A Schumpeterian Exploration of Gini and Top/Bottom Income Shares

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Abstract

Data show that an increase in the Gini coefficient is associated with a falling bottom p_B% income share and an increasing top p_T% income share where, e.g. $p_B = 40$ and $p_T = 1$. This relationship, which we call the X inequality relationship, is pervasive in the sense that it is observed in many countries, including the U.S., the U.K., France and others. The purpose of this paper is (i) to construct a Schumpeterian growth model to explain the relationship, and (ii) to identify/quantify factors behind it via calibration of the U.S economy. Our model gives rise to a double-Pareto distribution of income as a result of entrant and incumbent innovations. Its advantage is that it allows us to develop iso-Gini loci and iso-income share schedules in a tractable way. Using a double-Pareto distribution as an approximation of an underlying income distribution, calibration analysis revels that a declining business dynamism and fiscal policy changes in the past decades played a significant role in generating the X inequality relationship in the U.S.

JEL Code: O40, O30, D31

Keywords: Growth, Innovation, Gini, Income Shares, Double-Pareto Distribution

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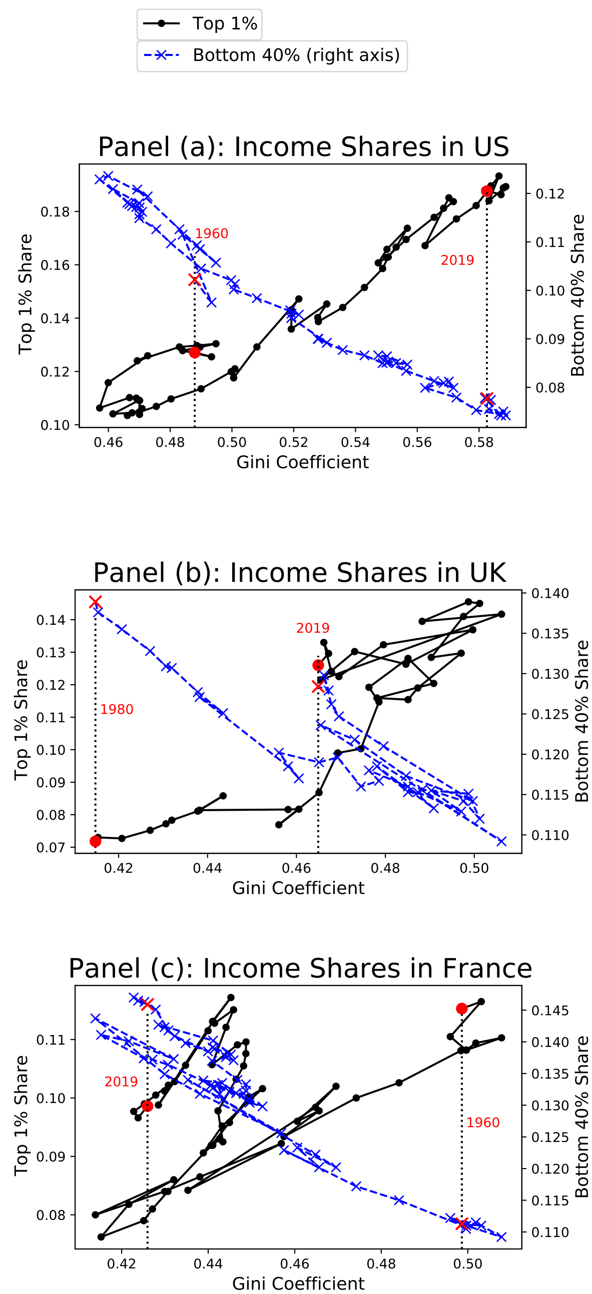
1 Introduction

In the literature on inequality, the Gini coefficient and the top/bottom income shares are often used to show how inequality evolves over time and to make comparison among countries. Although they show different aspects of inequality, they seem to move in a certain systematic way. A clue is provided by Leigh (2007) who demonstrates that the Gini and the top income shares in particular have a strong positive relationship in 13 countries.[1] Using data, Atkinson, Piketty and Saez (2011) also show that the top income shares can have sizable impacts on the Gini coefficient for the whole economy, despite that the number of income earners in the top 1% is very small relative to the total population. Such a close link between the Gini and the income shares is intuitive. However, it is not clear what economic forces drive them to move in a way data show. Viewed this way, several interesting questions arise. Are such co-movements inevitable? What economic mechanisms are working to make their relationship so strong? What about the bottom income shares? Do they move along with the Gini coefficient as well? If so (it is indeed as shown below), how one can explain the triangle relationship among the Gini coefficient and the top/bottom income shares? In addition, top incomes are known to follow a Pareto distribution. What role does it play in forming such relationships? The present paper represents an effort to approach those questions from the Schumpeterian perspective, pioneered by Aghion and Howitt (1992) and others, with a focus on the role of innovation.[2]

Figure 1: The X inequality relationship in the U.S., the U.K. and France. Data Source: World Income Inequality Database.



Panel (a) of Figure 1 shows how the Gini coefficient is related to the top 1% and bottom 30% income shares in the U.S. The former is about twice as large as the latter. The starting and end years, 1962 and 2019, are located near the left and right axes, respectively, meaning that the Gini coefficient increases in that pe-

[1] The author even argues that the income shares are "a good substitute" of the Gini coefficient if the latter is not available.

[2] See Aghion, Akcigit and Howitt (2014) for a survey on the literature.

Correlation Coefficients with the Gini coefficient

			Top Income Shares					Bottom Income Shares				
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
	Period	Gini Trend	0.1%	1%	5%	10%	20%	10%	20%	30%	40%	50%
Australia	1960–2019	+	0.933	0.978	0.992	0.998	1.000	−0.980	−0.993	−0.994	−0.994	−0.994
Canada	1960–2019	+	0.895	0.918	0.954	0.977	0.997	−0.905	−0.926	−0.924	−0.933	−0.949
China	1978–2019	+	0.970	0.979	0.986	0.993	0.999	−0.994	−0.994	−0.996	−0.997	−0.997
Denmark	1980–2019	+	0.948	0.955	0.969	0.983	0.997	−0.893	−0.908	−0.932	−0.959	−0.974
Finland	1980–2019	+	0.783	0.858	0.921	0.965	0.997	−0.916	−0.900	−0.923	−0.935	−0.946
France	1960–2019	−	0.276	0.548	0.869	0.960	0.993	−0.932	−0.939	−0.945	−0.959	−0.978
Germany	1980–2019	+	0.398	0.941	0.991	0.997	1.000	−0.992	−0.993	−0.996	−0.998	−0.999
Greece	1980–2019	−	0.441	0.658	0.866	0.943	0.977	−0.420	−0.633	−0.845	−0.956	−0.975
Ireland	1980–2019	+	0.933	0.903	0.874	0.908	0.992	−0.894	−0.904	−0.921	−0.932	−0.939
Italy	1980–2019	+	0.971	0.989	0.996	0.997	0.998	−0.991	−0.991	−0.993	−0.995	−0.998
Japan	1980–2019	+	0.609	0.808	0.971	0.988	0.994	−0.863	−0.852	−0.893	−0.943	−0.981
Korea	1976–2019	+	0.909	0.925	0.959	0.994	1.000	−0.986	−0.988	−0.990	−0.992	−0.995
Netherlands	1980–2019	+	0.856	0.900	0.946	0.971	0.992	−0.960	−0.963	−0.970	−0.981	−0.991
New Zealand	1960–2019	−	−0.334	−0.072	0.068	0.472	0.980	−0.937	−0.938	−0.941	−0.950	−0.963
Norway	1980–2019	+	0.918	0.950	0.975	0.990	1.000	−0.964	−0.971	−0.976	−0.982	−0.990
Poland	1980–2019	+	0.927	0.980	0.997	0.999	1.000	−0.986	−0.994	−0.997	−0.999	−0.999
Portugal	1980–2019	+	0.622	0.896	0.974	0.988	0.995	−0.909	−0.915	−0.952	−0.974	−0.986
Singapore	1969–2019	+	0.394	0.965	0.992	0.998	1.000	−0.994	−0.996	−0.996	−0.997	−0.998
Spain	1980–2019	−	−0.202	0.140	0.707	0.921	0.988	−0.821	−0.850	−0.891	−0.932	−0.959
Sweden	1980–2019	+	0.702	0.875	0.940	0.957	0.989	−0.850	−0.845	−0.875	−0.922	−0.963
Switzerland	1980–2019	+	0.829	0.898	0.957	0.987	0.998	−0.929	−0.936	−0.963	−0.979	−0.990
Taiwan	1977–2019	+	0.301	0.193	0.954	0.993	0.999	−0.984	−0.986	−0.987	−0.991	−0.994
USA	1960–2019	+	0.984	0.978	0.986	0.993	0.998	−0.948	−0.949	−0.966	−0.985	−0.994
United Kingdom	1980–2019	+	0.846	0.888	0.940	0.975	0.997	−0.869	−0.879	−0.892	−0.922	−0.960

Table 1: Correlation coefficients with the Gini coefficients. Negative values are shown in red. Grayed cells indicate that the null hypothesis of zero correlation cannot be rejected at a 5% significance level. The bordered cells correspond to Figure 1. Data Source: World Inequality Database.

riod with a dip in early years. The top 1% income share moves along with the Gini, so that the scatter plots over the period shows a positive trend. In sharp contrast, the bottom 30% income share is negatively and more tightly related to the Gini coefficient. Putting them together, what we may call the X inequality relationship clearly emerges. The U.K. data are shown in Panel (b) of Figure 1 over the 1980-2019 period. The X inequality relationship exits, though not as strongly as in the U.S. This result may not be surprising, given that those English-speaking countries are known to share an increasing trend in the Gini and the top income shares in particular. What is somewhat surprising is the case of France in Panel (c), which is often referred to as a contrasting case. Inequality measures of some continental European countries, including France, are known to move differently from those of the Anglo-Saxon countries. Despite this, the X inequality relationship is clearly visible.

To explore the relationship further, Table 1 shows correlation coefficients between the Gini coefficient and the top/bottom income shares for 24 countries. Data periods are given in column (1), and column (2) indicates whether the linear trend of the Gini coefficient over the period is positive or negative. Columns (3)-(7) show correlation coefficients between the Gini coefficient and the top income shares. Correlation between the Gini coefficient and the bottom income shares are given in columns (8)-(12). Negative values are shown in red. Grayed cells indicate that the null hypothesis of zero correlation cannot be rejected at a 5% significance level. Inspecting the table, four observations can be made. First, it is immediately clear that correlation is dominantly positive for the top income shares and negative for the bottom income shares, implying the X inequality relationship. It is so even if grayed cells are ignored. Second, the bordered cells for the U.S., the U.K. and France correspond to Figures 1. There are many countries with the X inequality relationship as strong as or even stronger than that of the three countries. This suggests that the X relationship is a widely observed phenomenon. Third, there is no grayed cells and "wrongly" signed cells in the bottom income shares. It implies that a negative correlation in the bottom income shares is more likely to occur than a positive correlation in the top income shares.[3] Fourth, the absolute values of correlation coefficients get higher, as the income share increases. While it is because the Gini coefficient is susceptible to changes in the middle income range, correlation coefficients in the bottom 10% are more than 0.8 in absolute value except for Greece.

At the backdrop of these observations, the present paper makes several contributions. First, we develop a Schumpeterian growth model which can account for the X inequality relationship. In the model, entrant and incumbent innovations drive growth and generate income inequality. In particular, we derive a double-Pareto distribution of income distribution, which is used as an *approximation* of an observed distribution. It consists of what we call the Left and Right distributions connected at mode. A double-Pareto distribution has two Pareto exponents, each for the Left and Right distributions.[4] Indeed, there are studies which provide evidence in support of such approximation (see below).

An advantage of this approximation-based approach is that we can derive iso-Gini loci, iso-top income share loci and iso-bottom income share loci in a tractable way in the space of those two Pareto exponents. These tools make it possible to examine how the top/bottom income shares are related to the Gini coefficient in an intuitive way. They also allow us to explore economic mechanisms working behind the X relationship in a simple way. Admittedly, any approximation, including ours, causes loss of information of the underlying phenomenon. To examine what

[3]In an earlier version of the paper, correlation coefficients of bottom 1% and 5% were included. However, many of the corresponding income shares are revised to zero as of writing (including the USA) as the dataset is updated.

[4]In the literature, a double-Pareto distribution would typically arise if income follows a geometric brownian motion with Poisson "death", matched by "birth" of entrepreneurs entering at a single point of a given profit. We depart from this perspective in that no geometric brownian motion is assumed.

information is retained/lost, we use the disaggregated data of 100% national income in the US developed by Piketty, Saez and Zucman (2018). The result shows that the trends of inequality indices, required to analyze the X inequality indices, are well preserved though their levels seem affected. In this sense, a double-Pareto distribution seems suited to approximate an observed distribution.

Our second contribution is related to identification/quantification of factors behind the X inequality relationship in calibration analysis. The model is calibrated to the U.S. economy, using innovation-related data. Our result shows that a declining business dynamism, captured by a fall in new firm entry rate as well as decreasing R&D productivity levels, is a major contributor to the X relationship. Falling corporate income taxes were also found important in line with Nallareddy, Rouen and Serrato (2018). Business dynamism is a driver of income growth via creating new products/jobs and reallocating resources from obsolete production units to more efficient ones. Its declining trend since the 1980s in the U.S. is a focus of several studies (e.g. see Decker, Haltiwanger, Jarmin and Miranda (2014), Decker, Haltiwanger, Jarmin and Miranda (2016b) and Akcigit and Ates (2019b)). It is feared that a falling business dynamism generates less opportunities to climb income ladders and less vibrant social mobility, exacerbating inequality (see Fikri, Lettieri and Reyes (2017) and Furman and Orszag (2018)). Our result indeed confirms such concern.

As our third contribution, we introduce a new approach of modeling incumbent innovations as a driver of income inequality. The top 1% (and even smaller percent) incomes follows a Pareto distribution, and one of its important characteristics is a heavy tail. If an innovation-driven growth model is to capture this property, profits of some firms stretch to near infinity, despite that the total profits are finite. This poses difficulties in building an otherwise standard R&D model with the homothetic production function. We solve the issue, resorting to a valuable insight of Klette and Kortum (2004) that the number of intermediate goods can be treated as countable, i.e. $1, 2, 3, \cdots$ in a continuum of product space. Countability implies that there are infinitely many products that are potentially produced by monopoly. Because of this property, some firms can earn disproportionately large profits in a finite product space, generating a double-Pareto distribution of profit *levels*. Acemoglu and Cao (2015) establishes a similar result. Their approach is to use a "lab-equipment" model where profits can be very large, but the Pareto distribution of profits arises relative to the mean profit level.[5] Jones and Kim (2018) sidestep the issue by assuming that profits are proportional to human capital that follows a stochastic growth process. More importantly, the study limits the role of innovations to causing firm exits, complemented with the assumption that incumbent firms do not conduct R&D.[6]

Turning to the description of our model, entrant and incumbent innovations drive growth, improving quality of intermediate goods. Upon successful entrant innovation, entrepreneurs start producing goods a given intermediate product industry. After entry, they engage in further R&D as incumbents. As long as entrant innovations do not arrive in their industries, their

[5] In terms of modeling approach, our model can be best viewed as complementing Acemoglu and Cao (2015) because we assume labour as an input for R&D. In models with R&D workers, the CES functions are often used to model expansion of variety goods (see Romer (1990)), and the Cobb-Douglas utility/production functions are often assumed for quality improvement of goods (see Aghion and Howitt (1992) and Grossman and Helpman (1991)). The CES function is used for quality improvement in Li (2001) for the first time, developed further by Li (2003) and used by others including Dinopoulos and Segerstrom (2010).

[6] Having said this, Jones and Kim (2018) and our work are both consistent with the observation of Smith, Yagan, Zidar and Zwick (2019) that a major source of top income is "pass-through" entrepreneurial profits, which accrue as returns to human capital, rather than capital income. Our approach of taking profits as an important source of increasing inequality is also supported by Barkai (2020). The study provides evidence that a large increase in pure profits contributed to a declining income shares of labour and capital in the U.S.

profits continue to increase without limit. This is the expanding force that stretches the income distribution in the direction of infinity. However, they exit the market and their products become obsolete if they are hit by entrant innovation. Some of those goods are replaced by new products and others become available as competitive goods. This is the contracting force which prevents the income distribution from collapsing in steady state, giving rise to a double-Pareto distribution of profit levels.

A double-Pareto distribution has two parameters, which we call the Left and Right exponents. They are endogenously determined and depends on Poisson rates of entrant and incumbent innovations. In turn, those Poisson rates are determined by incentives for R&D, entrant and incumbent, as in a standard Schumpeterian model. Based on iso-Gini, iso-top $p_T\%$ income share and iso-bottom $p_B\%$ income share loci (e.g. $p_T = 1$ and $p_B = 40$), we identify the areas where the X relationship emerges in the space spanned by the two Pareto exponents. Using those loci and the resulting equilibrium conditions, comparative statics analysis can be easily conducted, and they show how the Gini coefficient and the top/bottom income shares respond to parameter changes. In addition, our model can accommodate contrasting results of Jones and Kim (2018) and Aghion, Akcigit, Bergeaud, Blundell and Hémous (2019) regarding entrant innovations. The former predicts that entrant innovations reduce top income inequality because they destroy monopoly rents and induce exists of incumbent firms. According to the latter study, on the other hand, entrant innovations can increase top income inequality.[7] In our model, the both case can arise, depending upon parameters, at least on the theoretical level.

There are studies on a double-Pareto distribution. Reed (2001) argues that size distribution of some economic variables, including income, exhibits a double-Pareto distribution. In addition, using U.S. data drawn from the Current Population Survey (2000–2009) and the Panel Study of Income Dynamics (1968–1993), Toda (2012) establishes that personal labour income conditioned on education experiences follows a double-Pareto distribution. Toda (2011) also demonstrates that U.S. male wage in 1970-1993 appears to follow a double-Pareto distribution once its trend is removed. In addition, Toda and Walsh (2015) show that cross-sectional U.S. consumption (quarterly data in 1979-2004) obeys the power law in both the upper and lower tails. As far as the lower and upper tails of income are concerned, Reed (2003) and Reed and Wu (2008) argue that the lower and upper tails of incomes exhibit a Pareto distribution, though the middle range is best captured by log-normal distribution. In an early study, Champernowne (1953) considers that the lower tail follows a Pareto distribution.

Turning to the literature, our study is closely related to Aghion *et al.* (2019), Jones and Kim (2018) and Acemoglu and Cao (2015), as mentioned above, which focus on the right Pareto tail. In contrast, our model uses a double-Pareto distribution as an approximation of the entire income distribution and explores the X inequality relationship. Klette and Kortum (2004) which inspires our study is also Schumpeterian in that incumbent and entrant innovations shape firm dynamics. But in the model, firm profits do not grow fast enough, so that firm distribution follows a logarithmic rather than Pareto distribution.[8]In addition to those studies in the Schumpeterian framework, there are competitive models which account for a Pareto distribution of income. An important contribution is made by Aoki and Nirei (2017) and Nirei (2009). Gabaix and Landier (2008) can also cited in this vein, given that they develop a competitive assignment model. Gabaix,

[7]On the other hand, Aghion *et al.* (2019) and Aghion, Akcigit, Hyytinen and Toivanen (2017) show that innovation is inclusive in the sense that it promotes social mobility.

[8]The model is widely used in research. For example, Lentz and Mortensen (2008) use it to explore the link between growth and resource reallocation. Akcigit and Kerr (2018) is another study of entrant and incumbent innovation. They show that the firm distribution matters for long-run growth. More recently, Peters (2020) shows that entry mitigates misallocation of resources in a growing economy.

Lasry, Lions and Moll (2016) extends a random growth model to account for fast rise in top income.[9]

The structure of the paper is as follows. Section 2 gives a basic structure of the model, taking entrant and incumbent innovation as given. It shows the emergence of a double-Pareto distribution of profit income. In Section 3, we derive iso-Gini and iso-income share contours. Section 4 endogenizes entrant and incumbent innovations. We conduct comparative statics in Section 5. Calibration analysis is developed in Section 6 to identify contributing factors behind the X inequality relationship in the U.S. Section 7 concludes.

2 A Schumpeterian Profit Distribution with Pareto Tails

The purpose of this section is to demonstrate the emergence of a profit distribution with Pareto tails in our Schumpeterian model in the simplest possible setting. For this end, the model is developed here, taking the incentive structure of production and R&D activities as given. This allows us to highlight key mechanisms of the model.

2.1 The Basic Model Settings

Consumers are risk-neutral with no saving. Her instantaneous utility is given by

$$U = e^{\frac{1}{J}\int_0^J \ln Y_j dj}, \qquad J \geq 1 \tag{1}$$

where Y_j is differentiated final output j. We assume that Y_j is competitively produced with a continuum of intermediate goods y_{ji} according to

$$\ln Y_j = \int_0^1 \ln q_{ji} y_{ji} di, \qquad q_{ji} = \lambda^{k_{ji}}, \quad \lambda > 1, \quad k_{ji} = 0, 1, 2, ... \tag{2}$$

where q_{ji} is the quality level of y_{ji}. Intermediate product firms conduct R&D to increase k_{ji}, and the highest quality products are always used for final output production. Quality innovations allow monopoly firms to produce some intermediate goods and earn profits. In this section, we use π to denote (net) profit per intermediate product and take π as given.

Consider an intermediate goods industry i in Figure 2. y_{ji} is a product i used to produce final output j, and it is assumed to be specific to a product j. That is, y_{ji} is a different product from $y_{j'i}$, $j' \neq j$, and their quality improvement requires separate successful innovations. In an intermediate product industry i, a single monopoly firm, run by an entrepreneur, compete with competitive fringe firms. A monopoly firm initially produces a continuum of $h_i < J$ products when it enters an industry i after successful entrant R&D (replacing a previous incumbent firm). All other products in i are competitively produced (more explanation on the entry/exit process through creative destruction later). After entry, a firm conducts R&D to further increase the quality of competitive products in i to become their sole producer. Research activeties are entry are termed incumbent R&D. Let us use n_i to denote that number of products the firm produces in i. It is equivalent to

$$n_i = h_i + m_i \tag{3}$$

which consists of h_i and m_i, the latter of which is materialized through incumbent R&D. We also use c_i to denote the remaining competitive goods in i.

[9] There are studies on a Pareto distribution of wealth. For example, see Benhabib, Bisin and Zhu (2011).

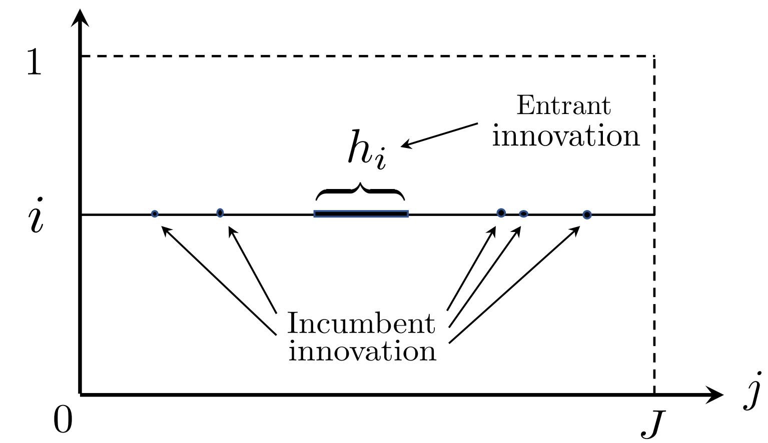


Figure 2: If an entrepreneur succeeds in entrant R&D, she starts with h_i number of monopoly goods in intermediate good industry i. After entry the entrepreneur generates further innovation as an incumbent.

To introduce a Pareto distribution of profits in this otherwise standard Schumpeterian model, we use an insight of Klette and Kortum (2004) that m_i can be treated as countable, i.e. $m_i = 0, 1, 2, 3, \cdots$ in a continuum of product space in i. Countability of m_i has two implications. First, there are infinitely many products that are potentially produced by monopoly, no matter how large it is. Second, "most" of products in i are competitively produced. This applies to all intermediate goods industries $i \in [0, 1]$. The state of a firm in industry i is completely characterized by n_i. Indeed, some monopoly firms are lucky enough to produce an exceptionally large number of products, earning overly huge profits. This is one of the prominent features necessary to generate a Pareto distribution.

Let N denote the number of monopoly products in the economy as a whole. Similarly, the number of competitive products across all intermediate goods industries is given by C. Then, $N = \int_0^1 n_i di$ and $C = \int_0^1 c_i di$ hold. Given that a single monopoly firm produces multiple products in each intermediate industry, N is equivalent to the average number of products produced by monopoly firms.[10] We also require

$$J = C + N. \tag{4}$$

Note that n_i can be exceptionally large to generate a Pareto distribution and this feature is accommodated in (4) because the average of n_i is finite, as will be established.

The economy is characterized by a turnover of monopoly firms through entry/exit, caused by creative destruction of intermediate products. Consider potential entrant firms conducting R&D. We assume that an entrant R&D success follows a Poisson process with an arrival rate of g_E, which is taken as given in this section. It is undirected in the sense that an industry is randomly chosen from $i \in [0, 1]$ to implement successful innovation. This assumption simplifies analysis, but it also captures in a simple way an unpredictable nature of R&D outcomes.[11] To introduce a drastic nature of creative destruction, we assume that all of the previous incumbent products in i

[10] An incumbent in i will be indifferent between incumbent and entrant R&D. We assume that incumbents invest in R&D in her own industry for simplicity.

[11] A well-known example of this type of uncertainty is a microwave oven, which was invented from radar technology for military purposes. The Internet and GPS are also byproducts of military R&D expenditure. Viagra is an example of a commercial product which was originally created for different purposes. A similar assumption is used in Kortum (1997) and Acemoglu, Akcigit, Alp, Bloom and Kerr (2018).

are rendered obsolete by entrant innovation in the same intermediate industry.[12] The "death" of firms due to entrant R&D is the contracting force of the income distribution, which prevents its collapse in steady state. A successful entrant in turn increases the quality level of a continuum of h_i products by a factor λ. We assume that h_i products are randomly allocated to entrant firms.[13] We proceed in two steps regarding the assumptions of h_i. First, we assume that the value of h_i is assumed to be constant for all intermediate goods industries, though its location in Figure 2 is random. In this case, a profit income follows a Pareto distribution with a single right tail. In the second step, the Pareto distribution in the first step is interpreted as the right part of the entire distribution. The left part arises once the value (or the length) of h_i is allowed to randomly change in addition to its random location in the figure.

After entry, incumbent firms engage in R&D to improve quality of competitive goods in their own industries. Incumbent R&D in industry i makes it possible to expand the portfolio of the firm's products stochastically with the Poisson arrival rate of g_I per product. In this section, we take g_I as given. The "per product" assumption plays a crucial role in generating a Pareto distribution in the right tail. To illustrate this point, consider an incumbent firm with n_i products. The arrival rate of incumbent innovation is now given by[14]

$$g_I n_i. \tag{5}$$

Its salient feature is that the more products are improved in quality, the higher the arrival rate of an additional innovation. This is the expanding force of the profit distribution.

There are two things worth mention regarding the assumption (5). First, it means that the rate of innovation is different across i. Initially, incumbent innovation occurs at a lower rate because n_i is low. But as more and more innovations are generated, income growth accelerates. This result is consistent with the finding of Piketty *et al.* (2018) who show that the average annual growth of income is increasing in income percentiles in the U.S. in 1980-2014 with a growth rate accelerating above the top 1%.[15] Second, as we will establish, the number of products n_i is distributed according to a double-Pareto distribution in equilibrium, and hence so is $g_I n_i$. This is in line with Guvenen, Karahan, Ozkan and Song (2015) which show that growth of earnings is double-Pareto distributed, using a large U.S. panel data set. Third, the assumption also captures heterogeneity of firm growth among, stressed by Luttmer (2011) and Gabaix *et al.* (2016).

2.2 Step I: A Constant $\overline{h}$

Taking π as given, define

$$z_i = n_i \pi \tag{6}$$

as the total (net) profit earned by an entrepreneur in an intermediate good industry i.

Assume that the value of h_i is fixed at $\overline{h}$, i.e.

$$\overline{z} = \overline{h}\pi \quad \forall i. \tag{7}$$

[12] The large creative destruction effect is reported by Guvenen, Ozkan and Song (2014). Using U.S. data, they show that the distribution of unfavorable shocks to the rich is left-skewed, meaning that the richer is more likely to be hit by shocks. Akcigit and Kerr (2018) and Aghion *et al.* (2019) also provide evidence for entrants' drastic innovations, using patent data.

[13] That is, the location of a continuum h_i in Figure 2 is random. Hence, some of the previous incumbent products may be included in the initial product portfolio of an entrant.

[14] As will be explained in more detail, n_i in (5) corresponds to the number of R&D projects rather than a positive externality.

[15] See Figure II on p.579 of Piketty *et al.* (2018).

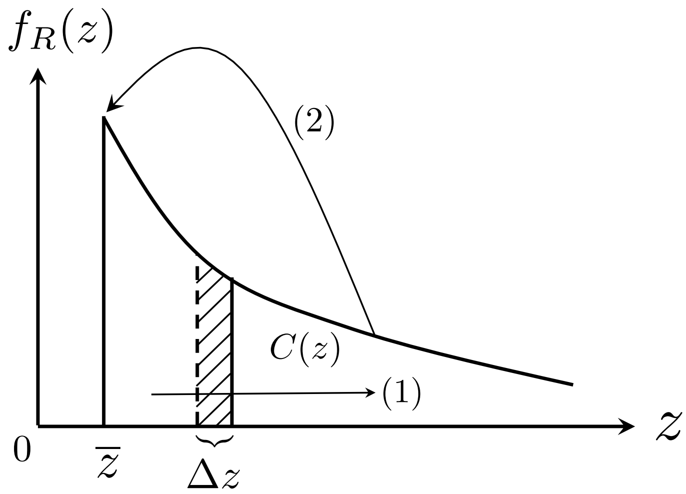


Figure 3: The Right distribution of entrepreneurial income. An arrow (1) indicates entrepreneurs earning more profits and moving rightward in the distribution.

All entrant firms start from the initial profit $\overline{z}$. After entry, firms engage in R&D to increase the range of products they produce according to (5). Whenever innovation occurs, the total profit increases by π, and its expected increase during Δt is given by

$$\Delta z_i = \pi \left\{ 1 \times n_i g_I \Delta t + 0 \times n_i \left(1 - g_I \Delta t\right) \right\} = z_i g_I \Delta t. \tag{8}$$

It shows that total profit geometrically grows and can be very large. Given this equation, this section derives the distribution of z_i. In Figure 3, (8) corresponds to the rightward movement of an entrepreneur, as indicated by arrow (1). On the other hand, there is always a possibility that entrant innovation occurs, causing exits of incumbents, which is captured by arrow (2) in the figure.

Given these assumptions, we denote the cumulative distribution function of z by $F_R(z,t)$ which depends on time t. Define its counter cumulative distribution as

$$C(z,t) = F_R(\infty,t) - F_R(z,t). \tag{9}$$

Now, consider how $C(n,t)$ changes during a small time interval Δt:[16]

$$C(z,t+\Delta t) - C(z,t) = \left[C(z-\Delta z,t) - C(z,t)\right] - g_E \Delta t C(z,t). \tag{10}$$

On the LHS is the total change in $C(n,t)$ which is decomposed into two terms on the RHS. The first term captures an inflow of firms into $C(z,t)$ due to an increase in z through incumbent R&D, captured by the shared area. The second term is a flow of existing firms due to entrant innovations. Rearranging the equation using (8), and then letting $\Delta t \to 0$ gives

$$\frac{dC(z,t)}{dt} = z g_I \left(-\frac{dC(z,t)}{dz}\right) - g_E C(z,t)$$

In steady state, $C(z,t)$ is constant. Therefore, solving the resulting differential equation using (9), we end up with

$$F_R(z) = F_R(\infty)\left[1 - \left(\frac{z}{\overline{z}}\right)^{-\zeta}\right], \qquad f_R(z) = F_R(\infty)\frac{\zeta}{\overline{z}^{\zeta}} z^{-\zeta-1} \tag{11}$$

[16] Derivation here is based on Jones and Kim (2018).

where

$$\zeta \equiv \frac{g_E}{g_I} > 1. \tag{12}$$

$f_R(z)$ gives the number of entrepreneurs earning z.[17] In particular, it is a Pareto distribution with the Pareto exponent ζ, which is assumed to be greater than one because it is required for a finite mean of z. This demonstrates that the power law exponent is determined by the two key variables in our Schumpeterian model. A higher Poisson rate g_E raises the exponent ζ, meaning that the right tail gets thiner. This is intuitive because $1/g_E$ is the average period of earning profits, which means that monopoly rents are lost more frequently for a higher g_E. On the other hand, a higher growth of incumbent profits via g_I reduces the Pareto exponent, making the right tail thicker. This is because entrepreneurs monopolize more products for a given period of time, moving faster rightward in Figure 3.

2.3 Step II: Randomizing $\overline{h}$

The Pareto distribution in the previous section is derived under the assumption that the initial profit $\overline{z}$ is the same for all entrant firms. Indeed, there is no reason why it should be the case, and it seems more natural to assume that the entry level of profits differ. It may be due to uncertainty of R&D activities in general or it may be caused by the timing of launching new products, regional characteristics and even business cycles. Due to those uncertain factors, the distribution of profits would extend below $\overline{z}$. Some entrepreneurs are lucky enough to start near $\overline{z}$, while unlucky ones are far off $\overline{z}$.[18]

To capture this observation, we introduce an additional uncertainty into R&D by randomizing initial profit levels of entrant firms. More specifically, dropping the subscript i for simplicity, we assume that the value of h is randomly drawn for entrant firms upon successful innovation according to

$$F_H(h;\Theta), \qquad f_H(h;\Theta), \qquad h \in \left(\underline{h}, \overline{h}\right], \qquad \underline{h} \geq 0 \tag{13}$$

where Θ is a set of parameters of the assumed distribution. $F_H(.)$ and $f_H(.)$ are the cumulative distribution and density functions of h. Note that $\overline{z}$ defined in (7) is now the maximum starting profit for entrant firms. In what follows, we derive the distribution of $z = h\pi$ for $z \leq \overline{z}$. Conveniently, the distribution (11) is still valid for $\overline{z} < z$.[19]

Making use of (6) and (13) and by changing the variables, let us rewrite $f_H(h)$ in terms of z as

$$\phi(z;\Theta) = \frac{f_H\left(\frac{z}{\pi};\Theta\right)}{\pi}, \qquad z \in (\underline{z}, \overline{z}]$$

where $\underline{z} = \underline{h}\pi$ and $\overline{z} = \overline{h}\pi$. Then,

$$\Phi(z;\Theta) = \int_{\underline{z}}^{z} \phi(s;\Theta)\, ds, \qquad \Phi(\overline{z};\Theta) = 1 \tag{14}$$

is the probability that the initial profit for entrant firms is equal to z or less.

Now, use $F_L(z,t)$ to denote the cumulative distribution function of z for $z < \overline{z}$. We call $F_L(z)$ the Left distribution because it is relevant to the range of $z \in (\underline{z}, \overline{z}]$. Similarly, $F_R(z)$ in (11) is

[17] Remembering that there is a single monopoly firm in each intermediate goods industry $i \in [0,1]$, we would have $F_R(\infty) = 1$ for a constant $\overline{h}$.

[18] Arguably the assumption is more realistic because many self-employed can also be found in the bottom part of the income distribution.

[19] Although $\overline{z}$ is not included, the validity of (11) does not change.

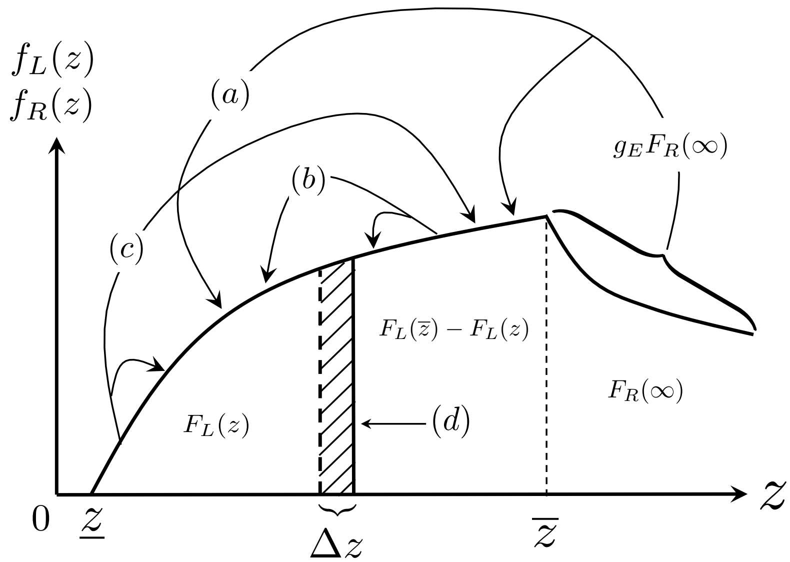


Figure 4: The Left distribution of entrepreneurial income. The arrows indicate exit of firms and entry of new firms. $\overline{z}$ is the boarder line net profit between the Left and Right distributions.

termed the Right distribution for $z \in (\overline{z}, \infty)$. To derive the exact expression of $F_L(z)$, consider how it changes during time interval Δt:

$$\begin{aligned} F_L(z, t+\Delta t) - F_L(z,t) = & \overbrace{g_E \Delta t F_R(\infty, t)\,\Phi(z,t;\Theta)}^{\text{(a)}} + \overbrace{g_E \Delta t\left[F_L(\tilde{z}, t) - F_L(z,t)\right]\Phi(z,t;\Theta)}^{\text{(b)}} \\ & - \underbrace{g_E \Delta t F_L(z,t)\left[1 - \Phi(z,t;\Theta)\right]}_{\text{(c)}} + \underbrace{\left[F_L(z - \Delta z, t) - F_L(z,t)\right]}_{\text{(d)}} \end{aligned} \tag{15}$$

This equation is best explained by using Figure 4 which shows the flows of firms.[20] Entrant firms always start at $z \leq \overline{z}$, and the exact entry profit is randomly determined by (13). After entry, firms move rightward in the distribution due to their own incumbent R&D as long as they are not hit by entrant innovation. Otherwise, they exit the market. Note that such exits can happen anywhere in the distribution of $z \in (\underline{z}, \infty)$. Also note that the total number of monopoly firms must be such that

$$1 = F_L(\overline{z}) + F_R(\infty)\,. \tag{16}$$

Let us explore sources of changes in $F_L(z)$ using 4. First consider the term (a) of (15). Due to entrant innovations, the number of exiting firms coming from $F_R(\infty)$ during Δt is $g_E \Delta t F_R(\infty, t)$. They are replaced with entrant firms, out of which a fraction $\Phi(z,t)$ flow into $F_L(z)$. Such flow of firms is indicated by the arrow (a). Similarly, the term (b) represents an inflow of entrant firms replacing exiting firms from $[F_L(\overline{z}, t) - F_L(z,t)]$, corresponding to the arrow (b) in the figure. In addition, there are two sources of firm outflows from $F_L(z)$. One is captured by the term (c) which corresponds to the arrow (c) in the figure. Exiting firms are replaced with entrants which start between z and $\overline{z}$. Another source is incumbent R&D, which moves firms rightward in the

[20]In the figure, $f_L(\underline{z};\Theta) > 0$ is also possible.

distribution out of $F_L(z)$. The term (d) of (15) represents this effect, which is captured by the shaded area in Figure 4.

As before, rearrangement and letting $\Delta t \to 0$ yield

$$\frac{dF_L(z,t)}{dt} = \frac{dF_L(z)}{dz} + \frac{\zeta}{z} F_L(z) - \frac{\zeta}{z}\Phi(z;\Theta)$$

where (8) and (16) are used. The last term captures the effect of "birth" of entrant firms. Utilizing the condition of the LHS being zero in steady state, one can confirm that the solution to the above differential equation is

$$F_L(z;\Theta) = \zeta \frac{B(z;\Theta)}{z^{\zeta}}, \qquad B(z;\Theta) = \int_{\underline{z}}^{z} s^{\zeta-1}\Phi(s;\Theta)\, ds \tag{17}$$

where Θ is made explicit in $F_L(z)$. Naturally, the Left distribution $F_L(z;\Theta)$ depends on $\Phi(z;\Theta)$, i.e. the distribution of initial profits of entrant firms. Its associated density is

$$f_L(z;\Theta) = \frac{\zeta}{z}\left[\Phi(z;\Theta) - F_L(z;\Theta)\right]. \tag{18}$$

Having derived (18), the question arises: how is it related to (11)? The answer is that the density is continuous at $\overline{z}$ in the following sense:

$$f_L(\overline{z};\Theta) = f_R(\overline{z}) \tag{19}$$

Indeed, it is easy to confirm this equality, using the second equation of (11), (16) and (18). In addition, note that $F_L(z;\Theta)$ and $F_R(z)$ must satisfy the condition that an inflow of firms into $F_L(\overline{z})$ must be matched by an outflow out of it in steady state. The former is given by $g_E F_R(\infty)$ during dt because all entrants must start at or below $\overline{z}$. Making use of this flow condition, Appendix A derives an outflow of firms crossing the borderline $\overline{z}$ from $F_L(\overline{z})$ to $F_R(\infty)$, and shows that equating those flows gives

$$1 = \int_{\underline{z}}^{\overline{z}} \left(\frac{1}{g_I \zeta}\left(\frac{z}{\overline{z}}\right)^{\zeta} + 1\right) f_L(z;\Theta)\, dz. \tag{20}$$

This is the condition which relates ζ and g_I to the distribution parameters Θ. It is an equilibrium condition which consistently links the Left and Right distributions of profit income. Note that Θ can consist of multiple parameters (e.g. mean and standard deviation for log-normal distribution, right-truncated at $\overline{z}$).

2.4 Double-Pareto Distribution

The exact shape of the left distribution $F_L(z;\Theta)$ depends on $\Phi(z;\Theta)$, and hence the assumed function of $f_H(h;\Theta)$. To fix our idea, let us consider

$$f_H(h) = \frac{\xi h^{\xi-1}}{\overline{h}^{\xi}}, \qquad \xi > 1, \quad \underline{h} = 0, \tag{21}$$

in what follows. This is a Pareto distribution of h with $\Theta \equiv \{\xi\}$. We assume $\xi > 1$ to ensure that the density is strictly increasing. It is strictly concave for $1 < \xi < 2$, linear for $\xi = 2$ and convex for $\xi > 2$. Therefore, entrant entrepreneurs are more likely start with lower profit income as ξ falls. In this sense, a lower ξ exacerbates inequality in the left tail.[21]

[21]This is reflected in the Gini coefficient calculated for $f_H(h)$ alone, which is $1/(2\xi+1)$.

Under this assumption, (20) is reduced to

$$\xi = \frac{1}{g_I} - \zeta. \tag{22}$$

This condition endogenously determines the value of ξ for given g_I and ζ such that flows of firms/entrepreneurs are consistent with the entire distribution of profit income. In particular, the Pareto exponents are negatively related, and its implications will be discussed below. Using (12) and (22), ξ can be expressed in terms of Poisson rates of innovation

$$\xi = \frac{1 - g_E}{g_I} \tag{23}$$

When (22) holds, z is now distributed according to

$$F(z) = \begin{cases} F_L(z) = \dfrac{\zeta}{\xi + \zeta}\left(\dfrac{z}{\overline{z}}\right)^{\xi} & 0 < z \leq \overline{z} \\ F_L(\overline{z}) + F_R(z) = 1 - \dfrac{\xi}{\xi + \zeta}\left(\dfrac{\overline{z}}{z}\right)^{\zeta} & \overline{z} < z < \infty \end{cases} \tag{24}$$

This is a double-Pareto distribution where z exactly obeys the Pareto law in both tails.[22][23] Note that $F_L(\overline{z}) = g_E$ and $F_R(\infty) = 1 - g_E$. This implies that the proportion g_E of all entrepreneurs are located in the Left distribution and others are on the other side. The result is intuitive because a higher g_E means that creative destruction occurs more often so that more firms tend be found in the Left distribution.

2.5 The Number of Monopoly Industries

In this section, we consider the determination of N which is the number of monopoly products in the economy. We derive it by examining the number of intermediate products flowing into and out of N. The approach will be turn out to be useful for later analysis.[24]

When an entrepreneur succeeds in entrant innovation in industry i, all incumbent products become obsolete in the industry and their number is $n_i g_E dt$ during time dt. Integrating it over i gives $\int_0^i n_i g_E dt di = N g_E dt$ which is the number of intermediate products flowing out of N. On the other hand, an entrant creates h number of products in i. Given that h is random, its average is

$$\int_0^{\overline{h}} h f_H(h)\, dh = \frac{\overline{h}\xi}{\xi + 1} \equiv \hat{h}\left(\xi; \overline{h}\right). \tag{25}$$

Note that $\hat{h}\left(\xi; \overline{h}\right)$ may include goods produced by the previous incumbent, and we count them as an inflow here. Therefore, $\hat{h}\left(\xi; \overline{h}\right) g_E dt$ is an inflow of goods due to entrant innovation during dt.

[22] $F(z)$ collapses to a right-tailed Pareto distribution $F_R(z)$ for $\xi \to \infty$.

[23] The associated density function is given by

$$f(z) = \begin{cases} \dfrac{\xi\zeta}{\xi + \zeta} \cdot \dfrac{z^{\xi - 1}}{\overline{z}^{\xi}} & 0 < z < \overline{z} \\ \dfrac{\xi\zeta}{\xi + \zeta} \cdot \dfrac{\overline{z}^{\zeta}}{z^{\zeta + 1}} & \overline{z} \leq z < \infty \end{cases}$$

[24] Alternatively, we can directly calculate N. Recall that n_i, the number of monopoly products in a given intermediate industry i, is related to the total profit z through (6). This relationship allows us to rewrite the density functions in footnote 23 in terms of n. This method also allows us to derive (27).

In addition, new intermediate goods are created via incumbent R&D with an average flow of $n_i g_I dt$ products being generated during dt in industry i. Integrating it over i gives $\int_0^1 (n_i g_I dt)\, di = N g_I$.

Equating inflows and outflows gives

$$N g_E dt = \left(\hat{h} g_E + N g_I\right) dt \tag{26}$$
$$\Downarrow$$
$$N = \frac{\overline{h}\xi\zeta}{(\xi+1)(\zeta-1)} \equiv N\left(\xi, \zeta; \overline{h}\right) \tag{27}$$

A higher $\bar{h}$ raises N because $\bar{h}$ determines the maximum number of monopoly products for entrant firms. To develop an intuitive explanation of how ξ and ζ affect N, recall that N is equivalent to the average number of products produced by monopoly firms. Consider ζ which is negatively related to N. A higher ζ, caused by a higher g_E for a given g_I (see (12)), means that a turnover of firms is relatively high, and hence the number of products per monopoly firm falls. On the other hand, N rises as ξ increases. An intuitive account for it can be easily developed, using (23). It shows that a higher ξ arises due to a lower g_E for a given g_I, implying a lower turnover of firms in contrast to ζ. A finite value of N requires $\zeta > 1$.

3 Inequality Measures

Using the double-Pareto distribution, we next demonstrate that the Gini coefficient and the top/bottom income shares can be expressed in terms of the two Pareto exponents.

3.1 Iso-Gini Contours

It is straightforward, though tedious, to show that the Gini coefficient of (24) is given by

$$G = \int_0^\infty F(z)(1-F(z)) = \frac{2\left(\xi^2+\xi\zeta+\zeta^2\right)+\xi-\zeta}{(2\xi+1)(\xi+\zeta)(2\zeta-1)}. \tag{28}$$

One can confirm that G is decreasing both in ξ and ζ . Figure 5 draws an iso-Gini locus, which are convex to $(1,1)$.[25] Inequality measured by G falls as we move northeastward. To interpret its slope, note that the Gini of the Right distribution would be $1/(2\zeta-1)$ if $F_R(z)$ alone was considered independently and calculated separately, and similarly $1/(2\xi+1)$ for $F_L(z)$ alone. This shows that a higher ζ and ξ reduces the Gini within each side, and increasing ζ and reducing ξ is akin to shifting inequality from the Right to the Left distribution. Following this interpretation, the slope of an iso-Gini curve is the marginal rate of substitution between inequalities in the Left and Right distributions. A unit increase ζ (falling inequality in the Right distribution) requires a fall in ξ (increasing inequality in the Left distribution) by the amount equivalent to the slope of an iso-Gini curve for a given level of the Gini coefficient. A downward-sloping iso-Gini locus implies that a fall in either ξ or ζ, taking the other Pareto exponent constant, necessarily increases the Gini coefficient.

3.2 Top/Bottom Income Shares

The top/bottom income shares can also be easily calculated in our framework. For this, define $100\bar{p}$ with $\bar{p} = F(\bar{z})$ as the percentile for $\bar{z}$, the threshold income between the Left and Right

[25] See Appendix B for proof.

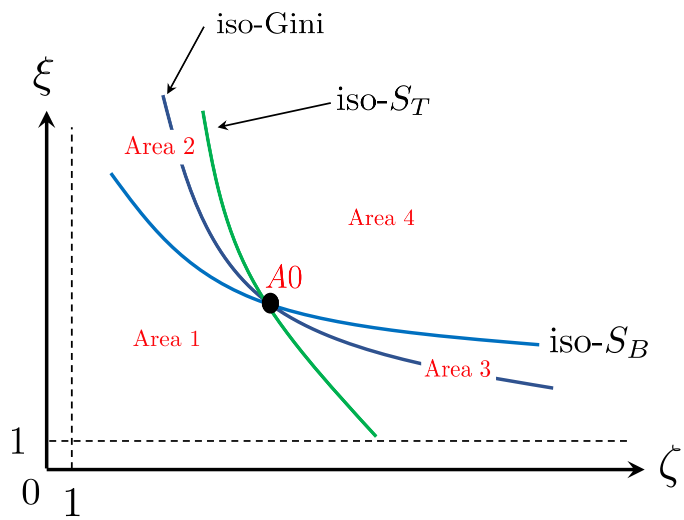


Figure 5: The four areas divided by iso-S_B and iso-S_T curves. Starting from $A0$, the X relationship holds in Areas 1 and 4.

distributions. First consider the bottom $100p_B\%$ income share for $p_B < \bar{p}$, which we use S_B to denote. Appendix C shows that it is defined by

$$S_B = p_B^{1+\frac{1}{\xi}} \left(1 - \frac{1}{\zeta}\right) \left(1 + \frac{\xi}{\zeta}\right)^{\frac{1}{\xi}}, \quad \text{for } p_B \leq \bar{p}, \qquad \frac{\partial S_B}{\partial \xi} > 0, \quad \frac{\partial S_B}{\partial \zeta} > 0. \tag{29}$$

This equation allows us to draw an iso-S_B curve in Figure 5, which shows different combinations of (ξ, ζ) for a given (S_B, p_B). S_B increases in ζ because its higher value means a thiner right tails. B_B is also increasing in ξ despite that the left tail gets thiner with it. Its reason is more involved, but it is basically driven by the fact that net profit at the $100p_B$ percentile increases with a higher ξ.[26] The signs of the derivatives in (29) imply that, holding p_B constant, S_B gets higher/lower in the northeast/southwest area. This property will be exploited to examine how the income share changes in response to parameter changes.

In the appendix, we also derive the top $100(1-p_T)\%$ income share (e.g. $1 - p_T = 0.01$), S_T for $p_T > \bar{p}$, which is defined by[27]

$$S_T = (1 - p_T)^{1-\frac{1}{\zeta}} \frac{1 + \dfrac{1}{\xi}}{\left(1 + \dfrac{\zeta}{\xi}\right)^{\frac{1}{\zeta}}}, \quad \text{for } p_T \geq \bar{p}, \qquad \frac{\partial S_T}{\partial \xi} < 0, \quad \frac{\partial S_T}{\partial \zeta} < 0. \tag{30}$$

This defines iso-S_T loci in Figure 5, giving different combinations of (ξ, ζ) for a given $(S_T, 1 - p_T)$. An intuition for the signs of the derivatives are similar to the one given for (29). The derivatives in (30) confirm that S_T is larger/smaller in the southwest/northeast area for a constant p_T.

Appendix C also demonstrates that an iso-S_T curve is steeper than an iso-S_B curve, giving rise to a unique intersection point $A0$, as illustrated in Figure 5. The two curves divide the (ξ, ζ) space

[26]The cumulative income up to $100p_B$ is given by $g_I \xi \zeta \int_0^{z_L(p_B)} \left(\frac{z}{\bar{z}}\right)^{\xi} dz$ where $z_L(p_B) = \bar{z} \left(\frac{\xi+\zeta}{\zeta} p_B\right)^{\frac{1}{\xi}}$ is the net profit at $100p_B$ percentile. It is increasing in ξ.

[27]One can easily confirm $S_B = \dfrac{\zeta - 1}{\xi + \zeta}$ and $S_T = \dfrac{\xi + 1}{\xi + \zeta}$ for $p = \bar{p}$.

into four areas. To interpret them, consider an economy located at $A0$. The following list gives what happens as the economy moves from $A0$ to four regions, holding p_B and $1-p_T$ constant:

Area 1: S_B falls, and S_T increases.

Area 2: S_B increases, and S_T increases (i.e. the income share of the middle income range between p_B and p_T falls).

Area 3: S_B falls, and S_T falls (i.e. the income share of the middle income range between p_B and p_T rises).

Area 4: S_B increases, and S_T falls.

Those four cases allow us to explore how income shares respond to parameter changes.

The next task is to locate an iso-Gini curve passing through $A0$. Appendix D shows that if p_B and p_T are sufficiently different from $\bar{p}$, an iso-Gini is steeper than an iso-S_B curve, but less so than an iso-S_T curve, as illustrated in Figure 5. A word "sufficiently" does not mean $p_B \to 0$ or $p_T \to 1$. In fact, an iso-Gini curve is sandwiched between two iso-income share curves for a large range of values of p_B and p_T. For example, Toda (2012) obtains estimates of $\zeta = 2.34$ and $\xi = 1.15$ using U.S. data from the Current Population Survey (2000-2009). He also shows that a calculated Gini coefficient based on those values is close to an actual value. Using the same numbers, we have $\bar{p} = 0.67$ and the sandwiched case arises for $p_B \leq 0.65$ and $1-p_T \leq 1-0.67$, which practically covers an almost entire range of values. In addition, our calibration analysis below suggests that the sandwiched case is a norm.[28]

The X inequality relationship between the Gini coefficient and the top/bottom income share in Table 1 corresponds to Areas 1 and 4 in Figure 5. Now consider again an economy at $A0$ in the figure. Further suppose that the economy moves to a random point in (ξ, ζ) space. It is clear from the figure that the economy is more likely to end up in Areas 1 and 4 than other ares because the iso-S_B and iso-S_T curves are downward-sloping, i.e. the X relationship is more likely to occur than otherwise. Having said this, however, the economy does not move randomly, but systematically according to economic incentives. To explore this, we need to endogenize R&D activity.

3.3 Data and a Double-Pareto Approximation

In general, an observed income distribution does not follow a particular distribution, and different distributions are proposed to model it. Examples include exponential, lognormal, gamma, Levy-stable, double-Pareto-lognormal, and they are considered as an approximation of a true distribution at best.[29] Similarly, we use a double-Pareto as an approximation of an underlying income distribution. But before we move on, we consider the extent to which the double-Pareto approximation is useful in understanding the X inequality relationship. For this, we use data of Piketty *et al.* (2018). Its advantage is that it allows us to calculate summary statistics based on disaggregated data of 100% national income in the U.S.

We first examine some properties relevant to our analysis. Panel (a) of Figure 6 shows a log-log plot of the data for five-year intervals in 1970-2015. The threshold is set to the 55th percentile and a tent map shape is clearly visible for all years.[30] This visual inspection suggests that a double-Pareto distribution is in the ballpark.

[28] Focussing upon the sandwiched case also allows us to avoid taxonomic analysis.

[29] See Toda (2012) for some references.

[30] The tent map shape does not dramatically change even if the 60th or 50th percentile is used.

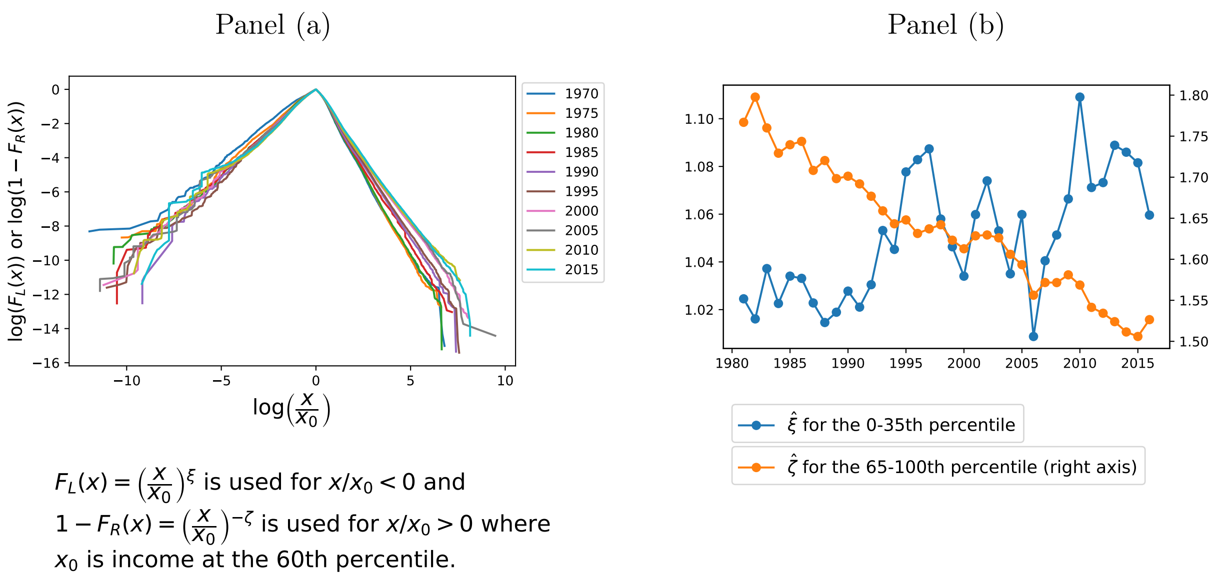


Figure 6: Both panels use an income variable called "ptinc" (pre-tax national income; non-negative values only) with a frequency varaible "dweght" in Piketty *et al.* (2018).

Having said this, however, approximation means loss of information by definition (more on this later). In addition, a question remains on how to calculate the Left and Right tail Pareto exponents. A possibility may be to choose a given threshold percentile which divides the entire distribution into the Left and Right parts and calculate exponents, respectively. However, this approach does not necessarily generate the best fit. Instead, we search different percentiles to calculate $\hat{\xi}$ and $\hat{\zeta}$ separately and choose the ones which meet the following conditions: (i) $\hat{\xi}$ and $\hat{\zeta}$ are both greater than 1, and (ii) prediction errors measured by the coefficient of variation of root mean square deviation is minimized in five percentile intervals. According to these criteria, we found the 35th percentile as the upper threshold for the Left distribution and the 65th percentile as the lower threshold for the Right distribution in the 1981-2016 period. Panel (b) shows the maximum likelihood estimates of ξ and ζ based on those percentiles. A clear downward trend is noticeable for $\hat{\zeta}$. This means that the Gini coefficient tends to increase according to (28). On the other hand, $\hat{\xi}$ has a positive trend. This tends to decrease the Gini coefficient according to (28). An increasing trend of the Gini coefficient in the U.S. means that the former effect dominates the latter.

Given those estimates, we compare the inequality indices calculated using data and predicted by a double-Pareto distribution with estimates of $\hat{\xi}$ and $\hat{\zeta}$. Panel (a) of Figure 7 (next page) shows the Gini coefficient. The predicted values closely follow the actual index, though slight deviation occurs in more recent years. As far as the Gini coefficient is concerned, a double-Pareto approximation works reasonably well.[31] Panels (b) and (c) show the top 1% and 5% income shares. It is immediately clear that predicted values again closely move together with the data, though there are differences in levels. Level differences are even clearer for the bottom 30% and 40% income shares in Panels (d) and (e). They are due to information loss caused by a double-Pareto approximation. What is remarkable, however, is that the tendency of changes, i.e. the slope, is preserved for all of those indices. Indeed, the null hypothesis that the slopes of a linear trend

[31] This confirms that the effect of a falling $\hat{\zeta}$ dominates that of an increasing $\hat{\xi}$ in Panel (b) of Figure 6. Also note that the Gini coefficient calculated using the data of Piketty *et al.* (2018) is somewhat higher than those reported in other sources, summarized in UNU-WIDER (2020).

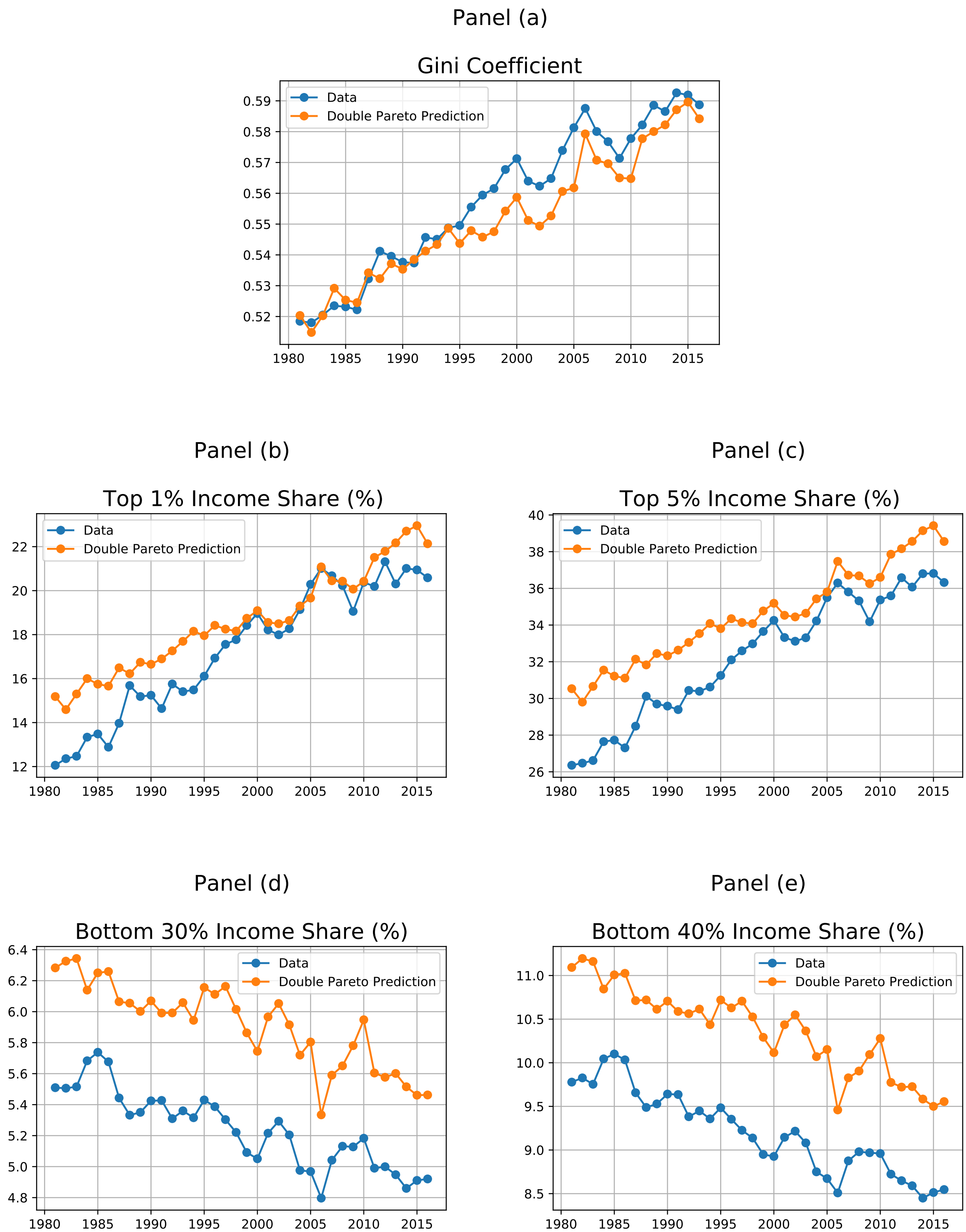


Figure 7: The plots labelled *Data* are calculated using the same variables in Figure 6. Those labelled *Double Pareto Prediction* use $\hat{\xi}$ and $\hat{\zeta}$ in Figure 6 to calculate (28), (29) and (30).

of data and predicted values are the same cannot be rejected even at the 10% level for the Gini coefficient and the bottom 30% and 40% income shares. Regarding the top 1% and 5% income shares, although a similar null hypothesis can be rejected at 1% level, the figures suggest that the data and the approximated series move closely together. Because the purpose of our analysis is *not* to explain levels, but changes or trend of the inequality indices, we take a Double-Pareto distribution as a reasonable approximation of the underlying income distribution for our purposes.

4 Endogenous Growth

The previous sections regarded g_E and g_I as exogenous and focused on how innovation affects inequality. In what follows, we endogenize those variables, introducing the reverse channels by which inequality affects innovation incentives.

4.1 Consumers

As mentioned above, consumers are risk-neutral, hence the interest rate r is equal to the rate of time preferences. We assume that the price index associated with (1) is one.[32] Given these, the demand for Y_j is

$$Y_j = \frac{E}{JP_j} \tag{31}$$

where E is consumption expenditure and P_j is the price of Y_j.

4.2 Demand for Intermediate Products

Consumption goods j is competitively produced according to (2). Profit maximization requires that $y_{ji}p_{ji} = Y_j P_J$ holds. Hence, a demand function of intermediate product y_{ji} is given by

$$y_{ji} = \frac{E}{Jp_{ji}} \tag{32}$$

where p_{ji} is the price of y_{ji}.

4.3 Profits

One unit of intermediate goods is produced with one worker. Intermediate goods are produced competitively or by monopoly firms. In the latter case, like other Schumpeterian models, firms charge the price with the quality step λ as a constant markup over the marginal cost, i.e. $p_{ji} = \lambda w$ where w is wage. Therefore, profit per product is $\Lambda \frac{E}{J}$ where $\Lambda \equiv 1 - \frac{1}{\lambda}$.

4.4 R&D Technology

To enter an intermediate good industry i as a monopoly, an entrepreneur has to be successful in R&D first. Entrant innovation occurs with a Poisson arrival rate of $\overline{\delta}_E \equiv \frac{\delta_E}{R_E^{1-\mu}}$ for each entrepreneur. R_E is the total number of entrepreneurs who engage in entrant R&D, and its presence in the denominator captures the negative congestion externality. Because of free entry,

[32] One can show that the price index associated with U is $e^{\frac{1}{J}\int_0^J \ln P_j dj} = 1$.

there are potentially many researchers, and each of them takes R_E as given. The Poisson rate for the economy as a whole is $g_E = \overline{\delta}_E R_E$ or

$$g_E = \delta_E R_E^{\mu}, \qquad \delta_E > 0, \quad 0 < \mu \leq 1. \tag{33}$$

It is equivalent to g_E in Section 2.

Recall that entrant R&D is undirected in the sense that an intermediate industry where innovation is implemented is randomly chosen from *all* industries $i \in [0,1]$ after innovation occurs. Once an industry i is chosen, then all goods produced by the previous incumbent monopoly firm in i, are rendered obsolete. Then, a range of goods with a measure $h_i < J$ is randomly selected for an entrant to start with, and their quality levels increases by a factor λ. An entrepreneur earns profits $h_i\pi$ at the time of entry. Any other goods in i are now competitively produced. h_i goods may include those produced by the previous incumbent.

After entry, an entrepreneur in industry i turns to incumbent R&D to increase profits further. If successful, a competitive product in i is randomly picked and its quality rises by a factor λ, increasing the entrepreneur's profit by π. We assume that incumbent R&D takes the form of multiple projects. A single project is financed out of profit arising from each intermediate goods production. That is, the number of R&D projects is equivalent to the number of products that firm produces. Specifically, innovation for each project follows a Poisson process with an arrival rate of

$$g_{Ii} = \delta_I R_{Ii}^{\gamma}, \qquad \delta_I > 0, \quad 0 < \gamma < 1 \tag{34}$$

where R_{Ii} is the number of workers in each project of a firm in industry i.

Now consider the expected change in n_i due to incumbent R&D. Given multiple R&D projects, it changes according to

$$dn_i = 1 \times n_i g_{Ii} dt + 0 \times n_i \left(1 - g_{Ii} dit\right) = n_i g_{Ii} dt. \tag{35}$$

Consider the first term of the first equality. A firm runs n_i multiple projects, and each generates an arrival rate g_{Ii}. Therefore, $n_i g_{Ii} dt$ is equivalent to a flow of innovations during dt, each of which improves the quality of a good. The second term represents the case where projects fail. (35) shows that n_i grows exponentially on average.

4.5 R&D Decisions

Let us consider an R&D decision facing an incumbent firm with n_i products. Let V_i denote the value of the incumbent firm. Given (35), define the value of that firm:

$$V_i\left(n_i\right) = \max_{R_{Ii}} \left\{ n_i \pi_i dt + \mathbb{E}\left[\left(1 - \rho dt\right)\left(1 - g_E dt\right) V_i\left(n_i\left(t + dt\right)\right) + V_i\left(t + dt\right)\big|_{n_i = \bar{n}_i} \right] \right\} \tag{36}$$

where

$$\pi_i \equiv \left(1 - \tau\right) \Lambda \frac{E}{J} - \left(1 - s_I\right) w R_{Ii} \tag{37}$$

is after-tax profit net of R&D expenditure, τ is the corporate tax rate and s_I is the rate of subsidy to incumbent R&D. The second term on the RHS of (36) is interpreted as follows. $V_i\left(n_i\left(t + dt\right)\right)$ is the value if an additional innovation occurs, and it is realized if no entrant innovation occurs with the probability of $\left(1 - g_E dt\right)$ during dt. The remaining term $\left(1 - \rho dt\right)$ discounts the future value. The last term captures capital gains due to growth of E and w which is realized irrespective of innovation. Appendix F shows that the optimal R_{Ii} is defined by

$$R_{Ii} = \left(\frac{\gamma \delta_I}{1 - s_I} \cdot \frac{V}{w} \right)^{\frac{1}{1-\gamma}} \equiv R_I \quad \forall\, i \in [0,1] \tag{38}$$

where

$$V \equiv \frac{\pi}{\rho + g_E - g_I - g_w} \tag{39}$$

is the value of an incumbent firm *per product* or $V \equiv V_i(n_i)/n_i$ and g_w is the growth rate of wage which captures capital gains. The presence of g_E in V represents the risk of losing profits, and g_I is an additional gain from incumbent innovation. Because R&D employment per product is the same for $i \in [0,1]$ (see (38)), we have

$$\pi_i = \pi, \qquad g_{Ii} = g_I \tag{40}$$

They are equivalent to π and g_I in Section 2. R_I in (38) is increasing in δ_I, s_I and V/w as expected.

Turning to entrant R&D decisions, recall that h_i is a random variable, and so is the value of an entrant firm because it depends on h_i. We therefore distinguish between its ex ante and ex post values. An ex ante or ex post value of innovation is the one before or after the result of uncertain R&D, i.e. the value of h_i becomes known. In fact, the ex post value per product is equivalent to V in (39), hence the ex post firm value is given by $h_i V$. Next, let us use v to denote the ex ante value of entrant innovation. Using the average of h_i in (25), v is given by

$$v = \hat{h}(\xi)\, V \tag{41}$$

because V is the same for all intermediate good industries. Free entry is assumed, and it leads to

$$\frac{\delta_E}{R_E^{1-\mu}} v = (1 - s_E)\, w \tag{42}$$

where s_E is the rate of subsidy for entrant R&D.

To explore how R&D incentives respond to research-related parameters, let us use (38), (41) and (42) to get

$$\frac{R_E}{R_I} = \left[\frac{1 - s_I}{1 - s_E} \cdot \frac{\delta_E}{\delta_I} \cdot \frac{\hat{h}(\xi)}{\gamma}\right]^{\frac{1}{1-\gamma}} \tag{43}$$

where we assume $\mu = \gamma$, i.e. the extent of the diminishing returns of entrant and incumbent R&D is the same for simplicity. This assumption is maintained in what follows.[33] The ratio of entrant-to-incumbent R&D workers R_E/R_I is increasing in $(1 - s_I)\,\delta_E/(1 - s_E)\,\delta_I$. An intuition goes as follows. An increase in δ_E or s_E encourages entrant R&D, but discourages incumbent innovations because the risk of losing all profits and existing the market rises. Regarding a higher δ_I or s_I, it indeed promotes incumbent *and* entrant innovations because the value of a monopoly product V increases. However, incumbent R&D incentives are larger than entrants' to the extent that R_E/R_I falls. (43) also shows that the relative R&D employment is increasing in $\hat{h}(\xi)$, the average value of h. This is because a higher $\hat{h}(\xi)$ raises the expected return from entrant R&D, leading to a greater employment in entrant R&D.

4.6 Labour Market

There are four sources that require workers. First, the number of entrepreneurs earning profits is 1. This is because there is a single monopoly firm run by an entrepreneur in each of intermediate goods industry $i \in [0,1]$. Second, workers are used for R&D. Those who engage in entrant R&D is

[33] Assuming $\mu \neq \gamma$ complicates a equilibrium condition without generating an additional insight.

$R_E = (g_E/\delta_E)^{\frac{1}{\gamma}}$ from (33). In addition, (22) and (23) allow us to write $g_E = \zeta/(\xi+\zeta)$. Therefore, $R_E = \{\zeta/[\delta_E(\xi+\zeta)]\}^{\frac{1}{\gamma}} \equiv R_E(\xi,\zeta)$ is equivalent to entrepreneurs trying to enter the intermediate goods industry. Incumbent R&D workers per product is similarly calculated from (34) and (22) as $R_I = (1/\delta_I(\xi+\zeta))^{\frac{1}{\gamma}} \equiv R_I(\xi,\zeta)$. The total number of workers used for incumbent research is $R_I N$ where N is the number of monopoly firms, given in (27). The remaining workers are used for manufacturing, which employs $CZ/J + NZ/J\lambda = (J - \Lambda N) Z/J$ where $Z \equiv E/w$. Workers are fully employed for

$$L = 1 + R_E(\xi,\zeta) + R_I(\xi,\zeta) N(\xi,\zeta) + [J - \Lambda N(\xi,\zeta)] \frac{Z}{J} \tag{44}$$

5 Steady State Equilibrium

5.1 Growth

Entrant and incumbent innovations improve quality of intermediate products. This manifests itself in the form of wage growth and utility growth.[34] Indeed, Appendix E shows that the following holds in steady state

$$g_Q \equiv \frac{\dot{Q}}{Q} = \frac{\dot{w}}{w} = \frac{\dot{U}}{U} \tag{45}$$

where $\ln(Q) = \frac{1}{J}\int_0^J \int_0^1 k_{ji} di dj \cdot \ln(\lambda)$ and Q is the overall quality index for the whole economy. Its changes during dt are given by

$$d\ln(Q) = \frac{1}{J}\int_0^J \int_0^1 \left(\begin{array}{l} \text{the number of quality} \\ \text{improvement during } dt \end{array} \right) \times [(k_{ji}+1) - k_{ji}]\, di dj \cdot \ln(\lambda)$$

Note that the number of innovations, entrant and incumbent, is equivalent to the right-hand side of (26). Therefore, we have

$$g_Q = \Big(\overbrace{g_E \hat{h}(\xi)}^{\substack{\text{Entrant} \\ \text{Contribution}}} + \overbrace{g_I N(\xi,\zeta)}^{\substack{\text{Incumbent} \\ \text{Contribution}}} \Big) \ln(\lambda) = \frac{\zeta}{\xi+\zeta} N(\xi,\zeta) \ln(\lambda) \tag{46}$$

using (22), (12) and (27). The first equality decomposes growth into entrant and incumbent contributions. The former depends only on the Pareto exponent of the Left distribution ξ because entrants start from there.

5.2 Equilibrium Conditions

To solve the model, let us rewrite (43) as

$$\zeta = A\left(\frac{\xi}{\xi+1}\right)^{\frac{\gamma}{1-\gamma}}, \qquad A \equiv \left(\frac{1-s_I}{1-s_E}\cdot\frac{\overline{h}}{\gamma}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{\delta_E}{\delta_I}\right)^{\frac{1}{1-\gamma}}. \tag{47}$$

We call it the *R&D incentive* condition. A positive relationship between ζ and ξ captures the following mechanism of the optimal R&D decisions of entrant and incumbent firms. To develop

[34] See Grossman and Helpman (1991) for example.

its intuition, note that ζ on the LHS captures the term R_E/R_I of (43), which can be verified using (12), (33) and (34). Also note that $\xi/(\xi+1)$ on the RHS comes from $\hat{h}(\xi)$ which is the average number of products with which entrant firms start with upon entry (see (25)). An increase in ξ means a greater number of products for entrants and makes entrant R&D more attractive for potential entrepreneurs. In turn, this leads to a higher ratio of entrant-to-incumbent R&D workers R_E/R_I, i.e. an increase in ζ.

The second condition is based on the ex ante value of successful entrant innovation. Rewriting the free entry condition (42) with (37), (39), (41), (44) and (46), one can derive what we call the *ex ante firm value* condition:

$$\frac{\delta_E}{R_E(\xi,\zeta)^{1-\gamma}} v(\xi,\zeta) = 1 - s_E \tag{48}$$

where

$$v(\xi,\zeta) \equiv \hat{h}(\xi)\frac{\Pi(\xi,\zeta)}{\Gamma(\xi,\zeta)} \tag{49}$$

$$\Pi(\xi,\zeta) \equiv (1-\tau)\Lambda\frac{L-1-R_E(\xi,\zeta)-R_I(\xi,\zeta)N(\xi,\zeta)}{J-\Lambda N(\xi,\zeta)}, \tag{50}$$

$$\Gamma(\xi,\zeta) \equiv \rho + \frac{(\zeta-1+\gamma)-\zeta N(\xi,\zeta)(\ln\lambda)}{\xi+\zeta}. \tag{51}$$

Although (49), (50) and (51) look complicated, they have clear interpretations. $\Pi(\xi,\zeta)$ is the after-tax (gross) profit and $\Gamma(\xi,\zeta)$ is the effective discount rate, both expressed in terms of the two Pareto exponents. [35] Therefore, $\Pi(\xi,\zeta)/\Gamma(\xi,\zeta)$ is the expected present value of a future steam of profits. Multiplied by $\hat{h}(\xi)$, it gives the *ex ante* value of a successful R&D firm, denoted by $v(\xi,\zeta)$.

(47) and (48) is the system of two equations with two unknowns (ξ,ζ). Note that g_E and g_I in equilibrium can be recovered once ξ and ζ are determined, using (22) and (23). Before considering comparative static analysis, we distinguish two cases, focusing upon a unique equilibrium.[36] The first case is illustrated in Figure 8 where the R&D incentive condition is steeper than the ex ante firm value condition at equilibrium.[37] In the second case (not shown), the relative slopes are reversed. In what follows, we focus on the first case because its equilibrium is stable in the following sense. The R&D incentive condition is based on the optimal R&D decisions. Incumbents optimally chose R_I, and R_E is determined via free entry. In particular, when firms consider whether to conduct entrant R&D, they take the *ex ante* value, which is given by $v \equiv \hat{h}(\xi)V$ in (41), as given. In this sense, the R&D incentive condition (47) determines ζ, taking ξ as given. On the other hand, the ex ante firm-value condition (48) essentially determines the Pareto exponent ξ of the Left distribution where entrant firms start, taking ζ as given. Viewed this way, the case of equilibrium illustrated in Figure 8 is stable. In addition, calibration analysis below shows that the stable case is a norm.

[35] $\Pi(\xi,\zeta)$ is equivalent to $(1-\tau)\frac{E}{Jw}$ expressed in terms of ξ and ζ. $\Gamma(\xi,\zeta)$ corresponds to the denominator of (39).

[36] Implicitly differentiating the labour market condition (44), we can show that the partial derivative of Z w.r.t. ξ is negative using the fact that the net profit (37) is positive. The partial derivative of Z w.r.t. ζ, on the other hand, turns out ambiguous, because a higher ζ raises R_E and manufacturing employment, but reduces R_I. In what follows, we assume $\frac{\partial Z}{\partial \zeta} \geq 0$. This assumption implies that a change in manufacturing employment in particular dominates and seems reasonable given the fact that R&D employment is about 1% in total employment in data (e.g. see calibration analysis below).

[37] The R&D incentives condition always satisfies $\zeta > 1$ for $\xi = 1$.

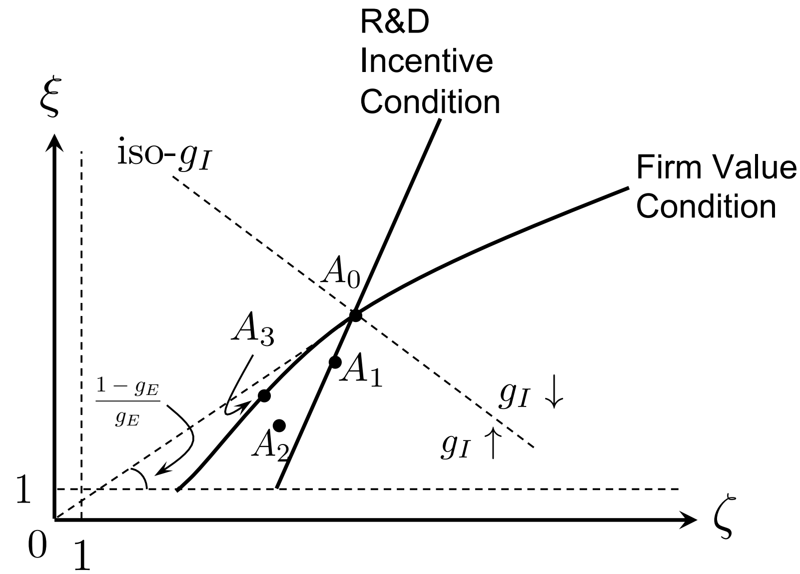


Figure 8: Steady state equilibrium as an intersection point between the R&D-incentive condition and the firm-value condition. Equilibrium is stable when the former is steeper than the latter. The line from the origin passing through the equilibrium point is equivalent to $(1-g_E)/g_E$. An iso-g_I curve is defined by (22).

5.3 Comparative Statics

Focusing on the stable case in Figure 8, the following summarizes comparative statics results:

Result 1: Following an increase in λ, L or a fall in J, ρ, τ,

- ξ and ζ decrease.
- g_E and g_I increase.
- the Gini coefficient increases, the bottom p_B income share falls, and the top p_T income share increases.

Result 2: Following an increase in δ_I or s_I,

- ξ and ζ decrease.
- g_E and g_I increase.
- the Gini coefficient increases, the bottom p_B income share falls, and the top p_T income share increases.

Result 3: Suppose that λ is not too large such that $\Gamma(\xi, \zeta) - \rho \geq 0$.
Following an increase in δ_E or s_E,

- ξ decreases and ζ increases.
- g_E and g_I change ambiguously.
- changes in the Gini coefficient increases, the bottom p_B income share falls, and the top p_T income share are ambiguous.

Result 4: Suppose $s_I = s_E$ initially. Following a simultaneous increase in s_I and s_E,

- ξ and ζ decrease.

- g_E and g_I increase.
- the Gini coefficient increases, the bottom p_B income share falls, and the top p_T income share increases.

To develop an intuition, note that $\xi/\zeta = (1-g_E)/g_E$ which is equivalent to the slope of a line from the origin to A_0 in Figure 8. Also note that (22) defines an iso-g_I contour with the slope of -1. As we move southwestward in the figure, g_I gets higher, and it becomes lower in the area above the iso-g_I locus. Now, consider Result 1. A fall in τ, for example, means that an after-tax profit per product is larger. This effect manifests itself in a downward shift of the ex ante firm value condition, moving an equilibrium from A_0 to A_1. It causes the ratio ξ/ζ to fall, and A_1 is located below the iso-g_I curve, given that the R&D-incentives condition unaffected. Intuitively, the tax reduction generates greater incentives for both types of R&D, but its impact on incumbent R&D is greater than entrant R&D because the effect is realized immediately for incumbents, but in future for entrants conditional on an R&D success. Such difference results in a fall in $\zeta \equiv g_E/g_I$. The result also reveals the inequality-worsening effects of a lower corporate tax, and it is also empirically supported (e.g. see Nallareddy *et al.* (2018)). Other parameters are similarly interpreted.

Turning to Result 2, as δ_I gets larger, the ex ante firm value condition shifts down and the R&D incentive condition moves left. As a result, equilibrium moves from A_0 to A_2. This is because a higher incumbent R&D productivity boosts an incentive for incumbent R&D. In fact, it also induces entrant R&D because a higher δ_I increases the ex ante value of entrant innovation. Those two effects reduce $\xi \equiv (1-g_E)/g_I$. In addition, the effect on g_I is strong to the extent that $\zeta \equiv g_E/g_I$ falls and inequality worsens for the same reason explained above. Regarding the subsidy rate of incumbent R&D s_I, its higher value affects the R&D incentive condition only, moving equilibrium to A_3 in Figure 8. g_E is positively affected for the same reason as in a higher δ_I.

Result 3 summarizes the effects of entrant R&D productivity improvement and a higher rate of entrant subsidy. They affect the Pareto exponents differently, hence changes in the inequality indices are ambiguous. Intuitively, a greater δ_E and s_E makes entrant R&D attractive, resulting in a higher g_E. On the other hand, it increases the risk of losing profits for incumbent firms. This discourages incumbent R&D. Those effects lead to opposite changes in ξ and ζ. This result leads us to an interesting observation regarding entry. Jones and Kim (2018) show that entry of new firms tends to reduce inequality via creative destruction, while entrants can increase inequality in Aghion *et al.* (2019) because of higher markups they enjoy. Aghion *et al.* (2019) also reported that entrant and incumbent innovations both are positively correlated with top 1% income share. These findings suggest that firm entry can increase or decrease inequality. Our model is in line with both of those findings.

In Result 4, a simultaneous increase in the rate of subsidies to incumbent and entrant R&D is considered. By assumption, the R&D-incentive condition is unaffected, while the ex ante firm value condition shifts downward. A new equilibrium is given by a point like A_1. It is the combination of the effects caused by a higher s_E and s_I in Results 2 and 3. This result qualitatively implies that more generous R&D subsidies may be behind the X inequality relationship.

According to these results, most parameter changes, ceteris paribus, moves the economy to Areas 1 and 4 in Figure 5. In this sense, our model predicts that those parameter shifts may have played a role in generating the X inequality relationship. While these are useful insights, they do not inform us about the extent to which each factor contributed to the X inequality relationship

Externally Set Parameters				Internally Set Parameters	
ρ	0.07	τ	$0.30 \to 0.20$ (changes linearly)	J	1.249
γ	0.35	s_E	$0.05 \to 0.20$ (changes linearly)	δ_E, δ_I $\overline{h}, \lambda$	Panels (c)-(f) of Figure 9
L	10.0	s_I	$0.05 \to 0.20$ (changes linearly)		

Table 2: Calibrated parameter values.

observed in many countries. To tackle this issue, we next resort to calibration analysis.[38]

6 Calibration

6.1 Strategy

The main purpose of this calibration exercise is to quantify the contribution of the key underlying factors of the Xinequality relationship in the U.S. in recent decades. Our strategy has two steps, following Akcigit and Ates (2019b).

In the first step, six parameters are set externally, and others are disciplined subject to data. In particular, we set the latter parameters on the basis of three types of data: (i) the entry rate of firms, (ii) the share of R&D workers in working population, and (iii) TFP growth. Given the series of $\hat{\xi}$ and $\hat{\zeta}$ in Panel (b) of Figure 6, parameters are matched with (i) and (ii) for the 1981-2016 period, and (iii) for the average in the period. By so doing, we basically allow parameter values to change to be consistent with $\hat{\xi}$ and $\hat{\zeta}$. In the second step, we let all parameters change as in the first step, *except for* a single parameter which is held fixed at the 1981 level. Shutting down the effect of a parameter makes it possible to identify the extent to which it contributed to the X inequality relationship. We conduct this exercise for six parameters of interest to quantify the individual contribution of parameters to changes in inequality indices.

6.2 Calibrated Values and the Model Fit

Six parameters in Table 2 are externally set. The subjective rate of time preference ρ is set to 0.07 to roughly mimic the long-run annual rate of return from the stock market. γ is the parameter which determines the degree of diminishing marginal product of R&D workers for entrant and incumbent firms. This parameter plays an important role in characterizing the nature of equilibrium. Kortum (1993) reports point estimates between 0.1 and 0.6. In a more recent attempt, Acemoglu, Akcigit, Hanley and Kerr (2016) runs a first-difference regression, reporting 0.286-0.455 with the the average of 0.35. They also conduct robustness checks, e.g. by restricting dataset, and obtain similar values. Acemoglu and Akcigit (2012) use those values for the analysis of IPR and innovation, and Acemoglu *et al.* (2018) use 0.5. Given those studies, we set $\gamma = 0.35$, and the result does not dramatically change as long as $\gamma \leq 0.5$. We set the working population $L = 10.0$ for the following reason. In our model there are always a measure one of entrepreneurs earning positive monopoly profits, and those profits can be lower than wage. In this sense, entrepreneurs in our model are more like self-employed in data. The US Bureau of Labor Statistics compiles data of self-employed, incorporated and unincorporated both starting in 2000. Its ratio to the

[38] We may also be interested in how inequality and TFP growth are related in the model. Unfortunately, g_Q is related to (ξ, ζ) in a highly non-linear way, and its change is ambiguous in response to parameters in general on the theoretical level. Calibration sheds light on this issue as well.

total employment is stable with the average of 10.7% in the 2000-2016 period, implying a roughly one in 10 are self-employed. $L = 10.0$ is used in line with this number.

Regarding a corporate profit tax rate and R&D subsidy rate, we borrow the values that Akcigit and Ates (2019b) use. They provide a brief historical account for changes in those rates. Setting the corporate tax at 30% and the subsidy rate at 5% in 1981 in their calibration, the authors examine declining business dynamism in the US by changing those rates to 20% in 2010, respectively. Although they use a sophisticated approach of changing those rates over the period, we adopt a simpler approach of linearly changing them. Incumbent and entrant subsidy rates are equalized. The remaining parameters are internally set in the following way.

Panel (a) of Figure 9 shows the entry rate of establishments in the U.S, constructed using the Business Dynamic Statistics compiled by the Census Bureau. It is the ratio of new establishments to the total number of active establishments in a given year. Its long-run trend is negative, although it increased in early 1980 and before a steep dive due to the financial crisis in 2008. In Panel (b), the share of R&D workers is plotted, using the data from the OECD Main Science and Technology Indicators. It is defined as the ratio of the full-time equivalent number of researchers to the total employment. In contrast to the rate of firm entry, it steadily increases over the period. Finally, we also use the TFP growth rates, adjusted for capital utilization and labour efforts, which are reported in Fernald (2014).

Using those values and given the (ξ, ζ) series in Panel (b) of Figure 5, we set up the system of four equations to determine four parameters δ_I, δ_E, $\overline{h}$ and λ. The first equation is the rate of firm entry, which is given by

$$\mathrm{Data}_{ER} = \frac{g_E \hat{h}\left(\xi\right)}{J} = \frac{\zeta\xi}{\left(\xi+\zeta\right)\left(\xi+1\right)} \cdot \frac{\overline{h}}{J}. \tag{52}$$

where Data_{ER} is in Panel (a) of Figure 9. At each moment, g_E number of innovations occur across $i \in [0, 1]$, and each innovation creates $\hat{h}\left(\xi\right)$ number of products. We take those products as establishments in data. $g_E \hat{h}\left(\xi\right)$ is divided by J to make it consistent with the definition of the data on the LHS, which corresponds to a series in Panel (a) of Figure 9. Rewriting the first equality using (12), (22) and (25), we can use (52) to pin down the value of $\overline{h}$, given ξ, ζ and $\overline{h}$.

Let Data_{RD} denote a series in Panel (b) of Figure 9. Then, the share of R&D workers satisfies the following condition

$$\mathrm{Data}_{RD} \times L = R_E\left(\xi, \zeta\right) + R_I\left(\xi, \zeta\right) N\left(\xi, \zeta\right). \tag{53}$$

The remaining two conditions are the R&D incentive condition (47) and the ex ante firm value condition (48) which we use to make parameter values data-consistent. Making use of those, we simultaneously determine the values of δ_I, δ_E, $\overline{h}$ and λ over the 1981-2016 period for a given J. Finally, given these parameter values and using g_Q in (46), we set $J = 1.249$ to match the average annual TFP growth rate over the period, which is 0.859 from the data. This gives us recalculated values of δ_I, δ_E, $\overline{h}$ and λ.

The results are shown in Panels (c)-(f) of Figure 9. A noticeable feature is that R&D productivity levels, entrant and incumbent both, steadily fell. Importantly, the rate of reduction in δ_E is 19.9% which is greater than 16.7% for incumbents. This has the following implication for inequality. (43) shows that its RHS falls, tending to reduce the ratio of entrant to incumbent R&D employment. This translates into a reduction of the Right exponent ζ, making the right tail thicker. In fact, this result captures a falling trend of ζ in Panel (b) of Figure 6. A similar observation can be made for the Left distribution. Its Pareto exponent is given in (23), which is increasing in the figure.

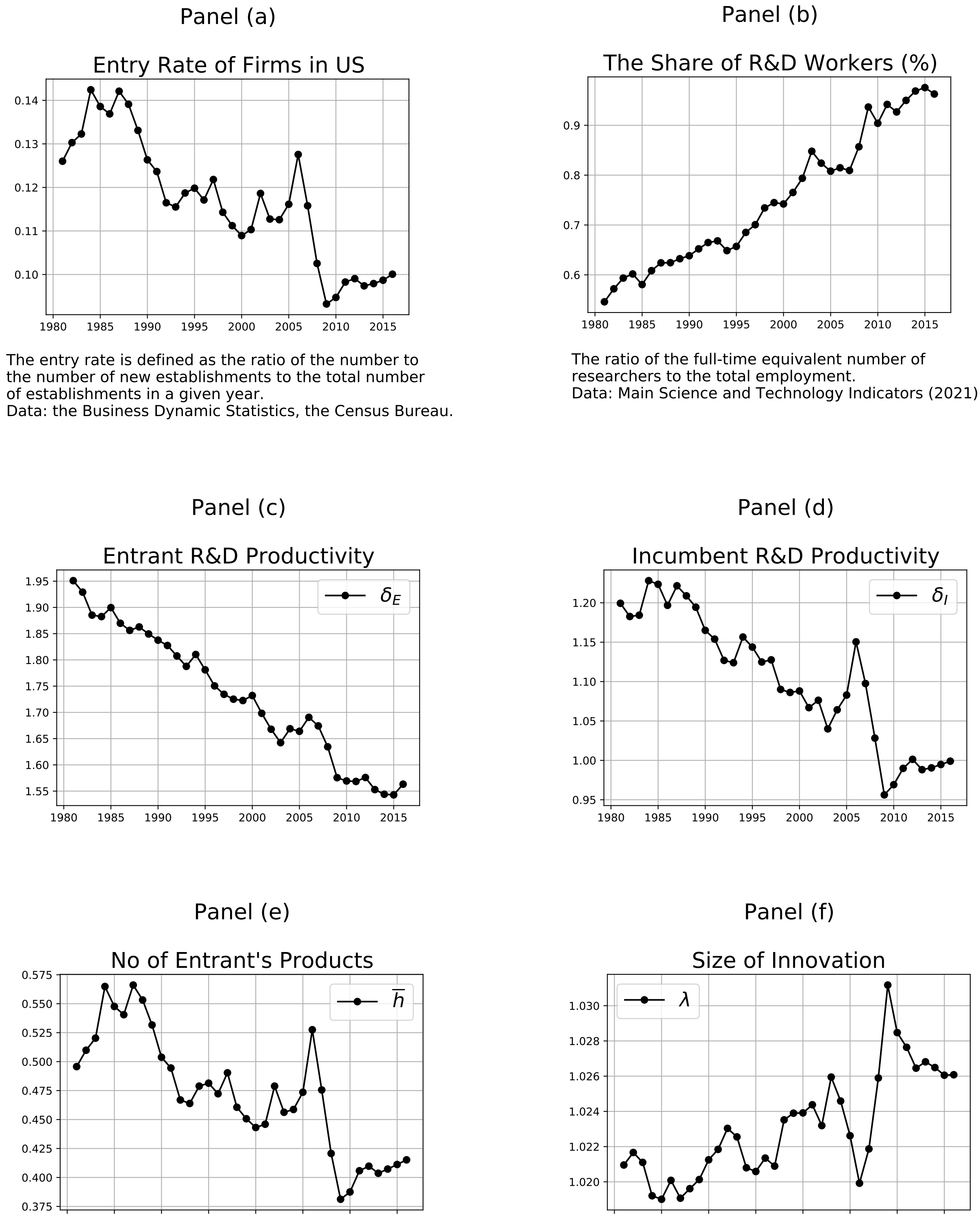


Figure 9: Panels (a) and (b) plot data, while calibrated values are shown in Panels (c)-(f).

		Model	Data	Source
(1)	TFP Growth	0.86%	0.86%	Fernald (2014)
(2)	Rate of Firm Entry	11.70%	11.70%	Business Dynamic Statistics, the US Census Bureau
(3)	Share of R&D Workers	0.75%	0.75%	OECD Main Science and Technology Indicators
(4)	Size of Innovation	1.023	1.075	Garcia-Macia, Hsieh and Klenow (2019)
(5)	Incumbent Contribution to TFP Growth	61.25%	75.17%	
(6)	Entrant Contribution to TFP Growth	38.75%	24.83%	

Table 3: The "Model" column shows the 1981-2016 averages. (1)-(3) in the "Data" columns are the average values in the 1981-2016 period, and those in (4)-(6) give the average of the three periods, 1983-1993, 1993-2003, 2003-2013.

$\overline{h}$ is the maximum initial number of products for entrants, and it fell by 16.2%, comparing the 1981 and 2016 values in Panels (e) of Figure 9. This follows the 20.6% reduction of the firm entry rate in data. An implication is that the income distribution becomes skew to the right. This tends to increase inequality, as will be discussed. The size of quality step λ is shown in Panel (f).

Table 3 summarizes the model fit on the basis of the average values. Note that parameter values are chosen so that the model fits the data for (1)-(3), while (4)-(6) compare the model prediction with values reportetd in Garcia-Macia *et al.* (2019). The size of innovation λ is slightly lower than the value reported, but it falls in the range considered "plausible" by Stokey (1995).[39] λ is also the monopoly price markup over marginal cost. Its increasing trend is consistent with the fact that the markup increased in the US in recent decades, though the level of markup predicted by our model is small.[40] (5) and (6) give the contribution of incumbent and entrant innovations to TFP growth. Though the model under- (or over-)predicts the incumbent (or entrant) contribution, those values are roughly in line with the data. In addition, Garcia-Macia *et al.* (2019) report that incumbent contribution increased while entrants' fell in the 1983-2013 period, and this trend is captured by our model.[41] Despite the parsimonious and stylized nature of the model, the fit of the model seems broadly reasonable. Figure 10 illustrates an equilibrium in 1981 based on calibrated parameter values. It shifts northeastward, generating the X inequality relationship, i.e. a higher Gini coefficient, a lower bottom income share and a higher top income share.

6.3 Quantifying Factors for the X Inequality Relationship

Calibrated parameter values used in Table 2 are data-consistent based on the Pareto exponents in Panel (b) of Figure 6. Put differently, our model can reproduce those series of $\hat{\xi}$ and $\hat{\zeta}$. More importantly, we can also reproduce Double-Pareto Prediction series of the Gini coefficient and

[39] Stokey (1995) considers $[1.02, 1.6]$ as a plausible range. Acemoglu and Cao (2015) use 1.1 and 1.2 for simulation, which are also in the range.

[40] Evidence cited in Akcigit and Ates (2019a) shows that the markup increases from 20% to 50% between 1980 and 2010.

[41] According to Garcia-Macia *et al.* (2019), entrant contribution is 32.3% in 1983-1993 and fell to 19.8% in 2003-2013. In our model, the corresponding percents is 41.6% and 36.3%.

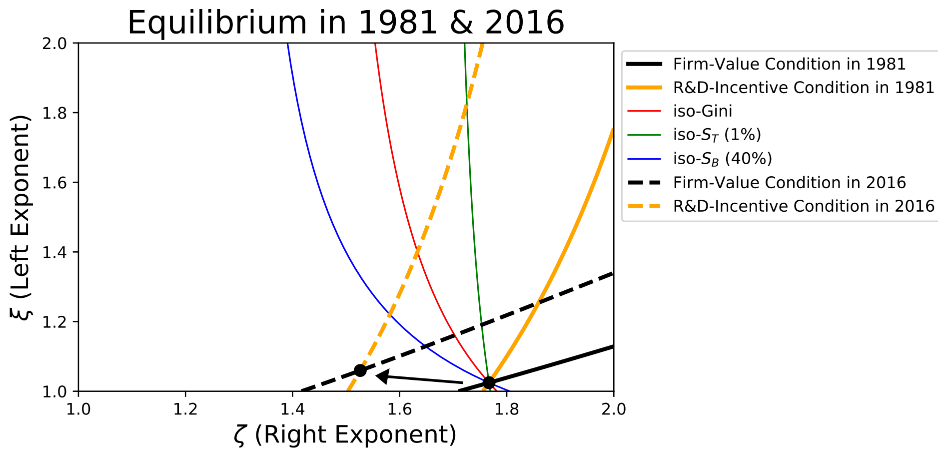


Figure 10: Based on calibrated values of the parameters, a movement of equilibrium from 1981 to 2016 is depicted with the R&D-incentive and firm-value conditions. Contours for a given Gini Coefficient, bottom 40% income shares and top 10% income shares for 1981 are also shown.

top/bottom income shares in Panels (a)-(e) of Figure 7, using the R&D-incentive and firm-value conditions with those calibrated parameter values. Viewed from the model's perspective, therefore, changes in ξ, ζ and the inequality indices are the results of changing all parameters *at the same time*.

Given this observation, we quantify the contribution of each parameter to the X relationship, using a method similar to the one employed in Akcigit and Ates (2019b). We conduct counterfactual experiments by holding one parameter at the 1981 level at a time, while other parameters change as documented in Table 2. Inevitably, the inequality indices deviate from the original series, and such deviation allows us to measure the contribution of a parameter held fixed. We repeat this process for δ_E, δ_I, $\overline{h}$, λ, τ and $s_I = s_E$. To quantify deviation, we use the following measures:

$$\Omega_1 = \frac{D_{2016} - D^k_{2016}}{D_{2016} - D_{1981}}, \qquad \Omega_2 = \frac{\frac{1}{d}\sum_{y=1981}^{y_{\text{end}}} \left(D_y - D^k_y\right)}{D_{y_{\text{end}}} - D_{1981}} \tag{54}$$

D refers to the Gini coefficient, the top 10% income share or the bottom 40% income share, and k is a variable fixed at the 1981 level.[42] For example, D_{2016} is the Gini coefficient in 2016 and D^k_{2016} is the Gini coefficient in 2016 with a variable k is fixed at the 1981 level. d is the number of years used in the numerator in Ω_2. Note that Ω_1 measures deviation in 2016, while Ω_2 uses the average of deviation as a measure of the contribution of a variable k.[43] Also note that the larger the value of Ω_1 and Ω_2, the greater the contribution made by a variable k. If Ω_1 or Ω_2 is negative, it means a negative contribution being made by a variable k.

Consider entrant R&D productivity δ_E. It is best explained using Panel (a) of Figure 11. The Gini coefficient, the bottom 40% income share and the top 1% income share are shown, and series labelled "Double-Pareto Prediction" and "Data" are equivalent to those in Panels (a), (b) and (e) of Figure 7. Series labelled "δ_E fixed" show what would happen if the parameter was held constant

[42] A similar index is used in Akcigit and Ates (2019b).

[43] In (54), y_{end} is the end year which may differ for the reason mentioned in Footnote 44.

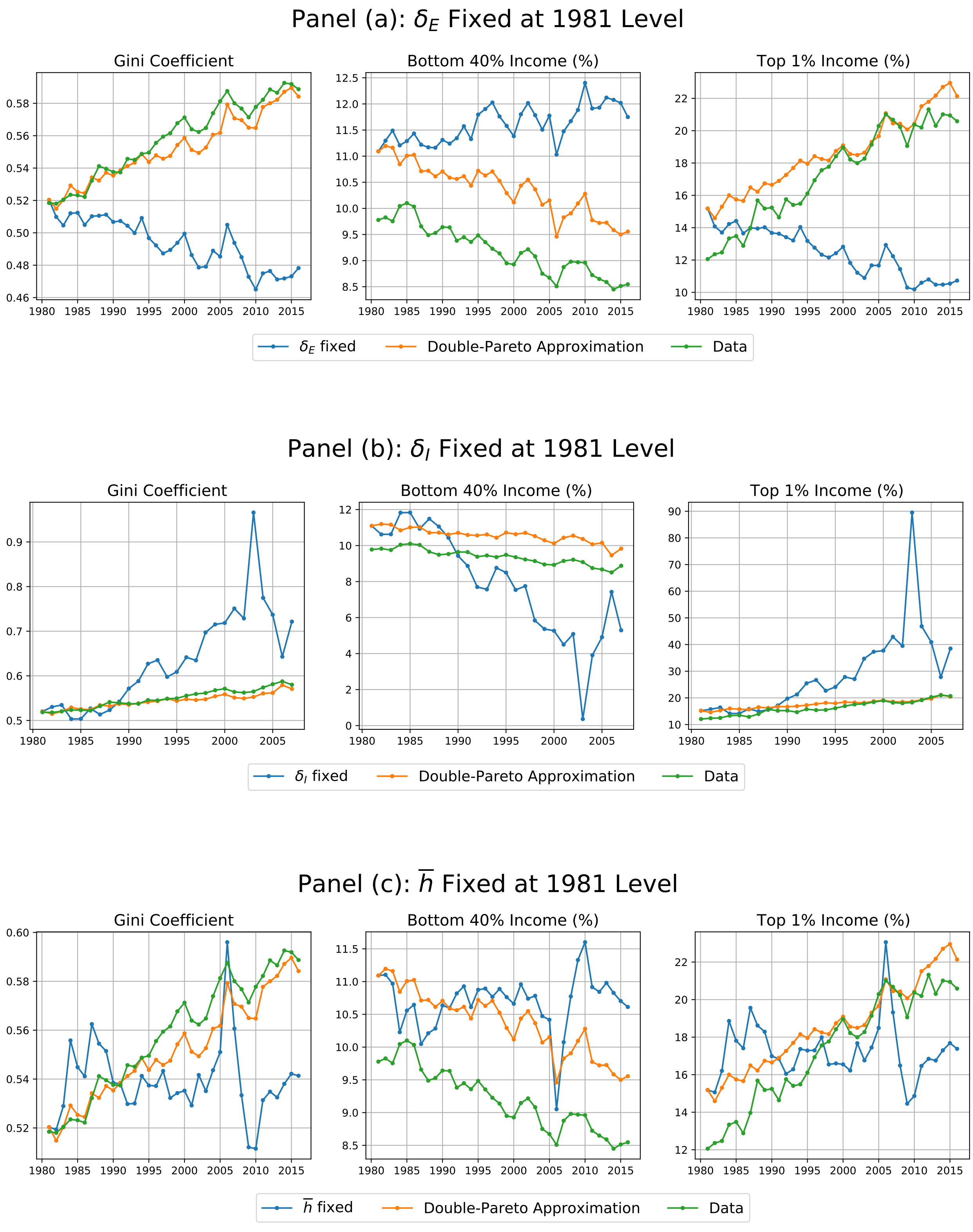
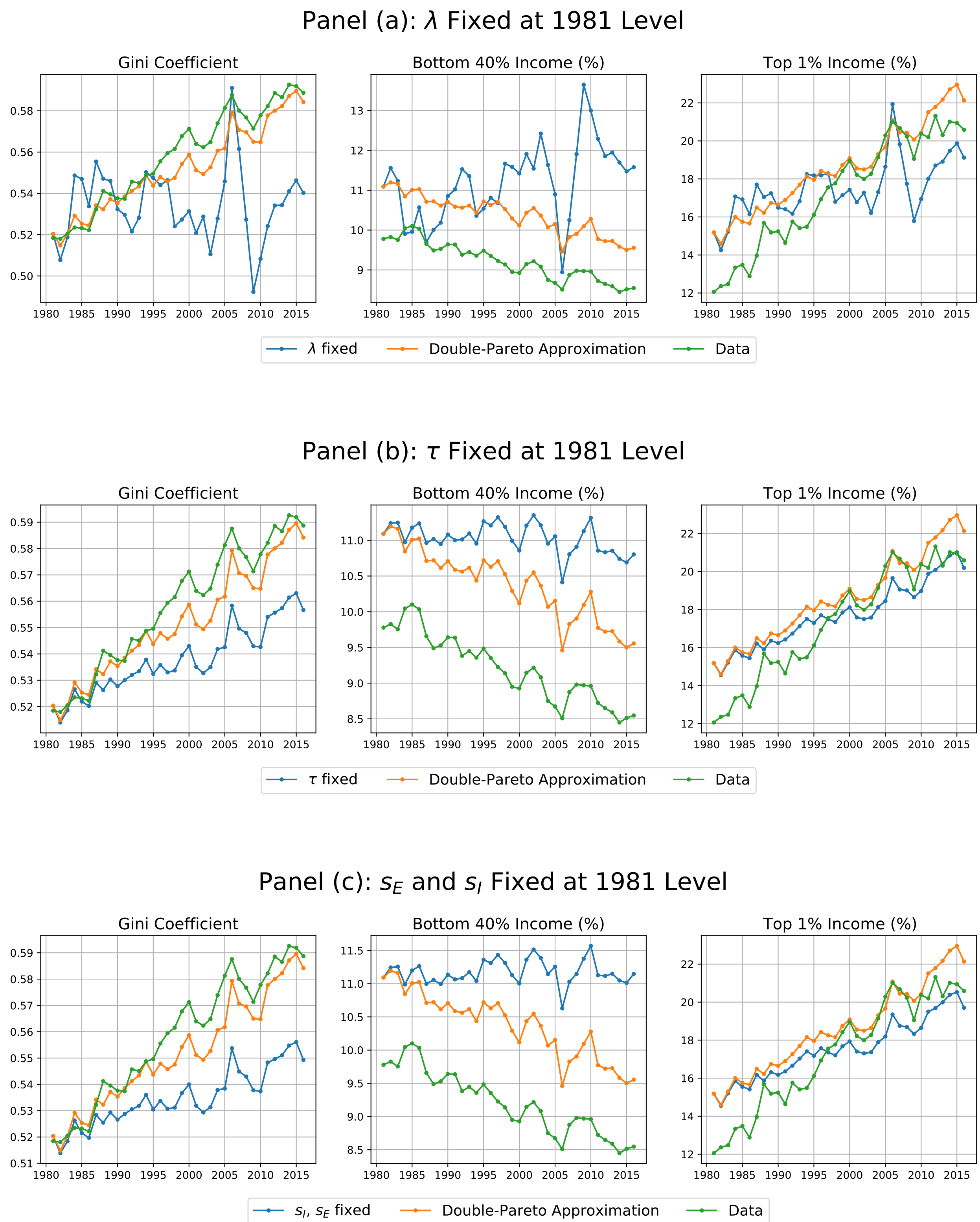

Panel (a): δ_E Fixed at 1981 Level
Gini Coefficient
Bottom 40% Income (%)
Top 1% Income (%)
δ_E fixed
Double-Pareto Approximation
Data
Panel (b): δ_I Fixed at 1981 Level
Gini Coefficient
Bottom 40% Income (%)
Top 1% Income (%)
δ_I fixed
Double-Pareto Approximation
Data
Panel (c): $\overline{h}$ Fixed at 1981 Level
Gini Coefficient
Bottom 40% Income (%)
Top 1% Income (%)
$\overline{h}$ fixed
Double-Pareto Approximation
Data

Figure 11:



Panel (a): λ Fixed at 1981 Level
Gini Coefficient
Bottom 40% Income (%)
Top 1% Income (%)
λ fixed
Double-Pareto Approximation
Data
Panel (b): τ Fixed at 1981 Level
Gini Coefficient
Bottom 40% Income (%)
Top 1% Income (%)
τ fixed
Double-Pareto Approximation
Data
Panel (c): s_E and s_I Fixed at 1981 Level
Gini Coefficient
Bottom 40% Income (%)
Top 1% Income (%)
s_I, s_E fixed
Double-Pareto Approximation
Data

Figure 12:

Table 4:

Measure: Ω_1	D: Double-Pareto Approximated Series								
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
	δ_E	δ_I	$\overline{h}$	λ	τ	$s_I = s_E$	δ_E and $\overline{h}$	τ and $s_I = s_E$	(7)/(8)
Gini Coefficient	1.66	−3.00	0.67	0.69	0.43	0.55	1.79	0.92	1.94
Top 1% Share	1.64	−3.43	0.68	0.43	0.28	0.35	1.83	0.57	3.24
Bottom 40% Share	1.43	−3.58	0.69	1.32	0.81	1.04	1.28	1.78	0.72

Table 5:

Measure: Ω_2	D: Double-Pareto Approximated Series								
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
	δ_E	δ_I	$\overline{h}$	λ	τ	$s_I = s_E$	δ_E and $\overline{h}$	τ and $s_I = s_E$	(7)/(8)
Gini Coefficient	0.91	−1.75	0.20	0.27	0.22	0.27	0.93	0.92	1.01
Top 1% Share	0.88	−2.07	0.20	0.16	0.13	0.16	0.93	0.57	1.64
Bottom 40% Share	0.80	−2.08	0.21	0.54	0.43	0.52	0.73	1.78	0.41

at the 1981 level. Consider the left graph. For a constant δ_E, the Gini coefficient falls rather than rises. It means that the effect of a reduction of δ_E is so strong that if it is removed, then the Gini coefficient follows a clear negative trend. In this sense, a falling δ_E made a significant contribution to an increase in the Gini coefficient. A similar pattern arises in the right graph of the top 1% income share. It would have fallen below 10% in 2010 if entrant R&D productivity had been left unchanged in 1981. The middle graph shows the bottom 40% income share. A δ_E-fixed series is trend-less or has a slightly positive trend, while the Data and Double-Pareto Prediction series fall. It confirms that a fall in δ_E had large impacts on different aspects of inequality.

Turning to Panel (b), it shows the case of fixing incumbent R&D productivity δ_I. In sharp contrast to δ_E, the trends of the δ_I-fixed series are all reversed.[44] For example, consider the Gini coefficient. If δ_I was fixed at the 1981 level, it would have increased as shown in the left graph. It means that a decreasing incumbent R&D productivity mitigated inequality measured by the Gini coefficient. The top 1% share in right graph is similarly interpreted. In the case of the bottom 40% income share in the middle graph, it would have been as low as 4% (ignoring an observation with less than 1% in 2003) with a δ_I fixed at the 1981 level. These imply that the worsening of inequality is mitigated due to a declining incumbent productivity.

An intuition for these results of entrant and incumbent R&D productivity levels is straightforward. Consider the case of δ_E being fixed first. If δ_E is kept at the 1981 level, changes in g_E become minimal, while a falling δ_I tends to reduce g_I. As a result, the left and right Pareto exponents $\xi = (1 - g_E)/g_I$ and $\zeta = g_E/g_I$ both tend to increase. In Figure 5, this means that an economy moves northeastward from $A0$ for a constant δ_E. Fixing δ_I is the opposite case where

[44] Data of years after 2008 are all dropped from the figure because they make either ξ less than one or the Gini coefficient greater than one.

equilibrium moves southwestward.

To quantify the contrasting results of fixing δ_E and δ_I, let us turn to Columns (1) and (2) of Table 4. It uses the end-year deviation Ω_1 as a measure of contribution with the Double-Pareto approximated series used for D in (54). The numbers in Column (1) are all positive, while negative in Column (2). The same pattern remains in Table 12 with the average cumulative measure Ω_2. These results concerning the relative roles of incumbent/entrant firms are in line Garcia-Macia *et al.* (2019) which find a dwindling role of entrant innovation in TFP growth in the 1983-2013 period.

The same quantifying approach is applied to $\overline{h}$, λ, τ and $s_E = s_I$. Panel (c) of Figure 11 shows that the $\overline{h}$-fixed series are trend-less, though they are more volatile compared with the δ_E-fixed series. It means that a falling $\overline{h}$ has a positive impact on the inequality indices, though its effect is less than δ_E, as confirmed in Column (3) of Tables 4 and 5. Panel (a) of Figure 12 shows the case of the quality step λ. It exhibits a volatile pattern similar to $\overline{h}$, and its impacts are also comparable to $\overline{h}$, as Column (4) of the tables confirm. In Panels (b) and (c) of the figure, the τ-fixed and $s_E = s_I$ fixed series follow more steady patterns. Visual inspection of the graphs indicates their significant impacts, which are confirmed in Columns (5) and (6) of Tables 4 and 5.

6.4 Declining Business Dynamism and Policy Changes

Having considered the impacts of each parameter separately, there are two issues that we consider next. First, how are those parameter changes interact in generating the X inequality relationship? Do they reinforce or contract each other? Second, the following difference seems to have emerged. The effects of δ_E and $\overline{h}$ on the three inequality indices, documented in Tables 4 and 5 are similar in magnitude, whereas λ, τ and $s_I = s_E$ affected the bottom 40% income share more because the magnitude of their impacts on the bottom share is about twice as large as the top 1% income share and the Gini coefficient. How do we interpret these results? To address those questions, we group those parameters (except λ) into two. One group consists of δ_E and $\overline{h}$ capturing an aspect of a declining business dynamism in the U.S., and another group of τ and $s_I = s_E$ consists of changing fiscal policy measures.

A declining business dynamism is characterized by a falling pace of startups and new businesses with an increasing share of older firms. As Acemoglu *et al.* (2018) argue, it would lead to adverse impacts on growth and productivity because it means a slower pace of reallocation of resources from less efficient to more efficient businesses. To the extent that new firms' innovations, involving job creation and destruction, are important to productivity growth, a declining business dynamism, observed in the U.S. at least since 1980, is a serious concern to policy makers.[45] Evidently, data show that an incentive for new firms to enter the market declined. In particular, according to Decker *et al.* (2014), a declining business dynamism is observed in almost all sectors and all geographic regions, though variations exist.[46] Whatever factors working behind the phenomenon, it is captured by a falling δ_E and $\overline{h}$ in our model. To assess the contribution of a declining business dynamism to the X inequality relationship, let us apply the method used above. That is, we fix those two parameters at the 1981 level and change others. The results are shown

[45] Startup firms account for about 20 percent of total job creation (see Decker *et al.* (2014)).

[46] As factors that are not sector-specific and region-specific, Decker *et al.* (2014) suggest regulation increasing adjustment costs (e.g.Gutiérrez, Jones and Philippon (2019)) and technological progress plus globalization favoring big businesses. Decker, Haltiwanger, Jarmin and Miranda (2016a) refer to network externalities which work in favor of big firms, and Akcigit and Ates (2019b) argue that a slower knowledge diffusion from frontier firms to lagging firms is a possible cause. Astebro, Braguinsky and Ding (2020) report that an increasing burden of knowledge in R&D and management discouraged startups, again favoring big firms, while population growth slowdown is cited as an important factor in Peters and Walsh (2019).

in Column (7) of Tables 4 and 5. The magnitude of the impacts are certainly large. However, compared with δ_E, an increase in the magnitude is not particularly dramatic. In addition, the magnitude even slightly fell for the bottom 40% income share. What it suggests is that δ_E and $\overline{h}$ have a relatively large "substitutability" in explaining the X inequality relationship, especially for the bottom income share.

Let us turn to the fiscal policy τ and $s_I = s_E$. Akcigit and Ates (2019b) consider changes in the fiscal policy as a possible cause for a declining business dynamism. According to the study, the U.S. went through major tax system overhauls in the 1980s with a substantial reduction of a statutory corporate tax rate. They also showed that an effective tax rate, which takes into account various tax benefits and determines actual tax bills, also dramatically fell. Akcigit and Ates (2019b) also explains an increasing intervention in supporting R&D in the period. The US government began a federal R&D tax credit in 1981, and in the next year state-level support started in Minnesota and spread to other states. Major recipients were incumbent firms because taxable profits were needed for the tax credit. In order to quantify their contribution to the X inequality relationship, τ and $s_I = s_E$ are held fixed at the 1981 level, letting other parameters change. Consider Column (8) of Table 4 first. The magnitude increases nearly in a linear way in the sense that summing the numbers in (5) and (6) approximately gives the magnitude in Column (8). In Table 5, on the other hand, the result is more dramatic because of the cumulative nature of the index Ω_2. The number in (8) is nearly twice as large as the sum of (5) and (6). In this sense, those policy measures are "complimentary" and their changes reinforce the effects of the other .

Given the above discussion, two results stand out. The effect of a declining business dynamism seems to have generated a larger impact on the Gini coefficient than the fiscal policy changes, though their impacts are comparable when the cumulative index Ω_2 is used in Table 5. Second, Column (9) shows the ratio of (7) over (8). It indicates that the top income share is more affected by a declining business dynamism, and the bottom income share by the policy changes. In this sense, the two factors operated on different aspects of inequality to a different degree.

7 Conclusion

Inequality can be measured in different ways. The Gini coefficient and the top/bottom income shares are often used in the literature. The Gini coefficient is a summary measure of the entire distribution and the income shares show how the chosen part of the distribution changes relative to the whole distribution. Although they show different aspects of inequality, data show that they are systematically related. That is, the Gini coefficient is negatively related to the bottom p_B% income share and positively to the top p_T% income share, giving rise to what we call the X inequality relationship. It is certainly intuitive that they are related in an observed way, but equations remain. How do we explain it? What economic forces are working behind?

We explore these issues by constructing a Schumpeterian growth model which gives rise to a double-Pareto distribution of income as a result of entrant and incumbent innovations. A double-Pareto income distribution allows us to develop iso-Gini loci and iso-income share schedules in a tractable way. In equilibrium, the rates of incumbent and entrant innovations determine the Left and Right Pareto exponents, which in turn characterize a market equilibrium. Comparative statics analysis shows that changes of most parameters generate the X inequality relationship. The results imply that incumbent and entrant innovations play an important role in generating the X inequality relationship.

We also used the model to quantify the underlying factors behind the relationship in the

U.S. via calibration. Making use of innovation-related data to pin down parameter values, we consider the impact of each parameter on equality indices. We found that the largest impact was caused by deterioration of entrant R&D productivity. Calibration also shows a fall in incumbent R&D productivity, which was found to mitigate inequality. These contrasting results highlight the important roles played by different types of innovations behind the X inequality relationship. In addition, we also grouped parameters into two; one capturing fiscal policy changes and the other for a declining business dynamism. Both are certainly important in understanding the X inequality relationship. But the latter seems to be have a particularly important implication, because some studies (e.g. Fikri *et al.* (2017) and Furman and Orszag (2018)) point out that a declining business dynamism is behind an increasing inequality in the U.S.

Our calibration analysis focuses upon the U.S. only. But, the X inequality relationship holds in other countries. Furthermore, Calvino, Criscuolo and Verlhac (2020) provides evidence of a declining business dynamism being "pervasive" in many countries. Our result indicates the possibility that those two phenomena are related in those economies as well.

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Appendix A Derivation of (22)

Footnote 23 gives $f_L(z) = C_L z^{\zeta-1}$ where $C_L = \frac{\xi\zeta}{(\xi+\zeta)\overline{z}^{\xi}}$. It is the number of entrepreneurs at z in the left distribution. Define T_z as the time period just required for them to reach $\overline{z}$, starting from z. Then their number after T_z falls to

$$f_L(z)\, e^{-g_E T_z} \tag{55}$$

because some exit due to entrant innovations. Now consider an entrant which innovates at t_z with h. Her profit at $t \geq t_z$ is given by $z = e^{g_I(t-t_z)} h\overline{z}$ where $e^{g_I(t-t_z)} h \leq 1$. After T_z, it increases to $z = \overline{z}$, at which

$$e^{g_I T_z} h = \overline{h} \quad \Rightarrow \quad T_z = -\frac{\ln \frac{z}{\overline{z}}}{g_I}. \tag{56}$$

Substituting (56) into (55) yields $\frac{C_L}{\overline{z}^{\zeta}} z^{\xi+\zeta-1}$, which is the number of entrepreneurs who reach $\overline{z}$ starting from z. Integrating it from 0 to $\overline{z}$ yields the flow of entrepreneurs reaching $\overline{z}$

$$\int_0^{\overline{z}} \frac{C_L}{\overline{z}^{\zeta}} z^{\xi+\zeta-1} dz = \frac{\xi\zeta}{(\xi+\zeta)^2}.$$

Equating it to $g_E F_R(z)$ and using (16) yields (22).

Appendix B Derivation of (28) and Iso-Gini Contours

Using (24), first calculate the total net profit which is also equal to the average net profit

$$Z_{\text{total}} = \int_0^{\infty} z f(z)\, dz = \frac{\xi\zeta}{(\xi+1)(\zeta-1)} \overline{z} \tag{57}$$

where $f(z)$ is given in footnote 23. Then, the Gini coefficient G is defined by

$$Z_{\text{total}} G = \int_0^{\infty} F(z)\left[1 - F(z)\right] dz = G_L(\overline{z}) + G_R(\overline{z})$$

where

$$\begin{aligned} G_L(\overline{z}) &= \int_0^{\overline{z}} \frac{\zeta}{\xi+\zeta} \left(\frac{z}{\overline{z}}\right)^{\xi} \left[1 - \frac{\zeta}{\xi+\zeta}\left(\frac{z}{\overline{z}}\right)^{\xi}\right] dz \\ &= \overline{z} \frac{\xi\zeta}{(\xi+\zeta)^2} \cdot \frac{2\xi+1+\zeta}{(\xi+1)(2\xi+1)} \end{aligned}$$

and

$$\begin{aligned} G_R(\overline{z}) &= \int_{\overline{z}}^{\infty} \left[1 - \frac{\xi}{\xi+\zeta}\left(\frac{z}{\overline{z}}\right)^{-\zeta}\right] \frac{\xi}{\xi+\zeta}\left(\frac{z}{\overline{z}}\right)^{-\zeta} dz \\ &= \overline{z} \frac{\xi\zeta}{(\xi+\zeta)^2} \cdot \frac{\xi+2\zeta-1}{(\zeta-1)(2\zeta-1)} \end{aligned}$$

after tedious rearrangement. Now, making use of $G_L(\overline{z})$ and $G_R(\overline{z})$, the Gini coefficient is re-expressed as (28) (again after tedious rearrangement).

To calculate the slope of an iso-Gini contour, note that

$$\begin{aligned} \frac{\partial G}{\partial \xi} &= -\frac{2\zeta(\zeta-1)(2\zeta+4\xi+1)}{(\xi+\zeta)^2(2\zeta-1)(2\xi+1)^2} < 0, \\ \frac{\partial G}{\partial \zeta} &= -\frac{2\xi(\xi+1)(4\zeta+2\xi-1)}{(\xi+\zeta)^2(2\zeta-1)^2(2\xi+1)} < 0. \end{aligned}$$

These allow us to derive the following:

$$\left.\frac{d\xi}{d\zeta}\right|_{G=\overline{G}} = -\frac{\xi(\xi+1)(2\xi+1)(4\zeta+2\xi-1)}{\zeta(\zeta-1)(2\zeta-1)(2\zeta+4\xi+1)} < 0 \tag{58}$$

To show convexity of an iso-Gini curve, define $b \equiv \frac{\xi}{\zeta}$ so that

$$\left.\frac{d\xi}{d\zeta}\right|_{G=\overline{G}} = -b^2 \frac{(\xi+1)(2\xi+1)\left[4\xi + b(2\xi-1)\right]}{(\xi-b)(2\xi-b)\left(2\frac{\xi}{b} + 4\xi + 1\right)}.$$

One can easily show $\frac{\partial}{\partial b}\left(-\left.\frac{d\xi}{d\zeta}\right|_{G=\overline{G}}\right) > 0$, establishing the desired result.

Appendix C Derivation of (29) and (30)

First note $\bar{p} = F(\bar{z}) = \frac{\zeta}{\xi+\zeta}$ and the total income Z_{total} is defined in (57). Now define the bottom percentile $p_B \leq \bar{p}$ such that $p_B = F(z(p_B))$ where $z(p_B)$ is net profit at p_B. This definition gives

$$z(p_B) = \bar{z}\left(\frac{\xi+\zeta}{\zeta}p_B\right)^{\frac{1}{\xi}}.$$

Using this result, calculate the cumulative income up to $z(p_B)$

$$Z(p_B) = \int_0^{z(p_B)} zf(z)\,dz = \frac{\xi\zeta}{(\xi+\zeta)(\xi+1)}\left(\frac{\xi+\zeta}{\zeta}p_B\right)^{1+\frac{1}{\xi}}\bar{z}.$$

Then, the bottom $100p_B\%$ income share is defined by $S_B = \frac{Z_L(p_B)}{Z_{\text{total}}}$, which gives (29). It is straightforward to calculate the slope of S_B

$$\left.\frac{d\xi}{d\zeta}\right|_{\text{Bottom}} = -\frac{\frac{\partial S_B}{\partial \zeta}}{\frac{\partial S_B}{\partial \xi}} = -\frac{\frac{\xi(\xi+1)}{\zeta(\zeta-1)}}{1+\left(1+\frac{\zeta}{\xi}\right)L_B(p_B,\xi,\zeta)} \tag{59}$$

where $L_B(p_B,\xi,\zeta) = \log\frac{1}{p\left(1+\frac{\xi}{\zeta}\right)} > 0$ because

$$p_B\left(1+\frac{\xi}{\zeta}\right) < \bar{p}\left(1+\frac{\xi}{\zeta}\right) = \frac{\zeta}{\xi+\zeta}\left(1+\frac{\xi}{\zeta}\right) = 1.$$

Convexity can also be shown, but omitted.

Next, define the top percentile $1-p_T$ for $p_T \geq \bar{p}$ such that $p_T = F(z(p_T))$ where $z(p_T)$ is net profit at p_T. This definition gives

$$z(p_T) = \bar{z}\left(\frac{\xi+\zeta}{\xi}(1-p_T)\right)^{-\frac{1}{\zeta}}.$$

Calculate the cumulative income up to $z(p_T)$

$$Z(p_T) = \int_0^{z(p_T)} zf(z)\,dz = \frac{\xi\zeta}{(\xi+\zeta)(\zeta-1)}\bar{z}\left\{\frac{\xi+\zeta}{\xi+1} - \left(\frac{\xi+\zeta}{\xi}(1-p_T)\right)^{1-\frac{1}{\zeta}}\right\}.$$

Then, the top $100(1-p_T)\%$ income share is defined by $S_T = 1 - \frac{Z(p_T)}{Z_{\text{total}}}$, which gives (30). Its slope is

$$\left.\frac{d\xi}{d\zeta}\right|_{\text{Top}} = -\frac{\frac{\partial S_T}{\partial \zeta}}{\frac{\partial S_T}{\partial \xi}} = -\frac{\xi(\xi+1)}{\zeta(\zeta-1)}\left[1+\left(1+\frac{\xi}{\zeta}\right)L_T(1-p_B,\xi,\zeta)\right] \tag{60}$$

where $L_T(1-p_T,\xi,\zeta) = \log\frac{1}{(1-p_T)\left(1+\frac{\zeta}{\xi}\right)} > 0$ because

$$(1-p_T)\left(1+\frac{\zeta}{\xi}\right) < (1-\bar{p})\left(1+\frac{\zeta}{\xi}\right) = \frac{\xi}{\xi+\zeta}\left(1+\frac{\zeta}{\xi}\right) = 1.$$

Convexity can also be shown, but omitted. Comparing (59) and (60) confirms

$$\left|\left.\frac{d\xi}{d\zeta}\right|_{\text{Bottom}}^{p_B<\bar{p}}\right| < \left|\left.\frac{d\xi}{d\zeta}\right|_{\text{Top}}^{p_T>\bar{p}}\right|.$$

Appendix D Relative Slopes of Iso-Gini, Iso-S_B and Iso-S_T Curves

Using (59), one can easily confirm that

$$\frac{\partial}{\partial p_B}\left(\left|\left.\frac{d\xi}{d\zeta}\right|_{\text{Bottom}}^{p_B<\overline{p}}\right|\right)>0. \tag{61}$$

It means that an iso-S_B curve pivots anti-clockwise around a given (ξ,ζ) with a lower p_B. Similarly, using (60),

$$\frac{\partial}{\partial p_T}\left(\left|\left.\frac{d\xi}{d\zeta}\right|_{\text{Bottom}}^{p_T>\overline{p}}\right|\right)>0 \tag{62}$$

which implies that an iso-S_T contour pivots clockwise around a given (ξ,ζ) as p_T becomes larger. Also note that

$$\left.\frac{d\xi}{d\zeta}\right|_{\text{Bottom}}^{p_B=\overline{p}}=\left.\frac{d\xi}{d\zeta}\right|_{\text{Top}}^{p_T=\overline{p}}=-\frac{\xi(\xi+1)}{\zeta(\zeta-1)}.$$

from (59) and (60).

Now, using (58)

$$\left|\left.\frac{d\xi}{d\zeta}\right|_{G=\overline{G}}\right|-\left|\left.\frac{d\xi}{d\zeta}\right|_{\text{Bottom}}^{p_B=\overline{p}}\right|=\frac{4\xi(\xi+1)(\xi+\zeta)}{\zeta(\zeta-1)(2\zeta-1)(2\zeta+4\xi+1)}(\zeta+1-\xi)$$

This shows that there are two possible cases:

$$\begin{aligned}
&\text{Case 1:} && \left|\left.\frac{d\xi}{d\zeta}\right|_{G=\overline{G}}\right|\geq\left|\left.\frac{d\xi}{d\zeta}\right|_{\text{Bottom}}^{p_B=\overline{p}}\right| && \text{for } \zeta+1\geq\xi\\
&\text{Case 2:} && \left|\left.\frac{d\xi}{d\zeta}\right|_{G=\overline{G}}\right|<\left|\left.\frac{d\xi}{d\zeta}\right|_{\text{Bottom}}^{p_B=\overline{p}}\right| && \text{for } \zeta+1<\xi
\end{aligned}$$

First consider Case 1. (62) means that an increase in p_T makes an iso-S_T curve pivots clockwise. Hence, if p_T increases sufficiently and p_B falls only slightly, then

$$\left|\left.\frac{d\xi}{d\zeta}\right|_{\text{Bottom}}^{p_B<\overline{p}}\right|<\left|\left.\frac{d\xi}{d\zeta}\right|_{G=\overline{G}}\right|<\left|\left.\frac{d\xi}{d\zeta}\right|_{\text{Bottom}}^{p_B>\overline{p}}\right| \tag{63}$$

arises. Turning to Case 2, if p_B decreases sufficiently and p_T rises only slightly, then (63) holds.

Appendix E Growth Rate

First, consider utility maximization with the Lagrangian equation

$$\mathcal{L}=e^{\frac{1}{J}\int_0^J \ln Y_j dj}+\mu\left[E-\int_0^J P_jY_jdj\right].$$

The F.O.C is $\frac{U}{JY_j}=\mu P_j$. Using this and the budget constraint gives (31). Substituting this back into (1) gives

$$U=\frac{E}{JP_U} \tag{64}$$

where

$$P_U=e^{\frac{1}{J}\int_0^J \ln P_j dj}=1 \tag{65}$$

is the price index which we normalize to one.

Next consider profit maximization of final output producers:

$$\Pi_{Yj} = P_j e^{\int_0^1 \ln q_{ij} y_{ij} di} - \int_0^1 p_{ij} y_{ij} di.$$

The F.O.C. is $\frac{P_j}{y_{ij}} e^{\int_0^1 \ln q_{ij} y_{ij} di} = p_{ij}$, which gives $P_j Y_j = p_{ij} y_{ij}$ and hence (32). Plug this into (2) with the highest quality levels to obtain the price index in final output industry j:

$$\ln P_j = \int_0^1 \ln p_{ij} di - \int_0^1 \ln q_{ij} di.$$

Substitute this into (65) and rewrite the resulting equation with $p_{ji} = \lambda w$ for monopoly products and $p_{ji} = w$ for competitive goods, we obtain

$$1 = \frac{w}{Q} e^{\frac{N}{J} \ln \lambda} \tag{66}$$

It means $\frac{\dot{Q}}{Q} = \frac{\dot{w}}{w}$. This together with (64) and (65) also means $\frac{\dot{Q}}{Q} = \frac{\dot{U}}{U}$.

Appendix F Endogenizing g_I

Rewrite (36) as

$$0 = \max_{n_i(t)} \left\{ \begin{array}{l} n_i(t) \left[(1-\tau) \Lambda \frac{E}{J} - (1-s_I) w R_{Ii}(t) \right] \\ + V_i'(n_i(t)) \delta_I R_{Ii}(t)^{\gamma} n_i(t) - (\rho + g_E) V_i(n(t)) + \dot{V}_i(t) \end{array} \right\}$$

The F.O.C. is

$$R_{Ii}(t) = \left(\frac{V_i'(n_i(t)) \gamma \delta_I}{(1-s_I) w} \right)^{\frac{1}{1-\gamma}}$$

Assume $V = V_i(n_i)/n_i$. Then, the F.O.C. is reduced to (38). Using (38), rewrite (36) to give (39).