständig: nur Entscheidung, ob Rauschen oder Signal + Rauschen
- Performance Analysis of Alamouti Code
  - Decision-variables after combining

$$r_1 = \sqrt{2} \frac{h_1^* y_1 + h_2 y_2^*}{|h_1|^2 + |h_2|^2}$$

$$r_2 = \sqrt{2} \frac{h_1^* y_1 - h_2 y_2^*}{|h_1|^2 + |h_2|^2}$$

because of symmetry it suffices to consider  $r_1$

$$r_1 = \sqrt{2} \frac{h_1^* \left( \frac{1}{\sqrt{2}} s_1 h_1 + \frac{1}{\sqrt{2}} h_2 s_2 + n_1 \right) + h_2 \left( -\frac{1}{\sqrt{2}} h_2 s_1^* + \frac{1}{\sqrt{2}} h_1 s_2^* + n_2 \right)^*}{|h_1|^2 + |h_2|^2}$$

$$= \sqrt{2} \frac{\frac{1}{\sqrt{2}} (|h_1|^2 + |h_2|^2) s_1 + h_1^* n_1 + h_2 n_2^*}{|h_1|^2 + |h_2|^2}$$

$$= 1 \cdot s_1 + n_{eq}$$

where

$$n_{eq} = \sqrt{2} \frac{h_1^* n_1 + h_2 n_2^*}{|h_1|^2 + |h_2|^2}$$

$$\text{SNR} \rightarrow \gamma_t = \frac{E_s \cdot 1^2}{\sigma_{eq}^2} \quad \text{with} \quad \mathcal{E}\{|s_1|^2\} = \mathcal{E}_s$$

$$\sigma_{eq}^2 = 2 \frac{|h_1|^2 \sigma_n^2 + |h_2|^2 \sigma_{eq}^2}{(|h_1|^2 + |h_2|^2)^2} = \frac{2\sigma_n^2}{|h_1|^2 + |h_2|^2}$$

- $\gamma_t = \frac{1}{2} \frac{E_s}{\sigma_n^2} (|h_1|^2 + |h_2|^2)$
- $\text{SNR}_{\text{Alamouti}} = \frac{1}{2} \text{SNR}_{\text{MRC}} = \frac{1}{2} \text{SNR}_{\text{MRT}}$
- Alamouti code has diversity gain  $G_d = 2$
- Transmission with Alamouti STBC requires 3dB higher SNR to achieve same performance as MRT  $\rightarrow$  3dB loss in coding gain  $G_c$
- Lack of CSI knowledge at transmitter “costs” 3dB in power efficiency
- General:
  - OSTBCs achieve a diversity gain of  $G_d = N_T$  if only one receive antenna is available
  - if  $N_R$  receive antennas are available, MRC can be used at the receiver to yield a diversity gain of  $\underline{G_d = N_T N_R}$
- Other STBCs:

- Quasi orthogonal STBCs
  - \* higher rate than OSTBCs
  - \* only subset of symbols have to be decoded jointly
  - \* Example:  $K = N_T = T = 4$

$$\mathbf{X} = \frac{1}{2} \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^* & s_1^* & -s_4^* & s_3^* \\ -s_3^* & -s_4^* & s_1^* & s_2^* \\ s_4 & -s_3 & -s_2 & s_1 \end{pmatrix}$$

- \* Anmerkung 1:  $\mathbf{X}$  ist ähnlich zu Alamouti Code
- \* Anmerkung 2:  $\mathbf{X}^H \mathbf{X}$ : viele Nicht-diagonal Elemente sind Null; die, die ungleich Null sind, zeigen, welche Symbole gemeinsam entschlüsselt werden müssen
- Golden Code for  $N_T = N_R = 2$ : achieves a rate of  $R_{STBC} = 2$  and full diversity of  $G_d = N_T, N_R = 4$
- Differential STBCs:  $\mathbf{X}_k = \mathbf{X}_{k-1} \cdot \mathbf{D}_k$ .  $\mathbf{X}_k$  is transmitted,  $\mathbf{D}_k$  is transmitted
- Linear dispersion codes: designed to achieve high mutual information
- noncoherent STBCs (On-Off-Keying)

### Space Time Code Design

Given:

- Code  $\mathcal{X} = \{\mathbf{X}_1, \dots, \mathbf{X}_{|\mathcal{X}|}\}$
- Channel: IID Rayleigh-fading:
  - $h_n \sim \mathcal{CN}(0, 1)$ ;  $n \in \{1, 2, \dots, N_T\}$
  - AWGN  $n \sim \mathcal{CN}(0, \sigma_n^2)$

Problem: How should we design codebook  $\mathcal{X}$ ?

- Need to derive error rate for general codebooks  $\mathcal{X}$ !
  - Codeword error rate

$$P_e = \frac{1}{|\mathcal{X}|} \sum_{i=1}^{|\mathcal{X}|} \Pr\{\mathbf{x}_i \neq \hat{\mathbf{x}}_i\}$$

where  $\hat{\mathbf{x}}_i$  is the detected codeword and we assume that all codewords are equally likely

Problem:  $\Pr\{\mathbf{x}_i \neq \hat{\mathbf{x}}_i\}$  is not tractable in general

- Use union bound to upper bound  $\Pr\{\mathbf{x}_i \neq \hat{\mathbf{x}}_i\}$  as upper sum over pairwise error probabilities (PEP)  $\Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\}$  where it is assumed that  $\mathbf{x}_i$  was transmitted and  $\mathbf{x}_i$  and  $\mathbf{x}_j$  are the only codewords in the codebook

$$P_e \leq \frac{1}{|\mathcal{X}|} \sum_{i=1}^{|\mathcal{X}|} \sum_{j=1}^{|\mathcal{X}|} \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} \text{ where } j \neq i$$

Calculation of PEPs

Recall:  $\hat{\mathbf{x}} = \underset{\mathbf{x} \in \mathcal{X}}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{x}\mathbf{h}\|^2$

Now,  $\mathbf{x}_i$  and  $\mathbf{x}_j$  are the only alternatives and an error is made if  $\|\mathbf{y} - \mathbf{x}_i\mathbf{h}\|^2 > \|\mathbf{y} - \mathbf{x}_j\mathbf{h}\|^2$  since  $\mathbf{x}_i$  was sent but  $\mathbf{x}_j$  was detected

$$\begin{aligned} \rightarrow \|\mathbf{x}_i\mathbf{h} + \mathbf{n} - \mathbf{x}_j\mathbf{h}\|^2 &> \|\mathbf{x}_i\mathbf{h} + \mathbf{n} - \mathbf{x}_j\mathbf{h}\|^2 \\ \|\mathbf{n}\| &> \|(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h} + \mathbf{n}\|^2 \\ \rightarrow \|\mathbf{n}\| &> \underbrace{\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h} + \mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{n} + \mathbf{n}^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h} + \|\mathbf{n}\|^2}_{\Delta} \\ &\rightarrow \underbrace{-\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)^H\mathbf{n} - \mathbf{n}^H(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h}}_z > \Delta \end{aligned}$$

for given  $\mathbf{h}$ ,  $z$  is a gaussian random variable

$$\begin{aligned} \sigma_z^2 &= \mathcal{E}\{|z|^2\} = \mathcal{E}\{2\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j) \underbrace{\mathbf{n}\mathbf{n}^H}_{\sigma_n^2 \mathbf{I}}(\mathbf{x}_i - \mathbf{x}_j)\mathbf{h} + 2\mathbf{h}^H(\mathbf{x}_i - \mathbf{x}_j)^H \underbrace{\mathbf{n}\mathbf{n}^T}_{=0}(\mathbf{x}_i - \mathbf{x}_j)^* \mathbf{h}^*\} \\ &= 2\sigma_n^2 \Delta + 0 \end{aligned}$$

$$\begin{aligned} \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} &= \int_{\Delta}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{z^2}{2\sigma_z^2}\right) dz, \quad t = \frac{z}{\sigma_z} \\ &= \frac{1}{\sqrt{2\pi}} \int_{\frac{\Delta}{\sigma_z}}^{\infty} e^{-\frac{t^2}{2}} dt = Q\left(\frac{\Delta}{\sigma_z}\right) = Q\left(\frac{\Delta}{\sqrt{2\sigma_n^2 \Delta}}\right) \\ &= Q\left(\sqrt{\frac{\Delta}{2\sigma_n^2}}\right) \end{aligned}$$

- $\Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} = \mathcal{E}\left\{Q\left(\sqrt{\frac{\Delta}{2\sigma_n^2}}\right)\right\}$ 
  - to avoid cumbersome Q-function we use Chernoff bound:

$$Q(x) \leq \frac{1}{2} e^{-\frac{x^2}{2}}$$

$$\begin{aligned} \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} &\leq \frac{1}{2} \mathcal{E}_h \left\{ \exp\left(-\frac{\mathbf{h}^H \mathbf{Q} \mathbf{h}}{4\sigma_n^2}\right) \right\} \\ \text{where } \mathbf{Q} &= (\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j) \end{aligned}$$

- Eigendecomposition:  $\mathbf{Q} = \mathbf{U}^H \mathbf{\Lambda} \mathbf{U}$  with  $\mathbf{\Lambda} = \text{diag}\{\lambda_1, \dots, \lambda_r, 0, \dots, 0\}$   $r = \text{rank}\{\mathbf{Q}\}$

- Elements  $\mathbf{h}$  are i.i.d. Gaussian

–  $\underline{\beta} = \mathbf{U}\mathbf{h}$  has also i.i.d. Gaussian random variables as elements since  $\mathbf{U}$  is unitary matrix

$$- \mathbf{h}^H \mathbf{Q} \mathbf{h} = \underbrace{\mathbf{h}^H \mathbf{U}^H}_{\underline{\beta}^*} \mathbf{\Lambda} \underbrace{\mathbf{U} \mathbf{h}}_{\underline{\beta}} = \sum_{i=1}^r \lambda_i |\beta_i|^2 \text{ with } \underline{\beta} = [\beta_1, \dots, \beta_{N_T}]$$

$$\begin{aligned} \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} &= \frac{1}{2} \mathcal{E}_{\underline{\beta}} \left\{ \exp \left( -\frac{\sum_{i=1}^r \lambda_i |\beta_i|^2}{4\sigma_n^2} \right) \right\} \\ &= \frac{1}{2} \mathcal{E}_{\underline{\beta}} \left\{ \prod_{i=1}^r e^{-\frac{\lambda_i}{4\sigma_n^2} |\beta_i|^2} \right\} \\ &= \frac{1}{2} \prod_{i=1}^r \mathcal{E}_{\beta_i} \left\{ e^{-\frac{\lambda_i}{4\sigma_n^2} |\beta_i|^2} \right\} \\ &= \frac{1}{2} \prod_{i=1}^r \mathcal{E}_{|\beta_i|^2} \left\{ e^{-\frac{\lambda_i}{4\sigma_n^2} |\beta_i|^2} \right\} \triangleq \text{MGF of exponentially distributed variable } \alpha_i = |\beta_i|^2 \end{aligned}$$

$$\rightarrow P_{\alpha_i}(x) = e^{-x}, \quad x \geq 0$$

$$\begin{aligned} \rightarrow \Pr\{\mathbf{x}_i \rightarrow \mathbf{x}_j\} &\leq \frac{1}{2} \prod_{i=1}^r \frac{1}{1 + \frac{\lambda_i}{4\sigma_n^2}} \\ &\leq \prod_{i=1}^r \frac{1}{\frac{\lambda_i}{4\sigma_n^2}} = 2^{2r-1} \frac{1}{\prod_{i=1}^r \lambda_i} \left( \frac{1}{\underbrace{\sigma_n^2}_{\triangleq SNR}} \right)^{-r} \end{aligned}$$

- upper bound on  $P_e$ :

$$\lambda_n(i, j) = n\text{th eigenvalue of } (\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j)$$

$$r(i, j) = \text{rank of } (\mathbf{x}_i - \mathbf{x}_j)^H (\mathbf{x}_i - \mathbf{x}_j)$$

$$\rightarrow P_e \leq \frac{1}{|\mathcal{X}|} \sum_{i=1}^{|\mathcal{X}|} \sum_{j=1}^{|\mathcal{X}|} 2^{2r(i,j)-1} \frac{1}{\prod_{n=1}^{r(i,j)} \lambda_n(i, j)} \left( \frac{1}{\sigma_n^2} \right)^{-r(i,j)}$$

- generally loose bound but offers significant insight for code design

**Two criteria:**

**Rank criterion:** The diversity gain of a ST code is given by  $G_d = \min_{i,j} (r(i,j)) = \min_{i,j} \text{rank}((\mathbf{x}_i - \mathbf{x}_j)^H(\mathbf{x}_i - \mathbf{x}_j))$   
→ Design code such that minimum rank of all possible matrices  $(\mathbf{x}_i - \mathbf{x}_j)^H(\mathbf{x}_i - \mathbf{x}_j)$  is maximized

$$T \overset{N_T}{\updownarrow} \mathbf{X}_i \Rightarrow r(i,j) = N_T \quad \forall i \neq j$$

**Determinant criterion:** To maximize the coding gain among all codes with  $r(i,j) = N_T$ ,

we need to maximize  $\max_{i,j} \min_{n=1}^{N_T} \lambda_n(i,j) = \max_{i,j} \min_{i,j} |(\mathbf{x}_i - \mathbf{x}_j)^H(\mathbf{x}_i - \mathbf{x}_j)| \quad \forall i \neq j$

- Rank and determinant criterion can be used for the search for good space-time block codes and space-time trellis codes. These two criteria were first derived by Tarokh, et. al. 1998.
- diversity increases to  $N_T N_R$  if  $N_K$  receive antennas are available
- Example: see B  ro, Bauch, Hansmann: Improved codes for space-time trellis coded modulation. IEEE Comm. Letters, 2000.