Some Topics In Mathematics

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Note :

I have added references to figures taken from internet. Consider them as sources for further reading.

Exercise

- Evaluate this integral

$$\int \sqrt{\sin x} \;\; dx$$

- Solve this differential equation

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0$$

Approximation for Differential Equation

- For small(?) θ, $\sin\theta \approx \theta$

$$\theta(t) = \theta_0 \cos t$$

- Maclaurin expansion of sine

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$

Sets

- A set is a well-defined collection of distinct objects.
- Examples :
  - $C = \{4, 2, 1, 3\}$
  - $D = \{5, 6, 7\}$
  - $T = \{+, -\}$
- $|C|$, $C \cup D$, $C \cap D$

Applications of Sets

VBS - Very Bad Syndrome

- X : Set of People in a clinical trial
- S : $\{x \in X : x \textit{ has VBS}\}$
- H : $\{x \in X : x \textit{ does not have VBS}\}$
- $X = S \cup H$
- $S \cap H = \phi$

- P : $\{x \in X : x \textit{ tests positive for VBS}\}$
- N : $\{x \in X : x \textit{ tests negative for VBS}\}$
- $X = P \cup N$
- $P \cap N = \phi$

Ideally

- $S = P$ and $H = N$

Consider

- $S \cap P, \;\; H \cap N, \;\; S \cap N, \;\; H \cap P$
- True Positives, True Negatives, False Negatives, False Positives

Numbers of Interest are

- $\frac{|S|}{|X|}, \frac{|H|}{|X|}$
- $\frac{|S \cap P|}{|S|}, \;\; \frac{|H \cap N|}{|H|}, \;\; \frac{|S \cap N|}{|S|}, \;\; \frac{|H \cap P|}{|H|}$

Functions

A function $f : A \longrightarrow B$ is a rule which transforms each a $\in$ A into $f(a) \in B$

Example :

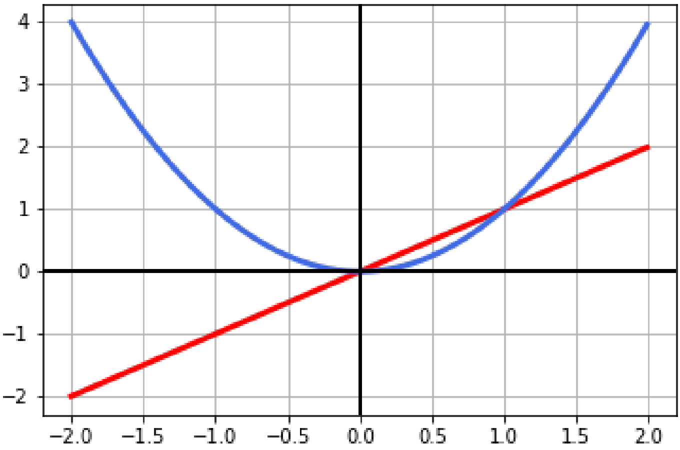
- let X : *All people in the VBS study*
- let Y : $\{+, -\}$
- Ideally *test* : $X \longrightarrow Y$,

$$test(x) = \begin{cases} + & : x \in S \\ - & : x \in H \end{cases}$$

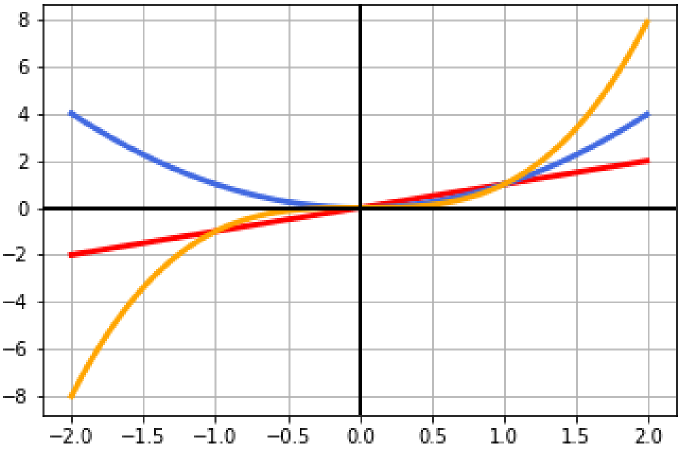
Some Common Functions

$f(x) = x$

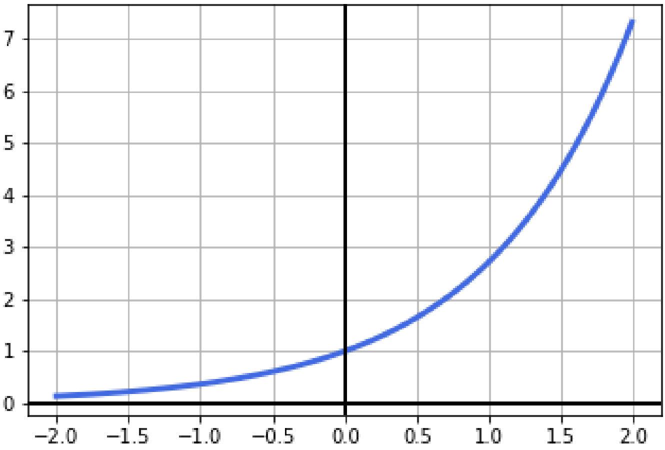
x, x^2



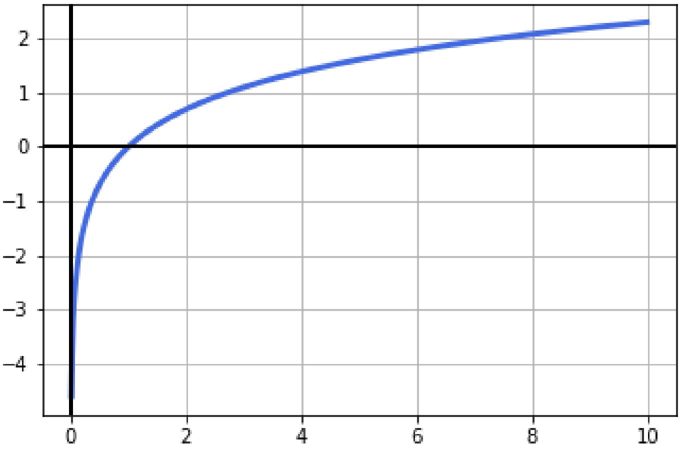
x, x^2, x^3



$f(x) = e^x$



f(x) = ln(x)



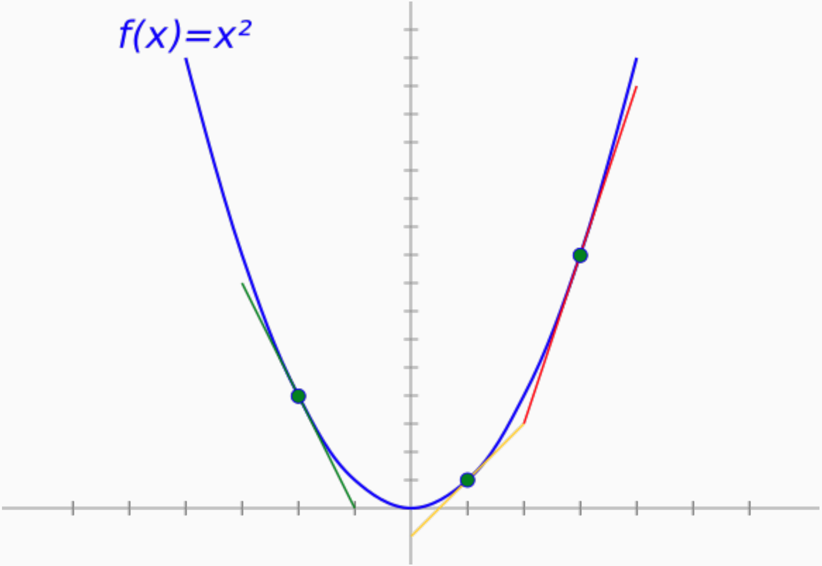
Derivative

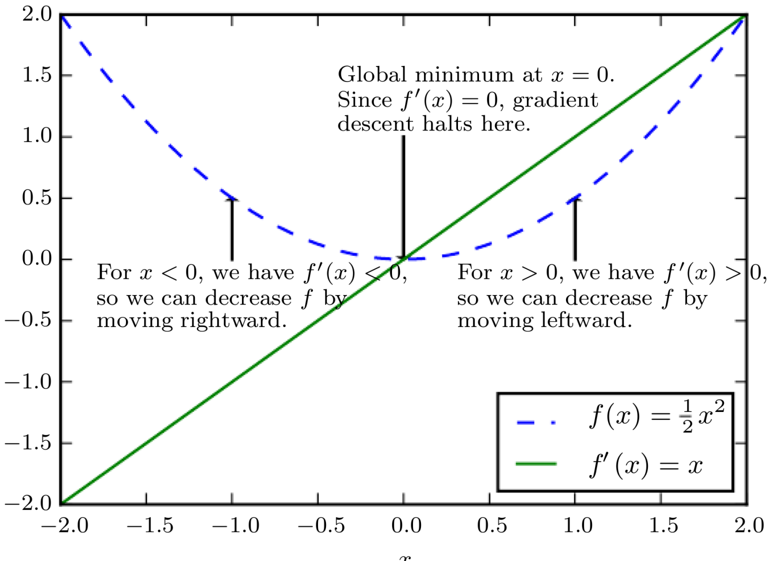
For a function f, the derivative is defined as

$$f^{'}(x) = \lim_{\epsilon \to \infty} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

So we have,

$$f(x+\epsilon) \approx f(x) + \epsilon\, f^{'}(x)$$


f(x)=x²

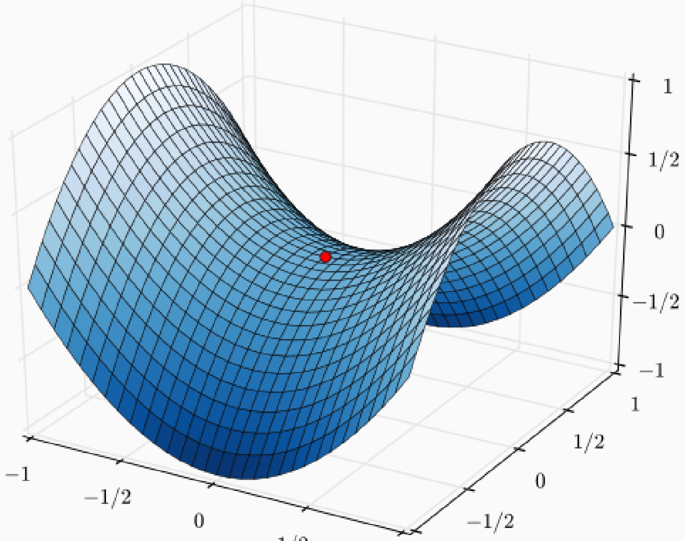

Global minimum at $x = 0$.
Since $f'(x) = 0$, gradient
descent halts here.
For $x < 0$, we have $f'(x) < 0$,
so we can decrease f by
moving rightward.
For $x > 0$, we have $f'(x) > 0$,
so we can decrease f by
moving leftward.
$f(x) = \frac{1}{2}x^2$
$f'(x) = x$
x

we find that,

$$f(x - \epsilon \, sign(f^{'}(x)) < f(x)$$

So we can reduce f by moving in x in small steps with opposite sign of the derivative.

Multivariate Functions



Directional Derivative

The derivative of a function along the direction of a vector $\vec{v}$ is called the directional derivative, given by

$$\mathcal{D}_{\vec{v}} f(x) = \lim_{h \to 0} \frac{f(\vec{x} + h\,\vec{v}) - f(\vec{x})}{h}$$

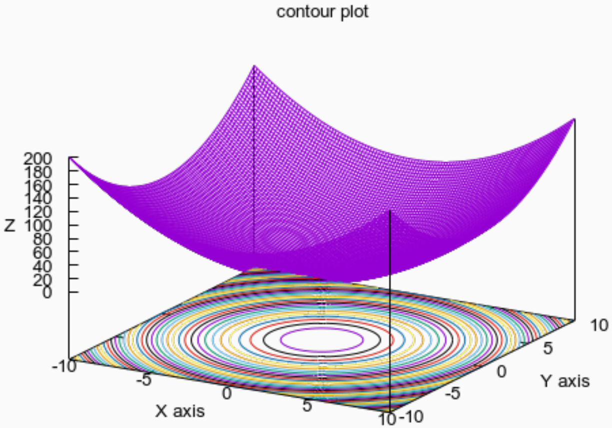
Gradient

$$\mathcal{D}_{\vec{v}} f(x) = \vec{\nabla} f(x) \cdot \vec{v}$$

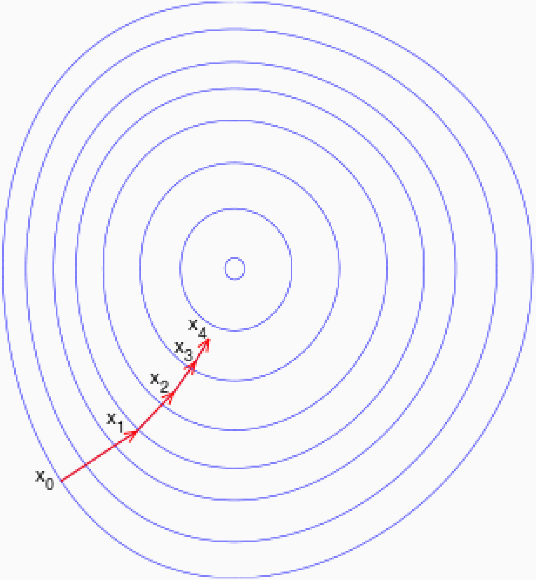
$$\vec{\nabla} = \frac{\partial}{\partial x_1}\hat{x_1} + \frac{\partial}{\partial x_2}\hat{x_2} + \dots$$

$$|a \cdot b| = |a||b| \cos(\theta)$$

The gradient gives the maximum space rate of change.


contour plot
200
180
160
140
120
Z 100
80
60
40
20
0
-10
-5
0
5
10
X axis
-10
-5
0
5
10
Y axis

Gradient Descent



The vector form

$$\vec{x} = \vec{x} - \alpha \vec{\nabla} F(\vec{x}), \quad n \geq 0$$

Each component can be treated separately,

$$\begin{aligned}
x_0 &= x_0 - \alpha \frac{\partial F(\vec{x})}{\partial x_0} \\
x_1 &= x_1 - \alpha \frac{\partial F(\vec{x})}{\partial x_1} \\
x_2 &= x_2 - \alpha \frac{\partial F(\vec{x})}{\partial x_2} \\
&\vdots
\end{aligned}$$

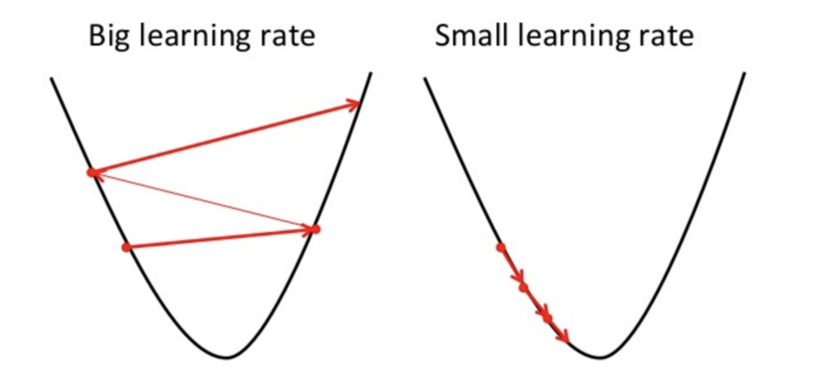
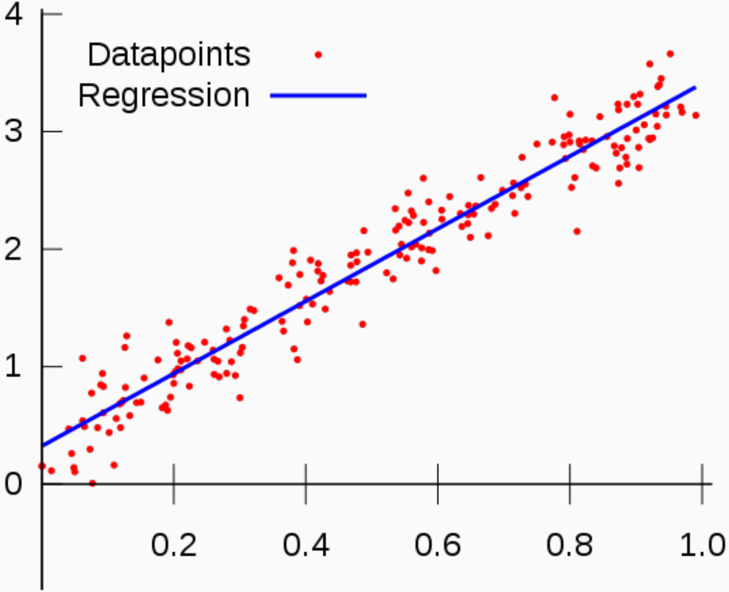


Figure 5: taken form : https://towardsdatascience.com/gradient-descent-in-a-nutshell-eaf8c18212f0

Example : Linear Regression



“Distance”

For all $a \in F$ and all $u, v \in V$,

- $p(av) = |a|\, p(v)$
- $p(u + v) \leq p(u) + p(v)$
- $p(v) \geq 0$
- If $p(v) = 0$ then $v=0$ is the zero vector

Examples

L2 norm

- $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

L1 norm

- $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \sum_{i=1}^{n} |x_i - y_i|$

Cost Function

Predicted value

- $$\mathbf{Y}_{pred} = \beta_0 + \beta_1\mathbf{x_1} + \beta_2\mathbf{x_2} + \ldots + \beta_n\mathbf{x_n}$$

RMSE

- $$\sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_{pred}^{(i)} - y^{(i)})^2}$$

Gradient Descent

repeat until convergence : $\{$

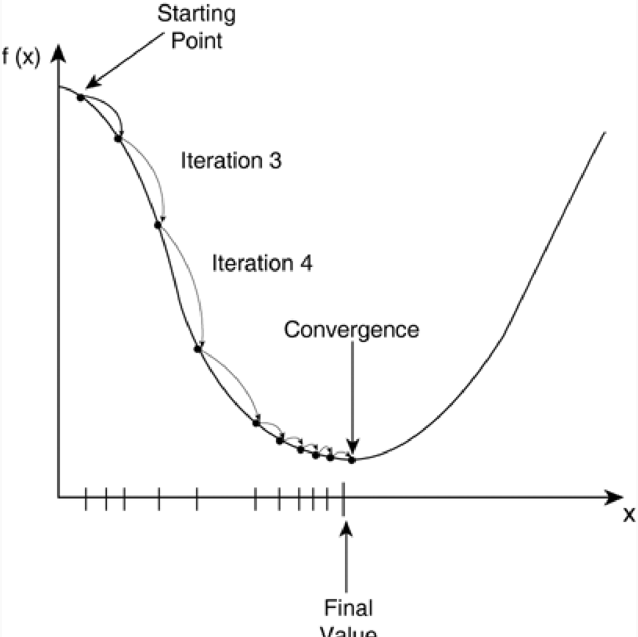
$$\beta_0 := \beta_0 - \alpha \frac{1}{n} \sum_{i=1}^{n} (y_{pred}^{(i)} - y^{(i)}).x_0^{(i)}$$

$$\beta_1 := \beta_1 - \alpha \frac{1}{n} \sum_{i=1}^{n} (y_{pred}^{(i)} - y^{(i)}).x_1^{(i)}$$

$$\vdots$$

$$\beta_n := \beta_n - \alpha \frac{1}{n} \sum_{i=1}^{n} (y_{pred}^{(i)} - y^{(i)}).x_n^{(i)}$$

$\}$


Starting
Point
f (x)
Iteration 3
Iteration 4
Convergence
x
Final