Linear Stability Analysis

One-Layer Shallow Water

In this subsection we consider the one-layer reduced gravity RSW model with topography below. We define the following:

- H: mean depth of the layer
- $z = \eta$: height of the free surface
- $z = -H + \eta_B$: height of the topography.
- $h = H + \eta - \eta_B$: total depth of layer
- (u, v): horizontal velocity
- g': reduced gravity
- ρ_0: reference density

The governing nonlinear equations are,

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - fv &= -g \frac{\partial}{\partial x} \left(h + \eta_B \right), \\
\frac{\partial v}{\partial t} + \vec{u} \cdot \vec{\nabla} v + fu &= -g \frac{\partial}{\partial y} \left(h + \eta_B \right), \\
\frac{\partial h}{\partial t} + \vec{\nabla} \cdot (h \vec{u}_1) &= 0.
\end{aligned}
$$

Basic State

To study shear flows in a meridional channel we consider solutions of the form,

$$
\begin{aligned}
u &= U_B(y), \\
v &= 0, \\
h &= H_B(y).
\end{aligned}
$$

For this to be an exact solution we require that the flow is in geostrophic balance,

$$
fU_B = -g \frac{d}{dy} \left(H_B + \eta_B \right).
$$

Perturbation

We perturb the basic state with infinitesimal quantities,

$$
\begin{aligned}
u &= U_B(y) + u', \\
v &= 0 + v', \\
h &= H_B(y) + h'.
\end{aligned}
$$

We substitute our perturbation into the governing equations and drop the primes (for brevity) and cancelling out the geostrophic terms

$$
\begin{aligned}
\frac{\partial u}{\partial t} + (u + U_B) \frac{\partial u}{\partial x} + v \frac{\partial}{\partial y} \left(u + U_B \right) - fv &= -g \frac{\partial h}{\partial x}, \\
\frac{\partial v}{\partial t} + (u + U_B) \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + fu &= -g \frac{\partial h}{\partial y}, \\
\frac{\partial h}{\partial t} + (u + U_B) \frac{\partial h}{\partial x} + v \frac{\partial}{\partial y} (H_B + h) + (H_B + h) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0.
\end{aligned}
$$

Now we neglect the quadratic terms to obtain the linearized equations,

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= -U_B \frac{\partial u}{\partial x} + \left(f - \frac{dU_B}{dy} \right) v - g \frac{\partial h_1}{\partial x}, \\
\frac{\partial v}{\partial t} &= -fu - U_B \frac{\partial v}{\partial x} - g \frac{\partial h_1}{\partial y}, \\
\frac{\partial h}{\partial t} &= -H_B \frac{\partial u}{\partial x} - v \frac{dH_B}{dy} - H_B \frac{\partial v}{\partial y} - U_B \frac{\partial h}{\partial x}.
\end{aligned}
$$

Finally, we assume a normal mode decomposition in the zonal direction and time,

$$
[u, v, h] = \mathrm{Re} \left\{ e^{ik(x-ct)} [\hat{u}, ik\hat{v}, \hat{h}] \right\}, \tag{1}
$$

which we can substitute into the above equations to yield in the inviscid limit

$$
\begin{aligned}
c\hat{u} &= U_B \hat{u} - (f - \frac{dU_B}{dy}) \hat{v} + g\hat{h}, \\
c\hat{v} &= -\frac{f}{k^2} \hat{u} + U_B \hat{v} - \frac{g}{k^2} \frac{dh}{dy}, \\
c\hat{h} &= H_B \hat{u} + \frac{d}{dy} (H_B \hat{v}) + U_B \hat{h}.
\end{aligned}
$$