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Three Notes About Linear Stochastic Oscillator Equations

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Abstract

This paper considers three notes about linear stochastic oscillator equations. First, we derive the exact and numerical approximations for the expectation of quadratic forms of solution of linear stochastic oscillator equations. This extends Strømmen Melbø and Higham's result (2004). The second note considers the same problem under measurement error assumptions and in the third one, we add a damping term to stochastic differential equations.

Keywords: Damping term; Ito lemma; Linear stochastic oscillator; Measurement error; Numerical approximation; Quadratic form

1 Introduction. Stochastic differential equations are strong tools for analyzing the continuous stochastic processes. They are used in modeling investment finance, population dynamics, biological waste treatment, neuronal models. They have been applied for modeling phenomena in some fields of engineering, for example, to model the fatigue damage of materials or fluid mechanical turbulence or modeling movement of a stochastic oscillator. There are many strong approaches to derive the analytical solution of a SDE. However, an important objective in this field is to achieve reasonable approximations to the solution of a SDE. There are many good approximation methods, for example the Euler method, the Milstein procedure and the Runge Kutta approach. These methods and their error estimates have been received considerable attentions in the literatures, see Kloeden and Platen (1999), Oksendal (2000) and Henderson and Plaschko (2006) among the others.

Strømmen Melbø and Higham (2004) (hereafter SMH) studied the expectation of

$$g_t = x_t^2 + y_t^2,$$

and its numerical approximations where $\mathbf{z}_t = (x_t, y_t)^{\mathbf{T}}$ satisfies the following two-dimensional stochastic differential equation

$$\begin{cases} dx_t = y_t dt \\ dy_t = -x_t dt + \sigma dw_t, \end{cases}$$

at which $\sigma > 0$ and w_t is standard Wiener process. Let $x_0 = 1, y_0 = 0$. Notation $\mathbf{T}$ stands for the transpose of a vector or matrix. This system of differential equations is referred to stochastic oscillator equation. In this note, we first study the exact and numerical approximation of the expectation of a quadratic form

$$f_t = \mathbf{z}_t^{\mathbf{T}} A \mathbf{z}_t,$$

where A is a symmetry, positive definite matrix. Then, we obtain the expectation of g_t under measurement error problems. Finally, we add a damping term to the stochastic differential equations. We first solve SMH's problem by our approach. The multivariate Ito lemma implies that

$$dg_t = g_x dx + g_y dy + (1/2)(g_{xx}(dx)^2 + g_{yy}(dy)^2 + 2g_{xy} dx dy),$$

where g_x is the partial derivative of g with respect to x_t. The g_y, g_{xx}, g_{yy} and g_{xy} are defined analogously. One can see that $(dx)^2 = dxdy = 0$, $(dy)^2 = \sigma^2 dt$ and $g_x = 2x$, $g_y = 2y$ and $g_{yy} = 2$. Therefore, we have

$$dg_t = 2(x_t dx_t + y_t dy_t) + \sigma^2 dt.$$

It is easy to see that

$$x_t dx_t + y_t dy_t = \sigma y_t dw_t,$$

and then $dg_t = 2\sigma y_t dw_t + \sigma^2 dt$. Then,

$$g_t = g_0 + \sigma^2 t + 2\sigma \int_0^t y_s dw_s.$$

Since $g_0 = x_0^2 + y_0^2 = 1 + 0 = 1$ then

$$g_t = 1 + \sigma^2 t + 2\sigma \int_0^t y_s dw_s.$$

Once can see that

$$E(g_t) = 1 + \sigma^2 t.$$

Theorem 1 in SMH (2004) states the same result. They solved the equations and derived the expectation, instead we used the Ito lemma. This approach helps us to extend this problem to other cases, as follows. A natural extension to g_t is f_t, the quadratic form of $\mathbf{z}_t$. Here, we derive the expectation of f_t under SMH's equations. Suppose that

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

Then,

$$f_x = 2(ax + by)\ ,\ f_y = 2(cy + bx),$$

and $f_{xx} = 2a$, $f_{yy} = 2c$ and $f_{xy} = 2b$. Therefore, using Ito lemma and integrating with respect to x, we have

$$f_t = a + c\sigma^2 t + 2\int_0^t h_1(x_s, y_s) ds + 2\sigma \int_0^t h_2(x_s, y_s) dw_s,$$

where

$$h_1(x_s, y_s) = (a - c)x_s y_s + b(y_s^2 - x_s^2) \text{ and } h_2(x_s, y_s) = bx_s + cy_s.$$

It is seen that

$$E(f_t) = a + c\sigma^2 t + 2E(\int_0^t h_1(x_s, y_s) ds),$$

where

$$E(\int_0^t h_1(x_s, y_s) ds) = (a - c)\int_0^t E(x_s y_s) ds + b\int_0^t E(y_s^2 - x_s^2) ds.$$

By letting $a = c = 1$ and $b = 0$, the previous result is obtained. Twice use of the Ito lemma proves that

$$E(x_s y_s) = -\cos(s)\sin(s) + \frac{\sigma^2}{2}(1 - \cos^2(s)),$$

and $E(y_s^2 - x_s^2) = -\cos(2s) - \frac{\sigma^2}{2}\sin(2s)$. However, note that calculating the $E(x_s y_s)$ and $E(y_s^2 - x_s^2)$ is straightforward. To see them, note that

$$\begin{aligned} E(x_s y_s) &= E[(\cos(s) + \sigma \int_0^s \sin(s-t) dw_t) \\ &\quad \times(-\sin(s) + \sigma \int_0^s \cos(s-t) dw_t)] \\ &= -\cos(s)\sin(s) + \sigma^2 \int_0^s \sin(s-t)\cos(s-t) dt \\ &= -\cos(s)\sin(s) + \frac{\sigma^2}{2}(1 - \cos^2(s)). \end{aligned}$$

Also, one can see that

$$E(y_s^2 - x_s^2) = -\cos(2s) - \frac{\sigma^2}{2}\sin(2s).$$

Remark. One can see that $f_t = ax_t^2 + 2bx_t y_t + cy_t^2$. Therefore, the direct calculation of $E(f_t)$ seems to be more easier than the above mentioned method. However, it is worth noting that this method uses only the multivariate version of Ito lemma. Authors believe that this approach may be useful for the other problems in stochastic differential equation field.

2 Extensions. In this section, we propose some other extensions to problem considered by SMH (2004). They include Measurement error and equations with damping terms cases. In each case, the exact value of expectations and their numerical approximations are given.

2.1 Measurement error. As another extension, we consider the measurement error case. Here, we assume that data process dx_t is observed with errors, that is dx_t, itself, obeys another stochastic differential equation, i.e.,

$$\begin{cases} dx_t = y_t dt + \delta dw_{1t} \\ dy_t = -x_t dt + \sigma dw_{2t}, \end{cases}$$

at which w_1 and w_2 are two correlated Brownian motions such that $dw_1 \cdot dw_2 = \rho dt$. This problem has been studied by many authors, for example see Baltazar-Larios and Sørensen (2009) and Habibi *et al.* (2010). The Ito lemma implies that

$$dg = g_x dx + g_y dy + (1/2)(g_{xx}(dx)^2 + g_{yy}(dy)^2 + 2g_{xy} dx dy).$$

One can see that $(dx)^2 = \delta^2 dt$, $(dy)^2 = \sigma^2 dt$, $dxdy = \delta\sigma\rho dt$ and $g_{xx} = g_{yy} = 2$ and $g_{xy} = 0$. Beside this,

$$g_x dx + g_y dy = 2(\delta x_t dw_{1t} + \sigma y_t dw_{2t}),$$

then, we have

$$g_t = 1 + (\delta^2 + \sigma^2)t + 2\delta \int_0^t x_s dw_{1s} + 2\sigma \int_0^t y_s dw_{2s}.$$

It is seen that

$$E(g_t) = 1 + (\delta^2 + \sigma^2)t.$$

For numerical approximation, let Δt be integral step-size required in Euler-Maruyama (EM) procedure and $t_n = n\Delta t$, $n = 1, 2, \ldots$. Here, the EM approximation is

$$\begin{cases} x_{n+1} = x_n + \Delta t y_n + \delta \Delta w_{1n} \\ y_{n+1} = y_n - \Delta t x_n + \sigma \Delta w_{2n}, \end{cases}$$

where $x_n = x(t_n)$, $y_n = y(t_n)$ and $\Delta w_{in} = w_i(t_{n+1}) - w_i(t_n)$, $i = 1, 2$. Following SMH (2004), since Δw_{in} and x_n and y_n (for each i) are independent, then

$$\begin{aligned} E(g_{n+1}) &= (1 + (\Delta t)^2)E(g_n) + (\delta^2 + \sigma^2)\Delta t \\ &\geq (1 + (\Delta t)^2)E(g_n). \end{aligned}$$

Hence,

$$E(g_n) \geq (1 + (\Delta t)^2)^n \geq e^{(\Delta t/2)t_n}.$$

This shows that for small Δt, the squared length of $\mathbf{z}_n$, the EM solution of stochastic differential equation, will grow exponentially with t_n. The Backward EM (BEM) procedure gives the following solutions

$$\begin{cases} x_{n+1} = x_n + \Delta t y_{n+1} + \delta \Delta w_{1n} \\ y_{n+1} = y_n - \Delta t x_{n+1} + \sigma \Delta w_{2n}, \end{cases}$$

One can see that

$$E(g_{n+1}) = \frac{1}{1 + (\Delta t)^2}\{E(g_n) + (\delta^2 + \sigma^2)\Delta t\}.$$

This fact implies that $E(g_n)/t_n \longrightarrow 0$ as n goes to infinity.

2.2 Equations with damping term. SMH (2004) ignored the damping term in second order stochastic differential equations. Here, we consider the following equations at which the damping term is added. They are given by

$$\begin{cases} dx_t = y_t dt \\ dy_t = (-x_t + \beta y_t)dt + \sigma dw_t. \end{cases}$$

Some algebra manipulations show that

$$g_t = 1 + \sigma^2 t + 2\beta \int_0^t y_s^2 ds + 2\sigma \int_0^t y_s dw_s.$$

It is seen that

$$E(g_t) = 1 + \sigma^2 t + 2\beta \int_0^t E(y_s^2) ds.$$

Suppose that g_n is the EM solution of above mentioned stochastic differential equation. Following SMH (2004), we can show that

$$E(g_n) \geq e^{(\Delta t/2)t_n},$$

and for BEM solution again it can be shown that $E(g_n)/t_n \longrightarrow 0$as n goes to infinity.

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