Black-Scholes and Normal Hybrid Stochastic Volatility Model

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Abstract

We propose an option pricing model where the dynamics of asset price gradually changes from normal model to Black-Scholes model as the asset price increases.

Keywords: stochastic volatility, Black-Scholes model, Bachelier model, SABR model

1. Introduction

The dynamics of asset price under the Black-Scholes model is given by the geometric Brownian Motion,

$$dF_t = \sigma F_t \, dW_t.$$

and that under the Bachelier (normal) model is given by

$$dF_t = \sigma_{\mathrm{N}} \, dW_t.$$

Interest rate is known to follow the BSM model when the rate F_t is well below h and the normal model when F_t is well above h for some reference rate h. Therefore, we want to construct a hybrid model.

Assume

$$dF_t = \sigma C(F_t) dW_t.$$

The local volatility function $C(x)$ should satisfy the following properties:

- $C(x) \approx x$ when x is small. Therefore, σ plays the role of the BSM volatility when x is small.
- $C(x) \to h$ when x is large. The parameter h is understood as a scale of F_t. Therefore, $\sigma_N = h\sigma$ plays the role of the normal volatility when x is large.

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2. Previously known methods

When $C(x) = x^\beta$, the model is known as the constant elasticity of variance (CEV) model:

$$\frac{dF_t}{F_t^\beta} = \sigma dW_t.$$

It is well known that the option price is given as

$$C_\beta(K, F_0, \sigma) = F_0\, Q_{\text{NCX2}}\left(\frac{q^2(K)}{\sigma^2 T};\, 2 + \frac{1}{1-\beta}, \frac{q^2(F_0)}{\sigma^2 T}\right) - K\, P_{\text{NCX2}}\left(\frac{q^2(F_0)}{\sigma^2 T};\, \frac{1}{1-\beta}, \frac{q^2(K)}{\sigma^2 T}\right),$$

where $q(x) = x^{1-\beta}/(1-\beta)$, and $P_{\text{NCX2}}(x; k, x_0)$ and $Q_{\text{NCX2}}(x; k, x_0)$ are the cumulative probability density (CDF) and survival functions, respectively, of the noncentral chi-squared distribution with degree of freedom k and noncentrality parameter x_0.

Hagan and Woodward (1999) derived the approximate equivalent BSM volatility for general $C(x)$. The solution is somewhat accurate near-the-money options.

3. Ideas

3.1. Candidate 1

For our first candidate, we consider

$$C(x) = \frac{hx}{h+x}.$$

This function satisfies the requirements. Ito's lemma leads to

$$\left(\frac{1}{F_t} + \frac{1}{h}\right) dF_t = \sigma dW_t$$

$$d(\log F_t + F_t/h) = \sigma dW_t - \frac{\sigma^2 h^2}{2(F_t+h)^2} dt$$

The integral in the left hand side is related to the Lambert W function $W(z)$ (Wikipedia) defined by

$$z = W(z\, e^z) \quad \text{or} \quad W^{-1}(z) = z\, e^z.$$

When z is small, $W(z) \approx z$ and, when z is large, $W(z) = \log z$.

To simplify the calculation, we assume that F_t is not far way from the initial point F_0.

$$d(\log F_t + F_t/h) = \sigma dW_t - \frac{\sigma^2 h^2}{2(F_0+h)^2} dt$$

Then,

$$F_T e^{F_T/h} = F_0 e^{F_0/h} \exp\left(\sigma W_T - \frac{\sigma^2 h^2 T}{2(F_0+h)^2}\right)$$

$$F_T = hW\left((F_0/h)e^{F_0/h} \exp\left(\sigma W_T - \frac{\sigma^2 h^2 T}{2(F_0+h)^2}\right)\right)$$

When $F_0 \ll h$, the dynamics follows the BSM model ($W(z) \approx z$):

$$F_T = F_0 \exp\left(\sigma W_T - \frac{\sigma^2 T}{2}\right).$$

When $F_0 \gg h$, the dynamics follows the normal model ($W(z) \approx \log z$):

$$F_T = F_0 + \sigma W_T.$$

Can we price the option quickly?

$$C(K) = E\left((F_T - K)^+\right)$$

3.2. Candidate 2

We consider

$$C(x) = h \tanh(x/h)$$

which also satisfies the requirements.

Using

$$\int \frac{dx}{h \tanh(x/h)} = \int \frac{\cosh(x/h)}{h \sinh(x/h)} dx = \log(\sinh(x/h)) + \text{const.},$$

we obtain:

$$\frac{\cosh(F_t/h)}{h \sinh(F_t/h)} dF_t = \sigma dW_t$$

$$d \log(\sinh(F_t/h)) = \sigma dW_t + \frac{\sigma^2}{2}(\tanh^2(F_t/h) - 1)dt.$$

[Check the computation.]

Again, assuming that $F_t \approx F_0$,

$$\sinh(F_T/h) = \sinh(F_0/h) \exp\left(\sigma W_T + \frac{\sigma^2 T}{2}(\tanh^2(F_0/h) - 1)\right)$$

$$F_T = h \sinh^{-1}\left(\sinh(F_0/h) \exp\left(\sigma W_T + \frac{\sigma^2 T}{2}(\tanh^2(F_0/h) - 1)\right)\right)$$

We find the similar asymptotic behavior in this candidate too. Reminded that $\sinh^{-1} z = \log(z + \sqrt{1+z^2})$. Therefore, $\sinh z \approx z$ when $z \ll 1$, $\sinh z \approx \log(2z)$ when $z \gg 1$. When $F_0 \ll h$, the dynamics follows the BSM model:

$$F_T = F_0 \exp\left(\sigma W_T - \frac{\sigma^2 T}{2}\right).$$

When $F_0 \gg h$, the dynamics follows the normal model:

$$F_T = F_0 + \sigma W_T.$$

4. Research questions

- How do we price the option? Any approximation?
- How can we run Monte-Carlo simulation efficiently? (It will be the subject of the *Applied Stochastic Processes* course.)

5. Extension

Eventually, I want to solve the stochastic volatility model with the local volatility function $C(x)$:

$$dF_t = \sigma_t C(F_t) dW_t, \quad \sigma_t = \alpha \sigma_t dZ_t, \quad \text{and} \quad dW_t dZ_t = \rho\, dt.$$

When $C(x) = x^\beta$, the model is known as stochastic-alpha-beta-rho (SABR) model (Hagan et al., 2002).

References

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