Layered quasigeostrophic model

The N-layer quasigeostrophic (QG) potential vorticity is

$$
\begin{aligned}
q_1 &= \nabla^2\psi_1 + \frac{f_0^2}{H_1}\left(\frac{\psi_2-\psi_1}{g_1'}\right), && i=1\,,\\
q_n &= \nabla^2\psi_n + \frac{f_0^2}{H_n}\left(\frac{\psi_{n-1}-\psi_n}{g_{n-1}'} - \frac{\psi_i-\psi_{n+1}}{g_i'}\right), && i=2,\mathrm{N}-1\,,\\
q_\mathrm{N} &= \nabla^2\psi_\mathrm{N} + \frac{f_0^2}{H_\mathrm{N}}\left(\frac{\psi_{\mathrm{N}-1}-\psi_\mathrm{N}}{g_{\mathrm{N}-1}'}\right) + \frac{f_0}{H_\mathrm{N}}h_b(x,y)\,, && i=\mathrm{N}\,,
\end{aligned}
\tag{1}
$$

where q_n is the n'th layer QG potential vorticity, and ψ_n is the streamfunction, f_0 is the inertial frequency, n'th H_n is the layer depth, and h_b is the bottom topography. (Note that in QG $h_b/H_\mathrm{N} << 1$.) Also the n'th buoyancy jump (reduced gravity) is

$$
g_n' \equiv g\frac{\rho_n-\rho_{n+1}}{\rho_n}\,, \tag{2}
$$

where g is the acceleration due to gravity and ρ_n is the layer density.

The dynamics of the system is given by the evolution of PV. We introduce a background flow that can vary in the horizontal. The streamfunction associated with this flow can be denoted with $\Psi_n(x,y)$ for each layer and geostrophy yields its corresponding velocity $\vec{V_n} = (U_n(x,y), V_n(x,y))$ where $\Psi_{ny} = -U_n$ and $\Psi_{nx} = V_n$. We can perturb the stream function in each layer into a background flow and deviations from that flow as,

$$
\psi_n^{\mathrm{tot}} = \Psi_n + \psi_n. \tag{3}
$$

With this basic decomposition we can than write out the corresponding decompositions in velocity

$$
\begin{aligned}
u_n^{\mathrm{tot}} &= U_n - \psi_{ny}\,,\\
v_n^{\mathrm{tot}} &= V_n + \psi_{nx}\,,
\end{aligned}
\tag{4}
$$

and

$$
q_n^{\mathrm{tot}} = Q_n + \delta_{n\mathrm{N}}\frac{f_0}{H_\mathrm{N}}h_b + q_n\,, \tag{5}
$$

where $Q_n + \delta_{n\mathrm{N}}\frac{f_0}{H_\mathrm{N}}h_b$ is n'th layer background PV, we obtain the evolution equations

$$
\begin{aligned}
q_{nt} + \mathsf{J}(\psi_n, q_n + \delta_{n\mathrm{N}}\frac{f_0}{H_\mathrm{N}}h_b) + U_n(q_{nx} + \delta_{n\mathrm{N}}\frac{f_0}{H_\mathrm{N}}h_{bx}) + V_n(q_{ny} + \delta_{n\mathrm{N}}\frac{f_0}{H_\mathrm{N}}h_{by}) +&\\
Q_{ny}\psi_{nx} - Q_{nx}\psi_{ny} = \mathrm{ssd} - r_{ek}\delta_{n\mathrm{N}}\nabla^2\psi_n\,, \qquad n = 1,\mathrm{N}\,,&
\end{aligned}
\tag{6}
$$

where ssd is stands for small scale dissipation, which is achieved by an spectral exponential filter or hyperviscosity, and r_{ek} is the linear bottom drag coefficient. The Dirac delta, δ_{nN}, indicates that the drag is only applied in the bottom layer.

Linear Stability Analysis

In order to study the stability of a jet in the context of our n-layer QG model we focus our attention on basic states that consist of zonal flows. i.e. $\Psi_n(y)$ only. If we assume that the quadratic quantities we can then linearize to obtain in the conservative limit over a flat bottom,

$$
q_{nt} + U_n q_{nx} + Q_{ny}\psi_{nx} = 0, \tag{7}
$$

for $n = 1, \cdots, N$.

We assume that the perturbations are normal modes in the zonal direction and time,

$$\psi_n = \mathrm{Re}[\hat{\psi}_n e^{i(kx-\omega t)}].$$

This implies that the PV will be modified appropriately and we denote it with $\hat{q}_n$.

We substitute this into the linear equations and then divide by the exponential to obtain,

$$c\hat{q}_n = U_n \hat{q}_n + Q_{ny}\hat{\psi}_n, \tag{8}$$

where the basic state only depends on y, and layer of course, and we have introduced the phase speed $c = \omega/k$. Note that the actual PVs are

$$\begin{aligned}
\hat{q}_1 &= (\partial_{yy} - k^2)\hat{\psi}_1 + \frac{f_0^2}{H_1}\left(\frac{\hat{\psi}_2 - \hat{\psi}_1}{g'_1}\right), && i = 1\,,\\
\hat{q}_n &= (\partial_{yy} - k^2)\psi_n + \frac{f_0^2}{H_n}\left(\frac{\hat{\psi}_{n-1} - \hat{\psi}_n}{g'_{n-1}} - \frac{\hat{\psi}_i - \hat{\psi}_{n+1}}{g'_i}\right), && i = 2, \mathrm{N}-1\,,\\
\hat{q}_\mathrm{N} &= (\partial_{yy} - k^2)\hat{\psi}_\mathrm{N} + \frac{f_0^2}{H_\mathrm{N}}\left(\frac{\hat{\psi}_{\mathrm{N}-1} - \hat{\psi}_\mathrm{N}}{g'_{\mathrm{N}-1}}\right), && i = \mathrm{N}\,,
\end{aligned} \tag{9}$$

Special case: one-layer model

In the one-layer case we have

$$c\hat{q}_1 = U_1\hat{q}_1 + Q_{1y}\hat{\psi}_1, \tag{10}$$

$$\hat{q}_1 = \left[\partial_{yy} - k^2 - \frac{f_0^2}{g'_1 H_1}\right]\hat{\psi}_1. \tag{11}$$

Special case: two-layer model

In the two-layer case we have

$$c\hat{q}_n = U_n\hat{q}_n + Q_{ny}\hat{\psi}_n, \tag{12}$$

$$\hat{q}_1 = \left[\partial_{yy} - k^2 - \frac{f_0^2}{g'_1 H_1}\right]\hat{\psi}_1 + \frac{f_0^2}{g'_1 H_1}\hat{\psi}_2, \tag{13}$$

$$\hat{q}_2 = \frac{f_0^2}{g'_1 H_2}\hat{\psi}_1 + \left[\partial_{yy} - k^2 - \frac{f_0^2}{g'_1 H_2}\right]\hat{\psi}_2. \tag{14}$$