Series of a function using Integration by parts

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Abstract

It is well known that some integrations can be solved by "integration by parts" using L-I-A-T-E method. But what if this sequence is not followed? Can some interesting results be derived using this? The present paper discuss the same. Using integration by parts, expression (1) is derived. The present paper is based on this expression.

Some Theorems

- This theorem section includes three theorems.
  - In the first theorem the development of series is done using integration by parts.
  - Second theorem gives the generalized version of first theorem.
  - The last theorem discusses the convergence of a series.

Theorem

Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous, real valued and n times differentiable function. Then

$$g(x) = g(0) + \sum_{r=1}^{n} (-1)^{r-1} \frac{x^r}{r!} \cdot g^{(r)}(x) + R_{n+1}(x); \qquad x \in \mathbb{R}, \qquad (1)$$

where $R_{n+1}(x) = (-1)^n \int_0^x g^{(n+1)}(t) \cdot \frac{t^n}{n!}\, dt.$

Proof.

By considering the term $\int_0^x g^{(1)}(x)\, dt$ and evaluating it using integration by parts the expression (1) can be obtained. □

Theorem

Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous, real valued and n times differentiable function. Let "a" be a fixed real number. Also, consider the term
$R_{k+1}(x) = (-1)^k \int_a^x \frac{(t-a)^k}{k!} \cdot g^{(k+1)}(t)\, dt.$ *Then*

$$g(x) = g(a) + \sum_{r=1}^{n} (-1)^{r-1} \frac{(x-a)^r}{r!} \cdot g^{(r)}(x) + R_{n+1}(x), \qquad x \in \mathbb{R}. \tag{2}$$

Proof.

From the assumption, $R_k(x)$ is obtained and evoluating it using integration by parts the following expression is obtained.

$$R_k(x) - R_{k+1}(x) = (-1)^{k-1} \frac{(x-a)^k}{k!} \cdot g^{(k)}(x)$$

Now, by putting $k = 1, 2, 3, \cdots, n$ and adding all the equations the expression (2) is achieved.
By putting $a = 0$ in the expression (2), it reduces to an expression (1). So, the expression (2) is the generalized version of expression (1).

$\square$

Theorem

The series of $g(x)$ given in equation (1) converges to $g(x)$ if and only if $R_n(x) \to 0$ as $n \to \infty$.

Proof.

Equation (1) can be written as

$$g(x) = S_n(x) + R_n(x), \tag{3}$$

where $R_n(x)$ is the remainder after n term described in equation (1) and

$$S_n(x) = g(0) + \sum_{r=1}^{n-1} (-1)^{(r-1)} \frac{x^r}{r!} g^{(r)}(x). \tag{4}$$

The right hand side of equation (3) converges to $g(x)$, then $S_n(x) \to g(x)$ as $n \to \infty$. This implies and is implied by $\lim_{n\to\infty} R_n(x) = 0$, when $S_n(x) = g(x) - R_n(x)$. $\square$

Examples

- This example section contains three examples.
  - In first two examples the analysis of error is done by taking some functions and the series expressions are derived for the same functions.
  - In last example the series expression of the trigonometric functions are shown.

a^x ...

Consider $g(x) = a^x$, $a \geq 1$. The r^{th} derivative of a^x can be express as,

$$g^{(r)}(x) = a^x (\log a)^r, \qquad r \in \mathbb{N}.$$

Now, substitute the above values of $g^{(r)}(x)$ in equation (4), we can have $S_{n+1}(x) = 1 + \sum_{r=1}^{n} (-1)^{(r-1)} \frac{x^r}{r!} \cdot a^x (\log a)^r$. So, from the equation (3)

$$a^x = 1 + \sum_{r=1}^{\infty} (-1)^{r-1} \frac{x^r}{r!} \cdot a^x (\log a)^r \tag{5}$$

because $R_{n+1}(x) \to 0$ as $n \to \infty$. Now, by taking $a = e$ in equation (5),

$$e^{-x} = \sum_{r=0}^{\infty} (-1)^r \frac{x^r}{r!}.$$

By putting $(-x)$ instead of x in this expression, we can have $e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!}$, which is the power series of e^x.

From the equation of remainder term, $R_{n+1}(x) = (-1)^n \int_0^x \frac{t^n}{n!} \cdot \frac{d^{n+1}}{dt^{n+1}} a^t \, dt.$

Error Analysis

Now, The remainder term can be evaluated using identity $|t|^n < |x|^n$, whenever $|t| < |x|$.

$$\begin{aligned} |R_{n+1}(x)| &\leq \frac{|\log a|^{n+1}}{n!} \int_0^x |t|^n \cdot |a|^t \, dt \\ &\leq \frac{|\log a|^{n+1} \cdot |x|^n}{n!} \int_0^x |a|^t \, dt = \frac{|\log a|^n}{n!} |x|^n \left(|a|^x - 1\right) \to 0 \text{ as } n \to \infty. \end{aligned}$$

For $a = e$ and $x = 1.5$, $g(1.5) = e^{1.5} \approx 4.4816891$. The values of $R_{n+1}(x)$ for $n = 2, 4, 6$ and 18 are as follows [2, 5] :

n	2	4	6	18
$S_{n+1}(x)$	2.6806334	4.2562272	4.4689324	4.4816891
$R_{n+1}(x)$	1.8010557	0.22546186	0.01275667	≈ 0

By considering $g(x) = \log(1+x), 0 \le x \le 1$, we can have the following expression

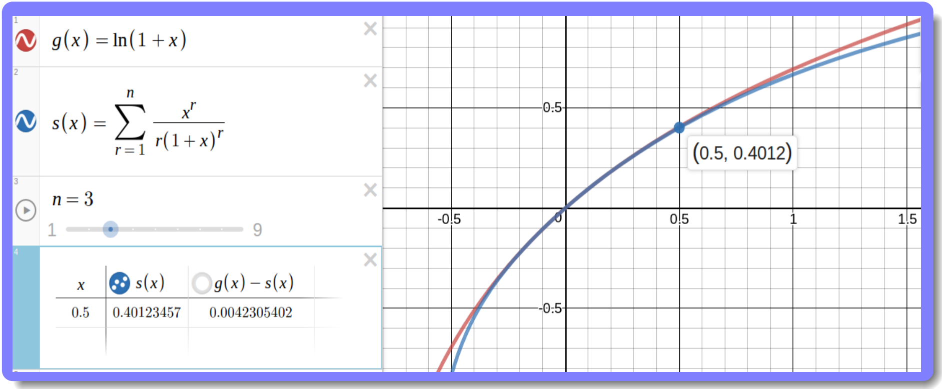
$$\log(1+x) = \sum_{r=1}^{\infty} \frac{x^r}{r(1+x)^r}.$$

Error Analysis

For $x = 1$, we have $g(1) = \log(1+1) = 0.6931471$. The values of $R_{n+1}(x)$ for $n = 2, 4, 6$ and 18 are as follows [2, 5] :

n	2	4	6	18
$S_{n+1}(x)$	0.625	0.6822917	0.6911458	0.69314699
$R_{n+1}(x)$	0.06814718	0.01085551	0.002001347	1.915792e-7

... $\log(1 + x)$



$g(x) = \ln(1 + x)$
$s(x) = \sum_{r=1}^{n} \frac{x^r}{r(1+x)^r}$
$n = 3$
1
9
x
$s(x)$
$g(x) - s(x)$
0.5
0.40123457
0.0042305402
0.5
-0.5
0
1
1.5
(0.5, 0.4012)

Trigonometric functions ...

By considering $g(x) = \sin x$ the series expressions for the all trigonometric functions can be derived.

$$\sin x = \frac{l_1}{\sqrt{l_1^2 + (1 - l_2)^2}} \qquad \cos x = \frac{1 - l_2}{\sqrt{l_1^2 + (1 - l_2)^2}},$$

$$\tan x = \frac{l_1}{1 - l_2} \qquad \cot x = \frac{1 - l_2}{l_1},$$

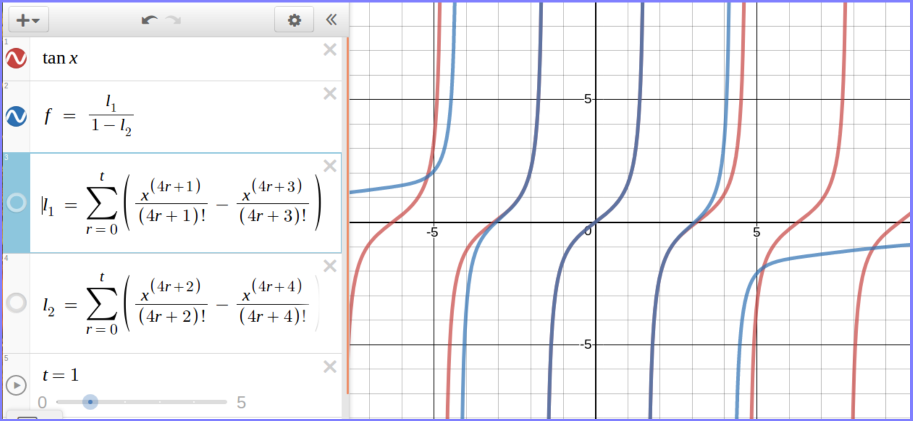
$$\sec x = \frac{\sqrt{l_1^2 + (1 - l_2)^2}}{1 - l_2} \qquad \csc x = \frac{\sqrt{l_1^2 + (1 - l_2)^2}}{l_1}.$$

.

where $l_1 = \sum_{r=0}^{\infty} \left(\frac{x^{4r+1}}{(4r+1)!} - \frac{x^{4r+3}}{(4r+3)!} \right)$ and $l_2 = \sum_{r=0}^{\infty} \left(\frac{x^{4r+2}}{(4r+2)!} - \frac{x^{4r+4}}{(4r+4)!} \right)$.

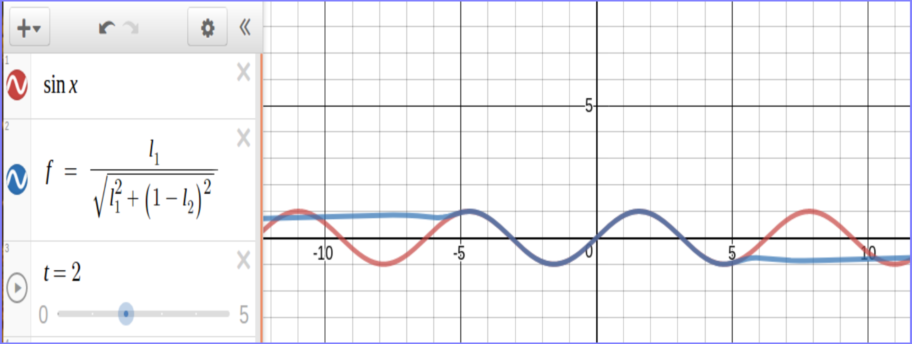
... Trigonometric functions ...

Graphically, the convergence of $\tan x = \dfrac{l_1}{1 - l_2}$ is as shown.



... Trigonometric functions

Similarly, the convergence of $\sin x = \dfrac{l_1}{\sqrt{l_1^2 + (1 - l_2)^2}}$ is as shown.



Result

The constant π can be derived using an expression (2) by taking $g(x) = \tan^{-1} x$ and $a = 1$,

$$\tan^{-1} x = \tan^{-1} 1 + \sum_{r=1}^{\infty} (-1)^{r-1} \frac{(x-1)^r}{r!} \left(\frac{d^r}{dx^r} \tan^{-1} x \right).$$

because $R_{n+1}(x) \to 0$ as $n \to \infty$. The expansion of $\tan^{-1} x$ is given by

$$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}, \qquad \text{where} \quad |x| \leq 1$$

and $\tan^{-1} 1 = \frac{\pi}{4}$. So,

$$\frac{\pi}{4} = \tan^{-1} x - \sum_{r=1}^{\infty} \left[(-1)^{r-1} \frac{(x-1)^r}{r!} \left(\frac{d^r}{dx^r} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \right) \right].$$

Now, the r^{th} derivative of $\tan^{-1} x$ is

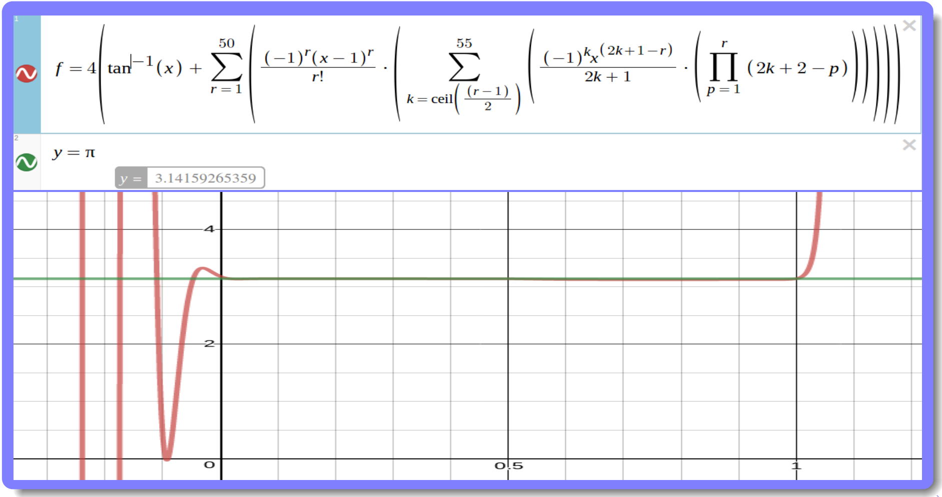
$$\frac{d^r}{dx^r}\left(\sum_{k=0}^{\infty}(-1)^k \cdot \frac{x^{2k+1}}{2k+1}\right) = \sum_{k=\lceil \frac{r-1}{2} \rceil}^{\infty}\left[\left\{\prod_{p=1}^{r}((2k+1)-(p-1))\right\} \cdot \frac{(-1)^k \cdot x^{(2k+1)-r}}{2k+1}\right].$$

So, the value of π can be written as

$$\pi = 4\left[\tan^{-1} x + \sum_{r=1}^{\infty}\left[(-1)^r \frac{(x-1)^r}{r!}\left[\sum_{k=\lceil \frac{r-1}{2} \rceil}^{\infty}\left(\frac{(-1)^k}{2k+1} \cdot x^{(2k+1)-r} \cdot \left\{\prod_{p=1}^{r}((2k+1)-(p-1))\right\}\right)\right]\right]\right]$$

for all $0 < x < 1$.

In the graph the value of f converges to value of π for $x \in (0, 1)$.



References

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THANK YOU