MIMO Skript - Wintersemester 2013
Kapitel 3

Inhaltsverzeichnis

3 Multiuser MIMO

- We distinguish two cases:
  - multipoint - to - point transmission
  - point - to - multipoint transmission
- Multipoint - to - point transmission
  - typical uplink scenario in cellular systems
  - information theoretical channel model: Multiple Access Channel (MAC)

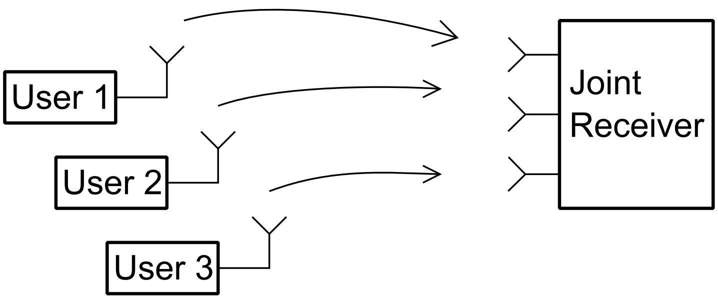


Abbildung 1: Multiple Access Channel

- Point - to -multipoint transmission

– typical downlink scenarion in cellular systems
– information theoretical channel model: Broadcast Channel (BC)

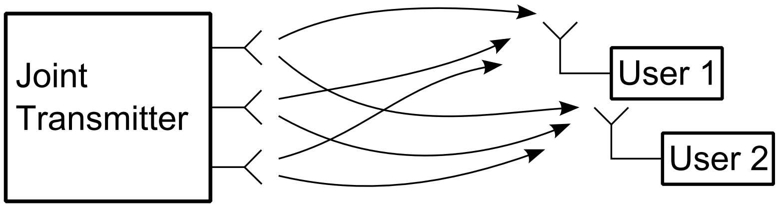


Abbildung 2: Broadcast Channel

- Advantage of multiuser MIMO compared to point-to-point MIMO
  - multiplexing gain can be exploited even if users have only single antenna
  - users are spatially distributed in cell → channels to different users are independent

3.1 Multiple Access Channel (MAC)

We consider two aspects:

- Detector structures
- Rate region

3.1.1 Detector structures

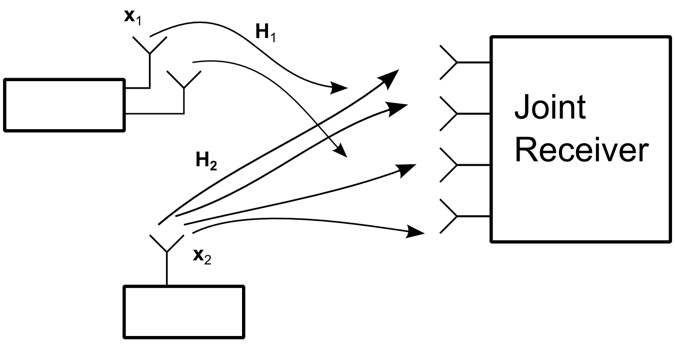


Abbildung 3: Block Diagramm of Channel Model

Channel model: → general MAC: $\mathbf{y} = \sum_{k=1}^{K} \mathbf{H}_k \mathbf{x}_k + \mathbf{n}$
with:

- K users
- user k has $N_{T,k}$ transmit antennas
- N_R receive antennas

- $\mathbf{H}_k \in \mathbb{C}^{N_R \times N_{T,k}}$

$$\mathbf{y} = \underbrace{\begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 & \dots & \mathbf{H}_k \end{bmatrix}}_{\mathbf{H}} \cdot \underbrace{\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_k \end{bmatrix}}_{\mathbf{x}} + \mathbf{n}$$

Observation:

- same equivalent channel model as for a point-to-point MIMO system transmitting $N_T = \sum_{k=1}^{K} N_{T,k}$ independent signal streams *(Anmerkung: kein Unterschied für Empfänger, ob Signale von einem Nutzer oder von mehreren)*
- the receiver (e.g. base station) can use detection schemes as for point-to-point MIMO systems
  - linear receiver
  - DFG
  - sphere decoder

Typical problems in uplink multiuser MIMO For given receiver structure:

- calculate SNR_k for all users k based on the expressions developed in Chapter 2.4
- optimize transmit power of users, $E_k = \mathcal{E}\{||x_k||^2\}$ for maximization of the sumrate or maximization of the minimum SNR_k *(Anmerkung: Maximierung der sumrate kann durch Maximierung des SNR des Users mit bestem Kanal erfolgen, aber: unfair anderen Usern gegenüber ⇒ starving)*

3.1.2 Rate region

For point-to-point links, we can decode error free, if the rate, R, meets

a) SISO $R < \log_2\left(1 + \frac{\mathcal{E}_s}{\sigma_n^2}\right)$

b) MIMO $R < \log_2 \underbrace{\left|\mathbf{I} + \frac{\mathcal{E}_s}{N_T \sigma_n^2} \mathbf{H}\mathbf{H}^H\right|}_{\det}$

Questions: What happens if there are multiple users?

Rate Region for Single Antenna Users and Receivers

- Gaussian channel
- $N_R = N_{T,k} = 1 \forall \quad k$

- received signal:

$$y = \sum_{k=1}^{K} x_k + n$$
$$*\mathcal{E}_k = \mathcal{E}\{||x_k||^2\}$$
$$*\sigma_n^2 = \{||n||^2\}$$

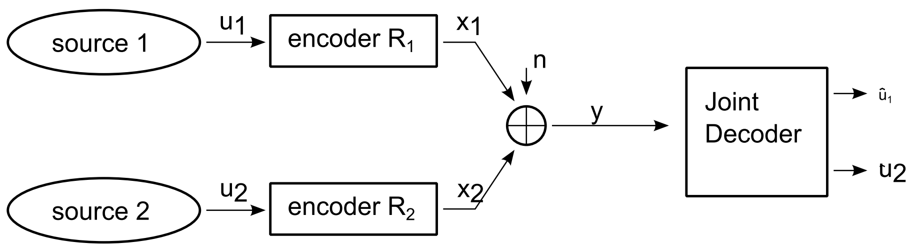


Abbildung 4: Block Diagramm

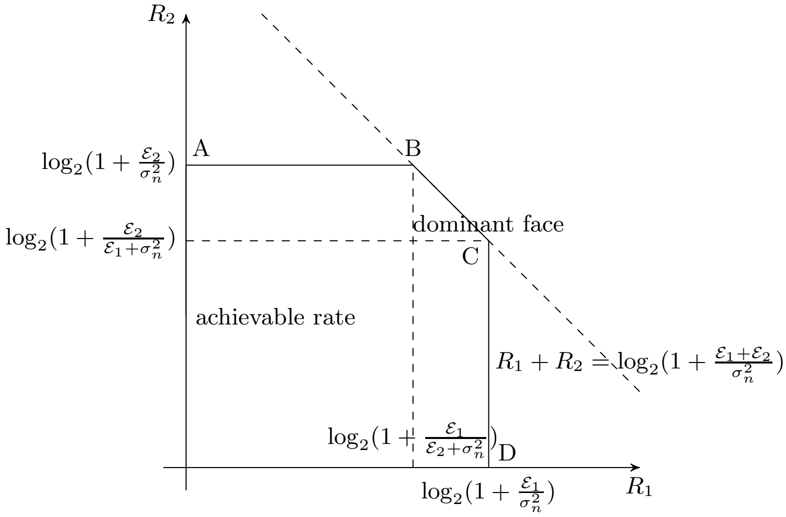
Example: 2 Users

- How should we choose R_1 and R_2 to ensure error free decoding of <u>both</u> signal streams?
- It is no longer sufficient to maximizie a single rate. Instead we have to consider rate pairs (R_1, R_2)
- All possible rate points, that allows error free decoding, define the rate region $\underline{C}$
- Possible desing goals of the system:
  - maximized sumrate $R_{\text{sum}} = \max\limits_{(R_1,R_2)\in\underline{C}} R_1 + R_2$
  - maximize minimum user rate: $R_{\text{max-min}} = \max\limits_{(R_1,R_2)\in\underline{C}} \min\limits_{i\in\{1,2\}} R_i$
- Rate Region of two user Gaussian MAC *Anmerkung: Einschränkungen*

$$R_1 < \log_2\left(1 + \frac{\mathcal{E}_1}{\sigma_n^2}\right) \tag{1}$$
$$R_2 < \log_2\left(1 + \frac{\mathcal{E}_2}{\sigma_n^2}\right) \tag{2}$$
$$R_1 + R_2 < \log_2\left(1 + \frac{\mathcal{E}_1 + \mathcal{E}_2}{\sigma_n^2}\right) \tag{3}$$

- Interpretation:
  - (1) and (2) (= single-to user constraint) are the „single-user bounds“, i.e., the maximum rates of user 1 and 2, if the other user was not there
  - (3) can be interpreted as the maximum rate if streams of users 1 and 2 were jointly encoded. The separate encoding in the MAC cannot yield a better performance

– Graphical represantation:



- Observations:
  - A - B is defined by (2)
  - C - D is defined by (1)
  - B - C is defined by (3)
  - A - B suggests that even if user 2 transmits with the same max. rate as in the single user case, user 1 can transmit with non-zero rate! $\rightarrow$ Multiuser communication enables "free rate gains!
  - Which point on A - B - C,- D we choose, depends on the design criterion
- How do we achieve points on A - B - C,- D?
  - Both user use Gaussian codebooks
  - B:
    * signal of user 1, x_1, is decoded first and x_2 is treated as noise:

$$y = x_1 + \underbrace{x_2 + n}_{\text{treat as noise}}$$
$$\rightarrow R_1 < \log_2\left(1 + \frac{\mathcal{E}_1}{\mathcal{E}_2 + \sigma_n^2}\right)$$

    * once x_1 is known, we form

$$y - x_1 = x_2 + n$$
$$\rightarrow R_2 < \log_2\left(1 + \frac{\mathcal{E}_2}{\sigma_n^2}\right)$$

    * this approach is referred to as successive interference cancellation (SIC) and is a direct result of the chain rule in information theory:
    $I(X_1, X_2, Y) = I(X_1, Y) + I(X_2; Y|X_1)$

- C: same as B, but X_1 and X_2 change rules
- Points on A - B, C - D can be achieved by decreasing the rate of users 1 and 2 respectively (not desirable)
- Points on B - C (dominant face): Achievable by "time-sharing, i.e., $\theta \cdot 100\%$ of the time we decode user 1 first and $(1-\theta)100\%$ of the time we decode user 2 first, $0 \leq \theta \leq 1$

$$
\begin{aligned}
R_1 &< \theta \log_2\big(1 + \frac{\mathcal{E}_1}{\mathcal{E}_2 + \sigma_n^2}\big) + \big(1-\theta\big) \log_2\big(1 + \frac{\mathcal{E}_1}{\sigma_n^2}\big) \\
R_2 &< \theta \log_2\big(1 + \frac{\mathcal{E}_2}{\sigma_n^2}\big) + \big(1-\theta\big) \log_2\big(1 + \frac{\mathcal{E}_2}{\mathcal{E}_1 + \sigma_n^2}\big) \\
\rightarrow R_1 + R_2 &< \theta\Big(\log_2\big(1 + \frac{\mathcal{E}_1}{\mathcal{E}_2 + \sigma_n^2}\big) + \log_2\big(1 + \frac{\mathcal{E}_2}{\sigma_n^2}\big)\Big) + \\
&+ \big(1-\theta\big)\Big(\log_2\big(1 + \frac{\mathcal{E}_1}{\sigma_n^2}\big) + \log_2\big(1 + \frac{\mathcal{E}_2}{\mathcal{E}_1 + \sigma_n^2}\big)\Big) = \\
&= \theta \log_2\Big(\frac{\mathcal{E}_1 + \mathcal{E}_2 + \sigma_n^2}{\mathcal{E}_2 + \sigma_n^2} \cdot \frac{\mathcal{E}_2 + \sigma_n^2}{\sigma_n^2}\Big) \cdot \big(1-\theta\big) \log_2\Big(\frac{\mathcal{E}_1 + \sigma_n^2}{\sigma_n^2} \cdot \frac{\mathcal{E}_1 + \mathcal{E}_2 + \sigma_n^2}{\mathcal{E}_1 + \sigma_n^2} = \\
&= \log_2\Big(1 + \frac{\mathcal{E}_1 + \mathcal{E}_2}{\sigma_n^2}\Big)
\end{aligned}
$$

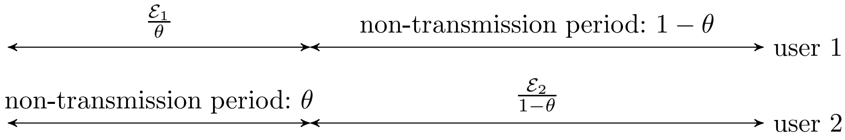
- Comparison with orthogonal transmission
  - User 1 transmits for $\theta \cdot 100\%$ of the time and user 2 transmits for $(1-\theta) \cdot 100\%$ of the time, $0 \leq \theta \leq 1$
  - to keep average transmit power independent of θ, the users transmit with powers $\frac{\mathcal{E}_1}{\theta}$ and $\frac{\mathcal{E}_2}{1-\theta}$
  - Rates:

$$
\begin{aligned}
R_1 &< \theta \log_2\Big(1 + \frac{\mathcal{E}_1}{\theta \sigma_n^2}\Big) \\
R_2 &< \big(1-\theta\big) \log_2\Big(1 + \frac{\mathcal{E}_2}{(1-\theta)\sigma_n^2}\Big)
\end{aligned}
$$

multiuser:

$\xleftrightarrow{\qquad\qquad\mathcal{E}_1\qquad\qquad}$ user 1

$\xleftrightarrow{\qquad\qquad\mathcal{E}_2\qquad\qquad}$ user 2 orthogonal:



– sumrate:

$$R_1 + R_2 < \theta \log_2\Big(1 + \frac{\mathcal{E}_1}{\theta\sigma_n^2}\Big) + \big(1-\theta\big)\log_2\Big(1 + \frac{\mathcal{E}_2}{(1-\theta)\sigma_n^2}\Big) = R_{\text{sum}}$$

– Which θ maximizes sumrate?

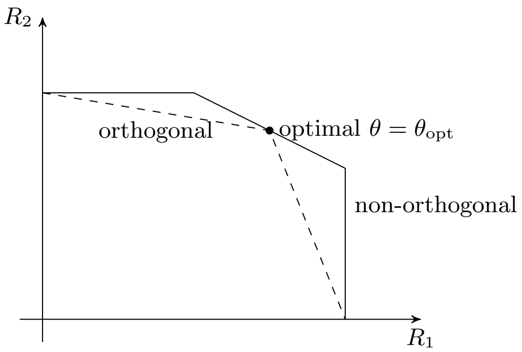
$$\frac{\delta R_{\text{sum}}}{\delta\theta} \overset{!}{=} 0 \text{ leads to } \theta_{\text{opt}} = \frac{\mathcal{E}_1}{\mathcal{E}_1 + \mathcal{E}_2}$$

– Maximum sumrate

$$\begin{aligned} R_{\text{sum}} &= \frac{\mathcal{E}_1}{\mathcal{E}_1 + \mathcal{E}_2}\log_2\Big(1 + \frac{\mathcal{E}_1 + \mathcal{E}_2}{\sigma_n^2}\Big) + \frac{\mathcal{E}_2}{\mathcal{E}_1 + \mathcal{E}_2}\log_2\Big(1 + \frac{\mathcal{E}_1 + \mathcal{E}_2}{\sigma_n^2}\Big) = \\ &= \log_2\Big(1 + \frac{\mathcal{E}_1 + \mathcal{E}_2}{\sigma_n^2}\Big) \end{aligned}$$

$\rightarrow$ same value as for general non-orthogonal transmission!

– <u>But:</u> In general, orthogonal transmission is suboptimal!



- 3 users case:

$$
\begin{aligned}
R_1 &< \log_2\Big(1 + \frac{\mathcal{E}_1}{\sigma_n^2}\Big) \\
R_2 &< \log_2\Big(1 + \frac{\mathcal{E}_2}{\sigma_n^2}\Big) \\
R_3 &< \log_2\Big(1 + \frac{\mathcal{E}_3}{\sigma_n^2}\Big) \\
R_i + R_j &< \log_2\Big(1 + \frac{\mathcal{E}_i + \mathcal{E}_j}{\sigma_n^2}\Big), \quad i \neq j \\
R_1 + R_2 + R_3 &< \log_2\Big(1 + \frac{\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3}{\sigma_n^2}\Big) \\
&\rightarrow \text{rate region } \mathcal{C} \text{ has } 3! = 6 \text{ corner points}
\end{aligned}
$$

- general case of K users
  - define all non-empty subsets of $\mathbf{K} = \big\{1, \ldots, K\big\}$ as $\mathbf{S} \in \mathbf{K}$, e.g. $K = 2$: $\mathbf{K} = \big\{1, 2\big\}, \mathbf{S} = \big\{\{1\}, \{2\}, \{1, 2\}\big\}$
- rate region $\mathcal{C}$ is defined by

$$
\sum_{k \in \mathbf{S}} R_k < \log_2\Big(1 + \frac{\sum_{k \in \mathbf{S}} \mathcal{E}_k}{\sigma_n^2}\Big) \quad \forall\, \mathbf{S}
$$

$\rightarrow$ $\mathcal{C}$ has $K!$ corner points which can all be achieved by successive interference cancellation (SIC)

Rate region for MIMO Users and Receivers

- Channel Model: $\mathbf{y} = \sum_{k=1}^{K} \mathbf{H}_k \mathbf{x}_k + \mathbf{n}$, with:
  - User k has $N_{T,k}$ transmit antennas
  - N_R receive antennas
  - $\mathbf{n}$: AWGN vector $\mathcal{N}(\mathbf{0}, \sigma_n^2 \mathbf{I})$
- 2 Users case:

$$
\mathbf{y} = \mathbf{H}_1 \mathbf{x}_1 + \mathbf{H}_2 \mathbf{x}_2 + \mathbf{n} \tag{4}
$$

  - Covariance matrix of the TX signal of user k: $\mathbf{Q}_k = \mathcal{E}\{\mathbf{x}_k \mathbf{x}_k^H\}$
  - transmit power: $\mathcal{E}_k = \mathrm{tr}\{\mathbf{Q}_k\}$
- rate region for 2 user case and given $\mathbf{Q}_k$
  - $\mathbf{Q}_k$ given, for example
    a) $\mathbf{Q}_k$ optimal for single user case $\rightarrow \mathbf{Q}_k = \mathbf{U}_k \mathbf{\Lambda}_k \mathbf{U}_k^H$, where:
      · $\mathbf{U}_k$ is an unitary matrix
      · obtained from $\mathbf{H}_k = \mathbf{U}_k \mathbf{\Sigma}_k \mathbf{V}_k^H$

· $\mathbf{\Lambda}_k = \text{diag}\{\mathcal{E}_{k,1}, \mathcal{E}_{k,2}, \ldots, \mathcal{E}_{k,N_T}\}$ with $\mathcal{E}_{l,i}$ obtained from waterfilling and $\sum_{i=1}^{N_{Tk}} \mathcal{E}_{k,i} = \mathcal{E}_k$

b) $\mathbf{Q}_k = \frac{\mathcal{E}_k}{N_{T,k}} \mathbf{I}_{N_{T,k}}$ if $\mathbf{H}_k$ is not known at transmitter

– for given $\mathbf{Q}_1$ and $\mathbf{Q}_2$ we can obtain the rate region as direct extension of the SISO case

$$R_1 < \log_2 \left| \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}_1 \mathbf{Q}_1 \mathbf{H}_1^H \right| \tag{5}$$

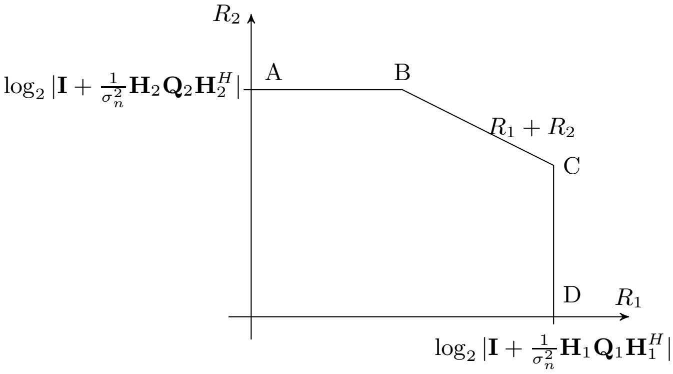
$$R_2 < \log_2 \left| \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}_2 \mathbf{Q}_2 \mathbf{H}_2^H \right| \tag{6}$$

$$R_1 + R_2 < \log_2 \left| \mathbf{I} + \frac{1}{\sigma_n^2} \sum_{i=1}^{2} \mathbf{H}_i \mathbf{Q}_i \mathbf{H}_i^H \right| \tag{7}$$

∗ equation 5 and equation 6 are the single user bounds,

∗ equation 7 is the bound for the joint encoding of both users

– graphical represantation



– Points on A - B - C - D can be achieved in a similar manner as for SISO case

– e.g. bound C can be achieved by SIC

– At B we have

$$R_2 = \log_2 \left| \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}_2 \mathbf{Q}_2 \mathbf{H}_2^H \right|$$

$$R_1 = \log_2 \left| \mathbf{I} + \frac{1}{\sigma_n^2} \sum_{i=1}^{2} \mathbf{H}_i \mathbf{Q}_i \mathbf{H}_i^H \right| - R_2$$

→ user 1 transmits with rate

$$R_1 = \log_2 \left| \mathbf{I} + \frac{1}{\sigma_n^2} (\mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}_2 \mathbf{Q}_2 \mathbf{H}_2^H)^{-1} \mathbf{H}_1 \mathbf{Q}_1 \mathbf{H}_1^H \right|$$

– How to achieve rates at B? → Treat $\mathbf{H}_2 \mathbf{x}_2 + \mathbf{n}$ in equation 8 as noise with covariance matrix $\mathbf{Q}_N = \mathbf{H}_2 \mathbf{Q}_2 \mathbf{H}_2^H + \sigma_n^2 \mathbf{I}$

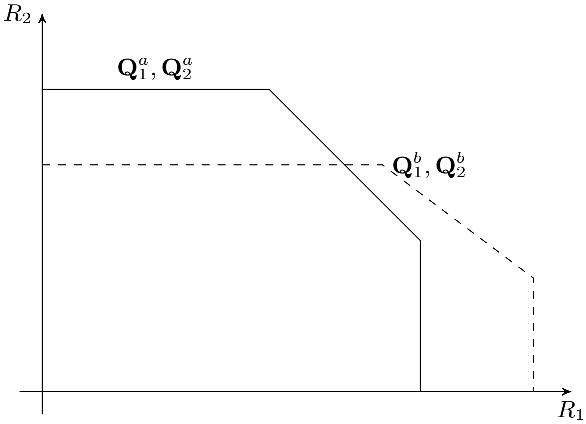
→ equivalent channel matrix with white noise: $\mathbf{r} = \mathbf{Q}_N^{-\frac{1}{2}} \mathbf{y} = \mathbf{Q}_N^{-\frac{1}{2}} \mathbf{H}_1 \mathbf{x}_1 + \tilde{\mathbf{n}}$ where $\tilde{\mathbf{n}}$ is white noise with covariance $\mathbf{I}$.

Anmerkung: Rauschen war vorher farbig, muss „geweißt“ werden.

$$\begin{aligned}\rightarrow R_1 &= \log_2\left|\mathbf{I} + \underbrace{\mathbf{Q}_N^{-\frac{1}{2}}\mathbf{H}_1}_{\mathbf{H}}\mathbf{Q}_1\underbrace{\mathbf{H}_1^H\mathbf{Q}_N^{-\frac{1}{2}}}_{\mathbf{H_{eq}^H}}\right| = \log_2\left(\left|\mathbf{Q}_N^{\frac{1}{2}} + \mathbf{Q}_N^{-\frac{1}{2}}\mathbf{H}_1\mathbf{Q}_1\mathbf{H}_1^H\right| \cdot \left|\mathbf{Q}_N^{-\frac{1}{2}}\right|\right) \\ &= \log_2\left|\mathbf{I} + \mathbf{Q}_N^{-1}\mathbf{H}_1\mathbf{Q}_1\mathbf{H}_1^H\right| = \\ &= \log_2\left|\mathbf{I} + \frac{1}{\sigma_n^2}\left(\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}_2\mathbf{Q}_2\mathbf{H}_2^H\right)^{-1}\mathbf{H}_1\mathbf{Q}_1\mathbf{H}_1^H\right|\end{aligned}$$

- $\rightarrow$ we can achieve R_1 in B by treating user 2 as noise
- $\rightarrow$ once user 1 is detected, we can subtract its contribution from the received signal and detect user 2
  - $\Rightarrow$ user 2 can transmit with maximum single user rate
- $\rightarrow$ bound C can be achieved by SIC similar to SISO case
- – points on B - C are achieved through time sharing

- extension to K user case $\rightarrow$ analogous to SISO case
- Note: Different choices for Q_k will lead to different rate regions



- Example: $K = 2$

$\rightarrow$ $\mathbf{Q}_1$ and $\mathbf{Q}_2$ can be optimized to achieve desired trade-off between performance of users 1 and 2

3.2 Broadcast Channel

We consider:

- uplink - downlink duality
- rate region

3.2.1 Multiplexing Gain - Degrees of freedom

Downlink scenarios:

- N_R antennas at transmitter, single antennas at the users
- user k receives: $y_k = \mathbf{h}_k^H \mathbf{x} + n_k$, with:
  - N_R dimensional channel vector of user k: $\mathbf{h}_k^H$
  - n_k: AWGN at user k
  - $\mathbf{x}$ transmit vector

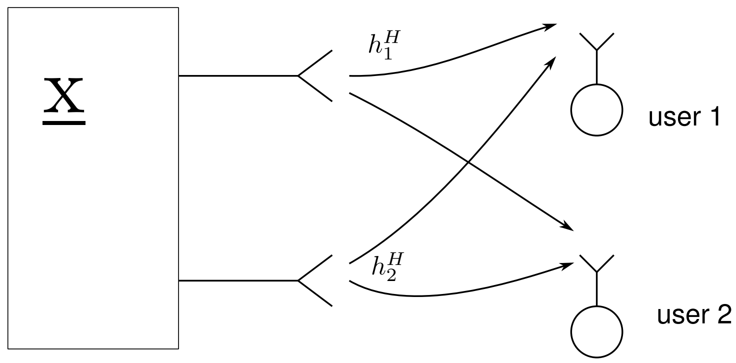


Abbildung 5: Scenario for Broadcast Channel

How many independent signal streams can we transmit?

- Consider transmit signal: $\mathbf{x} = \sum_{k=1}^{K} \mathbf{u}_k x_k$, with TX transmit vector $\mathbf{u}_k$ and symbol x_k is intended for user k
- received signal of user k: $y_k = \sum_{k=1}^{K} (\mathbf{h}_k^H \mathbf{u}_k) x_k + n_k$
- if all $\mathbf{h}_k$ were orthogonal and we chose $\mathbf{u}_k = \mathbf{h}_k$, the received signal would be:
  $y_k = ||\mathbf{h}_k||^2 x_k + n_k$
  Anmerkung: restliche Beiträge fallen in obiger Summe weg, da alle Vektoren orthogonal aufeinander sind.
- if $N_R \geq K$, we can transmit simultaneously and interference free to all K users
  $\Rightarrow$ multiplexing gain $= \min\{K, N_k\}$
- In practice, the $\mathbf{h}_k$ will not be orthogonal
  - $\rightarrow$ choose $\mathbf{n}_k$ such that it lies in the null space of $\begin{bmatrix} \mathbf{h}_1 & \dots & \mathbf{h}_{k-1} & \mathbf{h}_{k+1} & \dots & \mathbf{h}_K \end{bmatrix}$
  - $\rightarrow$ always possible if $\mathbf{h}_1, \dots, \mathbf{h}_K$ are linearly independent
  - $\rightarrow$ multiplexing gain (= degrees of freedom) is equal to $\min\{K, N_R\}$
- we can transmit interference free to $K \leq N_R$ users

3.2.2 Uplink - Downlink Duality

- How should we choose signature vectors to achieve a certain SNR at users?

- Difficult problem since optimal (in the SINR sense) $\mathbf{u}_k$ are not orthogonal $\rightarrow$ signature of user k, $\mathbf{u}_k$, influences SINR at all other users!
- On the other hand, the uplink problem was much easier to solve, since the receive filter of user k, $\mathbf{f}_k$, was not affected by receive filters of other users! (vgl. Point-to-Point $\rightarrow$ detection problem

$\Rightarrow$ We establish a duality between the uplink and downlink, that allows us to solve the more challenging downlink problem by solving an equivalent uplink problem.

Downlink:

- transmit signal

$$\mathbf{x}_{dl} = \sum_{k=1}^{K} \mathbf{u}_k x_{dl,k}$$

- received signal at user k

$$y_{dl,k} = \mathbf{h}_k^H \mathbf{u}_k x_{dl,k} + \sum_{j \neq k} \mathbf{h}_k^H \mathbf{u}_j x_{dl,j} + n_{dl,k}$$

- SINR of user k

$$\mathrm{SINR}_k^{dl} = \frac{\mathcal{E}_{dl,k} |\mathbf{u}_k^H \mathbf{h}_k|^2}{\sigma_n^2 + \sum\limits_{j \neq k} \mathcal{E}_{dl,j} |\mathbf{h}_j^H \mathbf{h}_k|^2}, \quad 1 \leq k \leq K$$

- where: $\mathcal{E}_{dl,k} = \mathcal{E}\big\{|x_{dl,k}|^2\big\}; \quad \sigma_n^2 = \mathcal{E}\big\{|n_{dl,k}|^2\big\}$
- Using:

$$a_k = \frac{\mathrm{SINR}_k^{dl}}{(1 + \mathrm{SINR}_k^{dl}) |\mathbf{h}_k^H \mathbf{u}_k|^2}$$

we can rewrite the SINR expressions as:

$$\boxed{\big(\mathbf{I}_K - \mathrm{diag}\{a_1, \ldots, a_k\} \mathbf{A}\big) \mathbf{p}_{dl} = \sigma_n^2 \mathbf{a}} \tag{8}$$

where:

$$\begin{aligned}
\mathbf{a} &= \big[a_1, \ldots, a_k\big]^T \\
\mathbf{A} &= \begin{bmatrix} |u_1^H h_1|^2 & |u_2^H h_1|^2 & \ldots & |u_K^H h_1|^2 \\ \vdots & & & \\ |u_1^H h_K|^2 & \ldots & & |u_K^H h_K|^2 \end{bmatrix} \\
\mathbf{p}_{dl} &= \big[\mathcal{E}_{dl,1}, \ldots, \mathcal{E}_{dl,K}\big]^T
\end{aligned}$$

$\rightarrow$ We can easily calculate transmit powers $\mathcal{E}_{dl,k}$ required to achieve desired $\mathrm{SINR}_k^{dl}, 1 \leq k_1 \leq K$, for given signature (precoding) vectores $u_k, 1 \leq k \leq K$

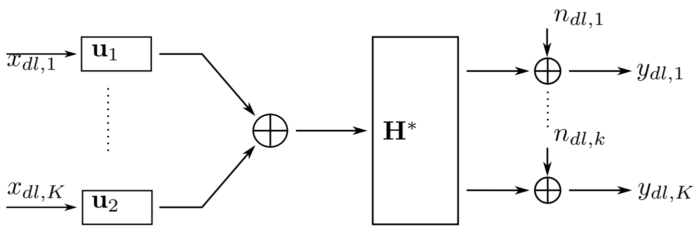


Abbildung 6: Block Diagramm of MIMO in Downlink

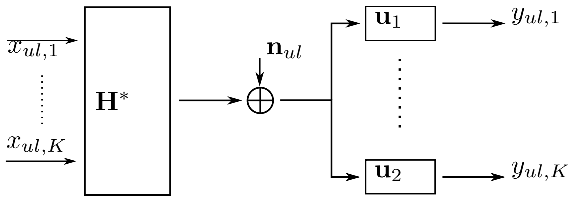


Abbildung 7: Block Diagramm of MIMO in Uplink

Uplink: Use downlink signatures, $\mathbf{u}_k$, as receive filters, $\mathbf{f}_k$

- block diagramm:
- Signal model:

$$y_{ul,k} = \mathbf{u}_k^H \big(\mathbf{H}\mathbf{x}_{ul} + \mathbf{n}_{ul}\big)$$

with

$$\mathbf{H} = \begin{bmatrix}\mathbf{h}_1 & \dots & \mathbf{h}_K\end{bmatrix}, \quad \mathbf{x}_{ul} = \begin{bmatrix}x_{ul,1} & \dots & x_{ul,K}\end{bmatrix}^T, \quad \mathbf{n}_{ul} = \begin{bmatrix}n_{ul,1} & \dots & n_{ul,K}\end{bmatrix}^T$$

$$\rightarrow y_{ul,k} = \mathbf{u}_k^H \mathbf{h}_k x_{ul,k} + \sum_{j\neq k} \mathbf{u}_k \mathbf{h}_j x_{ul,j} + \mathbf{u}_k^H \mathbf{n}_{ul} \tag{9}$$

$$\rightarrow \mathrm{SINR}_k^{ul} = \frac{\mathcal{E}_{ul,k}|\mathbf{u}_k^H \mathbf{h}_k|^2}{\sigma_n^2 + \sum_{j\neq k} \mathcal{E}_{ul,j}|\mathbf{u}_k^H \mathbf{h}_j|^2} \tag{10}$$

where we used $\mathbf{u}_k^H \mathbf{u}_k = 1$ and $\mathcal{E}_{ul,k} = \mathcal{E}\big\{|x_{ul,k}|^2\big\}$

- we define:

$$b_k = \frac{\mathrm{SINR}_k^{ul}}{(1 + \mathrm{SINR}_k^{ul})|\mathbf{u}_k^H \mathbf{h}_k|^2}$$

- we can rewrite SINR epression (10) as :

$$\sigma_n^2 + \sum_{j\neq k} \mathcal{E}_{ul,j}\big|\mathbf{u}_k^H \mathbf{h}_j\big|^2 = \frac{1}{\mathrm{SINR}_k^{ul}} \mathcal{E}_{ul,k}\big|\mathbf{u}_k^H \mathbf{h}_k\big|^2$$

$$\underbrace{\big(1 + \frac{1}{\mathrm{SINR}_k^{ul}}\big)\big|\mathbf{u}_k^H \mathbf{h}_k\big|^2}_{\frac{1}{b_k}} \mathcal{E}_{ul,k} - \sum_{j=1}^{K} \mathcal{E}_{ul,j}\big|\mathbf{u}_k^H \mathbf{h}_j\big|^2 = \sigma_n^2$$

$$\to \mathcal{E}_{ul,k} - b_k \sum_{j=1}^{K} \mathcal{E}_{ul,j} \left| \mathbf{u}_k^H \mathbf{h}_j \right|^2 = b_k \sigma_n^2$$

- matrix notation:

$$\boxed{\left(\mathbf{I}_K - \mathrm{diag}\{b_1, \ldots, b_K\} \mathbf{A}^T\right) \mathbf{p}_{ul} = \sigma_n^2 \cdot \mathbf{b}} \tag{11}$$

- where: $\mathbf{A}$ was defined for downlink case

$$\mathbf{p}_{ul} = \begin{bmatrix} \mathcal{E}_{ul,1} & \ldots & \mathcal{E}_{ul,K} \end{bmatrix}^T$$
$$\mathbf{b} = \begin{bmatrix} b_1 & \ldots & b_K \end{bmatrix}^T$$

- we can calculate power allocation vector $\mathbf{p}_{ul}$ for given SINR_1^{ul} and $\mathbf{u}_k, \quad 1 \leq k \leq K$

Comparison: Assume, we want to achieve same SINR in uplink and downlink

$\to$ $\mathrm{SINR}_k^{ul} = \mathrm{SINR}_k^{dl} \quad \forall \quad k$ or equivalently $a_k = b_k, \quad \forall \quad k$!

What sum power do we need in both cases?
From (8) on page 12 and (**??**) on page 14

$$\mathbf{p}_{dl} = \sigma_n^2 \left(\mathbf{I} - \mathrm{diag}\{a_1, \ldots, a_K\} \mathbf{A}\right)^{-1} \mathbf{a} =$$
$$= \sigma_n^2 \left(\mathbf{D}_a - \mathbf{A}\right)^{-1} \cdot \mathbf{1}$$

where: $\mathbf{D}_a = \mathrm{diag}\{\frac{1}{a_1}, \ldots, \frac{1}{a_K}\}$ and $\mathbf{1} = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \end{bmatrix}^T$

$$\mathbf{p}_{ul} = \sigma_n^2 \left(\mathbf{I} - \mathrm{diag}\{b_1, \ldots, b_K\} \mathbf{A}\right)^{-1} \mathbf{b} =$$
$$= \sigma_n^2 (\mathbf{D}_b - \mathbf{A}^T)^{-1} \cdot \mathbf{1}$$

where: $\mathbf{D}_b = \mathrm{diag}\{\frac{1}{b_1}, \ldots, \frac{1}{b_K}\}$

$$\begin{aligned}
\sum_{k=1}^{K} \mathcal{E}_{dl,k} &= \mathbf{1}^T \mathbf{p}_{dl} = \sigma_n^2 \mathbf{1}^T \left(\mathbf{D}_a - \mathbf{A}\right)^{-1} \cdot \mathbf{1} \\
&= \sigma_n^2 \mathbf{1}^T \left(\mathbf{D}_b - \mathbf{A}\right)^{-1} \mathbf{1} \\
&= \sigma_n^2 \left[\mathbf{1}^T \left(\mathbf{D}_b - \mathbf{A}\right)^{-1} \mathbf{1}\right]^T \\
&= \sigma_n^2 \mathbf{1}^T \left[\left(\mathbf{D}_b - \mathbf{A}\right)^{-1}\right]^T \mathbf{1} \\
&= \sigma_n^2 \mathbf{1}^T \Big(\underbrace{\mathbf{D}_b^T}_{\mathbf{D}_b} - \mathbf{A}^T\Big)^{-1} \cdot \mathbf{1} = \sum_{k=1}^{K} \mathcal{E}_{ul,k}
\end{aligned}$$

Conclusions:

- We can achieve any desired $\mathrm{SINR}_k^{dl}, \forall\, k$, in the downlink by using filters optimized for uplink transmission as signature (precoding) vectors and the same sum power as in the uplink

- suitable filters may be MMSE or ZF filters $\mathbf{u}_k = \mathbf{f}_k, \quad \forall\, k$
- Note that, in general, $\mathcal{E}_{ul,k} \neq \mathcal{E}_{dl,k}$, only the sum powers are equal!

→ $\boxed{\text{For linear precoding, we can solve the more challenging problem via solving an equivalent uplink problem!}}$

Extension: This concept can be extended to nonlinear receivers as well. In this case the DFE receiver in the uplink is dual to a nonlinear Tamlinson - Harashima (TH) precoder in the downlink.

3.2.3 Rate Region (only SISO)

- two user case: $y_k = h_k x + n_k\,, \quad k \in \{1, 2\}$
- Capacity can be achieved with so - called *Costa Cor (dirty paper)* precoding
- In this scheme, the signal of one user is processed such that it does not impair the receiver of the other user, while the signal of the other user is treated as Gaussian noise at the receiver of the first user.

→ Thus, the achievable rates would be

$$R_1 < \log_2\bigl(1 + \frac{|h_1|^2 \mathcal{E}_1}{\sigma_n^2}\bigr)$$
$$R_2 < \log_2\bigl(1 + \frac{|h_2|^2 \mathcal{E}_2}{\sigma_n^2 + |h_1|^2 \mathcal{E}_1}\bigr)$$

- $\mathcal{E}_1$ and $\mathcal{E}_2$ can be optimized as they both occure at the same transmitter