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RESEARCH ARTICLE

Alternatives to the normal model of stock returns: Gaussian mixture, generalised logF and generalised hyperbolic models

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Abstract Simple parametric models of the marginal distribution of stock returns are an essential building block in many areas of applied finance. Even though it is well known that the normal distribution fails to represent most of the "stylised" facts characterising return distributions, it still dominates much of the applied work in finance. Using monthly S&P 500 stock index returns (1871–2005) as well as daily returns (2001–2005), we investigate the viability of three alternative parametric families to represent both the stylised and empirical facts: the generalised hyperbolic distribution, the generalised logF distribution, and finite mixtures of Gaussians. For monthly return data, all three alternatives give reasonable fits for all sub-periods. However, the generalised hyperbolic distribution fails to describe some features of the marginal distributions in some sub-periods. The daily return data are much more symmetric and expose another problem for all three distributions: the parameters describing the behaviour of the tails also influence the scale so that simpler alternatives or restricted parameterisations are called for.

Keywords Stock returns · Non-normality · Gaussian mixtures · Generalised hyperbolic distribution · Generalised logF distribution

JEL Classification G12 · C16

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1 Introduction

It is a “stylised fact” that marginal distributions of stock returns are very poorly described by the normal distribution. It has been established that return distributions have thick tails, are skewed and leptokurtic (see for example Campbell et al. 1997; Eijgenhuijsen and Buckley 1999; Cont 2001). Despite these well-known facts, the finance literature is still very much dominated by the normal distribution. This is partly due to theoretical arguments: if returns follow a process with independent increments with finite second moment, time aggregated returns should be close to the normal distribution. In the early 1960s, this argument was extended and, dropping the assumption of finite second moments, symmetric stable distributions with index in $(1, 2)$ were put forward. Later, interest shifted to the study of the extremes of the return distribution, suggesting regularly varying distributions with power index close to three (Longin 1996; Guillaume et al. 1997). However, financial time series are characterised by dependencies of GARCH type that might well invalidate inferences for tail indices (Kearns and Pagan 1997). In response to these difficulties and in an attempt to generalise the Black–Scholes model (which implies normality of returns), Heston (1993) included a stochastic volatility term in the Black–Scholes model for the pricing of certain options. Assuming restrictions on the stochastic process describing the volatility, one can obtain the marginal distribution of returns after integrating out the volatility term for different time horizons. The solutions of the stochastic differential equations can only be written down as a Fourier integral (Drăgulescu and Yakovenko 2002) and imply exponential tails for time horizons between an hour and a month (Silva et al. 2004; Rehmer and Mahnke 2004) while shorter time horizons are governed by power type tails. It turns out that at a time horizon of a day the Laplace (or double exponential) distribution is a good approximation to the Heston model, while at considerably longer horizons distributions with Gaussian tails might fit. Takahashi (2004) and Vicente et al. (2006) refer to a simple Ehrenfest urn model to motivate the several types of distributional forms.

Partly inspired by these results and partly because of the practical need for flexible parametric distributions of returns that allow for the “stylised facts”, several families with exponential tails have been proposed. The most prominent is probably the generalised hyperbolic distribution. This five parameter family includes skew leptokurtic densities with thicker tails than the normal while still having moments of all orders (Barndorff-Nielsen 1977; Eberlein and Keller 1995; Küchler et al. 1999). However, the routine application of the generalised hyperbolic distribution is hampered by the non-existence of a closed form for the distribution function and rather slow simulation methods.

Kon (1984) examined daily returns from 30 different stocks and estimated mixtures of Gaussian distributions with two up to four components which were found to fit appropriately. Mixture models can be seen as a useful way of generalising a given family of distributions (see for example Badrinath and Chatterjee 1988; Peiro 1994; Aparacio and Estrada 2001). While there is no closed form for the distribution function of Gaussian mixtures, it can easily be approximated using well known formulae implemented in many well tested software routines. Simulation from Gaussian mixtures is also easy and fast and estimation can conveniently be carried out using the EM algorithm (McLachlan and Krishnan 1997).

Another flexible alternative to the normal and generalised hyperbolic distribution is the generalised logF distribution. It has exponential tails and can represent skew leptokurtic densities. In contrast to the generalised hyperbolic distribution there exist very efficient simulation algorithms. To our knowledge the generalised logF distribution has been applied very rarely in the finance context, the exception being Brown et al. (2002).

Our empirical investigation focuses on the monthly returns from the S&P 500 stock index. This data set has been examined extensively in the literature, using a host of models. Moreover, the data date back to 1871 thus enabling to study the stability of the distribution over time. We also examine daily returns in the period from 2001 to 2005 to investigate the appropriateness of the parametric families for shorter time horizons.

The result of our analysis is fourfold. First, we present evidence of the non-normality of returns for both the complete period 1871–2005 as well as for nearly all 10-year sub-periods. Secondly, we show that the generalised hyperbolic distribution fails to describe pertinent features of the distribution of returns for many sub periods. Thirdly, the adequacy of the mixed Gaussian and the generalised logF model is demonstrated for almost all sub periods where we use moving 10-year windows circumventing arbitrary choices of sub periods. Fourthly, for daily returns, the data suggest a nearly symmetric distribution. But in all models, the parameters that govern the tail behaviour and the skewness also influence the scale of the model. In consequence, the respective parameters are only barely identified. This may be remedied by restricting the parameters. However, while the peakedness of the empirical distributions favours the hyperbolic distribution, the parameter governing the peakedness tends to a boundary value and estimation becomes unstable. The two parameter Laplace distribution or its two-sided version, both special cases of the generalised hyperbolic distribution, are a reasonable alternative.

The paper is structured as follows. In Sect. 2, we present the distributions and their properties. In Sect. 3, we provide descriptive evidence and estimation results from analysing the distribution for the complete time span of 134 years. Section 4 contains the results from assessing the stability of the results in moving 10-year windows. Section 5 investigates whether results obtained for monthly returns continue to hold for daily return data. Section 6 concludes.

2 Three alternatives to the normal distribution

Throughout the paper, we denote the monthly return by x_t, calculated as the monthly difference of the logarithm of the price index X_t:

$$x_t = \log(X_t) - \log(X_{t-1})$$

2.1 The generalised hyperbolic distribution

The hyperbolic distribution has been used by geomorphologists to model the shape of dunes of windblown sand (Barndorff-Nielsen 1977). Due to its flexibility the hyperbolic model was found to provide a good model for the distribution of asset returns (Eberlein and Keller 1995; Küchler et al. 1999) and has been applied for value at risk modelling (e.g. Bauer 2000).

The generalised hyperbolic distribution is described by five parameters $(\lambda, \alpha, \beta, \delta, \mu) =: \Psi$. Its probability density function is given by:

$$f_{GH}(x; \Psi) = \kappa \left\{\delta^2 + (x-\mu)^2\right\}^{\frac{1}{2}\left(\lambda - \frac{1}{2}\right)} K_{\lambda - \frac{1}{2}}\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right) e^{\beta(x-\mu)} \quad (1)$$

with $0 \leq |\beta| < \alpha$, $\delta > 0$, where

$$\kappa = \frac{\left(\alpha^2 - \beta^2\right)^{\frac{\lambda}{2}}}{\sqrt{2\pi}\alpha^{\lambda - \frac{1}{2}}\delta^{\lambda} K_{\lambda}\left(\delta\sqrt{\alpha^2 - \beta^2}\right)}. \quad (2)$$

The function $K_\lambda(t)$ is the modified Bessel function of the third kind with index λ, also known as the MacDonald function. It is defined as

$$K_\lambda(t) = \frac{1}{2}\int_0^\infty u^{\lambda-1} e^{-\frac{1}{2}t\left(u+u^{-1}\right)}\, du\,, \quad t > 0. \quad (3)$$

The distribution function has no closed form expression and is generally found from numerically integrating the density. The density is unimodal, the distribution is infinitely divisible and moments of all order exist. The form of the density can accommodate all of the stylised facts about distributions of returns, allowing for leptokurtic, platykurtic, left and right skewed distributions. The tails are proportional to

$$f_{GH}(x; \Psi) \sim |x|^{\lambda-1} e^{-\alpha|x| + \beta x} \quad x \to \pm\infty \quad (4)$$

so that the tails are exponential unless $\alpha \to |\beta|$, in which one tail behaves as a power law.

Two special cases arise from $\lambda = 1$ (the hyperbolic distribution), and from $\delta \to 0$ (the variance-gamma distribution). Of particular interest in our applications is the case $\lambda = 1, \delta \to 0$ and $\beta = 0$. Using either formulae 9.6.9 and 10.2.17 of Abramowitz and Stegun (1965) or using the hyperbolic cosine version of (3), the density is seen to reduce to that of the Laplace (or double exponential) distribution with density

$$f_L(x; \alpha, \beta, \mu) = \frac{\alpha^2 - \beta^2}{2\alpha}\exp(-\alpha|x-\mu| + \beta(x-\mu)) \quad (5)$$

(see Kotz et al. 2001).

The generalised hyperbolic distribution can be simulated as a normal variance-mean mixture where the mixing distribution is the generalised inverse Gaussian distribution with any λ. However, this presupposes the ability to generate random variates from the generalised inverse Gaussian distribution, which is in itself rather slow.

2.2 The Gaussian mixture model

Gaussian mixture models as models of return distributions were suggested by Kon (1984). While it is difficult to maximise the likelihood of finite mixtures directly, the EM-algorithm provides a convenient estimation method. The algorithm was popularised by Dempster et al. (1977) in the context of missing data. The EM-algorithm is treated extensively by McLachlan and Krishnan (1997), details on mixture models are discussed by Everitt and Hand (1981).

We consider g component densities, assumed to be univariate normal densities $\phi(x; \mu_i, \sigma_i^2)$ with unknown means $\mu_1, \ldots, \mu_g$ and unknown variances $\sigma_1^2, \ldots, \sigma_g^2$. We denote the density for component i by $\phi(x; \theta_i)$, where $\theta_i = \left(\mu_i, \sigma_i^2\right)'$ and $\theta = \left(\mu_1, \ldots, \mu_g, \sigma_1^2, \ldots, \sigma_g^2\right)$. The vector Ψ of unknown parameters includes θ and the probabilities $\pi_1, \ldots, \pi_{g-1}$ for the g components ($g - 1$ probabilities suffice because $\pi_g = 1 - \sum_{i=1}^{g-1} \pi_i$).

The density is given by

$$f_{GM}(x; \Psi) = \sum_{i=1}^{g} \pi_i \phi(x; \mu_i, \sigma_i^2). \tag{6}$$

This is an unimodal density under some constraints on the variances and the distances between the means (see Block et al. 2005).

2.3 The generalised logF distribution

The density of the generalised logF distribution with parameters $\Psi = (a, b, \mu, \sigma)$ is given by

$$f_{LF}(x; \Psi) = \frac{a^a b^b}{B(a, b)} e^{a\frac{x-\mu}{\sigma}} \left(b + a e^{\frac{x-\mu}{\sigma}}\right)^{-(a+b)}, \quad a, b > 0 \tag{7}$$

making use of the beta function

$$B(a, b) = B(b, a) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}. \tag{8}$$

The logF distribution has a remarkable flexibility, incorporating a number of well known distributions as special cases. These include the normal, the log-Gamma and the extreme value distributions. Being a simple transform of the well-known

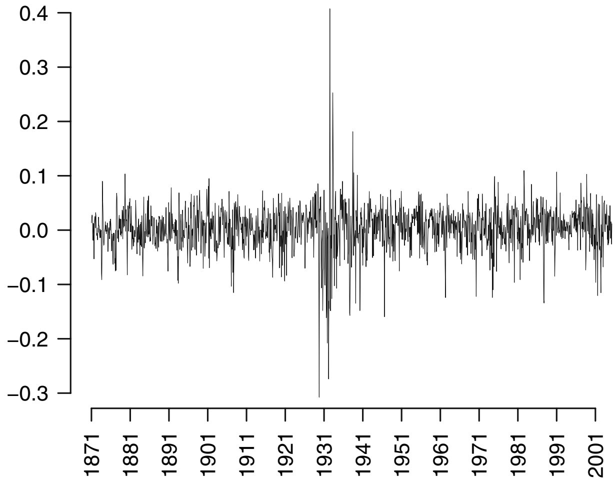


Fig. 1 S & P 500 stock index, monthly returns 1871–2005

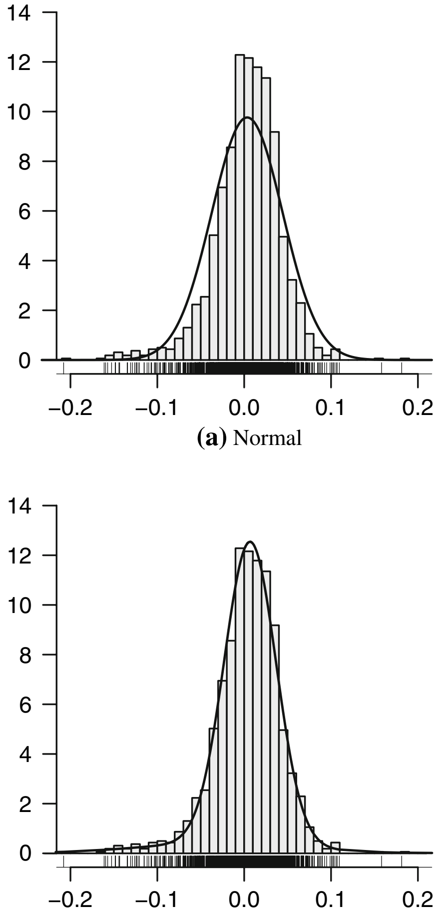
F-distribution, formulae for the approximation of the distribution function as well as algorithms for the computation of moments, quantiles, and other quantities of interest are readily available. Moreover, the density is always unimodal and moments of all order exist. Further properties have been described by Wu et al. (2000).

3 The distribution of S&P500 monthly rate of return 1871–2005

In this section, we provide descriptive evidence on the distribution of the S&P 500 monthly returns and model the distribution by means of Gaussian mixture, the generalised logF and the generalised hyperbolic distribution.

Figure 1 shows the time series of monthly returns of the S&P 500 stock index for the time span $1/1871 - 05/2005$.[1] The time series displays the well-documented high volatility around 1930 including the decline of 30.8% in November 1929 and the greatest increase of 40.7% in August 1932. Figure 2 shows that the empirical distribution is skewed to the left and reveals substantial kurtosis. The best fitting normal distribution ($\hat{\mu} = 0.0035$, $\hat{\sigma} = 0.0409$) cannot capture these features very well and consequently exhibits a very poor fit throughout the range of the data: at the centre of the data the frequencies are substantially underestimated, at a medium range from the centre frequencies are overestimated, especially so on the left, while in the tails, once again mainly on the left, frequencies are once again underestimated. Compared to the normal density, the estimated densities of the other three models provide a much superior fit. A formal test of the normality hypothesis is rejected by

[1] Data are taken from http://www.irrationalexuberance.com/index.htm.



(a) Normal

(b) Generalised Hyperbolic

(c) Two component Gaussian mixture

(d) Generalised logF

Fig. 2 Histogram and estimated densities of monthly returns

the Shapiro–Wilk, Jarque–Bera and Lilliefors[2] test, each test yielding p-values below 0.001.

The three alternative models were fitted using the Maximum Likelihood estimators of their parameters.[3] The estimated parameter vector for the generalised hyperbolic

[2] The classical Kolmogorov–Smirnov test statistics assume a fixed distribution to be tested against. Generalisations allowing for estimated parameters have been proposed only for a limited number of parametric families including the normal (see Dallal and Wilkinson 1986 for the Lilliefors-test). While results on generalisations to arbitrary smooth families are available (Shorack and Wellner 1986, Chap. 5.5) these are largely irrelevant in the present context. The dependence structure of return data will almost certainly upset any attempt to establish reasonable approximations to the null distribution of the test statistics. In consequence, also the formal test results for the normal distribution should not be taken literally.

[3] We estimated Gaussian models comprising between two and four components. Using more than two components led to only negligible fit improvements. Therefore we restrict the number of components in the following to two.

Table 1 Fit measures for estimated models

	d_K	d_I
Normal	0.0708	0.0070
Gen. hyperbolic	0.0228	0.0014
Gaussian mixture	0.0165	0.0014
Gen. logF	0.0230	0.0019

distribution is

$$\hat{\alpha} = 6.1515, \quad \hat{\beta} = -5.9524, \quad \hat{\delta} = 0.0577, \quad \hat{\mu} = 0.0123, \quad \hat{\lambda} = -2.1161.$$

The estimation results for the two component Gaussian mixture model are

$$f_{GM}\left(x\,;\ \hat{\pi}_1, \hat{\pi}_2, \hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1^2, \hat{\sigma}_2^2\right) = 0.9094 \cdot \phi\left(x\,;\ 0.0069, 0.0298^2\right) \\ +0.0906 \cdot \phi\left(x\,;\ -0.0307, 0.0907^2\right).$$

For the generalised logF distribution we obtain

$$\hat{\mu} = 0.0103, \quad \hat{\sigma} = 0.0120, \quad \hat{a} = 0.4165, \quad \hat{b} = 0.6161.$$

The two component mixture as well as the generalised logF and the generalised hyperbolic distributions lead to only minor differences between the empirical and estimated distributions. This is supported by a comparison of the Kolmogorov metric based on the maximal absolute difference of the empirical and estimated cumulative distribution function which we denote by d_K

$$d_K = \max_x \left|F^{\text{emp}}(x) - F^{\text{est}}(x)\right|. \tag{9}$$

We also calculate the integral of absolute deviations between $F^{\text{emp}}(x) - F^{\text{est}}(x)$ denoted by d_I:

$$d_I = \int_{-\infty}^{\infty} \left|F^{\text{emp}}(x) - F^{\text{est}}(x)\right|\, dx. \tag{10}$$

Table 1 gives the distances of the three estimated models. We find that the three flexible alternatives are doing about equally well in modelling the return distribution. The two component Gaussian mixture has the smallest maximal absolute difference between empirical and estimated distribution. The generalised logF has a slightly larger integral of absolute differences between empirical and estimated distribution compared to the mixture and the generalised hyperbolic distribution. Based on these two fit measures, the simple normal is once again seen to be inadequate for modelling the return distribution.

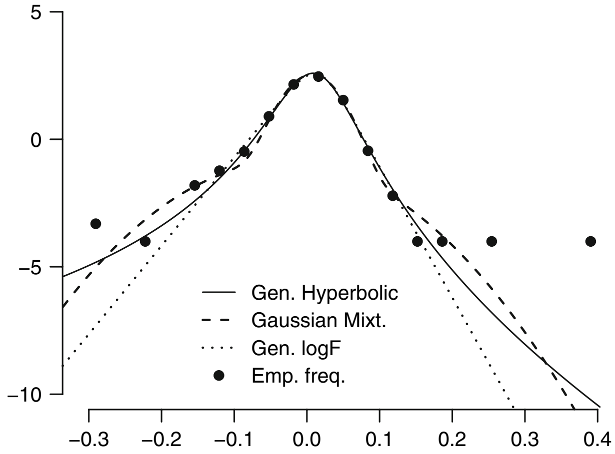


Fig. 3 Tail behaviour of the distributions

Table 2 Descriptives statistics of return distribution

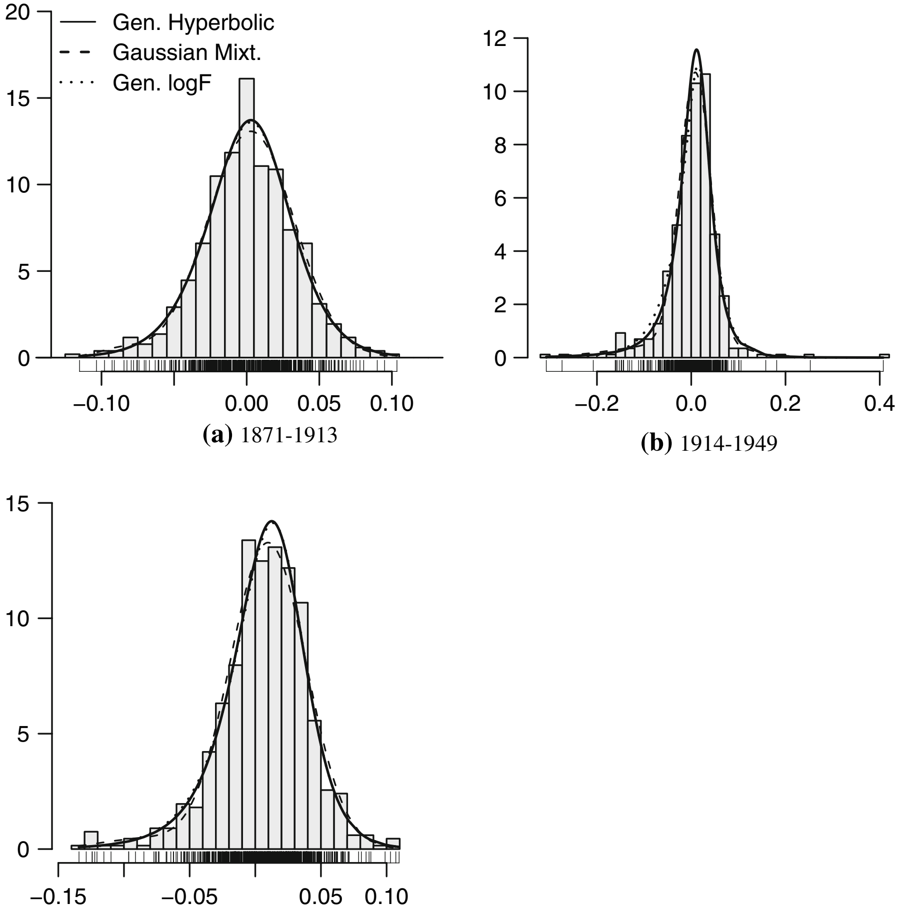
	Median	Mean	Std. dev.	Skewness	Kurtosis
1871–1913	0.0000	0.0012	0.0322	−0.19	0.70
1914–1949	0.0074	0.0017	0.0566	−0.12	10.39
1950–2005	0.0087	0.0064	0.0340	−0.71	2.04
1871–2005	0.0052	0.0035	0.0409	−0.32	11.47

Figure 3 shows the estimated densities on a log-linear scale. Additionally, the logarithms of the scaled relative frequencies in 21 bins of constant width are depicted. The figure makes evident the quadratic tail of the Gaussian mixture on the log-linear scale. The generalised logF as well as the generalised hyperbolic distributions both have exponential tails corresponding to linear tails on the log-linear scale. Note that the outermost two bins on both sides contain at most two observations so that the empirical support for a detailed analysis of tail behaviour is rather scarce.

3.1 Modelling the return distribution in three broad sub periods

When dividing the complete time period 1871–2005 into three broad sub periods, the first covering the time before World War I, the second period including World War I and II up to 1949, and finally, the period 1950–2005, the finding of non-normality is still evident for all sub periods.

The statistics (see Table 2) for the three broad time spans reveal that the second time period including the international economic crisis exhibits the strongest leptokurtic shape as well as a standard deviation about 70% higher than the first and last time period. The latest period, starting in 1950, shows the strongest evidence of skewness. The histograms and density estimates displayed in Fig. 4 make especially evident the heavy tales of the distribution pertaining to the period including World War I and II.



(a) 1871-1913 (b) 1914-1949 (c) 1950-2005

Fig. 4 Density estimates for three broad sub periods

Table 3 Hyperbolic distribution for broad subperiods

	n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\mu}$	$\hat{\lambda}$
1871–1913	515	67.5747	−5.1204	0.0475	0.0064	0.8907
1914–1949	432	8.4703	−4.7108	0.0477	0.0146	−1.1111
1950–2005	665	28.9025	−13.0831	0.0562	0.0201	−1.9422
1871–2005	1613	6.1491	−5.9507	0.0577	0.0123	−2.1161

Table 3 contains the estimates for the generalised hyperbolic distribution. Note that the estimates for α and β, influencing both the shape and skewness of the distribution as well as its tail behaviour, are rather large (and computationally unstable) for the first and last period.

Table 4 Two components Gaussian mixture for broad subperiods

	n	$\hat{\pi}_1$	$\hat{\pi}_2$	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\sigma}_1$	$\hat{\sigma}_2$
1871–1913	515	0.9808	0.0192	0.0029	−0.0864	0.0299	0.0153
1914–1949	432	0.8531	0.1469	0.0087	−0.0390	0.0332	0.1157
1950–2005	665	0.9656	0.0344	0.0097	−0.0869	0.0290	0.0311
1871–2005	1613	0.9094	0.0906	0.0069	−0.0307	0.0298	0.0907

Table 5 LogF distribution for broad subperiods

	n	$\hat{\mu}$	$\hat{\sigma}$	$\hat{a}$	$\hat{b}$
1871–1913	515	0.0029	0.0229	1.3247	1.5869
1914–1949	432	0.0145	0.0095	0.2291	0.3690
1950–2005	665	0.0132	0.0150	0.6140	1.0485
1871–2005	1613	0.0103	0.0120	0.4165	0.6161

Table 6 Fit measure d_1 for estimated models

		1871–1913	1914–1949	1950–2005
d_K	Normal	0.03038	0.10242	0.06291
	Gen. hyperbolic	0.03730	0.02983	0.02072
	Gaussian mixture	0.04046	0.03484	0.02241
	Gen. logF	0.03731	0.03591	0.02002
d_1	Normal	0.00178	0.01408	0.00407
	Gen. hyperbolic	0.00086	0.00376	0.00107
	Gaussian mixture	0.00097	0.00342	0.00105
	Gen. logF	0.00087	0.00402	0.00120

For the mixture distribution we find the estimates given in Table 4. The four parameters specifying the logF distribution for the three sub periods are given in Table 5.

To assess the fit of the three proposed models, we present in Table 6 the maxima of the absolute differences between observed and estimated cumulative distributions (d_K) and the integrals of these absolute differences (d_I) for the three sub periods. There is no uniformly best fitting model class. For the first period, the generalised hyperbolic and the logF do equally well. For the middle period, the generalised hyperbolic is best in terms of d_K, but the Gaussian mixture is best in terms of d_I. In the last period, the hyperbolic and the logF are best according to d_K, while according to d_I it is the Gaussian mixture and the hyperbolic distribution.

4 Assessing the stability of the models in different time periods

Because the analysis of three broad sub periods clearly revealed evidence of differences in the distribution of returns, in this section we assess whether one of these families

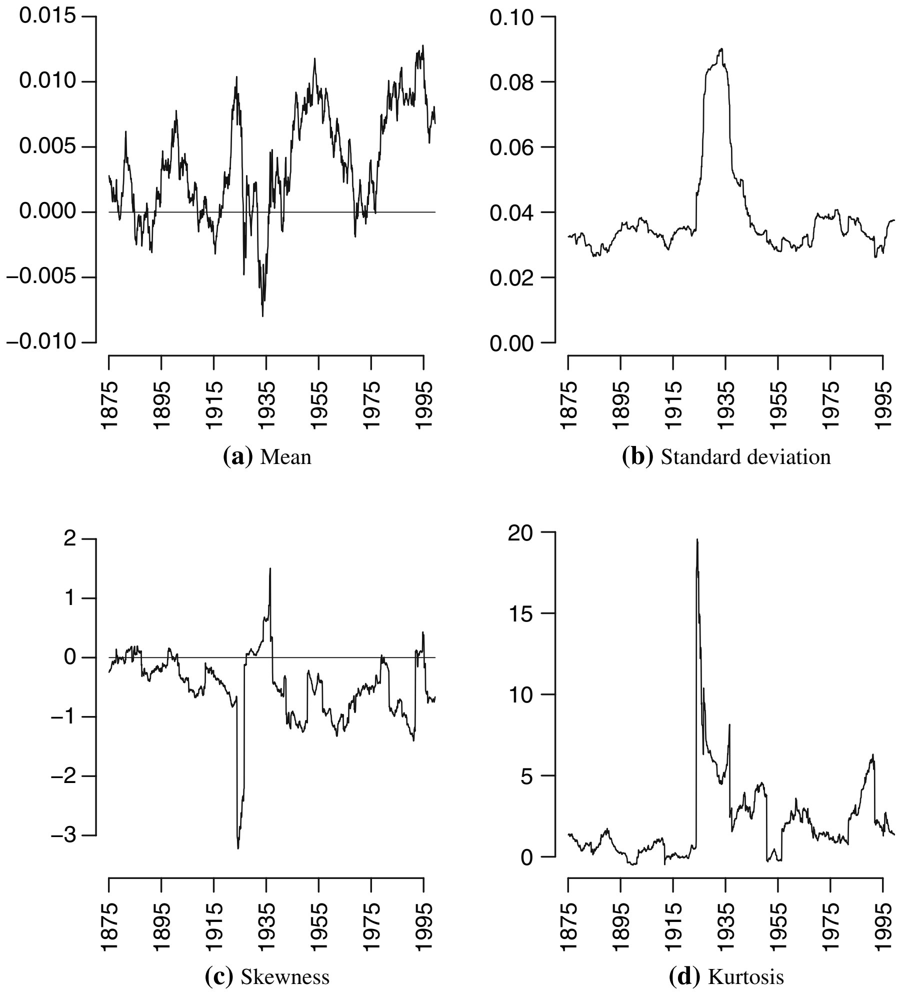


Fig. 5 Descriptive measures for rolling windows

is capable to represent the variety of shapes taken by the return distributions in 1,493 moving 10-year windows covering 134 years. Figure 5 contains descriptive measures for the rolling windows, revealing strong changes in the shape of the return distribution.

Figure 6 displays fit results in terms of d_I in 10 year sub periods for the models estimated only once for the complete time span. It is evident that especially around 1930 the models are not capable of adequately describing the return distribution. The results show that the Gaussian two-component mixture model and the generalised logF model are slightly superior to the generalised hyperbolic model on average over the 1,493 subsamples.

Since a single distribution can not be expected to represent adequately the distributions in all the subsamples, we now turn to a moving estimation strategy. For each

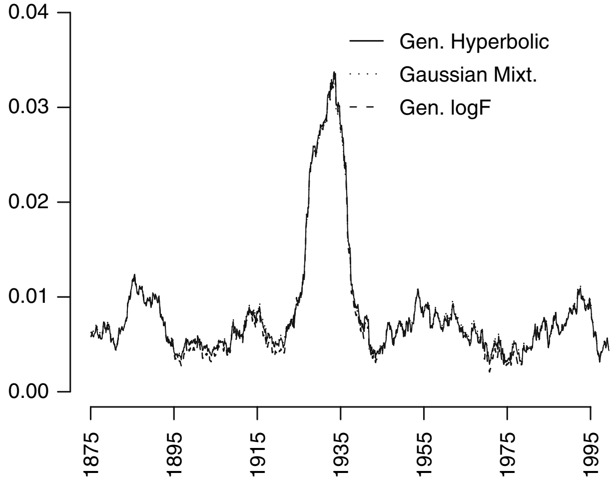


Fig. 6 Fit measure (d_I) for single parameter estimation

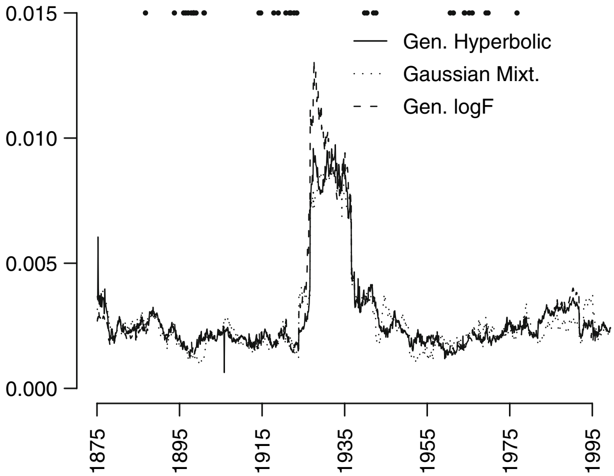


Fig. 7 Fit measure (d_I) for window specific parameter estimation

10-year window, we estimate the three models to account for the instability of the return distribution. Figure 7 summarises the results of the fit measure (d_I) applied to 1,493 windows.

Figure 7 reveals that in most windows the three flexible approaches are doing about equally well in modelling the returns distribution. Nevertheless, even with sub period specific estimation the return distributions around 1930 are hard to model despite of the flexibility of the models.

While the three models are doing about equally well in most sub periods, in 39 of the 1,493 rolling windows the generalised hyperbolic model could either not be estimated due to infinite values in the numerical optimisation routine or the cumulative

Table 7 Average fit measures in rolling windows

	Normal	Gen. hyperbolic	Mixture	Gen. logF
Single estimation	0.0101	0.0083	0.0084	0.0081
Window specific est.	0.0057	0.0028	0.0028	0.0029

distribution function could not be integrated due non convergence of the integral. These windows are indicated by dots at values of 0.03 in Fig. 7.

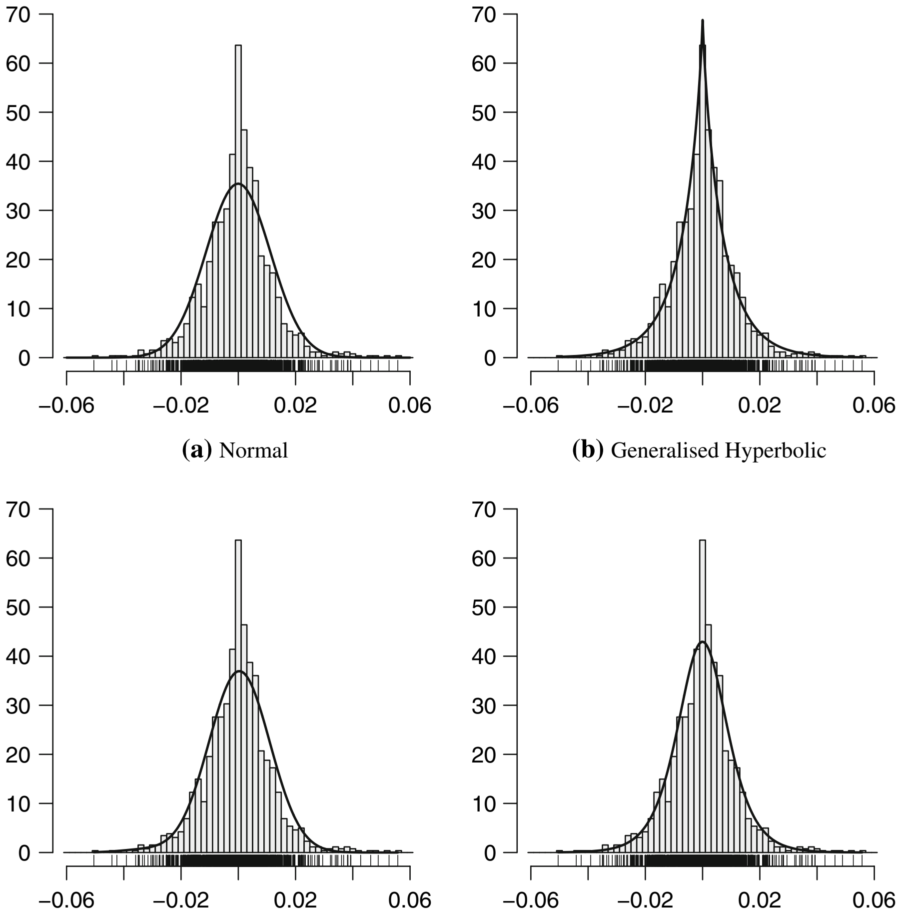
Table 7 shows that on average the generalised hyperbolic distribution as well as the Gaussian mixture and the generalised logF are doing equally well in modelling the return distribution. But it must be born in mind that the generalised hyperbolic distribution exhibited rather unstable estimation results so that in some 2.6% of the cases the computation of fit measures was impossible. This may in some cases be due to the naive use of Maximum Likelihood estimators with a re-parametrisation ensuring the constraint $|\beta| < \alpha$. However, a more common problem we detected when inspecting individual cases was divergence of α to infinity and/or convergence of δ to zero. In the latter case, the numerical evaluation of the likelihood becomes very unstable: On the one hand, the terms δ^λ and $K_\lambda(\delta(\alpha^2 - \beta^2))$, while theoretically of exactly opposite order for $\delta \to 0$, produce erratic numerical behaviour. On the other hand, also the terms $K_{\lambda-1/2}(\alpha\sqrt{\delta^2 + (x-\mu)^2}$ and $(\delta^2 + (x-\mu)^2)^{\lambda/2-1/4}$ should be of opposite order for $x_t \approx \mu$ and should cancel each other, but do not because of approximation errors. Moreover, the parameters $(\alpha, \beta, \delta, \lambda)$ simultaneously influence both shape and scale so that their separate identification may be difficult.

For the Gaussian mixture model, a well known problem with Maximum Likelihood estimators through the EM algorithm is their instability when either the variance or the proportion of a mixture component are close to 0. To remedy the problem, we restricted the smallest component of the Gaussian mixture model to generate in expectation at least two observations.

5 Modelling the distribution of daily returns

The results based on monthly returns in the previous section may be peculiar to the rather long aggregation period of a month, where central limit tendencies may be preponderant over short term characteristics of return distributions. We will next analyse daily S&P 500 return data for the 5-year period 2001–2005. The number of observations for these data is comparable to the number of observations for the monthly data. However, the autocorrelation structure and thus the statistical precision of estimates of marginal distributions are rather different.

Figure 8 shows a histogram of the daily returns with the estimated densities superimposed. The distribution is slightly skewed to the left and reveals substantial kurtosis. The estimated densities of the generalised logF and especially of the generalised



(a) Normal

(b) Generalised Hyperbolic

(c) Two component Gaussian mixture

(d) Generalised logF

Fig. 8 Histogram and estimated densities of daily returns

hyperbolic distribution provide a superior fit compared to both, the normal model and the two component Gaussian mixture.[4] Especially the very steep centre of the empirical distribution is well matched by the generalised hyperbolic distribution.

The estimated parameter vector for the generalised hyperbolic distribution is

$$\hat{\alpha} = 114.5074, \quad \hat{\beta} = -0.8683, \quad \hat{\delta} = 0.00004, \quad \hat{\mu} = 0.00002, \quad \hat{\lambda} = 0.8774$$

4 When testing the empirical distribution of daily returns, the normality hypothesis is again rejected by the Shapiro–Wilk, Jarque–Bera and Lilliefors tests at the 0.01 level.

Table 8 Fit measures for estimated models, daily returns

	d_K	d_I
Normal	0.0614	0.0015
Gen. hyperbolic	0.0374	0.0005
Gaussian mixture	0.0594	0.0012
Gen. logF	0.0378	0.0007

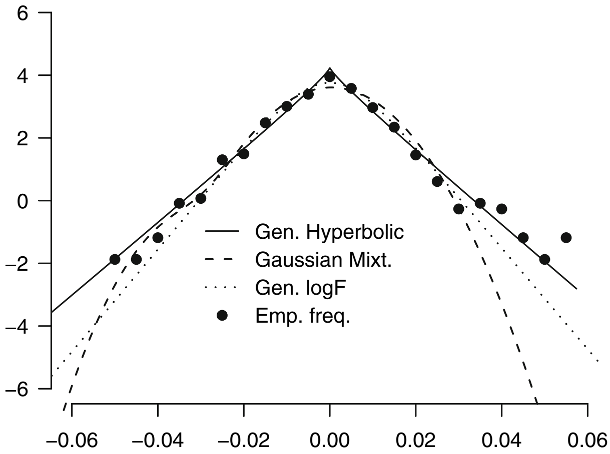


Fig. 9 Tail behaviour of the distributions

The estimation results for the two component Gaussian mixture model are

$$f_{GM}\left(x\,;\,\hat{\pi}_1,\hat{\pi}_2,\hat{\mu}_1,\hat{\mu}_2,\hat{\sigma}_1^2,\hat{\sigma}_2^2\right) = 0.9874\cdot\phi\left(x\,;\,0.0004, 0.0107^2\right) \\ +0.0126\cdot\phi\left(x\,;\,-0.0324, 0.0084^2\right).$$

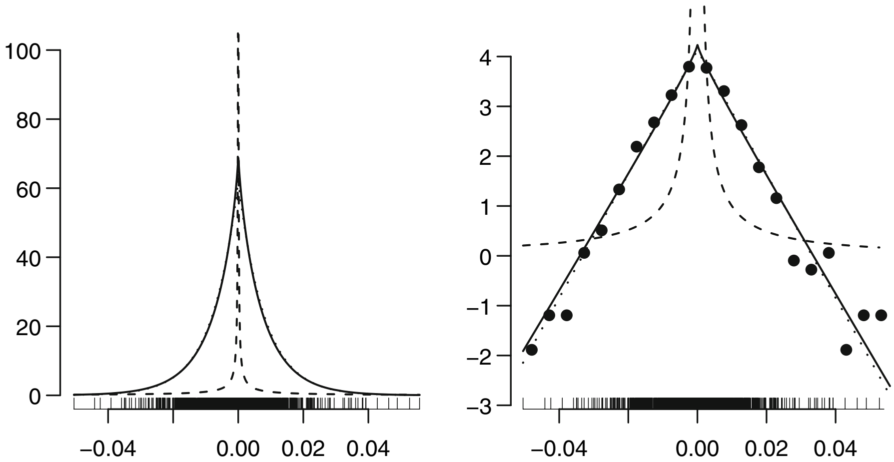
For the generalised logF distribution we obtain

$$\hat{\mu} = -0.00004,\quad \hat{\sigma} = 0.0053,\quad \hat{a} = 0.8649,\quad \hat{b} = 0.8546$$

Table 8 shows the fit measures which confirm the superior fit of the generalised hyperbolic and the generalised logF distribution in terms of d_I and the Kolmogorov metric d_K. Neither the normal model nor the two component Gaussian mixture model are able to capture the essential features of the empirical distribution.

Looking at the tail behaviour (Fig. 9), the generalised hyperbolic provides the best approximation. The quadratic form of the normal and Gaussian mixture distributions do not seem to be appropriate. And even the logF distribution with essentially the same tail behaviour does not fit too well.

For the centre of the distribution, however, the generalised hyperbolic distribution is the clear winner. But is it really? On closer inspection, there are several difficulties with the above results and particularly for the generalised hyperbolic distribution.



(a) Densities **(b)** log Densities

Fig. 10 Densities and log densities of the generalised hyperbolic (*solid*), the Laplace distribution (*dotted*), and the second solution of the score equations to the generalised hyperbolic distribution (*dashed*)

As we noted above, the maximum likelihood estimator of the generalised hyperbolic distribution becomes unstable for $\delta \to 0$. But clearly $\hat{\delta} = 0.00004$ is close to 0. Checking the result, it turns out that the solution given above is not the maximum likelihood estimator: The (numerically approximated) Hessian of the negative log-likelihood has a negative component.

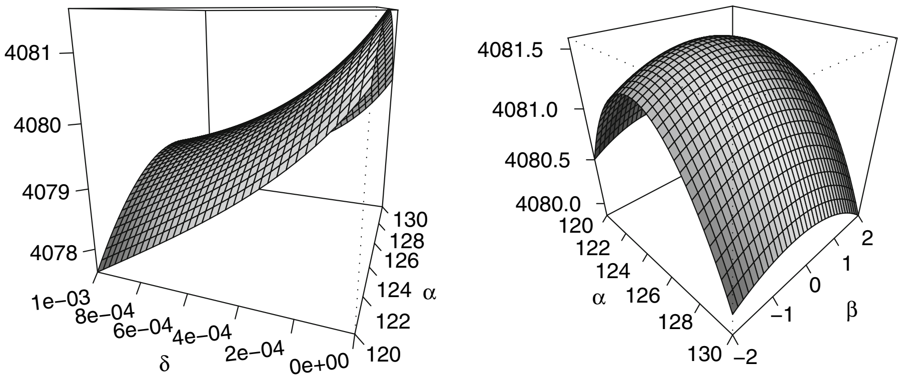
Using a more stable method for the computation of the likelihood function and a more reliable maximisation routine resulted in[5]

$$\hat{\alpha} = 2.0903, \quad \hat{\beta} = -1.5720, \quad \hat{\delta} = 2.1 \cdot 10^{-130}, \quad \hat{\mu} = 2.5 \cdot 10^{-135}, \quad \hat{\lambda} = 0.0113.$$

But this is still not the maximum likelihood solution, the Hessian once again not being definite. While the previous solutions provided a reasonable fit, this one does not: It is extremely steep and thus puts nearly all mass on the central few observations, basically neglecting all other information.

An interesting alternative suggested by the first solution, where λ is close to 1 and δ is close to 0, is to use the Laplace distribution with just 2 parameters. The fit of the Laplace distribution together with that of both solutions to the score equations of the generalised hyperbolic distribution are shown in Fig. 10. Relative frequencies of the data in equidistant intervals are represented by *bullet*.

[5] We used the ghFit function from the package fBasics by Diethelm Wuertz, version 251.70, with R, version 2.5.1. It uses the same function as the package HyperbolicDist, version 0.5–1, by David Scott for the computation of the log-likelihood function. When investigating the problems with some of the fits of the generalised hyperbolic model, we used the gsl package, version 1.6–10, by Robin K.S. Hankin, a wrapper for the GNU Scientific Library, version 1.9, and its function bessel_lnKnu() to compute the log-likelihood. That function approximates the logarithm of the MacDonald function quite well down to an argument of about 10^{-160}. We also used the Nelder-Mead algorithm for minimisation instead of a simple line search.



(a) Profile likelihood for (δ,α) **(b)** Profile likelihood for (α,β)

Fig. 11 Profile likelihoods for the hyperbolic distribution ($\lambda = 1$)

One may hope to alleviate the problem by fixing some of the parameters. This strategy is also suggested by the excellent fit of the Laplace distribution with only two parameters. Restricting attention to the hyperbolic distribution with $\lambda = 1$, however, reveals further problems. Figure 11 depicts the profile likelihoods for the hyperbolic distribution ($\lambda = 1$) for (α, β) and (α, δ) (the profile likelihood for (β, δ) is very similar to that for (α, δ)). Clearly, fixing λ at 1 helps to stabilise the estimating equations. In particular, the (α, β) profile likelihood is nearly quadratic. However, the profile likelihood for δ still indicates a boundary solution corresponding to the asymmetric Laplace distribution.

But the other two distributions meet similar problems. For the logF distribution, $\hat{a}$ nearly equals $\hat{b}$ so that the density is nearly symmetric. But $(\hat{a}, \hat{b})$ govern the tail behaviour and therefore influence the variance, and so does the scale parameter $\hat{\sigma}$. In fact, the correlations between $(\hat{a}, \hat{b})$ and $\hat{\sigma}$ are 0.86 and 0.83, respectively. Since the tails are exponential, the a and b parameters are themselves scale parameters for the left and right tail, respectively. If the data are nearly symmetric, the three parameters describe basically the same aspect of the data. As in the generalised hyperbolic case, one may try to remedy the situation by fixing one of the parameters. In fact, fixing σ at 0.01 results in $\hat{a} = 2.16$, $\hat{b} = 2.08$ and $\hat{\mu} = -0.00013$. This basically fixes the identifiability problem for the logF distribution.

For the mixture of Gaussians, the parameters are unidentified if either $\mu_1 = \mu_2$ or $p_2 = 0$. Problems of identification will not even show up automatically from diagnostics of numerical solutions since the EM algorithm was used. In consequence, the use of results from finite mixture models fitted by the EM algorithm should be carefully monitored for hints of over-parametrisation.

6 Conclusions

In this analysis we examined three flexible distributions, the generalised hyperbolic, the Gaussian mixture and the generalised logF distribution. The empirical evidence showed that the marginal distribution of monthly S&P 500 stock index returns for the very long time frame of 1871–2005 can be adequately described either by a generalised hyperbolic, a mixture of two Gaussian normal distributions or the generalised logF distribution.

To analyse the stability of the estimated models over time, a rolling window framework has been applied. Formal tests clearly reject the hypothesis of random draws from one estimated underlying distribution for a considerable fraction of the windows. A comprehensive analysis of all 10-year windows within the framework of a rolling window strategy reveals that period-specific estimated two-component Gaussian mixtures can adequately describe the empirical distributions in about 92% of 1,493 windows. The fit of the generalised logF model could be maintained in a slightly smaller share of periods only (about 88%). Given its flexibility and the comfortable estimation using the EM algorithm, Gaussian mixture models should be considered more frequently in empirical financial analysis. On the other hand, rather surprisingly the generalised hyperbolic distribution estimators did not even converge in 39 out of 1.493 sub periods and showed numerical instabilities in many more of the windows.

A different picture emerges when analysing daily return data. Here, on several accounts, the generalised hyperbolic distribution fits best, followed by the generalised logF. However, the very good fit of the generalised hyperbolic is once again counteracted by the practical difficulty of obtaining stable parameter estimates, a difficulty shared by the other two families. The daily return data are nearly symmetric so that the extra parameters governing the asymmetry may be superfluous. In fact, a simple two-parameter Laplace distribution fits almost as well as the five parameter generalised hyperbolic distribution.

While our aim here was to present parametric families that are capable to represent the spectrum of distributions encountered in medium to longer term return data, the results nevertheless seem to indicate a striking difference between daily and monthly return distributions. This is in accordance with the findings of Drăgulescu and Yakovenko (2002) and Rehmer and Mahnke (2004) as well as Silva et al. (2004).

From a practitioner's point of view, our findings suggest the use of either Gaussian mixtures or the logF distribution in day to day work for medium term time horizons. For shorter periods, a symmetric distribution like the double exponential might be a viable alternative. If short term data reveal substantial skewness, either a logF with restricted scale parameter or an asymmetric Laplace distribution seems appropriate. The generalised hyperbolic distribution does not offer such a superior fit for medium term data as to out-weight the numerical difficulties we experienced. For daily returns, it presumably is over-parameterised, at least for the data examined here.

References

Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. New York: Dover (1965)

Aparacio, F., Estrada, J.: Empirical distributions of stock returns: European security markets. Eur J Financ **7**, 1–21 (2001)

Badrinath, S.G., Chatterjee, S.: On measuring skewness and elongation in common stock return distributions: the case of the market index. J Bus **61**, 451–472 (1988)

Barndorff-Nielsen, O.: Exponentially decreasing distributions for the logarithm of particle size. Proc R Soc Lond A **353**, 401–419 (1977)

Bauer, C.: Value at risk using hyperbolic distributions. J Econ Bus **52**, 455–467 (2000)

Block, H.W., Li, Y., Savits, T.H.: Mixtures of normal distributions: modality and failure rate. Stat Probab Lett **74**, 253–264 (2005)

Brown, B.W., Spears, F.M., Levy, L.B.: The logF: a distribution for all seasons. Comput Stat **17**, 47–58 (2002)

Campbell, J.Y., Lo, A.W., MacKinlay, A.C.: Econometrics of Financial Markets. Princeton: Princeton University Press (1997)

Cont, R.: Empirical properties of asset returns: stylized facts and statistical issues. Quant Financ **I**, 223–236 (2001)

Dallal, G.E., Wilkinson, L.: An analytic approximation to the distribution of Lilliefors' test for normality. Am Stat **40**, 294–296 (1986)

Dempster, A.P., Laird, N.M., Rubin, D.B.: Maximum likelihood from incomplete data via the EM algorithm. J R Stat Soc B **39**, 1–38 (1977)

Drăgulescu, A.A., Yakovenko, V.M.: Probability distribution of returns in the Heston model with stochastic volatility. Quant Financ **2**, 443–453 (2002)

Eberlein, E., Keller, U.: Hyperbolic distributions in finance. Bernoulli **1**, 281–299 (1995)

Eijgenhuijsen, H., Buckley, A.: An overview of returns in Europe. Eur J Financ **5**, 276–297 (1999)

Everitt, B.S., Hand, D.J.: Finite Mixture Distributions. London: Chapman & Hall (1981)

Guillaume, D.M., Dacorogna, M.M., Davé, R.R., Müller, U.A., Olsen, R.B., Pictet, O.V.: From the bird's eye to the microscope: a survey of new stylized facts of the intra-day foreign exchange markets. Financ Stoch **1**, 95–130 (1997)

Heston, S.L.: A closed-form solution for options with stochastic volatility with applications to bond and currency options. Rev Financ Stud **6**, 327–343 (1993)

Kearns, P., Pagan, A.: Estimating the density tail index for financial time series. Rev Econ Stat **79**, 171–175 (1997)

Kon, S.: Models for stock returns—a comparison. J Financ **39**, 147–165 (1984)

Kotz, S., Kozubowski, T., Podgorski, K.: The Laplace Distribution and Generalizations: A Revisit with Applications to Communications, Economics, Engineering, and Finance. Boston: Birkhäuser (2001)

Küchler, U., Neumann, K., Sørensen, M., Streller, A.: Stock returns and hyperbolic distributions. Math Comput Model **29**, 1–15 (1999)

Longin, F.M.: The asymptotic distribution of extreme stock returns. J Bus **96**, 383–408 (1996)

McLachlan, G.J., Krishnan, T.: The EM Algorithm and Extensions. New York: Wiley (1997)

Peiro, A.: The distribution of stock returns: international evidence. Appl Financ Econ **4**, 431–439 (1994)

Rehmer, R., Mahnke, R.: Application of the Heston and Hull-White models to German DAX data. Quant Financ **4**, 685–693 (2004)

Shorack, G.R., Wellner, J.A.: Empirical Processes with Applications to Statistics. New York: Wiley (1986)

Silva, A.C., Prange, R.E., Yakovenko, V.M.: Exponential distribution of financial returns at mesoscopic time lags: a new stylized fact. Phys A **344**, 227–235 (2004)

Takahashi, H.: Ehrenfest model with large jumps in finance. Phys D **189**, 61–69 (2004)

Vicente, R., de Toledo, C.M., Leite, V.B.P., Caticha, N.: Underlying dynamics of typical fluctuations of an emerging market price index: the Heston model from minutes to months. Phys A **361**, 272–288 (2006)

Wu, J.-W., Hung, W.-L., Lee, H.-M.: Some moments and limit behaviors of the generalized logistic distribution with applications. Proc Natl Sci Counc Repub China Ser A **24**, 7–14 (2000)